Sh:123

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MONADIC THEORY OF ORDER AND TOPOLOGY IN ZFC*

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True first-order arithmetic is interpreted in the monadic theories of certain chains and topological spaces including the real line and the Cantor Discontinuum. It was known that existence of such interpretations is consistent with ZFC.

0. Introduction

The first-order theory of linear order is far from trivial. The monadic (second-order) theory of linear order is much stronger. Still surprising decidability results were proven in that direction. Rabin proved in [10] that the monadic theory of the rational chain is decidable. Hence the monadic theory of all countable linear orders is decidable. Büchi proved in [1] that the monadic theory of ordinals of cardinality at most $\aleph_1$ is decidable. The decision problems for the monadic theory of the real line R, the monadic theory of linear order, the monadic theory of $\aleph_2$ and the monadic theory of ordinals were long open. The last two theories are taken care of in [5] and will not be discussed here.

Let us recall the definition of the monadic theory of order. The *pure monadic* (second-order) *language* has two sorts of variables: for points and for sets of points. Its atomic formulas have the form $x_i \in X_j$. The rest of its formulas are built from the atomic ones by means of ordinary propositional connectives and quantifiers for variables of either sort. Augmenting the pure monadic language by the symbol $<$ for an order on points we get the *monadic language of order*; the new atomic formulas have the form $x_i < x_j$. For the sake of brevity linearly ordered chains are here called *chains*. The *monadic theory* of a chain C is the theory of C in the monadic language of order when the set variables range over all subsets of C.

One specific chain of special interest to us is the real line R. Recall that a set of reals is called meager if it is a union of $\leqslant\aleph_0$ nowhere dense sets. We call it *pseudo-meager* if it is a union of $<2^{\aleph_0}$ nowhere dense sets. The Continuum

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Hypothesis implies that every pseudo-meager set is meager and that R is not pseudo-meager. It is well known that neither "Every pseudo-meager subset of R is meager" nor "R is not pseudo-meager" can be proved or disproved in Zermelo–Fraenkel set theory with the axiom of choice (ZFC).

The first undecidability result about the monadic theory of R appeared in [11]. Assuming that R is not pseudo-meager, Shelah reduced true first-order arithmetic (i.e. the first-order theory of $\langle \omega, +, \cdot \rangle$) to the monadic theory of R. Litman noted that the monadic theory of R is easily interpretable in the monadic theory of linear order (Lemma 7.12 in [11]), thus both the monadic theory of the real line R and the monadic theory of linear order are undecidable if R is not pseudo-meager. Under a very weak set-theoretic assumption we reduced the full second-order logic to the monadic theory of linear order, see [7]. (Note that the monadic theory of linear order is easily interpretable in the second order logic.) In this paper we are interested only in those chains that embed neither ω_1 nor ω_1^* (the order dual to ω_1). We call them *short*. In what follows we use the term "chain" to mean "short chain".

In order to axiomatize the monadic theory of countable chains Gurevich introduced in [3] *p-modest* chains where p is a positive integer. The corresponding definition can be found in Section 7 below. For every p, p-modesty is expressible in the monadic theory of order. A chain is called *modest* if it is p-modest for every p. The modest chains are exactly the chains monadically equivalent to countable. Thus:

(i) The monadic theory of modest chains is decidable.

The statement (i) is proved directly in [6]. Moreover, generalizing Shelah's undecidability result the paper [6] proved:

(ii) Assume that every pseudo-meager subset of R is meager. There is a uniform in p algorithm that reduces true first-order arithmetic to the monadic theory of non-p-modest chains.

Note that (i) and (ii) together form a kind of dichotomy.

Now about topology. In [2] Grzegorczyk considered a topological space as a Boolean algebra of subsets with the closure operations (a closure algebra). The language of closure algebras looks poor; its expressive power is not trivial however, see [2–4]. Let us give a formal definition.

Augmenting the pure monadic language by a symbol for the closure operation we get the *monadic topological language*. The new atomic formulas have the form $X_i = \bar{X}_j$. The *monadic theory* of a topological space U (or the *monadic topology* of U) is the theory of U in the monadic topological language when the set variables range over all subsets of U. The monadic theory of the order topology of any chain C is easily interpretable in the monadic theory of C. The converse is not true generally (the monadic topology of a chain

$$n_0 + (\omega^* + \omega) + n_1 + (\omega^* + \omega) + \cdots$$

is decidable, the monadic theory can be undecidable). However the monadic

Sh:123

theory of the chain R is interpretable in the monadic topology of R. (Using connectivity express the relation "y is located between x and z" in topological terms. This relation allows the definition of order using parameters, and up to isomorphism it is the same order for any parameters. More details can be found in Section 4 of [9].)

Analyzing Shelah's reduction of true first-order arithmetic to the monadic theory of R, Gurevich noted the prominent role of topology. The same construction allows reduction of true first-order arithmetic to the monadic topology of the Cantor Discontinuum (under the same assumption that R is not pseudo-meager), which is a stronger result (and the negative answer to an old question in [2]).

Assuming Godel's Constructibility Axiom $V = L$ Gurevich reduced true third-order arithmetic to the monadic theory of any nonmodest chain and to the monadic topology of the Cantor Discontinuum or any other topological space in a certain class of topological spaces, see [4]. All these monadic theories are easily interpretable in true third-order arithmetic.

In spite of these undecidability results one could hope still to prove that "The monadic theory of R is decidable" is consistent with ZFC. In Shelah's original interpretation almost all set variables ranged over perfect sets. The assumption "R is not pseudo-meager" was used to build a diagonal set intersecting "bad" perfect sets and avoiding "good" ones. It seemed quite possible that the diagonal set may not exist. As for a possibility to eliminate the assumption "R is not pseudo-meager", a natural approach could be to define a special kind of perfect set such that R is not a union of $<2^{\aleph_0}$ special perfect sets and such that replacing "for every perfect set" by "for every special perfect set" does not change too much the meaning of formulas used in the interpretation. Shelah tried several possibilities (sets coding game strategies, and so on) till he came across the following question of Harvey M. Friedman: Is there a set $W \subset R$ of cardinality $2^{\aleph_0}$ such that for every perfect P there is a perfect $Q \subset P$ avoiding W? He answered the question positively, and the solution, described in Section 1 below, gave him an idea of an appropriate kind of special perfect set. He figured out how to interpret true first-order arithmetic in the monadic theory of R just in ZFC, published an abstract [12] and sent an amendment to [11] to the first author. In the hands of the first author it grew to this paper.

The main result of this paper about chains is proved in Section 7. It is the following theorem in ZFC completing the above mentioned dichotomy.

Theorem 0.1. *There is a uniform in p algorithm that reduces true first-order arithmetic to the monadic theory of non-p-modest chains.*

Corollary 0.2. *True first-order arithmetic is reducible to the monadic theory of linear order.*

The theory of R with quantification over countable subsets (i.e. the theory of R in the monadic language of order when the set variables range over countable

Sh:123

subsets of R) is decidable, see [20], [11] or [6]. Therefore the theory of R with quantification over sets of rational numbers is decidable. In contrast to this we prove in Section 4:

Theorem 0.3. *The set of sentences $F = \forall W F'(W)$ in the monadic language of order such that F holds in R, when the bound set variables of F' range over sets of rational numbers while W ranges over arbitrary sets of reals, is undecidable.*

By analogy with p-modest chains Gurevich introduced p-modest topological spaces in [3], here p is a positive integer. The corresponding definition can be found in Section 8 below. The Cantor Discontinuum is not even 1-modest. The main result of this paper about monadic topology is the following theorem proved in Section 8.

Theorem 0.4. *There is a uniform in p algorithm reducing true first-order arithmetic to the monadic theory of non-p-modest metrizable spaces.*

Most of this paper deals simultaneously with chains and topological spaces. If you are interested only in undecidability of the monadic theory of the real line R, note that the monadic topology of the Cantor Discontiuum is easily interpretable in the monadic theory of R. Read the paper having in mind the non-1-modest case and topological applications only.

