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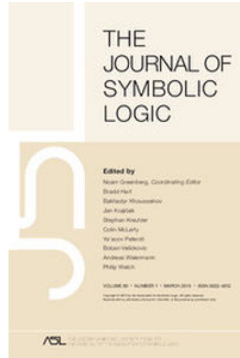


# Similar but not the same: various versions of ♣ do not coincide

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# SIMILAR BUT NOT THE SAME: VARIOUS VERSIONS OF ♣ DO NOT COINCIDE

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**Abstract.** We consider various versions of the ♣ principle. This principle is a known consequence of $\diamondsuit$. It is well known that $\diamondsuit$ is not sensitive to minor changes in its definition, e.g., changing the guessing requirement form "guessing exactly" to "guessing modulo a finite set". We show however, that this is not true for ♣. We consider some other variants of ♣ as well.

**§1. Introduction.** In this paper we consider various natural variants of ♣ principle. We answer questions of S. Fuchino and M. Rajagopalan.

The principle was introduced by A. Ostaszewski in [7]. It is easy to see that ♣ follows from $\diamondsuit$, and in fact it is true that $\diamondsuit$ is equivalent to ♣ + CH, by an argument of K. Devlin presented in [7]. By ([10, §5]) $\diamondsuit$ and ♣ are not equivalent, that is, it is consistent to have ♣ without having CH. Subsequently J. Baumgartner, in an unpublished note, gave an alternative proof, via a forcing which does not collapse $\aleph_1$ (unlike the forcing in [10]). P. Komjáth [5], continuing the proof in [10, §5] proved it consistent to have MA for countable partial orderings $+\neg$CH, and ♣. Then S. Fuchino, S. Shelah and L. Soukup [2] proved the same, without collapsing $\aleph_1$.

The original R. Jensen's formulation of $\diamondsuit$ ([3]) is about the existence of a sequence $\langle A_\delta : \delta < \omega_1 \rangle$ such that every $A_\delta$ is an unbounded subset of $\delta$, and for every $A \in [\omega_1]^{\aleph_1}$, we have $A \cap \delta = A_\delta$ stationarily often. Many equivalent reformulations can be obtained by using coding techniques (see [6]). As a well known example, we mention K. Kunen's proof ([6]) that $\diamondsuit^-$ is equivalent to $\diamondsuit$. Here $\diamondsuit^-$ is the version of $\diamondsuit$ which says that there is a sequence

$$\langle \{ A^\delta_n : n < \omega \} : \delta < \omega_1 \rangle,$$

each $A^\delta_n \subseteq \delta$, and for every $A \in [\omega_1]^{\aleph_1}$, we stationarily often have that $A \cap \delta = A^\delta_n$ for some $n$.

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We consider the question asking if ♣ has a similar invariance property. To be precise, we shall below formulate some versions of ♣, and ask if any two of them are equivalent. We are particularly interested in those versions of ♣ which have the property that the parallel version of $\diamondsuit$ is equivalent to $\diamondsuit$. The main result of the paper is that almost all of the ♣-equivalences we considered, are consistently false.

Versions of ♣ which are weaker than the ones we consider, are already known to be weaker than ♣. Namely, in his paper [4], I. Juhász considers the principle ♣′ claiming the existence of a sequence

$$\langle \langle A_n^\delta : n < \omega \rangle : \delta \text{ limit} < \omega_1 \rangle$$

where for any $\delta$ sets $\{ A_n^\delta : n < \omega \}$ are disjoint, and such that for every $A \in [\omega_1]^{\aleph_1}$ there is $\delta$ such that for all $n$ we have $\sup(A_n^\delta \cap \omega_1) = \delta$. I. Juhász shows that ♣′ is true in any extension by a Cohen real.

We heard of the question on the equivalence between ♣ and ♣$^\bullet$ from F. Tall, who heard it from J. Baumgartner. J. Baumgartner credited the question to F. Galvin, who credited it to M. Rajagopalan. And indeed, M. Rajagopalan asked this question in [8], where he introduced ♣$^\bullet$ (denoted there by ♣$_F$). In the same paper M. Rajagopalan also introduced ♣$^2$ (denoted there by ♣$^\infty$) and showed that CH + ♣$^2$ suffices to construct an Ostaszewski space. He also asked if ♣$^2$ was equivalent to ♣. The answer is negative by Theorem 2.1 below.

Most of the other equivalence questions we consider here were first asked by S. Fuchino.

We now proceed to give the relevant definitions.

DEFINITION 1.1. We define the meaning of the principle ♣$^l_\Upsilon$ for $l$ ranging in $\{0, 1, 2, \bullet\}$ and $\Upsilon$ a limit ordinal $< \omega_1$. (If $\Upsilon = \omega$ then we omit it from the notation.)

CASE 1. $l = 0$.

For some stationary set $S \subseteq \omega_1 \cap \text{LIM}$, there is a sequence $\langle A_\delta : \delta \in S \rangle$ such that

(a) $A_\delta$ is an unbounded subset of $\delta$.
(b) $\text{otp}(A_\delta) = \Upsilon$.
(c) For every unbounded $A \subseteq \omega_1$, there is a $\delta$ such that $A_\delta \subseteq A$.

CASE 2. $l = 1$.

For some stationary subset $S$ of $\omega_1 \cap \text{LIM}$, there is a sequence $\langle A_\delta : \delta \in S \rangle$ such that

(a) $A_\delta$ is an unbounded subset of $\delta$.
(b) $\text{otp}(A_\delta) = \Upsilon$.
(c) For every unbounded $A \subseteq \omega_1$, there is a $\delta$ such that $|A_\delta \setminus A| < \aleph_0$.

CASE 3. $l = 2$.

For some stationary $S \subseteq \omega_1 \cap \text{LIM}$, there is a sequence

$$\langle \{ A_n^\delta : n < \omega \} : \delta \in S \rangle$$

such that

(a) Each $A_n^\delta$ is an unbounded subset of $\delta$.

(b) $\mathrm{otp}(A^\delta_n) = \Upsilon$.
(c) For every unbounded $A \subseteq \omega_1$, there is a $\delta$ and an $n$ such that $A^\delta_n \subseteq A$.

CASE 4. $l = \bullet$.

For some stationary set $S \subseteq \omega_1 \cap \mathrm{LIM}$, there is a sequence

$$\langle \{ A^\delta_m : m \leq m^*(\delta) \} : \delta \in S \rangle$$

such that

(a) Each $A^\delta_m$ is an unbounded subset of $\delta$.
(b) $\mathrm{otp}(A^\delta_m) = \Upsilon$.
(c) For every unbounded $A \subseteq \omega_1$, there is a $\delta$ and an $m \leq m^*(\delta)$ such that $A^\delta_m \subseteq A$.
(d) For all relevant $\delta$, we have $m^*(\delta) < \omega$.

In the above, LIM stands for the class of limit ordinals.

REMARK 1.2.

(1) One could, of course, consider the previous definitions with $\omega_1$ replaced by some other uncountable ordinal, in fact an uncountable regular cardinal. As our proofs only deal with $\omega_1$, we only formulate our definitions in the form given above.

Also, we could consider principles of the form $\clubsuit^l_\Upsilon(T)$ in which $T$ is a stationary subset of $\omega_1$ and parameter $\delta$ in the above definitions is allowed to range only in $T$ (i.e., $S \cap T$).

(2) The definition that A. Ostaszewski [7] used for a $\clubsuit$-sequence $\langle A_\delta : \delta \in S \rangle$ requires that for each $A \in [\omega_1]^{\aleph_1}$ there is a stationary set of $\delta$ such that $A_\delta \subseteq A$. It is well known that this is equivalent to our definition of $\clubsuit^0$. Hence $\clubsuit^0$ is the usual $\clubsuit$ principle of Ostaszewski, and we shall often omit the superscript 0 when discussing this principle, and freely use the equivalence between the definitions.

It is obvious that $\clubsuit^0_\Upsilon \Longrightarrow \clubsuit^1_\Upsilon \Longrightarrow \clubsuit^2_\Upsilon$, and that $\clubsuit^0_\Upsilon \Longrightarrow \clubsuit^\bullet_\Upsilon \Longrightarrow \clubsuit^2_\Upsilon$. The result of the first Sections 2 and 3 of the paper is that, except for the following simple theorem, the above are the only implications that can be drawn.

THEOREM 1.3.

(1) *Suppose that* $\Upsilon_1, \Upsilon_2 < \omega_1$ *are limit ordinals and that* $\clubsuit_{\Upsilon_1}$ *and* $\clubsuit_{\Upsilon_2}$ *both hold. Then* $\clubsuit_{\Upsilon_1 \cdot \Upsilon_2}$ *holds.*
(2) $\clubsuit_{\Upsilon_1 \cdot \Upsilon_2} \Longrightarrow \clubsuit_{\Upsilon_1}$ *for* $\Upsilon_1$ *limit* $< \omega_1$ *and* $\Upsilon_2 < \omega_1$. *Similarly for the other versions of* $\clubsuit$ *considered.*

PROOF.

(1) Let $\langle A^l_\delta : \delta \in S_l \rangle$ for $l = 1, 2$ exemplify $\clubsuit_{\Upsilon_l}$. For $\delta \in \lim(S_1) \cap S_2$ we let

$$B_\delta \stackrel{\text{def}}{=} \bigcup_{\alpha \in A^2_\delta} A^1_\alpha.$$

Hence $B_\delta$ is an unbounded subset of $\delta$.

Suppose that $A \in [\omega_1]^{\aleph_1}$. For each $\alpha < \omega_1$, the set $A \setminus \alpha$ is an unbounded subset of $\omega_1$, hence contains stationarily many $A^1_\delta$ as subsets. So we can find an unbounded subset $T_1 = T_1[A]$ of $S_1$ such that

$$\alpha \in T_1 \Longrightarrow A^1_\alpha \subseteq A \setminus \sup(T_1 \cap \alpha).$$

Now we can find a $\delta \in \lim(S_1) \cap S_2$ such that $A^2_\delta \subseteq T_1$. Hence $B_\delta \subseteq A$ and $B_\delta$ is unbounded in $\delta$. Moreover, $\operatorname{otp}(B_\delta) = \Upsilon_1 \cdot \Upsilon_2$.

