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SIMPLE UNSTABLE THEORIES

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We point out a class of unstable theories which are simple, and develop for them an analog to the basic theorems on stable theories.

0. Introduction

In [4] the property 'T is stable' was investigated in detail, there were some theorems on classifying the unstable theories in II4 and III7. We proved that T is unstable iff it has the strict order property (i.e. some formula $\varphi(x, y)$ is a quasi-order with infinite chains), or it has the independence property (i.e. for some model M of T, $a_n \in M$ and $\varphi(x, y)$ any non trivial Boolean combination of the $\varphi(x, a_n)$ is satisfied in M). At earlier stage it seemed that unstable T without the independence property and unstable T without the strict order property are two incomparable classes of 'simple' unstable theory. The investigation of $K(\lambda, T) = \sup\{|S(A)|^+ : |A| \leq \lambda\}$ prefer the second one, but some facts from [4] pointed out to a subclass of the first:

Definition 0.1. T has the tree property if there are a formula $\varphi(\bar{x}, \bar{y})$, $k < \omega$ a model M of T, and sequences $\bar{a}_\eta \in M$ $(\eta \in {}^{\omega>}\omega)$ such that for any $\eta \in {}^{\omega>}\omega$, $\{\varphi(\bar{x}, \bar{a}_{\eta\frown\langle l\rangle}) : l < \omega\}$ is k-contradictory (i.e. no subset of cardinality k is satisfied in M) but for every $\eta \in {}^{\omega}\omega$, $\{\varphi(\bar{x}; \bar{a}_{\eta|n}) : n < \omega\}$ is consistent.

We shall call here theories without the tree property *simple*. In [4, III 7.7, 7.11] we proved:

Theorem 0.2. (1) *T has the tree property iff one of the following holds:*

(i) *there are $\varphi(x; \bar{y})$, $\bar{a}_\eta$, M as in definition* 0.1, *such that for $\eta, \nu \in {}^{\omega>}\omega$ no one an initial segment of the other $\varphi(x; \bar{a}_\eta)$, $\varphi(x; \bar{a}_\nu)$ are contradictory;*

(ii) *there are $\varphi(x; \bar{a}^l_n)$ $(l, n < \omega)$ such that $\Gamma_l = \{\varphi(x; \bar{a}^l_n) : n < \omega\}$ is 2-contradictory, but for any $\eta \in {}^{\omega}\omega$, $\{\varphi(x : \bar{a}^l_{\eta(l)}) : l < \omega\}$ is consistent.*

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(2) *The following conditions on T are equivalent:*

(i) *T is nonsimple;*

(ii) *for every λ, κ, such that $\lambda^{<\kappa} = \lambda$, there are λ^κ pairwise contradictory m-types of power κ over a set A of cardinality λ;*

(iii) *there is a set A, and a set S of m-types over A each of power χ such that $|S| > |A|^{|T|+\chi} + 2^{|T|+\chi}$ and no $\bar{c} \in \mathfrak{C}$ realizes $>\chi$ types from S.*

The example of a simple theory which we usually have in mind is T^*_{ind}, the model completion of the theory of one two-place symmetric irreflexive relation (see [4. Exercise 4.5 p. 79]) (or another varient, T^*_{ind} see [4. Definition II 4.8, p. 71]).

There were two cases in [4] in which investigating some property, we get a different answer for nonsimple theories and for T_{ind}.

Case I. Let $\mathrm{SP}_T(\lambda, \kappa)$ mean any model of T of power λ can be extended to a κ-saturated model of cardinality λ. Assume for simplicity $\lambda^{|T|} = \lambda \geq \kappa$. Then for T stable in λ $\mathrm{SP}_T(\lambda, \kappa)$ holds, and for non-simple T it is equivalent to $\lambda = \lambda^{<\kappa}$. The author thinks this will hold for any unstable T, and prove it under G.C.H., but by [4. VIII, Exercise 4.5] if $\mu = \mu^{<\kappa} < \lambda < 2^\mu$, $\mathrm{SP}_{T^*_{\text{ind}}}(\lambda, \kappa)$ holds. (And remember that it is consistent with ZFC (if ZFC consistent) that there are infinite cardinals μ, κ, λ such that $\mu = \mu^{<\kappa} < \lambda < 2^\mu$ and $\lambda \neq \lambda^{<\kappa}$.)

Case II. Investigating Keisler order, we prove it is consistent with ZFC, that T^*_{ind} is strictly smaller than any non-simple T. (It is always a minimal unstable theory.) We prove it by showing that Martin axiom implies there is an ultrafilter D over ω, such that for any $2^{\aleph_0}$-saturated model M of T^*_{ind}, M^ω/D is $2^{\aleph_0}$-saturated (see [4, VI 3.10]); but for any non-simple T, for any λ^+-universal M, and regular ultrafilter D over λ, M^λ/D is not λ^{++}-saturated.

The question was whether the simple unstable theories behave like T^*_{ind}, or we should weaken the tree property to get the right dividing line, or there is no comprehensible answer. In another sense the question was whether we can build a theory on simple T's.

What we succeed here to do is to show for Case I the *consistency* of 'for any simple T for some $\lambda < \lambda^{<\kappa}$, $\mathrm{SP}_T(\lambda, \kappa)$ holds'. We find the beginning of a theory on simple T which is the parallel of II, III in [4] (for stable T). There we analyze the Lindenbaum algebra of formulas over $\mathfrak{C}$, by finding when there are few ultrafilters. Here we try to find Boolean algebras with chain conditions. We use a kind of degree to prove facts on indiscernible, etc.

Problem. Is $\mathrm{SP}_T(\lambda, \kappa)$ equivalent to $\mathrm{SP}_{T^*_{\text{ind}}}(\lambda, \kappa)$ for any $\lambda = \lambda^{|T|} \geq \kappa$, and simple unstable T? By the above a negative answer can only be a consistency proof. We have an approximation to it in [6].

We prove in [6] that there may be simple unstable theories which behave differently for the questions of Cases I, II if we relativize the problem to a predicate.

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Notation. We use that of [4].. We let $\mathfrak{C}$ be a $\bar{\kappa}$-saturated model, and deal only with elementary submodels M, N of it of cardinality $<\bar{\kappa}$; $A, B, C \subseteq |\mathfrak{C}|$, $\bar{a}, \bar{b}, \bar{c}$ are finite sequences from $\mathfrak{C}$, I, J orders. $\boldsymbol{I}, \boldsymbol{J}$ sequences $\langle \bar{a}_i : i \in I \rangle$ of m-sequences from $\mathfrak{C}$.

We assume weak familiarity with [4, Chapters I.2, II.1, II.2, III.1].

Some of the assertions are presented without proofs. They are divided to two types. The first, those which are simple generalization of basic facts from [4] with almost the same proofs. The second, theorems which follow trivially from previous facts in this paper.

1. On types which divides

This section contains some results on the notion 'p divides over A' which hold in an arbitrary theory.

Definition 1.1. (1) The formula $\varphi(\bar{x}, \bar{a})$ *divides over* A if there is a sequence $\langle \bar{a}_i \mid i < \omega \rangle$ such that

(a) $\mathrm{tp}(\bar{a}, A) = \mathrm{tp}(\bar{a}_i, A)$ for all i,

(b) $\{\varphi(\bar{x}, \bar{a}_i) : i < \omega\}$ is n-contradictory for some n.

(2) p divides over A if $p \vdash \varphi$ for some φ which divides over A.

More malleable is the notion of p 'implicitly dividing' or forking over A.

Definition 1.2. The type p *forks over* A if there are formulas $\varphi_0(\bar{x}_0, \bar{a}_0), \ldots, \varphi_n(\bar{x}_n, \bar{a}_n)$ such that

(a) $p \vdash \bigvee_{k \leq n} \varphi_k(\bar{x}_k, \bar{a}_k)$;

(b) each $\varphi_k(\bar{x}_k, \bar{a}_k)$ divides over A.

[4, Chapter III] provides a detailed discussion of these notions with stable theories in mind. Here we give some further characterizations and refinements with unstable theories in mind.

Lemma 1.3. *For a fixed type p and set A the following are equivalent*

(i) *p divides over A;*

(ii) *there is a $k < \omega$ such that for every λ there are p_i $(i < \lambda)$, automorphic images of p over A which are k-contradictory;*

(iii) *there is a formula $\psi(\bar{x}, \bar{b})$ which is a conjunction of members of p an integer k and an infinite sequence $\boldsymbol{I}$, indiscernible over A, with $\bar{b} \in \boldsymbol{I}$ such that $\{\psi(\bar{x}, \bar{c}) : \bar{c} \in \boldsymbol{I}\}$ is k-contradictory.*

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Proof. The equivalence of (i) and (iii) (which follows easily by Ramsey's theorem) is [4, III 1.1(3)]. For (i)$\to$(ii) choose (by compactness) $\{\varphi(x, \bar{a}_i): \bar{a}_i \in \boldsymbol{I}\}$ with $|\boldsymbol{I}| = \lambda$, to be k-contradictory where all members of $\boldsymbol{I}$ realize the same type over A and $\bar{a} \in \boldsymbol{I}$. Let F_i be an automorphism of $\mathfrak{C}$ over A which takes $\bar{a}$ to $\bar{a}_i$. Then $\{F_i(p) \mid i < \lambda\}$ is the required set of images of p.

(ii)$\to$(i) Let $\{p_i: i < \lambda\}$ be a set of k-contradictory automorphic images of p where $\lambda > \beth_{\alpha+\omega}$ and $\alpha > |p| + |T| + |A|$. Suppose that the closure of p under con junction is enumerated by $\{\psi_j: j < |p|\}$ and let $F_i(p)$ be $\{\psi_{ij} = F_i(\psi_j): j < |p|\}$. For each function $f: k \to |p|$ let $C_f \subseteq \lambda^{[k]}$ be

$$\{\{i_0, \ldots, i_{k-1}\}: \{\psi_{i_0, f(0)}, \ldots, \psi_{i_{k-1}, f(k-1)}\} \text{ is contradictory, } i_0 < i_1, \ldots\}.$$

Since the p_i are k-contradictory $\lambda^{[k]} \subseteq \bigcup_{f \in |p|^k} C_f$; whence by the Erdos–Rado theorem there is an infinite set $S \subseteq \lambda$ and $f^* \in {}^k|p|$ such that for $i_0 < \cdots < i_{k-1} \in S$, $\{\psi_{i_0, f^*(0)} \cdots \psi_{i_{k-1}, f^*(k-1)}\}$ is contradictory. Let $\psi_i^* = \bigwedge_{m<k} \psi_{i, f^*(m)}$. Then each ψ_i^* is the image under F_i of $\psi^* = \bigwedge_{m<k} \psi_{f^*(m)}$ and the ψ_i^* are k-contradictory so ψ^* divides over A and $p \vdash \psi^*$ so p divides over A as required.

Lemma 1.4. *For any sequences $\bar{a}$ and $\bar{b}$ and set A the following are equivalent:*

(i) *the* $\mathrm{tp}(\bar{a}, A \cup \bar{b})$ *does not divide over* A:

(ii) *if $\boldsymbol{I}$ is an infinite indiscernible sequence over A with $\bar{b} \in \boldsymbol{I}$, then there is an automorphism F of $\mathfrak{C}$ fixing $A \cup \bar{b}$ such that $\boldsymbol{J} = F(\boldsymbol{I})$ is indiscernible over $A \cup \bar{a}$;*

(iii) *if $\boldsymbol{I}$ is an (infinite) indiscernible sequence over A with $\bar{b} \in \boldsymbol{I}$ there is an $\bar{a}'$ realizing* $\mathrm{tp}(\bar{a}, A \cup \bar{b})$ *such that $\boldsymbol{I}$ is indiscernible over $A \cup \{\bar{a}'\}$.*

Proof. If (iii) holds (ii) follows by letting F be an automorphism taking $\bar{a}'$ to $\bar{a}$, $F \restriction A$ the identity and a similar argument shows (ii)$\to$(iii). To see that (iii)$\to$(i) suppose that (i) fails. Then by the preceding lemma there are $k < \omega$ and a formula $\psi(\bar{x}, \bar{b}, \bar{c})$ with $\bar{c} \in A$ and an infinite indiscernible sequence $\boldsymbol{I}$ containing $\bar{b}$ such that $\{\psi(\bar{x}, \bar{b}, \bar{c}): \bar{b} \in \boldsymbol{I}\}$ is k-contradictory. Now if $\bar{a}'$ is chosen by (iii), $\vDash \psi(\bar{a}, \bar{b}, \bar{c})$ and $\bar{a}'$ realizes $\mathrm{tp}(\bar{a}, A \cup \bar{b})$ so $\vDash \psi(\bar{a}', \bar{b}, \bar{c})$. Since $\boldsymbol{I}$ is indiscernible over $A \cup \bar{a}'$ we deduce $\vDash \psi(\bar{a}', \bar{b}', \bar{c})$ for any $\bar{b}' \in \boldsymbol{I}$; but this violates the assertion, '$\{\psi(\bar{x}, \bar{b}', \bar{c}): \bar{b}' \in \boldsymbol{I}\}$ is k-contradictory'. Thus (iii)$\to$(i).

Now, assume (iii) fails and choose $\boldsymbol{I}$ witnessing that failure. Let $p(\bar{x}, \bar{b})$ denote the type of $\bar{a}$ over $A \cup \bar{b}$. Let $q = \bigcup_{\bar{b}' \in \boldsymbol{I}} p(\bar{x}, \bar{b}')$; then q is inconsistent. If not, letting $\Gamma(\bar{x})$ be the set of formulas which assert $\boldsymbol{I}$ is indiscernible over $A \cup \bar{x}$, we will show $q(\bar{x})$ consistent implies $q(\bar{x}) \cup \Gamma(\bar{x})$ is consistent. But any $\bar{a}'$ realizing $q(\bar{x}) \cup \Gamma(\bar{x})$ would verify (iii) for $\boldsymbol{I}$ and we have assumed $\boldsymbol{I}$ is a counterexample to (iii). To show $q(\bar{x}) \cup \Gamma \bar{x})$ is consistent note that for any finite $\Gamma^* \subseteq \Gamma$, if we can choose $\bar{c}$ realizing q, then by Ramsey's theorem we can find an infinite subsequence of $\boldsymbol{I}$ indiscernible for Γ^* over $A \cup \bar{c}$. Finally if q is inconsistent, then for some finite $\boldsymbol{J} \subseteq \boldsymbol{I}$ and $\psi(\bar{x}, \bar{b}) \in p(\bar{x}, \bar{b})$, $\{\psi(\bar{x}, \bar{b}'): b' \in \boldsymbol{J}\}$ is inconsistent. That is, since $\boldsymbol{I}$ is a sequence of indiscernibles over A, $\{\psi(\bar{x}, \bar{b}): \bar{b} \in \boldsymbol{I}\}$ is $|\boldsymbol{J}|$-inconsistent. But this shows $p(\bar{x}, \bar{b}) = \mathrm{tp}(\bar{a}, A \cup \bar{b})$ divides over A.