Unfortunately, topology fails to identify modest chains whereas this indentification is of local nature and all-important in the monadic thoery of chains. In order to provide the right frame to handle the monadic theory of chains the paper [4] introduces vicinity spaces. Vicinity spaces are defined in such a way that topological spaces form a special case; this allows a unified treatment of chains and topological spaces. We borrow the notion of vicinity spaces, the technique of towers and certain proofs from [4]. For the reader's convenience we make this paper self-contained however. (The oly exception is Theorem 2.1.)

Let us mention that a later paper [8] strengthens results of this paper and explains them in a way.

We thank the referee for the shorter proof of Claim 1.3.

1. A problem of Friedman

Theorem 1.1. *There is a subset W of the real line R such that W is of the cardinality of continuum and for every perfect set $P \subseteq R$ there is a perfect set $Q \subseteq P$ avoiding W.*

Theorem 1 answers positively a question of Harvey M. Friedman. We prove it in the rest of this section.

It suffices to prove Theorem 1 for the Cantor Discontinuum because it is homeomorphic to a perfect subset of R. For us here the Cantor Discontinuum is the set ${}^{\omega}2$ (the collection of functions from ω into $\{0, 1\}$) with the product topology.

We don't distinguish between ω and $\aleph_0$ but we do distinguish between ${}^{\omega}2$ and 2^{ω}, the latter is the cardinality of ${}^{\omega}2$ i.e., the cardinality of continuum. Here is some more notation. If α, β are ordinals, then ${}^{\alpha}\beta$ is the collection of functions from α into β,

$$^{<\alpha}\beta = \bigcup \{^{\gamma}\beta : \gamma < \alpha\}, \qquad ^{\leq\alpha}\beta = \bigcup \{^{\gamma}\beta : \gamma \leq \alpha\}.$$

Of course an ordinal is considered here as the set of smaller ordinals. Now, ${}^{\leq\omega}2$ with the inclusion relation forms a tree. For each $a \in {}^{\leq\omega}2$ let $[a] = \{x \in {}^{\leq\omega}2 : a \subseteq x\}$. We consider ${}^{\leq\omega}2$ as a topological space whose open subbasis consists of sets $\{a\}$ and $[a]$ where $a \in {}^{<\omega}2$. Thus ${}^{\omega}2$ is the collection of limit points of ${}^{\leq\omega}2$ and it suffices to prove Theorem 1.1 for ${}^{\leq\omega}2$ instead of R. We work below in ${}^{\leq\omega}2$. An element $a \in {}^{<\omega}2$ is considered also as sequence $a0, \ldots, a(l-1)$ where l is the domain of a, l is called the length of a and denoted $\mathrm{lh}(a)$. For $i < 2$, $a\hat{\ }i$ is the sequence $a0, \ldots, a(l-1), i$.

Let S be a subset of ω such that both S and $\omega - S$ are infinite. A perfect set P is *constant* on S if $x \restriction S = y \restriction S$ for every x, y in P. P is *one-to-one* on S if $x \restriction S = y \restriction S \rightarrow x = y$ for every x, y in P.

Claim 1.2. *For every perfect set P there is a perfect $Q \subseteq P$ which is either constant or one-to-one on S.*

Proof. Without loss of generality there is no clopen (closed and open) set K such that $K \cap P$ is non-zero and constant on S. Define $e : {}^{<\omega}2 \rightarrow {}^{<\omega}2$ such that

(i) every $[ea]$ meets P, and

(ii) for every $a \in {}^{<\omega}2$ there is $l \geq \mathrm{lh}(a)$ such that $l \in S$ and $e(a\hat{\ }0)$, $e(a\hat{\ }1)$ differ at l.

The set of limit points of the range of e is the desired set Q.

Claim 1.3. *Let $\kappa < 2^{\aleph_0}$ and for every $\alpha < \kappa$ let P_α be a perfect set constant or one-to-one on S. Then $\bigcup \{P_\alpha : \alpha < \kappa\} \neq {}^{\omega}2$.*

Proof. If P_α is constant on S, then there is $f_\alpha : S \rightarrow 2$ such that $x \restriction S = f_\alpha$ for all $x \in P_\alpha$. Since $\kappa < 2^{\aleph_0}$, there is $h : S \rightarrow 2$ such that $h \neq f_\alpha$ for any α. The set $X = \{x \in {}^{\omega}2 : x \restriction S = h\}$ has power $2^{\aleph_0}$. If the union of sets P_α exhausts ${}^{\omega}2$, then some P_α contains two distinct elements of X, say x and y. Since $x \restriction S = y \restriction S$, P_α is constant on S. But $h \neq f_\alpha$. Claim 1.3 is proved.

Proof of 1.1. Let $\langle P_\alpha : \alpha < 2^{\aleph_0}\rangle$ be a sequence of all perfect sets constant or one-to-one on S. By Claim 1.3 it is possible to select x_α in ${}^{\omega}2 - \bigcup \{P_\beta : \beta < \alpha\}$ for $\alpha < 2^{\aleph_0}$. We prove that $W = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ is the desired set.

Sh:123

By Claim 1.2 arbitrary perfect set P includes some P_α. But $|P_\alpha \cap W| \leq |\alpha| < 2^{\aleph_0}$. Any perfect set can be partitioned into continuum many disjoint perfect subsets. Hence some perfect subset of P_α avoids W.

2. Vicinity spaces and guard spaces

Vicinity spaces were defined in [4] in order to prove simultaneously results about topological spaces and chains. For the reader's convenience we repeat here the definition in slightly different form.

A *vicinity space* is a non-empty set (of *points*) together with a function (the *vicinity function*) associating a collection of non-empty points sets (*vicinities* of x) with each point x in such a way that

(V1) x doesn't belong to any vicinity of x,

(V2) if the intersection of two vicinities of x is not empty then it includes another vicinity of x,

(V3) the relation "X meets Y" on the vicinities of x is transitive, and

(V4) if x belongs to a vicinity X of another point and Y is a vicinity of x then X includes a vicinity of x meeting Y.

For each vicinity X of a point x the union of all vicinities of x meeting X will be called a *direction* around x. By (V3) different directions around x are disjoint. (V4) can be reformulated as follows: if x belongs to a vicinity X of another point then X includes a vicinity of x in every direction around x.

Example 2.1. U is a T_1 topological space. Isolated points of U have no vicinities. If x is not isolated, then $\{G-\{x\}: G$ is an open nbd of $x\}$ is the collection of vicinities of x. Thus there is at most one direction around any point.

Example 2.2. U is a chain. If x is the left (resp. right) end of U or x has a left (resp. right) neighbor in U, then x has no left (resp. right) vicinities. Otherwise $\{G \cap \{y: y < x\}: G$ is an open nbd of $x\}$ (resp. $\{G \cap \{y: x < y\}: G$ is an open nbd of $x\}$) is the collection of left (resp. right) vicinities of x. Every vicinity of x is either left or right. Thus there are at most two directions around each point of U. (Example 2 corrects the corresponding place in [4].)

The *monadic vicinity language* is obtained from the pure monadic language by adding a symbol Vic of the vicinity function and new atomic formulas $X_i \in \mathrm{Vic}(x_j)$. Its formulas will be called *vicinity formulas*. The *monadic theory of a vicinity space* U is the theory of U in the monadic vicinity language when the set variables range over all point sets in U. In Example 1 (resp. Example 2) the monadic theory of the vicinity space is easily interpretable in the monadic theory of the topological space (resp. the chain).

We define topology (the *natural topology*) in vicinity spaces as follows: a point set X is open iff it includes a vicinity of each point $x \in X$ in every direction around

Sh:123

x. This definition restores the original topology (resp. the order topology) in Example 1 (resp. Example 2). By (V4) any vicinity of any point is open.