We have shown that $\langle B_\delta : \delta \in \lim(S_1) \cap S_2 \ \& \ \operatorname{otp}(B_\delta) = \Upsilon_1 \cdot \Upsilon_2 \rangle$ witnesses that $\clubsuit_{\Upsilon_1 \cdot \Upsilon_2}$ holds (note that the fact that the set of relevant $\delta$ is stationary follows from the previous paragraph).

(2) Easy. $\dashv$

The questions considered in the paper are answered using the same basic technique, with some changes in the definition of the particular forcing used. A detailed explanation of the technique and the way it is used to prove that $\clubsuit^1$ does not imply $\clubsuit^0$, is given in §2. The changes needed to obtain the other two theorems are presented at the end of §2 and in §3.

## §2. Consistency of $\clubsuit^1$ and $\neg\clubsuit^0$.

THEOREM 2.1. $\mathrm{CON}(\clubsuit^1 + \neg\clubsuit)$.

PROOF. Throughout the proof, $\chi$ is a fixed large enough regular cardinal.

We start with a model $V$ of ZFC such that

$$V \models \diamondsuit(\omega_1) + 2^{\aleph_1} = \aleph_2,$$

and use an iteration $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2 \ \& \ \beta < \omega_2 \rangle$. The iteration is defined in the following definition.

DEFINITION 2.2.

(1) By a candidate for a $\clubsuit$, we mean a sequence of the form $\langle A_\delta : \delta < \omega_1 \text{ limit} \rangle$, such that $A_\delta$ is an unbounded subset of $\delta$, with $\operatorname{otp}(A_\delta) = \omega$.

(2) In $V$, we fix a continuously increasing sequence of countable elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$, call it $\bar{\bar{N}} = \langle N^0_i : i < \omega_1 \rangle$, such that

$$\mathscr{H}(\aleph_1) \subseteq \bigcup_{i<\omega_1} N^0_i$$

(this is possible by CH), and $\langle N^0_j : j \leq i \rangle \in N^0_i$ for $i < \omega_1$.

(3) During the iteration, we do a bookkeeping which hands us candidates for $\clubsuit$.

(4) Suppose that $\beta < \omega_2$, and let us define $Q_\beta$, while working in $V^{P_\beta}$.

(a) Suppose that CH holds in $V^{P_\beta}$ and the bookkeeping gives us a sequence $\bar{A}^\beta = \langle A^\beta_\delta : \delta < \omega_1 \text{ a limit ordinal} \rangle$ which is a candidate for $\clubsuit$. For some club $E_\beta$ of $\omega_1$ we choose a continuously increasing sequence $\bar{N}^\beta = \langle N^\beta_i : i \in E_\beta \rangle$ of countable elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$, such that we have

$$\mathscr{H}(\aleph_1) \subseteq \bigcup_{i \in E_\beta} N^\beta_i,$$

and such that for every $i \in E_\beta$ we have $N^\beta_i \cap V = N^0_i$, while

$$\langle N^\beta_j : j \leq i \rangle \in N^\beta_{\min(E_\beta \setminus (i+1))}.$$

Furthermore, $\bar{A}^\beta \in N^\beta_{\min(E_\beta)}$. Then $Q_\beta = Q_{\beta_{\bar{A},\bar{N}^\beta}}$ is defined by

$$Q_\beta \stackrel{\text{def}}{=} \{ f : \begin{array}{ll} \text{(i)} & f \text{ is a partial function from } \omega_1 \text{ to } \{0,1\} \\ \text{(ii)} & \operatorname{otp}(\operatorname{Dom}(f)) < \omega^\omega \\ \text{(iii)} & f \restriction (N_i^\beta \cap \omega_1) \in N^\beta_{\min(E_\beta \setminus (i+1))}, \text{ for } i \in E_\beta \\ \text{(iv)} & f^{-1}(\{1\}) \cap A^\beta_\delta = \emptyset \Longrightarrow |\operatorname{Dom}(f) \cap A^\beta_\delta| < \aleph_0 \\ \text{(v)} & f \in V \}. \end{array}$$

(b) If $\neg$CH, then $Q_\beta = \emptyset$. (Of course, our situation will be such that this case never occurs.)

In $Q_\alpha$, the order is given by

$$f \leq g \iff g \text{ extends } f \text{ as a function.}$$

(5) For $\alpha \leq \omega_2$, we define inductively

$$P_\alpha \stackrel{\text{def}}{=} \{ p : \operatorname{Dom}(p) \in [\alpha]^{\leq \aleph_0} \;\&\; (\forall \beta \in \operatorname{Dom}(p))$$
$$(p(\beta) \text{ is a canonical hereditarily countable over Ord}$$
$$P_\beta\text{-name of a member of } \underset{\sim}{Q}_\beta, \text{ and } p \restriction \beta \Vdash_{P_\beta} \text{“}p(\beta) \in \underset{\sim}{Q}_\beta\text{”}) \}.$$

The order in $P_\alpha$ is given by

$$p \leq q \iff \begin{array}{ll} \text{(i)} & \operatorname{Dom}(p) \subseteq \operatorname{Dom}(q). \\ \text{(ii)} & \text{For all } \beta \leq \alpha, \text{ we have } q \restriction \beta \Vdash \text{“}p(\beta) \leq q(\beta)\text{”}. \\ \text{(iii)} & \{ \gamma \in \operatorname{Dom}(p) : p(\gamma) \neq q(\gamma) \} \text{ is finite.} \end{array}$$

DEFINITION 2.3. Suppose $\alpha \leq \omega_2$, and $p \leq q \in P_\alpha$. Then

(1) We say that $q$ *purely extends* $p$, if $q \restriction \operatorname{Dom}(p) = p$. We write $p \leq_{\text{pr}} q$.
(2) We say that $q$ *apurely extends* $p$, if $\operatorname{Dom}(p) = \operatorname{Dom}(q)$. We write $p \leq_{\text{apr}} q$.
(3) The meaning of $p \geq_{\text{pr}} q$ and $p \geq_{\text{apr}} q$ is defined in the obvious way.

DEFINITION 2.4. Suppose that $\gamma < \omega_1$. A forcing notion $P$ is said to be *purely $\gamma$-proper* if:

For every $p \in P$ and a continuously increasing sequence $\langle N_i : i \leq \gamma \rangle$ of countable elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$ with $p, P \in N_0$, $\langle N_j : j \leq i \rangle \in N_{i+1}$, there is a $q \geq_{\text{pr}} p$ which is $(N_i, P)$-generic for all $i \leq \gamma$.

FACT 2.5. *A ccc forcing notion is purely $\gamma$-proper for every $\gamma < \omega_1$.*

PROOF OF THE FACT. This is because every condition in a *ccc* forcing is generic, see [9, III, 2.6 and 2.9]. ⊣

**General facts about the iterations like the one we are using.**

FACT 2.6. *Iterations with the support we are using, have the following general properties*:

(1) $\alpha \leq \beta \Longrightarrow P_\alpha \subseteq P_\beta$ *as ordered sets.*
(2) $(\alpha \leq \beta \;\&\; q \in P_\beta) \Longrightarrow (q \restriction \alpha \in P_\alpha \;\&\; q \restriction \alpha \leq q)$.

(3) $(\alpha \leq \beta \ \&\ p \in P_\beta \ \&\ p\restriction\alpha \leq q \in P_\alpha) \Longrightarrow q \cup (p\restriction[\alpha,\beta)) \in P_\beta$ *is the least upper bound of* $p$ *and* $q$.

(4) *If* $\alpha < \beta$, *then* $P_\alpha \lessdot P_\beta$. *Hence,* $G_{P_{\alpha+1}}/G_{P_\alpha}$ *gives rise to a directed subset of* $Q_\alpha$ *over* $V[G_{P_\alpha}]$.

(5) *If* $\langle p_i : i < i^* < \omega_1 \rangle$ *is a* $\leq_{\mathrm{pr}}$*-increasing sequence in* $P_{\alpha^*}$ *for some* $\alpha^* \leq \omega_2$, *then* $p \stackrel{\text{def}}{=} \bigcup_{i<i^*} p_i$ *is a condition in* $P_{\alpha^*}$ *and for every* $i < i^*$ *we have* $p_i \leq_{\mathrm{pr}} p$.

(6) *Pure properness is preserved by the iteration. Moreover, for any* $\gamma < \omega_1$, *pure* $\gamma$*-properness is preserved by the iteration.*

PROOF OF THE FACT.

(1)–(5) Just checking.

(6) The statement follows from some more general facts proved in [9, XIV]. A direct proof can be given along the lines of the proof that countable support iterations preserve properness, [9, III, 3.2]. $\dashv$

**Back to our specific iteration.**

CLAIM 2.7. *Suppose* $\alpha^* < \omega_2$. *In* $V^{P_{\alpha^*}}$, *the forcing* $Q_{\alpha^*}$ *has the ccc. Moreover, it has the property of Knaster.*

PROOF OF THE CLAIM. We fix such an $\alpha^*$ and work in $V^{P_{\alpha^*}}$. We assume CH, as otherwise we have defined $Q_{\alpha^*}$ as an empty set.

