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On some configurations related to the Shelah Weak Hypothesis

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Abstract. We show that some cardinal arithmetic configurations related to the negation of the Shelah Weak Hypothesis and natural from the forcing point of view are impossible.

1. Introduction

The Shelah Weak Hypothesis (SWH), formulated in [Sh:400A], states that for every cardinal λ the number of singular cardinals $\kappa < \lambda$ with $pp(\kappa) \geq \lambda$ is at most countable. The negation of SWH is one of the weakest statements on cardinal arithmetic whose consistency is unknown. Clearly, SWH follows from GCH and even from the Shelah Strong Hypothesis, which says that for every singular κ, $pp(\kappa) = \kappa^+$. On the other hand, as we shall now see, and as is shown in [Sh-g, VIII, 3.4 - Localization Theorem], the existence of a set a of regular cardinals with $\min(a) > |a|$ such that $|pcf(a)| > |a|$ implies $\neg$SWH. Suppose that $|pcf(a)| > |a|$ for some such set a. Let $\langle \kappa_\alpha \mid \alpha < |a|^+ \rangle$ be an increasing enumeration of the first $|a|^+$ elements of $pcf(a)$. Set $\lambda = \bigcup\{\kappa_\alpha \mid \alpha < |a|^+\}$. Clearly, for every $\beta < |a|^+$ we have $pcf\{\kappa_\alpha \mid \beta \leq \alpha < |a|^+\} \backslash \lambda \neq \emptyset$. Then, using the Localization Theorem, we define by induction an increasing sequence $\langle \beta_i \mid i < |a|^+ \rangle$ of ordinals below $|a|^+$ and a sequence $\langle \rho_i \mid i < |a|^+ \rangle$ of singular cardinals below λ with $pp(\rho_i) > \lambda$. Let β_0 be the least such that $pcf\{\kappa_\alpha \mid \alpha < \beta_0\} \backslash \lambda \neq \emptyset$. Set $\rho_0 = \bigcup_{\alpha < \beta_0} \kappa_\alpha$. Assume that for each $j < i$, β_j and ρ_j are defined. We define now β_i and ρ_i. Set $\beta'_i = \bigcup_{j<i} \beta_j$. Using the Localization Theorem, find least $\beta_i > \beta'_i$ so that $pcf\{\kappa_\alpha \mid \beta'_i \leq \alpha < \beta_i\} \backslash \lambda \neq \emptyset$. Set $\rho_i = \bigcup\{\kappa_\alpha \mid \beta'_i \leq \alpha < \beta_i\}$.

The forcing construction of [Gi-Sh] and [Gi-Ma] show that it is consistent that the order type of the set of κ's with $pp(\kappa) > \lambda$ is any finite or countable ordinal.

The present paper grew from an attempt made by the first author to force $\neg$SWH using a forcing of type of [Gi]. One of the features of this forcing is that it does

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not add new bounded subsets to a cardinal κ while increasing 2^κ. Here we show (in ZFC) that some configurations which are very natural from the forcing point of view are just impossible.

The first theorem, under stronger assumptions, was proved by the first author. The second author was able to weaken the assumptions and find a more elegant proof. Most of the generalizations are due to the second author. The second theorem is due solely to the second author.

2. Main results

Theorem 1. *The following (a)–(d) cannot hold together.*

(a) $\kappa_1 < \kappa_*$, $cf\kappa_1 = \aleph_0$, $cf\kappa_* > 2^{\aleph_0}$.

(b) *for every large enough* $\mu < \kappa_1$ *of cofinality* $(2^{\aleph_0})^+$ *we have* $pp(\mu) = \mu^+$.

(c) $\kappa_* = \sup\{\mu \mid \mu < \kappa_*, cf\mu = \aleph_0$ and $pp(\mu) > \kappa_*^+\}$.

(d) *there are a strictly increasing sequence* $\langle \lambda_\alpha \mid \alpha < cf\kappa_*\rangle$ *of regular cardinals between* κ_1 *and* κ_* *unbounded in* κ_**, a filter* D *on* ω *containing all cofinite subsets of* ω *and a sequence of functions* $\langle f_{\lambda_\alpha} \mid \alpha < cf\kappa_*\rangle$ *such that*

- (α) $f_{\lambda_\alpha} : \omega \to Reg \cap \kappa_1 \backslash (2^{\aleph_0})^+$.
- (β) $\lim_D f_{\lambda_\alpha} = \kappa_1$.
- (γ) $\lambda_\alpha = tcf(\prod_{n<\omega} f_{\lambda_\alpha}(n)/D)$.
- (δ) $f_{\lambda_\alpha} <_D f_{\lambda_\beta}$ *for* $\alpha < \beta < cf\kappa_*$.
- (ϵ) *if* $\alpha < \beta < cf\kappa_*$ *and* $\lambda \in Reg \cap \lambda_\beta \backslash \lambda_\alpha^+$ *then there is a function* $f_\lambda : \omega \to Reg \cap \kappa_1 \backslash (2^{\aleph_0})^+$ *such that* $f_{\lambda_\alpha} <_D f_\lambda <_D f_{\lambda_\beta}$ *and* $\lambda = tcf(\prod_{n<\omega} f_\lambda(n)/D)$.

Discussion

1. The assumption (c) is a form of $\neg$SWH which claims that there are more than $2^{\aleph_0}$ singular cardinals of cofinality $\aleph_0$ with pp above their supremum.
2. The assumption (d) holds naturally in forcing constructions with D = the filter of cofinite subsets of ω, but it seems to be problematic in ZFC. In [Sh-g,II§1] the proof of a weak related statement is a major result.
3. See [Sh-g, VI] for a version of (c) which handles also μ's with uncountable cofinalities.

Proof. Suppose otherwise. Wlog we can assume that $cf\kappa_* = (2^{\aleph_0})^+$. Just take $(2^{\aleph_0})^+$ λ_α's such that between any two of them there is a μ of cofinality $\aleph_0$ with $pp(\mu) > \kappa_*^+$. Also, replacing $\langle \lambda_\alpha \mid \alpha < (2^{\aleph_0})^+\rangle$ by its restriction to an unbounded subset and by restricting the domains of the functions f_{λ_α} in (d) to some D–positive set, we can assume that the following holds:

$(*)$ for every $n < \omega$, $\langle f_{\lambda_\alpha}(n) \mid \alpha < (2^{\aleph_0})^+\rangle$ is strictly increasing and, if $f_*(n) = \bigcup_{\alpha<(2^{\aleph_0})^+} f_{\lambda_\alpha}(n)$ then $f_*(n) < f_{\lambda_0}(n+1)$.

$(*)$ follows from [Sh-g, II, 1.2, 1.2A(3)]; we present here the argument.

Claim 1.1. *Let* I *be a filter on* ω *containing all finite subsets of* ω, $\langle f_\alpha \mid \alpha < (2^{\aleph_0})^+\rangle$ *be an* $<_I$*-increasing sequence of members of* ${}^{\aleph_0}On$. *Then there are* $S \subseteq (2^{\aleph_0})^+$, $|S| = 2^{\aleph_0}$ *and* $A \subseteq \omega$, $A \notin I$ *so that for every* $\alpha, \beta \in S$ *and* $n \in A$

$$\alpha < \beta \to f_\alpha(n) < f_\beta(n) .$$

Proof. Let D be an ultrafilter disjoint from I. Clearly, $\langle f_\alpha \mid \alpha < (2^{\aleph_0})^+\rangle$ is $<_D$-increasing. By [Sh-g, II, 1.2 and 1.2 A(3)] there is an $f \in {}^{\aleph_0}On$ for which the following (i)–(iii) hold.

(i) for every $\alpha < (2^{\aleph_0})^+$, $f_\alpha <_D f$.
(ii) if $g \in {}^{\aleph_0}On$, $g <_D f$ then for some α, $g <_D f_\alpha$.
(iii) $cf(f(n)) > \aleph_0$ for every $n < \omega$.