The number of directions around a point is the *degree* of this point. The *degree* of a vicinity space is the supremum of the degrees of its points. In the rest of this paper we restrict our attention to vicinity spaces satisfying the following conditions: the natural topology is regular and first-countable, the degree of the space is 1 or 2, and for every point x of the maximal degree any other point belongs to a vicinity of x.

Let U be a vicinity space of degree r. The *repletion* of $X \subseteq U$ is the set $\mathrm{rp}(X)$ of points x such that the degree of x is equal to r and every vicinity of x meets X. A set X is *replete* if $0 \neq X = \mathrm{rp}(X)$. X is *coherent* if $0 \neq X \subseteq \mathrm{rp}(X)$. If X meets every vicinity of every one of its points it forms a *subspace* of U in the following natural way: $\mathrm{Vic}_X(x) = \{X \cap Y: Y \in \mathrm{Vic}_U(x)\}$.

In topological applications

(i) the repletion of a set X is the set of nonisolated points in the closure of X,

(ii) replete means perfect,

(iii) coherent means dense in itself, and

(iv) X forms a vicinity subspace iff every point isolated in the topological subspace X is isolated in the whole space.

If you are interested in topological applications only, use this translation and think in toplogical terms. The vicinity approach does not give you any additional insight. The situation is different if you are interested in chains. Translating vicinity terms into the language of chains appears to be cumbersome and, what is more important, the order makes less natural some necessary definitions and constructions.

We say that a point set X is dense in a point set Y if $X \cap Y$ is dense in Y, i.e. if the closure of $X \cap Y$ includes Y. "Ewd" abbreviates "everywhere dense," and $\bar{X}$ is of course the closure of X.

Theorem 2.3. *Suppose $1 \leq \kappa \leq \omega$ and X_n is a coherent ewd point set for $n < \kappa$. Then there are a coherent countable $B \subseteq \bigcup \{X_n: n < \kappa\}$ and a family S of coherent subsets of B such that S is of the cardinality of continuum, and every $Y \in S$ is closed and nowhere dense in the subspace B, and every X_n is dense in every $Y \in S$, and $\bar{Y} \cap \bar{Z}$ is a scattered subset of B for any different Y, Z in S.*

Proof. See Theorem 1 in Section 1 of [4].

The monadic theory of a vicinity space of a positive degree can be decidable, see [6]. In order to prove undecidability of the monadic theory of U we want U to be rich. The following definition will be of help. Let p be a natural number. Point sets $X_0, \ldots, X_{p-1}$ form a *guard* if they are coherent, disjoint, ewd and there is no replete Y such that $Y \subseteq \bigcup \{X_q: q < p\}$ and every X_q is dense in Y. (When we say that a collection of sets is disjoint we mean it is pairwise disjoint.)

Sh:123

Suppose that $\{X_q : q < p\}$ is a guard of U. A set Y is *guarded* if every X_q is dense in Y.

Claim 2.4. *Every guarded replete set is of the cardinality of continuum.*

Proof. By contradiction suppose that Y is guarded and replete but $|Y| < 2^{\aleph_0}$. Without loss of generality $Y = U$. Let B, S be as in Theorem 2.3. $\{\bar{Z} - B : Z \in S\}$ is disjoint hence there is $Z \in S$ with $\bar{Z} = Z \subseteq B \subseteq \bigcup \{X_q : q < p\}$ which is impossible.

A vicinity space together with a guard consisting of p guardians will be called a *p-guard space.* The *monadic p-guard language* is obtained from the monadic vicinity language by adding set constant $Gd_0, \ldots, Gd_{p-1}$. Its formulas will be called *p-guard formulas.* The *monadic theory of a p-guard space* is the theory of it in the monadic p-guard language when the set variables range over all point sets.

3. Imposition and exploitation of a tree structure

How can one translate arithmetic into the monadic language of the real line? The line seems to be too homogeneous to help us. (Translating arithmetic into the monadic topology of the Cantor Discontinuum seems to be even more problematic.) The idea is to slice a countable everywhere dense set D into everywhere dense slices $A_0, A_1, \ldots$ and to code this decomposition of D by an appropriate parameter W. Later we will envision the decomposed D as a tower and the slices $A_0, A_1, \ldots$ as levels of the tower. These slices will represent natural numbers $0, 1, \ldots$ respectively. There is however no monadic formula with parameters that will define the slices up to, say, nowhere dense sets. (Nondefinability of slices will be proved in one of our forthcoming papers.) However there is a formula (the formula Storey in Section 4) defining slices locally: if Storey(X, D, W) holds, then every interval has a subinterval where X coincides with one of the slices. In this section we define a formula Code(X, D, W, G) and prove Theorem 3.1 stating that every nonempty open set inside G has a nonempty open subset where X is a part of a slice. The proof of Theorem 3.1 is the main novelty of this paper and its most sophisticated part. You may omit it at the first reading.

Let p be a positive natural number and U be a second-countable, zero-dimensional p-guard space with $r > 0$ directions around any point. Suppose every guardian of U is countable. Let $D^0 = \bigcup \{Gd_q : q < p\}$ and Code(X, D, W, G) be a p-guard formula saying the following: For every $G_1 \subseteq G$ and every $X_0 \subseteq Gd_0 \cap X, \ldots, X_{p-1} \subseteq Gd_{p-1} \cap X$, $X_p \subseteq X$ dense in G_1 there is a coherent $Y \subseteq D \cap G_1$ with $X_0, \ldots, X_p$ dense in Y and $|\mathrm{rp}(Y) \cap W| \leqslant 1$.

From here on letters G, H with or without subscripts denote non-empty open sets.

Sh:123

Theorem 3.1. *Let D be a countable subset of U such that $D^0 \subseteq D$ and $D - D^0$ is ewd. Let $P = \{P_n : n < \omega\}$ be a partition of D into ewd parts in such a way that D^0 is the union of guarded parts. Let F be a family of subsets of D such that each $A \in F$ is a union of members of P, each $A \in F$ meets D^0 and $\{A \cap D^0 : A \in F\}$ is disjoint. Then there is $W \subseteq U - D$ such that for every G and every guarded $X \subseteq D$ dense in G, Code(X, D, W, G) holds in U iff for every $G_0 \subseteq G$ there are $A \in F$ and $H \subseteq G_0$ with $H \cap X \subseteq A$.*

Theorem 3.1 is proved in the rest of this section. In Section 1 we spoke about a tree ${}^{<\omega}2$. Now we are interested in another tree, namely ${}^{<\omega}\omega$. An arbitrary element a of ${}^{<\omega}\omega$ is a function from some $l < \omega$ (i.e. from $\{k : k < l\}$) into ω, a can be considered as a sequence $a(0), \ldots, a(l-1)$ and with respect to this l is called the length of a and is denoted $\mathrm{lh}(a)$. For every $i < \omega$, $a\hat{\ }i$ is the sequence $a^0, \ldots, a(l-1), i$.

Let D, P, F be as in Theorem 3.1. We impose a tree structure on D. Let $\{Q_n : n < \omega\}$ be a partition of ω into infinite parts in such a way that

(i) if P_n is guarded, then Q_n avoids $\{i(p+1)+p : i < \omega\}$ and $Q_n \cap \{i(p+1)+q : i < \omega\}$ is infinite for $q < p$, and

(ii) if P_n is not guarded, then $Q_n \subseteq \{i(p+1)+p : i < \omega\}$.

Let $\{B_n : n < \omega\}$ be an open basis of U consisting of clopen sets. Order D by type ω.

