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GENERALIZED MARTIN'S AXIOM AND SOUSLIN'S HYPOTHESIS FOR HIGHER CARDINALS

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ABSTRACT
We consider different generalizations of Martin's Axiom to higher cardinals. For $\aleph_1$, assuming $CH + 2^{\aleph_1} > \aleph_2 + \square_{\aleph_1}$ we show that a generalized Martin's Axiom considered by Baumgartner settles the $\aleph_2$ Souslin Hypothesis ... the wrong way. We further show that, assuming $CH + 2^{\aleph_1} > \aleph_2$, a strengthening of this axiom implies $\square_{\aleph_1}$. Finally, we show that a seemingly innocuous further strengthening is inconsistent with $CH + 2^{\aleph_1} > \aleph_2$.

§1. Introduction

This paper grew out of our work on a "forcing principle" equivalent to morasses and most of the material here was first announced and circulated with preliminary versions of [6] (see (1.2) of [6] for the historical context). In §§4,5, we refer to notions and partial orderings introduced there, as well as to the results of [6].

In §2 we recall four versions of a generalized Martin's Axiom considered in [5]. In §3, we quickly sketch Jensen's method for forcing to obtain a $(\kappa, 1)$ morass from a $\square_\kappa$-sequence. In §4, we apply BA, the weakest axiom of §2, to the partial ordering of §3 to prove:

LEMMA 1. $CH + 2^{\aleph_1} > \aleph_2 + \square_{\aleph_1} + BA \Rightarrow$ *there's an* $(\aleph_1, 1)$-*morass.*

In virtue of (1), §1 of [6], we then have:

THEOREM 2. $CH + 2^{\aleph_1} > \aleph_2 + \square_{\aleph_1} + BA \Rightarrow$ *there's an* $\aleph_2$-*super-Souslin tree and thus* (viz. (2.3) of [6]) $\neg SH_{\aleph_2}$.

Since Jensen showed, [3],

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$$(1) \qquad \neg\square_{\aleph_1} \Rightarrow \aleph_2 \text{ is } (\text{Mahlo})^L,$$

we conclude

THEOREM 3. $CH + BA + SH_{\aleph_2} \Rightarrow \aleph_2$ *is* $(Mahlo)^L$.

If $2^{\aleph_1} > \aleph_2$, this is by Theorem 2 and (1); if $2^{\aleph_1} = \aleph_2$ this is by (1) and Gregory's result, [2],

$$(2) \qquad CH + 2^{\aleph_1} = \aleph_2 + \square_{\aleph_1} \Rightarrow \neg SH_{\aleph_2}.$$

Thus, in this context, BA does behave something like $2^{\aleph_1} = \aleph_2$. In the opposite direction, Laver and Shelah, [4], have obtained:

$$(3) \qquad \mathrm{Con}(ZF + \exists \text{ weakly compact } \kappa) \Rightarrow \mathrm{Con}(ZFC + CH + SH_{\aleph_2} + BA).$$

In their model $2^{\aleph_1} > \aleph_2$ and (necessarily) $\square_{\aleph_1}$ fails.

In §5 we introduce a stronger principle (S)″ and comment on how we were led to formulate it. We note that a slight modification of the relative consistency proof of [5] yields the relative consistency of (S)″. We then show:

LEMMA 4. $CH + 2^{\aleph_1} > \aleph_2 + (S)'' \Rightarrow \square_{\aleph_1}$.

Since $(S)'' \Rightarrow BA$, by Lemma 1, Theorem 2, we obtain:

THEOREM 5. $CH + 2^{\aleph_1} > \aleph_2 + (S)'' \Rightarrow$ *there's an* $(\aleph_1, 1)$-*morass and an* $\aleph_2$-*super-Souslin tree, hence* $\neg SH_{\aleph_2}$.

In §6 we show that (S)″ implies a generalized Martin's Axiom considered by Kunen, and used by him to prove analogues of Lemma 4 and Theorem 5.

In §7, we consider the question of generalizing the material of §§2–6 to regular $\kappa > \aleph_1$.

Finally, in §8, we prove a result of Shelah.

THEOREM 6. $CH + 2^{\aleph_1} > \aleph_2 \Rightarrow \neg(S)^*$.

Here (S)* is a seemingly innocuous strengthening of BA. This improves an earlier result of Devlin [1].

Our notation and terminology are intended to be either standard or self-explanatory.

§2. We recall the versions of a generalized Martin's Axiom for $\aleph_1$ considered in [5]. First some definitions.

DEFINITION. Let $\kappa > \omega$ be regular. A partial order **P** is κ-*linked* if $P =$

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$\bigcup_{\alpha<\kappa} P_\alpha$, where for $\alpha < \kappa$, $p, q \in P_\alpha \Rightarrow p, q$ are compatible in $\mathbf{P}$. $\mathbf{P}$ is *neatly-κ-linked* if $P = \bigcup_{\alpha<\kappa} P_\alpha$, where for $\alpha < \kappa$, $p, q \in P_\alpha \Rightarrow \{p, q\}$ has a lub in $\mathbf{P}$. $\mathbf{P}$ is κ^+-normal if whenever $(p_\alpha : \alpha < \kappa^+)$ is from P, there is C club $\subseteq \kappa^+$ and regressive $g : \kappa^+ \to \kappa^+$ such that if $(\alpha, \beta \in C$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \kappa$ and $g(\alpha) = g(\beta))$, then p_α, p_β are compatible in $\mathbf{P}$; the notion of *neatly κ^+-normal* is obtained by stengthening the conclusion to be: $\{p_\alpha, p_\beta\}$ has a lub in $\mathbf{P}$. Clearly (neatly) κ-linked $\Rightarrow$ (neatly) κ^+-normal. Finally, $\mathbf{P}$ is well-met if whenever p, q are compatible in $\mathbf{P}$ then $\{p, q\}$ has a lub in $\mathbf{P}$.