Subclaim 1.1.1. $A = \{n < \omega \mid cf(f(n)) = (2^{\aleph_0})^+\} \in D$.

Proof. Suppose otherwise. Let $B = \omega \backslash A \in D$. For $n \in B$ let δ_n be $cf(f(n))$. Let $\langle \delta_{n,i} \mid i < \delta_n\rangle$ be a sequence cofinal in $f(n))$. Consider $\prod_{n\in B}\delta_n/D$. Let $\delta = cf(\prod \delta_n/D)$. Then, $\delta \neq (2^{\aleph_0})^+$, since either $\{n \in B \mid \delta_n > (2^{\aleph_0})^+\} \in D$ or $\{n \in B \mid \delta_n \leq 2^{\aleph_0}\} \in D$. In the first case, clearly, $\delta > (2^{\aleph_0})^+$ (if $\{g_i \mid i < (2^{\aleph_0})^+\} \subseteq \prod_{n\in B}\delta_n$ then $h \in \prod_{n\in B}\delta_n$ where $h(n) = \bigcup_{i<(2^{\aleph_0})^+} g_i(n)$). In the second case, note that $|\prod_{n\in B}\delta_n| \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$.

Let $\langle g_i \mid i < \delta\rangle$ be a sequence witnessing $cf(\prod_{n\in B}\delta_n/D) = \delta$. We move g_i's to $\prod_{n\in B} f(n)$. For every $i < \delta$ define $h_i \in \prod_{n\in B} f(n)$ by $h_i(n) = \delta_{n,g_i(n)}$. Clearly, $\langle h_i \mid i < \delta\rangle$ is a $<_D$-increasing sequence unbounded in $\prod_{n\in B} f(n)$. But also $\langle f_\alpha \rightarrow B \mid \alpha < (2^{\aleph_0})^+\rangle$ is such a sequence. This is impossible unless $\delta = (2^{\aleph_0})^+$. ☐ of subclaim.

Now, for $n \in A$ we pick $\langle \delta_{n,i} \mid i < (2^{\aleph_0})^+\rangle$ to be a sequence cofinal in $f(n)$. Define $h_i(n) = \delta_{n,i}$ for every $i < (2^{\aleph_0})^+$ and $n \in A$. Then for every $i < j < (2^{\aleph_0})^+$ and $n \in A$ we have $h_i(n) < h_j(n)$. Also $\langle h_i \mid i < (2^{\aleph_0})^+\rangle$ is unbounded in $\prod_{n\in A} f_i(n)/D$. Define now by an easy induction two increasing sequences $\langle i_\nu \mid \nu < (2^{\aleph_0})^+\rangle$ and $\langle \alpha_\nu \mid \nu < (2^{\aleph_0})^+\rangle$ so that $h_{i_\nu} <_D f_{\alpha_\nu} < h_{i_{\nu+1}}$ holds for every $\nu < (2^{\aleph_0})^+$. Find a stationary $S \subseteq (2^{\aleph_0})^+$ and $B \in D$ so that for every $\alpha \in S$ and $n \in B$, $h_{i_\nu}(n) < f_{\alpha_\nu}(n)$. Then, for every $\nu_1, \nu_2 \in S$, $n \in B$, $\nu_1 < \nu_2$ implies $h_{i_{\nu_1}}(n) < f_{\alpha_{\nu_1}}(n) < h_{i_{\nu_1+1}}(n) \leq h_{i_{\nu_2}}(n) < f_{\alpha_{\nu_2}}(n)$. So, $\langle f_{\alpha_\nu} \rightarrow B \mid \nu \in S\rangle$ is an increasing sequence on B. Clearly, $B \notin I$. So, we are done. ☐

Now for every $\alpha < (2^{\aleph_0})^+$ and $\lambda \in Reg \cap \lambda_{\alpha+1}\backslash\lambda_\alpha$ we use (ϵ) and find a function $f_\lambda : \omega \rightarrow Reg \cap \kappa_1 \backslash (2^{\aleph_0})^+$ such that $\lambda = tcf(\prod_{n<\omega} f_\lambda(n)/D)$ and for every $n < \omega$, $f_{\lambda_\alpha}(n) < f_\lambda(n) < f_{\lambda_{\alpha+1}}(n)$.

Clearly, $\langle f_*(n) \mid n < \omega\rangle$ is strictly increasing with limit κ_1 and $cf(f_*(n)) = (2^{\aleph_0})^+$ for every $n < \omega$. Using (b), we can assume removing finitely many n's, if necessary, that $pp(f_*(n)) = (f_*(n))^+$ for every $n < \omega$. Let D_* by an ultrafilter on ω extending D. Let $\mu_* = tcf\prod_{n<\omega}((f_*(n))^+/D_*)$. It is well defined since D_* is an ultrafilter. By (c), w.l. of g., for every $\alpha < (2^{\aleph_0})^+$ there is κ_α, $\lambda_\alpha < \kappa_\alpha < \lambda_{\alpha+1}$, $cf\kappa_\alpha = \aleph_0$ and $pp(\kappa_\alpha) \geq \kappa_*^{++}$. Hence, there are $\tau^2_{\alpha,n} \in Reg \cap \kappa_\alpha \backslash \lambda_\alpha^{++}$ $(n < \omega)$ and a filter D_α on ω continuing all cofinite subsets of ω such that $\lim_{D_\alpha}\tau^2_{\alpha,n} = \kappa_\alpha$ and $\kappa_*^{++} = tcf(\prod_{n<\omega}\tau^2_{\alpha,n}/D_\alpha)$. By [Sh-g, II, 1.3], we can then find $\tau^1_{\alpha,n} \in Reg \cap \tau^2_{\alpha,n}\backslash\lambda^+_\alpha$ such that $\kappa^+_* = tcf(\prod_{n<\omega}\tau^1_{\alpha,n}/D_\alpha)$ (note that we are doing this separately for each $\alpha < (2^{\aleph_0})^+$). Let $a^{m,\ell}_\alpha = \{f_{\tau^\ell_{\alpha,n}}(m) \mid n < \omega\}$ for every $m < \omega$ and $\ell \in \{1, 2\}$. Set $a^m_\alpha = a^{m,1}_\alpha \cup a^{m,2}_\alpha$, $a^m = \bigcup_{\alpha<(2^{\aleph_0})^+} a^m_\alpha$ and $a = \bigcup_{m<\omega} a^m$. All these sets consist of regular cardinals above $(2^{\aleph_0})^+$, the

a^m_α's are countable, the a^m's and a have cardinality of at most $(2^{\aleph_0})^+$. Also $a^m_\alpha \subseteq [f_{\lambda_\alpha}(m), f_{\lambda_{\alpha+1}}(m))$. Clearly, a^m $(m < \omega)$ is an unbounded subset of $f_*(m) \cap Reg$ of order type $(2^{\aleph_0})^+$, since $\langle f_{\lambda_\alpha}(m) \mid \alpha < (2^{\aleph_0})^+ \rangle$ is increasing with limit $f_*(m)$. Then, $(f_*(m))^+ \in pcf(a^m) \subseteq pcf(a)$, as $pp(f_*(m)) = (f_*(m))^+$, for every $m < \omega$. Again, by [Sh-g, I, 1.12], $pcf(\{(f_*(m))^+ \mid m < \omega\}) \subseteq pcf(a)$. But $\mu_* = tcf(\prod_{n<\omega}(f_*(n))^+/D_*)$, hence $\mu_* \in pcf a$. Let $\langle b_\sigma \mid \sigma \in pcf a \rangle$ be a generating sequence for a (see [Sh-g, I, §3] or [Sh:506]). Wlog, if $\mu_* \neq \kappa_*^{+\ell}$ for $\ell \in \{1, 2\}$, then $b_{\mu_*} \cap b_{\kappa_*^{+\ell}} = \emptyset$. Let $\ell^* \in \{1, 2\}$ be such that $\mu_* \neq \kappa_*^{+\ell^*}$.

