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CATEGORICITY OF THEORIES IN $L_{\kappa\omega}$, WITH κ A COMPACT CARDINAL

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Introduction

In this paper, the following theorem is proved.

Theorem. *Assume κ is a strongly compact cardinal, $\kappa > \omega$, T is a theory in a fragment $\mathscr{F}$ of $L_{\kappa\omega}$ over a language L, and $\kappa' = \max(\kappa, |\mathscr{F}|)$. Assume T is categorical in the cardinal λ. Then*:

(i) *If λ is a successor cardinal and $\lambda > ((\kappa')^{<\kappa})^+$, then T is categorical in every cardinal greater than or equal to* $\min(\lambda, \beth_{(2^{\kappa'})^+})$.

(ii) *If $\lambda > \beth_{\kappa+1}(\kappa')$, then T is categorical in every cardinal of the form $\beth_\delta$ with δ divisible by $(2^{\kappa'})^+$ (i.e., $\delta = (2^{\kappa'})^+ \cdot \alpha$ (ordinal multiplication) for some ordinal $\alpha > 0$).*

Corollary. *If λ_1, λ_2 are two cardinals, and either both λ_1, λ_2 are successor cardinals $>((\kappa')^{<\kappa})^+$, or both are of the form $\beth_\delta$ with δ divisible by $(2^{\kappa'})^+$, then T as in the theorem is categorical in λ_1 if and only if it is categorical in λ_2.*

The result should be seen as belonging to the program of classification theory, undertaken by the first author in [9] (of which the second, enlarged edition is in print), [13], [10], [11], [15], etc. The present theorem is a partial extension of Morley's categoricity theorem [6] for finitary first-order logic to a particular kind of infinitary language, $L_{\kappa\omega}$ with κ a compact cardinal. In the context of 'large' infinitary languages, it is intended as a first step towards results of the kind characteristic of classification theory: dividing lines of structure/non-structure, determination of spectrum functions, the Main Gap, etc. Let us point out that the connections notwithstanding, the present paper is largely self-contained.

In order to stay on a desirable level of generality in most of the work, the categoricity assumption is used only sparingly; instead, in the main part of the work, more general global assumptions are used, and in Section 5, the conclusions concerning the problem of categoricity are summarized.

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We start, in Section 1, with a theory in a fragment $\mathcal{F}$ of $L_{\kappa\omega}$; very soon, the assumption of κ being a compact cardinal is introduced. At the beginning of Section 2, we introduce the assumption of the amalgamation property and the joint embedding property for the class K of all models of T of power $\geqslant\kappa'$, relative to the relation of '$\mathcal{F}$-elementary substructure'. In Section 4, four numbered assumptions (4.0, 4.4, 4.11, 4.16) are introduced. Each assumption is held valid for all the subsequent work, except the summary, Section 5. Each assumption is a consequence of hypotheses in the main results, as pointed out at the appropriate places, and in Section 5.

After a glance at the main definitions, the reader may profitably look at the short Section 5 where the various strands of the paper are brought together in the proof of the main result. Reading Section 5 will make it clear that the proofs of the two parts of the theorem are, to a large extent, disjoint from each other.

Section 1 aims at showing, in essence, the amalgamation property (familiar from model theory for finitary logic) for the class K of models of T of cardinality $\geqslant\kappa'$ (see the statement of the Theorem above), with respect to $\mathcal{F}$-embeddings, under either of the categoricity hypotheses in (i) and (ii). The proof for the case of (i) is easy (Proposition 1.9); for (ii), it is harder, and it occupies the rest of Section 1 after 1.9.

Section 2 is a study of types, both in an abstract sense, and in a more familiar formula-oriented sense, under the assumption of the amalgamation and joint embedding properties. This section is a more detailed restatement, for the context at hand, of Section II.3 of [15]. It is here that the need to extend the discussion from $L_{\kappa\omega}$ to $L_{\kappa\kappa}$ arises.

Section 3 collects the arguments needed using order-indiscernibles. Some of the material is folklore, and is included to fix notation and terminology. The more involved arguments, notably Propositions 3.3 and 3.6, are needed only for part (ii), not for part (i), of the Theorem.

Section 4, which is entirely for the purposes of part (i) of the Theorem (part (ii) is independent of this section), starts by building up an extension of a rather elementary part of stability theory for finitary stable theories, the theory of non-forking of types over models only. Therefore, the first part of this section should look familiar to people who have seen stability theory as given in [9]. The starting point of this section is the non-structure theorem 3.14 of Chapter III of [15], restated here as Proposition 4.3. It allows one to conclude that, under a suitable categoricity assumption, the class K is 1-stable (see 4.4), from which the existence of a good notion of independence (non-forking) over models is deduced. What this shows is that the axiomatic framework of Chapter II of [15] is reproduced to a considerable extent in the present context, although there are no direct references to that framework in this paper. Some arguments in this section have a place in a more general context of classification theory, notably in the still unpublished further chapters of [15].

The results of this paper are all due to the first author; the exposition is the

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work of the second author. The second author thanks the Lady Davis Fellowship Trust and The Hebrew University of Jerusalem for their support and hospitality during the work on this paper.

Finally, we would like to thank the referee for his careful reading of the paper. Following the referee's advice, we have expanded this introduction.

1. Preliminaries, and the amalgamation property

Throughout the paper, κ will denote a fixed infinite regular cardinal, which, for most of the paper, will be assumed to be (strongly) compact and greater than $\aleph_0$. (For the definition of 'compact', see below, or [4] which should serve as a reference for all undefined set-theoretic terminology.)

We let L be a set of finitary relation and operation symbols (a 'language'); L is fixed throughout the paper. $L_{\kappa\omega}$ denotes the least set of (infinitary) formulas ('logic') which contains the atomic formulas of L (including the ones using equality), and which is closed under applying the usual logical operators of finitary first-order logic and under taking the conjunction, or disjunction, of any set of size $<\kappa$ of formulas, provided the set of free variables of the conjunction (disjunction) is finite. Note that all formulas of $L_{\kappa\omega}$ have only finitely many free variables. $L_{\kappa\kappa}$ is the larger logic that also allows the formation of $\forall \boldsymbol{x}\, \varphi$, $\exists \boldsymbol{x}\, \varphi$, with $\boldsymbol{x}$ a sequence of length $<\lambda$ of variables; also, in forming conjunctions and disjunctions of sets of size $<\kappa$, the result is required to contain $<\kappa$ many free variables (rather than finitely many); all formulas in $L_{\kappa\kappa}$ have $<\kappa$ free variables.

For further details on infinitary logic, see e.g. [2].

As usual, a notation like $\varphi(\boldsymbol{x})$ indicates a formula with free variables among those in $\boldsymbol{x}$; $\boldsymbol{x}, \boldsymbol{y}, \ldots$ always denote sequences of distinct variables; when $\varphi(\boldsymbol{x})$ is assumed to be in $L_{\kappa\omega}$ (in $L_{\kappa\kappa}$), $\boldsymbol{x}$ is assumed to be finite (of length $<\kappa$).

We let $\mathcal{F}$ be a fragment of $L_{\kappa\omega}$: a set of formulas of $L_{\kappa\omega}$ containing all atomic L-formulas, and closed under finitary logical operators and under taking subformulas. Also, we let T be a set of sentences (formulas without free variables) of $\mathcal{F}$. $\mathcal{F}$ and T will also remain fixed throughout the paper.

By a model of T we always mean an L-structure that satisfies the axioms in T. From now on, a 'model' will mean a model of T, unless otherwise specified; the symbols $M, N, \ldots$ will denote models.

f is an *$\mathcal{F}$-elementary embedding* (or *$\mathcal{F}$-embedding*) *of M into N*, in notation $f: M \underset{\mathcal{F}}{\to} N$, or more simply, $f: M \to N$ (since $\mathcal{F}$ is fixed); if f is a function with domain $|M|$ (the underlying set of M) into $|N|$ and f preserves the meaning of all $\mathcal{F}$-formulas:

$$M \models \varphi[\boldsymbol{a}] \quad \Rightarrow \quad N \models \varphi[f(\boldsymbol{a})]$$

for all $\varphi(\boldsymbol{x}) \in \mathcal{F}$, all tuples $\boldsymbol{a}$ of elements M matching $\boldsymbol{x}$ (length($\boldsymbol{a}$) = length($\boldsymbol{x}$)) (if $\boldsymbol{a} = \langle a_i \rangle_{i<\alpha}$, $f(\boldsymbol{a}) = \langle f(a_i) \rangle_{i<\alpha}$). Since $\mathcal{F}$ is closed under negation, the last

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implication is in fact an equivalence. If, in particular, f is a set-inclusion $|M| \subset |N|$, we write $M \prec_{\mathscr{F}} N$ (or simply $M \prec N$) for $f: M \underset{\mathscr{F}}{\Rightarrow} N$, and we say that M is an *$\mathscr{F}$-elementary substructure (submodel) of N*, or *N is an $\mathscr{F}$-elementary extension of M*, or N is an *$\mathscr{F}$-extension of M*. We write $M \precsim_{\mathscr{F}} N$ to mean that there is an $\mathscr{F}$-embedding of M into N. We have the well-known Tarski Union Theorem:

Proposition 1.1. (TUT). (i) *The union of a $\prec_{\mathscr{F}}$-directed system of models is a model: if $(I, \leqslant)$ is a directed partial order, $\langle M_i \rangle_{i\in I}$ is a family of models satisfying $M_i \prec_{\mathscr{F}} M_j$ whenever $i \leqslant j$, then we have the model $\bigcup_{i\in I} M_i$ of T for which*

$$\left|\bigcup_{i\in I} M_i\right| = \bigcup_{i\in I} |M_i| \quad \textit{and} \quad M_j \prec_{\mathscr{F}} \bigcup_{i\in I} M_i \quad \textit{for every } j \in I.$$

(ii) *If, in the notation of* (i), *we have, in addition, that $M_i \prec_{\mathscr{F}} M$ for a fixed model M and for all $i \in I$, then*

$$\bigcup_{i\in I} M_i \prec_{\mathscr{F}} M.$$

(iii) *More generally than* (ii), *if, in addition to* (i), *we have $f_i: M_i \underset{\mathscr{F}}{\Rightarrow} M$ for all $i \in I$ with $f_i \subseteq f_j$ for $i \leqslant j$, then we have*

$$\bigcup_{i\in I} f_i : \bigcup_{i\in I} M_i \underset{\mathscr{F}}{\Rightarrow} M.$$

Usually, $(I, \leqslant)$ in 1.1 will be an ordinal, i.e. the set of ordinals less than a given ordinal, ordered by the standard ordering of ordinals. A *$\prec$-chain of models* is a sequence $\langle M_\beta \rangle_{\beta<\alpha}$ with α an ordinal and $M_\beta \prec M_\gamma$ for $\beta < \gamma < \alpha$; we speak of a *continuous* chain if $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$ for all limit ordinals $\beta < \alpha$; we write

$$M = \bigcup_{\beta<\alpha}^{\uparrow} M_\beta$$

to mean that $\langle M_\beta \rangle_{\beta<\alpha}$ is a $\prec$-chain and $M = \bigcup_{\beta<\alpha} M_\beta$; we write

$$M = \bigcup_{\beta<\alpha}^{c} M_\beta$$

to mean that $\langle M_\beta \rangle_{\beta<\alpha}$ is a continuous $\prec$-chain and $M = \bigcup_{\beta<\alpha} M_\beta$.

Another piece of notation fixed for the whole of the paper: $\kappa' = \max(\kappa, |L|)$.

Another well-known result is the downward Löwenheim–Skolem Theorem:

Proposition 1.2 (dLST). (i) *Given any model M, a subset $A \subset |M|$ and a cardinal λ with $\max(\kappa', |A|) \leqslant \lambda \leqslant \|M\|$, there is a model $N \models T$ with $A \subset |N|$, $N \prec_{\mathscr{F}} M$ and $\|N\| = \lambda$.*

(ii) *Any sentence in any $(L')_{\theta\omega}$ (in any $(L')_{\theta\theta}$) that has a model in a power λ_0 has a model in each power λ for which $\max(\theta, |L'|) \leqslant \lambda \leqslant \lambda_0$ (and for which $\lambda^{<\theta} = \lambda$).*

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We tend to use TUT and dLST without explicit reference to them.

A $\Sigma_1(\mathcal{F})$-formula, or more simply, a Σ_1-formula, is an $L_{\kappa\kappa}$-formula of the form

$$\exists \boldsymbol{x} \bigvee_{i\in I} \bigwedge_{j\in J_i} \psi_{ij} \quad \text{where each } \psi_{ij} \in \mathcal{F}.$$

If κ is (strongly) inaccessible (which is the case when κ is compact), then every $L_{\kappa\kappa}$-formula of the form $\exists \boldsymbol{x}\, \psi$ where ψ is an (infinitary) Boolean combination of $\mathcal{F}$-formulas is logically equivalent to a Σ_1-formula. In this case, any conjunction or disjunction of less than κ many Σ_1-formulas is logically equivalent to a Σ_1-formula. A *positive primitive* (p.p) formula is an $L_{\kappa\kappa}$-formula of the form $\exists \boldsymbol{x} \bigwedge_{i\in I} \psi_i$, with each $\psi_i \in \mathcal{F}$; every Σ_1-formula is logically equivalent to the disjunction of $<\kappa$ p.p. formulas.

$f: M \to N$ is a *Σ_1-embedding* (in notation: $f: M \xrightarrow[\Sigma_1]{} N$) if

$$M \models \varphi[\boldsymbol{a}] \quad \Leftrightarrow \quad N \models \varphi[f(\boldsymbol{a})]$$

for all Σ_1-formulas $\varphi(\boldsymbol{x})$ and appropriate $\boldsymbol{a} \in |M|$ [$\boldsymbol{a} \in |M|$ abbreviates $\boldsymbol{a}$ is a tuple of elements of $|M|$]. Note that the '$\Rightarrow$' direction of the last equivalence is automatic: a consequence of f being an $\mathcal{F}$-embedding. We write $M <_1 N$ if $|M| \subset |N|$ and the set inclusion of $|M|$ in $|N|$ is a Σ_1-embedding. Note that in the definition of Σ_1-embedding, or $<_1$, we may restrict attention to p.p. formulas φ.

We say that M is *existentially closed* (e.c.) if for all $N \models T$, $M <_{\mathcal{F}} N$ implies $M <_1 N$, or equivalently, every $\mathcal{F}$-embedding with domain M is a Σ_1-embedding. Let us call a sentence $\varphi(\boldsymbol{a})$ with parameters (individual constants) $\boldsymbol{a}$ in M (each a_i in $\boldsymbol{a}$ denoting itself) *consistent* with M if there is an $\mathcal{F}$-extension N of M such that $N \models \varphi[\boldsymbol{a}]$. Then M is e.c. if every Σ_1-sentence (or, every p.p. sentence) over M that is consistent with M is in fact true in M.

A well-known elementary argument gives

Proposition 1.3. *Every model has an existentially closed $\mathcal{F}$-elementary extension. More precisely, if λ is a cardinal such that $\lambda^{<\kappa} = \lambda$, $\lambda \geq \kappa$, and M is a model of power λ, then there is an e.c. model N of power λ such that $M <_{\mathcal{F}} N$.*

Proof. For any M' with $\|M'\| = \lambda$, for λ of the proposition, we list the pairs $\langle \varphi(\boldsymbol{x}), \boldsymbol{a} \rangle$ of Σ_1-formulas $\varphi(\boldsymbol{x})$ and matching tuples $\boldsymbol{a}$ from M' as $\langle \varphi_\alpha, \boldsymbol{a}_\alpha \rangle_{\alpha<\lambda}$ (note that the set of those pairs has cardinality $\leq\lambda$), and define, by induction on $\alpha \leq \lambda$ the models M_α by

$$M_0 = M',$$

$$M_\delta = \bigcup_{\alpha<\delta} M_\alpha \quad \text{for } \delta \text{ limit (see TUT)},$$

$$\begin{aligned} M_{\alpha+1} &= \text{some model of power } \lambda\ \mathcal{F}\text{-extending } M_\alpha \text{ and satisfying} \\ &\quad \varphi_\alpha(\boldsymbol{a}_\alpha) \quad \text{if the latter is consistent with } M_\alpha \text{ (see dLST)}, \\ &= M_\alpha \qquad \text{otherwise.} \end{aligned}$$

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Note that $(M')^* \stackrel{\text{def}}{=} M_\lambda$ is an $\mathscr{F}$-extension of M', and it satisfies the condition for 'e.c.' relative to parameters in M'. Now, let, for $\alpha \leqslant \kappa$, by induction

$$N_0 = M \qquad \text{(given in the Proposition)},$$

$$N_\alpha = \Big(\bigcup_{\beta<\alpha} N_\beta\Big)^* \quad (\alpha > 0).$$

Then, since any $<\kappa$-tuple of elements of $N \stackrel{\text{def}}{=} N_\kappa$ is in some N_α, $\alpha < \kappa$, it is clear that N is e.c.; clearly, $\|N\| = \lambda$. □

The following generalization of 'existentially closed' will also be used. Let $\mu \geqslant \kappa$ be any cardinal. M is called *$<\mu$-existentially closed* ($<\mu$-e.c.) if for any set $\Phi(\boldsymbol{x})$ of cardinality $<\mu$ of $\mathscr{F}$-formulas over M (with parameters in M), with free variables all in $\boldsymbol{x}$, $\boldsymbol{x}$ a tuple of variables of length $<\mu$, $M \models \exists \boldsymbol{x} \bigwedge \Phi(\boldsymbol{x})$ provided there is N, $M <_{\mathscr{F}} N$, such that $N \models \exists \boldsymbol{x} \bigwedge \Phi(\boldsymbol{x})$. We use '$\leqslant \mu$-e.c.' in the sense '$<\mu^+$-e.c.'.

The following generalization of 1.3 has the same proof as 1.3:

Proposition 1.3′. (i) *Suppose μ is regular, $\mu \geqslant \kappa$, $\lambda^{<\mu} = \lambda$, $\lambda \geqslant \kappa'$. Then any M of power λ can be $\mathscr{F}$-extended to a $<\mu$-e.c. model of power λ.*

(ii) *If in* (i), *in addition we have $M < M^*$ with M^* $<\mu$-e.c., then N as in* (i) *can be found as an $\mathscr{F}$-submodel of M^*.* □

The cardinal κ is *compact* if $(L')_{\kappa\kappa}$ satisfies the $<\kappa$-compactness theorem, for any L' of relation and operation symbols of arities $<\kappa$: for any set Σ of sentences of $(L')_{\kappa\kappa}$, if every subset of Σ of cardinality $<\kappa$ has a model [for which we say Σ is *$<\kappa$-consistent*], then Σ has a model. For a 'purely mathematical' definition, and further facts concerning compact cardinals, see [4].

$\aleph_0$ is compact; we are interested in compact κ greater than $\aleph_0$; of course, the existence of such is not provable, but hopefully consistent with, ZFC. At one point in the last section, the assumption that $\kappa > \aleph_0$ is of essential help. Let us make explicit our

Assumption for the rest of the paper. κ is a compact cardinal $\geqslant \aleph_0$.

Many facts of finitary model theory generalize to $L_{\kappa\kappa}$ with κ compact, by essentially the same 'compactness' arguments as in the finitary case. We develop some of these for later use.

For a model M, $\mathrm{Diag}_{\mathscr{F}}(M)$ denotes the set of sentences $\varphi(\boldsymbol{a})$, with $\varphi(\boldsymbol{x}) \in \mathscr{F}$, $\boldsymbol{a}$ a tuple of elements of M used as individual constants ($\boldsymbol{a}$ substituted for $\boldsymbol{x}$), such that $M \models \varphi[\boldsymbol{a}]$. The language of $\mathrm{Diag}_{\mathscr{F}}(M)$ is $L(M) \stackrel{\text{def}}{=} L \cup |M|$, with the elements of $|M|$ treated as new individual constants. Note that

$$N^* = (N, b_a)_{a\in|M|} \models \mathrm{Diag}_{\mathscr{F}}(M)$$

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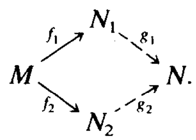
iff for $f = \{\langle a, b_a\rangle : a \in |M|\}$, we have $f: M \underset{\mathscr{F}}{\to} N$. Note in particular that the axioms of T are elements of $\mathrm{Diag}_{\mathscr{F}}(M)$, $T \subset \mathrm{Diag}_{\mathscr{F}}(M)$, since every element of T is in $\mathscr{F}$. Thus, a model of $\mathrm{Diag}_{\mathscr{F}}(M)$ is automatically a model of T. A familiar application of the $\mathscr{F}$-diagram $\mathrm{Diag}_{\mathscr{F}}(M)$ is the proof that if $\|M\| \geqslant \kappa$, then M has arbitrarily large $\mathscr{F}$-extensions (upward Löwenheim–Skolem–Tarski Theorem, uLSTT).

Definition 1.4. M is an *amalgamation base* (a.b.) (relative to $\mathscr{F}$-embeddings) if the following holds: whenever $f_i: M \to N_i$ $(i = 1, 2)$, there are N and $g_i: N_i \to N$ such that $g_1 f_1 = g_2 f_2$:



Remember that κ is compact.

Proposition 1.5. (i) *M is an a.b. iff for any two Σ_1-sentences σ_1, σ_2 over M that are separately consistent with M, $\sigma_1 \wedge \sigma_2$ is consistent with M as well.*

(ii) *Suppose M is a model which is not an a.b. Then there is a $<\kappa$-tuple $\boldsymbol{a} \in M$, and Σ_1-formulas $\sigma_1(\boldsymbol{x})$, $\sigma_2(\boldsymbol{x})$ without parameters such that each of $\sigma_1(\boldsymbol{a})$, $\sigma_2(\boldsymbol{a})$ is consistent with M, but $\sigma_1(\boldsymbol{a}) \wedge \sigma_2(\boldsymbol{a})$ is (logically) inconsistent (has no model).*

Remark. Note that (ii) is a strengthening of the 'if' part of (i): the inconsistency of $\sigma_1(\boldsymbol{a}) \wedge \sigma_2(\boldsymbol{a})$ with M is strengthened to a logical inconsistency.

Proof. The 'only if' part is easily seen, and it does not use the compactness of κ. To prove (ii) (and thus the 'if' part of (i) as well), suppose the conclusion of (ii) fails, and let f_1, f_2 be as in 1.4; we'll show they can be amalgamated. Without loss of generality, f_1 and f_2 are inclusions.

Consider the following set of sentences

$$\Sigma \overset{\text{def}}{=} \mathrm{Diag}_{\mathscr{F}}(N_1) \cup \mathrm{Diag}_{\mathscr{F}}(N_2)$$

where, in the two diagrams, we use the same individual constant for each $a \in |M|$, but make sure no individual constant other than those in $L(M)$, is common to the two diagrams. It is immediately seen that any model of Σ provides an amalgamation (g_1 and g_2 as in 1.4). On the other hand, the $<\kappa$-consistency of Σ reduces to the consistency of

$$\sigma_1(\boldsymbol{a}) \wedge \sigma_2(\boldsymbol{a})$$

with $\sigma_1(\boldsymbol{a})$, $\sigma_2(\boldsymbol{a})$ p.p. sentences with parameters $\boldsymbol{a}$ in M, such that $\sigma_1(\boldsymbol{a})$ is true in N_i, hence each $\sigma_i(\boldsymbol{a})$ is consistent with M. By our assumption, each instance of the latter consistency holds. □

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The following is an immediate consequence of 1.5(i):

Corollary 1.6. *If M is e.c., then it is an a.b.* □

Corollary 1.7. *Suppose $M \models T$ is not an a.b. Then there is $\boldsymbol{a} \in {}^{<\kappa}|M|$ such that for any $N < M$, if $\boldsymbol{a} \in N$, then N is not an a.b.*

Proof. Choose $\boldsymbol{a}$, σ_1, σ_2 as in 1.5(ii). If $N < M$ contains $\boldsymbol{a}$, then each of $\sigma_1(\boldsymbol{a})$, $\sigma_2(\boldsymbol{a})$ is consistent with N, and $\sigma_1(\boldsymbol{a}) \wedge \sigma_2(\boldsymbol{a})$ is inconsistent with N, since it is even logically inconsistent. Thus, by 1.5(i), N is not an a.b. □

Let K denote, for once and for all, the class of models of cardinality $\geqslant \kappa'$ (essentially, we are not interested in models of power $<\kappa'$). $K_\lambda = \{M \in K : \|M\| = \lambda\}$, $K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu$, etc. We assume throughout that K is non-empty.