Hence sequences $\bar{N}^{\alpha^*} \stackrel{\text{def}}{=} \langle N_i^{\alpha^*} : i \in E_{\alpha^*} \rangle$ and $\langle A_\delta^{\alpha^*} : \delta < \omega_1 \text{ limit} \rangle$ are given. Let

$$E \stackrel{\text{def}}{=} \{ \delta \in E_{\alpha^*} : N_\delta^{\alpha_*} \cap \omega_1 = \delta \},$$

so $E$ is a club of $\omega_1$. Suppose that $q_\alpha \in Q_{\alpha^*}$ for $\alpha < \omega_1$ are given. Let

$$A \stackrel{\text{def}}{=} \{ \delta \in E : \text{for some } \alpha \in E \setminus \delta \text{ we have } \delta > \sup(\delta \cap \mathrm{Dom}(q_\alpha)) \}.$$

$A$ contains a final segment of $\mathrm{acc}(E)$, as otherwise we can find an increasing sequence $\langle \delta_i : i < \omega^\omega \rangle$ from $\mathrm{acc}(E) \setminus A$. Choose $\alpha \geq \sup\{ \delta_i : i < \omega^\omega \}$ with $\alpha \in E$. Hence for all $i < \omega^\omega$ we have that $\delta_i = \sup[\mathrm{Dom}(q_\alpha) \cap \delta_i]$, which is in contradiction with $\mathrm{otp}(\mathrm{Dom}(q_\alpha)) < \omega^\omega$.

Let $C$ be a club such that $A \supseteq C$. For $\delta \in C$, we fix an ordinal $\alpha_\delta$ witnessing that $\delta \in A$. So $\alpha_\delta \in E \setminus \delta$ and $\delta > \sup(\delta \cap \mathrm{Dom}(q_{\alpha_\delta}))$.

For $\delta \in C$, let $g(\delta)$ be defined as the minimal ordinal $\in E$ such that $q_{\alpha_\delta} \in N_{g(\delta)}^{\alpha_*}$ (note that $g$ is well defined). Hence, the set of $\delta \in C$ which are closed under $g$, is a club of $\omega_1$. Call this club $C_1$.

Note that there is a stationary $S \subseteq C_1$ such that for some $\xi^*$ we have

$$\delta \in S \Longrightarrow \sup(\delta \cap \mathrm{Dom}(q_{\alpha_\delta})) = \xi^*.$$

Now notice that for $\delta_1 < \delta_2 \in C_1$, we have

$$\mathrm{Dom}(q_{\alpha_{\delta_1}}) \subseteq N_{\alpha_{\delta_2}}^{\alpha^*} \cap \omega_1 = \alpha_{\delta_2}.$$

So, if $\delta_1 < \delta_2 \in S$, we have

$$\mathrm{Dom}(q_{\alpha_{\delta_1}}) \cap \mathrm{Dom}(q_{\alpha_{\delta_2}}) \subseteq \alpha_{\delta_2} \cap \mathrm{Dom}(q_{\alpha_{\delta_2}}) \subseteq \xi^*.$$

Now let $\delta^* \stackrel{\text{def}}{=} \min(S)$, so $\delta^* > \xi^*$. By (iii) in the definition of $Q_{\alpha^*,\bar{N}^{\alpha^*}}$, for every $\delta \in S$ we have

$$q_{\alpha_\delta}\restriction(\mathrm{Dom}(q_{\alpha_\delta}) \cap \xi^*) = (q_{\alpha_\delta}\restriction(\mathrm{Dom}(q_{\alpha_\delta}) \cap \delta^*))\restriction\xi^* \in N^{\alpha^*}_{\min(E_{\alpha^*}\setminus(\delta^*+1))}.$$

So, there are only countably many possibilities, hence we can find an uncountable set of $\alpha_\delta$ such that $q_{\alpha_\delta}$ are pairwise compatible. ⊣

REMARK 2.8. *ccc* orders like the one above were considered by Abraham, Rubin and Shelah in [1].

CONCLUSION 2.9. *For all $\alpha \leq \omega_2$, the forcing $P_\alpha$ is purely $\gamma$-proper for all $\gamma < \omega_1$.*

[Why? By Fact 2.5, Fact 2.6 (6) and Claim 2.7.]

CLAIM 2.10. *The following hold for every $\alpha^* \leq \omega_2$:*

(1) *In $P_{\alpha^*}$, if $p \leq r$, then for some unique $q$ we have*

$$p \leq_{\mathrm{pr}} q \leq_{\mathrm{apr}} r \;\&\; (\alpha \in \mathrm{Dom}(q) \;\&\; q(\alpha) \neq r(\alpha) \Longrightarrow \alpha \in \mathrm{Dom}(p)).$$

(2) *The following is impossible in $P_{\alpha^*}$:*
*There is a sequence $\langle q_i : i < \omega_1 \rangle$ which is $\leq_{\mathrm{pr}}$-increasing, but for which there is an antichain $\langle r_i : i < \omega_1 \rangle$ such that $q_i \leq_{\mathrm{apr}} r_i$.*

(3) *If $p \in P_{\alpha^*}$ and $\underset{\sim}{\tau}$ is a $P_{\alpha^*}$-name of an ordinal, then there is $q \in P_{\alpha^*}$ with $p \leq_{\mathrm{pr}} q$, and a countable antichain $\underset{\sim}{I} \subseteq \{ r : q \leq_{\mathrm{apr}} r \}$ predense above $q$, such that each $r \in \underset{\sim}{I}$ forces a value to $\underset{\sim}{\tau}$.*

(4) *If $\alpha^* < \omega_2$, then $\Vdash_{P_{\alpha^*}}$ "$|\underset{\sim}{Q}_{\alpha^*}| = \aleph_1$".*

(5) *If $\alpha^* < \omega_2$, then $V^{P_{\alpha^*}} \models$ CH.*

(6) *$Q_{\alpha^*}$ is closed under finite unions of functions which agree on their common domain.*

(7) $V^{P_{\alpha^*}} \models 2^{\aleph_1} = \aleph_2$.

(8) *$P_{\alpha^*}$ satisfies $\aleph_2$-cc.*

PROOF OF THE CLAIM.

(1) Define $q$ by $q \stackrel{\text{def}}{=} p \cup (r\restriction(\mathrm{Dom}(r) \setminus \mathrm{Dom}(p))$.

(2) We prove this by induction on $\alpha^*$. The case $\alpha^* = 0$ is vacuous, and if $\alpha^*$ is a successor ordinal, the statement easily follows from the fact that each $Q_\alpha$ has the property of Knaster.

Suppose that $\alpha^*$ is a limit ordinal and $\langle q_i : i < \omega_1 \rangle$, $\langle r_i : i < \omega_1 \rangle$ exemplify a contradiction to (2). For $i < \omega_1$ let

$$w_i \stackrel{\text{def}}{=} \{ \alpha \in \mathrm{Dom}(q_i) : r_i(\alpha) \neq q_i(\alpha) \},$$

hence $w_i$ is a finite set. Without loss of generality, we can assume that sets $w_i$ $(i < \omega_1)$ form a $\Delta$-system with root $w^*$. Let $\beta^* \stackrel{\text{def}}{=} \mathrm{Max}(w^*) + 1$, so $\beta^* < \alpha^*$.

Now notice that

$$\alpha \in \mathrm{Dom}(r_i) \cap \mathrm{Dom}(r_j) \;\&\; \neg(\Vdash_{P_\alpha} \text{"}r_i(\alpha), r_j(\alpha) \text{ are compatible"})$$

implies that $\alpha \in w^*$, for any $i, j < \omega_1$. Hence, $\langle q_i\restriction\beta^* : i < \omega_1 \rangle$ and $\langle r_i\restriction\beta^* : i < \omega_1 \rangle$ exemplify that (2) fails at $\beta^*$, contradicting the induction hypothesis.

(3) We work in $V^{P_{\alpha^*}}$. Fix such $p$ and $\underset{\sim}{\tau}$. Let $J$ be an antichain predense above $p$, such that every $r \in J$ forces a value to $\underset{\sim}{\tau}$.

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We try to choose by induction on $i < \omega_1$ conditions $p_i$, $r_i$ such that

- $p_0 = p$,
- $j < i \Longrightarrow p_j \leq_{\mathrm{pr}} p_i$,
- $r_i \in J$,
- $p_i \leq_{\mathrm{apr}} r_i$,
- $j < i \Longrightarrow r_i \perp r_j$.

If we succeed, (2) is violated, a contradiction.

So, we are stuck at some $i^* < \omega_1$. We can let $q \stackrel{\mathrm{def}}{=} p_{i^*}$ and $I \stackrel{\mathrm{def}}{=} \{ r_i : i < i^* \}$.

(4) Obvious from the definition of $Q_{\alpha^*}$.

(5) Can be proved by induction on $\alpha^*$, using (3) and (4).

(6) Just check.

(7) Follows from the definition of $P_{\alpha^*}$, part (3) of this claim, and the fact that $V \models 2^{\aleph_1} = \aleph_2$.

(8) By 2.2(5) and part (4) of this claim (see [9, III, 4.1] for the analogue in the case of countable support iterations). ⊣

Claim 2.11. *It is possible to arrange the bookkeeping, so that* $\Vdash_{P_{\omega_2}} \neg\clubsuit$.

Proof of the Claim. As usual, using Claim 2.10(7), it suffices to prove that for every $\alpha^* < \omega_2$, in $V^{P_{\alpha^*}}$ we have

$$\Vdash_{Q_{\alpha^*}} \text{“}\langle A_\delta^{\alpha^*} : \delta < \omega_1 \rangle \text{ is not a } \clubsuit\text{-sequence.”}$$

Let $G$ be $Q_{\alpha^*}$-generic over $V^{P_{\alpha^*}}$, and let $F \stackrel{\mathrm{def}}{=} \bigcup G$. Let $A \stackrel{\mathrm{def}}{=} F^{-1}(\{0\})$. Suppose that $A \supseteq A_\delta^{\alpha^*}$ for some $\delta$. Then for every $f \in G$ we have $f^{-1}(\{1\}) \cap A_\delta^{\alpha^*} = \emptyset$, so $|\mathrm{Dom}(f) \cap A_\delta^{\alpha^*}| < \aleph_0$.

However, the following is true:

Subclaim 2.12. *The set*

$$\mathcal{I} \stackrel{\mathrm{def}}{=} \{ f \in Q_{\alpha^*} : |\mathrm{Dom}(f) \cap A_\delta^{\alpha^*}| = \aleph_0 \text{ or } f^{-1}(\{1\}) \cap A_\delta^{\alpha^*} \neq \emptyset \}$$

*is dense in* $Q_{\alpha^*}$.