Claim 1.2. *The set $A = \{m < \omega \mid$ for some $\alpha < (2^{\aleph_0})^+, \bigcup_{\beta \in [\alpha, (2^{\aleph_0})^+)} a^m_\beta \subseteq b_{\mu_*}\}$ is in D_*.*

Proof. Otherwise $\omega \backslash A \in D_*$ and for $m \in \omega \backslash A$, $f_*(m) = \sup(a^m \backslash b_{\mu_*})$. Hence $(f_*(m))^+ \in pcf(a \backslash b_{\mu_*})$. So, $pcf(\{(f_*(m))^+ \mid m \in \omega \backslash A\}) \subseteq pcf(a \backslash b_{\mu_*})$. But $\omega \backslash A \in D_*$ and $\mu_* = tcf(\prod_{m<\omega}(f_*(m))^+/D_*)$. Hence $\mu_* \in pcf(a \backslash b_{\mu_*})$. Contradicting the choice of b_{μ_*}. $\square$ of Claim 1.2.

For $m \in A$ let α_m be the minimal α such that $\bigcup_{\beta \in [\alpha, (2^{\aleph_0})^+)} a^m_\beta \subseteq b_{\mu_*}$. Set $\alpha_* = \bigcup_{m \in A} \alpha_m$. Clearly, $\alpha_* < (2^{\aleph_0})^+$. Let $a' = \bigcup\{a^m_\beta \mid m \in A, \beta \in [\alpha_*, (2^{\aleph_0})^+)\}$. Then $a' \subseteq b_{\mu_*}$ and hence $\kappa_*^{+\ell^*} \notin pcf a'$. However, $m \in A$ and $n < \omega$ imply that $f_{\tau^{\ell*}_{\alpha*,n}}(m) \in a^{m,\ell^*}_{\alpha_*} \subseteq a^m_{\alpha_*} \subseteq a'$. So, for each $n < \omega$ we have

$$\{f_{\tau^{\ell*}_{\alpha*,n}}(m) \mid m \in A\} \subseteq a' \ .$$

Hence $pcf\{f_{\tau^{\ell*}_{\alpha*,n}}(m) \mid m \in A\} \subseteq pcf a'$. But as $A \in D_*$, $\tau^{\ell^*}_{\alpha_*,n} = tcf(\prod_{m \in A} f_{\tau^{\ell*}_{\alpha*,n}}$ $(m)/D)$. So, for every $n < \omega$, $\tau^{\ell^*}_{\alpha_*,n} \in pcf a'$. Then by [Sh-g, I, 1.12], $pcf\{\tau^{\ell^*}_{\alpha^*,n} \mid n < \omega\} \subseteq pcf a'$. But $\kappa_*^{+\ell^*} = tcf(\prod_{n<\omega} \tau^{\ell^*}_{\alpha^*,n}/D_{\alpha^*})$. So, $\kappa_*^{+\ell^*} \in pcf a'$. Contradiction. $\square$

Remark 1.3

(1) We can replace in the statement (a) of Theorem 1 "$cf\kappa_* > 2^{\aleph_0}$" by "$cf\kappa_* > \aleph_0$" provided that (d) of the theorem is strengthened by adding the condition $(*)$ introduced in the beginning of the proof and $(2^{\aleph_0})^+$ is replaced by $\aleph_1$ in (b).

(2) It is possible to weaken "$pp(\mu) > \kappa_*^+$" in (c) of the theorem to "$pp(\mu) \geq \kappa_*$", replacing $(2^{\aleph_0})^+$ in (b) by $\aleph_1$. Just after $(*)$ is obtained using $cf\kappa_* \geq (2^{\aleph_0})^+$, we can replace $\kappa_*, \kappa_*^+, \kappa_*^{++}$ by the limit of the first $\aleph_1$ λ_α's, its successor and its double successor, provided that for every $\alpha < \omega_1$ there is κ_α, $\lambda_\alpha < \kappa_\alpha < \lambda_{\alpha+1}$ with $pp(\kappa_\alpha) \geq \lambda_*^{++}$, where $\lambda_* = \bigcup_{\alpha<\omega_1} \lambda_\alpha$. The condition "$pp(\mu) \geq \kappa_*$" can be easily used to construct such $\langle \lambda_\alpha \mid \alpha < \omega_1 \rangle$.

(3) It is possible to replace in (a) "$cf\kappa_* > 2^{\aleph_0}$" by "$\forall \alpha < cf\kappa_*$ $(|\alpha|^{\aleph_0} < \kappa_*)$". For this use $cf\kappa_*$ instead of $(2^{\aleph_0})^+$ in the proof.

The following is parallel to Solovay's result that SCH holds above a strongly compact cardinal.

Corollary 1.4. *Suppose that the following holds: κ is a cardinal such that*

(a) *for any given cardinal λ it is possible to force $2^\kappa \geq \lambda$ by a κ^{++}-c.c. forcing which does not add new bounded subsets to κ and adds λ ω-sequences $\langle f_\alpha \mid \alpha < \lambda\rangle$ to κ such that*
(i) *$\alpha < \beta \rightarrow f_\alpha < f_\beta$ (mod finite),* (ii) *for every $A \subseteq \lambda$ of cardinality $\aleph_1$ there is $B \subseteq A$ of the same cardinality and $n_0 < \omega$ such that for every $\alpha < \beta$ in B, $n \in \omega\backslash n_0$, $f_\alpha(n) < f_\beta(n)$, and* (iii) *$\delta \in (\kappa, \lambda]$ regular cardinal implies that $f_\delta(n)$ is regular cardinal for every $n < \omega$ and $\delta = tcf(\prod f_\delta(n)/finite)$.*

(b) *$pp(\mu) = \mu^+$ for every large enough $\mu < \kappa$ of cofinality $\aleph_1$.*

Then above κ the following version of SWH holds:
for every cardinal $\lambda > \kappa$ the set $\{\mu \mid \kappa < \mu < \lambda, cf\mu = \aleph_0, pp(\mu) > \lambda^+\}$ is at most countable.

Remark. The forcing notion of [Gi-Ma] and [Gi] satisfy (a).

Proof. Suppose otherwise. Let κ_* be the first cardinal such that the set $\{\mu \mid \kappa < \mu < \kappa_*, cf\mu = \aleph_0, pp(\mu) > \kappa_*^+\}$ is uncountable. Clearly, $cf\kappa_* = \aleph_1$. Now we force with the forcing of (a) and make $2^\kappa \geq \kappa_*$. The ω-sequences produced by such forcing will satisfy $(*)$ of the proof of Theorem 1 with D equal to the filter of cofinite sets. The chain condition of the forcing insures that the cardinal arithmetic does not change above κ. No new bounded subsets are added to κ, hence (b) of the statement of the corollary still holds. Now Theorem 1 (actually using 1.3(2)) provides a contradiction. □

Repeating the proof of Theorem 1 we can show the following generalization:

Theorem 1.5. *The following (a) – (d) cannot hold together.*

(a) $\kappa_1 < \kappa_*, cf\kappa_1 = \aleph_0, cf\kappa_* > 2^{\aleph_0}$.

(b) *there is ℓ, $1 \leq \ell < \omega$ such that for every $\mu < \kappa_1$ of cofinality $(2^{\aleph_0})^+$ we have $pp(\mu) \leq \mu^{+\ell}$.*

(c) $\kappa_* = \sup\{\mu \mid \mu < \kappa_*, cf\mu = \aleph_0$ and $pp(\mu) > \kappa_*^{+\ell}\}$.

(d) *As in Theorem* 1.

If we allow infinite gaps between μ and $pp(\mu)$ in (b) of 1.5, we the following theorem.