K has the *amalgamation property* (AP) if every $M \in K$ is an a.b.; K has the *joint embedding property* (JEP) if for any $M_1, M_2 \in K$ there are N and $f_1 : M_1 \to N$, $f_2 : M_2 \to N$. K is *categorical in* λ if, up to isomorphism, there is exactly one model in K_λ.

Proposition 1.8. *If K is categorical in $\lambda \geqslant \kappa'$, then K has the* JEP.

Proof. By an easy application of diagrams, if $K_{\kappa'}$ has the JEP, then so does K. By uLSTT, any $M \in K_{\kappa'}$ has an $\mathscr{F}$-extension of K_λ; now, the assertion is clear. □

Proposition 1.9. *Suppose $\lambda \geqslant \kappa'$, $\lambda^{<\kappa} = \lambda$ and K is categorical in λ. Then every $M \in K$ is e.c. (and hence an a.b. as well).*

Proof. Suppose $M \in K$ is not e.c. This means that there are $\boldsymbol{a}$ in M, $\varphi(\boldsymbol{x})$ a Σ_1-formula and N with $M <_{\mathscr{F}} N$ such that $N \models \varphi[\boldsymbol{a}]$ and $M \models \neg\varphi[\boldsymbol{a}]$. All this can be expressed by saying that the composite structure $(N, M, \boldsymbol{a})$ satisfies a certain sentence σ of $(L')_{\kappa\omega}$, with $L' = L \cup \{A\} \cup \{c_i : i < \alpha\}$, A a unary predicate (for $|M|$), c_i individual constants for a_i in $\boldsymbol{a} = \langle a_i \rangle_{i<\alpha}$). Since $\|M\| \geqslant \kappa' \geqslant \kappa$, σ has models in which the interpretation of A is of an arbitrary cardinality $\lambda \geqslant \kappa'$ (by $<\kappa$-compactness and dLST); applying this to our λ, we conclude that there is $M_1 \in K_\lambda$ which is not e.c. On the other hand, using $\lambda^{<\kappa} = \lambda$, by 1.3 we have some $M_2 \in K_\lambda$ which is e.c. Of course, this contradicts categoricity in λ. □

By a more sophisticated argument, we now show that categoricity of K in a sufficiently large, but otherwise arbitrary, cardinal implies that K has the AP. Since for $\kappa = \aleph_0$, by 1.9 we already know this, we may, and do, assume that κ (is compact and) $>\aleph_0$.

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Lemma 1.10. *Let μ be a cardinal $\geqslant\kappa'$, λ a limit cardinal $>2^\mu$ with cofinality* $\operatorname{cf}\lambda<\kappa$. *Suppose no $M\in K_\lambda$ is an a.b. Then there is a continuous $<$-chain $\langle M_i\rangle_{i<\operatorname{cf}\lambda}$ such that for every non-limit $i<\operatorname{cf}\lambda$, M_i is $\leqslant\mu$-e.c. and $\bigcup_{i<\operatorname{cf}\lambda}M_i$ is not an a.b.*

Proof. Let $\langle\lambda_i\rangle_{i<\operatorname{cf}\lambda}$ be a strictly increasing sequence of regular cardinals $>2^\mu$ with limit equal to λ. By induction on $i<\operatorname{cf}\lambda$, we construct M_i^* and $N_{\boldsymbol a}^i$, one for each $\boldsymbol a\in Q_i\stackrel{\text{def}}{=}\{\boldsymbol a=\langle\boldsymbol a_j\rangle_{j<i}:\boldsymbol a_j\in{}^{<\kappa}|M_j^*|\}$, such that for all $i<\operatorname{cf}\lambda$:

(i) $M_i^*\in K_{\lambda_i}$, $N_{\boldsymbol a}^i\in K_{2^\mu}$;
(ii) $\langle M_i^*\rangle_{i<\operatorname{cf}\lambda}$ is $<$-continuous;
(iii) each $N_{\boldsymbol a}^i$ $(\boldsymbol a\in Q_i)$ is $\leqslant\mu$-e.c., provided i is non-limit;
(iv) for any $\boldsymbol a\in Q_i$, the sequence $\langle N^j_{\boldsymbol a|j}\rangle_{j\leqslant i}$ is $<$-continuous;
(v) $N_{\boldsymbol a}^i<M_i^*$ for all $\boldsymbol a\in Q_i$;
(vi) if $\boldsymbol a\in Q_i$, then $\boldsymbol a$ is in $N_{\boldsymbol a}^i$ (meaning that every term of every term of the sequence $\boldsymbol a$ is in $|N_{\boldsymbol a}^i|$).

For $i=0$, we take $N_\emptyset^0$ to be any $\leqslant\mu$-e.c. model of power 2^μ (by 1.3′), and M_i^* any model of power λ_0 $\mathcal F$-extending $N_\emptyset^0$. For i limit, $i<\operatorname{cf}\lambda$, we put $M_i^*=\bigcup_{j<i}M_j^*$ and $N_{\boldsymbol a}^i=\bigcup_{j<i}N^j_{\boldsymbol a|j}$ for all $\boldsymbol a\in Q_i$; clearly, all relevant conditions are satisfied. Finally, let $i=j+1$. Let $\tilde M$ be any $\leqslant\mu$-e.c. extension of M_j^* of cardinality $\geqslant\lambda_i$ (by 1.3′). For each $\boldsymbol b\in{}^{<\kappa}|M_j|$ and $\boldsymbol a\in Q_j$, let $N^i_{\boldsymbol a\frown\langle\boldsymbol b\rangle}$ be a $\leqslant\mu$-e.c. $\mathcal F$-submodel of cardinality 2^μ of $\tilde M$ containing $N^j_{\boldsymbol a}$ and $\boldsymbol b$ (by 1.3′(ii)). Now, the cardinality of Q_i is $\leqslant|i|\times\lambda_i^{<\kappa}=\lambda_i$ (recall that for all regular $\nu\geqslant\kappa$, $\nu^{<\kappa}=\nu$; see [4]). Hence there is an $\mathcal F$-submodel M_i^* of $\tilde M$ of cardinality λ_i, containing $N_{\boldsymbol c}^i$ for all $\boldsymbol c=\boldsymbol a\frown\langle\boldsymbol b\rangle\in Q_i$. This completes the construction.

We have that $M^*=\bigcup_{i<\operatorname{cf}\lambda}M_i^*$ is a model of power λ, hence it is not an a.b. Let $\boldsymbol a\in{}^{<\kappa}|M^*|$ witness this fact, in the sense of 1.7. Let us write $\boldsymbol a$ in the form $\boldsymbol a={}^\frown\langle\boldsymbol a_j\rangle_{j<\operatorname{cf}\lambda}$ with $\boldsymbol a_j\in{}^{<\kappa}|M_j^*|$, and define, for each $i<\operatorname{cf}\lambda$, $M_i=N^i_{\boldsymbol a|i}$ (where we wrote $\boldsymbol a$ for $\langle\boldsymbol a_j\rangle_{j\in\operatorname{cf}\lambda}$). The M_i form a continuous $<$-chain of models of power 2^μ such that each M_i with i non-limit is $\leqslant\mu$-e.c. Moreover, $\bigcup_{i<\operatorname{cf}\lambda}M_i<M^*$ containing the witness $\boldsymbol a$, hence $\bigcup_{i<\operatorname{cf}\lambda}M_i$ is not an amalgamation base (see 1.7). □

Lemma 1.11. *Let μ be a cardinal, $\mu\geqslant\kappa'$, $\mu^{<\kappa}=\mu$. Let $\langle M_i:i<\sigma\rangle$ be a $<$-continuous chain such that for every non-limit $i<\sigma$, M_i is $\leqslant\mu$-e.c., and $M_\sigma\stackrel{\text{def}}{=}\bigcup_{i<\sigma}M_i$ is not an a.b. Assume that $\langle C_\alpha:\alpha\in S\rangle$ is a modified square-system on μ^+ (see Appendix). Then there is a $<$-continuous tree $\langle N_\eta:\eta\in{}^{<\mu^+}2\rangle$ of models of power μ such that for any $\eta\in{}^\alpha2$ with $\operatorname{otp}C_\alpha=\sigma$, $N_{\eta\frown\langle0\rangle}$ and $N_{\eta\frown\langle1\rangle}$ cannot be amalgamated over N_η.*

Proof. $\langle N_\eta\rangle_\eta$ being a $<$-continuous tree means that for all $\nu\in{}^{\kappa^+}2$, $\langle N_{\mu|\alpha}\rangle_{\alpha<\kappa^+}$ is a $<$-continuous chain.

Let us modify our system $\langle C_\alpha:\alpha\in S\rangle$ by simply discarding all $\alpha\in S$ for which $\operatorname{otp}C_\alpha>\sigma$; without loss, we may assume that for all $\alpha\in S$, $\operatorname{otp}C_\alpha\leqslant\sigma$.

The fact that M_σ is not an a.b. is witnessed by some $\boldsymbol{a} = {}^\wedge\langle \boldsymbol{a}_i : i < \sigma,\ i \text{ is a successor}\rangle$, with $\boldsymbol{a}_i \in {}^{<\kappa}|M_i|$ (see 1.7). For $\eta \in {}^{<\mu^+}2$, by induction on length(η), we define N_η as in the lemma, and also such that for η with length(η) non-limit, N_η is e.c. In addition, in case $\alpha = \text{length}(\eta) \in S$ (and hence $\text{otp}\, C_\alpha \leq \sigma$), we define the $\mathcal{F}$-embedding $h_\eta : N_\eta \to M_{\text{otp}\, C_\alpha}$ such that for $i < \text{otp}\, C_\alpha$, i a successor, $\boldsymbol{a}_i$ is in the range of h_η, and for $\beta \in C_\alpha$ (and hence $\beta \in S$), $h_{\eta \restriction \beta} \subseteq h_\eta$.

Suppose $\alpha < \mu^+$ and all items with indices η with length(η) $< \alpha$ have been defined. If α is a limit ordinal, the definition of items on level α is forced, and the new items will continue to satisfy the requirements.

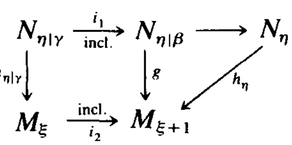
Suppose $\alpha = \beta + 1$.

Case 1: $\beta \in S$ *and* $\text{otp}\, C_\beta = \sigma$. In this case, by the induction hypothesis, for any $\eta \in {}^\beta 2$, $h_\eta'' N_\eta < M_\sigma$ contains the full sequence $\boldsymbol{a}$ of witnesses to the fact that M_σ is not an a.b.; hence $h_\eta'' N_\eta$, and also N_η, is not an a.b. We define $N_{\eta^\wedge\langle 0\rangle}$, $N_{\eta^\wedge\langle 1\rangle}$ to be two $\mathcal{F}$-extensions of N_η that cannot be amalgamated over N_η; by 1.3, we make sure that $N_{\eta^\wedge\langle 0\rangle}$, $N_{\eta^\wedge\langle 1\rangle}$ are both e.c. and of power μ. Note that now $\alpha \notin S$, and we have no obligation to define $h_{\eta^\wedge\langle 0\rangle}$, $h_{\eta^\wedge\langle 1\rangle}$.

Case 2: $\alpha \notin S$ and not Case 1. In this case, we may put $N_{\eta^\wedge\langle 0\rangle} = N_{\eta^\wedge\langle 1\rangle} = N_\eta$ for any $\eta \in {}^\beta 2$.

Case 3: $\alpha \in S$. Since C_α is closed in α, C_α has a last element γ; if $\xi = \text{otp}\, C_\gamma$, then $\text{otp}\, C_\alpha = \xi + 1$. We have $h_{\eta\restriction\gamma}$ as the last h defined before h_η to be defined now, and of course, $N_{\eta\restriction\beta}$ as the last N. Let us distinguish the subcases $\gamma = \beta$ (*Case* 3.1) and $\gamma < \beta$ (*Case* 3.2). In *Case* 3.1, our task is to extend $N_{\eta\restriction\beta}$ to an N_η, and to extend $h_{\eta\restriction\beta} : N_{\eta\restriction\beta} \to M_\xi$ to some $h_\eta : N_\eta \to M_{\xi+1}$ so that $\boldsymbol{a}_\xi$ is in the range of h_η. Since $M_{\xi+1}$ is (in particular) e.c., by 1.3′(ii) (with μ of 1.3′(ii) being κ, λ of 1.3′ being μ), there is $\hat{N} < M_{\xi+1}$ which is e.c., and contains $(h''_{\eta\restriction\beta} N_{\eta\restriction\beta}) \cup \{\boldsymbol{a}_\xi\}$; define N_η and $h_\eta : N_\eta \to M_{\xi+1}$ so that h_η extends $h_{\eta\restriction\beta}$ and $\hat{N} = h_\eta'' N_\eta$.

Finally, let us turn to *Case* 3.2. In this case, by the conditions on $\langle C_\alpha \rangle_{\alpha \in S}$ (see Appendix), γ is a successor ordinal, and thus $N_{\eta\restriction\gamma}$ is e.c., and an a.b. Consider the following diagram:



Let us amalgamate i_1 with $i_2 \circ h_{\eta\restriction\gamma}$ over $N_{\eta\restriction\gamma}$; we obtain an extension M of $M_{\xi+1}$ with a copy of $N_{\eta\restriction\beta}$ in it over $h''_{\eta\restriction\gamma} N_{\eta\restriction\gamma}$.

Since $M_{\xi+1}$ is $\leq\mu$-e.c., $N_{\eta\restriction\beta}$ is of power μ, we can realize the $\mathcal{F}$-diagram of $N_{\eta\restriction\beta}$ over $h''_{\eta\restriction\gamma} N_{\eta\restriction\gamma}$; in this way we get g making the square commute. Finally, as in Case 3.1, we can define N_η and $h_\eta : N_\eta \to M_{\xi+1}$ with range(h_η) containing $\boldsymbol{a}_\xi$ so that the triangle commutes. This completes the inductive definition, and the proof of the lemma. □

Lemma 1.12. *Assume $\langle N_\eta : \eta \in {}^{<\mu^+}2 \rangle$ is a $<$-continuous tree of models of power μ, and $S \subset \mu^+$ is a stationary set for which (weak) $\diamondsuit_S$ holds, and such that, for any*

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$\eta \in {}^{<\mu^+}2$ *with* length$(\eta) \in S$, $N_{\eta^\wedge\langle 0\rangle}$ *and* $N_{\eta^\wedge\langle 1\rangle}$ *cannot be amalgamated over* N_η. *Then there is no model* M *of power* μ^+ *such that every* $N_\nu \stackrel{\text{def}}{=} \bigcup_{\alpha<\mu^+} N_{\nu|\alpha}$, $\nu \in {}^{\mu^+}2$, *can be* $\mathcal{F}$*-embedded into* M.

Proof (See [11], proof of Theorem 3.5, especially the lower half of p. 436). With μ a cardinal, $S \subset \mu^+$, we consider the following set-theoretic principle:

Θ_S: For any system $\langle f_\nu \rangle_{\nu\in{}^{\mu^+}2}$ of functions $f_\nu : \mu^+ \to \mu^+$, there is $\nu \in {}^{\mu^+}2$ such that the set

$$\{\alpha \in S : (\exists \nu' \in {}^{\mu^+}2)(\nu' \mid \alpha = \nu \mid \alpha \mathbin{\&} f_{\nu'} \mid \alpha = f_\nu \mid \alpha \mathbin{\&} \nu'(\alpha) \neq \nu(\alpha))\}$$

is stationary.

Θ_S is a consequence of $\Diamond_S$ by arguments in [3]: the one showing that Φ follows from $\Diamond$ (p. 239 *loc.cit.*), and the one showing that a variant of Φ, 4.1(2) *loc.cit.*, implies Θ; see 6.1, p. 246 *loc.cit.* Thus, for our $S \subset \mu^+$ in the lemma, by the assumption of $\Diamond_S$, Θ_S holds.

Turning to the proof of the lemma, note first that we can easily arrange that the underlying set of each N_ν, $\nu \in {}^{\mu^+}2$, is identical to μ^+, the set of all ordinals less than μ^+. Suppose, contrary to the assertion of the lemma, that there is a model $M \in K$ with underlying set $|M| = \mu^+$ and for each $\mu \in {}^{\mu^+}2$ there is an $\mathcal{F}$-embedding $f_\nu : N_\nu \to M$. Then, in particular, $f_\nu : \mu^+ \to \mu^+$. By applying Θ_S, there is $\nu \in {}^{\mu^+}2$ such that the set displayed in the statement of Θ_S is stationary.

Now, note that the set

$$C = \{\alpha < \mu^+ : |N_{\nu|\alpha}| = \alpha\}$$

is a cub, by the continuity of the tree $\langle N_\eta \rangle_{\eta\in{}^{<\mu^+}2}$ of models. Intersecting C with the above stationary set, we see that there are $\alpha \in S$ and $\nu' \in {}^{\mu^+}2$ such that

$$\nu' \mid \alpha = \nu \mid \alpha, \qquad f_{\nu'} \mid \alpha = f_\nu \mid \alpha,$$
$$\nu'(\alpha) \neq \nu(\alpha), \qquad |N_{\nu|\alpha}| = \alpha.$$

If, e.g., $\nu(\alpha) = 0$, $\nu'(\alpha) = 1$, then for $\eta = \nu \mid \alpha = \nu' \mid \alpha$ and for $f^0 = f_\nu \parallel M_{\eta^\wedge\langle 0\rangle}|$, $f^1 = f_\nu \parallel M_{\eta^\wedge\langle 1\rangle}|$ we have $f^0 \mid M_\eta = f^1 \mid M_\eta$, $f^0 : M_{\eta^\wedge\langle 0\rangle} \to M$, $f^1 : M_{\eta^\wedge\langle 1\rangle} \to M$; that is, we have an amalgamation of $M_{\eta^\wedge\langle 0\rangle}$ and $M_{\eta^\wedge\langle 1\rangle}$ over M_η into M, contradicting length$(\eta) = \alpha \in S$, and the assumption on 'non-amalgamation' in the lemma. □

Proposition 1.13. *Suppose* $\lambda > \beth_{\kappa+1}(\kappa')$, *and* K *is categorical in* λ. *Then* K *has the* AP.

Proof. Let us write $\mu = \beth_\kappa(\kappa')$. If λ has cofinality $\geqslant\kappa$, the assertion is true by 1.9. Assume that λ is a limit cardinal of cofinality $\sigma < \kappa$.

Assume there is some $M \in K$ which is not an a.b. Then the model $M_\lambda \in K_\lambda$ is not an a.b. either as shown by the following argument. We have contradictory $\sigma_1(\boldsymbol{a})$, $\sigma_2(\boldsymbol{a})$ Σ_1-sentences, both consistent with M ($\boldsymbol{a} \in M$) (see 1.7); we can write down the theory in $(L')_{\kappa\omega}$ that expresses, using, in addition to L, $\boldsymbol{c}$ (for $\boldsymbol{a}$), new

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individual constants for the existentially quantified variables y_i of $\exists y_i\, \psi = \sigma_i$, and also using unary predicates $\tilde{M}$, $\tilde{M}_1$, $\tilde{M}_2$, that $\tilde{M}$, $\tilde{M}_1$, $\tilde{M}_2$ are models of T, $\tilde{M} < \tilde{M}_1$, $\tilde{M} < \tilde{M}_2$, c is in $\tilde{M}$, $\sigma_i(c)$ is true in $\tilde{M}_i$ witnessed by the appropriate constants for the y_i. This theory is satisfiable so that $|\tilde{M}| \geq \kappa$ (as given by our M above), hence by $<\kappa$-compactness and dLST, it has a model in which $\tilde{M}$ has cardinality equal to λ. This shows that there is a model in K_λ which is not an a.b.

By 1.10, we have $\langle M_i : i < \sigma \rangle$ satisfying the requirements in the second sentence of 1.11. Clearly, $\mu^{<\kappa} = \mu$. By the choice of μ and the Appendix, we have the modified square-system $\langle C_\alpha \rangle_{\alpha \in S}$ on μ^+ such that $S^* = \{\alpha \in S : \text{opt } C_\alpha = \sigma\}$ is stationary, and $\diamondsuit_{S^*}$ holds. By 1.11, we have $\langle N_\eta : \eta \in {}^{<\mu^+}2 \rangle$ as described there. We put $N_\nu = \bigcup_{\alpha < \mu^+} N_{\nu | \alpha}$ for each $\nu \in {}^{\mu^+}2$; $|N_\nu| \leq \mu^+$.

Since T has arbitrarily large models, there is an E-M defining scheme Φ over T (see the introductory part of Section 3) in a language $L' = L_{\mathrm{Sk}}$ of size $\leq \kappa'$ (see 3.1). Consider the model $M^* = \mathrm{EM}(\lambda, \Phi)$ (with λ denoting the well-ordering of type λ); of course, by categoricity, $M^* \mid L \cong M_\lambda$. By uLSTT, every N_ν can be embedded into $M^* \mid L$; hence for each ν, there is a subset X_ν of λ of size $\leq \mu^+$ such that N_ν can be $\mathcal{F}$-embedded into $\mathrm{EM}(X_\nu, \Phi) \mid L$. There a linear ordering I of size μ^+ such that every well-ordering of size μ^+ can be embedded in an order-preserving way into I (take I to be the set of finite sequences of ordinals $<\mu^+$, and define $<$ on I by $s < t \Leftrightarrow$ either t is a proper initial segement of s or $\exists \alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$ such that $s \mid \alpha = t \mid \alpha$ and $s(\alpha) < t(\alpha)$). Hence, every N_ν can be $\mathcal{F}$-embedded into $\mathrm{EM}(I, \Phi) \mid L$. But this contradicts 1.12. □

2. Saturation and types

We continue to work in the context of the fixed κ, L, $\mathcal{F}$, T, κ' and K. Let us formalize our assumptions for the rest of the paper (except the summary, Section 5).

Assumption. κ is compact; K has AP and JEP.

By convention, 'model' means an element of K. The compactness assumption is not used until 2.7.

First, we restate the essentially classical theory of universal-homogeneous models in our context. Then we introduce the abstract notions of type and saturation. Finally, using the compactness assumption, we arrive at the 'set-of-formulas' definition of type. In all this, AP (with JEP) is the basic tool. For a more general treatment, with less use of AP, see Part II, §3 of [15].

Definition 2.1. (i) Let λ be a cardinal, $\lambda > \kappa'$ and let $M \in K$. M is *λ-universal-homogeneous* (λ-u-h) if the following holds: whenever $N_1, N_2 \in K_{<\lambda}$, $f: N_1 \rightarrow N_2$

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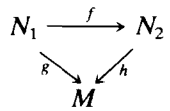
and $g: N_1 \to M$, then there is $h: N_2 \to M$ such that $g = h \circ f$:



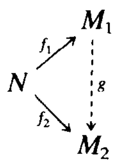
(remember: all arrows are $\mathscr{F}$-embeddings).

(ii) M is *universal-homogeneous* (u-h) if it is $\|M\|$-u-h.

Proposition 2.2 (essentially [5], [8]). (i) *A λ-u-h model is $<\lambda$-e.c. ($\lambda > \kappa'$).*

(ii) *If M is λ-u-h, then M is $\leq\lambda$-universal: any model of power $\leq\lambda$ has an $\mathscr{F}$-embedding into M.*

(iii) *If M_1, M_2 are both u-h, both of power λ, $\|N\| < \lambda$, then for any $f_1: N \to M_1$, $f_2: N \to M_2$, there is an isomorphism $g: M_1 \xrightarrow{\cong} M_2$ with $f_2 = g \circ f_1$:*



(iv) *For any $\lambda > \kappa'$, up to isomorphism, there is at most one u-h model of cardinality λ.*

(v) *If M is u-h, $N_1 < M$, $N_2 < M$, $\|N_1\| < \|M\|$, then any isomorphism $N_1 \xrightarrow{\cong} N_2$ can be extended to an automorphism of M.*

(vi) *If $\theta > \kappa'$, and either $\theta = \lambda^+ = 2^\lambda$ or θ is strongly inacessible, then there is a u-h model of power θ.*

Proof. (i) Suppose M is λ-u-h, $\Phi(\boldsymbol{x})$ is a set of $\mathscr{F}$-formulas over M, $|\Phi(\boldsymbol{x})| < \lambda$, $M <_{\mathscr{F}} N$, $N \models \exists \boldsymbol{x} \bigwedge \Phi(\boldsymbol{x})$. Let, by dLST, M_0 be such that $M_0 < M$, $\|M_0\| < \lambda$, and all parameters from M in Φ belong to M_0. Let N_0 be such that $N_0 < N$, $\|N_0\| < \lambda$, $M_0 < N_0$, and N_0 contains some $\boldsymbol{a}$ for which $N \models \bigwedge \Phi[\boldsymbol{a}]$. By M being λ-u-h, there is $h: N_1 \to M$ which is the identity on N_0. Clearly, $M \models \bigwedge \Phi[h(\boldsymbol{a})]$, hence $M \models \exists \boldsymbol{x} \bigwedge \Phi(\boldsymbol{x})$ as desired.