To each $a \in {}^{<\omega}\omega$ we assign a nonempty clopen subset $[a]$ of U and a point ea in $[a] \cap D$ as follows. If a is empty, then $[a] = U$. Suppose $[a]$ is chosen and $l = \mathrm{lh}(a) \in Q_n$. Choose ea to be the minimal $d \in D$ such that $d \in [a] \cap P_n$ and if P_n is guarded, then d belongs to Gd_q with $q \equiv l$ modulo $p+1$. Let $G_0, \ldots, G_{r-1}$ be the directions around ea in the subspace $[a]$ of U. Select clopen sets $[a] = H_0 \supset H_1 \supset \cdots$ such that

(i) $\{H_i : i < \omega\}$ is a neighborhood basis for ea,

(ii) every $H_i - H_{i-1}$ meets every G_s, and

(iii) if $ea \in B_l$ but $[a] - B_l$ meets G_s, then $G_s \cap H_1 = G_s \cap B_l$, and if $ea \notin B_l$ but B_l meets G_s, then $G_s \cap H_1 = G_s - B_l$.

Set

$$[a\hat{\ }(ir+s)] = G_s \cap (H_i - H_{i+1}) \quad \text{for } i < \omega, s < r.$$

The range of the map e is equal to D, we identify each $a \in {}^{<\omega}\omega$ with ea. Thus $[a] \cap D = \{b \in D : a \subseteq b\}$, arbitrary nbd of a includes $\bigcup \{[a\hat{\ }i] : i \geq j\}$ for some j, and arbitrary vicinity of a includes $\bigcup \{[a\hat{\ }(ir+s)] : i \geq j\}$ for some j, s.

Clopen sets $[a]$ and $(\bigcup \{[a\hat{\ }i] : i \geq j\}) \cup \{a\}$ form an open basis for U. For, let $x \in B_l$. If $x = a \in D$, then B_l includes some $\bigcup \{[a\hat{\ }i] : i \geq j\}$. If $x \notin D$, then there is $b \in D$ such that $\mathrm{lh}(b) = l+1$ and $x \in [b] \subseteq B_l$. Thus for every x, y in $U - D$ there is a with $[a] \cap \{x, y\} = \{x\}$.

For $X \subseteq D$ let $\log(X) = \{\mathrm{lh}(a) : a \in X\}$. Then $\log(Gd_q) = \{i(p+1)+q : i < \omega\}$ for $q < p$ and $\log(P_n) = Q_n$.

We adopt the following terminology. A subset X of D *varies* on level l if there are a, b in X with $\mathrm{lh}(a)$, $\mathrm{lh}(b) > l$ and $a(l) \neq b(l)$. X is *of color* $A \in F$ if X varies only on levels $l \in \log(A)$. X is *mono* if there is $A \in F$ such that X is of color A. X is *nowhere mono* if there is no G with non-empty mono $G \cap X$. X is *colorless* if for every mono $\{a, b\} \subseteq X$ either $a \subseteq b$ or $b \subseteq a$.

Claim 3.2. *For every coherent, guarded, nowhere mono $X \subseteq D$ there is a coherent, guarded, colorless $Y \subseteq X$.*

Proof. It suffices to build a map $E: {}^{<\omega}\omega \to X$ such that:

(i) $E(a\hat{\ }m) \cap E(a\hat{\ }n) = E(a)$ if $m \neq n$,

(ii) if $E(a\hat{\ }m) \in [(Ea)\hat{\ }n]$, then $m \equiv n$ modulo r,

(iii) $Ea \in Gd_q$ if $\mathrm{lh}(a) \equiv q$ modulo p, and

(iv) if $Ea \in A \in F$ and $m \neq n$, then $\{E(a\hat{\ }m), E(a\hat{\ }n)\}$ varies on some level $l \in \omega - \log(A)$.

We prove that the range Y of such map has all desired properties. By (i), E preserves the tree order.

An arbitrary vicinity V of Ea includes $\bigcup\{[(Ea)\hat{\ }(ir+s)]: i \geq j\}$ for some $s < r$ and j. By (i) and (ii), V contains $E(a\hat{\ }(ir+s))$ for sufficiently big i. Hence Y is coherent.

An arbitrary nbd G of Ea includes $\bigcup\{[(Ea)\hat{\ }i]: i \geq j\}$ for some j. By (i), G contains $\{E(a\hat{\ }i): i \geq k\}$ for some k. Given $q < p$ take $b \in D$ such that $a\hat{\ }k \subseteq b$ and $\mathrm{lh}(b) \equiv q$ modulo p. By (i) and (iii), $Eb \in G$. Hence Y is guarded.

If $\{Ea_1, Ea_2\}$ is of color $A \in F$ let $a = a_1 \cap a_2$. By (i), $\{Ea_1, Ea_2\}$ varies on level $\mathrm{lh}(a)$ hence $Ea \in A$. By (i) and (iv), $\{Ea_1, Ea_2\}$ varies on some level $l \in \omega - \log(A)$ hence it cannot be of color A. Therefore Y is colorless.

We build now a map E satisfying conditions (i)–(iv). Choose $E0 \in Gd_0 \cap X$. Suppose $b = Ea$ is already chosen and $M_0 = \{m: [b\hat{\ }m]$ meets $X\}$. $M_0 \cap \{ir+s: i < \omega\}$ is infinite for $s < r$ because X is coherent. If $b \notin \bigcup F$ choose $E(a\hat{\ }(ir+s))$ in $[b\hat{\ }m_i] \cap X \cap Gd_q$ where m_i is the ith element of $M_0 \cap \{jr+s: j < \omega\}$ and $q \equiv \mathrm{lh}(a)+1$ modulo p.

Suppose $b \in A \in F$. We define $E(a\hat{\ }i)$, $L_i \subseteq \omega$ and $M_{i+1} \subseteq \omega$ by induction on i. Suppose that $E(a\hat{\ }j)$, L_j, M_{j+1} are defined for $j < i$, and $Ea \subset E(a\hat{\ }j)$, L_j is finite for $j < i$, and $M_i \cap \{kr+s: k < \omega\}$ is infinite for $s < r$, and $m \in M_i$ implies that $[b\hat{\ }m]$ avoids $\{E(a\hat{\ }j): j < i\}$ and there is $c \in [b\hat{\ }m] \cap X$ with $c \mid L_j$ defined and different from $E(a\hat{\ }j) \mid L_j$.

Let $n = \min\{m \in M_i: m \equiv i$ modulo $r\}$, $c \in [b\hat{\ }n] \cap X$ and $c \mid L_j$ is defined and different from $E(a\hat{\ }j) \mid L_j$ for $j < i$. As X is nowhere mono there are $c_0, \ldots, c_r$ in $[c] \cap X$ and $L_i \subset \omega - \log(A)$ such that $c_0 \mid L_i, \ldots, c_r \mid L_i$ are defined and different. Let

$N_s = \{m \in M_i$: there is $d \in [b\hat{\ }m] \cap X$ with $d \mid L_j$ defined and different from $E(a\hat{\ }j)$ for $j < i$ and $d \mid L_i$ defined and different from $c_s\}$

for $s \leq r$. For any $r' < r$ there is at most one $s \leq r$ such that $N_s \cap \{kr + r' : k < \omega\}$ is finite. Choose $s \leq r$ such that $N_s \cap \{kr + r' : k < \omega\}$ is infinite for $r' < r$. Let $q < p$ and $q \equiv \mathrm{lh}(a) + 1$ modulo p. Choose $E(a^\frown i)$ in $[c_s] \cap Gd_q \cap X$ (which is not empty because X is guarded) and set $M_{i+1} = N_s - \{n\}$. Claim 3.2 is proved.