Lemma 1.5. *If for each* $i \leq n$ $\text{tp}(\bar{a}_i, A \cup \bar{b} \cup \bar{a}_0 \cup \cdots \cup \bar{a}_{i-1})$ *does not divide over* $A \cup \bar{a}_0 {}^\frown \cdots {}^\frown \bar{a}_{i-1}$, *then* $\text{tp}(\bar{a}_0 {}^\frown \bar{a}_1 {}^\frown \cdots {}^\frown \bar{a}_n, A \cup \bar{b})$ *does not divide over* A.

Proof. Fix $\boldsymbol{I}$ on indiscernible sequence over A with $\bar{b} \in \boldsymbol{I}$. By Lemma 1.4 (iii) $\rightarrow$ (i) it suffices to find $\bar{a}'_0, \ldots, \bar{a}'_n$ with

$$\text{tp}(\bar{a}_0 {}^\frown \cdots {}^\frown \bar{a}_n, A \cup \bar{b}) = \text{tp}(\bar{a}'_0 {}^\frown \cdots {}^\frown \bar{a}'_n, A \cup \bar{b})$$

such that $\boldsymbol{I}$ is indiscernible over $A \cup \bar{a}'_0 {}^\frown \cdots {}^\frown \bar{a}'_n$. We define the $\bar{a}'_i$ by induction. Suppose $\bar{a}'_0, \ldots, \bar{a}'_{k-1}$ have been appropriately chosen. Now choose $\bar{a}''_k$ so that

$$\text{tp}(\bar{a}''_k {}^\frown \bar{a}'_{k-1} {}^\frown \cdots {}^\frown \bar{a}'_0, A \cup \bar{b}) = \text{tp}(\bar{a}_k {}^\frown \bar{a}_{k-1} {}^\frown \cdots {}^\frown \bar{a}_0, A \cup \bar{b})$$

and hence that

$$p = \text{tp}(\bar{a}''_k, A \cup \bar{b} \cup \bar{a}'_{k-1} {}^\frown \cdots {}^\frown \bar{a}'_0)$$

does not divide over $A \cup \bar{a}'_0 {}^\frown \cdots {}^\frown \bar{a}'_{k-1}$. (Choose by the induction hypothesis an automorphism F of $\mathfrak{C}$ fixing $A \cup \bar{b}$ and mapping $\bar{a}_i$ to $\bar{a}'_i$ $(i < k)$; by the hypothesis of the Lemma $F(\bar{a}_k) = \bar{a}''_k$ works.) But now by (i) $\rightarrow$ (iii) of Lemma 1.4 we can choose $\bar{a}'_k$ so that $\boldsymbol{I}$ is indiscernible over $A \cup \bar{a}'_0 \cdots \bar{a}'_{k-1} \cup \bar{a}'_k$, and it realizes $\text{tp}(\bar{a}''_k, A \cup \bar{a}'_0 {}^\frown \cdots {}^\frown \bar{a}'_{k-1} \cup \bar{b})$.

The following claim lists some trivial facts.

Claim 1.6. (1) *If p divides over A, then p forks over A.*

(2) *The m-type p divides over A iff for some finite $q \subseteq p$ the formula $\bigwedge q$ divides over A; and $\{\varphi(\bar{x}; \bar{a})\}$ divides over A iff $\varphi(\bar{x}; \bar{a})$ divides over A; and p forks over A iff some finite $q \subseteq p$ forks over A.*

(3) *If $p \equiv q$ (i.e. for any $\bar{a}$, $\bar{a}$ realizes p iff $\bar{a}$ realizes q), then p forks over A iff q forks over A.*

2. Degrees and types which weakly divide

Definition 2.1. We define $D^m(p, \Delta, \lambda, k)$ (and ordinal, -1, or ∞) (p a set of m-formulas, Δ a set of formulas, λ a cardinal, k a natural number) by defining for every ordinal α when $D^m(p, \Delta, \lambda, k) \geq \alpha$ by induction on α.

(1) $D^m(p, \Delta, \lambda, k) \geq 0$ iff p is an m-type;

(2) $D^m(p, \Delta, \lambda, k) \geq \delta$ (δ a limit ordinal) iff $D^m(p, \Delta, \lambda, k) \geq \beta$ for every $\beta < \delta$;

(3) $D^m(p, \Delta, \lambda, k) \geq \alpha + 1$ if for every $\mu < \lambda$ and finite $q \subseteq p$ there are $\varphi(\bar{x}, \bar{y}) \in \Delta$, and sequences $\bar{a}_i$ $(i \leq \mu)$ such that:

(i) $D^m(q \cup \{\varphi(\bar{x}, \bar{a}_i)\}, \Delta, \lambda, k) \geq \alpha$

(ii) $\{\varphi(\bar{x}, \bar{a}_i) : i \leq \mu\}$ is k-contradictory, i.e. for every $w \subseteq (\mu + 1)$, $|w| = k$, $\models \neg(\exists \bar{x}) \bigwedge_{i \in w} \varphi(\bar{x}; \bar{a}_i)$

Claim 2.2. (1) *If* $p \vdash q$, *then*

$$D^m(p, \Delta, \lambda, k) \leq D^m(q, \Delta, \lambda, k).$$

(2) *If* $\Delta_1 \subseteq \Delta_2$, $\lambda_1 \geq \lambda_2$, $k_1 \leq k_2$, *then*

$$D^m(p, \Delta_1, \lambda_1, k_1) \leq D^m(p, \Delta_2, \lambda_2, k_2).$$

(3) *For every* p, Δ, λ, k *there is a finite* $q \subseteq p$,

$$D^m(p, \Delta, \lambda, k) = D^m(q, \Delta, \lambda, k).$$

(4) *For every finite* Δ, m, k *there is* $\theta(\bar{x}; \bar{y})$ *such that for every* m*-type* p *and infinite cardinal* λ,

$$D^m(p, \Delta, \lambda, k) = D^m(p, \theta(\bar{x}, \bar{y}), \lambda, k)$$

(5) *For infinite* λ

$$D^m\left(p \cup \left\{\bigvee_{i<n} \psi_i(\bar{x}, \bar{a}_i)\right\}, \Delta, \lambda, k\right) = \operatorname*{Max}_{i<n} D^m(p \cup \{\psi_i(\bar{x}, \bar{a}_i)\}, \Delta, \lambda, k).$$

Proof. Left to the reader (look at [4] for parallel claim).

Claim 2.3. *For every* m, Δ, *finite* m*-type* p, *cardinal* λ *and* $k, n < \omega$ *the following are equivalent:*

(1) $D^m(p, \Delta, \lambda^+, k) \geq n$;

(2) *there are* $\bar{a}_\eta (\eta \in {}^{n\geq}\lambda) \theta_\eta \in \Delta (\eta \in {}^{n>}\lambda)$ *such that:*

(i) *for every* $\eta \in {}^n\lambda$, $p \cup \{\theta_{\eta \restriction i}(\bar{x}, \bar{a}_{\eta \restriction (i+1)}) : 0 \leq i < n\}$ *is consistent,*

(ii) *for every* $\eta \in {}^{n>}\lambda$, $\{\theta_\eta(\bar{x}; \bar{a}_{\eta \frown \langle i \rangle}) : i < \lambda\}$ *is* k*-contradictory;*

(3)

$$\bigcup \{p(\bar{x}_\eta) : \eta \in {}^n\lambda\} \cup \left\{\neg(\exists \bar{x}) \bigwedge_{i \in w} \theta_\eta(\bar{x}; \bar{y}_{\eta \frown \langle i \rangle}) : \eta \in {}^{n>}\lambda, w \subseteq \lambda, |w| = k\right\}$$
$$\cup \{\theta_{\eta \restriction i}(\bar{x}_\eta, \bar{y}_{\eta \restriction (i+1)}) : \eta \in {}^n\lambda, i < n\}$$

is consistent for some $\theta_\eta \in \Delta (\eta \in {}^{n>}\lambda)$.

Proof. We prove it by induction on n (for all p's).

Claim 2.4. (1) *For* $\Delta = \{\theta\}$, *Claim* 2.3 *holds for not necessarily finite* p, *and* $\theta_\eta = \theta$.

(2) $D^m(p, \theta, \aleph_0, k) \geq \omega$ *iff* $D^m(p, \theta, n, k) \geq n$ *for every* $n < \omega$ *iff* $D^m(p, \theta, \infty, k) = \infty$. *For finite* λ, k, $D^m(p, \theta, \lambda, k) \geq \omega$ *iff* $D^m(p, \theta, \lambda, k) = \infty$.

(3) *For every* p, k, θ, m, *for all large enough* $l < \aleph_0$

$$D^m(p, \theta, \aleph_0, k) = D^m(p, \theta, l, k).$$

Proof. (1) By compactness.

(2), (3) Easy, by Claim 2.3.

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Definition 2.5. For an m-type $p = p(\bar{x})$, $\psi = \psi(\bar{x}_1, \ldots, \bar{x}_n; \bar{b})$ $(l(\bar{x}_i) = m))$ let $[p]^\psi$ be

$$\bigcup_{i=1}^{n} p(\bar{x}_i) \cup \{\psi(\bar{x}_1, \ldots, \bar{x}_n; \bar{b})\}.$$

Remark. Note that $[p \cap q]^\psi = [p]^\psi \cap [q]^\psi$, $[p \cup q]^\psi = [p]^\psi \cup [q]^\psi$.

Claim 2.6. (1) *For any finite Δ, k, n, m and finite λ there is a formula $\psi = \psi_{\Delta,k,n}(\bar{x}_1, \ldots, \bar{x}_{(\lambda^n)})$ such that: for any m-type p, $D^m(p, \Delta, \lambda+1, k) \geq n$ iff $[p]^\psi$ is consistent.*

(2) *For any finite Δ, k, n, m there are formulas $\psi^l = \psi^l_{\Delta,k,n}(\bar{x}_1, \ldots, \bar{x}_l)$ such that: for any m-type p, $D^m(p, \Delta, \aleph_0, k) \geq n$ iff $[p]^{\psi^l}$ is consistent for every $l < \omega$.*

Remark. Note that ψ is constructed from formulas of Δ by logical operation.

Proof. (1) by Claim 2.3(3). More exactly, let $\psi'(\ldots, \bar{x}, \ldots, \bar{y})$ be the conjunction of the set letting $\Delta = \{\theta\}$ for notational simplicity.

$$\Big\{\exists \bar{x} \bigwedge_{i \in w} \theta(\bar{x}, \bar{y}_{\eta \frown \langle i \rangle}) \colon \eta \in {}^{n>}\lambda,\ w \subseteq \lambda,\ |w| = k\Big\} \cup \{\theta(\bar{x}_\eta, \bar{y}_{\eta \restriction (i+1)}) \colon \eta \in {}^n\lambda,\ i < n\}$$

and $\psi = (\exists \bar{y})\psi'$. If $[p]^\psi$ is consistent there are $\langle \ldots, \bar{c}_\eta, \ldots \rangle_{\eta \in {}^n\lambda}$ realizing it, and there is $\bar{b}$ such that $\vDash \psi[\ldots, \bar{c}, \ldots, \bar{b}]$, so the $\bar{c}$'s and $\bar{b}$ shows Claim 2.3(3) holds, hence $D^m(p, \theta, \lambda, k) \geq n$. The other direction is easy too.

(2) By the first part and Claim 2.4(2).

Now we define the central notions:

Definition 2.7. (1) T is *simple* if for every θ, k $D^m(\bar{x} = \bar{x}, \theta, \aleph_0, k) < \omega$ (or equivalently, $< \infty$)

(2) For a set B and m-type r (not necessarily over B) we say the m-type p *weakly divides* over (r, B) if for some $\bar{b} \in B$, and $\psi = \psi(\bar{x}_1, \ldots, \bar{x}_n; \bar{b})$, $[r]^\psi$ is consistent but $[r \cup p]^\psi$ is inconsistent.

Remark. The idea behind the last definition is that 'p does not weakly divides over (r, B)' says that p is similar to r modulo formulas with parameters from B.

Claim 2.8. (1) *If $p \vdash q$, $A \subseteq B$, q weakly divides over (r, A), then p weakly divides over (r, B). Hence $p \equiv q$ implies p weakly divides over (r, B) iff q weakly divides over (r, B).*

(2) *For any m, $\varphi = \varphi(\bar{x}_1, \ldots, \bar{x}_{n(1)}; \bar{b}_1)$ $(l(\bar{x}_l) = m)$ and $\psi = \psi(\bar{x}^1, \ldots, \bar{x}^{n(2)}; \bar{b}_2)$ $(l(\bar{x}^l) = m \cdot n(1))$ there is a formula $\varphi * \psi$ that for any m-type p*

$$[[p]^\varphi]^\psi = [p]^{\varphi * \psi}.$$

(3) *If $[p]^\psi$ weakly divides over $([r]^\psi, B)$, then p weakly divides over (r, B) ψ with parameters from B).*

Proof. Easy.

Lemma 2.9. (1) *For m-types r, p and formula ψ such that $[r \cup p]^\psi$ is defined and consistent p weakly divides over (r, B) iff $[p]^\psi$ weakly divides over $([r]^\psi, B)$*

(2) *The following are equivalent (p, r are m-types):*

(i) *p weakly divides over (r, B);*

(ii) *for some $\psi = \psi(\bar{x}_1, \ldots, \bar{x}_n; \bar{b})(l(\bar{x}_1) = m)$ over B, and finite Δ, k, λ, let $l = m \cdot n$*

$$D^l([r]^\psi, \Delta, \lambda, k) > D^l([r \cup p]^\psi, \Delta, \lambda, k):$$

(iii) *as* (ii) *with λ replaced by $\aleph_0$.*

Proof. (1) by Claim 2.8(2), (3).

(2) (iii)$\to$(ii) by Claim 2.4(3). If (ii) holds, by Claim 2.4(2) $D^m([r \cup p]^\psi, \Delta, \lambda, k)$ is finite so by Claim 2.6(1) for some φ over ϕ, $[[r]^\psi]^\varphi$ is consistent, but $[[r \cup p]^\psi]^\varphi$ is inconsistent, so we get (i) by Claim 2.8(2). If (i) holds, then for some ψ over B, $[r]^\psi$ is consistent but $[r \cup p]^\psi$ is not, hence

$$D([r]^\psi, \Delta, \lambda, k) \geq 0 > -1 = D([r \cup p]^\psi, \Delta, \lambda, k)$$

so (iii) holds.

Conclusion 2.9. (3) *p, r are m-types, Δ, k finite,*

$$D^m(r, \Delta, \lambda, k) > D^m(r \cup p, \Delta, \lambda, k),$$

then p weakly divides over (r, B).

Claim 2.10. (1) *If p divides over A, then for some finite k_0, Δ_0, for every Δ and finite k such that $k_0 \leq k$, $\Delta_0 \subseteq \Delta$, and any λ:*

$$D^m(p, \Delta, \lambda, k) < \infty \quad \text{implies} \quad D^m(p, \Delta, \lambda, k) < D^m(p \restriction A, \Delta, \lambda, k).$$

(2) *For infinite λ the same holds when p forks over A.*

Proof. (1) Easy.

(2) (See Definition 1.2 and Claim 2.2(5)).