We shall now present four versions of a generalized Martin's Axiom for $\aleph_1$, a "weak" pair (S), and BA, considered by Shelah, and Baumgartner and a "strong" pair (S)' and (B.L.)'. See [8] for more on BA. In [5], p. 305, (3) Shelah already remarked on the possibility of such strengthenings and that the consistency proof for (S) goes through for (S)'. Strictly speaking, Shelah considered weakenings of (S), (S)' respectively, but he also observed in [5], p. 305, (2), that this distinction was empty. All of the principles have the same logical form: whenever $\mathbf{P}$ is a partial order such that card $P < 2^{\aleph_1}$, such that $\mathbf{P}$ satisfies certain additional hypotheses (H), and whenever $\mathcal{D}$ is a family of dense subsets of $\mathbf{P}$ with card$\mathcal{D} < 2^{\aleph_1}$, then there is a $\mathcal{D}$-generic subset of $\mathbf{P}$.

Hence, in stating the different versions, we shall limit ourselves to stating the different hypotheses on $\mathbf{P}$.

BA: $\mathbf{P}$ is $\aleph_1$-closed, $\aleph_1$-linked and well-met.

(S): $\mathbf{P}$ is $\aleph_1$-closed, $\aleph_2$-normal and well-met.

BA′: $\mathbf{P}$ is $\aleph_1$-closed and neatly-$\aleph_1$-linked.

(S)′: $\mathbf{P}$ is $\aleph_1$-closed and neatly-$\aleph_2$-normal.

Clearly then (S) $\Rightarrow$ BA, BA′ $\Rightarrow$ BA, and (S)′ $\Rightarrow$ (S), (S)′ $\Rightarrow$ BA′.

§3. We very rapidly sketch Jensen's conditions for obtaining a $(\kappa, 1)$-morass from a $\square_\kappa$-sequence where $\kappa \geqq \omega_1$ is regular. For a fuller development, and for proofs, see [7]. Again, we assume $\kappa^{<\kappa} = \kappa$. We indicate at which points we must assume $2^\kappa > \kappa^+$.

Fix $C = (C_\alpha : \alpha \in \mathrm{Lim} \cap \kappa^+)$ a $\square_\kappa$-sequence. Let

$$S_\kappa = \{\nu : \kappa < \nu < \kappa^+, \nu \text{ is a limit of ordinals } \tau \text{ such that } L_\tau^C \models \mathrm{ZF}^-\}.$$

For $\nu \in S_\kappa$, let $\mathfrak{A}_\nu = (L_\nu^{C|\nu}, \in, C \mid \nu, C_\nu)$, h_ν = the canonical Σ_1-Skolem function for $\mathfrak{A}_\nu$, so that $|\mathfrak{A}_\nu| = h_\nu''(\omega \times \kappa)$. Let

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$$B'_\nu = \{\alpha < \kappa : h''_\nu(\omega \times \alpha) \cap \kappa = \alpha\},$$

and for $\alpha \in B'_\nu$, let $X_{\alpha,\nu} = h''_\nu(\omega \times \alpha)$, and define $f_{\alpha,\nu}$, $\bar{\mathfrak{A}}_{\alpha,\nu}$ by taking $|\bar{\mathfrak{A}}_{\alpha,\nu}|$ transitive, $f_{\alpha,\nu} : \bar{\mathfrak{A}}_{\alpha,\nu} \tilde{\leftrightarrow} \mathfrak{A}_\nu \restriction X_{\alpha,\nu}$. Let

$$B_\nu = \{\alpha \in B'_\nu : \mathfrak{A}_\nu \restriction X_{\alpha,\nu} <_Q \mathfrak{A}_\nu\},$$

where Q is the "cofinally many ordinals" quantifier, a Q-formula is a formula of the form $Qx\varphi x$ where φ is Σ_1, and $<_Q$ means "substructure in which truth of Q-formulas with parameters from the substructure is absolute". The important property of $<_Q$ for $\in$-models having universe of the form L^A_α is that cofinal-Σ_0-elementary substructure $\Rightarrow <_Q$. As usual, B_ν, B'_ν are club subsets of κ.

Let $K = \{\bar{\mathfrak{A}}_{\alpha,\nu} : \nu \in S_\kappa, \alpha \in B'_\nu\}$, and for $\alpha \in \bigcup_{\nu \in S_\kappa} B'_\nu$, let $K_\alpha = \{\bar{\mathfrak{A}}_{\alpha,\nu} : \alpha \in B'_\nu\}$. If $\bar{\mathfrak{A}} \in K_\alpha$, note that $\bar{\mathfrak{A}}$ has the form $(L^{\bar{c}}_{\bar{\nu}}, \in, \bar{c}, \bar{c}_{\bar{\nu}})$ where $\bar{c} \cup \{(\bar{\nu}, \bar{c}_{\bar{\nu}})\}$ has the properties of a "$\square_\alpha$-sequence" defined out as far as $\bar{\nu}+1$. For $\bar{\mathfrak{A}}, \bar{\mathfrak{A}}' \in K_\alpha$, if $\bar{\mathfrak{A}} = (L^{\bar{c}}_{\bar{\nu}}, \in, \bar{c}, \bar{c}_{\bar{\nu}})$, $\bar{\mathfrak{A}}' = (L^{\bar{c}'}_{\bar{\nu}'}, \in, \bar{c}', \bar{c}_{\bar{\nu}'})$, set $\bar{\mathfrak{A}} \sqsubseteq_\alpha \bar{\mathfrak{A}}'$ iff $\bar{\nu} < \bar{\nu}'$, $\bar{c} \cup \{(\bar{\nu}, \bar{c}_{\bar{\nu}})\} = \bar{c}' \restriction \bar{\nu}+1$. Then $(K_\alpha, \sqsubseteq_\alpha)$ is a tree of height $\leqq (\operatorname{card}\alpha)^+$.