Claim 3.3. *There is $W \subseteq U - D$ such that for each coherent guarded $X \subseteq D$:*
(i) *$|rp(X) \cap W| \leq 1$ if X is mono, and*
(ii) *$rp(X)$ meets W if X is colorless.*

Proof. Arrange all coherent, guarded, colorless subsets of D into a sequence $\langle X_\alpha : \alpha < 2^{\aleph_0} \rangle$. A point $x_\alpha \in \mathrm{rp}(X_\alpha) - D$ is selected by induction. Suppose $\{x_\beta : \beta < \alpha\}$ is already selected. For every $x \in U - D$ and every $n < \omega$ there is $a \in D$ such that $\mathrm{lh}(a) = n$ and $x \in [a]$, this a will be denoted $x \mid n$. Let $Y_\beta = \{y \in U - D : x_\beta \mid n$ and $y \mid r$ form a mono pair for every $n < \omega\}$ for $\beta < \alpha$. Then $|\mathrm{rp}(X_\alpha) \cap Y_\beta| \leq 1$. (For, suppose x and y are different elements in $\mathrm{rp}(X_\alpha) \cap Y_\beta$. There is l such that $x \mid l \neq y \mid l$. Take $a \in [x \mid l] \cap X_\alpha$ and $b \in [y \mid l] \cap X_\alpha$ to form a nontrivial mono pair in X_α.) By Claim 2.5, $|\mathrm{rp}(X_\alpha)| = 2^{\aleph_0}$. Pick x_α in $(\mathrm{rp}(X_\alpha) - D) - \bigcup \{Y_\beta : \beta < \alpha\}$. Let $W = \{x_\alpha : \alpha < 2^{\aleph_0}\}$. Every $\mathrm{rp}(X_\alpha)$ meets W by choice of x_α. If $\beta < \alpha$, then $\{x_\alpha \mid n, x_\beta \mid n\}$ is not mono for some n hence $|\mathrm{rp}(X) \cap W| \leq 1$ for any mono X. Claim 3.3 is proved.

Let W be as in Claim 3.3, G be an arbitrary non-empty open set, $X \subseteq D$ be guarded and dense in G. We prove the equivalence stated in Theorem 1. First suppose that for every $G_0 \subseteq G$ there are $A \in F$ and $H \subseteq G_0$ with $H \cap X \subseteq A$. Let $G_1, X_0, \ldots, X_p$ be as in the formula Code. We look for an appropriate Y. W.l.o.g. $G_1 \cap X$ is included into some $A \in F$.

Construct a sequence $\langle a_n : n < \omega \rangle$ of elements of D such that $\mathrm{lh}(a_n) \leq n$ and for every $a \in D$, $q \leq p$ there is $n \equiv q$ modulo $p+1$ with $a_n = a$. We build $f : \omega \to \omega$ and $E : D \to G_1 \cap X$ as follows. Choose $E0$ in such a way that $[E0] \subseteq G_1$, set $f0 = \mathrm{lh}(E0)$. Suppose fn and $E \restriction {}^n\omega$ are defined and $h(Ea) = fn$ if $\mathrm{lh}(a) = n$. Let $q \leq p$ and $q \equiv n$ modulo $p+1$. Choose $a \in {}^n\omega$ extending a_n and $b \in [Ea] \cap X_q - \{Ea\}$, set $f(n+1) = \mathrm{lh}(b)$. For every $c \in {}^n\omega$ and $m < \omega$ define $E(c^\frown m) = Ec \cup \{(fn, m)\} \cup \{(l, bl) : fn < l < f(n+1)\}$.

Let Y be the range of E. Y is coherent because for any Ea it meets all $[(Ea)^\frown m]$. Each X_q is dense in Y because for any Ea there are $n \equiv q$ modulo $p+1$ with $a_n = a$ and $b \in [Ea] \cap X_q \cap \mathrm{range}(E \restriction {}^{n+1}\omega)$. Y is of color A because it varies only on levels fn and every $fn \in \log(G_1 \cap X) \subseteq \log(A)$. By Claim 3.3, $|\mathrm{rp}(Y) \cap W| \leq 1$. Thus $\mathrm{Code}(X, D, W, G)$ holds in U.

Now suppose that there is $G_0 \subseteq G$ such that any $H \subseteq G_0$ meets $X - A$ for any $A \in F$.

Lemma 3.4. *There is $G_1 \subseteq G_0$ and there are $X_0 \subseteq Gd_0 \cap X, \ldots, X_{p-1} \subseteq Gd_{p-1} \cap X$, $X_p \subseteq X$ dense in G_1 such that each $A \in F$ avoids some $G_1 \cap X_q$.*

Sh:123

Proof. If there are $A \in F$, $G_1 \subseteq G_0$, $q < p$ with $A \cap Gd_q \cap X$ dense in G_1 set $X_q = A \cap Gd_q \cap X$, $X_i = Gd_i \cap X$ for $q \neq i < p$ and $X_p = X - A$. X_i is dense in G_1 for $q \neq i < p$ because X is guarded and dense in G_1. X_p is dense in G_1 because any $H \subseteq G_1$ meets $X - A$. A avoids $G_1 \cap X_p$, any $B \in F - \{A\}$ avoids X_q.

Suppose that $A \cap G_0 \cap Gd_q \cap X$ is nowhere dense for $A \in F$, $q < p$. Construct a sequence $\langle a_n : n < \omega \rangle$ of points in $G_0 \cap X$ such that:

(i) $\{a_n : n \equiv q \text{ modulo } p+1\}$ is dense in G_0 for $q \leqslant p$,
(ii) $n \equiv q$ modulo $p+1$ implies $a_n \in Gd_q$ for $q < p$, and
(iii) if $A \in F$ meets $\{a_m : m < n\}$ then a_n avoids A.

Set $G_1 = G_0$ and $X_q = \{a_n : n \equiv q \text{ modulo } p+1\}$ for $q \leqslant p$. Lemma 3.4 is proved.

We prove that Code(X, D, W, G) fails. Let $G_1, X_0, \ldots, X_p$ be as in Lemma 3.4 and Y be a coherent subset of $D \cap G_1$ with $X_0, \ldots, X_p$ dense in Y.

Y is nowhere mono. For, suppose $A \in F$ and H meets Y. By Lemma 3.4, A avoids some $G_1 \cap X_q$. Let $a \in H \cap X_q \cap Y$, $l = \text{lh}(a)$. Then $H \cap Y$ varies on level a (because Y is coherent) and $l \in \omega - \log(A)$ hence $A \cap Y$ is not of color A.

By Claim 3.2 there is a coherent, guarded, colorless $Z \subseteq Y$. There are two coherent, guarded, disjoint subsets of Z. By Claim 3.3, $|\text{rp}(Z) \cap W| \geqslant 2$.

4. Pre-towers and interpreting a finitely axiomatizable arithmetic

We work in the p-guard space of Section 3. Let Storey(X, D, W) be a p-guard formula saying: $X \subseteq D$, and X is ewd. guarded, and Code(X, D, W, U) holds, and there are no G, $Y \subseteq G - X$ such that Y is dense in G and Code($X \cup Y, D, W, G$) holds.

Theorem 4.1. *Let D, P, F be as in Theorem* 3.1. *There is $W \subseteq U - D$ such that an arbitrary $X \subseteq D$ satisfies* Storey(X, D, W) *iff for every G there are $A \in F$ and $H \subseteq G$ with $A \cap H = H \cap X$.*

The proof is straightforward.

Recall that letters G, H denote non-empty open sets. Any G forms a *subspace* of U whose guard is $\{G \cap Gd_q : q < p\}$. Let $\varphi(V_1, \ldots, V_n)$ be a p-guard formula whose only free variables are the set variables $V_1, \ldots, V_n$. Let $X_1, \ldots, X_n$ be subsets of U. We say that $\varphi(X_1, \ldots, X_n)$ *holds* in G if $\varphi(G \cap X_1, \ldots, G \cap X_n)$ holds in the subspace G. The union of those G where $\varphi(X_1, \ldots, X_n)$ holds will be called the *domain* of $\varphi(X_1, \ldots, X_n)$ and denoted do($\varphi(X_1, \ldots, X_n)$).