(ii) Let N be any model of power $\leq\lambda$, and let us write it in the form $N = \bigcup^{c}_{i<\lambda} N_i$ with $\|N_i\| < \lambda$ for all $i < \lambda$ (this is possible by dLST). Let M_0^0 be any model with $M_0^0 < M$, $\|M_0^0\| < \lambda$. By JEP, there is M_0^1 with $M_0^0 \lesssim M_0^1$, $N_0 \lesssim M_0^1$. Since M is λ-u-h, there is $M_0 < M$ with $M_0 \cong M_0^1$, and hence we have some $f_0: N_0 \to M$. Now, by induction on i, $0 < i < \lambda$, we define $f_i: N_i \to M$ such that f_i extends f_j whenever $j < i < \lambda$. For limit i, we put $f_i = \bigcup_{j<i} f_j$ (see 1.1(iii)). For $i = j + 1$, we apply the condition in 2.1 with N_j in place of N_1, N_i in place of N_2, the inclusion as f, f_j as g, to obtain f_i as h. $\bigcup_{i<\lambda} f_i$ will be the required embedding of N into M.

(iii) We enumerate $|M_1|$, $|M_2|$, as $|M_l| = \{a_i^l : i < \lambda\}$ ($l = 1, 2$), and we define, by induction on $i < \lambda$, models $N_i^l < M_l$ of power $\kappa' + |i|$ with isomorphisms $g_i: N_i^1 \xrightarrow{\cong}$

N_i^2 such that $\langle N_i^l \rangle_{i<\lambda}$ is a $<$-continuous chain, and $a_i^l \in N_{i+1}^l$. For $i=0$, we put $N_0^l = f_l'' N$, $g_0 = f_2 \circ f_1^{-1}$. For i limit, $N_2^l = \bigcup_{j<i} N_j^l$, $g_i = \bigcup_{j<i} g_j$; note that $\|N_i^l\| \leqslant \kappa' + |i|$ as a consequence. For $i = j+1$, i even, by dLST choose $N_i^1 < M$ with $\|N_i^1\| \leqslant \kappa' + |i|$ such that $N_j^1 < N_i^1$, $a_j^1 \in |N_i^1|$; by M being λ-u-h and $\|N_i^1\| < \lambda$, we can choose $g_i : N_i^1 \to M_2$ extending g_j. For $i = j+1$, i odd, we do the similar thing with the roles of M_1, M_2 interchanged.

Clearly, $g = \bigcup_{i<\lambda} g_i$ is the desired isomorphism.

(iv) Let M_1, M_2 be u-h of cardinality λ. By dLST, let N be a model of power κ'. Since $\lambda > \kappa'$, by (ii) we have that $N \precsim M_1$, $N \precsim M_2$. Applying (iii), we obtain an isomorphism $M_1 \xrightarrow{\cong} M_2$.

(v) Apply (iii) with $M_1 = M_2 = M$.

(vi) The proof is similar to that of 1.3. First, given any $M \in K$ of power $\leqslant \theta$, we construct $M^* > M$ of power θ such that for any $N_1, N_2 \in K_{<\theta}$, any diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ M & \xrightarrow[l]{} & M^* \end{array}$$

with l inclusion can be completed as shown with some h to commute. To this end, note that the cardinality of all isomorphism types of triples $\langle N_1, N_2, f : N_1 \to N_2 \rangle$ with $N_1, N_2 \in K_{<\theta}$ is $\leqslant \theta$ ($\langle N_1, N_2, f : N_1 \to N_2 \rangle$ and $\langle N_1', N_2', f : N_1' \to N_2' \rangle$ are isomorphic if there are $f_l : N_l \xrightarrow{\cong} N_l'$ such that

$$\begin{array}{ccc} N_1 & \xrightarrow{f_1} & N_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ N_2 & \xrightarrow[f_2]{} & N_2' \end{array}$$

commutes). Let $\langle N_1^i, N_2^i, f^i \rangle_{i<\theta}$ enumerate a set of representatives of all of these isomorphism types. We define the $<$-continuous chain $\langle M_i \rangle_{i<\theta}$ by putting $M_0 = M$, $M_i = \bigcup_{j<i} M_j$ for i limit, and for $i = j+1 < \theta$, applying amalgamation: we have

$$\begin{array}{ccc} N_i^j & \xrightarrow{f^j} & N_2^j \\ {\scriptstyle g}\downarrow & \searrow^{l_j \circ g} & \\ M_0 & \xrightarrow[l_j]{\text{incl.}} & M_j; \end{array}$$

by amalgamating $l_j \circ g$ and f, find M_i with $M_j < M_i$, and $h : N_2^j \to M_i$; M_i can be chosen to be of power $<\theta$ by dLST. Having defined $\langle M_i \rangle_{i<\theta}$, $M^* = \bigcup_{i<\theta} M_i$ obviously satisfies the requirements.

Having defined M^* for each $M \in K_\theta$, we define the $<$-continuous chain $\langle M_i \rangle_{i<\theta}$ by choosing an arbitrary $M_0 \in K_\theta$, and putting $M_{j+1} = (M_j)^*$. Quite clearly, $\bigcup_{i<\theta} M_i$ is u-h of power θ. $\square$

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It is convenient to use a large u-h model $\mathscr{C}$ so that every model we might be interested in is an $\mathscr{F}$-submodel of $\mathscr{C}$. We assume that θ is a 'large' strongly inaccessible cardinal greater than any cardinal we are interested in, and we define $\mathscr{C}$ as the u-h model of power θ (see 2.2(vi)). By 2.2(ii), every model of power $<\theta$ has an isomorphic copy which is an $\mathscr{F}$-submodel of $\mathscr{C}$. By convention, from now on, a 'model' is an $\mathscr{F}$-submodel of $\mathscr{C}$ of power $<\theta$. Moreover, $A, B, \ldots$ denote subsets of $|\mathscr{C}|$ of cardinality $<\theta$.

Using $\mathscr{C}$, we define an abstract notion of 'type'. For any A ($\subset |\mathscr{C}|$, $|A| < \theta$), and any ordinal α, we define the equivalence relation $\overset{\alpha}{\underset{A}{\sim}}$ on ${}^{\alpha}|\mathscr{C}|$ by

$$\boldsymbol{b} \overset{\alpha}{\underset{A}{\sim}} \boldsymbol{c} \quad \Leftrightarrow \quad \exists h \in \mathrm{Aut}_A(\mathscr{C})\, h(\boldsymbol{b}) = \boldsymbol{c}$$

($\mathrm{Aut}_A(\mathscr{C})$ is the collection of all A-automorphisms of $\mathscr{C}: h: \mathscr{C} \xrightarrow{\cong} \mathscr{C}$ with $h \restriction A =$ identity). It is clear that $\overset{\alpha}{\underset{A}{\sim}}$ is an equivalence. An *α-type over A* is, by definition, an equivalence class of $\overset{\alpha}{\underset{A}{\sim}}$. The set of α-types over A is denoted by $S^{\alpha}(A)$;

$$S(A) = \bigcup_{\alpha \in \mathrm{Ord}} S^{\alpha}(A); \qquad S^{<\beta}(A) = \bigcup_{\alpha < \beta} S^{\alpha}(A);$$

$\underset{A}{\sim}$ is the union of all the $\overset{\alpha}{\underset{A}{\sim}}$, $\alpha \in \mathrm{Ord}$:

$$\boldsymbol{b} \underset{A}{\sim} \boldsymbol{c} \quad \Leftrightarrow \quad \exists \alpha \in \mathrm{Ord}\, \boldsymbol{b} \overset{\alpha}{\underset{A}{\sim}} \boldsymbol{c}.$$

A type $p = \boldsymbol{c}/\overset{\alpha}{\underset{A}{\sim}} \in S(A)$ is *realized* in $B \supseteq A$ if there is $\boldsymbol{b} \in B$ (meaning: $\boldsymbol{b} \in {}^{\alpha}B$) with $p = \boldsymbol{b}/\overset{\alpha}{\underset{A}{\sim}}$ (i.e., $\boldsymbol{b} \overset{\alpha}{\underset{A}{\sim}} \boldsymbol{c}$). Given $p \in S(A)$, and $B \subseteq A$, then $p \restriction B = \boldsymbol{c}/\underset{B}{\sim}$ for any (some) $\boldsymbol{c}$ for which $p = \boldsymbol{c}/\underset{A}{\sim}$ (clearly, $p \restriction B$ is well-defined). Note that, in our context, any type $p \in S(A)$ can be extended, usually in more than one way, to a type over any superset B of A: for any $B \supseteq A$, there is $q \in S(B)$ with $q \restriction A = p$.

Suppose $f: A \xrightarrow{\cong} B$ is a bijection that can be extended to an automorphism of $\mathscr{C}$ (such an f may be called an 'elementary mapping'). Then f acts on $S(A)$: for every $p \in S(A)$, $f(p) \in S(B)$ can be defined by $f(p) = h(\boldsymbol{c})/\underset{B}{\sim}$, for any (some) $h \in \mathrm{Aut}(\mathscr{C})$ extending f (this is easily seen). This applies, in particular, if $f: M \xrightarrow{\cong} N$ is any ($\mathscr{F}$-) isomorphism of models (see 2.2(v)).

We write $p = \mathrm{tp}(\boldsymbol{b}/A)$ instead of $p = \boldsymbol{b}/\underset{A}{\sim}$.

Proposition 2.2′. *Suppose $\langle M_n \rangle_{n<\omega}$ is an increasing $<$-chain of models, $M = \bigcup_{n<\omega} M_n$, $p_n \in S(M_n)$, $p_n = p_{n+1} \restriction M_n$ for $n < \omega$. Then there is a type $p \in S(M)$ such that $p \restriction M_n = p_n$ for all $n < \omega$.*

Proof. Let $\boldsymbol{c}$ realize p_0. By induction on $n < \omega$, we construct an elementary mapping $f_n: M_n \xrightarrow{\cong} M_n^*$, f_{n+1} extending f_n, such that $\boldsymbol{c}$ realizes $f_n(p_n)$ for each $n < \omega$. For $n = 0$, we put $f_0 =$ identity on M_0. Suppose we have constructed $f_n: M_n \xrightarrow{\cong} M_n^*$ such that $\boldsymbol{c}$ realizes $f_n(p_n)$. Let $\boldsymbol{c}_{n+1}$ realize p_{n+1}. Since $p_n = p_{n+1} \restriction M_n$, $\boldsymbol{c}_{n+1}$ realizes p_n. Thus, we have that the mapping that is defined on M_n as f_n, and takes $\boldsymbol{c}_{n+1}$ to $\boldsymbol{c}$, is elementary. Let h_{n+1} be an automorphism of $\mathscr{C}$ extending the latter mapping; $h_{n+1} \restriction M_n = f_n$, $h_{n+1}(\boldsymbol{c}_{n+1}) = \boldsymbol{c}$. Define $f_{n+1} = h_{n+1} \restriction M_{n+1}$. Clearly, $f_n \subset f_{n+1}$. Also, since $\boldsymbol{c}_{n+1}$ realizes p_{n+1}, $\boldsymbol{c} = h_{n+1}(\boldsymbol{c}_{n+1})$ realizes $h_{n+1}(p_{n+1}) = f_{n+1}(p_{n+1})$. This completes the recursive construction.

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The mapping $f=\bigcup_{n<\omega}f_n : M \overset{\equiv}{\Rightarrow} M^*$ is elementary, and c realizes $f(p_n)$ for every $n<\omega$. Hence, for any automorphism h of $\mathscr{C}$ extending f, $h^{-1}(c)$ realizes p_n for each $n<\omega$. $p=\text{tp}(h^{-1}(c)/M)\in S(M)$ will then satisfy $p \mid M_n = p_n$ for all $n<\omega$. □

Definition 2.3. *Let $\lambda>\kappa'$. M is λ-saturated* if for all $N<M$, $\|N\|<\lambda$, every $p\in S^1(N)$ is realized in M. M is *saturated* if it is $\|M\|$-saturated.

Proposition 2.4. *Let $\lambda>\kappa'$, M any model. M is λ-saturated iff it is λ-universal-homogeneous. Moreover, if M is λ-u-h, for every $A\subset|M|$ of power $<\lambda$, every type in $S^{\leq\lambda}(A)$ is realized in M.*

Proof (see Chapter II, §3 of [15]). Suppose M is λ-u-h, $A\subset|M|$, $|A|<\lambda$, $p\in S^{\leq\lambda}(A)$. Let $M_0<M$ be of power $<\lambda$ with $A\subset|M_0|$, and let $N>M_0$ contain a realization $\boldsymbol{b}$ of p, $\|N\|\leq\lambda$. By 2.2(ii), N has an $\mathscr{F}$-embedding f into M over M_0; f can be extended to an automorphism h of $\mathscr{C}$; clearly, $h(\boldsymbol{b})$ is a realization of p, $h(\boldsymbol{b})\in M$.

Conversely, assume that M is λ-saturated, and, to show that M is λ-u-h, let N_1, N_2 be models ($<\mathscr{C}$) of power $<\lambda$, $N_1<N_2$, $N_1<M$. Let $|N_2|=\{a_i : i<\alpha\}$, $\alpha<\lambda$. By induction on $i<\alpha$, we define $b_i\in M$ and the $<$-continuous chain $\langle M_i\rangle_{i<\alpha}$ of $\mathscr{F}$-submodels of M such that $\|M_i\|\leq|\alpha|+|i|$, $b_i\in M_{i+1}$, and also the isomorphisms $f_i : M_i \overset{\cong}{\Rightarrow} M_i'$ with $f_j\subset f_i$ for $j<i$ such that $f_i(b_i)=a_i$, as follows. We let $M_0=M_0'=N_1$, $f_0=\text{id}_{N_1}$; for limit i, we let

$$M_i=\bigcup_{j<i}M_j,\qquad M_i'=\bigcup_{j<i}M_j',\qquad f_i=\bigcup_{j<i}f_j.$$

Having defined M_j, M_j', f_j, we define b_j as an element realizing the type $f_j^{-1}(\text{tp}(a_j/M_j'))\in S^1(M_j)$ (b_j exists since M is λ-saturated), define M_{j+1} to be a submodel of M of cardinality $\leq|\alpha|+|j+1|$ containing M_j and b_j, and define f_{j+1} as the restriction to M_{j+1} of an automorphism of $\mathscr{C}$ extending the mapping $f_j\cup\{b_j\mapsto a_j\}$. Clearly, $(\bigcup_{i<\alpha}f_i)^{-1}$ maps N_2 into M over N_1. □

Stability, in the usual sense, allows us to construct saturated (hence u-h) models in prescribed cardinalities.

Definition 2.5. Let $\mu\geq\kappa'$. T (or K) is *μ-stable* (*stable in μ*) if for all $M\in K_\mu$, $S^1(M)$ has cardinality μ.

Proposition 2.6. *Suppose $\kappa'<\mu$, μ is regular, $\mu\leq\lambda$, and K is stable in λ. Then there is $M\in K_\lambda$ which is μ-saturated (hence μ-u-h).*

Proof. Given any $N\in K_\lambda$, we let N^* be a model $\in K_\lambda$ such that $N<N^*$ and all 1-types over N are realized in N^*; this is clearly possible by λ-stability. Next,

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define a $<$-continuous chain $\langle N_i \rangle_{i<\mu}$ of length μ such that $N_{i+1} = (N_i)^*$, and put $M = \bigcup_{i<\mu} N_i$. Given any $N < M$ of cardinality $<\mu$, and $p \in S^1(N)$, by the regularity of μ, there is $i < \mu$ with $N < N_i$; let $q \in S^1(N_i)$ extend p; by the construction, q, and hence also p, is realized in $N_{i+1} < M$. $\square$

So far, the compactness of κ was not used; using it, we can relate types with formulas in the usual style. In fact, every type $p \in S(A)$ can be specified by a set of Σ_1-formulas over A. To see this, we make some preparations.

Let us call a mapping f with $A \overset{\text{def}}{=} \text{dom}(f) \subset |\mathscr{C}|$ and $B \overset{\text{def}}{=} \text{range}(f) \subset |\mathscr{C}|$ Σ_1-*elementary* if ($|A|$, $|B| < \theta$, and) for every Σ_1-formula $\varphi(\boldsymbol{x})$ (without parameters) and any $\boldsymbol{a} \in A$ (appropriate for $\boldsymbol{x}$), $\mathscr{C} \models \varphi[\boldsymbol{a}]$ implies $\mathscr{C} \models \varphi[f(\boldsymbol{a})]$. (Henceforth, we write $\models \varphi[\boldsymbol{a}]$ for $\mathscr{C} \models \varphi[\boldsymbol{a}]$, etc.) We write $f: A \xrightarrow[\Sigma_1]{\cong} B$ to indicate that f is a Σ_1-elementary map, $\text{dom}(f) = A$, $\text{range}(f) = B$. Clearly, any restriction of an automorphism (to a subset of $\mathscr{C}$ of power $<\theta$) is a Σ_1-elementary map. In fact, we have the converse.

Proposition 2.7. *Any Σ_1-elementary map can be extended to an automorphism of $\mathscr{C}$.*

Proof. Suppose $f: A \xrightarrow{\cong} B$ is Σ_1-elementary. Let M, N be such that $A \subset M$, $B \subset N$. Consider the set

$$\Sigma = \text{Diag}_{\mathscr{F}}(M) \cup \text{Diag}_{\mathscr{F}}(N)$$

of sentences, where we use the same constant for $a \in A$ and $f(a) \in B$ in the two diagrams, for each $a \in A$, but otherwise, use distinct constants. The $<\kappa$-consistency of Σ follows immediately from the assumption that f is Σ_1-elementary, by quantifying out the constants in $|M| - A$, and modelling the resulting Σ_1-sentences in N. By the compactness of κ, Σ has a model, say P; P is a model of T, but not yet an $\mathscr{F}$-submodel of $\mathscr{C}$. At any rate, we have $\mathscr{F}$-embeddings $g: M \to P$ and $h: N \to P$ with $g(a) = h(f(a))$ for $a \in A$. By $\mathscr{C}$ being u-h, P can be mapped into $\mathscr{C}$ over N, i.e., we may assume that $P < \mathscr{C}$, and that h is an inclusion $N < P$. Hence, $g: M \to N$ extends f. By 2.2(iv), $g: M \xrightarrow{\cong} g''M < N$ can be extended to an automorphism of $\mathscr{C}$. $\square$

Corollary 2.8. *If $f: A \xrightarrow[\Sigma_1]{\cong} B$, then $f^{-1}: B \xrightarrow[\Sigma_1]{\cong} A$ (the notion of Σ_1-elementary map is symmetric).* $\square$

It follows from the last proposition, and a remark made above, that any Σ_1-elementary mapping f acts on types over its domain: if $p \in S(\text{dom}(f))$, then $f(p) \in S(\text{range}(f))$ is well-defined ($f(p) = \text{tp}(h(\boldsymbol{a})/\text{range}(f))$ if $p = \text{tp}(\boldsymbol{a}/\text{dom}(f))$ and $h \in \text{Aut}(\mathscr{C})$ extending f). If $A \subset \text{dom}(f)$ and $p \in S(A)$, $f(p) = (f \restriction A)(p)$.

It also follows that types are determined by the Σ_1-formulas they 'imply'. For $p \in S^\alpha(A)$, $p = \text{tp}(\boldsymbol{b}/A)$, let $\boldsymbol{x}$ be a tuple of length α of variables, fixed once α is

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given, not depending on A or p, and let $\Phi_p \stackrel{\text{def}}{=} \{\varphi(\boldsymbol{x}) \in (\Sigma_1)_{\boldsymbol{x}}(A) : \models \varphi[\boldsymbol{b}]\}$. (Here, $(\Sigma_1)_{\boldsymbol{x}}(A)$ denotes the set of Σ_1-formulas with parameters from the set A, and with the free variables $\boldsymbol{x}$ at most.). It is clear that Φ_p is well-defined (it does not depend but on p). Also, for $p, q \in S(A)$, $\Phi_p = \Phi_q$ implies $p = q : \Phi_p = \Phi_q$ says that, with $\boldsymbol{b} = \langle b_i \rangle_{i<\alpha}$, $\boldsymbol{c} = \langle c_i \rangle_{i<\alpha}$, realizing p and q, respectively, $f \stackrel{\text{def}}{=} \mathrm{id}_A \cup \{\langle b_i, c_i \rangle : i < \alpha\}$ is a Σ_1-elementary mapping, hence by 2.7, it can be extended to an automorphism h of $\mathscr{C}$; which means that $h(\boldsymbol{b}) = \boldsymbol{c}$ for an A-automorphism h of $\mathscr{C}$, i.e., $p = q$.

For $A \subset \mathscr{C}$, $\mathrm{Diag}_{\Sigma_1}(A)$ denotes the set of Σ_1-sentences over A that are true in $\mathscr{C}$. In particular, always $T \subset \mathrm{Diag}_{\Sigma_1}(A)$.

Lemma 2.9. *Suppose $\Phi \subset (\Sigma_1)_{\boldsymbol{x}}(A)$ is $<\kappa$-consistent, and $\mathrm{Diag}_{\Sigma_1}(A) \subset \Phi$. Then there is $\boldsymbol{b} \in \mathscr{C}$ such that $\Phi \subset \Phi_{\mathrm{tp}(\boldsymbol{b}/A)}$.*

Proof. Let M be any model, $A \subset |M|$, $M < \mathscr{C}$. Consider the following set of formulas:

$$\mathrm{Diag}_{\mathscr{F}}(M) \cup \Phi.$$

This is consistent, since any $<\kappa$-subset of $\mathrm{Diag}_{\mathscr{F}}(M)$ gives rise, after quantifying out the constants in $|M| - A$, to a Σ_1-sentence in $\mathrm{Diag}_{\Sigma_1}(A)$, hence in Φ. Let $(N, \bar{a}, \boldsymbol{b})_{a \in |M|}$ model $\mathrm{Diag}_{\mathscr{F}}(M) \cup \Phi$, with $\boldsymbol{b}$ standing for $\boldsymbol{x}$. Using the universal-homogeneity of $\mathscr{C}$, we may arrange that $N < C$ and $\bar{a} = a$ for all $a \in |M|$. Then $N \models \varphi[\boldsymbol{b}]$ for $\varphi(\boldsymbol{x}) \in \Phi$, and since φ is Σ_1 and $N < \mathscr{C}$, $\mathscr{C} \models \varphi[\boldsymbol{b}]$, hence $\Phi \subset \Phi_{\mathrm{tp}(\boldsymbol{b}/A)}$ as desired. □

Next, we characterize the sets arising as Φ_p.

$\Phi \subset (\Sigma_1)_{\boldsymbol{x}}(A)$ is *maximal consistent over* A if $\mathrm{Diag}_{\Sigma_1}(A) \subset \Phi$, and for any $\psi(\boldsymbol{x}) \in (\Sigma_1)_{\boldsymbol{x}}(A)$, if $\Phi \cup \{\psi(\boldsymbol{x})\}$ is consistent (satisfiable), then $\psi(\boldsymbol{x}) \in \Phi$.

Proposition 2.10. *$\Phi \subset (\Sigma_1)_{\boldsymbol{x}}(A)$ is equal to Φ_p for a (necessarily unique) $p \in S(A)$ iff Φ is maximal consistent over A.*

Proof. Assume first that $\Phi = \Phi_p$, $p = \mathrm{tp}(\boldsymbol{b}/A)$. It is clear that $\mathrm{Diag}_{\Sigma_1}(A) \subset \Phi$. Suppose $\psi(\boldsymbol{x}) \in (\Sigma_1)_{\boldsymbol{x}}(A)$ and $\Phi \cup \{\psi(\boldsymbol{x})\}$ is consistent. By 2.9, there is $\boldsymbol{c} \in \mathscr{C}$ satisfying all formulas in $\Phi \cup \{\psi(\boldsymbol{x})\}$. It follows that the mapping f for which $f \mid A = \mathrm{id}_A$ and $f(\boldsymbol{b}) = \boldsymbol{c}$ is Σ_1-elementary. But then, by 2.8, f^{-1} is Σ_1-elementary as well. Since $\models \psi[\boldsymbol{c}]$, it follows that $\models \psi[\boldsymbol{b}]$, i.e. $\psi \in \Phi$.