Set $(s, w, u) \in P^C$ iff $s \subseteq \bigcup_{\nu \in S} B_\nu$ is closed, $\operatorname{card} s < \kappa$, w is a function, $\operatorname{dom} w = s$, for $\alpha \in s$, w_α is an initial segment of a branch of $(K_\alpha, \sqsubseteq_\alpha)$, $\operatorname{card} w_\alpha < \kappa$, $u \subseteq \bigcup_{\alpha \in s} \{(\alpha, \nu) : \alpha \in B_\nu\}$, $\operatorname{card} u < \kappa$ such that

(i) if $\alpha \in s$, then $\{\nu : (\alpha, \nu) \in u\} \neq \emptyset$,

(ii) if $(\alpha, \nu) \in u$, then $\bar{\mathfrak{A}}_{\alpha,\nu} \in w_\alpha$,

(iii) if $(\alpha, \nu) \in u$, and $\alpha' \in s$, $\alpha < \alpha'$, then $(\alpha', \nu) \in u$.

If $(s, w, u), (s', w', u') \in P^C$, set $(s, w, u) \leqq (s', w', u')$ iff s' is an end-extension of s, $w' \supseteq w$, $u' \supseteq u$. $\mathbf{P}^C = (P^C, \geqq)$.

The important properties of $\mathbf{P}^C$ are:

LEMMA. *$\mathbf{P}^C$ is κ-closed; if $(s, w, u), (s, w, u') \in P^C$ then $(s, w, u), (s, w, u')$ are compatible in $\mathbf{P}^C$; if $(s, w, u), (s', w', u')$ are compatible in $\mathbf{P}^C$, then $(s \cup s', w \cup w', u \cup u') \in P^C$ and is the* lub *in $\mathbf{P}^C$ of $\{(s, w, u), (s, w, u')\}$.*

PROOF. See [7].

§4. Thus we have:

LEMMA 1. $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{BA} + \square_{\aleph_1} \Rightarrow$ *there is an $(\aleph_1, 1)$-morass.*

PROOF. Let C be a $\square_{\aleph_1}$-sequence. CH guarantees that $\{(s, w) : (\exists u)((s, w, u) \in P^C)\}$ has power $\aleph_1$. Hence by §3, $\mathbf{P}^C$ satisfies the hypotheses of BA. Let $\mathscr{D} = \{D^\alpha_\nu : \nu \in S_{\omega_1}$ and $\alpha < \omega_1\}$, where

$$D^\alpha_\nu = \{(s, w, u) \in P^C : (\exists \alpha')((\alpha', \nu) \in u \text{ and } \alpha \leqq \alpha')\}.$$

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It is proved in [7] that each D_ν^α is dense, and that if there's a $\mathcal{D}$-generic subset of $\mathbf{P}^C$ then there's an $(\aleph_1, 1)$-morass (actually this is proved assuming full genericity, but it's clear from the proof that $\mathcal{D}$-genericity suffices). Also, $\operatorname{card} P^C = \aleph_2$, again, since we're assuming CH. But then if $2^{\aleph_1} > \aleph_2$, $\operatorname{card} \mathcal{D}$, $\operatorname{card} P^C < 2^{\aleph_1}$.

Accordingly, by (1), §1 of [6] we have:

THEOREM 2. $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{BA} + \square_{\aleph_1} \Rightarrow$ *there's an $\aleph_2$-super-Souslin tree.*

As we've already mentioned in §1, Laver and Shelah proved that Con(ZF + $\exists$ weakly compact κ) $\Rightarrow$ Con(ZFC + CH + $2^{\aleph_1} > \aleph_2 + \mathrm{SH}_{\aleph_2}$ + BA). By Theorem 2 we get a partial converse:

THEOREM 3. $\mathrm{CH} + \mathrm{SH}_{\aleph_2} + \mathrm{BA} \Rightarrow \aleph_2$ *is* $(Mahlo)^L$.

PROOF. By Gregory's result, [2], (2) of the Introduction, we may assume $2^{\aleph_1} > \aleph_2$. By Theorem 2, $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{SH}_{\aleph_2} + \mathrm{BA} \Rightarrow \neg \square_{\aleph_1}$, but it's well-known, see e.g. [3], §5, that $\neg \square_{\aleph_1} \Rightarrow \aleph_2$ is $(\mathrm{Mahlo})^L$.

§5. We had originally claimed that $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + (\mathrm{S}) \Rightarrow \square_{\aleph_1}$. Then Velleman pointed out to us that if this were correct, then we'd also be able to show that $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{BA} \Rightarrow \square_{\aleph_1}$. But, then, by Theorem 3, $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{BA}$ would also imply the existence of $(\aleph_1, 1)$-morasses, and hence $\aleph_2$-super-Souslin trees. This however would contradict the theorem of Laver–Shelah [4] that it's consistent, relative to the existence of a weakly compact cardinal, that $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 + \mathrm{BA} + \mathrm{SH}_{\aleph_2}$ hold.