Proof of the Subclaim. Given $f \in Q_{\alpha^*}$. If $\mathrm{Dom}(f) \cap A_\delta^{\alpha^*}$ is infinite, then $f \in \mathcal{I}$. Otherwise, let $\beta \stackrel{\mathrm{def}}{=} \min(A_\delta^{\alpha^*}) \setminus \mathrm{Dom}(f)$. Let $g \stackrel{\mathrm{def}}{=} f \cup \{(\beta, 1)\}$, hence $g \geq f$ and $g \in \mathcal{I}$. ⊣

We obtain a contradiction, hence $A$ is not a superset of $A_\delta^{\alpha^*}$. ⊣

Definition 2.13. Suppose that

(a) $\gamma < \omega_1$,

(b) $\bar{N} = \langle N_i : i \leq \gamma \rangle$ is a continuous increasing sequence of countable elementary submodels of $\langle \mathscr{H}(\chi), \in, <_\chi^* \rangle$,

(c) $\underset{\sim}{\tau}, \bar{Q} \in N_0$ and $p \in P_{\omega_2} \cap N_0$,

(d) $p \Vdash$ “$\underset{\sim}{\tau} \in [\omega_1]^{\aleph_1}$” and

(e) $\bar{N} \restriction (i+1) \in N_{i+1}$ for $i < \gamma$.

We say that $\varepsilon \leq \gamma$ is *bad* for $(\bar{N}, \underset{\sim}{\tau}, p, \bar{Q})$ if $\varepsilon$ is a limit ordinal, and there are no $r_n$, $\beta_n \in N_\varepsilon$ $(n < \omega)$ such that

(1) $r_n \Vdash_{P_{\omega_2}}$ “$\beta_n \in \underset{\sim}{\tau}$”,

(2) $\bigcup_{n\in\omega} \beta_n = N_\varepsilon \cap \omega_1$,
(3) $r_n \geq p$ for all $n$,
(4) $\beta_n$ increase with $n$,
(5) for some $n_0 \in \omega$ the set $\{ r_n : n \geq n_0 \}$ has an upper bound in $P_{\omega_2}$,
(6) $\bar{r}_{\bar{N}\restriction\varepsilon,p,\underset{\sim}{\tau}} \stackrel{\text{def}}{=} \langle r_n : n < \omega \rangle$ and $\bar{\beta}_{\bar{N}\restriction\varepsilon,p,\underset{\sim}{\tau}} \stackrel{\text{def}}{=} \langle \beta_n : n < \omega \rangle$ are definable in $(\mathscr{H}(\chi)^V, \in, <^*_\chi)$ from the isomorphism type of $(\langle N_\xi : \xi \leq \varepsilon \rangle, p, \underset{\sim}{\tau}, \bar{Q})$ (we shall sometimes abbreviate this by saying that these objects are defined in a *canonical* way).

MAIN CLAIM 2.14. *Suppose that $\bar{N}$, $\gamma$, $p$ and $\underset{\sim}{\tau}$ are as in Definition 2.13. Then the set*

$$B \stackrel{\text{def}}{=} \{ \varepsilon \leq \gamma : \varepsilon \textit{ bad for } (\bar{N}, \underset{\sim}{\tau}, p, \bar{Q}) \}$$

*has order type $< \omega^\omega$.*

PROOF OF THE MAIN CLAIM. We start by

SUBCLAIM 2.15. *Let $\bar{N}$, $\gamma$, $p$ and $\underset{\sim}{\tau}$ be as in the hypothesis of Claim 2.14. Then, we can choose canonically a sequence $\bar{p} = \langle p_j : j < \omega\gamma \rangle$ such that*
(1) *$\bar{p}$ is $\leq_{pr}$-increasing.*
(2) *$p_0 = p$.*
(3) *For $i < \gamma$ and $n < \omega$, we have that $p_{\omega i+n} \in N_{i+1}$.*
(4) *For each $i < \gamma$, for every formula $\psi(x, y)$ with parameters in $N_i$, there are infinitely many $n$ such that one of the following occurs:*
   ($\alpha$) *For no $p' \geq p_{\omega i+n}$ do we have that for some $y$, the formula $\psi(p', y)$ holds.*
   ($\beta$) *For the $<^*_\chi$-first $r \geq p_{\omega i+n}$ such that $\psi(r, y)$ holds for some $y$, we have $r \geq_{\text{apr}} p_{\omega i+n+1}$.*
(5) *For $j < \omega\gamma$ a limit ordinal, we have $p_j = \bigcup_{i<j} p_i$.*

PROOF OF THE SUBCLAIM. We prove this by induction on $\gamma$, for all $\bar{N}$ and $p$.

If $\gamma = 0$, there is nothing to prove.

If $\gamma < \omega_1$ is a limit ordinal, we fix an increasing sequence $\langle \gamma_k : k < \omega \rangle$ which is cofinal in $\gamma$, such that $\gamma_0 = 0$ (we are taking the $<^*_\chi$-first sequence like that). By induction on $k$ we define $\langle p_j : \omega\gamma_k < j \leq \omega\gamma_{k+1} \rangle$. We let $p_0 \stackrel{\text{def}}{=} p$. At the stage $k$ of the induction we use the induction hypothesis with $p_{\omega\gamma_k}$, $\langle N_j : \omega\gamma_k < j \leq \omega\gamma_{k+1} \rangle$ here standing for $p$, $\bar{N}$ there, obtaining $\langle p_j : \omega\gamma_k < j \leq \omega\gamma_{k+1} \rangle$, noticing that $p_{\omega\gamma_k} \in N_{\omega\gamma_{k+1}}$. We define $p_{\omega\gamma_{k+1}} \stackrel{\text{def}}{=} \bigcup_{j<\omega\gamma_k} p_j$. We thus obtain

$$\langle p_j : \omega\gamma_k < j \leq \omega\gamma_{k+1} \rangle$$

in $V$. As the parameters used are in $N_{\omega\gamma_k+1}$, by the fact that our choice is canonical, we have that $\langle p_j : \omega\gamma_k < j \leq \omega\gamma_{k+1} \rangle \in N_{\omega\gamma_{k+1}+1}$.

Suppose that $\gamma = \gamma' + 1$. By the induction hypothesis, we can find a sequence $\langle p_j : j < \omega\gamma' \rangle$ satisfying the subclaim for $p$ and $\bar{N}\restriction\gamma'$. As $\bar{N}\restriction\gamma \in N_\gamma$, again we have that the sequence $\langle p_j : j < \omega\gamma' \rangle$ is in $N_\gamma$. Let $p_{\omega\gamma'} \stackrel{\text{def}}{=} \bigcup_{j<\omega\gamma'} p_j$.

We list as $\langle \psi^\gamma_n = \psi_n : n < \omega \rangle$ all formulas $\psi(x, y)$ with parameters in $N_{\gamma'}$, so that each formula appears infinitely often, picking the $<^*_\chi$-first such enumeration. By induction on $n < \omega$, we choose $p_{\omega\gamma'+n}$. We have already chosen $p_{\omega\gamma'}$.

At the stage $n+1$ of the induction, we consider $\psi_n$. If $(\alpha)$ holds, we just let $p_{\omega\gamma'+n+1} \stackrel{\text{def}}{=} p_{\omega\gamma'+n}$. Otherwise, there is a condition $r \geq p_{\omega\gamma'+n}$ such that $\psi_n(r,y)$ for some $y$. By elementarity, the $<^*_\chi$-first such $r$ is in $N_{\gamma'+1}$. By Claim 2.10 (1), there is a unique $q$ such that $r \geq_{\text{apr}} q \geq_{\text{pr}} p_{\omega\gamma'+n}$ and

$$\alpha \in \mathrm{Dom}(q) \;\&\; r(\alpha) \neq q(\alpha) \Longrightarrow \alpha \in \mathrm{Dom}(p).$$

Hence, $q \in N_{\gamma'+1}$ and we set $p_{\omega\gamma+n+1} \stackrel{\text{def}}{=} q$. ⊣

We now choose $\bar{p}$ as in the Subclaim, using our fixed $\gamma$, $\bar{N}$, $\underset{\sim}{\tau}$ and $p$.

NOTE 2.16. For every limit $\varepsilon < \gamma$ we have that $\mathrm{Dom}(p_{\omega\varepsilon}) = N_\varepsilon \cap \omega_2$.

[Why? Let $i < \omega\varepsilon$ be given, and let $\alpha \in N_i \cap \omega_2$. Consider the formula $\psi(x,y)$ which says that $x = y \in P_{\omega_2}$ and $\alpha \in \mathrm{Dom}(x)$. This is a formula with parameters in $N_i$. Option $(\alpha)$ from item 2.15 of Subclaim 2.15 does not occur, so there is $m$ and $r \geq_{\text{apr}} p_{\omega i+m}$ such that $\psi(r,y)$ holds for some $y$. Hence

$$\alpha \in \mathrm{Dom}(r) = \mathrm{Dom}(p_{\omega i+m}) \subseteq \mathrm{Dom}(p_{\omega(i+1)}).$$

So $N_i \cap \omega_2 \subseteq \mathrm{Dom}(p_{\omega(i+1)})$, and hence $N_\varepsilon \cap \omega_2 \subseteq \mathrm{Dom}(p_{\omega\varepsilon})$.

On the other hand, if $\alpha \in \mathrm{Dom}(p_{\omega\varepsilon})$, there is $i < \varepsilon$ such that $\alpha \in \mathrm{Dom}(p_{\omega i}) \subseteq N_{i+1} \subseteq N_\varepsilon$.]

OBSERVATION 2.17. Suppose $\alpha \leq \omega_2$, while $q \in P_\alpha$ and $w \in [\mathrm{Dom}(q)]^{<\aleph_0}$. Then there is $q^+ \geq q$ in $P_\alpha$ such that

$(*)^\alpha$ If $i \in w \cup \{\, j \in \mathrm{Dom}(q) : q(j) \neq q^+(j)\,\}$, then $q^+(i) \in V$ (an object), and not just $q^+{\restriction} i \Vdash$ "$q^+(i) \in V$" (not just a name).