Conversely, assume $\Phi \subset (\Sigma_1)_{\boldsymbol{x}}(A)$ is maximal consistent over A. By 2.9, there is $\boldsymbol{b}$ such that $\Phi \subset \Phi_{\mathrm{tp}(\boldsymbol{b}/A)}$. Clearly, $\Phi_{\mathrm{tp}(\boldsymbol{b}/A)}$ is consistent; by the maximality assumption, $\Phi = \Phi_{\mathrm{tp}(\boldsymbol{b}/A)}$. □

From now, we will identify p with Φ_p, for every $p \in S(A)$, $A \subset \mathscr{C}$. For any tuple $\boldsymbol{x}$ of (distinct) variables, $S_{\boldsymbol{x}}(A)$ denotes the set of all $p \subset (\Sigma_1)_{\boldsymbol{x}}(A)$ that are

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maximal consistent over A. $S^{\alpha}(A)$ is the same as $S_x(A)$, for a (definite) $\boldsymbol{x}$ with length$(\boldsymbol{x}) = \alpha$. $\operatorname{tp}(\boldsymbol{b}/A) = \{\varphi(\boldsymbol{x}) \in (\Sigma_1)_x(A) : \vDash \varphi[\boldsymbol{b}]\}$; of course, this is the same as $\Phi_{\operatorname{tp}(\boldsymbol{b}/A)}$, with $\operatorname{tp}(\boldsymbol{b}/A)$ in the previous sense. For $p \in S_x(A))$ and $\boldsymbol{b}$ matching $\boldsymbol{x}$ (for length), to say that $\boldsymbol{b}$ *realizes* p is to say that $\boldsymbol{b}$ satisfies all formulas in p; this corresponds exactly to the previous 'abstract' notion because of the maximal consistency over A of each $p \in S(A)$.

For $p \in S_x(A)$ and $B \subset A$, $p \mid B = p \cap (\Sigma_1)_x(B)$, and this is in agreement with the 'abstract' notion of $p \mid B$. Finally, for a Σ_1-elementary f, and $p \in S_x(A)$, $f(p) = \{\varphi(\boldsymbol{x}, f(\boldsymbol{a})) : \varphi(\boldsymbol{x}, \boldsymbol{y}) \in (\Sigma_1)_{x,y}(\emptyset)$, $\boldsymbol{y}$ any tuple of variables, $\boldsymbol{a} \in A$ matching $\boldsymbol{y}\}$.

Corollary 2.11. *A set $p \subset (\Sigma_1)_x(A)$ belongs to $S_x(A)$ iff for all subsets $\boldsymbol{y}$ of $\boldsymbol{x}$ of size $<\kappa$ and for all $B \subset A$ of size $<\kappa$, we have that $(p \mid \boldsymbol{y}) \mid B \in S_y(B)$. $((p \mid \boldsymbol{y}) \mid B \stackrel{\text{def}}{=} p \cap (\Sigma_1)_y(B).)$*

Proof. The 'only if' direction is clear. Assume the condition after 'iff', and we show that p is maximal consistent over A. Since $\operatorname{Diag}_{\Sigma_1}(A) = \bigcup \{\operatorname{Diag}_{\Sigma_1}(B) : B \in \mathcal{P}_{<\kappa}(A)\}$, we clearly have $\operatorname{Diag}_{\Sigma_1}(A) \subset p$. Since

$$p = \bigcup \{(p \mid \boldsymbol{y}) \mid B : \boldsymbol{y} \in \mathcal{P}_{<\kappa}(\boldsymbol{x}),\ B \in \mathcal{P}_{<\kappa}(A)\}$$

and every subset of p of size $<\kappa$ is contained in a term of the union, p is $<\kappa$-consistent. If $p \cup \{\psi(\boldsymbol{x})\}$ is consistent, $\psi(\boldsymbol{x}) \in (\Sigma_1)_x(A)$, then $\psi(\boldsymbol{x}) = \psi(\boldsymbol{y}) \in (\Sigma_1)_y(B)$ for some $\boldsymbol{y} \in \mathcal{P}_{<\kappa}(\boldsymbol{x})$, $B \in \mathcal{P}_{<\kappa}(A)$, and since $(p \mid \boldsymbol{y}) \mid B \cup \{\psi(\boldsymbol{y})\}$ is consistent, by $(p \mid \boldsymbol{y}) \mid B \in S_y(B)$, we have $\psi(\boldsymbol{x}) = \psi(\boldsymbol{y}) \in (p \mid \boldsymbol{y}) \mid B \subset p$ as desired. $\square$

The next proposition relates the truth of Σ_1-formulas in arbitrary models with truth in $\mathscr{C}$.

Proposition 2.12. *Let $M < \mathscr{C}$, $\boldsymbol{a} \in M$, $\varphi(\boldsymbol{x})$ a Σ_1-formula without parameters, $\boldsymbol{x}$ matching $\boldsymbol{a}$. Then the following are equivalent*:

(i) $\mathscr{C} \vDash \varphi[\boldsymbol{a}]$,

(ii) $\operatorname{Diag}_{\mathscr{F}}(M) \cup \{\varphi(\boldsymbol{a})\}$ *is consistent,*

(iii) *for any Σ_1-formula $\psi(\boldsymbol{x})$ (without parameters), if $M \vDash \psi[\boldsymbol{a}]$, then $\varphi(\boldsymbol{a}) \wedge \psi(\boldsymbol{a})$ is consistent.* (Note that in (ii) and (iii), $\boldsymbol{a}$ is used as a tuple of individual constants; those constants are the same as the ones denoting the terms in $\boldsymbol{a}$ in $\operatorname{Diag}_{\mathscr{F}}(M)$.)

Proof. (i)$\Rightarrow$(iii). This is clear since $\mathscr{C}$ with $\boldsymbol{a}$ witnesses consistency; note that $M \vDash \psi[\boldsymbol{a}]$ implies $\mathscr{C} \vDash \psi[\boldsymbol{a}]$ since ψ is Σ_1.

(iii)$\Rightarrow$(ii). Any $<\kappa$-subset Φ of $\operatorname{Diag}_{\mathscr{F}}(M)$ gives rise, after taking its conjunction and quantifying out existentially all constants not in $\boldsymbol{a}$, to a Σ_1-sentence $\psi(\boldsymbol{a})$ true in M; the consistency of $\Phi \cup \{\varphi(\boldsymbol{a})]$ is implied by that of $\varphi(\boldsymbol{a}) \wedge \psi(\boldsymbol{a})$. This shows the assertion.

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(ii)$\Rightarrow$(i). Let $(N, \bar{a})_{a\in|M|}$ be a model of $\mathrm{Diag}_{\mathscr{F}}(M)\cup\{\varphi(\boldsymbol{a})\}$; by $\mathscr{C}$ being u-h, we may assume $M<N<\mathscr{C}$, $\bar{a}=a$ for all $a\in|M|$. Since $N\models\varphi[\boldsymbol{a}]$ and φ is a Σ_1-formula, $\mathscr{C}\models\varphi[\boldsymbol{a}]$. □

Corollary 2.13. *For* $\boldsymbol{a}, \boldsymbol{b}\in M<\mathscr{C}$, *if* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *satisfy the same* Σ_1*-formulas in* M, *then they satisfy the same* Σ_1*-formulas in* $\mathscr{C}$ *(i.e.,* $\mathrm{tp}(\boldsymbol{a}/\emptyset)=\mathrm{tp}(\boldsymbol{b}/\emptyset)$).

Proof. Immediate by the equivalence of (i) and (iii) in 2.12. □

In what follows, we sometimes neglect to denote tuples in bold type. Thus '$a\in M$' may mean that a is a tuple of elements of M.

Moreover, the tuples $a, b, c, \ldots$ are tuples of elements of $\mathscr{C}$, $a\in\mathscr{C}, \ldots$

Definition 2.14 (compare [9]). (i) $\langle a_\beta\rangle_{\beta<\alpha}$ is a *sequence of indiscernibles over* the set B (a_β: tuples in $\mathscr{C}$) if for all $k\in\omega$, $\beta_1<\beta_2<\cdots<\beta_k<\alpha$, $\gamma_1<\gamma_2<\cdots<\gamma_k<\alpha$, we have

$$a_{\beta_1}{}^\wedge a_{\beta_2}{}^\wedge\cdots{}^\wedge a_{\beta_k}\underset{B}{\sim} a_{\gamma_1}{}^\wedge a_{\gamma_2}{}^\wedge\cdots{}^\wedge a_{\gamma_k}$$

(if the latter is required only for $k=2$, we talk about a sequence of 2-indiscernibles).

(ii) Let $B\subset A\subset|\mathscr{C}|$ and $p\in S(A)$. *p splits strongly over B* if there are tuples $a_0, a_1\in A$ of length $<\kappa$ such that $a_0\underset{c}{\not\sim}a_1$, with c some (any) realization of p [$a_0\underset{c}{\sim}a_1$ means $a_0\underset{C}{\sim}a_1$ with $C=$ range of c], but there is a sequence $\langle a_\alpha\rangle_{\alpha<\kappa}$ of indiscernibles (in $\mathscr{C}$) over B with the prescribed first two members a_0, a_1. (If the indiscernibility is replaced by 2-indiscernibility, we talk about 2-strong splitting.)

Proposition 2.15. *Let λ be a strong limit cardinal, $\lambda>\kappa'$. Suppose T is stable in λ. Then for any M, every type $p\in S^{<\kappa}(M)$ does not split strongly (even, does not split 2-strongly) over some $N<M$ of cardinality κ'.*

Proof. We show the contrapositive. Suppose $p=\mathrm{tp}(c/M)\in S^{<\kappa}(M)$ is a counterexample to the conclusion. Let $\theta=\mathrm{cf}(\lambda)$, and $\langle\lambda_i\rangle_{i<\theta}$ a strictly increasing sequence of regular cardinals limiting to λ, with $2^{\lambda_i}\leq\lambda_{i+1}$. Using θ times the fact that p splits 2-strongly over all $N<M$ with $\|N\|=\kappa'$, we easily obtain, by induction on $i<\theta$, items

$$M_i,\quad a_0^i,\quad a_1^i,\quad I^i$$

such that

$$\begin{aligned}&M_i<M,\quad \|M_i\|=\kappa';\qquad \langle M_i\rangle_{i<\theta}\text{ is }<\text{-continuous};\\ &a_0^i\underset{c}{\not\sim}a_1^i;\qquad a_0^i, a_1^i\in{}^{<\kappa}|M_{i+1}|\,;\end{aligned}\tag{1}$$

and $I^i=\langle a_\alpha^i\rangle_{\alpha<\lambda}$ is a sequence of 2-indiscernibles over M_i [note that a_0^i, a_1^i are the first two members of I^i; note also that by $<\kappa$-compactness, we can make the I^i to be of length λ instead of κ].

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Next, we define, for every $\eta \in {}^{<\theta}\{\langle\alpha,\beta\rangle \in \lambda\times\lambda : \alpha<\beta\}$ a Σ_1-elementary mapping f_η with domain M_i, and another one, g_η, with domain $M_i \cup I^i$ (here $i = \text{length}(\eta)$; I^i also denotes the set of elements involved in all a^i_α) such that

$$f_\eta \subset g_\eta; \qquad \eta_1 \text{ is an initial sequment of } \eta_2 \Rightarrow f_{\eta_1} \subset f_{\eta_2};$$

$$f_{\eta^\wedge\langle\alpha,\beta\rangle}(a^i_0) = g_\eta(a^i_\alpha), \qquad f_{\eta^\wedge\langle\alpha,\beta\rangle}(a^i_1) = g_\eta(a^i_\beta)$$

where $i = \text{length}(\eta)$ and $\alpha<\beta<\lambda$.

The definition is by induction on $\text{length}(\eta) \in \theta$. For $\eta = \emptyset$, we make $f_\emptyset$, $g_\emptyset$ both identities. Let $i<\theta$. Having defined f_η for η of length $i<\theta$, we take g_η to be an arbitrary extension of f_η to $M_i \cup I^i$. As for the definition of the f_η, if $\text{length}(\eta) = i$ is a limit ordinal $<\theta$, we are forced to put $f_\eta = \bigcup_{j<i} f_{\eta|j}$. Finally, if $\text{length}(\eta) = j$, $\alpha<\beta<\lambda$, and f_η, g_η have been defined, we define $f_{\eta^\wedge\langle\alpha,\beta\rangle}$ as follows. Since g_η is Σ_1-elementary and extends f_η, we have by 2-indiscernibility

$$\begin{aligned} a^j_0{}^\wedge a^j_1{}^\wedge \boldsymbol{M}_j &\sim a^j_\alpha{}^\wedge a^j_\beta{}^\wedge \boldsymbol{M}_j \\ &\sim g_\eta(a^j_\alpha)^\wedge g_\eta(a^j_\beta)^\wedge f_\eta(\boldsymbol{M}_j) \end{aligned}$$

with $\boldsymbol{M}_j$ a tuple enumerating M_j. This means that the mapping with domain $M_j \cup \{a^j_0, a^j_1\}$ which is f_η extended by $a^j_0 \mapsto g_\eta(a^j_\alpha)$, $a^j_1 \mapsto g_\eta(a^j_\beta)$ is Σ_1-elementary. We define $f_{\eta^\wedge\langle\alpha,\beta\rangle}$ to be an arbitrary $\mathcal{F}$-extension of the latter mapping to M_{j+1}.

This completes the definition of the f_η, g_η; they clearly satisfy the requirements.

With $U \overset{\text{def}}{=} {}^\theta\{\langle\alpha,\beta\rangle \in \lambda\times\lambda : \alpha<\beta\}$, for $\nu \in U$, let f_ν be $\bigcup_{i<\theta} f_{\nu|i}$; note that since the domains of the $f_{\nu|i}$ form a $<$-chain, f_ν is a Σ_1-elementary mapping. Let c_ν be a realization of the type $f_\nu(p \mid \bigcup_{i<\theta} M_i)$. Note the following property of c_ν: for any $i<\theta$ and for $\eta = \nu \mid i$, $\langle\alpha,\beta\rangle = \eta(i)$, we have

$$g_\eta(a^i_\alpha) \underset{c_\nu}{\not\sim} g_\eta(a^i_\beta); \tag{2}$$

This follows from (1).

Let $A = \bigcup\{g''_\eta I^i : i = \text{length}(\eta) < \theta\}$. A is a set of cardinality λ. We claim that the elements c_ν realize more than λ many distinct types over A, which will show the desired unstability in λ.

Suppose not, and let $X \subset U$ be a set of cardinality $<\lambda$ such that for any $\nu \in U$ there is $\nu' \in X$ with $c_\nu \underset{A}{\sim} c_{\nu'}$. Let us write X as $X = \bigcup_{i<\theta} X_i$ with $|X_i| \leq \lambda_i$, and define a specific $\hat\nu \in U$ as follows. Suppose $\eta = \hat\nu \mid i$ has been defined, and consider the set $B = \{c_\nu : \nu \in X_i\}$ of cardinality $\leq\lambda_i$. The number of $<\kappa$-types over B is $\leq 2^{\lambda_i} \leq \lambda_{i+1}$ (since $\mu^{<\kappa} = \mu$ for all regular $\mu \geq \kappa$), hence there are $\alpha<\beta<\lambda$ such that

$$g_\eta(a^i_\alpha) \underset{c_\nu}{\sim} g_\eta(a^i_\beta) \tag{3}$$

for all $\nu \in X_i$. Put $\hat\nu(i) = \langle\alpha,\beta\rangle$. This defines $\hat\nu \in U$. By assumption, there is $\nu \in X$ with

$$c_{\hat\nu} \underset{A}{\sim} c_\nu. \tag{4}$$

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There is $i < \theta$ such that $\nu \in X_i$. Let $\eta = \hat{\nu} \restriction i$, $\langle \alpha, \beta \rangle = \eta(i)$. (2) gives that

$$g_\eta(a^i_\alpha) \not\sim_{c_{\hat{\nu}}} g_\eta(a^i_\beta). \tag{5}$$

Since $g_\eta(a^i_\alpha)$, $g_\eta(a^i_\beta) \in A$, (3), (4) and (5) give a contradiction. □

3. Indiscernibles

First, to fix notation, we review the basics of the well-known Ehrenfeucht–Mostowski method of building models by using order-indiscernibles.

Let L^* be a finitary language: a set of finitary relation and operation symbols. If M^* is an L^*-structure, $X \subset |M^*|$, the substructure of M^* generated by X is called the *Skolem hull of* X (in M^*), and is denoted by $H_{M^*}(X)$, or more simply, $H(X)$.

An *Ehrenfeucht–Mostowski* (E-M) *scheme in* L^* is a set Φ of atomic and negated atomic formulas over L^* with free variables all in the fixed set $\{v_i : i < \omega\}$ of variables such that

(i) Φ is maximal consistent: Φ has a model (an L^*-structure with distinguished elements interpreting the variables v_i), and for any atomic formula $\varphi(v_0, \dots, v_{n-1})$, either $\varphi(v_0, \dots, v_{n-1}) \in \Phi$, or $\neg\varphi(v_0, \dots, v_{n-1}) \in \Phi$.

(ii) Φ is homogeneous: for any atomic formula $\varphi(v_0, \dots, v_{n-1})$ and any $i_0 < i_1 < \cdots < i_{n-1} < \omega$,

$$\varphi(v_0, \dots, v_{n-1}) \in \Phi \quad \Leftrightarrow \quad \varphi(v_{i_0}, \dots, v_{i_{n-1}}) \in \Phi.$$

(iii) Φ is non-degenerate: $v_i \neq v_j \in \Phi$ for all $i \neq j$ in ω.

Given any E-M scheme Φ in L^*, and any linear ordering $I = (I, <)$, we can define the L^*-strucure $\mathrm{EM}(I, \Phi)$ so that $I \subset |\mathrm{EM}(I, \Phi)|$ [because of the non-degeneracy assumption, Φ does not collapse elements of I], $\mathrm{EM}(I, \Phi)$ is generated by I, and $\mathrm{EM}(I, \Phi) \models \varphi[a_0, \dots, a_{n-1}]$ whenever $\varphi(v_0, \dots, v_{n-1}) \in \Phi$, and $a_0 < a_1 < \cdots < a_{n-1}$ in I. $\mathrm{EM}(I, \Phi)$ is uniquely determined by Φ and I: any two structures answering the description of $\mathrm{EM}(I, \Phi)$ have a unique isomorphism which is the identity on I.

Clearly, whenever J is a sub-ordering of I, the Skolem hull $H(J)$ of J in $\mathrm{EM}(I, \Phi)$ is (isomorphic to) $\mathrm{EM}(J, \Phi)$. Obviously, $\|\mathrm{EM}(I, \Phi)\| = \|I\|$, whenever $\|I\| \geq |L^*|$.

Let $I = (I, <)$ be a linear ordering. For $\boldsymbol{a} = \langle a_\beta \rangle_{\beta<\alpha}$, $\boldsymbol{b} = \langle b_\beta \rangle_{\beta<\alpha}$, tuples of elements of I, we write $\boldsymbol{a} \equiv_0 \boldsymbol{b}$ in I if $a_\beta < a_\gamma$ iff $b_\beta < b_\gamma$ for all $\beta, \gamma < \alpha$. An important, but obvious, fact about E-M-models is the order-indiscernibility of elements of I in $M^* = \mathrm{EM}(I, \Phi)$ with respect to atomic formulas: if $\boldsymbol{a} \equiv_0 \boldsymbol{b}$ in I, then for any atomic L^*-formulas $\varphi(\boldsymbol{x})$, $M^* \models \varphi[\boldsymbol{a}]$ iff $M^* \models \varphi[\boldsymbol{b}]$. The reason for this is that for a strictly increasing $\boldsymbol{a}$, $M^* \models \varphi[\boldsymbol{a}]$ iff $\varphi(\boldsymbol{v}) \in \Phi$. Notice that the above order-indiscernibility automatically extends to φ any quantifier-free formula.

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With also writing α for the standard well-ordering of ordinals $<\alpha$, we have, in particular, the E-M models $\mathrm{EM}(\alpha, \Phi)$, for $\alpha \in \mathrm{Ord}$. Note that Φ is nothing but the atomic diagram of $\mathrm{EM}(\omega, \Phi)$.

An abstract version of Morley's omitting types theorem can be stated as follows.

Proposition 3.1 [7]. *Let $\mu = |L^*| + \aleph_0$. Suppose $\mathcal{M}$ is a family of L^*-structures such that for every $\alpha < (2^\mu)^+$, there is $M \in \mathcal{M}$ with $\|M\| > \beth_\alpha$ (e.g., $\mathcal{M} = \{M\}$ with $\|M\| \geq \beth_{(2^\mu)^+}$). Then there is an E-M scheme Φ in L^* such that for every $n < \omega$, $\mathrm{EM}(n, \Phi)$ is isomorphic to a substructure of some $M \in \mathcal{M}$.*

Remark. In the notation of 3.1, suppose T^* is a universal theory in $(L^*)_{\infty\omega}$ (that is, the axioms of T^* are sentences $\forall \boldsymbol{x}\, \varphi(\boldsymbol{x})$, with $\varphi(\boldsymbol{x})$ quantifier-free in $(L^*)_{\infty\omega}$) such that every $M \in \mathcal{M}$ is a model of T^*. Then, for Φ given by 3.1, every E-M model $\mathrm{EM}(I, \Phi)$ will be a model of T^*: this is true for $\mathrm{EM}(n, \Phi)$ if $n < \omega$, since it is isomorphic to a submodel of a model of T^*; since $\mathrm{EM}(I, \Phi)$ is the directed union of copies of $\mathrm{EM}(n, \Phi)$'s, $n < \omega$, and the truth of universal axioms in $(L^*)_{\infty\omega}$ is obviously preserved under such unions, $\mathrm{EM}(I, \Phi) \models T^*$.

Let us say that the E-M scheme is *over* T if all $\mathrm{EM}(I, \Phi)$ are models of T.

To draw conclusions from 3.1 for models of our theory $(T, \mathcal{F})$ of Sections 1 and 2, we use Skolem functions. Let us summarize the well-known method of Skolem functions. Depending on the fragment $\mathcal{F}$, there is a canonically constructed finitary language L_{Sk} of cardinality $\leq |\mathcal{F}|$, $L \subset L_{\mathrm{Sk}}$, and there is a universal theory T_{Sk} in $(L_{\mathrm{Sk}})_{\kappa\omega}$ depending also on T satisfying (i) to (v) below.

(i) The L-reduct of any L_{Sk}-model of T_{Sk} is a model of T.

(ii) Whenever N^* is an L_{Sk}-model of T_{Sk}, M^* is a submodel of N^*, then $M^* | L <_{\mathcal{F}} N^* | L$.

(iii) Any L-model of T can be expanded to an L_{Sk}-model of T_{Sk}.

(iv) Given $M <_{\mathcal{F}} N$, there are L_{Sk}-expansions M^*, N^* of M, N, respectively, such that $M^* \subseteq N^* \models T_{\mathrm{Sk}}$.

(v) To any formula $\varphi(\boldsymbol{x})$ of $\mathcal{F}$, there corresponds a quantifier-free formula $\varphi^*(\boldsymbol{x})$ of $(L_{\mathrm{Sk}})_{\kappa\omega}$ which is equivalent to $\varphi(\boldsymbol{x})$ under T_{Sk} (i.e., $T_{\mathrm{Sk}} \models \forall \boldsymbol{x}\, (\varphi(\boldsymbol{x}) \leftrightarrow \varphi^*(\boldsymbol{x}))$).

Let us call a formula a Σ_1^*-formula if it is of the form $\exists \boldsymbol{y}\, \psi^*(\boldsymbol{x}, \boldsymbol{y})$, where $\psi^*(\boldsymbol{x}, \boldsymbol{y})$ is a quantifier-free formula of $(L_{\mathrm{Sk}})_{\kappa\kappa}$. It follows from (v) that every Σ_1-formula (over L) is equivalent in T_{Sk} to a Σ_1^*-formula.