In fact, we must strengthen not only (S), but (S)$'$ to obtain the desired result. So, we consider that version (S)$''$ of generalized Martin's Axiom which applies to those $\mathbf{P}$ such that $\operatorname{card} P < 2^{\aleph_1}$ and

(S)$''$: there are binary relations R_{ij} on P ($i \leqq j < \omega_2$, $\operatorname{cf} i = \operatorname{cf} j = \aleph_1$) such that:

(a) whenever $p \in P$, there is $i_0 < \omega_2$ such that whenever $i_0 \leqq j < \omega_2$ and $\operatorname{cf} j = \aleph_1$, then $R_{jj}(p, p)$,

(b) whenever $(p_\alpha : \alpha < \omega_2)$ is a family of elements of P, there is regressive $g : \omega_2 \to \omega_2$ and club $C \subseteq \omega_2$ such that if $\operatorname{cf} \alpha = \operatorname{cf} \beta = \aleph_1$, $\alpha, \beta \in C$ and $g(\alpha) = g(\beta)$, then $R_{\alpha\beta}(p_\alpha, p_\beta)$,

(c) suppose that $(p^n : n < \omega)$, $(q^n < \omega)$ are increasing sequences from P, $i \leqq j < \omega_2$, $\operatorname{cf} i = \operatorname{cf} j = \aleph_1$, and for all $n < \omega$, $R_{ij}(p^n, q^n)$; then $\{p^n : n < \omega\} \cup \{q^n : n < \omega\}$ has an upper bound in $\mathbf{P}$.

Note that (a) together with (c) imply that $\mathbf{P}$ is $\aleph_1$-closed, choosing $i = j$ sufficiently large and taking $p_n = q_n$. Further, $R_{ij}(p, q)$ implies that p, q are compatible in **P** because in (c) we can take $p_n = p$, $q_n = q$. Thus (S)″ is seen to be a strengthening of (S)′. In fact, even (a) can be dropped and then (b), (c) are just what is needed to push through the relative consistency proof for (S) in [5].

The motivation for (S)″ is to find a principle similar to (S)′ but where we no longer require *neat* $\aleph_2$-normality; (c) represents a weak version of "neatness", and we will see below that we cannot entirely dispense with some form of "neatness".

THEOREM. *Let* **P** *be the countable conditions for forcing* $\square_{\aleph_1}$ *of* §4 *of* [6]. *Then, if* $\mathrm{CH}+2^{\aleph_1}>\aleph_2$, **P** *satisfies the hypotheses of* (S)″, *and so*:

$$\mathrm{CH}+2^{\aleph_1}>\aleph_2+(\mathrm{S})'' \Rightarrow \square_{\aleph_1},$$

and thus, since (S)″ $\Rightarrow$ BA, $\mathrm{CH}+2^{\aleph_1}>\aleph_2+(\mathrm{S})'' \Rightarrow$ *there's an* $(\aleph_1,1)$*-morass and hence an* $\aleph_2$*-super-Souslin tree.*

PROOF. The reader should refer to §4 of [6] for the definition of **P** and $\mathcal{T}$. Let $i \leqq j < \omega_2$, $\mathrm{cf}\, i = \mathrm{cf}\, j = \aleph_1$. Set $R_{ij}(p,q)$ iff there are $\tau \in \mathcal{T}$, $s, s' \in [\omega_2]^{\lg \tau}$, and f such that $p = \tau(s)$, $q = \tau(s')$, $s' = f \circ s$, f is increasing, $f \restriction i = \mathrm{id} \restriction i$, $f(i) = j$, and range $s \subsetneqq i$.

The reader can then easily check (a), (c), using the arguments of §4. We verify (b). The argument yields a much more general conclusion: for example, it applies to any **P** which, in the terminology of [6], is $\aleph_1$-special and which satisfies the Amalgamation Property.

Let $p_i = (a^i, c^i) \in P$ for $i < \omega_2$. Let π be a reasonable pairing function. It's easy enough to find club $C \subseteq \omega_2$ such that for $j \in C$, $\pi'' j \times j \subseteq j$ and if $i < j$, then $a^i \subseteq j$. For $i < \omega_2$ if $\mathrm{cf}\, i = \aleph_1$, $\bar{g}(i) = \sup(a^i \cap i) < i$. By CH, $\mathcal{T}$, the set of terms for **P**, has power $\aleph_1$, so fix a 1-1 enumeration $(\tau_\zeta : \zeta < \omega_1)$ of $\mathcal{T}$. For $i < \omega_2$, let $\zeta(i)$ be the unique $\zeta < \omega_1$ such that for some (unique) s, $p_i = \tau_\zeta(s)$. Also, by CH, if $\alpha < \omega_2$, then card $[\alpha]^{<\omega_1} = \aleph_1$, so fix a system of 1-1 enumerations of $[\alpha]^{<\omega_1}$, $(x^\alpha_\xi : \xi < \omega_1)$, for $\alpha < \omega_2$. For $i < \omega_2$, and $\mathrm{cf}\, i = \aleph_1$, let $\xi(i)$ be the unique $\xi < \omega_1$ such that $a^i \cap i = X^{\bar{g}(i)}_\xi$. Then, for such i, $g(i) = \pi(\bar{g}(i), \pi(\zeta(i), \xi(i))) < i$. For $i < \omega_2$, $i = 0$, i successor or $\mathrm{cf}\, i = \omega$, set $g(i) = 0$. This g suffices.