Definition 4.2. Let t be a sequence $(D, D^1, \ldots, D^l, W)$ of subsets of U where $1 \leqslant l < \omega$. A set $X \subseteq D$ is a *storey* of t if Storey(X, D, W) holds in U. t is a *pre-tower* if:

(i) $D^0 \subset D$, $D - D^0$ is ewd, $W \subseteq U - D$,
(ii) $D^1, \ldots, D^l$ are ewd subsets of $D - D^0$,

Sh:123

(iii) $\mathrm{do}(A = B$ or $A \cap B \cap D^0 = 0)$ is ewd for every t-storeys A, B, and

(iv) $\mathrm{do}(A = B$ or $A \cap D^1 \subset B$ or $B \cap D^1 \subset A)$ is ewd for every t-storeys A, B.

(Writing $X \subset Y$ we mean $X \subseteq Y$ and $X \neq Y$.

We need some more definitions. Let $t = (D, D^1, \ldots, D^l, W)$ be a pre-tower and A, B, C range over storeys of t. $A \leqslant B$ modulo t means that $\mathrm{do}(A \cap D^1 \subseteq B)$ is ewd. $A < B$ modulo t means that $A \leqslant B$ modulo t and $(B - A) \cap D^1$ is ewd. A collection S of t-storeys is a *skeleton* of t if $\mathrm{do}(A = B)$ is empty for every distinct A, B in S, and $\bigcup \{\mathrm{do}(A = B) \colon A \in S\}$ is ewd for any B.

Claim 4.3. *There are a pre-tower $t = (D, D^1, D^2, D^3, W)$ and a skeleton $\{A_n \colon n < \omega\}$ of t such that for every $i < j < k < \omega$*

(i) *$A_i \cap A_j \cap A_k \cap D^2$ is ewd if $i + j = k$ and is empty otherwise, and*

(ii) *$A_i \cap A_j \cap A_k \cap D^3$ is ewd if $i \cdot j = k$ and is empty otherwise.*

Proof. Let D be a countable subset of U such that $D^0 \subset D$ and $D - D^0$ is ewd. Partition $D - D^0$ into ewd parts D^1, D^2, D^3. Partition D^0 into ewd guarded parts D^0_n with $n < \omega$. Partition D^1 into ewd parts D^1_n. Partition D^2 into ewd parts D^2_{ijk} where $i < j < k < \omega$ and $i + j = k$. Partition D^3 into ewd parts D^3_{ijk} where $i < j < k < \omega$ and $i \cdot j = k$. Let A_n be the subset of D such that $A_n \cap D^0 = D^0_n$, and $A_n \cap D^1 = \bigcup \{D^1_m \colon m < n\}$, and $A_n \cap D^\varepsilon = \bigcup \{D^\varepsilon_{ijk} \colon n \in \{i, j, k\}\}$ for $\varepsilon = 2, 3$. Set $F = \{A_n \colon n < \omega\}$ and use Theorem 4.1.

Let L be the first-order language whose only non-logical constants are ternary predicate symbols Add and Mlt. Let $M = \langle \omega, \mathrm{Add}, \mathrm{Mlt} \rangle$ be the model for L with $\mathrm{Add} = \{(i, j, k) \colon i < j < k$ and $i + j = k\}$ and $\mathrm{Mlt} = \{(i, j, k) \colon i < j < k$ and $i \cdot j = k\}$

Claim 4.4. *There is an* L*-sentence φ_0 such that φ_0 holds in M and the theory in* L*, whose only non-logical axiom is φ_0, is essentially undecidable.*

The proof is clear.

Given a pre-tower $t = (D, D^1, D^2, D^3, W)$ we interpret variables of L as subsets of D and define t-domains of L-formulas as follows:

$$\mathrm{do}_t(\mathrm{Add}(X, Y, Z)) = \mathrm{do}(X \cap D^1 \subset Y \text{ and } Y \cap D^1 \subset Z \text{ and } X \cap Y \cap Z \cap D^2 \text{ is ewd}),$$

$$\mathrm{do}_t(\mathrm{Mlt}(X, Y, Z)) = \mathrm{do}(X \cap D^1 \subset Y \text{ and } Y \cap D^1 \subset Z \text{ and } X \cap Y \cap Z \cap D^3 \text{ is ewd}),$$

$$\mathrm{do}_t(\varphi \text{ or } \psi) = \mathrm{do}_t(\varphi) \cup \mathrm{do}_t(\psi),$$

$\mathrm{do}_t(\sim\varphi)$ is the complement of the closure of $\mathrm{do}_t(\varphi)$,

$$\mathrm{do}_t(\exists X \varphi(X, Y_1, \ldots, Y_n)) = \bigcup \{\mathrm{do}_t \varphi(X, Y_1, \ldots, Y_n) \colon X \text{ is a storey of } t\}.$$

Claim 4.5. *Suppose t is a pre-tower. The collection of* L*-sentences with ewd $\mathrm{do}_t(\varphi)$ is a theory in* L*. In other words this collection includes the logical axioms and is closed under the rules of inference.*

Sh:123

The proof is easy.

Claim 4.6. *Let $t, A_0, A_1, \ldots$ be as in Claim 4.3. Let $\varphi(x_1, \ldots, x_n)$ be an L-formula whose only free variables are $x_1, \ldots, x_n$. If $\varphi(i_1, \ldots, i_n)$ holds in M then* $\mathrm{do}_t(\varphi(A_{i_1}, \ldots, A_{i_n}))$ *is ewd. If $\varphi(i_1, \ldots, i_n)$ fails in M then* $\mathrm{do}_t(\varphi(A_1, \ldots, A_n)$ *is empty.*

Proof. An easy induction on φ simultaneously for U and any (open) subspace of U.

Theorem 4.7. *The monadic theory of U is undecidable.*

Proof. By Claim 4.4 there is an L-sentence φ_0 such that φ_0 holds in M and the theory T_0 in the language L, whose only non-logical axiom in φ_0, is undecidable. Let T be the collection of L-sentences φ such that for every pre-tower t, if $\mathrm{do}_t(\varphi_0)$ is ewd, then $\mathrm{do}_t(\varphi)$ is ewd. By Claim 4.5, T is a theory and $\mathrm{T}_0 \subseteq \mathrm{T}$. By Claim 4.6, M is a model of T, hence T is consistent. Any consistent extension of an essentially undecidable theory is undecidable, hence T is undecidable. But T is obviously interpretable in the monadic theory of U.

It is easy to see that we have proved actually:

Theorem 4.8. *Let D be a countable subset of U such that $D^0 \subset D$ and $D - D^0$ is ewd. Then the set of p-guard formulas $F(D, W)$, such that $\forall W F(D, W)$ holds in U, when the bound set variables of F range over subsets of D while W ranges over arbitrary subsets of U, is undecidable.*

Corollary 4.9. *The set of sentences $F = \forall W F'(W)$ in the monadic language of order such that F holds in the real line R, when the bound set variables of F' range over sets of rational numbers while W ranges over arbitrary sets of reals, is undecidable.*

Proof. Consider R as a vicinity space with respect to Example 2.2. Let D be the set of rational numbers and D^0 be an ewd subset of D such that $D - D^0$ is ewd. D^0 forms a 1-guard of R. Now use Theorem 4.8.

5. Towers and interpreting true arithmetic

We still work in the p-guard space U of Section 3. A pre-tower $t = (D, D^1, \ldots, D^l, W)$ will be called a *tower* if there are no G and $W' \subseteq G - D$ such that $t' = (D \cap G, D^1 \cap G, W')$ is a pre-tower in G, and t' has at least one storey, and for every t'-storey A' there is a t-storey A with $A' = A \cap G$, and for every t'-storey A' there is a t'-storey B' with $B' < A'$ modulo t'. As in [4] we can prove that every tower has a well ordered skeleton. However, a simpler result suffices for us in this paper.