Conclusion 2.11. *Let T be simple.*

(1) *If p forks over A, then p weakly divides over $(p \restriction A, \emptyset)$.*

(2) *If p is over A, then p does not fork over A.*

(3) *If p does not fork over A, p is an m-type over B, $A \subseteq B$, thus there is $q \in S^m(B)$, $p \subseteq q$, q does not fork over A.*

Proof. (1) By Claim 2.10(2), and 2.9(3).

(2) By Claim 2.10(2).

(3) By Claim 2.2(5).

Conclusion 2.12. *For any simple T axioms II.4, VIII, X.1, X.2, X1.1, X1.2 hold for* F^f_λ *any* λ. *Note that axioms I, II.2–II.3, III.1, III.2, IV, VIII and IX hold for* F^f_λ *in any theory.*

Proof. Left to the reader; this will not be used.

Lemma 2.13. *Suppose* $\mathrm{tp}(\bar a, A \cup \bar b)$ *does not weakly divide over* $(\mathrm{tp}(\bar a, A), A)$. *Then for any* A' *there is* $\bar b'$ *such that:*

$$\bar a' \in A', \quad \mathrm{tp}(\bar a', A) = \mathrm{tp}(\bar a, A) \textit{ implies } \mathrm{tp}(\bar b' {}^\frown \bar a', A) = \mathrm{tp}(\bar b {}^\frown \bar a, A).$$

Proof. Let $\{\bar a_i : i < \alpha\}$ be a list of the $\bar a' \in A'$ realizing $\mathrm{tp}(\bar a, A)$, and w.l.o.g. $A' = A \cup \bigcup_{i<\alpha} \bar a_i$. Let

$$p(\bar x) = \mathrm{tp}(\bar a, A \cup \bar b),$$
$$\Gamma = \{\varphi(\bar x_{i(0)}, \ldots, \bar x_{i(n-1)}, \bar c) : n < \omega, \bar c \in A$$
$$\models \varphi[\bar a_{i(0)}, \ldots, \bar a_{i(n-1)}; \bar c]\}.$$

It suffices to prove $\Gamma \cup \bigcup_{i<\alpha} p(\bar x_i)$ is consistent. For if the assignment $\bar x_i \mapsto \bar a_i'$ satisfies it, let F be an automorphism of $\mathfrak{C}$, $F \restriction A$ the identity and $F(\bar a_i') = \bar a_i$. Then let $\bar b' = F(\bar b)$.

If $\Gamma \cup \bigcup_{i<\alpha} p(x_i)$ is inconsistent, there is a finite inconsistent subset, so, by easy manipulation, it is contained in a set of the form

$$\{\psi(\bar x_{i(0)}, \ldots, \bar x_{i(n-1)}; \bar c_0)\} \cup \bigcup_{l<n} p(\bar x_{i(l)})$$

where $\psi = \psi(\bar x_{i(0)}, \ldots, \bar x_{i(n-1)}; \bar c_0) \in \Gamma$.

This means $[p(\bar x)]^\psi = [\mathrm{tp}(\bar a, A \cup \bar b)]^\psi$ is inconsistent, but clearly $[\mathrm{tp}(\bar a, A)]^\psi$ is consistent ($\bar a_{i(0)} {}^\frown \cdots {}^\frown \bar a_{i(n-1)}$ realizes it), so by Definition 2.7 $\mathrm{tp}(\bar a, A \cup \bar b)$ weakly divides over $(\mathrm{tp}(\bar a, A), A)$, contradicting the hypothesis.

Lemma 2.14 (The Weak Symmetry lemma). *If* $\mathrm{tp}(\bar b, A \cup \bar a)$ *divides over* A, *then* $\mathrm{tp}(\bar a, A \cup \bar b)$ *weakly divides over* $(\mathrm{tp}(\bar a, A), A)$.

Proof. As $\mathrm{tp}(\bar b, A \cup \bar a)$ divides over A, there is an indiscernible sequence $\boldsymbol{I} = \langle \bar a_n : n < \omega \rangle$ over A, $\bar a_0 = \bar a$ and $\varphi(\bar x, \bar a) \in \mathrm{tp}(\bar b, A \cup \bar a)$ such that $\{\varphi(\bar x, \bar a_n) : n < \omega\}$ is k-contradictory. Letting $A' = A \cup \bigcup_{n<\omega} \bar a_n$ there cannot exist $\bar b'$ as in the previous lemma, hence necessarily the hypothesis there fail, i.e. $\mathrm{tp}(\bar a, A \cup \bar b)$ weakly divides over $(\mathrm{tp}(\bar a, A), A)$, as required.

3. Boolean algebras, essentially of formulas

Let T be simple in this section.

Definition 3.1. For any set B, and type r let $W(r, B) = (W(r, B), \leq)$ be the following partially ordered set. The elements of $W(r, B)$ are the formulas with parameters from $\mathfrak{C}$ (identified up to logical equivalence) which do not weakly divide over (r, B). $\varphi \leq \psi$ if $\varphi \vdash \psi$. Thus $W(r, B)$ is a partial subalgebra of the Lindenbaum algebra of $\mathfrak{C}$. In particular it is easy to see that $W(r, B)$ is closed under $\vee$ and that if $\varphi, \psi, \varphi \wedge \psi$ are all in $W(r, B)$, then $\varphi \wedge \psi$ is the greatest lower bound of φ and ψ. Now, φ, ψ are incompatible means there is no $\theta \in W(r, B)$ such that $\theta \vdash \varphi$ and $\theta \vdash \psi$. But, since for any formulas φ_1, φ_2 if $\varphi_1 \vdash \varphi_2$ and $\varphi_1 \in W(r, B)$, then $\varphi_2 \in W(r, B)$, φ and ψ are incompatible just if $\varphi \wedge \psi \notin W(r, B)$. We define $W_A(r, B)$ similarly, restricting ourselves to formulas $\psi(\bar{x} : \bar{a})$, $a \in A$.

Remark. For motivation of the above definition see the remark after Definition 2.7 and think about Lemma 4.9 (which proves Theorem 4.10).

Lemma 3.2. *Fix r and B. Let $\lambda = (2^{|B|+|T|})^+$. Then $W(r, B)$ satisfies the λ-chain condition.*

Proof. If not, let $\langle \varphi_i : i < \lambda \rangle$ be a sequence of elements of $W(r, B)$ such that for each i, j there is a formula ψ_{ij} with parameters from B such that $\varphi_i \wedge \varphi_j$ weakly divides over (r, B) by ψ_{ij}. By the Erdos–Rado theorem, we may assume that for $i < j < \omega$ all the ψ_{ij} are the same formula $\psi = \psi(\bar{x}_1, \ldots, \bar{x}_m ; \bar{b})$, and all the φ_i are of the form $\varphi(\bar{x}, \bar{a}_i)$. Now, for each $i < \omega$ $[r \cup \{\varphi(\bar{x}, \bar{a}_i)\}]^\psi$ is consistent but $[r \cup \{\varphi(\bar{x}, \bar{a}_i), \varphi(\bar{x}, \bar{a}_j)\}]^\psi$ is inconsistent if $i \neq j$, $i, j < \omega$. For any finite Δ,

$$D([r \cup \{\varphi(\bar{x}, \bar{a}_i)\}]^\psi, \Delta, \aleph_0, 2) \leq D([r]^\psi, \Delta, \aleph_0, 2]) = l < \omega$$

(by simplicity of T). Now let $\Delta = \bigcup_{i=1}^m \{\varphi(\bar{x}_i, \bar{y})\}$. If for each i, $D([r \cup \{\varphi(x, a_i)\}]^\psi, \Delta, \aleph_0, 2)$ were equal to l it would follow by Definition 2.1(3) that $D([r]^\psi, \Delta, \aleph_0, 2) > l$; thus for some i

$$D([r \cup \{\varphi(x, a_i)\}]^\psi, \Delta, \aleph_0, 2) < D([r]^\psi, \Delta, \aleph_0, 2).$$

But then by Lemma 2.9(2) $\varphi(x, a_i)$ weakly divides over (r, B) contrary to hypothesis.

Lemma 3.3. *If $p \in S^m(C)$, then for some $A \subseteq C, |A| \leq |T|$ p does not weakly divide over $(p \restriction A, A)$.*

Proof. We first show that for any set B there is a B' such that p does not weakly divide over $(p \restriction B', B)$, $B \subseteq B'$, $|B'| \leq |T| + |B| + \aleph_0$. Then setting $A_0 = \emptyset$, $A_{n+1} = (A_n)'$, $A = \bigcup_{n<\omega} A_n$ is as required. Now for each $\psi = \psi(\bar{x}, \bar{b})$ $\bar{b} \in B$ such that p^ψ is inconsistent choose $\theta_\psi \in p$ such that $[\theta_\psi]^\psi$ is inconsistent. Let B' be the set of parameters which occur in any θ_ψ. The number of ψ is $\leq |T| + |B| + \aleph_0$.

Definition 3.4. For any partially ordered set $(P, <)$ there is a unique complete

Boolean algebra RO(P) and a homomorphism e of P into RO(P) such that

(i) p, q are compatible in P iff $e(p), e(q)$ are compatible (i.e. have a common lower bound) in RO(P).

(ii) $e(P)$ is dense in RO(P)

(iii) e commutes with $\wedge$ when defined, so (i) holds for finite sets.

So chain conditions are preserved. We will call RO($W(r, B)$), $B^c(r, B)$. (This is used e.g. in passing from forcing to Boolean-valued models.)

Lemma 3.5. (i) *If p does not weakly divide over (r, B), then the image of p in $B^c(r, B)$ has the finite intersection property.*

(ii) *If F is an ultrafilter on $B^c(r, B)$, then $p = \{\varphi \in W(r, B): e(\varphi) \in F\}$ is a type which does not weakly divide over (r, B).*

Proof. (i) Let $a_1 \cdots a_k$ be in the image of p. Then there exist $\varphi_1 \cdots \varphi_k \in p$ such $e(\varphi_i) = a_i$. But then $\bigwedge_{i=1}^k \varphi_i \in p$ and $e(\bigwedge_{i=1}^k \varphi_i) = \bigwedge_{i=1}^k a_i$ so (i) holds.

(ii) Suppose $\varphi_1 \cdots \varphi_k \in p$. Then $e(\varphi_1), \ldots, e(\varphi_k)$ are compatible in $B^c(r, B)$ so by the third condition on e $\varphi_1, \ldots, \varphi_k$ are compatible in $W(B, r)$, that is, $\bigwedge_{i=1}^k \varphi_i$ does not weakly divide so p is as required.

Notation 3.6. (1) In the sequel, we will frequently drop the e and denote $e(\varphi(\bar{x}, \bar{c}))$ by $\varphi(\bar{x}, \bar{c})$. We denote elements of $B^c(r, B)$ by $\boldsymbol{\varphi}$.

(2) Any automorphism F of $\mathfrak{C}$, $F \restriction (B \cup \operatorname{Dom} r) = \mathrm{id}$ induce an automorphism of $W(r, B)$, $B^c(r, B)$ which we also denote by F.

Definition 3.7. (1) A set $A \subseteq \mathfrak{C}$ is a *support* of an element $\boldsymbol{\varphi}$ of $B^c(r, B)$ if for every automorphism F of $\mathfrak{C}$,

$$F \restriction (A \cup B \cup \operatorname{Dom} r) = \mathrm{id} \Rightarrow F(\boldsymbol{\varphi}) = \boldsymbol{\varphi}.$$

(2) A set $A \subseteq \mathfrak{C}$ is a *weak support* of an element φ of $B^c(r, B)$ if

$$\{F(\boldsymbol{\varphi}): F \text{ an automorphism of } \mathfrak{C},\ F \restriction (A \cup B \cup \operatorname{Dom} r) = \mathrm{id}\}$$

has cardinality $< \|\mathfrak{C}\|$.

(3) Let $B^p(r, B)$ be the set of elements of $B^c(r, B)$ with finite weak support.

Claim 3.8. (1) *Any element $\boldsymbol{\varphi}$ of $B^c(r, B)$ has a support of cardinality $< (2^{|B|+|T|})^+$.*

(2) *The set of elements $\boldsymbol{\varphi}$ of $B^c(r, B)$ such that A is a support of $\boldsymbol{\varphi}$ [A is a weak support of $\boldsymbol{\varphi}$] is a complete subalgebra.*

(3) *$B^p(r, B)$ is a subalgebra of $B^c(r, B)$, including $W(r, B)$.*

Proof. (1) Clearly every element of $W(r, B)$ has finite support (the set of parameters, of any formula representing it.) Let $\lambda = (2^{|B|+|T|})^+$, so by Lemma 3.2 and Definition 3.4 $B^c(r, B)$ satisfy the λ-chain condition. As $W(r, B)$ is dense, any

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$\varphi \in B^c(r, B)$ is $\sup\{\varphi_i : i < i_0 \leqslant \lambda\}$ $\varphi_i \in W(r, B)$, so if A_i is a finite support of φ_i, $\bigcup_{i<i_0} A_i$ is a support of φ.

(2) Easy.

(3) If A_l is a support of φ_l ($l = 0, 1$), then $A_0 \cup A_1$ is a support of $\varphi_0 \wedge \varphi_1$, $\varphi_0 \vee \varphi_1$, and A_0 is a support of $-\varphi_0$, so there are no problems.

4. Extending to quite saturated models

Definition 4.1. (1) Let $\kappa r_{\mathrm{cdt}}(T)$ be the first regular cardinal κ, such that there are no A_i ($i \leqslant \kappa$) increasing with i, $p \in S^m(A_\kappa)$, $p \restriction A_{i+1}$ fork over A_i.

(2) Let $\kappa r_1(T)$ be for simple T the first regular κ, such that for every $p \in S^m(A)$, there is $B \subseteq A$, $|B| < \kappa$ such that p does not weakly divide over $(p \restriction B, \emptyset)$. For non simple T we stipulate $\kappa r_1(T) = \infty$.

Claim 4.2. (1) *$\kappa r_{\mathrm{cdt}}(T)$ is the first κ such that for every $p \in S^m(A)$, p does not fork over some $B \subseteq A$, $|B| < \kappa$, and κ is regular; a similar assertion holds for $\kappa r_1(T)$.*

(2) *For simple T*

$$\kappa r_{\mathrm{cdt}}(T) \leqslant \kappa r_1(T) \leqslant |T|^+.$$

(3) *In Definition 4.1 (1) we can take $m = 1$.*

Theorem 4.3. (1) *In Definition 4.1 (1) we can replace 'forks' by 'divides' and for a fix λ, assume there is an indiscernible $\boldsymbol{I} \subseteq A_{i+1}$ witnessing it, $|\boldsymbol{I}| \geqslant \lambda$ (i.e. there are $k < \omega$, $a \in A_i$, $b \in \boldsymbol{I}$, and $\varphi(\bar{x}; \bar{y}; \bar{z})$ such that $\varphi(\bar{x}; \bar{b}; \bar{a}) \in p$ and $\{\varphi(\bar{x}; \bar{c}; \bar{a}) : \bar{c} \in \boldsymbol{I}\}$ is k contradictory; in fact we can omit $\bar{a}$).*

(2) *It is equivalent also to the definition of $\kappa r_{\mathrm{cdt}}(T)$ in [4, III, Definition 7.2].*

(3) *If T is not simple $\kappa r(T) = \kappa r_{\mathrm{cdt}}(T) = \infty$, if T is stable it is simple, for stable T, $\kappa r(T) = \kappa r_{\mathrm{cdt}}(T) = \kappa r_1(T)$. If T is simple unstable, T has the independence property (see [4, II, Definition 4.2.]) but not the strict order property. T is simple iff it does not have the tree property [4, III 7.2].*

Proof of Claim 4.2. (1) Immediate.