Finally, assuming CH and $2^{\aleph_1} > \aleph_2$, card $P = \aleph_2 < 2^{\aleph_1}$, and card $\mathcal{D} = \aleph_2 < 2^{\aleph_1}$, where, for $\alpha < \omega_2$ and $\mathrm{cf}\, \alpha = \aleph_1$, and for $\xi < \omega_1$,

$$D_{\alpha,i} = \{(a,c) \in P : \alpha \in a \text{ and o.t. } C_\alpha \geqq \xi\}, \qquad \text{and}$$

$$\mathcal{D} = \{D_{\alpha,\xi} : \alpha < \omega_2, \mathrm{cf}\, \alpha = \aleph_1, \xi < \omega_1\}.$$

By the argument of (4.7) of [6] the existence of a $\mathcal{D}$-generic subset of **P** yields $\square_{\aleph_1}$. This completes the proof.

Unfortunately **P** is not well-met, nor even neatly $\aleph_2$-normal because of (4.2) (iii) of [6]; this is already implicit in (4.4) of [6]. On the other hand, if we drop requirement (iii) of (4.2) of [6], **P** no longer has enough properties of $\aleph_1$-closure (even though it will have an $\aleph_1$-closed dense subset — namely the set of conditions which do satisfy (iii) of (4.2)).

In fact, (S)″ can be applied directly to Velleman's morass partial-order of §5 of [6].

PROPOSITION. *Let* **P** *be the morass partial-order of* [6] §5 *for* $\mu = \aleph_1$. *Then* **P** *satisfies the hypotheses of* (S)″, *assuming* $\mathrm{CH} + 2^{\aleph_1} > \aleph_2$.

PROOF. The reader should refer to §5 of [6] for the definition of **P**.

Let $i \leqq j < \omega_2$, $\mathrm{cf}\, i = \mathrm{cf}\, j = \aleph_1$. Then set $R_{ij}(p, q)$ iff $p \restriction \aleph_1 = q \restriction \aleph_1$, $S^p_{\aleph_1} \subseteq j$, and there is increasing g such that $S^q_{\aleph_1} = g'' S^p_{\aleph_1}$, $g \restriction i = \mathrm{id} \restriction i$, $g(i) = j$. Once again the verification of (a), (c) is easy, using arguments of §5 of [6]. The verification of (b) is similar to the above verification for $\square_{\aleph_1}$.

§6. Kunen has considered a generalized Martin's Axiom (see [10]) which, assuming CH and $2^{\aleph_1} > \aleph_2$, also implies $\square_{\aleph_1}$. We show here that (S)″ implies Kunen's axiom.

If a is a set of ordinals, let $\bar{a}$ = the closure of a; i.e., $\bar{a} = a \cup a^*$. If p is a set of finite sets of ordinals, let $d(p) = \bigcup p$. If $\overline{d(p)}$ and $\overline{d(q)}$ have the same order type, let f_{pq} be the unique order isomorphism from $\overline{d(p)}$ onto $\overline{d(q)}$. If $A \subseteq \mathrm{OR}$, set $p \cong_A q$ iff o.t. $(\overline{d(p)}) = \text{o.t.}(\overline{d(q)})$, $f_{pq} \restriction A = \mathrm{id} \restriction A$ and $q = \{f''_{pq} a : a \in p\}$. These notions are due to Kunen, as well as the following definition:

DEFINITION. **P** is *admissible* iff

(1) $p \in P \Rightarrow p$ is a countable set of finite sets of ordinals.

(2) $p \leqq q \Rightarrow p \subseteq q$ (we are keeping with our convention that $p \leqq q$ means q gives more information).

(3) If $(p_n : n < \omega)$ is increasing then $\bigcup_n p_n$ is an upper bound.

(4) If $p \cong_{(\omega_1 \cup (d(p) \cap d(q)))} q$ and if: $\overline{d(p)} \cap \omega_2$, $\overline{d(q)} \cap \omega_2$ are such that there are e, a, b such that $\overline{d(p)} \cap \omega_2 = e \cup a$, $\overline{d(q)} \cap \omega_2 = e \cup b$, all members of e precede all members of $a \cup b$, and either all members of a precede all members of b, or conversely (i.e., p, q have the strong Δ-property), then p, q are compatible.

PROPOSITION. (CH): *If* **P** *is admissible, then* **P** *satisfies the hypotheses of* (S)″. *Thus, letting* (A) *be that form of* GMA *for* $\aleph_1$ *which applies to admissible* **P**, $\mathrm{CH} + (\mathrm{S})'' \Rightarrow \mathrm{CH} + \mathrm{A}$.

PROOF. Assuming $\mathbf{P}$ is admissible, we must define binary relations R_{ij} for $i \leqq j$, $\mathrm{cf}\, i = \mathrm{cf}\, j = \aleph_1$, $i, j < \omega_2$, which satisfy (a), (b), (c) of (S)$''$.

Define $R_{ij}(p, q)$ iff $p \cong_{i \cup (d(p) \cap d(q))} q$, $\overline{d(p)} \cap \omega_2 \subseteq j$, $\overline{d(q)} \cap j \subseteq i$.

Clearly (a) is satisfied if we choose i_0 with $\mathrm{cf}\, i_0 = \aleph_1$, $i_0 < \omega_2$ such that $\overline{d(p)} \cap \omega_2 \subseteq i_0$. For (c), we use that by admissibility $\bigcup_n p^n$ is an upper bound for $(p^n : n < \omega)$, $\bigcup_n q^n$ is an upper bound for $(q^n : n < \omega)$, so it suffices to see that $\bigcup_n p^n$, $\bigcup_n q^n$ are compatible. In fact, it's easily seen that $R_{ij}(\bigcup_n p^n, \bigcup_n q^n)$, since $\mathrm{cf}\, i = \mathrm{cf}\, j = \aleph_1$ and for all n $R_{ij}(p^n, q^n)$. But then clearly, setting $p^* = \bigcup_n p^n$, $q^* = \bigcup_n q^n$, we have $p^* \cong_{(\omega_1 \cup (d(p^*) \cap d(q^*)))} q^*$, so that by (4) of admissibility, p^*, q^* are compatible.