[Why? By induction on $\alpha$. The induction is trivial for $\alpha = 0$, and in the case of $\alpha$ a limit ordinal it follows from the finiteness of $w$. Suppose that $\alpha = \beta + 1$. We have $q{\restriction}\beta \Vdash$ "$q(\beta) \in V$", so we can find $r \in P_\beta$ such that $r \geq q{\restriction}\beta$, and $A$ such that $r \Vdash$ "$q(\beta) = A$". Now apply $(*)^\beta$ with $r$ in place of $q$ and

$$(w \cap \beta) \cup \{\, j : r(j) \neq q(j)\,\}$$

to obtain $q^+_\beta$. Let $q^+ \stackrel{\text{def}}{=} q^+_\beta \frown \{\langle \beta, A\rangle\}$.]

**Continuation of the proof of 2.14.** Since $\bar{p}$ is $\leq_{\text{pr}}$-increasing, the limit of $\bar{p}$ is a condition, say $p_*$. Now let $q^* \geq p_*$ be the $<^*_\chi$-first such that $q^* \Vdash$ "$\beta \in \underset{\sim}{\tau}$" for some $\beta > N_\gamma \cap \omega_1$, and with the property

$$[\alpha \in \mathrm{Dom}(p_*) \;\&\; p_*(\alpha) \neq q^*(\alpha)] \Longrightarrow q^*(\alpha) \text{ an object},$$

which exists by Observation 2.17. Let $w^* \stackrel{\text{def}}{=} \{\, \alpha \in \mathrm{Dom}(p_*) : p_*(\alpha) \neq q^*(\alpha)\,\}$.

We now define

$$b \stackrel{\text{def}}{=} \left\{ \varepsilon \leq \gamma : \left( \bigcup_{\alpha \in w^*} \mathrm{Dom}(q^*(\alpha)) \cap (N_\varepsilon \cap \omega_1) \right) \text{ is unbounded in } N_\varepsilon \cap \omega_1 \right\}.$$

NOTE 2.18. $\mathrm{otp}(b) < \omega^\omega$.

[Why? Suppose that $\varepsilon_j$ for $j < \omega^\omega$ are elements of $b$, increasing with $j$. Now, for every $j < \omega^\omega$ we know that $N_{\varepsilon_j} \cap \omega_1$ is bounded in $N_{\varepsilon_{j+1}} \cap \omega_1$, but

$$\bigcup_{\alpha \in w^*} \mathrm{Dom}(q^*(\alpha)) \cap (N_{\varepsilon_{j+1}} \cap \omega_1)$$

is unbounded in $N_{\varepsilon_{j+1}} \cap \omega_1$. Hence

$$\bigcup_{\alpha \in w^*} \mathrm{Dom}(q^*(\alpha)) \cap [N_{\varepsilon_j} \cap \omega_1, N_{\varepsilon_{j+1}} \cap \omega_1) \neq \emptyset.$$

However, by the definition of the forcing,

$$\mathrm{otp}\Big(\bigcup_{\alpha \in w^*} \mathrm{Dom}(q^*(\alpha))\Big) < \omega^\omega,$$

a contradiction.]

**Continuation of the proof of 2.14.** Our aim is to show that $B \subseteq b$ ($B$ was defined in the statement of the Main Claim). So, let $\varepsilon^* \in (\gamma + 1) \setminus b$ be a limit ordinal. We show that $\varepsilon^* \notin B$. We have to define $\bar{r} \stackrel{\text{def}}{=} \bar{r}_{\bar{N} \restriction \varepsilon^*, p, \underset{\sim}{\tau}}$ and $\bar{\beta} \stackrel{\text{def}}{=} \bar{\beta}_{\bar{N} \restriction \varepsilon^*, p, \underset{\sim}{\tau}}$ so to satisfy (1)–(5) from the definition of $B$, and to do so in a canonical way, to be able to prove Subclaim 2.19 below, hence showing that (6) from Definition 2.13 holds.

Let

$$\xi \stackrel{\text{def}}{=} \Big[\sup\Big(\bigcup_{\alpha \in w^*} \mathrm{Dom}(q^*(\alpha)\Big) \cap N_{\varepsilon^*} \cap \omega_1\Big] + 1,$$

so $\xi < N_{\varepsilon^*} \cap \omega_1$. We enumerate $N_{\varepsilon^*} \cap w^*$ as $\{\alpha_0, \ldots, \alpha_{n^*-1}\}$. By Note 2.16, we can fix $j^* < \varepsilon^*$ such that $\{\alpha_0, \ldots, \alpha_{n^*-1}\} \subseteq \mathrm{Dom}(p_{\omega j^*})$. Let $j^*$ be the first such. Also let $\delta \stackrel{\text{def}}{=} N_{\varepsilon^*} \cap \omega_1$. Now we observe that for all $l < n^*$, we have $q^*(\alpha_l) \restriction \xi \in N_{\varepsilon^*}$.

[Why? Clearly, there is $\varepsilon' < \varepsilon^*$ such that $\{\alpha_0, \ldots, \alpha_{n^*-1}, \xi\} \subseteq N_{\varepsilon'}$. With $\bar{\bar{N}}$ defined in Definition 2.2 (2), we have that $\bar{\bar{N}} \in N_0$. Also, we have that

$$\emptyset \Vdash_{\alpha_{n^*-1}} \text{“}\underset{\sim}{E} \stackrel{\text{def}}{=} \bigcap_{l<n^*} \underset{\sim}{E}_{\alpha_l} \text{ is a club of } \omega_1\text{”},$$

(cf. Definition 2.2 (4) (a). Hence, by properness and the choice of $\bar{N}$, we have that for every $\varepsilon \in [\varepsilon', \gamma]$, we have that

$$\emptyset \Vdash_{\alpha_{n^*-1}} \text{“}N_\varepsilon \cap \omega_1 \in \underset{\sim}{E}\text{”}.$$

Let $i \stackrel{\text{def}}{=} N_{\varepsilon'} \cap \omega_1$, hence $N_i^0 \in N_{\varepsilon'+1}$. In particular, we have $\emptyset \Vdash_{\alpha_{n^*-1}}$ “$i \in \underset{\sim}{E}$” and $N_i^0 \cap \omega_1 < N_{\varepsilon^*} \cap \omega_1$. So for all $l < n^*$ we have

$$q^*(\alpha_l) \restriction \xi = q^*(\alpha_l) \restriction (N_i^0 \cap \omega_1),$$

but

$$\emptyset \Vdash_{\alpha_l} \text{“}N_i^0 \cap \omega_1 = \underset{\sim}{N}_i^{\alpha_l} \cap \omega_1\text{”},$$

hence by Definition 2.2 (4) (a) (iii), we have

$$q^* \restriction \alpha_l \Vdash \text{“}q^*(\alpha_l) \restriction \xi \in N^0_{\min(\underset{\sim}{E}_{\alpha_l} \setminus (i+1))}\text{”}.$$

But

$$\emptyset_{\alpha_l} \Vdash \text{“}\min(\underset{\sim}{E}_{\alpha_l} \setminus (i+1)) \in N_{\varepsilon'+1}[\underset{\sim}{G}]\text{”},$$

hence

$$q\restriction\alpha_l \Vdash \text{“}q(\alpha_l)\restriction\xi \in N_{\varepsilon'+1}[\underset{\sim}{G}]\text{”}.$$

By properness and the fact that $q^*(\alpha_l) \in V$, we have $q^*(\alpha_l)\restriction\xi \in N_{\varepsilon'+1}$.]

Let us pick the $<^*_\chi$-first increasing sequence $\langle \varepsilon_n : n < \omega \rangle$ such that $\varepsilon^* = \bigcup_{n<\omega} \varepsilon_n$, while $\omega j^* + 1 < \varepsilon_0$ and $\xi \in N_{\varepsilon_0}$, in addition to $(\forall l < n^*)[q^*(\alpha_l)\restriction\xi \in N_{\varepsilon_0}]$.

**Defining $r_n$ and $\beta_n$.** We do this by induction on $n$. If $n = 0$, we set $r_0 \overset{\text{def}}{=} p_{\omega\varepsilon_0}$, and also let $m_0 = 0$, $\xi_0 = \xi$.

At stage $n + 1$, we assume that at stage $n$ we have chosen $r_n \in N_{\varepsilon_n+1} \cap P_{\omega_2}$ and $m_n < \omega$ so that $r_n \geq_{\text{apr}} p_{\omega\varepsilon_n+m_n}$. We also have chosen $\xi_n, \beta_n \in N_{\varepsilon_n+1}$.

We define a formula $\varphi_n(x, y)$ which says

(1) $x \in P_{\omega_2}$ and $y$ is an ordinal $> \text{Max}\{\beta_n, N_{\varepsilon_n} \cap \omega_1\}$.
(2) $x \Vdash \text{“}y' \in \underset{\sim}{\tau}\text{”}$ for some $y' > y$.
(3) If $l < n^*$, then $x(\alpha_l)$ is an object, not a name, and $x(\alpha_l)\restriction\xi = q^*(\alpha_l)\restriction\xi$.
(4) For $l < n^*$, we have $x(\alpha_l)\restriction\xi \in N_{\varepsilon_0}$ and $\text{Dom}(x(\alpha_l)) \setminus \xi \subseteq \omega_1 \setminus \xi_n$.
(5) For all $\alpha$ we have

$$\alpha \in \text{Dom}(x) \cap \text{Dom}(p_{\omega\varepsilon_n+m_n}) \;\&\; x(\alpha) \neq p_{\omega\varepsilon_n+m_n}(\alpha) \\ \Longrightarrow \alpha \in \{\alpha_0, \ldots \alpha_{n^*-1}\}.$$

Hence, $\varphi_n$ is a formula with parameters in $N_{\varepsilon_n+1} \subseteq N_{\varepsilon_{n+1}}$. Also, we have that $\varphi_n(q^*, \delta)$ holds.