Theorem 1.6. *Assume that*

(a) $\kappa_1 < \kappa_*, cf\kappa_1 = \aleph_0, cf\kappa_* = \theta > \aleph_0, \alpha^* < \kappa_1, cf\alpha^* > \aleph_0$.

(b) *for every large enough $\mu < \kappa_1$ of cofinality θ we have $pp(\mu) < \mu^{+\alpha^*}$.*

(c) *for some β^*, $\kappa_* = \sup\{\mu \mid \mu < \kappa_*, cf\mu = \aleph_0$ and $pp(\mu) \geq \kappa_*^{+\beta^*}\}$.*

(d) *the condition (d) of Theorem* 1 *and $(*)$ of its proof.*

Then $\beta^* < \sigma^{+4}$ for some $\sigma < \alpha^*$.

Sketch of the proof. Suppose otherwise. We define $f_*(n)$'s as in Theorem 1. Now $cff_*(n) = \theta$ and so $pp(f_*(n)) < (f_*(n))^{+\alpha^*}$ for every $n < \omega$. Find $\sigma < \alpha^*$ such that for every $n < \omega$, $pp(f^*(n)) \leq (f_*(n))^{+\sigma}$. Here we use that $cf\alpha^* > \aleph_0$.

Instead of one μ_* in the proof of Theorem 1 (or finitely many cardinals in 1.6) we consider $pcf\{(f_*(n))^{+\sigma'} \mid n < \omega, \sigma' \leq \sigma\} \cap (\kappa_*, \kappa_*^{+\beta^*}]$. By the assumption we made, $\beta^* \geq \sigma^{+4}$. Then there should be $\kappa_*^{+\ell^*} \notin pcf\{(f_*(n))^{+\sigma'} \mid n < \omega, \sigma' \leq \sigma\}$ for some ℓ^*, $1 \leq \ell^* \leq \beta^*$. This follows by results of [Sh:g, IX], see also [Sh:g, Analytical Guide, 4.18 (b)]. The rest of the proof is as those of Theorem 1, only we use [Sh:g, I, 3.2(5)] to include $pcf\{(f_*(n))^{+\sigma'} \mid n < \omega, \sigma' \leq \sigma\}$ into a union of finitely many pcf-generators. □

Now we turn to another theorem which provides a different proof of Theorem 1 and some of its generalizations.

Theorem 2. *Suppose that*

(a) $\kappa_0 < \kappa_1 < \kappa_*$, $1 \leq n^* < \omega$, $n^* < \gamma^* < \theta$ *and* γ^* *is a successor ordinal*
(b) $\theta = cf\aleph_0 < \theta < \kappa_0$ *and for every* $\alpha < \theta$, $|\alpha|^{\aleph_0} < \theta$
(c) $cf\kappa_1 = \aleph_0$ *and* $pp(\kappa_1) \geq \kappa_*^{+\gamma^*}$
(d) *if* $\mu \in (\kappa_0, \kappa_1)$ *and* $cf\mu = \theta$ *then* $pp(\mu) \leq \mu^{+n^*}$.

Then *the following holds*

(1) *For every nonprincipal ultrafilter* D *on* ω *and a sequence* $\overline{\sigma}^* = \langle \sigma_\ell^* \mid \ell < \omega \rangle$ *with* $\kappa_1 = \lim_D \overline{\sigma}^*$ *and* σ_ℓ^* $(\ell < \omega)$ *a limit cardinal of cofinality* $\geq \theta$ *in the interval* (κ_0, κ_1) **there are** *a set* $w \subseteq \gamma^* + 1$ *consisting of at most* n^* *elements and a sequence* $\overline{\sigma}^{**} = \langle \sigma_\ell^{**} \mid \ell < \omega \rangle$, $\kappa_0 < \sigma_\ell^{**} < \sigma_\ell^*$ $(\ell < \omega)$ **such that**

   $(*)_1$ *if* $a \in [R_{D,\overline{\sigma}^*,\overline{\sigma}^{**}}]^{\aleph_0}$, $\beta \leq \gamma^*$ *and* $\kappa_*^{+\beta} \in pcf a$ *then* $\beta \in w$,
   where $R_{D,\overline{\sigma}^*,\overline{\sigma}^{**}} = \{tcf(\Pi\overline{\sigma}/D) | \overline{\sigma} = \langle \sigma_n \mid n < \omega \rangle$, $\sigma_n^{**} \leq \sigma_n = cf\sigma_n < \sigma_n^*$ $(n < \omega)\} \cap [\kappa_1, \kappa_*)$.

(2) *There are* $\alpha^* < \theta$ *and a sequence* $\langle R_\alpha \mid \alpha < \alpha^* \rangle$ *with* $\bigcup_{\alpha<\alpha^*} R_\alpha = Reg \cap \kappa_* \backslash \kappa_1$ **so that**

   $(*)_2$ *for every* $\alpha < \alpha^*$ *there is* $w \subseteq \gamma^* + 1$ *consisting of at most* n^* *elements such that*
   if $a \in [R_\alpha]^{\aleph_0}$, $\beta \leq \gamma^*$ *and* $\kappa_*^{+\beta} \in pcf a$ *then* $\beta \in w$.

(3) *Let* D *be a nonprincipal ultrafilter on* ω. *There is a partition* $\langle I_\rho \mid \rho < \rho^* \rangle$, $\rho^* < \theta$ *of* $Reg \cap \kappa_1 \backslash \kappa_0$ *into closed open intervals (i.e. of the form* $[x, y)$*) with* $\langle \min I_\rho \mid \rho < \rho^* \rangle$ *strictly increasing such that*

   $(*)_3$ *for every sequence* $\langle \rho_n \mid n < \omega \rangle$ *of ordinals below* ρ^* *with* $\lim_D \langle \min I_{\rho_n} \mid n < \omega \rangle = \kappa_1$

$$\{tcf(\prod_{n<\omega} \sigma_n/D) \mid \sigma_n \in I_{\rho_n} \text{ for } n < \omega\} \cap [\kappa_1, \kappa_*)$$

   is included in one of R_α*'s* $(\alpha < \alpha^*)$ *from a sequence* $\langle R_\alpha \mid \alpha < \alpha^* \rangle (\alpha^* < \theta)$ *satisfying* $(*)_2$.

Remark 2.1. Part (1) is close to [Sh:g, IX 1.x].

Proof of (2) and (3) from (1). As $\kappa_*^{+\gamma^*} \leq pp(\kappa_1)$ there are a countable unbounded $a \subseteq \kappa_1 \cap Reg \backslash \kappa_0$ and an ultrafilter D_0 on a containing all cobounded subsets of a with $\kappa_*^{+\gamma^*} = tcf(\Pi a/D_0)$. Let $a = \{\lambda_n \mid n < \omega\}$ and $D = \{A \subseteq \omega \mid \{\lambda_n \mid n \in A\} \in D_0\}$. Now, by [Sh:g, II], for every regular $\tau \in \kappa_*^{+\gamma^*} \backslash \kappa_1$ we can find $\overline{\sigma} = \langle \sigma_n \mid n < \omega \rangle$, $\sigma_n \in Reg \cap \kappa_1 \backslash \kappa_0$ $(n < \omega)$, $\lim_D \overline{\sigma} = \kappa_1$ such that $\tau = tcf(\Pi \overline{\sigma}/D)$.