Proposition 3.2. *Suppose $\mu \geq \kappa'$, $M \in K$, $\|M\| \geq \beth_{(2^\mu)^+}$, $M_0 < M$, $\|M_0\| = \mu$, $p \in S^{<\omega}(M_0)$, and M omits p. Then p is omitted in arbitrarily large $\mathcal{F}$-extensions of M_0.*

Proof. With $(L_{\mathrm{Sk}}, T_{\mathrm{Sk}})$ the Skolem theory associated to $(T, \mathcal{F})$, let M_0^*, M^* be expansions of M_0, M respectively, which are L_{Sk}-models of T_{Sk}, so that M_0^* is a submodel of M^*.

Let us apply 3.1 with $L^* = L_{\mathrm{Sk}} \cup |M_0|$ (the elements of $|M_0|$ used as individual constants), and with $\mathcal{M} = \{(M^*, a)_{a\in|M_0|}\}$; note that the cardinality assumptions of 3.1 are satisfied. Let Φ be the E-M scheme given by 3.1. For any linear ordering I, and for $N^{**} = \mathrm{EM}(I, \Phi)$, we may assume that every $a \in |M_0|$ denotes itself in N^{**} [in every $\mathrm{EM}(n, \Phi)$, $n < \omega$, the denotations of distinct $a \in |M_0|$ are distinct, since $\mathrm{EM}(n, \Phi)$ is isomorphic to a substructure of $(M^*, a)_{a\in|M_0|}$]; hence, for $N = N^{**} \restriction L$, $M_0 \prec N$. If $\boldsymbol{x}$ is a finite subset of I, $|\boldsymbol{x}| = n$, then $H_{N^{**}}(\boldsymbol{x}) \cong \mathrm{EM}(n, \Phi)$, which is isomorphic to a substructure of $(M^*, a)_{a\in|M_0|}$. In the latter, p is omitted; since the isomorphisms involved are identities on $|M_0|$, p is omitted in $H_{N^{**}}(\boldsymbol{x})$. Since $|N|$ is the union of all the sets $|H_{N^{**}}(\boldsymbol{x})|$, $\boldsymbol{x}$ a finite subset of I, p is omitted in N. $\|N\|$ is at least as large as $\|I\|$, which is arbitrary. □

The next proposition is proved by similar but more complicated arguments than those for 3.1 (not given) and 3.2. The improvement in it with respect to 3.2 is the better bound on $\|M\|$.

Some terminology, to facilitate the proof of 3.3. Let B, M^* be L_{Sk}-structures, $B \subseteq M^*$ (B a submodel of M^*), $n < \omega$, $\boldsymbol{a}$ an n-tuple of elements of M^*. The *quantifier-free type of* $\boldsymbol{a}$ *over* B *in* M^*, $\mathrm{tp}_{\mathrm{qf}}^{M^*}(\boldsymbol{a}/B)$ (or just $\mathrm{tp}_{\mathrm{qf}}(\boldsymbol{a}/B)$) is the set of atomic and negated atomic formulas $\varphi(v_0, \ldots, v_{n-1})$ with parameters in B for which $(M^*, b)_{b\in B} \models \varphi[\boldsymbol{a}]$. A *quantifier-free* n*-type over* B is any $\mathrm{tp}_{\mathrm{qf}}^{M^*}(\boldsymbol{a}/B)$ for any M^* extending B and $\boldsymbol{a} \in {}^n|M^*|$. The set of all quantifier-free n-types over B is denoted by $S^n_{\mathrm{qf}}(B)$. If $q \in S^n_{\mathrm{qf}}(B)$, and $f : B \xrightarrow{\cong} B'$, then $f(q) \in S^n_{\mathrm{qf}}(B')$ is defined in the obvious way, by replacing each parameter $b \in |B|$ in any formula in q by $f(b)$. If $q \in S^n_{\mathrm{qf}}(B)$ and $s \subset n$, $s = \{i_0, \ldots, i_{m-1}\}$, $i_0 < \cdots < i_{m-1} < n$, then $q \restriction s \in S^m_{\mathrm{qf}}(B)$ is defined by

$$\varphi(v_0, \ldots, v_{m-1}) \in q \restriction s \quad \Leftrightarrow \quad \varphi(v_{i_0}, \ldots, v_{i_{m-1}}) \in q.$$

For the same s, and an n-tuple $\boldsymbol{a}$, $\boldsymbol{a} \restriction s \overset{\mathrm{def}}{=} \langle a_{i_0}, \ldots, a_{i_{m-1}} \rangle$; a *subtuple* of $\boldsymbol{a}$ is any $\boldsymbol{a} \restriction s$, with $s \subset n$.

Proposition 3.3. *Suppose* $\mu \geq \kappa'$, $M \in K$, $\|M\| \geq \beth_{(2^{\kappa'})^+}(\mu)$, $M_0 \prec M$, $\|M_0\| = \mu$, $p \in S^{<\omega}(M_0)$, *and* M *omits* p. *Then there are a model* M_0', $\|M_0'\| = \kappa'$, *and a type* $p' \in S^{<\omega}(M_0')$ *such that* p' *is omitted in arbitrarily large* $\mathcal{F}$*-extensions of* M_0'.

Proof. Let M_0^*, M^* be expansions of M_0, M, respectively, which are L_{Sk}-models of T_{Sk}, and also such that M_0^* is a submodel of M^*. We define, for every finite tuple $\boldsymbol{a}$ of elements of M, a submodel $B_{\boldsymbol{a}}$ of M_0^* of cardinality κ' such that $B_{\boldsymbol{b}} \subseteq B_{\boldsymbol{a}}$ for every subtuple $\boldsymbol{b}$ of $\boldsymbol{a}$ and $H(\boldsymbol{a} \cup B_{\boldsymbol{a}}) \restriction L$ omits the type $p \restriction B_{\boldsymbol{a}}$. [$\boldsymbol{a} \cup B_{\boldsymbol{a}}$ abbreviates $\mathrm{range}(\boldsymbol{a}) \cup |B_{\boldsymbol{a}}|$; H refers to Skolem hull in M^*.] To construct the $B_{\boldsymbol{a}}$, we proceed by induction on $\mathrm{length}(\boldsymbol{a})$. For a fixed $\boldsymbol{a}$, assuming that $B_{\boldsymbol{b}}$ has been defined for every proper subtuple $\boldsymbol{b}$ of $\boldsymbol{a}$, we let $B^0_{\boldsymbol{a}}$ be a submodel of M_0^* of cardinality κ' containing $B_{\boldsymbol{b}}$ for every proper subtuple $\boldsymbol{b}$ of $\boldsymbol{a}$, and by induction on $n > 0$, we let $B^n_{\boldsymbol{a}}$ be a submodel of M_0^* of power κ' such that $B^{n-1}_{\boldsymbol{a}} \subseteq B^n_{\boldsymbol{a}}$, and

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$H(\boldsymbol{a} \cup B_{\boldsymbol{a}}^{n-1}) \mid L$ omits $p \mid B_{\boldsymbol{a}}^{n}$: since for every $b \in H(\boldsymbol{a} \cup B_{\boldsymbol{a}}^{n-1})$, there is $A_b \subset |M_0|$, $|A_b| < \kappa$, such that $M \nVdash (p \mid A_b)[b]$, we may let $B_{\boldsymbol{a}}^{n}$ be any submodel of M_0^* of cardinality κ' containing the set

$$|B_{\boldsymbol{a}}^{n-1}| \cup \bigcup \{A_b : b \in H(\boldsymbol{a} \cup B_{\boldsymbol{a}}^{n-1})\}.$$

Finally, let $B_{\boldsymbol{a}} = \bigcup_{n<\omega} B_{\boldsymbol{a}}^{n}$, Since $H(\boldsymbol{a} \cup B_{\boldsymbol{a}}) = \bigcup_{n<\omega} H(\boldsymbol{a} \cup B_{\boldsymbol{a}}^{n})$, it is clear that $B_{\boldsymbol{a}}$ satisfies the requirements.

For a linear ordering X, let $X^{(<,n)}$ denote the set of all strictly increasing n-tuples of elements of X.

Let $<$ be a linear ordering of $|M|$. For every $n < \omega$, we construct a family of subsets I_α^n of $|M|$ $(\alpha < (2^{\kappa'})^+)$, of cardinality $|I_\alpha^n| \geq \beth_\alpha(\mu)$ with the following properties and additional items:

(i) For any fixed $\alpha < (2^{\kappa'})^+$, $B_{\boldsymbol{a}}$ (see above) has a constant value on $(I_\alpha^n)^{(<,n)}$: for all $\boldsymbol{a} \in (I_\alpha^n)^{(<,n)}$, $B_{\boldsymbol{a}} = B_\alpha^n$; and also, $\mathrm{tp}_{\mathrm{qf}}(\boldsymbol{a}/B_\alpha^n)$ is constant:

$$\mathrm{tp}_{\mathrm{qf}}(\boldsymbol{a}/B_\alpha^n) = q_\alpha^n \quad \text{for all } \boldsymbol{a} \in (I_\alpha^n)^{(<,n)}.$$

(ii) For varying $\alpha < (2^{\kappa'})^+$, the B_α^n are all isomorphic to each other: we have the L_{Sk}-structure B^n, and isomorphisms $f_\alpha^n : B^n \xrightarrow{\cong} B_\alpha^n$ $(\alpha < (2^{\kappa'})^+)$; moreover, the isomorphism $(f_\alpha^n)^{-1}$ takes q_α^n into a constant type q^n over B^n:

$$f_\alpha^n(q^n) = q_\alpha^n,$$

and $(f_\kappa^n)^{-1}$ takes the type $p \mid (B_\alpha^n \mid L)$ into a constant type p^n over $B^n \mid L$:

$$(f_\alpha^n)(p^n) = p \mid (B_\alpha^n \mid L).$$

(iii) Furthermore, for any $0 < n < \omega$,

(a) $B^{n-1} \subseteq B^n$,

(b) for any $\alpha < (2^{\kappa'})^+$ there is $\beta < (2^{\kappa'})^+$ such that $I_\alpha^n \subset I_\beta^{n-1}$, and the diagram

$$\begin{array}{ccc} B^{n-1} & \xrightarrow{\text{incl.}} & B^n \\ {\scriptstyle f_\beta^{n-1}}\downarrow & & \downarrow{\scriptstyle f_\alpha^n} \\ B_\beta^{n-1} & \xrightarrow[\text{incl.}]{} & B_\alpha^n \end{array}$$

commutes (note that $B_\beta^{n-1} \subseteq B_\alpha^n$ since $B_\beta^{n-1} = B_{\boldsymbol{b}}$, $B_\alpha^n = B_{\boldsymbol{a}}$ for any $\boldsymbol{a} \in (I_\alpha^n)^{(<,n)}$ and for $\boldsymbol{b}$, any subtuple of $\boldsymbol{a}$ of length $n-1$; recall that $B_{\boldsymbol{b}} \subseteq B_{\boldsymbol{a}}$).

For $n = 0$, put $I_\alpha^0 = |M|$ for all α, $B^0 = B_\alpha^0 = B_\emptyset$, $f_\alpha^0 = \text{identity}$, $q^0 = \mathrm{tp}_{\mathrm{qf}}^{M^*}(\emptyset/B^0)$, $p^0 = p \mid B^0$. Suppose $n > 0$, and that for $n-1$, all items have been defined.

Given $\alpha < (2^{\kappa'})^+$, let, by the Erdös–Rado theorem, $\hat{I}_\alpha^n$ be a subset of $I_{\alpha+\omega}^{n-1}$ of cardinality $\beth_\alpha(\mu)$ such that the functions

$$\boldsymbol{a} \mapsto B_{\boldsymbol{a}}, \qquad \boldsymbol{a} \mapsto \mathrm{tp}_{\mathrm{qf}}^{M^*}(\boldsymbol{a}/B_{\boldsymbol{a}})$$

defined for $\boldsymbol{a} \in (\hat{I}_\alpha^n)^{(<,n)}$ are constant. Since these functions, defined on $(I_{\alpha+\omega}^{n-1})^{(<,n)}$, have ranges of size $\leq \mu^{\kappa'}$, and $|I_{\alpha+\omega}^{n-1}| \geq \beth_{\alpha+\omega}(\mu)$, this is possible.

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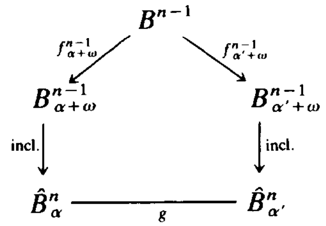
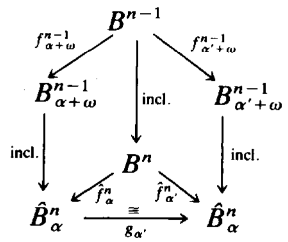
Define $\hat{B}^n_\alpha$ to be the constant value of $B_{\boldsymbol{a}}$ for $\boldsymbol{a} \in (\hat{I}^n_\alpha)^{(<,n)}$, and let q^n_α be the constant type $\mathrm{tp}_{\mathrm{qf}}(\boldsymbol{a}/\hat{B}^n_\alpha)$ $(\boldsymbol{a} \in (\hat{I}^n_\alpha)^{(<,n)})$.

Let $\alpha < (2^{\kappa'})^+$, and pick any $\boldsymbol{a} \in (\hat{I}^n_\alpha)^{(<,n)}$, and $\boldsymbol{b}$, subtuple of $\boldsymbol{a}$ of length $n-1$. Then $B_{\boldsymbol{b}} = B^{n-1}_{\alpha+\omega}$, $B_{\boldsymbol{a}} = \hat{B}^n_\alpha$, and since $B_{\boldsymbol{b}} \subseteq B_{\boldsymbol{a}}$, we have $B^{n-1}_{\alpha+\omega} \subseteq \hat{B}^n_\alpha$. For $\alpha, \alpha' < (2^{\kappa'})^+$, let us write $\alpha \sim \alpha'$ if there is an isomorphism g that makes the diagram



commute, and for which also $g(q^n_\alpha) = q^n_{\alpha'}$ and $g(p \mid (\hat{B}^n_\alpha \mid L)) = p \mid (\hat{B}^n_{\alpha'} \mid L)$. Clearly, the number of equivalence classes of $\sim$ is $\leqslant 2^{\kappa'}$ (since $\|\hat{B}^n_\alpha\| = \kappa'$). Let $X \subset (2^{\kappa'})^+$ be an equivalence class of $\sim$ of cardinality $(2^{\kappa'})^+$. Let us pick a fixed $\alpha \in X$, and define the L_{Sk}-structure B^n with the isomorphism $\hat{f}^n_\alpha : B^n \xrightarrow{\cong} \hat{B}^n_\alpha$ so that $B^{n-1} \subseteq B^n$, and in the diagram



the left-hand side quadrangle commutes. For any other $\alpha' \in X$, there is an isomorphism $g_{\alpha'}$ that makes the outer pentagon commute, and for which $g_{\alpha'}(q^n_\alpha) = q^n_{\alpha'}$ and $g_{\alpha'}(p \mid (\hat{B}^n_\alpha \mid L)) = p \mid (\hat{B}^n_{\alpha'} \mid L)$. Define $\hat{f}^n_{\alpha'}$ to make the lower triangle commute; this will make the right-hand side quadrangle commute as well.

Let $q^n = (\hat{f}^n_\alpha)^{-1}(q^n_\alpha) = (\hat{f}^n_{\alpha'})^{-1}(q^n_{\alpha'})$ (independent of α'), $p^n = (\hat{f}^n_\alpha)^{-1}(p \mid (\hat{B}^n_\alpha \mid L))$. Let X be enumerated as $X = \{\gamma_\alpha : \alpha < (2^{\kappa'})^+\}$ with $\alpha \leqslant \gamma_\alpha$, and put $B^n_\alpha \overset{\mathrm{def}}{=} \hat{B}^n_{\gamma_\alpha}$, $f^n_\alpha = \hat{f}^n_{\gamma_\alpha}$. It is clear that all requirements in (i) to (iii) are satisfied: $\beta = \gamma_\alpha + \omega$ can be taken in (iii)(b).

As a consequence mainly of (iii)(b), we have

$$q^{n-1} = (q^n \mid B^{n-1}) \mid s \tag{1}$$

whenever $s \subset n$, $|s| = n-1$, in particular, $q^{n-1} \subset q^n$: the reason is that, for α and β as in (iii)(b), and for $\boldsymbol{a} \in (I^n_\alpha)^{(<,n)}$, $\boldsymbol{b} = \boldsymbol{a} \mid s$, we have $\boldsymbol{b} \in (I_\beta)^{(<,m)}$, and $q^{n-1} = (f^{n-1}_\beta)^{-1}(\mathrm{tp}_{\mathrm{qf}}(\boldsymbol{b}/B^{n-1}_\beta))$, $q^n = (f^n_\alpha)^{-1}(\mathrm{tp}_{\mathrm{qf}}(\boldsymbol{a}/B^n_\alpha))$.

For the same reason, we also have $p^{n-1} \subset p^n$ for all $n \geqslant 1$.

Consider the set $\Phi = \bigcup_{n<\omega} q^n$ of formulas. Φ is a set of atomic and negated atomic formulas with variables all in $\{v_i : i < \omega\}$ over the language $L_{\mathrm{Sk}}(B^\omega) \overset{\mathrm{def}}{=} L_{\mathrm{Sk}} \cup \bigcup_{n<\omega} |B^n|$ (the elements of $\bigcup_{n<\omega} |B^n|$ being used as individual constants).

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Since each q^n is consistent, $q^{n-1} \subseteq q^n$, Φ is consistent; the 'maximality' and 'non-degeneracy' of Φ are also clear. Finally, it is easily seen by (1) that Φ is homogeneous. Thus, Φ is an E-M scheme in $L_{\text{Sk}}(B^\omega)$.

Let $M_0'^* = \bigcup_{n<\omega} B^n$, an L_{Sk}-structure; $M_0' = M_0'^* \mid L$. By 2.14, let $p' \in S^{<\omega}(M_0')$ be a type extending each p^n, $n < \omega$.

Let I be any linear ordering, $N^* = \text{EM}(I, \Phi) \mid L_{\text{Sk}}$, $N = N^* \mid L$.

We may assume that the interpretation in $\text{EM}(I, \Phi)$ of each $b \in B^\omega$ is b itself; this ensures that $M_0' < N$. We will show that N omits p'.

Let $\boldsymbol{x} = \langle x_i \rangle_{i<n} \in I^{(<,n)}$. Then the definition of Φ as $\bigcup_{n<\omega} q^n$ tells us that $q^n = \text{tp}_{\text{qf}}^{N^*}(\boldsymbol{x}/B^n)$. Take any $\alpha < (2^{\kappa'})^+$ and $\boldsymbol{a} = \langle a_i \rangle_{i<n} \in (I_\alpha^n)^{(<,n)}$. Then $q_\alpha^n = \text{tp}_{\text{qf}}^{M^*}(\boldsymbol{a}/B_\alpha^n)$. The isomorphism $f_\alpha^n : B^n \xrightarrow{\cong} B_\alpha^n$ takes q^n into q_α^n; this means that we have an isomorphism

$$H_{N^*}(\boldsymbol{x} \cup B^n) \xrightarrow{\cong} H_{M^*}(\boldsymbol{a} \cup B_\alpha^n)$$

taking x_i to a_i $(i < n)$, and acting on B^n as f_α^n. Since $p \mid B_\alpha^n$ is omitted in $H_{M^*}(\boldsymbol{a} \cup B_\alpha^n)$ $(B_\alpha^n = B_{\boldsymbol{a}}$; this is the defining property of $B_{\boldsymbol{a}})$, $p^n = p' \mid B^n$ is omitted in $H_{N^*}(\boldsymbol{x} \cup B^n)$; in particular, there is no realization of p' in the subset $|H_{N^*}(\boldsymbol{x} \cup B^n)|$ of $|N|$. But, of course,

$$|N| = \bigcup_{\substack{n<\omega \\ \boldsymbol{x} \in I^{(>,n)}}} H^{N^*}(\boldsymbol{x} \cup B^n).$$

Thus, p' is omitted in N. □

Given any linear ordering $I = (I, <)$ and tuples $\boldsymbol{a}, \boldsymbol{b} \in I$, we write

$$\boldsymbol{a} \equiv_\kappa \boldsymbol{b} \quad \text{in } I$$

if for all $\boldsymbol{c} \in {}^{<\kappa}I$ there is $\boldsymbol{d} \in {}^{<\kappa}I$ with $\boldsymbol{a}^\wedge\boldsymbol{c} \equiv_0 \boldsymbol{b}^\wedge\boldsymbol{d}$, and for all $\boldsymbol{d} \in {}^{<\kappa}I$ there is $\boldsymbol{c} \in {}^{<\kappa}I$ with $\boldsymbol{a}^\wedge\boldsymbol{c} \equiv_0 \boldsymbol{b}^\wedge\boldsymbol{d}$.

Lemma 3.4. *Assume Φ is an E-M scheme in L_{Sk}, and $\boldsymbol{a} \equiv_\kappa \boldsymbol{b}$ in I. Then $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the same Σ_1^*-formulas in* $\text{EM}(I, \Phi)$.

Proof. Let $M^* = \text{EM}(I, \Phi)$. Let $\varphi(\boldsymbol{x})$ be a Σ_1^*-formula, $\varphi(\boldsymbol{x}) = \exists \boldsymbol{y}\, \psi(\boldsymbol{x}, \boldsymbol{y})$, ψ quantifier-free, and assume $M^* \models \varphi[\boldsymbol{a}]$. Since M^* is generated by I, there are L_{Sk}-terms $\boldsymbol{t}(\boldsymbol{z})$ matching $\boldsymbol{x}$, and elements $\boldsymbol{c} \in I$ matching $\boldsymbol{z}$ such that $M^* \models \psi(\boldsymbol{x}, \boldsymbol{t}(\boldsymbol{z}))[\boldsymbol{a}/\boldsymbol{x}, \boldsymbol{c}/\boldsymbol{z}]$. Find $\boldsymbol{d}$ such that $\boldsymbol{a}^\wedge\boldsymbol{c} \equiv_0 \boldsymbol{b}^\wedge\boldsymbol{d}$ in I. By the order-indiscernibility of elements of I in M^* with respect to quantifier-free formulas, $M^* \models \psi(\boldsymbol{x}, \boldsymbol{t}(\boldsymbol{z}))[\boldsymbol{b}/\boldsymbol{x}, \boldsymbol{d}/\boldsymbol{z}]$, hence $M^* \models \varphi[\boldsymbol{b}]$. □

For $A \subset I$, $\boldsymbol{b}, \boldsymbol{c} \in I$, let us write $\boldsymbol{b} \equiv_\kappa \boldsymbol{c} \pmod{A}$ if $\boldsymbol{A}^\wedge\boldsymbol{b} \equiv_\kappa \boldsymbol{A}^\wedge\boldsymbol{c}$ for some (any) tuple $\boldsymbol{A}$ enumerating A.

The following proposition is a varient of Morley's theorem [6] stating that categoricity in uncountable powers implies stability in $\aleph_0$, in the case of countable first-order theories.

Proposition 3.5. *Suppose $\lambda > \kappa'$, T is categorical in λ. Then for any μ, $\kappa' \leq \mu < \lambda$, T is stable in μ.*

Proof. Since K has arbitrarily large models, an application of 3.1 (with the remarks following it) gives an E-M scheme Φ in L_{Sk} over T. Consider the models $M^* = \mathrm{EM}(\lambda, \Phi)$, $M = M^* \mid L$. Let $A \subset |M|$ be of power $\leq \mu$. We claim that the number of types in $S^{<\omega}(A)$ realized in M is at most μ. Let $B \subset \lambda$ be such that $|B| \leq \mu$ and $A \subset H(B)$. If $\boldsymbol{c}, \boldsymbol{d} \in {}^{<\omega}(\lambda)$, and $\boldsymbol{c} \equiv_\kappa \boldsymbol{d} \,(\mathrm{mod}\, B)$ in λ, then $\boldsymbol{c}$ and $\boldsymbol{d}$ satisfy in M^* the same Σ_1^*-formulas with parameters in B, by 3.4; hence, for any finite tuple $\boldsymbol{t}(\boldsymbol{x})$ of L_{Sk}-terms, $\boldsymbol{t}^M[\boldsymbol{c}]$ and $\boldsymbol{t}^M[\boldsymbol{d}]$ satisfy in M the same Σ_1-formulas with parameters in $A \subset H(B)$, and hence, by 2.13, $\boldsymbol{t}^M[\boldsymbol{c}]$ and $\boldsymbol{t}^M[\boldsymbol{d}]$ have the same type over A in the sense of $\mathscr{C}$. Since, as it is easily seen (see below), the number of equivalence classes of $\equiv_\kappa \,(\mathrm{mod}\, B)$ restricted to finite tuples is at most μ, and the number of finite tuples $\boldsymbol{t}(\boldsymbol{x})$ is $\leq \kappa'$, the claim follows. [The equivalence class of $\boldsymbol{c} = \langle c_i \rangle_{i<n}$ with respect to $\equiv_\kappa \,(\mathrm{mod}\, B)$ is determined by which Dedekind cut C of B in I the c_i fall in, and by the order-type modulo κ of the two parts of C into which c_i cuts C; two order-types are equal modulo κ if either they are equal and both are smaller than κ, or else both are at least κ.]