By the choice of $\bar{p}$, there is $m_{n+1} > m_n$ (we pick the first one) such that for the $<^*_\chi$-first $r \geq p_{\omega(\varepsilon_{n+1})+m_{n+1}-1}$ for which there is $y$ for which $\varphi_n(r, y)$ holds, we have $r \geq_{\text{apr}} p_{\omega(\varepsilon_{n+1})+m_{n+1}}$. We let

$$r_{n+1} \overset{\text{def}}{=} r \cup (p_{\omega\varepsilon_{n+1}+m_{n+1}} \restriction \text{Dom}(p_{\omega\varepsilon_{n+1}+m_{n+1}}) \setminus \text{Dom}(r)).$$

Note that $r_{n+1} \in N_{\varepsilon_{n+1}+1}$ and that $\varphi_n(r_{n+1}, y)$ must hold for some $y$. The $<^*_\chi$-first such $y$ is an element of $N_{\varepsilon_{n+1}+1}$, and we choose it to be $\beta_{n+1}$.

Finally, we define $\xi_{n+1} \overset{\text{def}}{=} \min(N_{\varepsilon_{n+1}} \setminus \sup\{\bigcup_{l<n^*} \text{Dom}(r_{n+1}(\alpha_l)) \setminus \xi\})$.

At the end, we obtain (canonically chosen) sequences $\langle r_n : n < \omega \rangle$, $\langle \beta_n : n < \omega \rangle$, $\langle \xi_n : n < \omega \rangle$ and $\langle m_n : n < \omega \rangle$ such that

(1) $r_n \geq_{\text{apr}} p_{\omega\varepsilon_n+m_n}$.
(2) $\xi_0 = \xi$ and $\xi_n$ are strictly increasing with $n$.
(3) For all $l < n^*$, we have $\text{Dom}(r_n(\alpha_l)) \setminus \xi \subseteq (\xi_n, \xi_{n+1})$ and $r_n(\alpha_l)$ is an object.
(4) $r_n \Vdash_{P_{\omega_2}} \text{“}\beta_n \in \underset{\sim}{\tau}\text{”}$.
(5) $\beta_{n+1} > \beta_n$.
(6) $\bigcup_{n<\omega} \beta_n = N_{\varepsilon^*} \cap \omega_1$.
(7) $r_n \in N_{\varepsilon^*}$.
(8) For $l < n^*$, we have $r_n(\alpha_l)\restriction\xi = r_1(\alpha_l)\restriction\xi$.
(9) $\alpha \in \{\beta \in \text{Dom}(r_n) : r_n(\beta) \neq p_{\omega\varepsilon_n+m_n}(\beta)\} \Longrightarrow \alpha \in \{\alpha_0, \ldots \alpha_{n^*} - 1\}$.

[Why? By item 2 in the definition of $\varphi_n$.]

We will use $r_n$, $\beta_n$ $(n < \omega)$ to witness that $\varepsilon^* \notin B$. It is true that $r_n \geq p$ and $\beta_n$ increase with $n$, and their limit is $N_{\varepsilon^*} \cap \omega_1$. We need to show that for some $n_0$, the sequence $r_n$ $(n \geq n_0)$ has an upper bound in $P_{\omega_2}$. The natural choice to use would be $\bigcup_{n<\omega} r_n$, but this is not necessarily a condition!

[Why? By item 2 above, all $r_n$ for $n > 0$ agree on $\alpha$ such that $\alpha \notin \{\alpha_0, \dots, \alpha_{n^*-1}\}$. By items 2, 2. and 2 above, we even know that for every $l < n^*$, the union

$$\bigcup_{n<\omega} r_n(\alpha_l)$$

is a function. If $\delta' < N_{\varepsilon^*} \cap \omega_1$, then for all $l < n^*$ we have

$$\bigcup_{n<\omega} r_n(\alpha_l)\restriction\delta' = \bigcup_{n<n'} r_n(\alpha_l)\restriction\delta'$$

for some $n' < \omega$, so this is a condition in $\underset{\sim}{Q}_{\alpha_l}$ (by Claim 2.10 (6)). If $\delta' > N_{\varepsilon^* \cap \omega_1}$, then $\bigcup_{n<\omega} r_n(\alpha_l) \cap \delta'$ is finite. However, it is possible that for some $\alpha_l$ it is forced that the intersection of the set

$$\bigcup_{n\in\omega} \mathrm{Dom}(r_n(\alpha_l))$$

with $\underset{\sim}{A}^{\alpha_l}_{N_{\varepsilon^*} \cap \omega_1}$ is infinite, so $\bigcup_{n<\omega} r_n(\alpha_l)$ might fail to be a condition in $\underset{\sim}{Q}_{\alpha_l}$.]

(We remark that it is because of this point that we are getting $\clubsuit^1$ and not $\clubsuit$ in $V^P$.)

Now, we define conditions $q^*_l$ for $l \leq n^*$ as follows. First set $\alpha_{n^*} \overset{\text{def}}{=} \omega_2$. By induction on $l \leq n^*$ we choose $q^*_l \in P_{\alpha_l}$, so that

(a) $q^*_l \leq q^*_{l+1}$,

(b) $q^*_l\restriction\alpha_l$ is above $r_n\restriction\alpha_l$ for all $n$ large enough.

This clearly suffices, as $q^*_{n^*} \cup q^*\restriction(\mathrm{Dom}(q^*) \setminus \mathrm{Dom}(q^*_{n^*}))$ is a condition in $P_{\omega_2}$ which is above all but finitely many $r_n$.

**The choice of** $q^*_l$**.** Let $q^*_0 \overset{\text{def}}{=} q^*\restriction\alpha_0 = p_*\restriction\alpha_0$. Given $q^*_l \in P_{\alpha_l}$, with $l < n^*$. We can find $q^{**}_l \geq q^*_l$ in $P_{\alpha_l}$, such that

$$q^{**}_l \Vdash \text{“}\min(\underset{\sim}{A}^{\alpha_l}_{N_{\varepsilon^*} \cap \omega_1} \setminus \xi_0) = \zeta_l\text{”}$$

for some ordinal $\zeta_l$. By item 3, above, the ordinal $\zeta_l$ belongs to $\mathrm{Dom}(r_n(\alpha_l))$ for at most one $n$. Let $n_l$ be greater than this $n$. Hence there is a condition $q^+_l$ in $P_{\alpha_l+1}$ such that $q^+_l(\alpha_l)$ is an object and

$$q^+_l\restriction\alpha_l = q^{**}_l \;\&\; q^+_l(\alpha_l) \geq \bigcup_{n\geq n_l} r_n(\alpha_l) \;\&\; q^+_l(\alpha_l)(\zeta_l) = 1.$$

Now let

$$q^*_{l+1} \overset{\text{def}}{=} q^+_l \cup \bigcup_{n\geq n_l} r_n\restriction[\alpha_l + 1, \alpha_{l+1}).$$

Note that $q^*_{l+1}(\alpha)$ is forced to be a function, for any $\alpha \in \mathrm{Dom}(q_l)$, as all $r_n$ agree on $[\alpha_l + 1, \alpha_{l+1})$. Also, $q^*_{l+1}(\alpha)$ is forced to be in $V$.

Now, the sequence $\langle q^*_{\alpha_l} : l \leq n^* \rangle$ is as required.

To finish the proof of the Main Claim, we need to observe

SUBCLAIM 2.19. *Suppose that $\bar{N}$ and $\bar{M}$ are two equally long countable continuously increasing sequences of countable elementary submodels of $\langle \mathscr{H}(\chi), \in, <^*_\chi, p, \underset{\sim}{\tau}, \bar{Q} \rangle$ with $\bar{Q}^N = \bar{Q}^M = \bar{Q}$, and $F = \langle f_i : i < \lg(\bar{N}) \rangle$ is an increasing sequence of isomorphisms $f_i : N_i \to M_i$.*

*Then, if $\bar{\beta}_{\bar{N},p,\underset{\sim}{\tau}}$ and $\bar{r}_{\bar{N},p,\underset{\sim}{\tau}}$ are defined, so are $\bar{\beta}_{\bar{M},F(p),F(\underset{\sim}{\tau})}$ and $\bar{r}_{\bar{M},F(p),F(\underset{\sim}{\tau})}$. Moreover,*

$$\bar{\beta}_{\bar{M},F(p),F(\underset{\sim}{\tau})} = \bar{\beta}_{\bar{N},p,\underset{\sim}{\tau}} \quad \textit{and} \quad \bar{r}_{\bar{M},F(p),F(\underset{\sim}{\tau})} = F(\bar{r}_{\bar{N},p,\underset{\sim}{\tau}}).$$

Proof of the Subclaim. Check, looking at the way $\bar{\beta}, \bar{r}$ were defined. $\dashv$

To finish the proof of the Theorem, we prove

Claim 2.20. $\Vdash_{P_{\omega_2}} ♣^1$.

Proof of the Claim. We use the following equivalent reformulation of $\diamondsuit$ in $V$: There is a sequence

$$\langle \bar{N}^\delta = \langle N_i^\delta : i < \delta \rangle : \delta < \omega_1 \rangle,$$

such that

(1) Each $\bar{N}^\delta = \langle N_i^\delta : i < \delta \rangle$ is a continuously increasing sequence of countable elementary submodels of $\langle \mathscr{H}(\chi), \in, <_\chi^*, p, \underset{\sim}{\tau}, \bar{Q} \rangle$, with $N_i^\delta \cap \omega_1 < \delta$ and $\bar{N}^\delta \restriction (i+1) \in N_{i+1}^\delta$ for $i < \delta$. Here, $p$, $\bar{Q}$ and $\underset{\sim}{\tau}$ are constant symbols. In addition, $\bar{Q}^{N_0^\delta} = \bar{Q}$.