Fix χ to be a large enough cardinal. Let $M \prec (H(\chi), \in)$ be such that $|M| < \theta$, ${}^{\omega}M \subseteq M$, $\{\kappa_0, \kappa_1, \theta, D, \kappa_*\} \in M$ and $M \cap \theta \in \theta$. There is such M since we assumed (b). Consider the following set $\Phi = \{\overline{\sigma}^* \mid \overline{\sigma}^* = \langle \sigma_n^* \mid n < \omega \rangle, \lim_D \overline{\sigma}^* = \kappa_1$ and for every $n < \omega, \sigma_n^* \in M \cap [\kappa_0^+, \kappa_1)$ is a limit cardinal of cofinality $\geq \theta\}$. Clearly, $\Phi \subseteq M$ since ${}^{\omega}M \subseteq M$. Now, by (1), applied with D defined above for each $\overline{\sigma}^* \in \Phi$ there is a $\overline{\sigma}^{**}$ for which $(*)_1$ holds. By elementarity, there is such $\overline{\sigma}^{**}$ in M. Denote it by $\overline{\sigma}^{**}[\overline{\sigma}^*]$. Define $\langle R_\alpha \mid \alpha < \alpha^* \rangle$ to be an enumeration of the set $\{R_{D,\overline{\sigma}^*,\overline{\sigma}^{**}[\overline{\sigma}^*]} \mid \overline{\sigma}^* \in \Phi\} \cup \{\{tcf(\prod_{n<\omega} \sigma_n/D)\} \mid \sigma_n \in M \cap \kappa_1 \cap Reg \backslash \kappa_0$ and $\lim_{n<\omega\ D} \sigma_n = \kappa_1\}$. Then $\alpha^* < \theta$ since $|M| < \theta$. Clearly here $(*)_1$ implies $(*)_2$. So, in order to complete the proof of (2) it remains to show that $Reg \cap \kappa_* \backslash \kappa_1 = \bigcup_{\alpha<\alpha^*} R_\alpha$. Let $\tau \in Reg \cap \kappa_* \backslash \kappa_1$. Then for some $\overline{\sigma} = \langle \sigma_n \mid n < \omega \rangle$, $\sigma_n \in Reg \cap \kappa_1 \backslash \kappa_0 (n < \omega)$, $\lim_D \overline{\sigma} = \kappa_1$, $\tau = tcf(\Pi \overline{\sigma}/D)$. Let $A = \{n < \omega \mid \sigma_n \in M\}$.

Case 1. $A \in D$.

Then, wlog we can assume that $A = \omega$ (if $\sigma_n \notin M$ replace it by κ_0^+). But then τ appears in the second part of the union defining $\langle R_\alpha \mid \alpha < \alpha^* \rangle$.

Case 2. $A \notin D$.

Clearly $\kappa_1 \geq \kappa_0^{+\theta}$, since otherwise $\kappa_1 \cap Reg \subseteq M$ and Case 2 cannot occur. So wlog we can assume that $A = \emptyset$. Let for $n < \omega$, $\sigma_n^* = \min(M \cap \kappa_1 \backslash \sigma_n)$. Such σ_n^* is well defined since $\kappa_1 \in M$, $cf \kappa_1 = \aleph_0$ and hence $\kappa_1 = \sup(\kappa_1 \cap M)$. Also, σ_n^* has to be a limit cardinal of cofinality $\geq \theta$ as $M \cap \theta \in \theta$. So $\overline{\sigma}^* = \langle \sigma_n^* \mid n < \omega \rangle \in \Phi$. Let $\overline{\sigma}^{**} = \overline{\sigma}^{**}[\overline{\sigma}^*]$. Now, for every $n < \omega$, $\kappa_0^+ \leq \sigma_n^{**} < \sigma_n^*$ and $\sigma_n^{**} \in M$. Hence, $\sigma_n^{**} < \sigma_n < \sigma_n^*$ for every $n < \omega$. Then $tcf(\Pi \overline{\sigma}/D) = \tau \in R_{D,\overline{\sigma}^*,\overline{\sigma}^{**}}$ by $(*)_1$ and we are done.

This completes the proof of (2) from (1).

Let us turn now to (3). Here we are given a nonprincipal ultrafilter D. Define M and $\langle R_\alpha \mid \alpha < \alpha^* \rangle$ as above using this D. For every $\nu \in M \cap \kappa_1 \backslash \kappa_0$ which is a limit cardinal of cofinality $\geq \theta$ denote $\sup(M \cap \nu)$ by $\nu(M)$. Let $\langle I_\rho \mid \rho < \rho^* \rangle$ be the increasing enumeration of the following disjoint intervals:
$\{Reg \cap [\nu(M), \nu] \mid \nu \in M \cap \kappa_1$ is a limit cardinal of cofinality $\geq \theta\} \cup \{\{\nu\} \mid \nu \in M, cf \nu = \nu\}$.

Clearly, $\rho^* < \theta$, since $|M| < \theta$. Let us check that $(*)_3$ holds. So let $\langle \rho_n \mid n < \omega \rangle$ be a sequence of ordinals below ρ^* with $\lim_D \langle \min I_{\rho_n} \mid n < \omega \rangle = \kappa_1$ and let $\sigma_n \in I_{\rho_n}$ for $n < \omega$. Consider $\tau = tcf(\prod_{n<\omega} \sigma_n/D)$. Let $A = \{n < \omega \mid \sigma_n \in M\}$. As above we can concentrate on the situation when $A = \emptyset$ (i.e. Case 2). Define $\overline{\sigma}^*$ and $\overline{\sigma}^{**}$ as in Case 2. Then for every $n < \omega$, $\sigma_n^{**} < \sigma_n^*$ and $\sigma_n^{**} \in M$. But $\sigma_n^* = \min(M \cap \kappa_1 \backslash \sigma_n)$ is a limit cardinal of cofinality $\geq \theta$ in M. Let $\tilde{\rho}_n$ denote the left side of the interval I_{ρ_n}. Then $\sigma_n^* = \tilde{\rho}_n$, since $\tilde{\rho}_n \in M$ is a limit cardinal of cofinality $\geq \theta$ and $\sigma_n \in I_{\rho_n} = (\sup(M \cap \tilde{\rho}_n), \tilde{\rho}_n) \cap Reg$. Also the last equality implies that $\sigma_n > \sigma_n^{**}$. Then $\tau = tcf(\prod_{n<\omega} \sigma_n/D) \in R_{D,\overline{\sigma}^*,\overline{\sigma}^{**}}$ and we are done.

Proof of (1). Suppose otherwise. Let D be a nonprincipal ultrafilter on ω and $\overline{\sigma}^* = \langle \sigma_n^* \mid n < \omega \rangle$ a sequence of limit cardinals of cofinality $\geq \theta$ in the interval (κ_0, κ_1) with $\kappa_1 = \lim_D \overline{\sigma}^*$ witnessing the failure of (1). We choose by induction on $\xi < \theta$ cardinals $\sigma_{\xi,n}, \tau_\xi^k, \sigma_{\xi,n}^k$ $(n, k < \omega)$ so that

(α) $\kappa_0^+ \leq \sigma_{\xi,n} < \sigma_n^*$.
(β) $\xi < \xi'$ implies $\sigma_{\xi,n}^k < \sigma_{\xi',n}$.
(γ) $\tau_\xi^k \in Reg \cap \kappa_* \backslash \kappa_1$.
(δ) $\kappa_*^{+\gamma} \cap pcf(\{\tau_\xi^k \mid k < \omega\}) \backslash \kappa_*$ has at least $n^* + 1$ members.
(ϵ) $\sigma_{\xi,n} < \sigma_{\xi,n}^k < \sigma_n^*$ and $\sigma_{\xi,n}^k$ is regular.
(ξ) $tcf(\prod_{n<\omega} \sigma_{\xi,n}^k/D) = \tau_\xi^k$.
(η) $\xi < \xi'$ implies that $\sigma_{\xi,n} < \sigma_{\xi',n}$.