If we had $|S^{<\omega}(A)| > \mu$ for some set $A \subset |\mathscr{C}|$ of power μ, then realizing $\mu^+ \leq \lambda$ many of the types in $S^{<\omega}(A)$, and including these realizations in a model M' of power λ, we would have the situation of some $A \subset |M'|$ of power μ, with M' realizing more than μ many types in $S^{<\omega}(A)$; thus, certainly, M' is not isomorphic to the above M, contradiction to the assumed categoricity in λ. □

Proposition 3.6. *Let λ, λ_1 be singular cardinals such that $\kappa' < \lambda_1 < \lambda$, $\mathrm{cf}\,\lambda_1 = \mathrm{cf}\,\lambda < \kappa$, λ_1 is strong limit. Assume T is stable in λ_1. Then there is $M \in K_\lambda$ such that $|S^{<\omega}(M)| = \lambda$.*

Proof. Following the referee's advice, we give a preview of the proof (which, in the referee's words, "is long but not hard"). The required model M will be an Ehrenfeucht–Mostowski model based on an ordered sum $I = \sum_{i<\mathrm{cf}\,\kappa} I_i$ of saturated dense linear orderings I_i of appropriate powers, without endpoints; $|I| = \lambda$. M will inherit all the numerous partial and total automorphisms of I; we will consider only partial automorphisms, that is, order-preserving mappings g of subsets of I into I that respect levels: for $x \in I_i$, $g(x) \in I_i$. We call a type p over M *$*$-definable over* a set $A \subset I$ if any two tuples $\boldsymbol{b}, \boldsymbol{c}$ of length $<\kappa$ of elements of I behave with respect to p in the same way in the natural sense provided they behave the same way with respect to the ordering of I, the elements of A, and with respect to membership in the sets I_i (see the precise definition before Claim 3). Claim 3 below asserts that every type (in finitely many variables) is $*$-definable over some set of cardinality $\leq \kappa'$; this is a consequence of the non-strong splitting provided by 2.15. $*$-definability allows forming the *translation* $g(p)$ of any type p along a partial automorphism g of the ordering if g

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is defined at least on the set over which p is $*$-definable; this is the content of Claim 5.

Now, the important question concerning a type is whether it is $*$-definable over a *bounded* set of size $\leqslant\kappa'$, where a set is bounded if it is a subset of $\sum_{i<\alpha} I_i$ for some $\alpha < \operatorname{cf}\lambda$; if the answer is "Yes", we call the type *bounded*. It is easy to see that the number of bounded types is $\leqslant\lambda$ (Claim 6 and following passages). Assuming now that the conclusion of the proposition fails, there is an unbounded type p; p is still $*$-definable over a necessarily unbounded, set A of size $\leqslant\kappa'$. The main point is Claim 7 which asserts that for any partial automorphism g of the ordering that moves all but a bounded part of A to a position entirely above A, we have that $g(p) \neq p$; the reason is that the assumption $g(p) = p$ would lead to the conclusion that p is $*$-definable over the bounded unmoved part of A.

The last conclusion gives us a tool of an orderly construction of many distinct types, in virtue of the existence of many appropriate g's. In particular, we can construct a 'tree' of distinct types for which the fact that they are distinct is witnessed on a subset I' of I having a small intersection with each I_i. We have that $|I'| \leqslant \lambda_1$, and using the restrictions of the types to the Skolem hull of I', we get a conclusion contradicting the stability assumption on λ_1. This is the subject of the rest of the proof.

Assume the hypotheses. Let $(L_{\mathrm{Sk}}, T_{\mathrm{Sk}})$ be the Skolem theory associated with $(T, \mathscr{F})$; $|L_{\mathrm{Sk}}| \leqslant \kappa'$. Since (by $<\kappa$-compactness) T has arbitrarily large models, a 'weak' application of 3.1 gives us an E-M defining scheme Φ over $(T, \mathscr{F})$ in the language L_{Sk}.

If J is a $\leqslant\mu$-saturated dense linear ordering without endpoints, in the usual sense of the model theory of finitary logic, then, since the finitary first-order theory of $(J, <)$ admits elimination of quantifiers, $\boldsymbol{a} \equiv_0 \boldsymbol{b}$ in J implies $(J, \boldsymbol{a}) \equiv_{\infty\mu} (J, \boldsymbol{b})$ (elementary equivalence in $(L_0)_{\infty\mu}$, $L_0 = \{<\}$) provided $\boldsymbol{a}, \boldsymbol{b}$ are of length $\leqslant\mu$. In particular, $\boldsymbol{a}, \boldsymbol{b} \in {}^{\leqslant\mu}J$ and $\boldsymbol{a} \equiv_0 \boldsymbol{b}$ imply that for any $\boldsymbol{c} \in {}^{\leqslant\mu}J$ there is $\boldsymbol{d}$ such that $\boldsymbol{a}{}^\wedge\boldsymbol{c} \equiv_0 \boldsymbol{b}{}^\wedge\boldsymbol{d}$. As a special case, $\boldsymbol{a} \equiv_0 \boldsymbol{b}$ implies $\boldsymbol{a} \equiv_\kappa \boldsymbol{b}$.

The general existence theorem for saturated models (see [1]) gives, for every infinite cardinal ρ such that $\rho^\mu = \rho$, a $\leqslant\mu$-saturated ordering of cardinality ρ.

Let $\theta = \operatorname{cf}\lambda = \operatorname{cf}\lambda_1$, and let $\langle \mu_i \rangle_{i<\theta}$ be a strictly increasing sequence of regular cardinals limiting to λ, $\mu_0 \geqslant \lambda_1^+$ (note that $\lambda_1^+ = 2^{\lambda_1}$ since λ_1 is strong limit, $\lambda_1 > \kappa$; see [4]). By Theorem 4 in [16], $\mu_i^{\lambda_1} = \mu_i$. Let I_i be a $\leqslant\lambda_1$-saturated dense linear ordering without endpoints of power μ_i, and let $I = \sum_{i<\theta} I_i$, the ordered sum of the I_i's; the linear orderings I_i are disjoint subsets of I, and $I_i < I_j$ (meaning for all $x \in I_i$ and $y \in I_j$. $x < y$ in I) whenever $i < j < \theta$. Define $M^* = \mathrm{EM}(I, \Phi)$, and $M = M^* \upharpoonright L$; we may, and do, assume that $M < \mathscr{C}$.

A function f with $\operatorname{dom}(f) \subset I$, $\operatorname{range}(f) \subset I$ is *proper* if f is one-to-one, order-preserving, and for all $i < \theta$, $a \in \operatorname{dom}(f)$, $a \in I_i$ iff $f(a) \in I_i$. We also write $f : A \xrightarrow[1]{\cong} B$ for f proper, $\operatorname{dom}(f) = A$, $\operatorname{range}(f) = B$. We write $\boldsymbol{b} \equiv_1 \boldsymbol{c} \pmod A$ when $\mathrm{Id}_A \cup \{\langle b_\beta, c_\beta \rangle : \beta < \alpha\}$ is proper; here $\boldsymbol{b} = \langle b_\beta \rangle_{\beta<\alpha}$, $\boldsymbol{c} = \langle c_\beta \rangle_{\beta<\alpha}$.

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Claim 1. *Any proper function preserves the meaning of Σ_1^*-formulas: if $f: A \xrightarrow[1]{\cong} B$, $\boldsymbol{a} \in {}^{<\kappa}A$, $\varphi(\boldsymbol{x})$ is a Σ_1^*-formula, then $M^* \models \varphi[\boldsymbol{a}] \Leftrightarrow M^* \models \varphi[f(\boldsymbol{a})]$.*

Proof. This follows from 3.4, once we see that $\boldsymbol{a} \equiv_\kappa f(\boldsymbol{a})$, for $\boldsymbol{a} \in {}^{<\kappa}A$. This follows easily from the fact $\boldsymbol{a} \equiv_0 \boldsymbol{b}$ implies $\boldsymbol{a} \equiv_\kappa \boldsymbol{b}$ in $<\kappa$-saturated orderings (separate $\boldsymbol{a}$ into parts, each in one I_i). □ (Claim 1)

Claim 2. *For any $A \subset I$ of cardinality $\leq \kappa'$ and any $\boldsymbol{b}, \boldsymbol{c}$ in ${}^{<\kappa}I$ such that $\boldsymbol{b} \equiv_1 \boldsymbol{c} \pmod A$, there are $\boldsymbol{d}_\alpha$ $(1 \leq \alpha < \kappa)$ such that both*

$$\boldsymbol{b}^\wedge \langle \boldsymbol{d}_\alpha \rangle_{1 \leq \alpha < \kappa} \quad \text{and} \quad \boldsymbol{c}^\wedge \langle \boldsymbol{d}_\alpha \rangle_{1 \leq \alpha < \kappa}$$

are sequences of indiscernibles with respect to the relation $\equiv_1 \pmod A$, i.e.

$$\boldsymbol{b}^\wedge \boldsymbol{d}_{\alpha_1}{}^\wedge \cdots {}^\wedge \boldsymbol{d}_{\alpha_k} \equiv_1 \boldsymbol{c}^\wedge \boldsymbol{d}_{\alpha_1}{}^\wedge \cdots {}^\wedge \boldsymbol{d}_{\alpha_k} \pmod A$$
$$\equiv_1 \boldsymbol{d}_{\beta_0}{}^\wedge \cdots {}^\wedge \boldsymbol{d}_{\beta_k} \pmod A$$

whenever $k < \omega$, $\alpha_1 < \cdots < \alpha_k < \kappa$, $\beta_0 < \cdots < \beta_k < \kappa$.

Proof. It is easily seen that the special case of the claim for the case when $\boldsymbol{b}, \boldsymbol{c}$ and A are all in a fixed I_i implies the general case; one uses the properness involved in the assumption $\boldsymbol{b} \equiv_1 \boldsymbol{c} \pmod A$. Accordingly, assume $i < \theta$, $\boldsymbol{b}, \boldsymbol{c} \in {}^{<\kappa}I_i$, $A \subset I_i$, $\boldsymbol{b} \equiv_0 \boldsymbol{c} \pmod A$. Let $\boldsymbol{b} = \langle b_j \rangle_{j<\gamma}$, $\boldsymbol{c} = \langle c_j \rangle_{j<\gamma}$; we want to define the $\boldsymbol{d}_\alpha = \langle d_j^\alpha \rangle_{j<\gamma}$ $(\alpha < \kappa)$. Whenever $b_j \in A$, and hence $c_j = b_j$, we put $d_j^\alpha = b_j = c_j$ $(\alpha < \kappa)$.

Let C be a Dedekind cut of A, i.e. a maximal convex subset of $I_i - A$ ('X is convex' means that $x, y \in X$, $x < z < y$ imply $z \in X$). Consider the set $\Gamma_C = \{j < \gamma : b_j \in C\} = \{j < \gamma : c_j \in C\}$ of indices. For $j \in \Gamma_C$, we define d_j^α $(\alpha < \kappa)$, so that for all $\alpha < \kappa$,

$$d_j^\alpha \in C,$$
$$\langle d_j^\alpha \rangle_{j \in \Gamma_C} \equiv_0 \langle b_j \rangle_{j \in \Gamma_C} \equiv_0 \langle c_j \rangle_{j \in \Gamma_C},$$
$$d_j^\alpha > b_{j'},\ d_j^\alpha > c_{j'} \quad \text{for all } j, j' \in \Gamma_C,$$

and when $\beta < \alpha < \kappa$,

$$d_j^\alpha > d_{j'}^\beta \quad \text{for all } j, j' \in \Gamma_C.$$

By the $\leq \kappa'$-saturativity of I_i, this is easily done. Since for every $j < \gamma$, there is a unique Dedekind cut C of A such that $b_j \in C$ and $c_j \in C$, unless $b_j = c_j \in A$, the above will specify d_j^α for all $\alpha < \kappa$, $j < \gamma$. It is clear by inspection that the d_j^α so determined will satisfy the requirements. □ (Claim 2)

Let f be a function with $\mathrm{dom}(f) \subset I$, $\mathrm{range}(f) \subset I$, and let $p, q \in S_{\boldsymbol{x}}(M)$. We say f is a *partial similarity between p and q*, in notation $f: p \approx q$, if the following holds: whenever $\varphi(\boldsymbol{x}, \boldsymbol{y})$ is a Σ_1-formula (in L) and $\boldsymbol{t}(\boldsymbol{z})$ is a tuple of L_{Sk}-terms of

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the same length as $\boldsymbol{y}$, then

$$\varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{a}]) \in p \quad \Leftrightarrow \quad \varphi(\boldsymbol{x}, \boldsymbol{t}^M[f(\boldsymbol{a})]) \in q$$

for all tuples $\boldsymbol{a}$ matching $\boldsymbol{z}$ of elements of dom(f). We also write $(\boldsymbol{a}, p) \approx (\boldsymbol{b}, q)$ if, with $\boldsymbol{a} = \langle a_i \rangle_{i<\alpha}$, $\boldsymbol{b} = \langle b_i \rangle_{i<\alpha}$, $f = \{\langle a_i, b_i \rangle : i < \alpha\}$, we have $f : p \approx q$. We say p is $*$-*definable over* $A \subset I$ if for all $\boldsymbol{b}, \boldsymbol{c}$ from I,

$$\boldsymbol{b} \equiv_1 \boldsymbol{c} \ (\mathrm{mod}\, A) \quad \Rightarrow \quad (\boldsymbol{b}, p) \approx (\boldsymbol{c}, p).$$

Claim 3. *Every* $p \in S^{<\omega}(M)$ *is* $*$-*definable over some* $A \subset I$ *of cardinality* $\leqslant \kappa'$.

Proof. By 2.15 (applied to λ_1 as the λ of 2.15), p does not split strongly over some $A_0 \subset |M|$ of cardinality $\leqslant \kappa'$. Let $A \subset I$ be a set of cardinality $\leqslant \kappa'$ such that A_0 is a subset of the Skolem hull of A. We show that p is $*$-definable over A.

Assume $\boldsymbol{b}, \boldsymbol{c}$ are $<\kappa$-tuples in I, and $\boldsymbol{b} \equiv_1 \boldsymbol{c} \ (\mathrm{mod}\, A)$. Use Claim 2 to get $\boldsymbol{d}_\alpha$, $1 \leqslant \alpha < \kappa$, as there. By Claim 1, the tuples

$$\boldsymbol{b}^\wedge \boldsymbol{d}_{\alpha_1}{}^\wedge \cdots {}^\wedge \boldsymbol{d}_{\alpha_k},$$
$$\boldsymbol{c}^\wedge \boldsymbol{d}_{\alpha_1}{}^\wedge \cdots {}^\wedge \boldsymbol{d}_{\alpha_k},$$
$$\boldsymbol{d}_{\beta_0}{}^\wedge \boldsymbol{d}_{\beta_1}{}^\wedge \cdots {}^\wedge \boldsymbol{d}_{\beta_k}$$

all satisfy the same Σ_1^*-formulas over A, whenever $k < \omega$, $\alpha_1 < \cdots < \alpha_k < \kappa$, $\beta_0 < \beta_1 < \cdots < \beta_k < \kappa$. Now, let $\varphi(\boldsymbol{x}, \boldsymbol{y})$ be any Σ_1-formula (over L), $\boldsymbol{t}(\boldsymbol{z})$ a tuple of L_{Sk}-terms matching $\boldsymbol{y}$. Using the given $\boldsymbol{t}$, we conclude from the last-made statement that the tuples

$$\boldsymbol{t}^M[\boldsymbol{b}]^\wedge \boldsymbol{t}^M[\boldsymbol{d}_{\alpha_1}]^\wedge \cdots {}^\wedge \boldsymbol{t}^M[\boldsymbol{d}_{\alpha_k}],$$
$$\boldsymbol{t}^M[\boldsymbol{c}]^\wedge \boldsymbol{t}^M[\boldsymbol{d}_{\alpha_1}]^\wedge \cdots {}^\wedge \boldsymbol{t}^M[\boldsymbol{d}_{\alpha_k}],$$
$$\boldsymbol{t}^M[\boldsymbol{d}_{\beta_0}]^\wedge \boldsymbol{t}^M[\boldsymbol{d}_{\beta_1}]^\wedge \cdots {}^\wedge \boldsymbol{t}^M[\boldsymbol{d}_{\beta_k}]$$

all satisfy the same Σ_1^*-formulas over A, whenever $k < \omega$, $\alpha_1 < \cdots < \alpha_k < \kappa$, $\beta_0 < \cdots < \beta_k < \kappa$. Since the L_{Sk}-translation of a Σ_1-formula is a Σ_1^*-formula (see the introductory part of this section), and A_0 is in the Skolem hull of A, the last-listed three tuples satisfy the same Σ_1-formulas over A_0 in M, hence also the same Σ_1-formulas in $\mathscr{C}$, by 2.11. This says that the two sequences

$$\boldsymbol{t}^M[\boldsymbol{b}], \boldsymbol{t}^M[\boldsymbol{d}_1], \ldots, \boldsymbol{t}^M[\boldsymbol{d}_\alpha], \ldots \quad (\alpha < \kappa),$$
$$\boldsymbol{t}^M[\boldsymbol{c}], \boldsymbol{t}^M[\boldsymbol{d}_1], \ldots, \boldsymbol{t}^M[\boldsymbol{d}_\alpha], \ldots \quad (\alpha < \kappa)$$

are both sequences of indiscernibles over A_0 in $\mathscr{C}$, in the sense of 2.14. Since p does not split strongly over A_0,

$$\varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{b}]) \in p \quad \Leftrightarrow \quad \varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{d}_1]) \in p,$$
$$\varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{c}]) \in p \quad \Leftrightarrow \quad \varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{d}_1]) \in p.$$

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It follows that

$$\varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{b}]) \in p \quad \Leftrightarrow \quad \varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{c}]) \in p.$$

Since φ and $\boldsymbol{t}$ were arbitrary, this proves that $(\boldsymbol{b}, p) \approx (\boldsymbol{c}, p)$. □ (Claim 3)

Claim 4. *For any $f: A \xrightarrow[1]{\cong} B$ with $|A| \leqslant \kappa'$, and any set $C \subset I$ of cardinality $\leqslant \kappa'$, there is an extension $g: A \cup C \xrightarrow[1]{\cong} B \cup D$ of f.*

Proof. Easy; left to the reader. □ (Claim 4)

Claim 5. *Suppose $f: A \xrightarrow[1]{\cong} B$, and $p \in S_{\boldsymbol{x}}(M)$ is $*$-definable over A. Then there is a unique type, denoted by $f(p)$, in $S_{\boldsymbol{x}}(M)$, which is $*$-definable over B and for which the following holds:*

$$(*) \qquad g: A \cup C \xrightarrow[1]{\cong} B \cup D, f \subseteq g \quad \Rightarrow \quad g: p \approx f(p).$$

Proof. Let us first see that there exists a set $f(p) \subset (\Sigma_1)_{\boldsymbol{x}}(M)$ satisfying the conclusion $(*)$, regardless of whether it is a type. The problem is whether for any Σ_1-formula $\varphi(\boldsymbol{x}, \boldsymbol{y})$ and tuple $\boldsymbol{v}$ of elements of M, the requirements in the claim can give contradictory answers to the question "$\varphi(x, \boldsymbol{v}) \in f(p)$". To show this cannot happen, suppose

$$g: A \cup C \xrightarrow[1]{\cong} B \cup D, \tag{1}$$

$$g': A \cup C' \xrightarrow[1]{\cong} B \cup D', \tag{2}$$

both extending f, $\boldsymbol{t}(\boldsymbol{z})$, $\boldsymbol{t}'(\boldsymbol{z}')$ are tuples of L_{Sk}-terms, $\boldsymbol{c} \in C$ matching $\boldsymbol{z}$, $\boldsymbol{c}' \in \boldsymbol{C}$ matching $\boldsymbol{z}'$,

$$\boldsymbol{v} = \boldsymbol{t}^M[g(\boldsymbol{c})] = \boldsymbol{t}'^M[g'(\boldsymbol{c}')]; \tag{3}$$

we need to show

$$\varphi(x, \boldsymbol{t}^M[\boldsymbol{c}]) \in p \quad \Leftrightarrow \quad \varphi(x, \boldsymbol{t}'^M[\boldsymbol{c}']) \in p. \tag{4}$$

To do so, let, by Claim 4, C'' and g'' be such that

$$g'': B \cup D' \cup D'' \xrightarrow[1]{\cong} A \cup C'' \tag{5}$$

extending f. Then for $\boldsymbol{c}_1 = g''(g(\boldsymbol{c}))$ and $\boldsymbol{c}_1' = g''(g'(\boldsymbol{c}'))$, we have

$$\boldsymbol{c}_1 \equiv_1 \boldsymbol{c} \pmod{A} \tag{6}$$

by (1) and (5),

$$\boldsymbol{c}_1' \equiv_1 \boldsymbol{c}' \pmod{A} \tag{7}$$

by (2) and (5), and

$$g(\boldsymbol{c})^\frown g'(\boldsymbol{c}') \equiv_1 \boldsymbol{c}^\frown \boldsymbol{c}_1' \tag{8}$$

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by (5) alone. Thus

$$\varphi(x, t^M[c]) \in p \quad\Leftrightarrow\quad \varphi(x, t^M[(c_1)] \in p \tag{9}$$

by (6) and the $*$-definability of p over A,

$$\varphi(x, t'^M[c']) \in p \quad\Leftrightarrow\quad \varphi(x, t'^M[c_1']) \in p \tag{10}$$

by (7). Now, by (8), (3), and indiscernibility,

$$t^M[c_1] = t'^M[c_1']. \tag{11}$$

Thus, on the right-hand sides of (9) and (10), we have identical formulas, hence (4) follows from (9) and (10).

We have shown that $f(p)$ as a set of formulas is well-defined and satisfies $(*)$. To show $f(p) \in S_x(M)$, let V be an arbitrary subset of M of size $<\kappa$; we'll show that $f(p) \mid V \in S_x(V)$. For every $v \in V$, let us choose a term $t_v(z_v)$ and a finite tuple d_v of elements of I such that $v = t_v^M[d_v]$. Let D be the set of all members of all the tuples d_v, $v \in V$, and let, by Claim 4, g be an extension of f such that $g: A \cup C \overset{\equiv}{\underset{1}{\Rightarrow}} B \cup D$ $(C = g^{-1}(D))$. The mapping $h: U \overset{\equiv}{\Rightarrow} V$ defined by

$$h^{-1}(v) = t_v^M[g^{-1}(d_v)]$$

is a Σ_1^*-elementary map in M^* by Claim 1, hence, by 2.13, it is a Σ_1-elementary map in the sense of $\mathscr{C}$. Therefore, $h(p \mid U) \in S_x(V)$. However, $h(p \mid U) = f(p) \mid V$: by $(*)$, $g: p \approx f(p)$; this, applied to an arbitrary φ, and the particular tuple $t = \langle t_v \rangle_{v \in V}$ gives the desired equality.

Since $f(p) \mid V$ is a type, for any V of cardinality $<\kappa$, by 2.11, $f(p) \in S_x(M)$. The uniqueness of $f(p)$, and its definability over B, are clear. □ (Claim 5)

Claim 6. *For any set $A \subset I$ of cardinality $\leqslant \kappa'$, the number of types in $S^{<\omega}(M)$ that are $*$-definable over A is $\leqslant 2^{(\kappa'^+)}$.*

Proof. Using that κ is strongly inaccessible, $\theta < \kappa$, we easily see that the number of equivalence classes of $<\kappa$-tuples with respect to the relation $\equiv_1 \pmod A$ is $\leqslant \kappa$. The number of Σ_1-formulas is $\leqslant (\kappa')^{<\kappa} \leqslant \kappa'^+$. Notice that a type $*$-definable over A is determined by a function whose arguments are pairs, each of an equivalence class of $\equiv_1 \pmod A$ and a Σ_1-formula, and whose values are Σ_1-formulas, hence the number of such types is $\leqslant (\kappa'^+)^{\kappa \times (\kappa'^+)} = 2^{(\kappa'^+)}$. □ (Claim 6)

The number of bounded subsets (those in some $\bigcup_{j<i} I_j$, $i < \theta$) of I of cardinality $\leqslant \kappa'$ is

$$\sup_{i<\theta} \mu_i^{\kappa'} \leqslant \sup_{i<\theta} \mu_i^+ = \lambda$$

(by Theorem 4 of [16] quoted above). Hence, by Claim 6 (and $2^{(\kappa'^+)} < \lambda_1 < \lambda$), the number of types in $S^{<\omega}(M)$ $*$-definable over some bounded subset of I of cardinality $\leqslant \kappa'$ is $\leqslant \lambda$.