(2) For every continuously increasing sequence $\bar{N} = \langle N_i : i < \omega_1 \rangle$ of countable elementary submodels of $\langle \mathscr{H}(\chi), \in, <_\chi^*, p, \underset{\sim}{\tau}, \bar{Q} \rangle$ such that $\bar{Q}^{N_0} = \bar{Q}$, there is a stationary set of $\delta$ such that for all $i < \delta$ the isomorphism type of $N_i$ and $N_i^\delta$ is the same, as is witnessed by some sequence of isomorphisms $\langle f_i^\delta : i < \delta \rangle$ which is increasing with $i$.

For each limit ordinal $\delta$, let $N^\delta \overset{\text{def}}{=} \bigcup_{i<\delta} N_i^\delta$. We define $A_\delta$:

> If $\bar{\beta}_{\bar{N}^\delta, p^{N_\delta}, \underset{\sim}{\tau}^{N_\delta}}$ is well defined, then we let $A_\delta \overset{\text{def}}{=} \operatorname{Rang}(\bar{\beta}_{\bar{N}^\delta, p^{N_\delta}, \underset{\sim}{\tau}^{N_\delta}})$. Otherwise, we let $A_\delta$ be the range of any cofinal $\omega$-sequence in $\delta$. Note that in any case $A_\delta$ is an unbounded subset of $\delta$ of order type $\omega$.

We claim that $\langle A_\delta : \delta < \omega_1 \rangle$ exemplifies that $V^P \models ♣^1(\omega_1)$. We have to check that for every unbounded subset $A$ of $\omega_1$ in $V^{P_{\omega_2}}$, there is a $\delta < \omega_1$ with $|A_\delta \setminus A| < \aleph_0$.

Suppose this is not true. So, there are $p^*, \underset{\sim}{\tau}^*$ exemplifying this, that is

$$p^* \Vdash \text{“}\underset{\sim}{\tau}^* \in [\omega_1]^{\aleph_1} \text{ and for all } \delta \text{ we have } |A_\delta \setminus \underset{\sim}{\tau}^*| = \aleph_0\text{”}.$$

We fix in $V$ a continuously increasing sequence $\bar{N} = \langle N_i : i < \omega_1 \rangle$ of countable elementary submodels of $\langle \mathscr{H}(\chi), \in, <_\chi^*, p, \underset{\sim}{\tau}, \bar{Q} \rangle$ such that $p^{N_0} = p^*$, while $\underset{\sim}{\tau}^{N_0} = \underset{\sim}{\tau}^*$ and $\bar{Q}^{N_0}$ is our iteration $\bar{Q}$. In addition, $\bar{N} \restriction (i+1) \in N_{i+1}$ for all $i$. For every $\gamma < \omega_1$, we can apply Claim 2.14 to $\bar{N} \restriction (\gamma + 1)$. Using this, we can easily conclude that the set

$$C \overset{\text{def}}{=} \{\delta < \omega_1 : \begin{array}[t]{l} \text{(a) } N_\delta \cap \omega_1 = \delta \\ \text{(b) } \delta \text{ is a limit ordinal} \\ \text{(c) } \bar{\beta}_{\bar{N} \restriction \delta, p^*, \underset{\sim}{\tau}^*} \text{ and } \bar{r}_{\bar{N} \restriction \delta, p^*, \underset{\sim}{\tau}^*} \text{ are defined} \} \end{array}$$

is a club of $\omega_1$. Let $\delta \in C$ be such that sequences $\bar{N} \restriction \delta$ and $\langle N_i^\delta : i < \delta \rangle$ have the same isomorphism type. Let this be exemplified by $F = \langle f_i : i < \delta \rangle$, an increasing sequence of isomorphisms $f_i \colon N_i \to N_i^\delta$. By our choice of constant symbols, we also have that $F(\bar{Q}) = \bar{Q}$, $F(p^*) = p^{N_0^\delta}$ and $F(\underset{\sim}{\tau}^*) = \underset{\sim}{\tau}^{N_0^\delta}$. By Subclaim 2.19, we have that

$$\bar{\beta}_{\bar{N}^\delta, p^{N_0^\delta}, \underset{\sim}{\tau}^{N_0^\delta}} = \bar{\beta}_{\bar{N}\restriction\delta, p^*, \underset{\sim}{\tau}^*}, \quad \text{and} \quad \bar{r}_{\bar{N}^\delta, p^{N_0^\delta}, \underset{\sim}{\tau}^{N_0^\delta}} = F(\bar{r}_{\bar{N}\restriction\delta, p^*, \underset{\sim}{\tau}^*}).$$

We now let $\langle \beta_n : n < \omega \rangle \stackrel{\text{def}}{=} \bar{\beta}_{\bar{N}^\delta, p^{N_0^\delta}, \underset{\sim}{\tau}^{N_0^\delta}}$. By the definition of $\bar{r}$ and $\bar{\beta}$, there is $n_0$ and condition $q$ such that $q \Vdash$ "$\beta_n \in \underset{\sim}{\tau}^*$" for all $n \geq n_0$, and $q \geq p^*$. Hence $q \Vdash$ "$|A_\delta \setminus \underset{\sim}{\tau}^*| < \aleph_0$", which is in contradiction with the fact that $q \geq p^*$. $\dashv$

Note 2.21.

(1) We note that the present result clearly implies that $\clubsuit$ and $\dagger$ are not the same (even without CH).

Clearly, $V^{P_{\omega_2}} \models 2^{\aleph_0} = \aleph_2$. One of the ways to see this is to notice that under CH the full $\clubsuit$ and $\clubsuit^1$ agree (while $V^{P_{\omega_2}} \models 2^{\aleph_0} \leq \aleph_2$ obviously).

(2) Note that the sequence $\langle A_\delta : \delta < \omega_1 \rangle$ exemplifying $\clubsuit^1$ in $V^P$, is in fact a sequence in $V$.

For clarity of presentations we decided to give details of the proof of Theorem 2.1 rather than Theorem 2.22 below, which is of course stronger than Theorem 2.1. Now the obvious changes to the proof of Theorem 2.1 (just change the definition of $\underset{\sim}{Q}_\beta$) give

Theorem 2.22. CON($\clubsuit^1 + \neg\clubsuit^\bullet$).

In the next section we encounter another similar proof, where the changes needed to the proof of Theorem 2.1 are more significant, and we spell them out.

## §3. Consistency of $\clubsuit^\bullet$ and $\neg\clubsuit^1$.

Theorem 3.1. CON($\clubsuit^\bullet + \neg\clubsuit^1$).

Proof. The proof is a modification of the proof from §2, so we shall simply explain the changes, keeping all the non-mentioned conventions and definitions in place.

Our iteration is again called $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, but $\underset{\sim}{Q}_\beta$ will be redefined below.

Definition 3.2.

(1) A candidate for a $\clubsuit^1$ is a synonym for a candidate for $\clubsuit$.

(2) Suppose that $\beta < \omega_2$, and let us define $Q_\beta$, while working in $V^{P_\beta}$. It is defined the same way as in Definition 2.2 (3), but we change the condition (a) (iv) into

(iv′) $\mathrm{Dom}(f) \cap A_\delta^\beta$ infinite $\Longrightarrow (\exists^\infty \gamma \in \mathrm{Dom}(f) \cap A_\delta^\beta)[f(\gamma) = 0]$.

Note 3.3. The following still hold with the new definition of the iteration

(1) Claim 2.7.

(2) Conclusion 2.9.

(3) Claim 2.10.

[Why? The same proofs.]

Claim 3.4. *It is possible to arrange the bookkeeping, so that* $\Vdash_{P_{\omega_2}} \neg\clubsuit^1$.

PROOF OF THE CLAIM. It suffices to prove that for every $\alpha^* < \omega_2$, in $V^{P_{\alpha^*}}$ we have

$$\Vdash_{Q_{\alpha^*}} \text{"}\langle A_\delta^{\alpha^*} : \delta < \omega_1 \rangle \text{ is not a } \clubsuit^1\text{-sequence."}$$

Let $G$ be $Q_{\alpha^*}$-generic over $V^{P_{\alpha^*}}$, and let $F \stackrel{\text{def}}{=} \bigcup G$. Let $A \stackrel{\text{def}}{=} F^{-1}(\{1\})$. Suppose that $|A_\delta^{\alpha^*} \setminus A| < \aleph_0$. We can find $p^* \in G$ which forces this, in fact without loss of generality for some $\varepsilon < \delta$ we have

$$p^* \Vdash \text{"}A_\delta^{\alpha^*} \setminus \underset{\sim}{A} \subseteq \varepsilon\text{"}.$$

But consider

$$\mathcal{J} \stackrel{\text{def}}{=} \{ q \geq p^* : (\exists \gamma \in (A_\delta^{\alpha^*} \setminus \varepsilon) \cap \operatorname{Dom}(q))[q(\gamma) = 0] \}.$$

This set is dense above $p^*$: if $r \geq p^*$ is such that $\operatorname{Dom}(r) \cap A_\delta^{\alpha^*}$ is infinite, then $r \in \mathcal{J}$. Otherwise, let

$$\gamma = \min(A_\delta^{\alpha^*} \setminus (\operatorname{Dom}(r) \cup \varepsilon))$$

and let $q \stackrel{\text{def}}{=} r \cup \{(\gamma, 0)\}$. Contradiction. ⊣

DEFINITION 3.5. Suppose that

(a) $\gamma < \omega_1$,
(b) $\bar{N} = \langle N_i : i \leq \gamma \rangle$ is a continuous increasing sequence of countable elementary submodels of $\langle \mathscr{H}(\chi), \in, <_\chi^* \rangle$,
(c) $\underset{\sim}{\tau}, \bar{Q} \in N_0$ and $p \in P_{\omega_2} \cap N_0$,
(d) $p \Vdash \text{"}\underset{\sim}{\tau} \in [\omega_1]^{\aleph_1}\text{"}$ and
(e) $\bar{N} \restriction (i+1) \in N_{i+1}$ for $i < \gamma$.