In order to carry out the construction we choose first at stage ξ, a $\sigma_{\xi,n}$ satisfying (α), (β). This is possible, since σ_n^* is a limit cardinal $> \kappa_0$ of cofinality $\geq \theta$. Second, as $\langle \sigma_{\xi,n} \mid n < \omega \rangle$ cannot serve as $\overline{\sigma}^{**}$ in $(*)_1$ by our assumption, there are $\tau_\xi^k \in R_{D,\overline{\sigma}^*,\langle \sigma_{\xi,n} \mid n<\omega \rangle}$ for $k < \omega$ such that $pcf(\{\tau_\xi^k \mid k < \omega\}) \cap (\kappa_*, \kappa_*^{+\gamma^*}]$ has at least $n^* + 1$ members. So clauses (γ), (δ) hold. By the definition of $R_{D,\overline{\sigma}^*,\langle \sigma_{\xi,n} \mid n<\omega \rangle}$, we can find for each $k < \omega$, $\sigma_{\xi,n}^k \in Reg \cap \sigma_n^* \backslash \sigma_{\xi,n}$ such that $tcf(\prod_{n<\omega} \sigma_{\xi,n}^k, D) = \tau_\xi^k$. So clauses (ϵ) and (ξ) hold. The clause (η) is implied by the previous ones. So, we have finished the inductive construction.

Now, for every $n < \omega$, as $\langle \sigma_{\xi,n} \mid \xi < \theta \rangle$ is strictly increasing, its limit $\sigma_n = \bigcup_{\xi<\theta} \sigma_{\xi,n}$ is a singular cardinal of cofinality θ. Also, clearly, $\sigma_n \in [\kappa_0^+, \kappa_1)$. Hence, by the assumption (d) of the theorem, $pp(\sigma_n) \leq \sigma_n^{+n^*}$. For $\ell = 1, \ldots, n^*$ let $\lambda_\ell = tcf(\prod_{n<\omega} \sigma_n^{+\ell}/D)$. Set $w^* = \{\alpha \leq \gamma^* \mid \kappa_*^{+\alpha} = \lambda_\ell$ for some $\ell, 1 \leq \ell \leq n^*\}$. Then w^* is a set of $\leq n^*$ ordinals below $\gamma^* + 1$. Let $a_n = \{\sigma_{\xi,n}^k \mid k < \omega, \xi < \theta\}$ and $a = \bigcup_{n<\omega} a_n \cup \{\sigma_n^{+\ell} \mid n < \omega, 1 \leq \ell \leq n^*\}$. So, a is a set of $\leq \theta < \kappa_0 < \min a$ regular cardinals. By [Sh:g, VIII §2] or [Sh:506, §2] a has a generating sequence $\langle b_\tau \mid \tau \in pcf a \rangle$. For each $\xi < \theta$ we can find a successor ordinal $\gamma_\xi \leq \gamma^*$ so that $\kappa_*^{+\gamma_\xi} \in pcf(\{\tau_\xi^k \mid k < \omega\}) \backslash \{\lambda_\ell \mid 1 \leq \ell \leq n^*\}$. So, for some successor ordinal $\gamma^{**} \leq \gamma^*$ there is an unbounded in θ set Y consisting of ξ's such that $\xi < \theta$ and $\gamma_\xi = \gamma^{**}$. Clearly, $\lambda_\ell \in pcf a$ for $\ell = 1, \ldots, n^*$ and $\kappa_*^{+\gamma^{**}} \in pcf a$. Then, wlog, we can assume that $b_{\kappa_*^{+\gamma^{**}}}$ is disjoint from each b_{λ_ℓ} for $\ell = 1, \ldots, n^*$. Set $A = \{n < \omega \mid b_{\kappa_*^{+\gamma^{**}}} \cap \sigma_n$ is unbounded in $\sigma_n\}$.

Claim 2.2. $A \in D$.

Proof. If this does not hold, then there is $\xi(*) < \theta$ such that for every $n \in \omega \backslash A$ $b_{\kappa_*^{+\gamma^{**}}} \cap [\sigma_{\xi(*)}, \sigma_n) = \emptyset$. Wlog $\xi(*) \in Y$. Also, $n \in \omega \backslash A$ implies that $\{\sigma_{\xi(*),n}^k \mid k < \omega\} \cap b_{\kappa_*^{+\gamma^{**}}} = \emptyset$, since for every $k < \omega$, $\sigma_{\xi(*)} < \sigma_{\xi(*),n}^k < \sigma_n$.

Hence $\{\sigma^k_{\xi(*),n} \,|\, k < \omega, n \in \omega\backslash A\}$ is disjoint from $b_{\kappa_*^{\gamma^{**}}}$. Now, each $\tau^k_{\xi(*)} \in pcf(\{\sigma^{k'}_{\xi(*),n} \,|\, k' < \omega, n \in \omega\backslash A\})$. Here we use the assumption that $A \notin D$ and so $\omega\backslash A \in D$. $\kappa_*^{+\gamma^{**}} \in pcf(\{\tau^k_{\xi(*)} \,|\, k < \omega\})$. Hence $\kappa_*^{+\gamma^{**}} \in pcf(\{\sigma^k_{\xi(*),n} \,|\, k < \omega, n \in \omega\backslash A\}) \subseteq pcf(a\backslash b_{\kappa_*^{+\gamma^{**}}})$, which is impossible by the choice of generators. □ of the claim.

Let $n \in A$. Then $b_{\kappa_*^{+\gamma^{**}}} \cap \sigma_n$ is unbounded in σ_n. Hence $pcf(b_{\kappa_*^{+\gamma^{**}}} \cap \sigma_n)\backslash\sigma_n \neq \emptyset$. But $pp(\sigma_n) \leq \sigma_n^{+n^*}$, hence for some $\ell(n) \in \{1, \dots, n^*\}$ we have $\sigma_n^{+\ell(n)} \in pcf(b_{\kappa_*^{+\gamma^{**}}} \cap \sigma_n) \subseteq pcf(b_{\kappa_*^{+\gamma^{**}}})$. Then for some $\ell(*) \in \{1, \dots, n^*\}$ the set $A^* = \{n \in A \,|\, \ell(n) = \ell(*)\}$ belongs to D. So, $\lambda_{\ell(*)} \in pcf(\{\sigma_n^{+\ell(*)} \,|\, n \in A^*\}) \subseteq pcf(b_{\kappa_*^{+\gamma^{**}}})$. But $b_{\kappa_*^{+\gamma^{**}}} \cap b_{\lambda_{\ell(*)}} = \emptyset$. Contradiction. □

Using (3) of Theorem 2 we shall now give another proof of Theorem 1.

2.3. Second proof of Theorem 1

Wlog $cf\kappa_* = (2^{\aleph_0})^+$. Let $\theta = (2^{\aleph_0})^+$ and $\kappa_0 = \theta^+$. Assume also, wlog, that D is a nonprincipal ultrafilter on ω. For every $f : \omega \to Reg \cap \kappa_1\backslash\kappa_0$ we define $g_f : \omega \to \rho^* < \theta$ as follows:

$$g_f(n) = \rho \quad \text{iff} \quad f(n) \in I_\rho \ .$$

Then, $f_1 \geq_D f_2$ implies $g_{f_1} \geq_D g_{f_2}$ since the sequence $\langle \min I_\rho \mid \rho < \rho^* \rangle$ is strictly increasing. Consider $\langle f_{\lambda_\alpha} \,|\, \alpha < \theta \rangle$ of (d) of Theorem 1. This is a strictly increasing sequence modulo D. Now, the total number of g_f's is $(\rho^*)^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Hence there are $g^* : \omega \to \rho^*$ and $\alpha^* < \theta$ such that for every $\alpha, \theta > \alpha \geq \alpha^*$, every $f : \omega \to Reg \cap \kappa_1\backslash\kappa_0$ such that $f_{\lambda_\alpha} \leq_D f <_D f_{\lambda_{\alpha+1}}$

$$f(n) \in I_{g^*(n)}, \ \text{for almost each } n < \omega \text{ mod } D \ .$$

Apply $(*)_3$ to $\langle g^*(n) \,|\, n < \omega \rangle$ with $\gamma^* = 2$. Then for some $\ell^* \in \{1, 2\}$ the following holds:

if $a \in [\{tcf(\prod_{n<\omega} \sigma_n/D) \,|\, \sigma_n \in I_{g^*(n)} \text{ for } n < \omega\} \cap [\kappa_1, \kappa^*)]^{\aleph_0}$ then $\kappa_*^{+\ell^*} \notin pcf a$. Let $\alpha, \theta > \alpha \geq \alpha^*$. Pick κ_α, $\lambda_\alpha < \kappa_\alpha < \lambda_{\alpha+1}$, $cf\kappa_\alpha = \aleph_0$ and $pp(\kappa_\alpha) \geq \kappa_*^{++}$ (by (c) of Theorem 1 we can assume, wlog, that it exists). Then, by [Sh-g], there are $\tau_{\alpha,n} \in Reg \cap \kappa_\alpha\backslash\lambda_\alpha^{++}$ $(n < \omega)$ and a filter D_α on ω containing all cofinite sets such that $\kappa_*^{+\ell^*} = tcf(\prod_{n<\omega} \tau_{\alpha,n}/D_\alpha)$. Consider $\langle f_{\tau_{\alpha,n}}(m) \,|\, m < \omega \rangle$ for every $n < \omega$. It is a sequence of regular cardinals such that $\tau_{\alpha,n} = tcf(\prod_{m<\omega} f_{\tau_{\alpha,n}}(m)/D)$ and $f_{\lambda_\alpha} <_D f_{\tau_{\alpha,n}} <_D f_{\lambda_{\alpha+1}}$. Then $\{m < \omega \,|\, f_{\tau_{\alpha,n}}(m) \in I_{g^*(m)}\} \in D$. Hence $\tau_{\alpha,n} \in \{tcf(\prod_{m<\omega} \sigma_m/D) \,|\, \sigma_m \in I_{g^*(m)}, m < \omega\}$ for every $n < \omega$. Take $a = \{\tau_{\alpha,n} \,|\, n < \omega\}$. Then $k_*^{+\ell^*} \notin pcf a$, but $\kappa_*^{+\ell^*} = tcf(\prod_{n<\omega} \tau_{\alpha,n}/D_n)$. Contradiction. □

The following is parallel to 1.6.

Theorem 2.4. *Suppose that*

(a) $\kappa_0 < \kappa_1 < \kappa_*$.

(b) *$\theta_1, \theta_2 < \kappa_0$ are such that $cf\theta_1 > \aleph_0$, $\theta_2 = \theta_1^{+3}$ or θ_2 is regular $\geq \theta_1^{+3}$ and for every $\alpha < \theta_2$ $cf([\alpha]^{<\theta_1}, \supseteq) < \theta_2$.*

(c) *$cf\kappa_1 = \aleph_0$ and $pp(\kappa_1) \geq \kappa_*^{+\theta_2}$.*

(d) *θ_3 is a regular cardinal between θ_2 and κ_0.*

(e) *θ_4 is cardinal between θ_3 and κ_0 of cofinality $\geq \theta_3$.*

(f) *$\theta_5 \in [\theta_4, \kappa_0)$ is a cardinal such that $cf([\theta_5]^{\leq\aleph_0}, \subseteq) = \theta_5$.*

(g) *D is an $\aleph_1$-complete filter on $\theta_4 + 1$*

(Notice that we allow D to be principal. For example, generated by $\{\theta_4\}$).

(h) *if $\langle \mu_\alpha \mid \alpha \leq \theta_4 \rangle$ is a strictly increasing continuous sequence of singular cardinals between κ_0 and κ_1, then*

$$\{\alpha \leq \theta_4 \mid \alpha \text{ limit}, \ cf\mu_\alpha \geq \theta_4 \text{ and } pp(\mu_\alpha) < \mu_\alpha^{+\theta_1}\} \in D\ .$$

(Thus, if $\{\theta_4\} \in D$ then the condition means $pp(\mu) < \mu^{+\theta_1}$ for every limit cardinal $\mu \in (\kappa_0, \kappa_1)$ of cofinality θ_4.)

Then

(1) *For every sequence $\overline{\sigma}^* = \langle \sigma_n^* \mid n < \omega \rangle$ of limit cardinals of cofinality $\geq \theta_4$ between κ_0^+ and κ_1 there are $\beta < \theta_2$ and a sequence $\overline{\sigma}^{**} = \langle \sigma_n^{**} \mid n < \omega \rangle$, $\kappa_0^+ \leq \sigma_n^{**} < \sigma_n^*$ $(n < \omega)$ such that*

$(\tilde{*})_1$ if $a \in [R_{\overline{\sigma}^*, \overline{\sigma}^{**}}]^{\aleph_0}$ then $pcf(a) \cap [\kappa_*^{+\beta}, \kappa_*^{+\theta_2}) = \emptyset$, where $R_{\overline{\sigma}^*, \overline{\sigma}^{**}} = \{\tau \in (\kappa_0^+, \kappa_1) \mid$ there is a sequence $\langle \sigma_n \mid n < \omega \rangle$, with $\sigma_n \in Reg \cap [\sigma_n^{**}, \sigma_n^*)$ such that $\tau \in pcf\{\sigma_n \mid n < \omega\}\}$.

(2) *There are $\alpha^* \leq \theta_5$ and a sequence $\langle R_\alpha \mid \alpha < \alpha^* \rangle$ with $\bigcup_{\alpha < \alpha^*} R_\alpha = Reg \cap \kappa_* \backslash \kappa_1$ so that*

$(\tilde{*})_2$ for every $\alpha < \alpha^*$ there is $\beta < \theta_2$ such that for every $a \in [R_\alpha]^{\aleph_0}$ we have $pcf(a) \cap [\kappa_*^{+\beta}, \kappa_*^{+\theta_2}) = \emptyset$.

(3) *There are $\rho^* < \theta_5^+$ and a partition $\langle I_\rho \mid \rho < \rho^* \rangle$ of $Reg \cap \kappa_1 \backslash \kappa_0$ into closed open intervals (i.e. of the form $[x, y)$) with $\langle \min I_\rho \mid \rho < \rho^* \rangle$ strictly increasing such that*

$(\tilde{*})_3$ for every sequence of ordinals $\langle \rho_n \mid n < \omega \rangle$ below ρ^* there is $\beta < \theta_2$ such that for every $a \in [\{tcf(\prod_{n<\omega} \sigma_n / \widetilde{D}) \mid \sigma_n \in I_{\rho_n}$ for $n < \omega$, $\widetilde{D}$ is a nonprincipal ultrafilter on ω with $\lim_{n<\omega\ D}(\min I_{\rho_n}) = \kappa_1\}]^{\aleph_0}$

$$pcf(a) \cap [\kappa_*^{+\beta}, \kappa_*^{+\theta_2}) = \emptyset\ .$$

Proof of (2) and (3) from (1). Let χ be a large enough cardinal. Pick $M \prec (H(\chi), \epsilon)$ so that $|M| = \theta_5, \kappa_0, \kappa_1, \theta_5 \in M$, $M \cap \theta_5^+ \in \theta_5^+$ and $(\forall X \in [M]^{\aleph_0})(\exists Y \in M)(X \subseteq Y \wedge |Y| = \aleph_0)$.

This is possible since by (f) $cf([\theta_5]^{\leq\aleph_0}, \subseteq) = \theta_5$. Define the set Φ now to be $\{\overline{\sigma}^* \in M \mid \overline{\sigma}^* = \langle \sigma_n^* \mid n < \omega \rangle$ is a sequence of limit cardinals between κ_0 and κ_1 with $cf\sigma_n^* \geq \theta_4$ $(n < \omega)\}$.