(2) Clearly Conclusion 2.11(1) implies $\kappa r_{\mathrm{cdt}}(T) \leqslant \kappa r_1(T)$ and Lemmas 2.9 and 3.3 imply $\kappa r_1(T) \leqslant |T|^+$.

(3) Clearly $\kappa r^1(T) \leqslant \kappa r^m(T)$ because we can add dummy variables. Assume that $\kappa < \kappa r^{m+1}(T)$, then there are a sequence $\bar{a} = \langle a_0, \ldots, a_m \rangle$, and an increasing sequence of sets A_i ($i < \kappa$) such that $\mathrm{tp}(\bar{a}, A_{i+1})$ divides over A_i (by Theorem 4.3(1)).

Now by the assumption and Lemma 1.5 it is impossible that $\mathrm{tp}(\langle a_0, \ldots, a_m \rangle, A_{i+1})$ does not fork over A_i and $\mathrm{tp}(a_m, A_{i+1} \cup \{a_0, \ldots, a_{m-1}\})$ does not divide over $A_i \cup \{a_0, \ldots, a_{m-1}\}$.

For all $i<\kappa$ $\operatorname{tp}(\langle a_0,\ldots,a_m\rangle, A_{i+1})$ divides over A_i or $\operatorname{tp}(a_m, A_i\cup\{a_0,\ldots,a_{m-1}\})$ divides over $A_i\cup\{a_0,\ldots,a_{m-1}\}$.

One of the cases happens κ times:

In the first case $\langle a_0,\ldots,a_{m-1}\rangle$ and a subsequence of $\{A_i\}_{i<\kappa}$ exemplifies $\kappa<\kappa r^m(T)$ (remember monotonicity of dividing).

In the second case a_m and a subsequence of $\langle A_i\cup\{a_0,\ldots,a_{m-1}\}: i<\kappa$ exemplifies $\kappa<\kappa r^1(T)\rangle$.

Proof of Theorem 4.3. (1) If T is not simple, by Theorem 4.3(3) (proved below) T has the tree property, and then it is easy to prove that in both variants of Definition 4.1(1) we get ∞.

If $p\restriction A_{i+1}$ divides over A_i clearly it forks over A_i (see Claim 2.2) hence one inequality is clear. For the other direction suppose A_i $(i\leq\kappa)$ is increasing, $p\in S^m(A_\kappa)$, and $p\restriction A_{i+1}$ fork over A_i, and for $l<n^i$ $\boldsymbol{I}^i_l=\langle\bar a^i_{l,\alpha}: \alpha<\lambda\rangle$ is indiscernible over A_i $\{\varphi_l(x:\bar a^i_{l,k}): \alpha<\lambda\}$ is m^i_l-contradictory and

$$p\restriction A_{i+1}\vdash\bigvee_l\varphi_l(\bar x:\bar a^i_{l,0});$$

w.l.o.g. $\bar a^i_{l,0}=\bar a^i_{0,0}$. We define elementary mappings F^i, F^{i+1}_l, such that:

(i) $\operatorname{Dom} F^{i+1}=A_{i+1}\cup\bar a^i_{0,0}$, $\operatorname{Dom} F_\delta=A_\delta$ (for limit δ), $\operatorname{Dom} F^{i+1}_l=A_{i+1}\cup\boldsymbol{I}^i_l$.

(ii) $F^i\restriction A_i$ is increasing.

(iii) $F^{i+1}_l(\boldsymbol{I}^i_l)$ is indiscernible over $B_i=\bigcup\{\operatorname{Range} F^j_l: j\leq i, l<n^i\}$.

We define by induction on i; *for* $i=0$, i limit there are no problems. For $i+1$ first define a sequence $\bar b^i_{0,0}=\bar b^i_{l,0}$ $(l<n^i)$ such that:

(a) $\operatorname{tp}(\bar b^i_{0,0}, F_i(A_i))=F_i(\operatorname{tp}(a^i_{0,0}, A_i))$;

(b) $\operatorname{tp}_*(B_i, F^i(A_i)\cup\bar b^i_{0,0})$ does not fork over $F^i(A_i)$ (hence does not divide).

This is possible by Conclusion 2.11(2), (3) so by Lemma 1.4 we can define F^{i+1}_l. In the end let $\bar c$ realize $F_\kappa(p)$, then B_i $(i\leq\kappa)$ is increasing, for each i $F^{i+1}_l(\boldsymbol{I}^i_l)$ is indescernible over B_i (and infinite). $\bar c$ realizes $F^{i+1}(p\restriction A_{i+1})$, hence satisfies $\bigvee_l\varphi^i_l(\bar x:\bar a^i_{l,0})$ hence for some $l(i)$, $\varphi^i_{l(i)}(\bar x;\bar a^i_{l(i)0})\in\operatorname{tp}(\bar c, B_{i+1})$. However, $\{\varphi^i_{l(i)}(\bar x,\bar a): \bar a\in F^i_{l(i)}(\boldsymbol{I}^i_{l(i)})\}$ is m^i_l-contradictory, so $\operatorname{tp}(\bar c, B_{i+1})$ divides over B_i, and the indiscernible sequence witnessing it is $\subseteq B_{i+1}$, and has power $\geq\lambda$ so we finish.

(2) Is left to the reader.

(3) Start by proving that T is not simple iff it has the tree property (see Definitions 0.1, 2.1, 2.7(1)) and the rest also easy.

Definition 4.4. (1) Let $\mathrm{SP}(T)$ (saturation pairs of T) be $\{(\lambda,\kappa): \lambda\geq|T|$, and every model of T of cardinality $\leq\lambda$, has a κ-saturated elementary extension of cardinality $\leq\lambda\}$.

(2) $\mathrm{CP}(T)$ (compactness pairs of T) is defined similarly, with κ-compact replacing κ-saturated.

(3) Let $\mathrm{SP}'(T)=\mathrm{SP}(T)\cap\{(\lambda,\kappa): \lambda^{|T|}=\lambda\geq\kappa\}$, and $\mathrm{CP}'(T)$ is defined similarily.

Claim 4.5. (1) $\mathrm{SP}(T) \subseteq \mathrm{CP}(T)$.

(2) *The following are equivalent:*

(i) $(\lambda, \kappa) \in \mathrm{SP}(T)$;

(ii) *for every T_1, $T \subseteq T_1$, $|T_1| \leq \lambda$, and model M of T_1, $\|M\| \leq \lambda$ there is a model N of T_1, $M \prec N$, $\|N\| \leq \lambda$, and the L-reduct of N is κ-saturated;*

(iii) *for every A, $|A| \leq \lambda$, there is a set $S \subseteq S(A)$ of cardinality $\leq \lambda$, such that for every 1-type p over A, $|\mathrm{Dom}\, p| < \kappa$, there is $q \in S$, $p \subseteq q$.*

(3) *Like (2) replacing SP by CP in (i) and $|\mathrm{Dom}\, p| < \kappa$ by $|p| < \kappa$ in (iii).*

(4) *$(\lambda, \aleph_0) \in \mathrm{CP}(T)$ whenever $\lambda > |T|$, and $(\lambda, \aleph_0) \in \mathrm{SP}(T)$ iff $\lambda \geq |D(T)|$.*

Proof. Left to the reader.

Conclusion 4.6. (1) *If $\lambda = \lambda^{<\kappa} \geq |T|$, then $(\lambda, \kappa) \in \mathrm{CP}(T)$.*

(2) *If $\lambda = \lambda^{<\kappa} \geq 2^{|T|}$ or at least $\lambda = \lambda^{<\kappa} \geq |D(T)|$, then $(\lambda, \kappa) \in \mathrm{SP}(T)$.*

Proof. (1) Use Claim 4.5(3)(iii).

(2) Use Claim 4.5(2)(iii).

By [4]:

Theorem 4.7. (1) *If T is not simple, or $\kappa < \kappa r_{\mathrm{cdt}}(T)$, then $(\lambda, \kappa) \in \mathrm{CP}(T)$ iff $\lambda = \lambda^{<\kappa} \geq |T|$ and $(\lambda, \kappa) \in \mathrm{SP}(T)$ iff $\lambda = \lambda^{<\kappa} \geq |D(T)|$.*

(2) *If T is stable, $(\lambda, \kappa) \in \mathrm{SP}(T)$ iff $\lambda = \lambda^{<\kappa} \geq |D(T)|$ or T is stable in λ.*

(3) *If T is stable, $\lambda \geq |D(T)|$, then $(\lambda, \kappa) \in \mathrm{CP}(T)$ iff $\lambda = \lambda^{<\kappa(0)}$ where $\kappa(0) =$ $\min\{\kappa, \kappa r_{\mathrm{cdt}}(T)\}$.*

(4) *If T is unstable, λ strong limit with cofinality $< \kappa$ or $\lambda < 2^{<\kappa}$, then $(\lambda, \kappa) \notin \mathrm{CP}(T)$.*

(5) *Suppose $\kappa > |T|$. Then $\mathrm{CP}(\lambda, \kappa)$ iff $\mathrm{SP}(\lambda, \kappa)$.*

Proof. (1) The 'if' parts follows by Conclusion 4.6, and the 'only if' parts by [4, VIII 4.9, p. 456].

(2) The 'if' part follows from [4, VIII 4.7, p. 455] and the 'only if' part by [4, VIII 4.9, p. 456].

(3) Left to the reader.

(4) By the proof of [4, VIII 4.8, p. 456].

(5) Easy.

Theorem 4.8. (1) *$(\lambda, \kappa) \in \mathrm{SP}(T^*_{\mathrm{ind}})$ iff*

$(*1)^{\kappa}_{\lambda}$ *$\lambda \geq \kappa$, and there are λ functions $f_i : \lambda \to \{0, 1\}$, such that every partial function g from λ to $\{0, 1\}$, $|\mathrm{Dom}\, g| < \kappa$ is included in some f_i.*

(2) *If $\mu = \mu^{<\kappa} \leq \lambda \leq 2^{\mu}$, then $(\lambda, \kappa) \in \mathrm{SP}(T^*_{\mathrm{ind}})$.*

(3) *If T is simple unstable, then $\mathrm{SP}(T) \subseteq \mathrm{SP}(T^*_{\mathrm{ind}})$.*

Proof. (1) Easy by Claim 4.5(2)(iii).

(2) By part (1) and Engelking and Karlowicz [2] (see also [6]).

(3) Easy.

Remark. It is not clear whether $(\lambda, \kappa) \in \mathrm{SP}(T_{\mathrm{ind}})$ implies that for some μ, $\mu^{<\kappa} = \mu \leqslant \lambda \leqslant 2^{\mu}$; this is a set theoretic question. In a counterexample necessarily for some strong limit μ, $\mu < \lambda < \mu^{<\kappa} = 2^{\mu}$, so $0^{\#}$ exists (and much more). We shall try, however, to reverse Theorem 4.8(3) with a partial success.

Lemma 4.9. *Suppose T is simple, $\kappa \geqslant \kappa(1) = \kappa r_1(T)$, $\lambda = \lambda^{<\kappa(1)} \geqslant |T| + \kappa$.*

(1) *For proving $(\lambda, \kappa) \in \mathrm{CP}(T)$, it suffices to prove:*

(St. 1) *For every A, $|A| \leqslant \lambda$, and type r over A, $|\mathrm{Dom}\, r| < \kappa(1)$, there are λ ultrafilters $D_i (i < \lambda)$ over $B^c(r, \emptyset)$, such that each family D of $<\kappa$ elements of $W_A(r, \emptyset)$ with the finite intersection property, is included in one of them.*

(2) *In* (1) *we can replace $\mathrm{CP}(T)$ by $\mathrm{SP}(T)$, provided that we replace* (st. 1) *by* (st. 1′) *by allowing D to have the form $p = \{\varphi_i(x, \bar{a}_i): i < i_0\}$, $|\bigcup_i \bar{a}_i| < \kappa$, p does not weakly divide over $(r, \emptyset)$.*

(3) *If $\mu = \mu^{<\kappa} \leqslant \lambda \leqslant 2^{\mu}$, then the following implies* (St. 1):

(St. 2) *For every $A, |A| \leqslant \lambda$, and type r over A, $|\mathrm{Dom}\, r| < \kappa(1)$ there are μ ultrafilters D_i $(i < \mu)$ over $B^c(r, \emptyset)$, such that $\bigcup_{i<\mu} D_i = W_A(r, \phi) - \{0\}$.*

(4) *If $\mu = \mu^{<\kappa} \leqslant \lambda \leqslant 2^{\mu}$, then the following implies* (St. 1′):

(St. 2′) *For every $A, |A| \leqslant \lambda$ and type r over A. $|\mathrm{Dom}\, r| < \kappa(1)$ there are μ-ultrafilters D_i $(i < \mu)$ over $B^c_A(r, \emptyset)$ such that: every set $D \subseteq W_A(r, \emptyset)$ with the finite intersection property of the form $\{\varphi_i(x, \bar{a}): i < i_0\}$ is included in some D_i.*

Remark. We could use $\mathrm{RO}(W_A(r, B))$ instead $B^c_A(r, B)$ and make other similar non-essential changes.

Proof. (1), (2) Left to the reader.

(3) Let A, r be as in (St. 1), we should find D_i $(i < \lambda)$ satisfying the demands of (St. 1). By (St. 2) there are ultrafilters D_i $(i < \mu)$ of $B^c(r, \phi)$ such that $\bigcup_{i<\mu} D_i = B^c(r, \phi) - \{0\}$.

For every $\chi < \lambda$ let $I_\chi = \{t$: t a finite subset of $\chi\}$. As $\{\{t \in I_\chi: \alpha \in t\}: \alpha < \chi\}$ is a family of subsets of I_χ with the finite intersection property there is an ultrafilter E_χ over I_χ extending this family.

For each function $f: I_\chi \to \mu$ let $D_f = \{a \in B^c(r, \phi): \{t \in I_\chi: a \in D_{f(t)}\} \in E_\chi\}$.

Now the following three facts finish the proof:

Fact A: Each D_f $(f: I_\chi \to \lambda, \chi < \kappa)$ is an ultrafilter of $B^c(r, \phi)$.

Fact B: $\boldsymbol{D}\{D_f: f: I_\chi \to \mu, \chi < \kappa\}$ is a set of μ ultrafilters of $B^c(r, \phi)$ (as its power is $\sum_{\chi<\kappa} \mu^{|I_\chi|} = \sum_{\chi<\kappa} \mu^{\chi} \leqslant \sum_{\chi<\kappa} \mu^{<\kappa} = \sum_{\chi<\kappa} \mu \leqslant \kappa\mu = \mu$).

Fact C: If D is a family of $<\lambda$ elements of $B^c(r, \phi)$, then for some $D_0 \in \boldsymbol{D}$, $D \subseteq D_0$.