Assume now, contrary to the conclusion of the proposition, that $|S^{<\omega}(M)| > \lambda$. Hence, there is $p \in S^{<\omega}(M)$ that is not $*$-definable over any bounded subset of I. Let us fix such a p. Let $A \subset I$ be a set of cardinality $\leqslant \kappa'$ such that p is $*$-definable over A (by Claim 4). For any set $X \subset I$, let $X_{<i}$ mean $X \cap \bigcup_{j<i} I_j$, $X_i = X \cap I_i$.

For proper functions f, g, and $i < \theta$, let us write $f <_i g$ to mean:

$$f <_i g \quad \Leftrightarrow \quad f \restriction I_{<i} = g \restriction I_{<i} \text{ and } f \restriction I_k < g \restriction I_k \text{ for all } k,\ i \leqslant k < \theta.$$

($f \restriction J = f \cap (J \times I)$, even if $J \not\subset \operatorname{dom} f$; for subsets A, B of I, $A < B$ means for all $x \in a$, $y \in B$, we have $x < y$; $I_{<_i} = \bigcup_{j<i} I_j$.)

Claim 7. *Let $i < \theta$, g a proper function with domain A such that* $\mathrm{id}_A <_i g$. *Then* $p \neq g(p)$.

Proof. Suppose $p = g(p)$. We derive from this that p is $*$-definable over $A_{<_i}$ ($= g''A_{<_i}$), in contradiction to the choice of p. The tool for this derivation will be the fact that p is $*$-definable over both A and $g''A$ (see Claim 5). Let $\boldsymbol{b} = \langle b_\beta \rangle_{\beta<\alpha}$, $\boldsymbol{c} = \langle c_\beta \rangle_{\beta<\alpha}$, and assume $\boldsymbol{b} \equiv_1 \boldsymbol{c} \pmod{A_{<_i}}$, to show that $(\boldsymbol{b}, p) \approx (\boldsymbol{c}, p)$.

Let $S_{<_i} \overset{\text{def}}{=} \{\beta < \alpha : b_\beta \in I_{<_i}\} = \{\beta < \alpha : c_\beta \in I_{<_i}\}$, and for any k such that $i \leqslant k < \theta$, let $S_k \overset{\text{def}}{=} \{\beta < \alpha : b_\beta \in I_k\} = \{\beta < \alpha : c_\beta \in I_k\}$. For each $\beta \in S_k$, use the saturativity of I_k, and choose an element $d_\beta \in I_k$ such that

$$A_k < d_\beta < g''A_k,$$

and

$$\langle d_\beta \rangle_{\beta \in S_k} \equiv_0 \langle b_\beta \rangle_{\beta \in S_k} \equiv_0 \langle c_\beta \rangle_{\beta \in S_k}.$$

Define

$$\boldsymbol{d} = \langle b_\beta \rangle_{\beta \in S_{<_i}} \cup \bigcup_{i \leqslant k < \theta} \langle d_\beta \rangle_{\beta \in S_k}.$$

We are going to show that

$$(\boldsymbol{b}, p) \approx (\boldsymbol{d}, p) \tag{1}$$

and

$$(\boldsymbol{c}, p) \approx (\boldsymbol{d}, p), \tag{2}$$

which will suffice. To see (1), let

$$S_k^1 = \{\beta \in S_k : b_\beta < g''A_k\},$$
$$S_k^2 = S_k - S_k^1.$$

We then have

$$\langle d_\beta \rangle_{\beta \in S_k^1} \cup \langle b_\beta \rangle_{\beta \in S_k^2} \equiv_0 \langle b_\beta \rangle_{\beta \in S_k^1} \cup \langle b_\beta \rangle_{\beta \in S_k^2} \pmod{g''A_k}$$
$$= \langle b_\beta \rangle_{\beta \in S_k},$$

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and hence, for

$$d' \stackrel{\text{def}}{=} \langle b_\beta \rangle_{\beta \in S_{<i}} \cup \bigcup_{i \leq k < \theta} \langle d_\beta \rangle_{\beta \in S_k^1} \cup \bigcup_{i \leq k < \theta} \langle b_\beta \rangle_{\beta \in S_k^2},$$

we have

$$d' \equiv_1 b \pmod{g''A}. \tag{3}$$

But also,

$$\begin{aligned} \langle d_\beta \rangle_{\beta \in S_k^1} \cup \langle b_\beta \rangle_{\beta \in S_k^2} &\equiv_0 \langle d_\beta \rangle_{\beta \in S_k^1} \cup \langle d_\beta \rangle_{\beta \in S_k^2} \pmod{A_k} \\ &= \langle d_\beta \rangle_{\beta \in S_k} \end{aligned}$$

and hence

$$d' \equiv_1 d \pmod{A}. \tag{4}$$

Since p is $*$-definable over both $g''A$ and A, (3) and (4) give that $(d', p) \approx (b, p)$ and $(d', p) \approx (d, p)$, hence (1). For proving (2), we use

$$\langle c_\beta \rangle_{\beta \in S_{<i}} \equiv_1 \langle b_\beta \rangle_{\beta \in S_{<i}} \pmod{A_{<i}}$$

(which follows from $c \equiv_1 b \pmod{A_{<i}}$), but otherwise, the proof of (2) is the same as that of (1). □ (Claim 7)

In Claim 5, we defined $f(p)$ for any proper function f with $\mathrm{dom}(f) = A$. If f is a proper function with $A \subset \mathrm{dom}(f)$, $f(p)$ will mean $(f \restriction A)(p)$.

Claim 8. *There is a set B of cardinality $\leq \kappa'$ such that $A \subset B \subset I$, and such that for any two proper functions f, g with* $\mathrm{dom}(f) = \mathrm{dom}(g) = B$, *and any $i < \theta$, if $f <_i g$, then $f(p) \restriction H(f''B \cup g''B) \neq g(p) \restriction H(f''B \cup g''B)$.* ($H(X)$ is the Skolem hull of X.)

Proof. For every $i < \theta$, choose $C_i \subset I_i$ and $h_i^0 : A_i \underset{1}{\overset{\cong}{\to}} C_i$ such that $A_i < C_i$; let

$$C \stackrel{\text{def}}{=} \bigcup_{i<\theta} C_i, \qquad h^0 = \bigcup_{i<\theta} h_i^0.$$

For any $i < \theta$, let $h^{(0,i)}$ be the proper function with domain A for which $h^{(0,i)} \restriction A_{<i} = \text{identity}$, $h^{(0,i)} \restriction A_k = h_k^0$ for $i \leq k < \theta$. By Claim 7, there is $D^{(i)} \subset I$ such that $|D^{(i)}| < \kappa$ and $p \restriction H(D^{(i)}) \neq h^{(0,i)}(p) \restriction H(D^{(i)})$. Let $D = \bigcup_{i<\theta} D^{(i)}$. Thus $D \subset I$, $|D| < \kappa$, and

$$p \restriction H(D) \neq h^{(0,i)}(p) \restriction H(D) \tag{1}$$

for all $i < \theta$. By Claim 4, choose $B^0 \subset I$ and $h : A \cup B^0 \underset{1}{\overset{\cong}{\to}} C \cup D$ extending h^0; let $h_k = h \restriction I_k$. Define $B = A \cup B^0 \cup D$; clearly, $|B| \leq \kappa'$; we show B satisfies the requirements of the claim.

Let $i < \theta$, and let f, g be proper functions with domain B, $f <_i g$. Let $h^{(i)}$ be the function with $\mathrm{dom}\, h^{(i)} = B$, $h^{(i)} \restriction B_{<i} = \text{identity}$, $h^{(i)} \restriction B_k = h \restriction B_k$; $h^{(i)}$ extends

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$h^{(0,i)}$. By (1), there are $\boldsymbol{d} \in {}^{<\kappa}D$, a Σ_1-formula $\varphi(\boldsymbol{x}, \boldsymbol{y})$, and a tuple $\boldsymbol{t}(\boldsymbol{z})$ of L_{Sk}-terms such that

$$\varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{d}]) \in p \quad \not\Leftrightarrow \quad \varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{d}]) \in h^{(i)}(p) \tag{2}$$

By possibly rearranging $\boldsymbol{d}$ and $\boldsymbol{y}$, we may assume that $\boldsymbol{d} = \boldsymbol{d}^0 {}^\wedge \boldsymbol{d}^1 {}^\wedge \boldsymbol{d}^2$, and $\boldsymbol{d}^1 = {}^\wedge\langle \boldsymbol{d}_k^1 \rangle_{i \leq k < \theta}$, $\boldsymbol{d}^2 = {}^\wedge\langle \boldsymbol{d}_k^2 \rangle_{i \leq k < \theta}$ with $\boldsymbol{d}^0 \in I_{<i}$, $\boldsymbol{d}_k^1 \in I_k$, $\boldsymbol{d}_k^2 \in I_k$, and moreover

$$\boldsymbol{d}_k^1 < \boldsymbol{d}_k^2, \tag{3}$$

$$A_k < \boldsymbol{d}_k^2, \tag{4}$$

$$\boldsymbol{d}_k^1 < C_k \tag{5}$$

for all k, $i \leq k < \theta$; this is possible since $A_k < C_k$. (Of course, e.g. (3) means that $d < d'$ for all members d, d' of the tuples $\boldsymbol{d}_k^1, \boldsymbol{d}_k^2$, respectively.) Let $\boldsymbol{b}_k^2 = (h^{(i)})^{-1}(\boldsymbol{d}_k^2)$, $\boldsymbol{b}^2 = {}^\wedge\langle \boldsymbol{b}_k^2 \rangle_{i \leq k < \theta}$. We have the following two relations:

$$\boldsymbol{A}^\wedge \boldsymbol{d}^{0\wedge} \boldsymbol{d}^{1\wedge} d^2 \equiv_1 f(\boldsymbol{A})^\wedge f(\boldsymbol{d}^0)^\wedge f(\boldsymbol{d}^1)^\wedge g(\boldsymbol{b}^2), \tag{6}$$

$$h^{(i)}(\boldsymbol{A})^\wedge \boldsymbol{d}^{0\wedge} \boldsymbol{d}^{1\wedge} \boldsymbol{d}^2 \equiv_1 g(\boldsymbol{A})^\wedge f(\boldsymbol{d}^0)^\wedge f(\boldsymbol{d}^1)\, {}^\wedge g(\boldsymbol{b}^2). \tag{7}$$

Concerning (6): within $I_{<i}$, it reduces to

$$\boldsymbol{A}_{<i}{}^\wedge \boldsymbol{d}^0 \equiv_1 f(\boldsymbol{A}_{<i})^\wedge f(\boldsymbol{d}^0)$$

which is obviously true; and within I_k $(i \leq k < \theta)$, it reduces to

$$\boldsymbol{A}_k{}^\wedge \boldsymbol{d}_k^{1\wedge} \boldsymbol{d}_k^2 \equiv_0 f(\boldsymbol{A}_k)^\wedge f(\boldsymbol{d}_k^1)^\wedge g(\boldsymbol{b}_k^2)$$

which is true since $\boldsymbol{A}_k{}^\wedge \boldsymbol{d}_k^1 < \boldsymbol{d}_k^2$ by (3) and (4), and $f(\boldsymbol{A}_k)^\wedge f(\boldsymbol{d}_k^1) < g(\boldsymbol{b}_k^2)$ by $f <_i g$. Concerning (7): within $I_{<i}$, it reduces to

$$h^{(i)}(\boldsymbol{A}_{<i})^\wedge \boldsymbol{d}^0 \equiv_1 g(\boldsymbol{A}_{<i})^\wedge f(\boldsymbol{d}^0),$$

which is true since $h^{(i)}$ is the identity on $I_{<i}$, and g and f agree on $I_{<i}$; within I_k $(i \leq k < \theta)$, it reduces to

$$h^{(i)}(\boldsymbol{A}_k)^\wedge \boldsymbol{d}_k^{1\wedge} h^{(i)}(\boldsymbol{b}_k^2) \equiv_0 g(\boldsymbol{A}_k)^\wedge f(\boldsymbol{d}_k^1)^\wedge g(\boldsymbol{b}_k^2)$$

(since $\boldsymbol{d}_k^2 = h^{(i)}(\boldsymbol{b}_k^2)$), which is true since $\boldsymbol{d}_k^1 < h^{(i)}(\boldsymbol{A}_k)^\wedge h^{(i)}(\boldsymbol{b}_k^2)$ by (3) and (5) and $f(\boldsymbol{d}_k^1) < g(\boldsymbol{A}_k)^\wedge g(\boldsymbol{b}_k^2)$ by $f <_i g$.

Let us write $\boldsymbol{e} = f(\boldsymbol{d}^0)^\wedge f(\boldsymbol{d}^1)^\wedge g(\boldsymbol{b}^2)$. Note that $\boldsymbol{e} \in f''B \cup g''B$. (6) and (7) say, respectively, that $\boldsymbol{A}^\wedge \boldsymbol{d} \equiv_1 f(\boldsymbol{A})^\wedge \boldsymbol{e}$, and $h^{(i)}(\boldsymbol{A})^\wedge \boldsymbol{d} \equiv_1 g(\boldsymbol{A})^\wedge \boldsymbol{e}$. By the definition of $f(p)$ (see Claim 5), the first of these relations implies that

$$(\boldsymbol{d}, p) \approx (\boldsymbol{e}, f(p)) \tag{8}$$

and similarly, the second implies that

$$(\boldsymbol{d}, h^{(i)}(p)) \approx (\boldsymbol{e}, g(p)). \tag{9}$$

(for the latter, note that $g(p) = (g \circ (h^{(i)})^{-1})(h^{(i)}(p))$ in the sense of the construction $f(p)$ of Claim 5). (8), (9) and (2) imply that

$$\varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{e}]) \in f(p) \quad \not\Leftrightarrow \quad \varphi(\boldsymbol{x}, \boldsymbol{t}^M[\boldsymbol{e}]) \in g(p).$$

Since $e \in f''B \cup g''B, f(p) \restriction H(f''B \cup g''B) \neq g(p) \restriction H(f''B \cup g''B)$. □ (Claim 8)

Now, we complete the proof of the proposition as follows. Find strictly increasing cardinals μ_i^* for $i<\theta$ such that $\prod_{j<i}\mu_j^* \leq \mu_i^*$ $(i<\theta)$, and $\lambda_i = \sup_{i<\theta}\mu_i^*$.

For any $j<i<\theta$, and $\eta, \eta' \in \prod_{k<i}\mu_k^*$, let us write $\eta <_j^i \eta'$ for: $\eta(j) < \eta'(j)$ and $\eta(k)=\eta'(k)$ for all $k<j$. Then the relation $<^i$ on $\prod_{k<i}\mu_k^*$ defined by

$$\eta <^i \eta' \quad \Leftrightarrow \quad \text{there is } j<i \text{ such that } \eta <_j^i \eta'$$

is the lexicographic linear ordering.

By induction on $i<\theta$, we define, for all $\eta \in \prod_{j<i}\mu_j^*$, a proper function with $\operatorname{dom} f_\eta = B_{<i}$, with B from Claim 8, such that

$$\eta <_j^i \eta' \quad \Rightarrow \quad f_\eta <_j f_{\eta'}$$

(where the latter means, as before, $f_\eta \restriction I_{<j} = f_{\eta'} \restriction I_{<j}$, $f_\eta'' B_k < f_{\eta'}'' B_k$ for $j \leq k < i$). For $\eta = \emptyset$, $f_\emptyset = \emptyset$; for i limit, length$(\eta)=i$, $f_\eta = \bigcup_{j<i} f_{\eta\restriction j}$; the induction hypothesis clearly persists.

Let us assume that $i<\theta$, and f_η has been defined for all $\eta \in \prod_{j<i}\mu_j^*$; we define $f_{\eta\hat{}\langle\alpha\rangle}$ for all $\alpha \in \mu_i^*$. We claim that there is a family of functions $\langle h_{\langle\eta,\alpha\rangle}\rangle_{\eta\in\prod_{j<i}\mu_j^*,\ \alpha\in\mu_i^*}$,

$$h_{\langle\eta,\alpha\rangle} : B_i \underset{1}{\overset{\cong}{\to}} C_{\langle\eta,\alpha\rangle},$$

such that $C_{\langle\eta,\alpha\rangle} < C_{\langle\eta',\alpha'\rangle}$ whenever $\eta <^i \eta'$ or ($\eta=\eta'$ and $\alpha<\alpha'$). Indeed, the claim is equivalent to saying that a certain (quantifier-free) type over $L_0 = \{<\}$ in $|\prod_{j<i}\mu_j^*| \times \nu_i^* = \mu_i^*$ variables can be realized in I_i; since I_i is $<\lambda_1$-saturated, this is possible [so far, we used only $\leq(\kappa')$-saturation of the I_i; this is the place where $<\lambda_1$-saturation is used]. Define

$$f_{\eta\hat{}\langle\alpha\rangle} = f_\eta \cup h_{\langle\eta,\alpha\rangle}.$$

It is clear that we have satisfied the requirements. This completes the definition of the f_η, for $\eta \in \prod_{j<i}\mu_j^*$, $i<\theta$.

For $\nu \in \prod_{i<\theta}\mu_i^*$, let $f_\nu = \bigcup_{i<\theta} f_{\nu\restriction i}$. Then each f_ν is a proper function with domain B, and $\nu <_i^\theta \nu'$ implies that $f_\nu <_i f_{\nu'}$; thus, Claim 8 can be applied to f_ν and $f_{\nu'}$. We conclude that for any $\nu, \nu' \in \prod_{i<\theta}\mu_i^*$, if $\nu \neq \nu'$ (hence $\nu <_i^\theta \nu'$, or $\nu' <_i^\theta \nu$, for some $i<\theta$), then

$$f_\nu(p) \restriction H(f_\nu''B \cup f_{\nu'}''B) \neq f_{\nu'}(p) \restriction H(f_\nu''B \cup f_{\nu'}''B).$$

Let $N = H(\bigcup\{f_\eta''B : \eta \in \prod_{j<i}\mu_j^*, i<\theta\}) \restriction L$; clearly, $\|N\| = \lambda_1$, and $N < M$. By the last inequality, the types $f_\nu(p) \restriction N \in S^{<\omega}(N)$, $\nu \in \prod_{i<\theta}\mu_i^*$, are pairwise distinct. Since, by König's inequality, $|\prod_{i<\theta}\mu_i^*| > \lambda_1$, we obtain that T is unstable in λ_1, contrary to the hypothesis of the proposition. □

Formally, for the next conclusion, the assumptions of Section 2 are lifted, but those of Section 1 are still in force. In particular κ is a compact cardinal.

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Conclusion 3.7. *Suppose $\lambda > \beth_{\kappa+1}(\kappa')$ and K is categorical in λ. Then all $M \in K$ are existentially closed.*

Proof. By 1.13, K has the amalgamation property; by 1.8, it has the joint embedding propety. Thus, Sections 2 and 3 are applicable.

If $\mathrm{cf}\,\lambda \geq \kappa$, the conclusion is known by 1.9. Assume $\theta = \mathrm{cf}\,\lambda < \kappa$. Then, let $\lambda_1 = \beth_\theta(\kappa')$. By 3.5, T is stable in λ_1. Thus, 3.6 is applicable, and we get, using again that T is categorical in λ, that T is stable in λ. By 2.6, the model $M \in K_\lambda$ is μ-saturated, for all regular $\mu \leq \lambda$, *i.e.*, M is saturated. By 2.2(i), M is existentially closed. Using the argument of 1.9, it follows that every model in K is existentially closed. □

4. Stability

We continue with the conventions and assumptions introduced in Sections 1 and 2. Another assumption we make is

Assumption 4.0. Every $M \in K$ is existentially closed.

Note that, by 1.9 and 3.7, if K is categorical in λ, and either $\lambda > \kappa'$ and $\mathrm{cf}\,\lambda \geq \kappa$, or $\lambda > \beth_{\kappa+1}(\kappa')$, the last Assumption holds.

For a sentence σ of $L_{\infty\infty}$, let us write $K \models \sigma$ to mean that $M \models \sigma$ for all $M \in K$. Let T' be the set of all Σ_1-sentences σ such that $\mathscr{C} \models \sigma$; T' includes T. Then, for any sentence $\tau \in L_{\kappa\kappa}$, or more generally, any τ of the form $\forall \boldsymbol{x}\,((\bigwedge \Psi(\boldsymbol{x})) \to \varphi(\boldsymbol{x}))$ with $\Psi(\boldsymbol{x}) \subset L_{\kappa\kappa}$, $\varphi(\boldsymbol{x}) \in L_{\kappa\kappa}$, $K \models \tau$ iff $T' \models \tau$, and in fact, the 'if' direction holds for any $\tau \in L_{\infty\infty}$. Since every $M \in K$ is (isomorphic to) a Σ_1-substructure of $\mathscr{C}$, $K \models T'$, hence the 'if' direction follows for any $\tau \in L_{\infty\infty}$. Conversely, if $K \models \tau$ then, picking any fixed $M_0 \in K$, $\mathrm{Diag}_{\mathscr{F}}(M_0) \models \tau$, since every $\mathscr{F}$-extension of M_0 is in K. Hence, by applying $<\kappa$-compactness, we find $\sigma \in \Sigma_1$, such that $M_0 \models \sigma$, hence $\sigma \in T'$, and $\sigma \models \tau$; this shows $T' \models \tau$. By another compactness argument, we obtain

Proposition 4.1. (i) *For any Σ_1-formula $\varphi(\boldsymbol{x})$, there is a Σ_1-formula $\psi(\boldsymbol{x})$ such that $\neg\varphi(\boldsymbol{x})$ is equivalent to $\psi(\boldsymbol{x})$ in K: $K \models \forall \boldsymbol{x}\,((\neg\varphi(\boldsymbol{x})) \leftrightarrow \psi(\boldsymbol{x}))$.*

(ii) *For any $\varphi(\boldsymbol{x}) \in L_{\kappa\kappa}$, there is a Σ_1-formula $\psi(\boldsymbol{x})$ such that $\varphi(\boldsymbol{x})$ is equivalent to $\psi(\boldsymbol{x})$ in K.*

(iii) *For any M, N in K, $M < N \Leftrightarrow M <_1 N \Leftrightarrow M <_{\kappa\kappa} N$ (where $M <_{\kappa\kappa} N$ means: $M \models \varphi[\boldsymbol{a}] \Leftrightarrow N \models \varphi[\boldsymbol{a}]$ for all $\varphi(\boldsymbol{x}) \in L_{\kappa\kappa}$ and $\boldsymbol{a} \in |M|$).*

Proof. (i) Let $\varphi(\boldsymbol{x})$ be a Σ_1-formula, and let $\Psi \overset{\text{def}}{=} \{\neg\psi(\boldsymbol{x}) : \psi(\boldsymbol{x}) \in \Sigma_1,\ \models \forall \boldsymbol{x}\,(\varphi(\boldsymbol{x}) \to \neg\psi(\boldsymbol{x}))\}$. We claim that $K \models \forall \boldsymbol{x}\,((\bigwedge \Psi(\boldsymbol{x})) \to \varphi(\boldsymbol{x}))$. Suppose, to the contrary, that there is $M \in K$ and $\boldsymbol{a} \in |M|$ such that $M \models \bigwedge \Psi[\boldsymbol{a}]$ and $M \models \neg\varphi[\boldsymbol{a}]$. Consider the set $\Sigma = \mathrm{Diag}_{\mathscr{F}}(M) \cup \{\varphi(\boldsymbol{a})\}$ of sentences. If there were $\Sigma' \in$

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$\mathscr{P}_{<\kappa}(\mathrm{Diag}_{\mathscr{F}}(M))$ such that $\Sigma' \cup \{\varphi(\boldsymbol{a})\}$ had no model, then, by existentially quantifying out the $|M|$-constants other than those in $\boldsymbol{a}$ in the conjunction $\bigwedge \Sigma'$, we would get a Σ_1-sentence $\psi(\boldsymbol{a})$ with $M \models \psi(\boldsymbol{a})$ and $\models \forall \boldsymbol{x}\, (\varphi(\boldsymbol{x}) \rightarrow \neg\psi(\boldsymbol{x}))$, hence $\psi(\boldsymbol{x}) \in \Psi$, contrary to the choice of M. But, any model N of Σ is an $\mathscr{F}$-extension of M, $M \prec N$, with $N \models \varphi[\boldsymbol{a}]$; since $M \models \neg\varphi[\boldsymbol{a}]$, this means that $M \not\prec_1 N$. This contradicts the assumption that M is e.c.