For each $\overline{\sigma}^* \in \Phi$ we choose $\overline{\sigma}^{**} = \overline{\sigma}^{**}[\overline{\sigma}^*]$ in M satisfying $(\tilde{*})_1$. Define $\langle R_\alpha \mid \alpha < \alpha^* \rangle$ to be an enumeration of the set $\{R_{\overline{\sigma}^*, \overline{\sigma}^{**}[\overline{\sigma}^*]} \mid \overline{\sigma}^* \in \Phi\} \cup \{pcf(\{\sigma_n \mid n < \omega\}) \mid \langle \sigma_n \mid n < \omega \rangle \in \Phi$ and for every $n < \omega$ $cf\sigma_n = \sigma_n\}$.

Now we proceed as in Theorem 2.

Proof of (1). Assume toward contradiction that for some $\overline{\sigma}^*$ there is no $\overline{\sigma}^{**}$ satisfying (1). We choose by induction on $\xi < \theta_4$ cardinals $\sigma_{\xi,n}$, $\tau_\xi^{i,k}$, $\sigma_{\xi,n}^{i,k}$ $(k, n < \omega, i < \theta_2)$ such that

(α) $\kappa_0^+ \leq \sigma_{\xi,n} < \sigma_n^*$
(β) $\xi < \xi'$ implies that $\sigma_{\xi,n}^i < \sigma_{\xi',n}$
(γ) $\tau_\xi^{i,k} \in Reg \cap \kappa_* \backslash \kappa_1$
(δ) $pcf(\{\tau_\xi^{i,k} \mid k < \omega\}) \cap [\kappa_*^{+1}, \kappa_*^{+\theta_2}) = \emptyset$
(ϵ) $\tau_\xi^{i,k} \in pcf(\{\sigma_{\xi,n}^{i,k} \mid n < \omega\})$
(ξ) $\sigma_{\xi,n} < \sigma_{\xi,n}^{i,k} = cf\sigma_{\xi,n}^{i,k} < \sigma_n^*$
(η) $\langle \sigma_{\xi,n} \mid \xi < \theta_4 \rangle$ is an increasing continuous sequence of singular cardinals.

Such a construction is possible as seen in the proof of (1) of Theorem 2.

Let $\sigma_n = \sigma_{n,\theta_4} = \bigcup_{\xi<\theta_4} \sigma_{\xi,n}$ for each $n < \omega$. Applying the condition (h) of the statement of the theorem to $\langle \sigma_{\xi,n} \mid \xi \leq \theta_4 \rangle$ we find for every $n < \omega$ a set $Y_n \in D$ such that $\xi \in Y_n$ implies that $pp(\sigma_{\xi,n}) < \sigma_{\xi,n}^{+\theta_1}$. By $\aleph_1$-completeness of D, the set $Y = \bigcap_{n<\omega} Y_n \in D$. Choose some $\delta^* \in Y$. Let $pp(\sigma_{\delta^*,n}) = (\sigma_{\delta^*,n})^{+\beta_n}$ for some $\beta_n < \theta_1$ $(n < \omega)$.

Consider sets $a_n = \{\sigma_{\xi,n}^{i,k} \mid \xi < \delta^*, i < \theta_2, k < \omega\}$ and $a = (\bigcup_{n<\omega} a_n) \cup a^*$, where $a^* = \{(\sigma_{\delta^*,n})^{+\beta} \mid n < \omega, \beta \leq \beta_n$ is a successor ordinal $\}$. Then a is a set of regular cardinals of cardinality $\leq \theta_4 + \theta_2 < \kappa_0 < \min a$. Let $\langle b_\tau \mid \tau \in pcf a \rangle$ be a generating sequence. As each $\beta_n < \theta_1$ and $cf\theta_1 > \aleph_0$, $|a^*| < \theta_1$. By [Sh:g, IX] or [Sh:g, Analytical Guide, 4.18(b)] $c = pcf(a^*) \cap [\kappa_*, \kappa_*^{+\theta_2})$ is bounded in $\kappa_*^{+\theta_2}$, since $\theta_2 \geq \theta_1^{+3} \geq |a^*|^{+4}$. Also $pcf(c) = c$. For each $\xi < \delta^*$ for some $i(\xi)$ we have $pcf(\{\tau_\xi^{i(\xi),k} | k < \omega\}) \cap [\kappa_*, \kappa_*^{+\theta_2})$ is not bounded by $\sup c$. So, choose $\kappa_*^{+\rho(\xi)} \in pcf(\{\tau_\xi^{i(\xi),k} \mid k < \omega\}) \cap [\kappa_*, \kappa_*^{+\theta_2}) \backslash \sup c$. Clearly, $\rho(\xi) < \theta_2$ is a successor ordinal. As, $\theta_2 < \theta_3 = cf\theta_3$, and $\delta^* \in Y$ implies either $(cf\delta^* = \theta_3)$ or ($\delta^* = \theta_4$ and then also $cf\delta_1^* \geq \theta_3$), necessary, for some $\rho^* < \theta_2$ the set $Z = \{\xi < \delta^* \mid \rho(\xi) = \rho^*\}$ is unbounded in δ^*. Let $J_n = J_{a_n}^{bd}$. So J_n is an ideal on a_n and, clearly, for every $c_n \in J_n$ $(n < \omega)$ we have $\kappa_*^{+\rho^*} \in pcf(\bigcup_{n<\omega}(a_n \backslash c_n))$.

By *pcf* theory (see [Sh:g, VIII, 1.5] or [Sh:g, Analytical Guide]) there are finite sets $e_n \subseteq \cap\{pcf(a_n \backslash c_n) \mid c_n \in J_n\}$ $(n < \omega)$ such that $\kappa_*^{+\rho^*} \in pcf(\bigcup_{n<\omega} e_n)$. But $\cap\{pcf(a_n \backslash c_n) \mid c_n \in J_n\} \subseteq \{\sigma_{\delta^*,n}^{+\beta} \mid \beta < \beta_n$ is a successor ordinal$\}$ for every $n < \omega$. So $\bigcup_{n<\omega} e_n \subseteq \cup\{\sigma_{\delta^*,n}^{+\beta} \mid \beta < \beta_n$ is a successor ordinal and $n < \omega\} = a^*$. Hence, $\kappa_*^{+\rho^*} \in pcf(a^*)$. But then $\kappa_*^{+\rho^*} \in pcf(a^*) \cap [\kappa_*, \kappa_*^{+\theta_2}) = c$, which is impossible by the choice of ρ^*. Contradiction. $\square$

Let us conclude with a question which is most natural, taking into account the results above.

Question. Is the following situation possible:

(a) $\kappa_1 < \kappa_*$, $cf\kappa_1 = \aleph_0$, $cf\kappa_* = \aleph_1$.
(b) for every singular $\mu < \kappa_1$, $pp(\mu) = \mu^+$ (or if one likes only for μ's of countable cofinality).

(c) $\kappa_* = \sup\{\mu \mid \mu < \kappa_*, cf\,\mu = \aleph_0 \text{ and } pp(\mu) = \kappa_*^+\}$.
(d) the same as (d) of Theorem 1 or even add $(*)$ of the proof of Theorem 1.

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References

[Gi] Gitik, M.: Blowing-up power of a singular cardinal-wider gaps, To appear in Annals of Pure and Applied Logic
[Gi-Ma] Gitik, M., Magidor, M.: Extender based forcings, JSL **59**, 450–460 (1994)
[Gi-Sh] Gitik, M., Shelah, S.: On certain indestructibiltiy of strong cardinals and a question of Hainal, Arch. Math. Logic., **28** 35–42 (1989)
[Sh:400A] Shelah, S.: Cardinal arithmetic for skeptics, Bulletin of the AMS **26** 197–210 (1992)
[Sh:g] Shelah, S.: Cardinal arithmetic. Oxford Science Publ., Oxford Logic Guides 29. CVlarendon Press, Oxford (1994)
[Sh:506] Shelah, S.: The pcf-theorem revisited. In The Mathematics of Paul Erdös II (R.L. Graham and J. Neŝetr̂il, editors), Springer Verlag, 420–459 (1997)