(For let $D=\{a_i: i<\chi<\kappa\}$ and to $t\in I_\chi$ let $a_t=\bigcap_{\alpha\in t} a_\alpha$, so by the assumption on D, $a_t\neq 0$. Define $f(t)=\min\{i<\mu: a_t\in D_i\}$ it is well-defined by the choice of the D_i's. It is easy to check that $D\subseteq D_f$, and obviously $D_f\in \mathbf{D}$.

(4) Left to the reader.

Remark. From Lemma 4.9 we have reduced the problem of proving $(\lambda,\kappa)\in \mathrm{SP}(T)$ (for simple T, and e.g. $\mu<\lambda<2^\mu$, $\mu^{<\kappa}+\mu^{|T|}=\mu$) to showing that certain Boolean algebras of power λ, satisfying the $(2^{|T|})^+$-chain condition is the union of μ^+-ultrafilters.

Theorem 4.10. *It is consistent with ZFC (provided that ZFC is consistent) that:*

(1) *for stable T, $\mathrm{SP}'(T)=K^1_{|T|}=^{\mathrm{def}}\{(\lambda,\kappa): \lambda^{|T|}=\lambda\geq\kappa\}$;*

(2) *for simple unstable T,*

$$\mathrm{SP}'(T)=K^2_{|T|}=^{\mathrm{def}}\{(\lambda,\kappa): \lambda^{|T|}=\lambda\geq\kappa,\ \text{and } \lambda \text{ is not strong limit}\},$$

(3) *for non-simple T, $\mathrm{SP}'(T)=K^3_{|T|}=^{\mathrm{def}}\{(\lambda,\kappa): \lambda^{<\kappa}=\lambda\}$;*

(4) *for every λ the classes K^2_λ, K^3_λ are distinct.*

Remark. This theorem shows in some sense the distinction between simple and not simple theories is significant.

Proof of Theorem 4.10. Follows from Theorem 4.12(2), and Lemma 4.13 below, by Lemma 4.9 and Theorem 4.7(4). In Theorem 4.12(2) we choose (μ_α,χ_α) such that $\chi_\alpha>\aleph_{\mu_\alpha}$ to ensure part (1).

Definition 4.11. $Ax_0(\lambda)$ for λ regular is the following statement: Let P be a partial order, $|P|<2^\lambda$, and D_i $(i<i_0<2^\lambda)$ dense subsets. There is a $G\subseteq P$ generic for $\{D_i: i<i_0\}$ provided that:

(1) P is λ-complete, i.e. if $c_i\in P$ $(i<i_0<\lambda)$, $i<j\Rightarrow c_i\leq c_j$, then form some $c\in P$, $c_i\leq c$;

(2) for any sequence $\langle c_i: i<\lambda^+\rangle$ of sequences of P, there is a closed unbounded $S\subseteq\lambda^+$ and sets V_i $(i<\lambda^+)$ increasing $|V_i|\leq\lambda$, and a function $f:\lambda^+\to\bigcup_i V_i$, $f(i)\in V_i$, such that

(i) $cf i=\lambda$, $i\in S$ implies $V_i=\bigcup_{j<i} V_j$,

(ii) $i,j\in S$, $cf i=cf j=\lambda$, $F(i)=F(j)$, implies c_i, c_j has a least upper bound $c_i\vee c_j$.

Theorem 4.12. (1) *Suppose V as model of ZFC satisfies GCH, μ is regular $\chi>\mu$, χ regular. Then in some generic extension V' of V:*

(i) $2^\mu=\chi$, $(\forall\lambda<\mu)2^\mu=\lambda^+$, $(\forall\lambda\geq\mu)2^\lambda=\lambda^++\chi$, *so* $\mu=\mu^{<\mu}$;

(ii) (Ax_0) *holds*;

(iii) *Cardinality and cofinality are preserved by the extensions.*

(2) *We can do the same simultaneously for $\{(\mu_\alpha,\chi_\alpha): \alpha$ an ordinal$\}$ provided that all μ_α, χ_α are regular, $\alpha<\beta\Rightarrow\chi_\alpha<\mu_\beta$.*

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Proof. See [5].

Lemma 4.13. *Suppose* $(Ax_0\mu)$, $\mu^{<\mu}=\mu$, $\boldsymbol{B}$ *a Boolean algebra*, $|\boldsymbol{B}|<2^\mu$, B *satisfies the* μ*-chain condition. Then* $\boldsymbol{B}-\{0\}$ *is the union of* μ *ultrafilters. (We identify* $\boldsymbol{B}$ *and it's universe).*

Proof. Let us define the partial order P. The members of P are partial functions f from $\boldsymbol{B}-\{0\}$ into μ such that $|\text{Range}(f)|<\mu$ and for every $\varphi_1,\ldots,\varphi_n\in \text{Dom} f$, $f(\varphi_1)=\cdots=f(\varphi_n)$ implies $\bigcap_{i=1}^n\varphi_i\neq 0$. The order is inclusion. For every $\varphi\in|B|-\{0\}$ let $D_\varphi=\{f\colon \varphi\in\text{Dom} f\}$. It is easy to see each D_φ is dense (for every $f\in P$ choose $i<\mu$ $i\notin\text{Range} f$, and extend f by $\varphi\mapsto i$), and if $G\subseteq P$ is generic for $\{D_\varphi\colon \varphi\in B-\{0\}\}$, clearly $f^*=\bigcup\{f\colon f\in G\}$, give our conclusion (for each i, extend $\{\varphi\colon f^*(\varphi)=i\}$ to an ultrafilter E_i of B, (possibly as it has the finite intersection property), and clearly $\bigcup_{i<\mu}E_i=|B|-\{0\}$). So we have to check conditions (1), (2) from Definition 4.11. Now (1) is obvious. Let us prove (2). Let $f_\alpha\in P(\alpha<\lambda^+)$, $\text{Dom} f_\alpha=\{\varphi_i^\alpha\colon i<i_\alpha<\mu\}$, $f_\alpha(\varphi_i^\alpha)=j_i^\alpha$. Now for every α, j and finite $w\subseteq i_\alpha$, such that, $i\in w\Rightarrow f_\alpha(\varphi_i^\alpha)=j$ we define by induction on $\zeta<\mu$, an ordinal $\beta(\alpha,w,\zeta)<\alpha$ such that

(a) $i_{\beta(\alpha,w,\zeta)}=i_\alpha$, $\quad f_{\beta(\alpha,w,\zeta)}(i)=j$ for $i\in w$;

(b) $\{\bigcap_{i\in w}\varphi_i^\beta\colon \beta=\beta(\alpha,w,\xi), \xi\leq\zeta, \text{ or } \beta=\alpha\}$ is a family of pairwise disjoint (non-zero by (a)) elements of $|B|$;

(c) $\beta(\alpha,w,\xi)$ $(\xi\leq\zeta)$ is strictly increasing: and

(d) $\beta(\alpha,w,\zeta)$ is the first ordinal for which (a), (b), (c) holds. As B satisfies the μ-chain condition, by (b) for some (first) $\zeta=\zeta(\alpha,w)<\mu$ $\beta(\alpha,w,\zeta)$ is not defined. We let $F(\alpha)$ be the sequence of the following:

(α) $\langle j_i^\alpha\colon i<i_\alpha\rangle$;

(β) $\langle\langle w,\zeta,\beta(\alpha,w,\zeta)\rangle\colon \zeta<\zeta(\alpha,w)\rangle$;

(γ) $\langle\langle\beta_i^\alpha,\gamma_i^\alpha\rangle\colon i<i_\alpha\rangle$, where for $i<i_\alpha$, if for some $\beta<\alpha$, $j<i_\beta$, $\varphi_j^\beta=\varphi_i^\alpha$, then $\langle\beta_i^\alpha,\gamma_i^\alpha\rangle=\langle\alpha+1,j+1\rangle$ otherwise $\beta_i^\alpha=\gamma_i^\alpha=0$.

We let $V_\alpha=\{F(\beta)\colon \beta\leq\alpha\}$, and left the checking to the reader.

Remark 4.14. We can decompose Lemma 4.13 to two:

(i) if $(|B|-\{0\},\geq)$ satisfies condition (2) from Definition 4.11, then it is the union of μ ultrafilters:

(ii) if $\boldsymbol{B}$ satisfies the μ-chain condition, then $(\boldsymbol{B}-\{0\},\geq)$ satisfies condition (2) from Definition 4.11.

Remark 4.15. By [7] if $\boldsymbol{B}$ satisfies the μ chain condition $|\boldsymbol{B}|=\lambda^+$, $\lambda^{<\mu}=\lambda$, then $\boldsymbol{B}-\{0\}$ is the union of λ ultrafilters.

5. Indiscernible sequences based on sets

Definition 5.1. (1) Let $\boldsymbol{I}_0,\boldsymbol{I}_1$ be infinite indiscernible sequences over A of m-sequences. Let $d_A(\boldsymbol{I}_0,\boldsymbol{I}_1)$ be the minimal natural number n such that there are infinite indiscernible sequences over A of m-sequences, $\boldsymbol{J}_l = \langle \bar{a}_t : t \in J_l \rangle$, $(l \leq 2n)$ (the J_l pairwise disjoint infinite, ordered sets, called the witnesses such that:

(i) $\boldsymbol{J}_0 \supseteq \boldsymbol{I}_0, \boldsymbol{J}_{2n} \supseteq \boldsymbol{I}_1$;

(ii) $\langle a_t : t \in J_{2l} + J_{2l+1} \rangle$ is an indiscernible sequence over A for $l < n$, and $\langle a_t : t \in J_{2l+2} + J_{2l+1} \rangle$ is an indiscernible sequence over A for $l+1 \leq n$. If there is no such n, $d_A(\boldsymbol{I}_0, \boldsymbol{I}_1) = \infty$.

(2) $d'_A(\boldsymbol{I}_0, \boldsymbol{I}_1)$ is defined similarly, except that we replace (ii) by

(ii)' $\boldsymbol{J}_l \cap \boldsymbol{J}_{l+1}$ is infinite for $l < 2n$.

(3) $\boldsymbol{I}_0 \approx_A \boldsymbol{I}_1$ if $\boldsymbol{I}_0, \boldsymbol{I}_1$ are infinite indiscernible sequences of m-sequences over A, and $d_A(\boldsymbol{I}_0, \boldsymbol{I}_1) < \infty$.

(4) $C_A(\boldsymbol{I})$ where $\boldsymbol{I}$ is an infinite indiscernible sequence over A, is $\bigcup \{\boldsymbol{J} : \boldsymbol{J} \approx_A \boldsymbol{I}\}$, and $C^n_A(\boldsymbol{I}) = \bigcup \{\boldsymbol{J} : d_A(\boldsymbol{J}, \boldsymbol{I}) \leq n\}$. (So $C_A(\boldsymbol{I}) = \bigcup_{n<\omega} C^n_A(\boldsymbol{I})$.)

Definition 5.2. (1) The infinite indiscernible sequence $\boldsymbol{I}$ is *weakly based over A if* it is indiscernible sequence over A, and the number of images of $C_A(\boldsymbol{I})$ by automorphisms of $\mathfrak{C}$ fixing A is $< \|\mathfrak{C}\|$.

(2) The infinite indiscernible sequence $\boldsymbol{I}$ is *based on* A if there are no automorphism F_i $(i < \|\mathfrak{C}\|)$ of $\mathfrak{C}$, $F_i \restriction A$ = the identity, and for $i \neq j$, $F_i(\boldsymbol{I}) \not\approx_A F_j(\boldsymbol{I})$.

Claim 5.3. *Let $\boldsymbol{I}_0, \boldsymbol{I}_1$ be infinite indiscernible sequences over A of m_0-sequences. The following are equivalent:*

(1) $d_A(\boldsymbol{I}_0, \boldsymbol{I}_1) \leq n$;

(2) *for every $m < \omega$ and finite $\Delta \subseteq L$ and finite $A' \subseteq A$ and $\bar{a}^l_i \in \boldsymbol{I}_l$ $(i < m)$ increasing (in the order of the sequence) $(l = 0, 1)$ there are $\bar{b}^k_i$ $(k \leq 2n, i < m)$ such that:*

(i) $\bar{b}^0_i = \bar{a}^0_i$, $\bar{b}^{2n}_i = \bar{a}^1_i$ $(i < m)$,

(ii) $\langle \bar{b}^{2l}_i : i < m \rangle {}^\frown \langle \bar{b}^{2l+1}_i : i < m \rangle$ *is (Δ, m)-indiscernible over A' and* $\langle \bar{b}^{2l+2}_i : i < m \rangle {}^\frown \langle \bar{b}^{2l+1}_i : i < m \rangle$ *is (Δ, m)-indiscernible over A', for $l < n$.*

Proof. Left to the reader.

Conclusion 5.4. (1) For any A, n, and m, there is a type

$$r^{m,n}_A = r^{m,n}_A(\bar{x}_0, \bar{y}_0, \ldots, \bar{x}_k, \bar{y}_k, \ldots, A) \quad (l(\bar{x}_k) = l(\bar{y}_k) = m),$$

such that if $\boldsymbol{I} = \langle \bar{a}_k : k < \omega \rangle$, $\boldsymbol{J} = \langle \bar{b}_k : k < \omega \rangle$ are indiscernible sequences of m-sequences over A, then $d_A(\boldsymbol{I}, \boldsymbol{J}) \leq n$ *iff* $\bar{a}^0_0 {}^\frown \bar{b}^0_0 {}^\frown \cdots {}^\frown \bar{a}_k {}^\frown \bar{b}_k {}^\frown \cdots$ realizes $r^{m,n}_A$.

(2) For any A, n, m there is a type

$$r^{m,n}_A(\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_k, \ldots, A) \quad (l(\bar{x}) = l(\bar{y}_k) = m)$$

such that if $\boldsymbol{I} = \langle \bar{b}_k : k < \omega \rangle$ is an indiscernible sequence of m-sequences over A, $l(\bar{a}) = m$, then $\bar{a} \in C^n_A(\boldsymbol{I})$ iff $\bar{a} {}^\frown \bar{b}_0 {}^\frown \bar{b}_1 {}^\frown \cdots {}^\frown \bar{b}_k {}^\frown \cdots$ realizes $r^{m,n}_A$.

Claim 5.5. (1) *d_A, d'_A are distance functions and $\approx_A$ is an equivalence relation.*

(2) *Let $\boldsymbol{I}_0, \boldsymbol{I}_1$ be infinite indiscernible sequences over A, then $d_A(\boldsymbol{I}_0, \boldsymbol{I}_1) < \infty$ iff $d'_A(\boldsymbol{I}_0, \boldsymbol{I}_1) < \infty$*

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(3) *If* $I_0 \approx_A I_1$, *then for any increasing* $\bar{a}^l_0, \ldots, \bar{a}^l_n \in I_l$ $(l = 0, 1)$ $\mathrm{tp}(\bar{a}^0_0 {}^\frown \cdots {}^\frown a^0_n, A) = \mathrm{tp}(\bar{a}^1_0 {}^\frown \cdots {}^\frown \bar{a}^1_n, A)$.

Proof. (1) Let I_l $(l = 0, 1, 2)$ be infinite indiscernible sets over A.

(i) Reflexivity: $d_A(I_0, I_0) = d'_A(I_0, I_0) = 0$ because we can choose $J_0 = J_0$, $n = 0$.