Definition 5.1. Let $t = (D, D^1, \ldots, D^l, W)$ be a tower, and A, B, C be storeys of t. $A = 0$ modulo t means $A \leqslant B$ modulo t for any B. $A + 1 = B$ modulo t means that $A < B$ modulo t and

$$\mathrm{do}(C \leqslant A \text{ modulo } t) \cup \mathrm{do}(B \leqslant C \text{ modulo } t)$$

is ewd for any C. t is *arithmetical* if there is $A = 0$ modulo t, and for every A there is $B = A + 1$ modulo t, and for every B there is A such that

$$\mathrm{do}(A + 1 = B \text{ modulo } t) \cup \mathrm{do}(B = 0 \text{ modulo } t) \quad \text{is ewd.}$$

Theorem 5.2. *Suppose $t = (D, D^1, \ldots, D^l, W)$ is an arithmetical tower. Then there is a skeleton $\{A_n : n < \omega\}$ of t such that $A_0 = 0$ modulo t and $A_n + 1 = A_{n+1}$ modulo t for every n.*

Proof. Build a sequence $A_0, A_1, \ldots$ of t-storeys such that $A_0 = 0$ modulo t and $A_n + 1 = A_{n+1}$ modulo t for every n. By contradiction suppose that this is not a skeleton. Then there are a t-storey B_0 and some G such that $\mathrm{do}(A_n = B_0)$ avoids G for any n. Build a sequence $B_0, B_1, B_2, \ldots$ of t-storeys such that $\mathrm{do}(B_{n+1} + 1 = B_n)$ is dense in G for every n. By Theorem 4.1 there is $W' \subseteq G - D$ such that arbitrary $X \subseteq D \cap G$ satisfies $\mathrm{Storey}(X, D \cap G, W')$ in G iff for every $H \subseteq G$ there are n and $H' \subseteq H$ with $B_n \cap H' = H' \cap X$. Then $t' = (D \cap G, D^1 \cap G, W')$ is a pre-tower in G contradicting the fact that t is a tower.

Let L_1 be the first-order language with equality whose non-logical constants are a unary predicate symbol Ze, a binary predicate symbol Suc and ternary predicate symbols Add, Mlt. Thus L_1 extends the language L of Section 4. Let M_1 be the model for L_1 such that the reduction of M_1 into L is the model M of Section 4, and $\mathrm{Ze}(i)$ holds in M_1 iff $i = 0$, and $\mathrm{Suc}(i, j)$ holds in M_1 iff $j = i + 1$.

Claim 5.3. *There is an L_1-sentence φ_1 such that φ_1 holds in M_1 and for every model N of φ_1:*

(i) *there is exactly one element in N satisfying* Ze,

(ii) *for every $x \in N$ there is exactly one $y \in N$ such that* $\mathrm{Suc}(x, y)$ *holds in N, and*

(iii) *the minimal submodel of N, containing the element satisfying* Ze *and closed under* Suc, *is isomorphic to M_1.*

The proof is easy.

We have defined in Section 4 t-domans of L-formulas where t is a pre-tower.

The following clauses extend the definition of t-domain for L_1-formulas.

$$\mathrm{do}_t(X=Y)=\mathrm{do}(X=Y),$$

$$\mathrm{do}_t(\mathrm{Ze}(X))=\mathrm{do}(x=0 \text{ modulo } t),$$

$$\mathrm{do}_t(\mathrm{Suc}(X,Y))=\mathrm{do}(X+1=Y \text{ modulo } t).$$

Claim 5.4. *Suppose that t is an arithmetical tower (D, D^1, D^2, D^3, W), φ_1 is as in Claim 5.3 and $\mathrm{do}_t(\varphi_1)$ is ewd. Then there is a skeleton $\{A_n : n<\omega\}$ of t such that $A_0=0$ modulo t and $A_n+1=A_{n+1}$ modulo t and for every $i<j<k<\omega$:*
(i) *$A_i \cap A_j \cap A_k \cap D^2$ is ewd of $i+j=k$ and is empty otherwise, and*
(ii) *$A_i \cap A_j \cap A_k \cap D^3$ is ewd if $i \cdot j=k$ and is empty otherwise.*

Proof. Use Theorem 5.2 and Claim 5.3.

Theorem 5.5. *True arithmetic is interpretable in the monadic theory of U.*

Proof. True arithmetic is the first-order theory of the standard model of Peano Arithmetic. True arithmetic is easily interpretable in the theory of model M. Thus is suffices to check that an arbitrary L-sentence φ holds in M iff $\mathrm{do}_t(\varphi)$ is ewd for every arithmetical tower. Now use Claim 4.6 and Claim 5.4.

6. Non-modest vicinity spaces

Let p be a positive integer. We recall the definition of p-modest vicinity spaces. A vicinity space U is *perfunctorily p-modest* if for every coherent everywhere dense subsets $X_0, \ldots, X_{p-1}$ of U there is a replete subset Y of U such that $Y \subseteq X_0 \cup \cdots \cup X_{p-1}$ and every X_q is dense in Y. U is *p-modest* if every coherent subspace of U is perfunctorily p-modest. Let r be a positive integer. All vicinity spaces in this section are of degree r.

Definition 6.1. Let U be a vicinity space. Subsets $X, Y_0, \ldots, Y_{p-1}$ of U *witness* that U is not p-modest if X is coherent, and $Y_0, \ldots, Y_{p-1}$ are coherent subsets of X dense in X, and there is no $Z \subseteq Y_0 \cup \cdots \cup Y_{p-1}$ such that Z is replete in the subspace X and $Y_0, \ldots, Y_{p-1}$ are dense in Z.

Lemma 6.2. *Suppose U is a second-countable vicinity space and $X, Y_0, \ldots, Y_{p-1}$ witness that U is not p-modest. Then there are countable disjoint sets $Z_0, \ldots, Z_{p-1}$ such that $Z_q \subseteq Y_q$ for $q<p$ and $X, Z_0, \ldots, Z_{p-1}$ witness that U is not p-modest*

Proof. *Let $\{G_i : i<\omega\}$ be an open basis of X. Build a sequence $x_0, x_1, \ldots$ of different points such that if $k=ip+q$ and $q<p$ then $x_k \in G_i \cap Y_q$. Set $Z_q=$* $\{x_k : k \equiv q \text{ modulo } p\}$.

Definition 6.3. Let T be a theory in a language L and K' be a collection of models for a language L$'$. T is *uniformly interpretable* in the theory of K' if there is an algorithm assigning an L$'$-sentence φ' with each L-sentence φ in such a way that for every $M' \in K'$, φ is a theorem of T iff φ' holds in M'.

Theorem 5.5 can be reformulated as follows. Let Ar be true first-order arithmetic.

Claim 6.4. Ar *is uniformly interpretable in the monadic theory of coherent, second-countable, zero-dimensional p-guard spaces with countable guardians.*

Our aim in this section is:

Theorem 6.5. Ar *is uniformly interpretable in the monadic theory of second-countable, zero-dimensional, non-p-modest vicinity spaces.*

Proof. Given a sentence φ in the language of Ar compute a p-guard sentence φ' interpreting φ with respect to Claim 6.4. φ' is $\varphi''(Gd_0, \ldots, Gd_{p-1})$ for some vicinity formula φ''. Let φ^* be a vicinity sentence saying that there are $X, Y_0, \ldots, Y_{p-1}$ witnessing non-p-modesty and such that $\varphi''(Z_0, \ldots, Z_{p-1})$ holds in the subspace X for every coherent $Z_0 \subseteq Y_0, \ldots, Z_{p-1} \subseteq Y_{p-1}$ dense in X. We check that φ^* is an appropriate interpretation of φ.

Let U be a second-countable zero-dimensional vicinity space which is not p-modest. First suppose that φ is a theorem of Ar. Let $X, Y_0, \ldots, Y_{p-1}$ witness that U is not p-modest. In virtue of Lemma 6.2 we can suppose that $Y_0, \ldots, Y_{p-1}$ are countable and disjoint. If Z_q is a coherent subset of Y_q dense in X for $q<p$, then $Z_0, \ldots, Z_{p-1}$ form a p-guard of X and $\varphi''(Z_0, \ldots, Z_{p-1})$ holds in the subspace X.