It's in proving (b) that we use CH.

Let $(p_\alpha : \alpha < \omega_2)$ be a family of elements of P, and let π be a reasonable pairing function. Let $X = \bigcup_{\alpha < \omega_2} \overline{d(p)}$. Thus $\mathrm{card}\, X \leqq \aleph_2$. Let h be a bijection of X with the set of odd ordinals less than ω_2. For $i < \omega_2$, let $a^i = (h''\overline{d(p_i)}) \cup \{2 \cdot \xi : \xi \in \overline{d(p_i)} \cap \omega_2\}$. As in the proof of the Theorem in §5, we can easily find club $C \subseteq \omega_2$ such that for $j \in C$, $\pi'' j \times j \subseteq j$ and if $i < j$ then $a^i \subseteq j$. For $i \in C$, $\mathrm{cf}\, i = \aleph_1$ set $\bar{g}(i) = \sup(a^i \cap i)$, so $\bar{g}(i) < i$. By CH the following set has power $\aleph_1 : \{(\eta, x) : \eta < \omega_1$, x a set of finite subsets of $\eta\}$, so let $((\eta_\zeta, x_\zeta) : \zeta < \omega_1)$ enumerate it without repetitions. For $i < \omega_2$, define p'_i as follows: let $\sigma_i : \theta_i \to d(p_i)$ be the unique increasing function, where $\theta_i = \mathrm{o.t.}\,(d(p_i))$; then $p'_i = \{\sigma_i^{-1}(x) : x \in p_i\}$. Let $\zeta(i) =$ the unique ζ such that $\theta_i = \eta_\zeta$, $p'_i = x_\zeta$. Also let $\xi(i)$ be as in the proof of the Theorem in §5. Then, as there, g suffices.

§7. The material of §§3,4 generalizes easily to arbitrary regular uncountable μ replacing $\aleph_1$. We take BA_μ in the form which guarantees $\mathcal{D}$-generic subsets for those $\mathbf{P}$ and $\mathcal{D}$ such that card P, card $\mathcal{D} < 2^\mu$, $\mathcal{D}$ a family of dense subsets of $\mathbf{P}$, $\mathbf{P}$ μ-closed, μ-linked, and well-met, and then, exactly as in §§3,4, $\mu^{<\mu} = \mu + 2^\mu > \mu^+ + \mathrm{BA} + \Box_\mu \Rightarrow$ there is a $(\mu, 1)$-morass. For the application to SH_{μ^+} via super-Souslin trees and (1), §1 of [6] we require that μ be a successor cardinal.

Concerning the material of §5, the situation is different depending on whether we seek the application to morasses, or the application to $\Box$. For the former, the generalization is straightforward. We take (S)$''_\mu$ in the form that guarantees $\mathcal{D}$-generic subsets of P for those $\mathcal{D}$ and $\mathbf{P}$ such that card $\mathcal{D}$, card $P < 2^\mu$, $\mathcal{D}$ is a family of dense subsets of $\mathbf{P}$, and

there are binary relations R_{ij} on P ($i \leqq j < \mu^+$, $\mathrm{cf}\, i = \mathrm{cf}\, j = \mu$) such that:

(a)$_\mu$ whenever $p \in P$ there is $i_0 < \mu^+$ such that whenever $i_0 \leqq j < \mu^+$ and $\mathrm{cf}\, j = \mu$ then $R_{jj}(p)$, (b)$_\mu$ whenever $(p_\alpha : \alpha < \mu^+)$ is a family of elements of P, there is a regressive $g : \mu^+ \to \mu^+$ and club $C \subseteq \mu^+$, etc. (as in (b)),

(c)$_\mu$ suppose that $\theta < \mu$ and that $(p^\alpha : \alpha < \theta)$, $(q^\alpha : \alpha < \theta)$ are increasing sequences from P, $i \leqq j < \mu^+$, $\operatorname{cf} i = \operatorname{cf} j = \mu$ and for all $\alpha < \theta$, $R_{ij}(p^\alpha, q^\alpha)$. Then $\{p^\alpha : \alpha < \theta\} \cup \{q^\alpha : \alpha < \theta\}$ has an upper bound in $\mathbf{P}$.

Then once again we have: $\mu^{<\mu} + 2^\mu > \mu^+ + (\mathrm{S})''_\mu \Rightarrow$ there's a $(\mu, 1)$-morass, applying $(\mathrm{S})''_\mu$ to Velleman's morass-order for μ. Once again, μ must be a successor cardinal for the application to SH_{μ^+}.