(ii) Symmetry: $d'_A(I_0, I_1) = d'_A(I_1, I_0)$ because if $\langle J_l : l \leq 2n \rangle$ is a witness for $d'_A(I_0, I_1) = n$, then $\langle J_{n-l} : l \leq 2n \rangle$ is a witness for $d'_A(I_0, I_1) \leq n$ hence $d'_A(I_0, I_1) \leq n$ implies $d'_A(I_1, I_0) \leq n$. By the symmetry, equality follows; $d_A(I_0, I_1) = d_A(I_1, I_0)$ is proved similarly.

(iii) The triangle law: Suppose $d_A(I_l, I_{l+1}) = n_l < \infty$ $(l = 0, 1)$ and $\langle J^l_i : i \leq 2n_l \rangle$ is a witness for this. Without loss of generality $J^0_{2n_0} = I_1 = J^1_0$ (see Definition 5.1) and let

$$J_i = \begin{cases} J^0_i & i \leq 2n_0, \\ J^1_{i-2n_0} & 2n_0 \leq i \leq 2n_0 + 2n_1. \end{cases}$$

Then $\langle J_i : i \leq 2(n_0 + n_1) \rangle$ is a witness for $d_A(I_0, I_2) \leq n_0 + n_1$. So

$$d_A(I_0, I_2) \leq d_A(I_0, I_1) + d_A(I_1, I_2).$$

(If the right-hand side is finite by the previous arguments, otherwise trivially.) The triangle law for d'_A is proved similarly.

(2), (3) Left to the reader.

Claim 5.6. (1) *If* I *is based on* A, *then* I *is weakly based on* A.

(2) *If Definition* 5.2(1) *we can replace* $< \|\mathfrak{C}\|$ *by* $\leq 2^{|A|+|T|}$. *Similarly in Definition* 5.2(2) *it is equivalent to demand there are no such* F_i *for* $i < (2^{|A|+|T|})^+$. *Also if we change in Definition* 5.2(2) $F_i(I) \not\approx_A F_j(I)$ *to* $d_A(F_i(I), F_j(I)) > 1$ *the above mentioned assertion remain true.*

(3) *The number of* $C_A(I)$, I *weakly based on* A *is* $\leq 2^{|A|+|T|}$.

Proof. (1) Trivial

(2) Let us concentrate for example on Definition 4.2(2): similar argument appear in McKenzie and Shelah [3]. Suppose F_i is an automorphism of $\mathfrak{C}$, $F_i \restriction A =$ the identity for $i < (2^{|A|+|T|})^+$, $I_i = F_i(I)$ *and* $I_i \not\approx_A I_j$ for $i < j$. We can assume $I = \langle \bar{a}_k : k < \omega \rangle$, and let $\bar{a}^i_k = F_i(\bar{a}_k)$. By a variant of Erdos–Rado theorem we can assume $\mathrm{tp}(\bar{a}^i_0 {}^\frown \bar{a}^j_0 {}^\frown \cdots {}^\frown \bar{a}^i_k {}^\frown \bar{a}^j_k, \ldots, A)$ for $i < j < (|A| + |T|)^+$ depends on i only. For $i < j$, and $n < \omega$ as $d(I_i, I_j) > n$, there is $\psi^n_{i,j}(\bar{x}_0, \bar{y}_0, \ldots, \bar{x}_k \bar{y}_k, \bar{b}) \in r^{m,n}_A$ $(k = k(i, j, n))$, $(\bar{b} = \bar{b}^n_{i,j})$ (see Conclusion 5.4(1) for its definition) such that $\vDash \neg \psi^n_{i,j}(\bar{a}^i_0, \bar{a}^j_0, \ldots, \bar{a}^i_k, \bar{a}^j_k, \bar{b})$. By the above mentioned assumption we can assume $\psi^n_{i,j} = \psi^n_i$, $k(i, j, n) = k(i, n)$, $\bar{b}^n_{i,j} = \bar{b}^n_i$. The number of possible $\langle \psi^n_i, \bar{b}^n_i, k(i, n) \rangle$ is $|T| + |A|$, hence we can choose by induction on n, ψ^n, $\bar{b}^n$, $k(n)$ and $S_n \subseteq (|A| + |T|)^+$ (i.e. S_n is a set of ordinals) such that:

(i) $|S_n| = (|A| + |T|)^+$, $S_{n+1} \subseteq S_n$;

(ii) for every $i \in S_n$, $\psi^n_i = \psi^n$, $\bar{b}^n_i = \bar{b}^n$, $k(n, i) = k(n)$.

Now by compactness argument, there are $\bar{a}^\alpha_k$ $(\alpha < \|\mathfrak{C}\|)$ such that

(a) $\operatorname{tp}(\bar{a}^\alpha_0 {}^\frown \bar{a}^\alpha_1 {}^\frown \cdots {}^\frown \cdots, A) = \operatorname{tp}(\bar{a}_0 {}^\frown \bar{a}_1 {}^\frown \cdots {}^\frown \bar{a}_k {}^\frown \cdots, A)$

and

(b) for $\alpha < \beta, n < \omega$, $\models \neg\psi^n[\bar{a}^\alpha_0, \bar{a}^\beta_0, \bar{a}^\alpha_1, \bar{a}^\beta_1, \ldots, \bar{a}^\alpha_{k(n)}, \bar{a}^\beta_{k(n)}, \bar{b}^n]$

hence

$$d_A(\boldsymbol{J}^\alpha, \boldsymbol{J}^\beta) \geq n \quad \text{where } \boldsymbol{J}^\alpha = \langle \bar{a}^\alpha_k \colon k < \omega \rangle.$$

so by (a) for each $\alpha < \|\mathfrak{C}\|$ there is an automorphism F'_α of $\mathfrak{C}$, $F'_\alpha \restriction A$ = the identity, $F_\alpha(\boldsymbol{I}) = \boldsymbol{J}_\alpha$, and by (b) for $\alpha \neq \beta$, $d_A(\boldsymbol{J}_\alpha, \boldsymbol{J}_\beta) = \infty$, so we finish.

For Definition 5.2(1) we use conclusion 5.4(2).

Lemma 5.7. *For any $p \in S^m(|M|)$ there is an indiscernible sequence $\boldsymbol{I}$ of sequences realizing p which is based on $|M|$ in fact $\boldsymbol{J} \approx_A \boldsymbol{I}$ iff $d_A(\boldsymbol{I}, \boldsymbol{J}) \leq 1$ iff for any increasing $\bar{a}_0, \ldots, \bar{a}_n \in \boldsymbol{I}$, $\bar{b}_0, \ldots, \bar{b}_n \in \boldsymbol{J}$*

$$\operatorname{tp}(\bar{a}_0 {}^\frown \cdots {}^\frown \bar{a}_n, |M|) = \operatorname{tp}(\bar{b}_0 {}^\frown \cdots {}^\frown \bar{b}_n, |M|).$$

Proof. This is essentially from [4, VII, §4]. Let D_0 be the filter over ${}^m|M|$ generated by the family $\{\varphi(M, \bar{a}) \colon \varphi(\bar{x}; \bar{a}) \in p\}$ $(\varphi(M; \bar{a}) = \{b \in |M| \colon M \models \varphi([b; a]\})$. Let D be an ultrafilter extending D_0 and for every B, let $\operatorname{Av}(D, B) = \{\varphi(\bar{x}; \bar{b}) \colon \bar{b} \in B, \{\bar{a} \in |M| \colon \models \varphi[\bar{a}; \bar{b}]\} \in D\}$. It is in $S^m(B)$. Define inductively $\bar{a}_n$: $\bar{a}_n$ realizes $\operatorname{Av}(D, |M| \cup \bigcup_{l<n} \bar{a}_l)$. $\boldsymbol{I} = \langle \bar{a}_n \colon n < \omega \rangle$ is indiscernible over $|M|$, and if $\boldsymbol{J}$ is as in the lemma, define $\bar{c}_n$ to realize $\operatorname{Av}(D, |M| \cup \boldsymbol{I} \cup \boldsymbol{J})$, and then $\boldsymbol{J}_0 = \boldsymbol{I}$, $\boldsymbol{J}_1 = \langle \bar{c}_n : n < \omega \rangle$, $\boldsymbol{J}_1 = \boldsymbol{J}$ are witnesses for $d_A(\boldsymbol{I}, \boldsymbol{J}) \leq 1$. For the last iff, apply Claim 5.5(3).

6. Existence of indiscernible sequences based on a set

Definition 6.1. (1) An infinite indiscernible sequence $\boldsymbol{I} = \langle \bar{a}_t \colon t \in I \rangle$ is called *based on* (A, B), where $A \subseteq B$, if

(i) I is indiscernible over B:

(ii) for every $t \in I$, $\operatorname{tp}(\bar{a}_t, B \cup \bigcup_{s<t} \bar{a}_s)$ does not fork over A.

Claim 6.2. *If $\boldsymbol{I}$ is based on (A, B), $A \subseteq A_1 \subseteq B_1 \subseteq B$, then $\boldsymbol{I}$ is based on (A_1, B_1). Also if $\boldsymbol{I} = \boldsymbol{I}_0 + \boldsymbol{I}_1$, then $\boldsymbol{I}_1$ is based on $(A, B \cup \boldsymbol{I}_0)$.*

Remark. Note the difference between this definition and Definition 5.2(2).

Lemma 6.3. *If $A \subseteq B$, $p \in S^m(B)$, and p does not fork over A, then there are $\bar{a}_n$ realizing p such that $\langle \bar{a}_n \colon n < \omega \rangle$ is based on (A, B).*

Proof. Let $\lambda = 2^{|B|+|T|}$, and we define by induction on $\alpha < \beth_\lambda$, m-sequences $\bar{b}_\alpha$ such that $\operatorname{tp}(\bar{b}_\alpha, B \cup \bigcup_{\beta<\alpha} \bar{b}_\beta)$ extend p and does not fork over A. This is possible

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by Conclusion 2.11(3). Now by the method Morley proved his omitting type theorem (see [1, Theorem 7.2.2]) there are m-sequences $\bar a_n$ $(n<\omega)$ such that:

(i) for every n there are $\alpha(0)<\cdots<\alpha(n)$ for which

$$\operatorname{tp}(\bar a_0{}^\frown\cdots{}^\frown\bar a_n, B)=\operatorname{tp}(\bar b_{\alpha(0)}{}^\frown\cdots{}^\frown\bar b_{\alpha(n)}, B);$$

(ii) $\langle\bar a_n: n<\omega\rangle$ is indiscernible over B.

By (i) each $\bar a_n$ realizes p and $\operatorname{tp}(\bar a_n, B\cup\bigcup_{k<n}\bar a_k)$ does not fork over A, hence remembering (ii), $\langle\bar a_n: n<\omega\rangle$ is as required.

Conclusion 6.4. *For simple T, $p\in S^m(A)$ there is an indiscernible sequence based on (A, A) of sequences realizing p.*

Proof. By Conclusion 2.11(2) p does not fork over A, so apply Lemma 6.3.

Lemma 6.5. *For simple T, every indiscernible set $\boldsymbol I$ based on (A, B) is based on A.*

Proof. Suppose not, let $\lambda=2^{|A|+|T|}$, so by Claim 5.6(2) there are automorphisms F_i of $\mathfrak{C}$, $F_i\restriction A=$ the identity, $(i<\beth_{\lambda^+})$ such that $d_A(\boldsymbol I_\alpha,\boldsymbol I_\beta)>1$ for $\alpha\neq\beta$, where $\boldsymbol I_\alpha=F_\alpha(\boldsymbol I)$, and w.l.o.g. let $\boldsymbol I=\langle\bar a_k: k<\omega\rangle$, $\bar a_k^\alpha=F_\alpha(\bar a_k)$.

Claim 6.5. (1) *There are b_k^α, $k<\omega$, $\alpha<\omega$ such that*

(i) $\operatorname{tp}(\bar b_0^\alpha{}^\frown\cdots{}^\frown\bar b_k^\alpha{}^\frown\cdots, A)=\operatorname{tp}(\bar a_0{}^\frown\cdots{}^\frown\bar a_k{}^\frown\cdots, A)$

(ii) *for every k, $\langle\bar b_0^\alpha{}^\frown\cdots{}^\frown\bar b_k^\alpha: \alpha<\omega\rangle$ is an indiscernible sequence over A.*

(iii) *for every $n<l<\omega$ there are $\alpha<\beta<\beth_{\lambda^+}$ such that*

$$\begin{aligned}&\operatorname{tp}_*(\bar b_0^n{}^\frown\bar b_0^l{}^\frown\bar b_1^n{}^\frown\bar b_1^l{}^\frown\cdots{}^\frown\bar b_k^n{}^\frown\bar b_k^l{}^\frown\cdots, A)\\ &=\operatorname{tp}_*(\bar a_0^\alpha{}^\frown\bar a_0^\beta{}^\frown\bar a_1^\alpha{}^\frown\bar a_1^\beta{}^\frown\cdots{}^\frown\bar a_k^\alpha{}^\frown\bar a_k^\beta{}^\frown\cdots, A).\end{aligned}$$

Proof of Claim 6.5. (1) Define by induction on $n<\omega$ a type $p_n=p(\bar x^l_{m l, m\leq n})$, $l(\bar x_m)=l(\bar a.)$ such that:

(A) $p_n\subseteq p_{n+1}$.

(B) For each $\gamma<\lambda^+$ there is an increasing sequence $\bar a^r=\langle\alpha_i^r: i<\beth_\gamma\rangle$ of ordinals less than $\beth_{\lambda^+}$ such that:

$(1^*)_n$ $\langle\bar a_0^{\alpha_i}\cdots\bar a_{n-1}^{\alpha_i}: i<\beth_\gamma\rangle$ is an indiscernible sequence over A.

$(2^*)_n$ if $i_1<i_2<\cdots<i_n$, then $\operatorname{tp}(\bar a_m^{\alpha_{i_l}}, m<n)=p_n$.

We shall prove the existence by induction on n:

For $n=0$, $p_n=\emptyset$.

For $n+1$ for each there is a sequence $\langle\alpha_i: i<\beth_{\gamma+n+1}\rangle$ as mentioned above in (B) for n. Applying Erdos–Rado we get an appropriate sequence of length $\beth_\gamma$ for the type p^γ_{n+1}.

Because there are only λ types there is p_{n+1} s.t. $|\{r: p^r_{n+1}=p_{n+1}\}|=\lambda^+$. It is easy to verify all the other conditions. $\bigcup_{n<\omega}p_n$ is consistent, hence $\langle b_k^\alpha: \alpha<\omega, k<\omega\rangle$ realize it. It is easy to check that they are as required. Now we define by induction on $l<\omega$, m-sequences $\bar{c}_l$ such that

(i)$_l$ $\mathrm{tp}(\bar{b}_0^\alpha{}^\frown\cdots{}^\frown\bar{b}_k^\alpha{}^\frown\bar{c}_0{}^\frown\cdots{}^\frown\bar{c}_{l-1}, A)$ does not depend on α and is equal to $\mathrm{tp}(\bar{b}_0^\alpha{}^\frown\cdots{}^\frown\bar{b}_k^\alpha{}^\frown\bar{b}_{k+1}^\alpha,\ldots,\bar{b}_{k+l}^\alpha, A)$ (for every $k<\omega$);

(ii)$_l$ for every k $\langle\bar{b}_0^\alpha{}^\frown\cdots{}^\frown\bar{b}_k^\alpha: \alpha<\omega\rangle$ is an indiscernible sequence over $A\cup\bigcup_{n<l}\bar{c}_n$.