Now suppose that φ^* holds in U and $X, Y_0, \ldots, Y_{p-1}$ are as in φ^*. By Lemma 6.2 there are countable disjoint $Z_0 \subseteq Y_0, \ldots, Z_{p-1} \subseteq Y_{p-1}$ dense in X. Then $\varphi''(Z_0, \ldots Z_{p-1})$ holds in X, hence φ is a theorem of Ar.

We will use the following:

Lemma 6.6. *Every non-p-modest vicinity space has a coherent, separable, non-p-modest subspace.*

Proof. Suppose that $A, A_0, \ldots, A_{p-1}$ witness that the space is not p-modest. Without loss of generality A is the whole space. By Theorem 2.3 there is a countable coherent $B \subseteq A_0 \cup \cdots \cup A_{p-1}$ such that every A_q is dense in B. The repletion of B is the desired subspace.

Sh:123

7. Non-modest chains

A *chain* is a linearly ordered set. A chain is *short* if it embeds neither ω_1 nor ω_1^*. Only short chains are considered here.

With respect to Example 2.2 we assign a vicinity space to each chain U, it will be called the vicinity space of U. Warning: let X be a subchain of a chain U, the order topology of X (which is the topology of the vicinity space of X) can differ from the inherited topology of X. For example, let P be a nowhere dense perfect subset of the real line R. Form $X \subset R-P$ by choosing one point from every maximal interval of $R-P$. The subchain X has no isolated points, in the inherited topology every point of X is isolated.

Suppose $1 \leq p < \omega$. We say that a chain U is *p-modest* if the vicinity space of any subchain of U is p-modest.

Lemma 7.1. *Every non-p-modest chain has a subchain whose vicinity space is second-countable, zero-dimensional and not p-modest.*

Proof. Let U be a non-p-modest chain. It has a subchain whose vicinity space is not p-modest. Without loss of generality the vicinity space of U is not p-modest. In virtue of Lemma 6.6 we can suppose that the vicinity space of U is coherent and separable. It means in particular that U is densely ordered. Every densely ordered separable chain is embeddable into the real line. Hence U is second-countable. Let $A, B_0, \ldots, B_{p-1}$ witness that U is not p-modest. Without loss of generality $A = U$. Let $B = B_0 \cup \cdots \cup B_{p-1}$. There is $C \subset B$ such that C and $B-C$ are dense in B, and every B_q is dense in C as well as in $B-C$. The subchain $U-C$ is the desired one.

Theorem 7.2. Ar *is uniformly interpretable in the monadic theory of non-p-modest chains.*

Proof. Given a sentence φ in the language of Ar compute a monadic vicinity sentence φ' coding φ according to Theorem 6.5. Let φ^* be a sentence in the monadic language of order saying that there is a subchain X such that the vicinity space of X is zero-dimensional and not p-modest, and every non-p-modest vicinity subspace of X satisfies φ'.

8. Non-modest topology

Speaking about topological spaces we mean first-countable regular T_1 spaces. We consider them as vicinity spaces with respect to Example 2.1. In particular a topological space is *p-modest* if the corresponding vicinity space is p-modest. Sup-

Sh:123

pose $1 \leq p < \omega$. Theorem 6.5 gives:

Corollary 8.1. Ar *is uniformly interpretable in the monadic theory of second-countable, zero-dimensional, non-p-modest topological spaces.*

The next lemma can be easily generalized for more general vicinity spaces.

Lemma 8.2. *Every second-countable non-p-modest topological space has a zero-dimensional non-p-modest subspace.*

Proof. Let $A, X_0, \ldots X_{p-1}$ witness that the space is not p-modest. Without loss of generality A is the whole space. Let $\{B_i: i < \omega\}$ be an open basis of A. To every sequence $a \in {}^{<\omega}\omega$ we assign an open set $[a] \subseteq A$ and a point $ea \in [a]$ as follows. If a is empty, then $[a] = A$ and ea is an arbitrary point in A. Suppose that $l = \mathrm{lh}(a)$ and $[a]$, ea are already chosen. Select a basis $H_0 \supset H_1 \supset \cdots$ of open neighbourhoods of ea such that $\bar{H}_0 \subseteq [a]$ and $\bar{H}_{i+1} \subset H_i$ for $i < \omega$. As usual $a\hat{\ }k$ is the sequence $a0, \ldots, a(l-1), k$. Suppose $k = ip + q$ where $q < p$. Choose $[a\hat{\ }k]$ in such a way that the closure of $[a\hat{\ }k]$ is included into $H_i - \bar{H}_{i+1}$ and either $[a\hat{\ }k] \subseteq B_l$ or $[a\hat{\ }k] \subseteq A - B_l$. Choose $e(a\hat{\ }k)$ in $[a\hat{\ }k] \cap X_q$. Let E be the range of e and C be the closure of E. C and $E \cap X_0, \ldots, E \cap X_{p-1}$ witness that C is not p-modest. We check that C is zero-dimensional.

By induction on $\mathrm{lh}(a)$ prove that $[a] \cap C$ is clopen in C. Let G be an open subset of C and $x \in G$. We have to find a clopen subset K of C such that $x \in K \subseteq G$. If $x = ea$ for some a, then there is n such that $[a\hat{\ }k] \subseteq G$ for $k > n$. Choose $K = [a] \cap C - \bigcup \{[a\hat{\ }k]: k \leq n\}$. If $x \in C - E$ take l with $x \in B_l \cap C \subseteq G$. Clopen sets $[a]$ with $\mathrm{lh}(a) = l+1$ partition $C - \{eb: \mathrm{lh}(b) \leq l\}$, choose $K = [a] \cap C$ with $\mathrm{lh}(a) = l+1$ and $x \in [a]$. By the construction $[a] \subseteq B_l$.

Theorem 8.3. Ar *is uniformly interpretable in the monadic theory of second-countable non-p-modest topological spaces.*

Proof. Use Corollary 8.1 and Lemma 8.2.

Lemma 8.4. Every non-p-modest metrizable space has a second-countable non-p-modest subspace.

Proof. Use Lemma 6.6 and the fact that every separable metric space is second-countable.

Theorem 8.5. Ar *is uniformly interpretable in the monadic theory of non-p-modest metrizable spaces.*

Proof. Given a sentence φ in the language of Ar compute a monadic topological

Sh:123

sentence φ' coding φ according to Theorem 8.3. Let φ^* say that there is a non-p-modest subspace X such that every non-p-modest subspace $Y \subseteq X$ satisfies φ'.

References

[1] J.R. Büchi, The monadic second-order theory of ω_1, Springer Lecture Notes in Math. 328 (1973) 1–127.
[2] A. Grzegorczyk, Undecidability of some topological theories, Fund. Math. 38 (195.) 137–152.
[3] Y. Gurevich, Monadic theory of order and topology I, Israel J. Math. 27 (1977) 299–319.
[4] Y. Gurevich, Monadic theory of order and topology II, Israel J. Math. 34 (1979) 45–71.
[5] Y. Gurevich, M. Magidor and S. Shelah, The monadic theory of ω_2, J. Symbolic Logic, to appear.
[6] Y. Gurevich and S. Shelah, Modest theory of short chains II, J. Symbolic Logic 44 (1979) 491–502.
[7] Y. Gurevich and S. Shelah, Complexity of the monadic theory of order, J. Symbolic Logic, to appear.
[8] Y. Gurevich and S. Shelah, Monadic theories and the 'next' world, Israel J. Math., to appear.
[9] C.W. Henson, C.G. Jockusch, Jr., L.A. Rubel and G. Takeuti, First order topology, Dissertationes Math., CXLIII (1977).
[10] M.O. Rabin, Decidability of second order theories and automata on infinite trees, Trans. AMS 141 (1969) 1–35.
[11] S. Shelah, The monadic theory of order, Ann. Math. 102 (1975) 379–419.
[12] S. Shelah, On the decidability of the monadic theory of order, Abstract AMS 1 (1980) 174.