If we seek to obtain $\square_\mu$ from some generalization of $(\mathrm{S})''$ to μ, we must weaken (c)$_\mu$ above, thus strengthening the principle. This is because, in the order analogous to the countable conditions for forcing $\square_{\aleph_1}$ of §4 of [6], we must require, e.g., that a be closed. So that, in particular, if $p \in P$, $p = (a, c)$ then a has a largest element, $\sigma(p)$. Thus $\mathbf{P}$ will no longer be μ-closed: $\mathbf{P}$ will be $\aleph_1$-closed, and if $\operatorname{cf} \theta > \omega$ and $(p^\alpha : \alpha < \theta)$ is an increasing sequence with an upper bound, then $(p^\alpha : \alpha < \theta)$ has a least upper bound. $(p_\alpha : \alpha < \theta)$ will have an upper bound just in case there is club $C \subseteq \theta$ such that setting $p^\alpha = (a^\alpha, c^\alpha)$, $\sigma_\alpha = \sigma(p_\alpha)$ for $\alpha < \theta$, we have:

$$\alpha < \beta, \quad \alpha, \beta \in C \Rightarrow \sigma_\alpha \in (c^\beta_{\sigma_\beta})^*.$$

Thus, $\mathbf{P}$ is strategically μ-closed (see (1.1) of [6]) where NON-EMPTY's strategy is to extend $p^{\lambda+2n+1}$ in such a way as to make $\sigma_{\lambda+2n}$ a limit point of $c^{\lambda+2n+2}_{\sigma_{\lambda+2n+2}}$ and for $0 < \lambda$, NON-EMPTY takes $\sigma_\lambda = \sup_{\alpha<\lambda} \sigma_\alpha$, and sets

$$c^\lambda_{\sigma_\lambda} = \bigcup_{\substack{\alpha<\lambda \\ \alpha \text{ even}}} c^\alpha_{\sigma_\alpha}.$$

Thus, we strengthen $(\mathrm{S})''_\mu$ by dropping (a)$_\mu$ but requiring that $\mathbf{P}$ be strategically μ-closed, say with winning strategy s, and weakening (c)$_\mu$ to:

(c)$'_\mu$ Suppose that $i \leqq j < \mu^+$, $\operatorname{cf} i = \operatorname{cf} j = \mu$, that $\lambda < \mu$ and that $(p^\alpha : \alpha < \lambda)$, $(q^\alpha : \alpha < \lambda)$ are runs of the length-λ-game where NON-EMPTY follows s and such that for $\alpha < \lambda$, $R_{ij}(p^\alpha, q^\alpha)$. Then $\{p^\alpha : \alpha < \lambda\} \cup \{q^\alpha : \alpha < \lambda\}$ has an upper bound in $\mathbf{P}$.

All of the above principles can be proved relatively consistent by appropriate generalizations of the consistency proof for (S) in [5].

§8. In this section, we show that there is a limit to how far we can go in strengthening such principles. Some time ago Devlin proved, [1], that assuming

$\mathrm{CH}+2^{\aleph_1}>\aleph_2$, there are $(\mathbf{P},\mathscr{D})$ such that card P, card $\mathscr{D}<2^{\aleph_1}$, $\mathscr{D}$ a family of dense subsets of $\mathbf{P}$, $\mathbf{P}$ $(\aleph_0,\infty)$-distributive and $\aleph_2$-c.c., but there is no $\mathscr{D}$-generic subset of $\mathbf{P}$.

DEFINITION. Let (S)* be that version of generalised Martin's Axiom which applies to those $\mathbf{P}$ which are $\aleph_1$-closed with least upper bounds and $\aleph_1$-linked.

THEOREM 6 (Shelah). $\mathrm{CH}+2^{\aleph_1}>\aleph_2 \Rightarrow \neg\,(\mathrm{S})^*$.

PROOF. For f, g functions with domain $\aleph_1$, let

$$\delta(f,g)=\inf\{\alpha\leqq\omega_1 : f(\alpha)\neq g(\alpha)\text{ or }\alpha=\omega_1\}.$$

Let $\boldsymbol{f}$, $\boldsymbol{g}$ be such that $\boldsymbol{f}=(f_i : i<\omega_2)$, $\boldsymbol{g}=(g_i : i<\omega_2)$ where f_i, g_i are functions with domain ω_1 $(i<\omega_2)$. Set $\boldsymbol{f}<^*\boldsymbol{g}$ iff

(*): for all $i<j<\omega_2$, $\delta(f_i,f_j)<\delta(g_i,g_j)$, or $\delta(f_i,f_j)=0$.

It's easily seen that $<^*$ is a partial order on ${}^{\omega_2}({}^{\omega_1}\omega_1)$.

LEMMA (CH). *$<^*$ is well-founded on ${}^{\omega_2}({}^{\omega_1}\omega_1)$.*

PROOF. Suppose $\boldsymbol{f}^{n+1}<^*\boldsymbol{f}^n$ $(n<\omega)$, where $\boldsymbol{f}^n=(f_i^n : i<\omega_2)$ and $f_i^n : \omega_1\to\omega_1$. For $i<\omega_2$, let $f_i=(f_i^n : n<\omega)$, and let $f_i(0)=(f_i^n(0) : n<\omega)$. Then $\mathrm{card}\{f_i(0) : i<\omega_2\}\leqq\aleph_1^{\aleph_0}=\aleph_1$, so there are $i<j<\omega_2$ such that for all $n<\omega$, $f_i^n(0)=f_j^n(0)$. Thus, for $n<\omega$, $\delta(f_i^n,f_j^n)>0$ and since $\boldsymbol{f}^{n+1}<^*\boldsymbol{f}^n$, clearly $\delta(f_i^{n+1},f_j^{n+1})<\delta(f_i^n,f_j^n)$. But then $(\delta(f_i^n,f_j^n) : n<\omega)$ is a decreasing sequence of ordinals, contradiction.