For $l=0$, (i)$_l$, (ii)$_l$ reduce to (i), (ii) hence holds. So suppose (i)$_l$, (ii)$_l$ hold, $\bar{c}_0,\ldots,\bar{c}_{l-1}$ are defined, and we shall define $\bar{c}_l$ such that (i)$_{l+1}$, (ii)$_{l+1}$ holds. Note that (i)$_l$ is equivalent to: for every $\alpha<\omega$, $\langle\bar{b}_k^\alpha: k<\omega\rangle{}^\frown\langle\bar{c}_n: n<l\rangle$ in an indiscernible sequence over A. Let, for $\alpha<\omega$

$$p_\alpha=\{\varphi(\bar{x},\bar{c}_{l-1},\bar{c}_{l-2},\ldots,\bar{c}_0,\bar{b}_{k-1}^\alpha,\ldots,\bar{b}_0^\alpha,\bar{a}): \bar{a}\in A,\ \models\varphi[b_{k+l}^\alpha,\bar{b}_{k+l-1}^\alpha,\ldots,\bar{b}_k^\alpha,\bar{b}_{k-1}^\alpha,\ldots,\bar{b}_0^\alpha,\bar{a}]\}.$$

Clearly $\bar{c}$ realizes p_α iff $\langle\bar{b}_k^\alpha: k<\omega\rangle{}^\frown\langle\bar{c}_0,\bar{c}_1,\ldots,\bar{c}_{l-1},\bar{c}\rangle$ is an indiscernible sequence over A.

Let Γ be a set of formulas, with the free variable in $\bar{x}$ only, over $A\cup\{\bar{b}_k^\alpha: \alpha<\omega, k<\omega\}\cup\{\bar{c}_0,\ldots,\bar{c}_{l-1}\}$, such that $\bar{c}$ realizes Γ iff for every k

$$\langle\bar{b}_0^\alpha{}^\frown\cdots{}^\frown\bar{b}_k^\alpha: \alpha<\omega\rangle \text{ is an indiscernible sequence over}$$
$$A\cup\textstyle\bigcup_{n<l}\bar{c}_n\cup\bar{c}.$$

Clearly such Γ exists.

Now a sequence $\bar{c}_l$ satisfies (i)$_{l+1}$ iff $\bar{c}$ realizes $\bigcup_{\alpha<\omega}p_\alpha$, and (ii)$_{l+1}$ iff it realizes Γ. So it suffices to show $\bigcup_{\alpha<\omega}p_\alpha\cup\Gamma$ is consistent.

First we show $\bigcup_{\alpha<\omega}p_\alpha$ is consistent. Otherwise there are $\alpha(n)<\omega (n<n(0))$, $\varphi_n\in p_{\alpha(n)}$ such that $\{\varphi_n: n<n(0)\}$ is contradictory. As we can replace φ_n by any conjunction of formulas from $p_{\alpha(n)}$ in which it appears, and increase $\{\alpha(n): n<n(0)\}$, we can assume $\alpha(n)=n$ for $n<n(0)$ and $\varphi_n=\varphi(\bar{x},\bar{c}_{l-1},\ldots,\bar{c}_0,\bar{b}_k^n,\bar{b}_{k-1}^n,\ldots,\bar{b}_0^n,\bar{a})$, $\bar{a}\in A$, so φ_n is defined naturally for every n. By (ii)$_l$, for every $n_0<n_l<\cdots<n_{n(0)-1}<\omega$ $\{\varphi_{n_0},\varphi_{n_1},\ldots,\varphi_{n_{n(0)-1}}\}$ is contradictory, and in fact, for every distinct $n_0,\ldots,n_{n(0)}<\omega$. In other words $\{\varphi_n: n<\omega\}$ is $n(0)$-contradictory. Again by (ii), $\langle\bar{c}_{l-1}{}^\frown\cdots{}^\frown\bar{c}_0{}^\frown\bar{b}_k^n{}^\frown\bar{b}_{k-1}^n{}^\frown\cdots{}^\frown\bar{b}_0^n{}^\frown\bar{a}: n<\omega\rangle$ is an indiscernible sequence over A, but in the nth place appears the sequence of parameters of φ_n. Hence by Definition 1.1(1) φ_0 divides over A. But by (i)$_l$ this implies that $\varphi(\bar{x},\bar{b}_{k+l}^0,\ldots,\bar{b}_{k+1}^0,\bar{b}_k^0,\bar{b}_{k-1}^0,\ldots,\bar{b}_0^0,\bar{a})$ divides over A. But $\bar{b}_{k+l+1}^0$ satisfies it. So by (i), $\varphi(\bar{x},\bar{a}_{k+l}^0,\ldots,\bar{a}_0,\bar{a})$ divides over A and $\bar{a}_{k+l+1}$ satisfies it, so $\mathrm{tp}(\bar{a}_{k+l+1}, A\cup\bigcup_{i\le k+l}\bar{a}_i)$ divides (hence fork) over A, contradicting an hypothesis.

So $\bigcup_{n<\omega}p_n$ is consistent. Using (ii)$_l$ and Ramsey theorem, it is not hard to prove that $\bigcup_{n<\omega}p_n\cup\Gamma$ is consistent, and choose $\bar{c}_l$ as any sequence realizing it. So as said before, (i)$_{l+1}$, (ii)$_{l+1}$ holds, hence this completes the inductive definition of the $\bar{c}_l$'s. Now $\boldsymbol{J}_0=\langle\bar{b}_k^0: k<\omega\rangle$, $\boldsymbol{J}_1=\langle\bar{c}_k: k<\omega\rangle$, $\boldsymbol{J}_2=\langle\bar{b}_k^1: k<\omega\rangle$ are witnesses for $d_A(\boldsymbol{J}_0,\boldsymbol{J}_2)\le 1$ (as $\boldsymbol{J}_0+\boldsymbol{J}_1$, $\boldsymbol{J}_2+\boldsymbol{J}_1$ are indiscernible sequences over A by (i)$_l$ $(l<\omega)$.

So by (iii) (from the conditions on the $\bar{b}$'s) for some $\alpha \neq \beta$, $d_A(\boldsymbol{I}_\alpha, \boldsymbol{I}_\beta) \leq 1$ contradiction.

In fact the proof shows (remembering the proof of Claim 5.6) e.g.

Lemma 6.6. *If $\boldsymbol{I}_\alpha = \langle \bar{a}_t^\alpha : t \in I_\alpha \rangle$ is infinite indiscernible sequence based on (A, A) for $\alpha < (2^{|A|+|T|})^+$, T simple, then for some $\alpha \neq \beta$ $d_A(\boldsymbol{I}_\alpha, \boldsymbol{I}_\beta) \leq 1$.*

Conclusion 6.7. *If T is simple, $p \in S^m(A)$, then there is an infinite indiscernible sequence based on A of sequences realizing p.*

Proof. Combine Lemmas 6.5 and 6.6.

7. Summing on an indiscernible sequences

In this section T is simple.

Definition 7.1. For a $B^c(r, B)$ clear from the context, r an m-type, a formula $\varphi(\bar{x}, \bar{y})$, a set A and an infinite sequence $\boldsymbol{I}$ indiscernible over $A_1 = A \cup B \cup \operatorname{Dom} r$, $\bar{b} \in \boldsymbol{I} \Rightarrow l(\bar{b}) = l(\bar{y})$, let

$$\operatorname{cn}_A^\varphi(\boldsymbol{I}) = \operatorname{cn}(\boldsymbol{I}, \varphi, A) = \bigvee \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in C_{A_1}(\boldsymbol{I})\} \in B^c(r, B).$$

If $B^c(r, B)$ is not clear in the content, we write $\operatorname{cn}(\boldsymbol{I}, \varphi, A, r, B)$ or $\operatorname{cn}_A^\varphi(\boldsymbol{I}, \boldsymbol{B})$ where $\boldsymbol{B} = B^c(r, B)$.

Remark. Note $\operatorname{cn}_A^\varphi(\boldsymbol{I})$ is well-defined because the disjunction $\vee$ is taken in the complete Boolean algebra $B^c(r, B)$.

Claim 7.2. *If $B \cup \operatorname{Dom} r \subseteq A$, $\boldsymbol{I}$ an infinite indiscernible sequence over A, weakly based on A, then $\operatorname{cn}_A^\varphi(\boldsymbol{I}, \boldsymbol{B})$ $(\boldsymbol{B} = B^c(r, B))$ is weakly supported by A.*

Proof. Trivial (see Definition 3.7).

Theorem 7.3. *If $B \cup \operatorname{Dom} r \subseteq A$, $\boldsymbol{I}$ infinite, indiscernible sequence over A, $|\boldsymbol{I}| \geq \kappa r_{\text{cdt}}(T)$, then in $B^c(r, B)$*

$$\operatorname{cn}_A^\varphi(\boldsymbol{I}) = \bigvee_{\bar{c} \in \boldsymbol{I}} \varphi(\bar{x}; \bar{c}).$$

The proof is decomposed to claims, $\boldsymbol{B} = \boldsymbol{B}(r, B)$ is fixed, and λ the first cardinal such that B satisfy the λ-chain condition (exists by Lemma 3.2 and Definition 3.4).

Fact 7.4. *If $I=\langle \bar a_t: t\in I\rangle$ is indiscernible over A, and infinite, then for any ordered set J, $I\subseteq J$, we can define $\bar a_t$ $(t\in J-I)$ such that $\langle \bar a_t: t\in J\rangle$ is indiscernible over A.*

Claim 7.5. *If $B\cup \operatorname{Dom} r\subseteq A$, $\boldsymbol{I}=\langle \bar a_t: t\in I\rangle$ is an indiscernible sequence over A, and $J\subseteq I$, $|J|\geqslant\lambda$, then*

$$\varphi_J \overset{\text{df}}{=} \bigvee_{t\in J}\varphi(\bar x: \bar a_t)=\varphi_I \overset{\text{df}}{=} \bigvee_{t\in I}\varphi(\bar x: \bar a_t)$$

Proof. Trivially $\varphi_J\leqslant\varphi_I$. To prove the converse it suffices to prove $\varphi(\bar x, \bar a_{s(0)})\leqslant\varphi_J$ for any $s(0)\in I$. Let us define for $s\in I$, $J_s=\{t\in J: t<s\}$, $\boldsymbol{J}_s=\langle \bar a_t: t\in J_s\rangle$. Clearly $J_{s(0)}=\{t\in J: t<s(0)\}$ or $\{t\in J: t\geqslant s(0)\}$ has cardinality λ, by symmetry (we can invert the order) we can assume $|J_{s(0)}|\geqslant\lambda$. So for any

$$s\in J_{s(0)}\varphi(\bar x; \bar a_{s(0)})\not\leqslant \bigvee_{t\in J_s}\varphi(\bar x, \bar a_t)$$

(in $B^c(r, B)$), but clearly

$$\operatorname{tp}(\bar a_s, A\cup \boldsymbol{J}_s)=\operatorname{tp}(\bar a_{s(0)}, A\cup \boldsymbol{J}_s).$$

So $\varphi(\bar x; \bar a_s)\not\leqslant \bigvee_{t\in J_s}\varphi(\bar x: \bar a_t)$. Let $\varphi_s=\varphi(\bar x: \bar a_s)-\bigvee_{t\in J_s}\varphi(\bar x: \bar a_t)$, so $\varphi_s\in B^c(r, B)$, $\varphi_s\neq 0$, and clearly for $s(1)\neq s(2)$ in J, $\varphi_{s(1)}, \varphi_{s(2)}$ are disjoint. So $\{\varphi_s: s\in J\}$ is a family of $\geqslant\lambda$ non-zero, pairwise disjoint elements of $B^c(r, B)$ contradicting the choice of λ.

Claim 7.6. (1) *If $B\cup\operatorname{Dom} r\subseteq A$, $\boldsymbol{I}_0$ and $\boldsymbol{I}_1$ are indiscernible sequences over A such that, $\boldsymbol{I}_0\approx_A\boldsymbol{I}_1$, $|\boldsymbol{I}_0|, |\boldsymbol{I}_1|\geqslant\lambda$,* then

$$\bigvee_{a\in I_0}\varphi(\bar x; \bar a)=\bigvee_{a\in \boldsymbol{I}_1}(\bar x: \bar a).$$

(2) *Theorem 7.3 holds when $|\boldsymbol{I}|\geqslant\lambda$.*

Proof. (1) By Definition 5.1(3) there are infinite indiscernible sequences $\boldsymbol{J}_l$ $(l<2n, n<\omega)$, $\boldsymbol{I}_0=\boldsymbol{J}_0$, $\boldsymbol{I}_1=\boldsymbol{J}_{2n}$, $\boldsymbol{J}_{2l}+\boldsymbol{J}_{2l+1}$, $\boldsymbol{J}_{2l+2}+\boldsymbol{J}_{2l+1}$ are indiscernible over A. We can assume $\|\boldsymbol{J}_l\|\geqslant\lambda$ (add predicates for the $\boldsymbol{J}_l$ is, extend to a λ-saturated model and embedd in $\mathscr{C}$ over $A\cup\boldsymbol{I}_0\cup\boldsymbol{I}_1$). We now prove by induction on l that $\bigvee_{\bar a\in\boldsymbol{I}_l}\varphi(x; \bar a)=\bigvee_{\bar a\in\boldsymbol{I}_0}\varphi(x; \bar a)$. For $l=0$ it is trivial, for $l+1$ use Claim 7.5 twice, and for $l=2n$ we get the conclusion.

(2) Easy by (1).

Claim 7.7. *If $\boldsymbol{I}$ is indiscernible over $A=B\cup\operatorname{Dom} r$ and $\psi(\bar x, \bar b)$ is a formula such that for some $\bar a_0\in\boldsymbol{I}$ $\{\varphi(\bar x, \bar a_0), \psi(\bar x, \bar b)\}$ does not weakly divide over (r, B), while for some $\bar a_1\in\boldsymbol{I}$ $\{\varphi(\bar x, \bar a_1), \varphi(\bar x, \bar b)\}$ weakly divides over (r, B), then $\operatorname{tp}(\bar b, A\cup\bar a_0\cup\bar a_1)$ divides over A hence $\operatorname{tp}(\bar b, A\cup\boldsymbol{I})$ does.*

Proof. Without loss of generality we may assume $\boldsymbol{I}$ is $\langle \bar{a}_i: i<\lambda_1\rangle$ with $\bar{a}_0, \bar{a}_1$ satisfying the hypotheses, $\lambda_1 \geq \lambda$. Let F_i be an automorphism of $\mathscr{C}$ fixing A while mapping $\bar{a}_0$ to $\bar{a}_{2i}$, $\bar{a}_1$ to $\bar{a}_{2i+1}$ and let r_i be $F_i(\mathrm{tp}(\bar{b}, A \cup \{\bar{a}_0, \bar{a}_1\}))$. For some k, the r_i are k-contradictory. Otherwise by the indiscernibility of $\bar{a}_i$, there is an element $\bar{b}^*$ which realizes all the r_i. But $\psi(\bar{x}, \bar{b}^*) \wedge \varphi(\bar{x}, \bar{a}_{2i+1}) = 0$ in $B^c(r, B)$ for all i. But then $\psi(\bar{x}, \bar{b}^*) \wedge \bigvee_i \varphi(\bar{x}, \bar{a}_{2i}) = 0$ and since by the previous lemma $\bigvee_i \varphi(\bar{x}, \bar{a}_{2i}) = \bigvee_i (\bar{x}, \bar{a}_i)$ this implies $\psi(x, \bar{b}^*) \wedge \bigvee_{i<\lambda_1} \varphi(\bar{x}, \bar{a}_i) = 0$. In particular, $\psi(x, b^*) \wedge \varphi(\bar{x}, \bar{a}_0) = 0$ contrary to hypothesis.