We say that $\varepsilon \leq \gamma$ is *bad* for $(\bar{N}, \underset{\sim}{\tau}, p, \bar{Q})$ if $\varepsilon$ is a limit ordinal, and there is no $m(\varepsilon) = m(\bar{N} \restriction \varepsilon, p, \underset{\sim}{\tau}) < \omega$ and sequences $\langle r_n^m : n < \omega \rangle$ and $\langle \beta_n^m : n < \omega \rangle$ for $m \leq m(\varepsilon)$ such that $r_n^m, \beta_n^m \in N_\varepsilon$ and

(1) $r_n^m \Vdash_{P_{\omega_2}} \text{"}\beta_n^m \in \underset{\sim}{\tau}\text{"}$,
(2) $\bigcup_{n \in \omega} \beta_n^m = N_\varepsilon \cap \omega_1$,
(3) $r_n^m \geq p$ for all $n, m$,
(4) $\beta_n^m$ increase with $n$,
(5) for some $m \leq m(\varepsilon)$ the set $\{ r_n^m : n < \omega \}$ has an upper bound in $P_{\omega_2}$
(6) $m(\varepsilon)$ and

$$\bar{r}_{\bar{N} \restriction \varepsilon, p, \underset{\sim}{\tau}} \stackrel{\text{def}}{=} \langle \langle r_n^m : n < \omega \rangle : m < m(\varepsilon) \rangle$$

and

$$\bar{\beta}_{\bar{N} \restriction \varepsilon, p, \underset{\sim}{\tau}} \stackrel{\text{def}}{=} \langle \langle \beta_n^m : n < \omega \rangle : m < m(\varepsilon) \rangle$$

are definable in $(\mathscr{H}(\chi)^V, \in, <_\chi^*)$ from the isomorphism type of $(\langle N_\xi : \xi \leq \varepsilon \rangle, p, \underset{\sim}{\tau}, \bar{Q})$ (we shall sometimes abbreviate this by saying that these objects are defined in a *canonical* way).

MAIN CLAIM 3.6. *Suppose that $\bar{N}$, $\gamma$, $p$ and $\underset{\sim}{\tau}$ are as in Definition 3.5. Then the set*

$$B \stackrel{\text{def}}{=} \{ \varepsilon \leq \gamma : \varepsilon \text{ bad for } (\bar{N}, \underset{\sim}{\tau}, p, \bar{Q}) \}$$

*has order type $< \omega^\omega$.*

PROOF OF THE MAIN CLAIM. Fix such $\bar{N}$, $\gamma$, $p$ and $\underset{\sim}{\tau}$. We define $\bar{p} = \bar{p}(\gamma, \bar{N}, \underset{\sim}{\tau}, p)$ as in Subclaim 2.15 and $p_*$, $q^*$, $w^*$, $b$ as in the proof of Main Claim 2.14. We shall show that $B \subseteq b$, by taking any limit ordinal $\varepsilon^* \in (\gamma+1) \setminus b$ and showing that it is not in $B$.

Given $\varepsilon^*$, we define $n^*$, $\xi$ and $\langle r_n : n < \omega \rangle$ and $\langle \beta_n : n < \omega \rangle$the way we did in the proof of Main Claim 2.14. We let $m(\varepsilon^*) = 2^{n^*} - 1$. For $m \leq m(\varepsilon^*)$, we let $\{ i_n^m : n < \omega \}$ be the increasing enumeration of

$$\{ i < \omega : i = m \ (\mathrm{mod}\ 2^{n^*}) \}$$

and let $r_n^m = r_{i_n^m}$ and $\beta_n^m = \beta_{i_n^m}$. We shall show that for some $m \leq m(\varepsilon^*)$, the sequence $\langle r_n^m : n < \omega \rangle$ has an upper bound in $P_{\omega_2}$. Recall the definition of $\alpha_l$ for $l \leq n^*$ from the proof of Main Claim 2.14. Notice that it is not a priori clear that $\bigcup_{n<\omega} r_n^m$ is a condition, as it may happen that for some $l < n^*$ it is forced that

$$\underset{\sim}{X}_l \stackrel{\mathrm{def}}{=} \bigcup_{n<\omega} \mathrm{Dom}(r_n^m)(\alpha_l) \cap \underset{\sim}{A}^{\alpha_l}_{N_{\varepsilon^*} \cap \omega_1}$$

is infinite, yet $\bigcup_{n<\omega} r_n^m(\alpha_l) \restriction \underset{\sim}{X}_l$ is 0 only finitely often.

By induction on $l \leq n^*$ we choose $q_l^* \in P_{\alpha_l}$ and $k_l < 2^l$, so that

(a) $q_l^* \geq p_* \restriction \alpha_l$,
(b) $(\forall n < \omega)[n = k_l (\mathrm{mod}\ 2^l) \Longrightarrow r_n \restriction \alpha_l \leq q_l^*]$.
(c) $q_l^* \leq q_{l+1}^*$.

This clearly suffices, as we have that $q_{n^*} \in P_{\omega_2}$ is a common upper bound of $\{ r_n^{k_{n^*}} : n < \omega \}$.

Let $q_0^* \stackrel{\mathrm{def}}{=} q^* \restriction \alpha_0 = p_* \restriction \alpha_0$.

Given $q_l^* \in P_{\alpha_l}$ and $k_l < 2^l$ for some $l < n^*$. Let

$$\Gamma \stackrel{\mathrm{def}}{=} \{ n < \omega : n = k_l \ (\mathrm{mod}\ 2^l) \}.$$

Let $k_1' \stackrel{\mathrm{def}}{=} k_l$ and $k_2' \stackrel{\mathrm{def}}{=} k_l + 2^l$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are infinite disjoint and defined by the following, for $j \in \{1,2\}$.

$$\Gamma_j \stackrel{\mathrm{def}}{=} \{ n \in \Gamma : n = k_j' \ (\mathrm{mod}\ 2^{l+1}) \}.$$

If

$$q_l^* \Vdash \text{“} \bigcup_{n \in \Gamma_j} \mathrm{Dom}(r_n(\alpha_l)) \cap \underset{\sim}{A}^{\alpha_l}_{N_{\varepsilon^*} \cap \omega_1} \text{ finite”}$$

for at least one $j \in \{1,2\}$, let $j^*$ be the smallest such $j$ and let $k_{l+1} \stackrel{\mathrm{def}}{=} k_{j^*}'$. Let

$$q_{l+1}^* \stackrel{\mathrm{def}}{=} q_l^* \frown \left\{ \left( \alpha_l, \bigcup_{n \in \Gamma_{j^*}} r_n(\alpha_l) \right) \right\} \frown p_* \restriction (\alpha_l, \alpha_{l+1}).$$

Otherwise, we can find some $q_l' \in P_{\alpha_l}$ such that $q_l' \geq q_l^*$ and

$$q_l' \Vdash \text{“} \bigcup_{n \in \Gamma_2} \mathrm{Dom}(r_n(\alpha_l)) \cap \underset{\sim}{A}^{\alpha_l}_{N_{\varepsilon^*} \cap \omega_1} \text{ infinite”}.$$

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Let $j^* \stackrel{\text{def}}{=} 1$ and $k_{l+1} \stackrel{\text{def}}{=} k'_1$, and let

$$q^*_{l+1} \stackrel{\text{def}}{=} q'_l \frown \left\{ \left( \alpha_l, \bigcup_{n \in \Gamma_1} r_n(\alpha_l) \cup 0_{\bigcup_{n \in \Gamma_2} \mathrm{Dom}(r_n(\alpha_l)) \setminus \xi} \right) \right\} \frown p_* \restriction (\alpha_l, \alpha_{l+1}).$$

(Remember that for $n_1 \neq n_2$, we have that $\mathrm{Dom}(r_{n_1}(\alpha_l)) \setminus \xi$ and $\mathrm{Dom}(r_{n_2}(\alpha_l)) \setminus \xi$ are disjoint.)

Observe, similarly to Subclaim 2.19, that the choice of $\bar{r}$ and $\bar{\beta}$ in this proof was canonical. $\dashv$

Claim 3.7. $\Vdash_{P_{\omega_2}} ♣^{\bullet}$.

Proof of the Claim. Let $\langle \bar{N}^\delta = \langle N^\delta_i : i < \delta \rangle : \delta < \omega_1 \rangle$ be as in the proof of Claim 2.20, as well as $N^\delta$ for limit ordinal $\delta < \omega_1$.

For limit $\delta < \omega_1$, we define $n^*(\delta)$ and $\langle A^m_\delta : m \leq m^*(\delta) \rangle$ as follows.

If $\bar{\beta}_{\bar{N}^\delta, p^{N_\delta}, \underset{\sim}{\tau}^{N_\delta}}$ and $\bar{r}_{\bar{N}^\delta, p^{N_\delta}, \underset{\sim}{\tau}^{N_\delta}}$ are well defined, then we let $m^*(\delta) \stackrel{\text{def}}{=} m_{\bar{N}^\delta, p^{N_\delta}, \underset{\sim}{\tau}^{N_\delta}}$ and for $m \leq m^*(\delta)$ we let $A^m_\delta \stackrel{\text{def}}{=} \{ \beta^m_n : n < \omega \}$. Otherwise, we let $m^*_\delta = 0$ and $A^0_\delta$ be the range of any cofinal $\omega$-sequence in $\delta$.

We claim that

$$\langle \langle A^m_\delta : m \leq m^*(\delta) \rangle : \delta < \omega_1 \rangle$$

exemplifies that $V^P \models ♣^{\bullet}(\omega_1)$.

Suppose that

$$p^* \Vdash \text{“}\underset{\sim}{\tau}^* \in [\omega_1]^{\aleph_1} \text{ and for all } \delta, m \text{ we have } A^m_\delta \setminus \underset{\sim}{\tau}^* \neq \emptyset\text{”}.$$

Let $\bar{N}$, $C$, $\delta$ and $F$ be as in the proof of Claim 2.20. It is easily seen that $q_{n^*}$ obtained as in the proof of Main Claim 3.6 exemplifies a contradiction. $\dashv$

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