LEMMA (CH). *Let $\boldsymbol{f}\in{}^{\omega_2}({}^{\omega_1}\omega_1)$. There is a partial order $\mathbf{P}=\mathbf{P}_f$ and $\mathscr{D}=(D_\alpha : \alpha<\omega_2)$ such that each D_α is a dense subset of $\mathbf{P}$ and:*

(a) card $P=\aleph_2$, *$\mathbf{P}$ is $\aleph_1$-closed with least upper bounds,*

(b) *$\mathbf{P}$ is $\aleph_1$-linked,*

(c) *if G is $\mathscr{D}$-generic then G gives rise to a $\boldsymbol{g}\in{}^{\omega_2}({}^{\omega_1}\omega_1)$ with $\boldsymbol{g}<^*\boldsymbol{f}$.*

PROOF. $p\in P$ iff $p=(\xi,a,\boldsymbol{g})$ where $\xi<\omega_1$, $a\in[\omega_2]^{<\omega_1}$, $\boldsymbol{g}=(g_i : i\in a)$, each $g_i:\xi\to\omega_1$ and for $i<j\in a$, if $\xi>0$, then either $(\delta(f_i,f_j)=0$ and $g_i(0)=g_j(0))$ or there is $\delta_{i,j}<\inf(\xi,\delta(f_i,f_j))$ such that $g_i(\delta_{i,j})\neq g_j(\delta_{i,j})$.

Set $(\xi,a,\boldsymbol{g})\leqq(\xi',a',\boldsymbol{g}')$ iff $\xi\leqq\xi'$, $a\subseteq a'$ and for $i\in a$, $g_i=g'_i\restriction\xi$.

Then (a) is clear. For (b), first suppose that $(\xi,a,\boldsymbol{g}),(\xi,a',\boldsymbol{g}')\in P$, that h is a bijection of a and a', that $h\restriction a\cap a'=\mathrm{id}\restriction a\cap a'$, that for $i\in a$, $g_i=g'_{h(i)}$ and $f_{h(i)}\restriction\xi+1=f_i\restriction\xi+1$.

We show that $(\xi,a,\boldsymbol{g}),(\xi,a',\boldsymbol{g}')$ are compatible. Let $a''=a\cup a'$; for $i\in a$, set

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$g''_i = g_i \cup \{(\xi, 0)\}$; for $i \in a' \setminus a$, set $g''_i = g'_i \cup \{(\xi, 1)\}$. Then $(\xi+1, a'', g'') \in P$, $(\xi + 1, a'', g'') \geqq (\xi, a, g)$, (ξ, a', g').

Now we use, without proof, the following:

PROPOSITION (CH). *There is a partition* $(X_\alpha : \alpha < \omega_1)$ *of* $[\omega_2]^{<\omega_1}$ *such that for all* $\alpha < \omega_1$, $a, b \in X_\alpha \Rightarrow$ o.t. $a =$ o.t. b *and* $a \cap b$ *is an initial segment of* $a \cup b$.

This will permit us to define P_α as required. Note that by CH, if η, $\theta < \omega_1$, then card $({}^{\eta}({}^{\theta}\omega_1)) = \aleph_1$, and hence $\{(f_i \mid \theta : i < \text{o.t.}\, a) : a \in [\omega_2]^{<\omega_1}\}$ has power $\aleph_1$.

So, set $(\xi, a, g) \sim (\xi', a', g')$ iff $\xi = \xi'$, a, a', come from the same X_α (as in the Proposition above), and letting $\eta =$ o.t. $a =$ o.t. a', letting $(\alpha_i : i < \eta)$, $(\alpha'_i : i < \eta)$ increasingly enumerate a, a' respectively, for $i < \eta$, $g_{\alpha_i} = g'_{\alpha'_i}$, $f_{\alpha_i} \mid \xi + 1 = f_{\alpha'_i} \mid \xi + 1$. Then, it's easily seen that $\sim$ is an equivalence relation, and by the above there are $\aleph_1$ equivalence classes. So, let $(P_\alpha : \alpha < \aleph_1)$ enumerate the equivalence classes.

For (c), for $i < \omega_2$, $\zeta < \omega_1$, set $(\xi, a, g) \in D_{i,\zeta}$ iff $i \in a$, $\zeta < \xi$. Then $D_{i,\zeta}$ is dense, and $\mathcal{D} = \{D_{i,\zeta} : i < \omega_2, \zeta < \omega_1\}$ has power $\aleph_2$. And clearly, if G is $\mathcal{D}$-generic, then setting $g^*(i)(\zeta) = e$ iff there is $(\xi, a, g) \in G \cap D_{i,\zeta}$ s.t. $g_i(\zeta) = e$ $(i < \omega_2$, $\zeta < \omega_1$, $e = 0, 1)$, $g^* <^* f$.

REMARK. By a more careful analysis, we can strengthen (b) to:

(b′) $P = \bigcup_{\alpha < \omega_1} P_\alpha$ such that if $\alpha < \omega_1$ and $Y \subseteq P_\alpha$ is countable then Y has an upper bound in **P**.

Part of the additional work to be done will be to require that $(\xi, a, g) \sim (\xi', a', g')$ iff $\xi = \xi'$, a, a', come from the same X_α and (with the above notation) for $i < \eta$, $g_{\alpha_i} = g'_{\alpha'_i}$, $f_{\alpha_i} \mid \xi + \omega = f_{\alpha'_i} \mid \xi + \omega$.

Thus, assuming CH, **P** satisfies the hypotheses of (S)*, and so, assuming $2^{\aleph_1} > \aleph_2$, (S)* $\Rightarrow$ for all $f \in {}^{\omega_2}({}^{\omega_1}\omega_1)$ there is $g \in {}^{\omega_2}({}^{\omega_1}\omega_1)$ with $g <^* f$, which contradicts the well-foundedness of $<^*$. Hence $\mathrm{CH} + 2^{\aleph_1} > \aleph_2 \Rightarrow \neg\,(\mathrm{S})^*$.

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