Clearly all the r_i's are automorphic images of r_0 over A so by Lemma 1.3 r_0 divides over A.

Proof of Theorem 7.3. Let $\boldsymbol{I} = \langle a_t: t \in I\rangle$, and remember $|I| \geq \kappa r_{\mathrm{cdt}}(T)$. Choose a $(\lambda + |I|)^+$-saturated dense order J, with no first or last element, $I \subseteq J$, and $\bar{a}_t$ $(t \in J - I)$ such that $\boldsymbol{J} = \langle \bar{a}_t: t \in J\rangle$ is indiscernible over A. (exists by Fact 7.4).

Trivially $\mathrm{cn}^{\varphi}_{A}(\boldsymbol{I}) \geq \bigvee_{t \in I} \varphi(\bar{x}; \bar{a}_t)$, and for proving the converse inequality it suffices to prove for any $\bar{b} \in C_A(\boldsymbol{I}) = C_A(\boldsymbol{J})$ that $\varphi(x, \bar{b}) \leq \bigvee_{t \in I} \varphi(x; \bar{a}_t)$. By recalling the meaning of $\bigvee_{t \in I}$, this is equivalent to: no nonzero $\boldsymbol{\psi} \leq \mathrm{cn}^{\varphi}_{A}(\boldsymbol{I})$ is disjoint from all $\varphi(\bar{x}; \bar{a}_t)$ $(t \in I)$ (everything is in $B^c(r, B)$). As $W(r, B)$ is dense in $B^c(r, B)$, it suffices to assume $\boldsymbol{\psi} = \psi(\bar{x}; \bar{b})$.

So we assume $\boldsymbol{\psi} = \psi(\bar{x}; \bar{b}) \in B^c(r, \phi) - \{0\}$ is disjoint to every $\varphi(\bar{x}; \bar{a}_t)$, $t \in I$, $\boldsymbol{\psi} \leq \mathrm{cn}^{\varphi}_{A}(\boldsymbol{I})$ and we shall get a contradiction. The first assumption means $\{\psi(\bar{x}; \bar{b}), \varphi(\bar{x}; \bar{a}_t)\}$ weakly divides over (r, B) for every $t \in I$. As $\boldsymbol{\psi} \leq \mathrm{cn}^{\varphi}_{A}(\boldsymbol{I}) = \mathrm{cn}^{\varphi}_{A}(\boldsymbol{J})$, by Claim 7.6 for all but $<\lambda$ $t \in J$, $\boldsymbol{\psi} \wedge \varphi(\bar{x}; \bar{a}_t) \neq 0$ in $B^c(r, B)$ or equivalently, $\{\psi(\bar{x}, \bar{b}), \varphi(\bar{x}; \bar{a}_t)\}$ does not weakly divide over (r, B). Now by Claim 4.2(7) (and monotoniality of weak dividing, see Claim 2.8) for some $I_0 \subseteq J$, $|I_0| < \kappa r_{\mathrm{cdt}}(T)$, $\mathrm{tp}(\bar{b}, A \cup \boldsymbol{J})$ does not divide over $A \cup \boldsymbol{I}_0$ $(\boldsymbol{I}_0 = \{\bar{a}_t: t \in I_0\})$. Choose $t_0 \in I - I_0$, let $J_0 = \{t \in J$: for every $s \in I_0$, $s < t \equiv s < t_0$ and $t \notin I_0 \cup I\}$. As J is $(|\boldsymbol{I}| + \lambda)^+$-saturated, J_0 is $(|I| + \lambda)^+$-saturated too. Hence $|J_0| \geq \lambda$, and clearly $\boldsymbol{J}_0 = \langle \bar{a}_t: t \in J_0\rangle$ is indiscernible over $A \cup \boldsymbol{I}_0$. We can choose $t_1 \in J_0$ such that $\{\psi(\bar{x}, \bar{b}), \varphi(\bar{x}; \bar{a}_{t_1})\}$ does not weakly divide over (r, B) (see above-all but $<\lambda$ member of J_0 are suitable). So $A \cup \boldsymbol{I}_0, \boldsymbol{J}_0, \psi(\bar{x}; \bar{b})$ and $\bar{a}_{t_0}, \bar{a}_{t_1}$, contradict Claim 7.7, hence we finish.

Theorem 7.8. *Suppose $p = \mathrm{tp}(\bar{a}, A \cup \bar{b})$ does not fork over A, $\mathrm{Dom}\, r \cup B \subseteq A$, and $\varphi_1(\bar{x}, \bar{a}), \varphi_2(\bar{x}, \bar{b})$ are contradictory in $B^c(r, B)$. Then there is an element $\boldsymbol{\psi}$ in $B^c(r, B)$ such that $\boldsymbol{\psi}$ is almost over A and separates φ_1 and φ_2 (i.e. $\varphi_1 \leq \psi$ and $\varphi_2 \leq \neg\psi$).*

Proof. By Theorem 6.3, there is an $\boldsymbol{I}$ with $\bar{a} \in \boldsymbol{I}$ based on $(A, A \cup \bar{b})$ hence by Lemma 6.5, Claim 5.6 weakly based on A so $C_A(\boldsymbol{I})$ has few images under automorphisms fixing A. In particular $\bigvee_{\bar{c} \in \boldsymbol{I}} \varphi_1(\bar{x}, \bar{c}) = \bigvee_{\bar{c} \in C_A(I)} \varphi_1(\bar{x}, \bar{c})$ is almost over A. Since $\varphi_1(x, \bar{a}), \varphi_2(x, \bar{b})$ are contradictory and $\boldsymbol{I}$ is a set of indiscernibles over $A \cup \bar{b}$ containing $\bar{a}, \varphi_1(\bar{x}, \bar{c}), \varphi_2(\bar{x}, \bar{b})$ are contradictory for all $\bar{c} \in \boldsymbol{I}$. Thus $\boldsymbol{\psi} = \bigvee_{\bar{c} \in C_A(\boldsymbol{I})} \varphi_1(\bar{x}, \bar{c})$ and $\varphi_2(\bar{x}, \bar{b})$ are disjoint and $\boldsymbol{\psi}$ is as required, by Claim 7.2.

Theorem 7.9. *If* $\mathrm{tp}(\bar a_i, A \cup \bigcup_{j<i} \bar a_j)$ *does not divide over* $A \supseteq B \cup \mathrm{Dom}\, r$ *and* D *is a ultrafilter in* $B^c_A(r, B)$ *(the subalgebra of* $B^c(r, B)$ *containing those elements weakly supported by* A*) such that for each* $i < \alpha$ $D \cup \{\varphi(\bar x, \bar a_i)\}$ *is consistent, then* $D \cup \{\varphi(\bar x, \bar a_i) : i < \alpha\}$ *is consistent (i.e. generate a (proper) filter, in* $B^c(r, B)$*).*

Proof. Without loss of generality we may assume α is finite as the hypothesis holds if even we replace $\langle a_i : i < \alpha\rangle$ by a finite subsequence, by monoticity of dividing (Claim 1.6) and the conclusion for all α follows from the conclusion for all finite subsequences of $\langle a_i : i < \alpha\rangle$ by the finite character of generating a filter and the argument above.

Now we work by induction on α. Let $\bar a = \bar a_\alpha$, $\bar b = \bar a_0 {}^\frown \cdots {}^\frown \bar a_{\alpha-1}$ and $\psi(x, \bar b) = \bigwedge_{i<\alpha} \varphi(\bar x, \bar a_i)$. By the induction hypothesis $\psi(x, \bar b)$ is consistent with D and by hypothesis $\mathrm{tp}(\bar a, A \cup \bar b)$ does not divide over A and $\varphi(\bar x; \bar a)$ is consistent with. By Theorem 6.4, there is an indiscernible set $\boldsymbol{I}'$ based on (A, A) with $\bar b \in \boldsymbol{I}'$. Now by Lemma 1.4 (ii) there is an automorphism F fixing $A \cup \bar a \cup \bar b$ such that $\boldsymbol{I} = F(\boldsymbol{I}')$ is indiscernible over $A \cup \{\bar a\}$. If $\{\psi(x, \bar b), \varphi(x, \bar a)\} \cup D$ is inconsistent, then for every $\bar b' \in \boldsymbol{I}$ $\{\psi(\bar x, \bar b'), \varphi(\bar x, \bar a)\} \cup D$ is inconsistent. Thus for some $\theta \in D$ and every $\bar b' \in \boldsymbol{I}$ $B^c(B, r) \vDash (\varphi(\bar x; \bar a) \wedge \theta) \wedge \psi(\bar x, \bar b') = 0$ which implies $B^c(B, r) \vDash (\varphi(\bar x; \bar a) \wedge \theta) \bigvee_{b' \in \boldsymbol{I}} \psi(\bar x; \bar b') = 0$. But $\bigvee_{b' \in \boldsymbol{I}} \psi(\bar x; \bar b')$ is almost over A so either $\bigvee_{b' \in \boldsymbol{I}} \psi(\bar x; \bar b') \in D$ or $\bigvee_{b' \in \boldsymbol{I}} \psi(\bar x; \bar b) \notin D$. But $\bigvee_{b' \in \boldsymbol{I}} \psi(\bar x; \bar b') \notin D$ is impossible since each $\{\varphi(\bar x, \bar a')\} \cup D$ is consistent. So $(\varphi(\bar x; \bar a) \wedge \theta) \wedge \bigvee_{b' \in \boldsymbol{I}} \psi(\bar x; \bar b') = 0$ is impossible so $D \cup \{\varphi(\bar x, \bar a_i) : i \leq \alpha\}$ is consistent as required.

Added in proof

Until now we established a theory for a class of first order theories which included in the class of the theories with the independence property.

It is interesting to check what happens in the other side of the unstable theories. Therefore we added the following theorem which gives us some information on the theories without the independence property.

The following is a slight improvement of a theorem of Poizat, using a model theoretic proof, and answers a question of his.

(1) For an ultrafilter D over a set A we let

$$\mathrm{Av}_\Delta(D, B) = \{\varphi(x, \bar a) : \{b \in A : \vDash \varphi(b, \bar a)\} \in D, \bar a \in B, \varphi \in \Delta\}.$$

If $\Delta = L$ we omit it; if $\Delta = \{\varphi, \neg\varphi\}$ we write φ.

(2) *Hypothesis.* T does not have the independence property.

(3) *Main Lemma.* Suppose D_1, D_2 are ultrafilters on a model M. Suppose $a^l_i (i < \omega)$ are defined by induction on i such that a^l_i realizes $\mathrm{Av}(D_l, M \cup \{a^l_j : j < i\})$. Then

(i) if $\mathrm{tp}(\langle a^1_i : i < \omega\rangle, M) = \mathrm{tp}(\langle a^2_i : i < \omega\rangle, M)$, then for every B, $\mathrm{Av}(D_1, B) = \mathrm{Av}(D_2, B)$;

(ii) for every $\varphi(x, \bar y)$ there are finite $\eta_\varphi < \omega$, $\Delta_\varphi \subseteq L$ (not depending on D_l, M)

such that if $\mathrm{tp}_{\Delta_\varphi}(\langle a_i^1 : i < \eta_\varphi\rangle, M) = \mathrm{tp}_{\Delta_\varphi}(\langle a_i^2 : i < \eta_\varphi\rangle, M)$, then for every B, $\mathrm{Av}_\varphi(D_1, B) = \mathrm{Av}_\varphi(D_2, B)$.

Proof. (i) Suppose the assumptions hold, but not the conclusion. So there are B, $\bar{c}$ such that

$$\varphi(x, \bar{c}) \in \mathrm{Av}(D_1, B), \neg\varphi(x, \bar{c}) \in \mathrm{Av}(D_2, B).$$

It is known that $\langle a_i^1 : i < \omega\rangle$ is an indiscernible sequence over M. Now for any finite Δ and n, we can define by downward induction on $m \leq n$, an element $b_m \in M$ such that:

$$(*)(1) \quad \mathrm{tp}_\Delta(b_m, \{a_0^1, \ldots, a_{m-1}^1, b_{m+1}, \ldots, b_n\}) = \mathrm{tp}_\Delta(a_m^1, \{a_0^1, \ldots, a_{m-1}^1, b_{m+1}, \ldots, b_n\}),$$

$(*)(2)$ $\quad \varphi(b_m, \bar{c})$ holds iff m is even.

Why can we define b_m? First suppose m is even; then $p = \mathrm{Av}(D_1, M \cup \{a_0^1, \ldots, a_{m-1}^1\} \cup \bar{c})$ include $\mathrm{tp}_\Delta(a_m^1, \{a_0^1, \ldots, a_{m-1}^1, b_{m+1}, \ldots, b_n\})$ and $\{\varphi(x, \bar{c})\}$, but as both sets are finite and p is finitely satisfiable in M (by definition) there is b_m as required. If m is odd, in $(*)(1)$ we can replace a_l^1 by a_l^2 as by an assumption $\mathrm{tp}(\langle a_0^1, a_2^2, \ldots\rangle, M) = \mathrm{tp}(\langle a_0^2, a_2^2, \ldots\rangle, M)$, and we look for $b_m \in M$. Now use D_2. Clearly $\mathrm{tp}_\Delta(\langle a_l : l < n\rangle, \emptyset) = \mathrm{tp}_\Delta(\langle b_l : l < n\rangle, \emptyset)$. Hence $\{\varphi(a_m^1, \bar{y})^{\text{if } (m \text{ even})} : m < \omega\}$ is consistent contradictory to "T does not have the independence property".

(ii) Same proof essentially.

(4) *Conclusion.* If $M \subseteq B$, then

(j) for every finite Δ, $\{p \in S_\Delta^m(B) : P$ finitely satisfiable in $M\}$, has power $\leq \mathrm{Ded}_r(|M|)$.

(jj) $\{p \in S^m(B)$: p finitely satisfiable in $M\}$ has power $\leq \Pi_m |S^m(M)|$.

Proof. The change from 1-types to m types is trivial.

(j) Use the obvious fact that a 1-type p is in a set A iff for some ultrafilter D over A, $p \subseteq \mathrm{Av}(D, A)$. By (ii) of the Main Lemma (3) the averages are determined by types of finite sequences (of length n_φ) and by a theorem from Section 4 Ch. II from [4] the number of complete n_φ types over B is bounded by $\mathrm{Ded}_n(|M|)$.

(jj) A similar argument using the result of (3)(i) that types are determined by ω-types over M.

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