The claim having been proved, we have $T' \models \forall \boldsymbol{x}\, ((\bigwedge \Psi(\boldsymbol{x})) \rightarrow \varphi(\boldsymbol{x}))$ (by remarks made above), hence by compactness, $T' \models \forall \boldsymbol{x}\, ((\bigwedge \Psi'(\boldsymbol{x})) \rightarrow \varphi(\boldsymbol{x}))$ for some $\Psi'(\boldsymbol{x}) \subset \Psi(\boldsymbol{x})$ of cardinality $<\kappa$. The conjunction $\bigwedge \Psi'(\boldsymbol{x})$ is equivalent to $\neg\psi(\boldsymbol{x})$ for a Σ_1-formula $\psi(\boldsymbol{x})$; the definition of Ψ' gives $T' \models \forall \boldsymbol{x}\, (\varphi(\boldsymbol{x}) \rightarrow \neg\psi(\boldsymbol{x}))$; the way Ψ' was obtained gives $T' \models \forall \boldsymbol{x}\, ((\neg\psi(\boldsymbol{x})) \rightarrow \varphi(\boldsymbol{x}))$. Thus, $T' \models \forall \boldsymbol{x}\, (\varphi(\boldsymbol{x}) \leftrightarrow \neg\psi(\boldsymbol{x}))$, which suffices.

(ii) This follows from (i) by an induction on the complexity of formulas in $L_{\kappa\kappa}$.

(iii) Immediate from Assumption 4.0 and (ii). □

Definition 4.2. *K has a long definable order,* or simply, *K has order,* if there are a formula $\varphi(\boldsymbol{x}, \boldsymbol{y}) \in L_{\kappa\kappa}$, with $\mathrm{length}(\boldsymbol{x}) = \mathrm{length}(\boldsymbol{y})(<\kappa)$, and a sequence $\langle \boldsymbol{a}_i \rangle_{i<\kappa}$ of length κ of tuples in $\mathscr{C}$ such that for all $i, j < \kappa$,

$$\models \varphi[\boldsymbol{a}_i, \boldsymbol{a}_j] \quad \text{iff} \quad i<j.$$

Proposition 4.3. *If K has order, then for all* $\lambda > \kappa'$, $I(\lambda, K)$(= *the number of isomorphism types of models in* K_λ) *is equal to* 2^λ.

Proof. This is essentially a special case of Theorem 3.14, part (2) of Chapter III in [15]. In detail, let $\varphi(\boldsymbol{x}, \boldsymbol{y})$ be a formula defining a long order in $\mathscr{C}$. By 2.1, φ and $\neg\varphi$ are equivalent in K to Σ_1-formulas, say, $\exists \boldsymbol{z}_1\, \psi_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1)$ and $\exists \boldsymbol{z}_2\, \psi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_2)$, respectively; here, ψ_1, ψ_2 are Boolean combinations of formulas in $\mathscr{F} \subset L_{\kappa\omega}$. Introduce new and disjoint tuples $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ of individual constants; let the similarity types τ_1, τ_2 be obtained by adjoining the constant $\boldsymbol{c}_1$, respectively $\boldsymbol{c}_2$, to L; $\tau_1 \cap \tau_2 = L$. Let $\varphi_i(\boldsymbol{x}, \boldsymbol{y}) = \psi_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}_i)$ $(i = 1, 2)$. Take ψ of *loc.cit.* to be the conjunction of the axioms of T; $\psi \in L_{(\kappa')^+,\omega}$, thus χ of *loc.cit.* can be taken to be κ'. $\sigma \overset{\mathrm{def}}{=} |\mathrm{length}(\boldsymbol{x})| < \kappa$, hence $\sigma^+ < \kappa \leqslant \kappa'$. Notice that, by $<\kappa$-compactness, once we have a φ-order of length κ in $\mathscr{C}$, we have φ-orders in $\mathscr{C}$ of arbitrary lengths. The assumption $(*)$ in *loc.cit.* is satisfied: every large enough model of T is in K, so the orders defined by φ translate into sequences related to φ_1, φ_2 as needed for $(*)$. The conclusion of *loc.cit.* is the assertion of the proposition. □

Since we are interested in the case when K is categorical in some $\lambda > \kappa'$, it is reasonable to make the following assumption for the rest of this section:

Assumption 4.4. K does not have (long, definable) order.

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In the case $\kappa = \aleph_0$, this is the usual condition of *stability* for T' (for T', see above before 4.1; it is a complete theory in the usual sense if $\kappa = \aleph_0$). We also call K *1-stable* if the last Assumption holds for it. On the basis of the assumption of 1-stability, we develop a generalization of a part of the theory of forking of types originally developed in [9] for stable first-order theories. The part in question is the one that concerns forking of types over models, rather than types over general sets. Although, for technical convenience, our definition of non-forking (independence) will be for a general base-set, we will be essentially restricted to using the notion with base-sets that are models, because of similar restrictions in the properties we can prove for the notion. It is to be noted that the notions to be introduced will coincide with the 'usual' ones for $\kappa = \aleph_0$ only if the base-set is a model.

In the rest of this section, a 'formula over A' means one of $L_{\kappa\kappa}(A)$, that is, an $L_{\kappa\kappa}$-formula with parameters in A.

Definition 4.5. (i) For sets A, B, C (subsets of $\mathscr{C}$, of cardinality $<\|\mathscr{C}\|$ as always), we say that A and B are independent over C, and write $\mathrm{NF}(C, A, B)$, or $A \downarrow_C B$, if the following holds: whenever $\varphi(\boldsymbol{x})$ is a formula over $C \cup B$ (with parameters in $C \cup B$), $\boldsymbol{a} \in A$, then $\models \varphi[\boldsymbol{a}]$ implies there is $\boldsymbol{a}'$ in C such that $\models \varphi[\boldsymbol{a}']$.

(ii) For $C \subseteq B$ and $p \in S_{\boldsymbol{x}}(B)$, we say that p *does not fork* (dnf) *over* C if for all $\varphi(\boldsymbol{x}) \in p$, there is $\boldsymbol{a}' \in C$ such that $\models \varphi[\boldsymbol{a}']$.

Clearly, $A \downarrow_C B$ iff $\mathrm{tp}(\boldsymbol{A}/C \cup B)$ dnf over C (where $\boldsymbol{A}$ is any tuple enumerating A). The notation $\boldsymbol{a} \downarrow_C \boldsymbol{b}$ (or $\mathrm{NF}(C, \boldsymbol{a}, \boldsymbol{b})$, or $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent over C) is used in the natural sense: $\models \varphi[\boldsymbol{a}, \boldsymbol{b}]$ implies $\models \varphi[\boldsymbol{a}', \boldsymbol{b}]$ for some $\boldsymbol{a}' \in C$, whenever $\varphi(\boldsymbol{x}, \boldsymbol{y})$ is a formula over C; $\boldsymbol{a} \downarrow_C \boldsymbol{b}$ is equivalent to saying $A \downarrow_C B$ with A, B the ranges of the tuples $\boldsymbol{a}$, $\boldsymbol{b}$, respectively. The meaning of $\boldsymbol{a} \downarrow_C B$ should be clear.

In the next proposition, $a, b, c, \ldots$ may denote (not just elements of $|\mathscr{C}|$, but also) tuples of length $<\kappa$ of elements of $|\mathscr{C}|$.

Proposition 4.6. *Suppose* $a \downarrow_C b$, $b \downarrow_C a'$, *and* $a \underset{C}{\sim} a'$ [*i.e.* $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$]. *Then*

$$a^\frown b \underset{C}{\sim} a'^\frown b.$$

Proof. Suppose the hypotheses, and assume that, contrary to the conclusion,

(i) $\models \varphi(a, b)$,

(ii) $\models \neg\varphi(a', b)$

for some $\varphi(x, y)$ over C. By induction on $i < \kappa$ define tuples a_i, b_i from C satisfying the following:

(iii) $\models \varphi(a_i, b)$,

(iv) $\models \neg\varphi(a_i, b_j)$ when $j < i$,

(v) $\models \neg\varphi(a, b_i)$,

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or equivalently to (v), since $a \underset{C}{\sim} a'$ and $b_i \in C$

(v′) $\vDash \neg\varphi(a', b_i)$,

(vi) $\vDash \varphi(a_j, b_i)$ when $j \leq i$.

Let $i < \kappa$ and assume a_j, b_j have been defined for all $j < i$ satisfying the relevant parts of (iii) to (vi). Note that, by (i) and (v) (for $j < i$ rather than i),

$$\vDash \varphi(a, b) \wedge \bigwedge_{j<i} \neg\varphi(a, b_j).$$

Here, we have a formula $\psi(a, b)$ over C, hence by $a \perp_C b$, there is $a_i \in C$ with

$$\vDash \varphi(a_i, b) \wedge \bigwedge_{j<i} \neg\varphi(a_i, b_j),$$

that is, we have satisfied (iii) and (iv). Next, by (ii), (iii) and the choice of a_i, we have

$$\vDash \neg\varphi(a', b) \wedge \bigwedge_{j\leq i} \varphi(a_j, b).$$

Therefore, by $b \perp_C a'$, there is $b_i \in C$ such that

$$\vDash \neg\varphi(a', b_i) \wedge \bigwedge_{j\leq i} \varphi(a_j, b_i),$$

that is, we have satisfied (v′) ($\equiv$(v)) and (vi).

Having completed the construction, (iv) and (vi) show that there is a long order defined by φ, in contradiction to Assumption 4.4. □

Proposition 4.7. *Given sets A, B and a model M(!), there is a Σ_1-elementary mapping f with domain $M \cup A$ such that $f \upharpoonright M$ = identity, and $f''A \perp_M B$.*

Proof. Let us introduce a new individual constant $\bar{b}$ for each $b \in M \cup B$, and another one, $\bar{a}$ for each $a \in A - M$ such that $\bar{b} \neq \bar{b}'$, $\bar{a} \neq \bar{a}'$, $\bar{b} \neq \bar{a}$ whenever $b \neq b'$, both in $M \cup B$, and $a \neq a'$, both in $A - M$. Let

$$\Sigma^{(1)} = \{\varphi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{c}}) : \varphi(\boldsymbol{x}, \boldsymbol{y}) \in L_{\kappa\kappa},\ \boldsymbol{a} \in A - M,\ \boldsymbol{c} \in M,\ \vDash \varphi[\boldsymbol{a}, \boldsymbol{c}]\},$$
$$\Sigma^{(2)} = \{\varphi(\bar{\boldsymbol{b}}) : \varphi(\boldsymbol{x}) \in L_{\kappa\kappa},\ \boldsymbol{b} \in M \cup B,\ \vDash \varphi[\boldsymbol{b}]\},$$
$$\Sigma^{(3)} = \{\psi(\bar{\boldsymbol{a}}, \boldsymbol{b}) : \psi(\boldsymbol{x}, \boldsymbol{y}) \in L_{\kappa\kappa},\ \boldsymbol{a} \in A - M,\ \boldsymbol{b} \in M \cup B,$$
$$\text{and for all } \boldsymbol{c} \in M \text{ matching } \boldsymbol{x},\ \vDash \psi[\boldsymbol{c}, \boldsymbol{b}]\}.$$

Let $\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)} \cup \Sigma^{(3)}$; we claim that Σ is $<\kappa$-consistent. Let Σ' be a subset of Σ of cardinality $<\kappa$. Let $A_0 = \{a \in A - M : \bar{a}$ occurs in some formula in $\Sigma'\}$. Since $M <_{\kappa\kappa} \mathscr{C}$, there is an assignment $a \mapsto a^*$, $a \in A_0 \Rightarrow a^* \in M$, such that ($\mathscr{C}$, a^* for $\bar{a}$, c for $\bar{c})_{a\in A-M, c\in M} \vDash \Sigma' \cap \Sigma^{(1)}$. Thus ($\mathscr{C}$, a^* for $\bar{a}$, b for $\bar{b}) \vDash \Sigma'$ since in $\Sigma^{(2)}$, no $\bar{a}$ occurs, and the definition of $\Sigma^{(3)}$ ensures that ($\mathscr{C}$, a^* for $\bar{a}$, b for $\bar{b}) \vDash \Sigma^{(3)}$ with *any* $a^* \in M$. We have shown that Σ is $<\kappa$-consistent.

Let (N, a^{**} for $\bar{a}$, b^* for $\bar{b})_{a\in A-M, b\in M\cup B}$ be a model of Σ. By the universality of $\mathscr{C}$, we may assume that $N < \mathscr{C}$. Because of the fact that (N, b^* for $\bar{b})_{b\in M\cup B}$ is a

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model of $\Sigma^{(2)}$, the mapping $b^* \mapsto b$ ($b \in M \cup B$) is a Σ_1-elementary mapping; let g be an automorphism of $\mathscr{C}$ extending this mapping. By passing from N to $g''N$, from a^{**} to $g(a^{**})$, and from b^* to $g(b^*) = b$, we may assume that $M \cup B \subset N < \mathscr{C}$ and $b^* = b$ for all $b \in M \cup B$. Let the function f with domain $M \cup A$ be defined by $f(a) = a^{**}$ for $a \in A - M$, $f(c) = c$ for $c \in M$. Since $(N, a^{**}, b)_{a \in A-M, b \in M \cup B} \models \Sigma^{(1)}$, f is Σ_1-elementary, and of course, $f \mid M = \text{identity}$. If $\psi(\boldsymbol{x}, \boldsymbol{y}) \in L_{\kappa\kappa}$, $\boldsymbol{a} \in A - M$, $\boldsymbol{b} \in M \cup B$, and $\models \psi(f(\boldsymbol{a}), \boldsymbol{b})$, then $\neg\psi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \notin \Sigma^{(3)}$, since otherwise, $(N, a^{**}, b)_{a \in A-M, b \in M \cup B} \models \Sigma^{(3)}$ would say that $\models \neg\psi(f(\boldsymbol{a}), \boldsymbol{b})$. The definition of $\Sigma^{(3)}$ says that there is $\boldsymbol{c} \in M$ such that $\not\models \neg\psi[\boldsymbol{c}, \boldsymbol{b}]$, i.e. $\models \psi[\boldsymbol{c}, \boldsymbol{b}]$. This shows that $f''A \downarrow_M B$. □

Proposition 4.8. *Properties of the independence relation* ($\boldsymbol{A}$ *denotes a tuple enumerating* A, *etc.*)

(I) (Invariance under Σ_1-elementay maps)

$$\boldsymbol{A}^\frown \boldsymbol{B}^\frown \boldsymbol{C} \sim \boldsymbol{A}'^\frown \boldsymbol{B}'^\frown \boldsymbol{C}',\ A \underset{C}{\downarrow} B \quad \Rightarrow \quad A' \underset{C'}{\downarrow} B'$$

(M) (Monotonicity)

$$A' \subseteq A,\ C \subseteq C' \subseteq B' \subseteq B,\ A \underset{C}{\downarrow} B \quad \Rightarrow \quad A' \underset{C'}{\downarrow} B'$$

(T) (Transitivity)

$$C \subseteq C',\ A \underset{C'}{\downarrow} B,\ C' \underset{C}{\downarrow} B \quad \Rightarrow \quad A \underset{C}{\downarrow} B$$

$(C)_{<\kappa}$ ($<\kappa$-continuity)

(i) $[\forall A' \in \mathscr{P}_{<\kappa}(A), \forall B' \in \mathscr{P}_{<\kappa}(B)\ A' \underset{C}{\downarrow} B'] \quad \Rightarrow \quad A \underset{C}{\downarrow} B,$

(ii) $[\langle A_i \rangle_{i<\kappa}, \langle C_i \rangle_{i<\kappa}$ *are increasing*, $\forall i < \kappa\ A_i \underset{C'}{\downarrow} B] \quad \Rightarrow \quad \bigcup_{i<\kappa} A_i \underset{\bigcup_{i<\kappa} C_i}{\downarrow} B.$

(E) (Existence) *For any* A, B, M *there is* A' *such that*

$$\boldsymbol{A}' \underset{M}{\sim} \boldsymbol{A} \quad \text{and} \quad A' \underset{M}{\downarrow} B.$$

(S) (Symmetry)

$$A \underset{M}{\downarrow} B \quad \Rightarrow \quad B \underset{M}{\downarrow} A.$$

(U) (Uniqueness)

$$\boldsymbol{A}' \underset{M}{\sim} \boldsymbol{A},\ A' \underset{M}{\downarrow} B,\ A \underset{M}{\downarrow} B \quad \Rightarrow \quad \boldsymbol{A}' \underset{M \cup B}{\sim} \boldsymbol{A}.$$

$(B)_\mu$ *Suppose* $\mu^{<\kappa} = \mu \geq \kappa'$.

(i) *For any* A, B, *if* $|B| \leq \mu$, *then there is* $C \in \mathscr{P}_{\leq\mu}(A)$ *such that* $A \downarrow_C B$.

(ii) *For any* A, M, *if* $|A| \leq \mu$, *then there is* $N < M$ *such that* $\|N\| \leq \mu$ *and* $A \downarrow_N M$.

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Proof. (I), (M). Clear.

(T). Suppose $\varphi(x)$ is a formula over $C \cup B$, and $\models \varphi[a]$, $a \in A$ (again, here and below, $a, b, \ldots$ may denote tuples of length $<\kappa$ of elements of $\mathscr{C}$, $x, y, \ldots$ tuples of variables). Since $C \subseteq C'$ and $A \perp_{C'} B$, there is $a' \in C'$ such that $\models \varphi[a']$. Since $C' \perp_C B$, there is $a'' \in C$ such that $\models \varphi[a'']$, as required.

$(C)_{<\kappa}$. Clear.

(E). This is 4.7.

(S). This follows from the special case $a \perp_M b \Rightarrow b \perp_M a$ (by $(C)_{<\kappa}$); in turn, the latter follows from 4.6 and (E): assume $a \perp_M b$; by (E), let b' be such that $b' \underset{M}{\approx} b$, $b' \perp_M a$; by 4.6, $a^\wedge b \underset{M}{\approx} a^\wedge b'$; thus, by (I), $b \perp_M a$ follows from $b' \perp_M a$.

(U). By $(C)_{<\kappa}$, this follows from the special case

$$a' \underset{M}{\approx} a,\ a' \underset{M}{\perp} b,\ a \underset{M}{\perp} b \quad \Rightarrow \quad a'^\wedge b \underset{M}{\approx} a^\wedge b.$$

In turn, this follows by 4.6 once we note that, by (S), $a' \perp_M b$ implies $b \perp_M a'$.

$(B)_\mu$ (i). Define, by induction on $i < \kappa$, subsets C_i of A such that $|C_i| \leqslant \kappa'$ as follows. Having defined C_j for $j < i$, consider all formulas $\varphi(x)$ over $B \cup \bigcup_{j<i} C_j$; since $\mu^{<\kappa} = \mu$, there are $\leqslant\mu$ many such formulas; for each such $\varphi(x)$ such that $\models \varphi[a]$ for some $a \in A$, choose a particular such $a = a_\varphi$, and define C_i as the union of all the tuples a_φ, φ as above. This completes the definition of $\langle C_i \rangle_{i<\kappa}$; put $C = \bigcup_{i<\kappa} C_i$. Any formula $\varphi(x)$ over $B \cup C$ is over $B \cup \bigcup_{j<i} C_j$ for some $i < \kappa$; if $\models \varphi[a]$, $a \in A$, then $a_\varphi \in C$ witnesses that $A \perp_C B$.

(ii). In the proof of (i), if A is a model, then, by dLST, each C_i can be made into a model. This means that, under the conditions in (ii), we can find $N < M$, $\|N\| \leqslant \mu$, such that $M \perp_N A$. By (S), $A \perp_N M$ as follows. □

Remark. Most of the properties of $\perp$ in 4.8 have useful forms in terms of non-forking of types:

(I). $C \subseteq B$, f is a Σ_1-elementary map with domain B, $p \in S(B)$, p dnf over $A \Rightarrow f(p)$ dnf over $f''A$.

(M). $p \in S(B)$, $C \subseteq C' \subseteq B' \subseteq B$, p dnf over $C \Rightarrow p \mid B'$ dnf over C' (and $q \mid B'$ dnf over C' for any subtype $q \in S(B)$ of $p: q \in S_{\boldsymbol{x}'}(B)$ for some $\boldsymbol{x}' \subset \boldsymbol{x}$ if $p \in S_{\boldsymbol{x}}(B)$).

$(C)_{<\kappa}$. For $p \in S(B)$, $C \subseteq B$, if $p \mid (B' \cup C)$ dnf over C for all $B' \in \mathscr{P}_{<\kappa}(B)$, then p dnf over C.

(T). $C \subseteq C' \subseteq B$, $p \in S(B)$, p dnf over C', $p \mid C'$ dnf over $C \Rightarrow p$ dnf over C.

(E). For any $M \subseteq B$, $p \in S(M)$, there is $q \in S(B)$ such that $q \mid M = p$ and q dnf over M.

(U). If $M \subseteq B$, $p, q \in S(B)$ both dnf over M, $p \mid M = q \mid M$, then $p = q$.

$(B)_{\leqslant\kappa''}$. Any $p \in S^{\leqslant\kappa''}(M)$ dnf over some $N < M$ of cardinality $\leqslant\kappa''$ $(\kappa'' = (\kappa')^{<\kappa})$.

Proposition 4.9. *Suppose* $p \in S^{<\omega}(\bigcup^{\uparrow}_{i<\alpha} M_i)$, *where* $\alpha < \kappa$. *Then* p *dnf over* M_i *for some* $i < \alpha$.

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Proof. Let U be a κ-complete ultrafilter on a set I. For any model M, the ultrapower $M^U = \prod_{i\in I} M/U$ of M is a model in K again, with the canonical $L_{\kappa\kappa}$-elementary embedding $M \to M^U$ mapping each $a \in |M|$ into $\langle a\rangle \overset{\text{def}}{=} \langle a\rangle_{i\in I}/U$. If $M < \mathscr{C}$, we may take M^U to be an $\mathscr{F}$-substructure of $\mathscr{C}$, by applying an $\mathscr{F}$-embedding $f : M^U \to \mathscr{C}$ over M, that is, for which $f(\langle a\rangle) = a$. Below, we will mean by M^U the copy $f''M^U$ for such an f; note that the canonical embedding has become an inclusion $M < M^U$; also, by $\langle a_i\rangle_{i\in I}/U$ (a typical element of M^U) we will mean $f(\langle a_i\rangle_{i\in I}/U)$, with $\langle a_i\rangle_{i\in I}/U$ here in the original sense. Note, however, that M^U is not uniquely determined within $\mathscr{C}$; it is determined up to an M-isomorphism.

For $A \subset \mathscr{C}$ ($|A| < \|\mathscr{C}\|$), we define A^U such that $A \subseteq A^U \subset \mathscr{C}$ (A^U is somewhat ambiguous as M^U above is) as follows: we consider any M with $A \subset M < \mathscr{C}$; we take

$$A^U \overset{\text{def}}{=} \{\langle a_i\rangle_{i\in I}/U : a_i \in A \text{ for all } i \in I\} \subset M^U,$$

with M^U meant in the original sense; we finally let A^U be $f''A^U$ for any $f : M^U \to \mathscr{C}$ over M.

Suppose $A \subseteq B$; consider the ultrapower B^U, and the canonical copy of A^U inside B^U; by A^U below, we mean this canonical copy. Then, we claim, we have

$$A^U \underset{A}{\perp} B.$$

Indeed, assume that $\vDash \varphi[\boldsymbol{c}, \boldsymbol{b}]$, with $\boldsymbol{b} \in B$, $\boldsymbol{c} \in A^U$, $\boldsymbol{c} = \langle c_\beta\rangle_{\beta<\alpha}$, $c_\beta = \langle a_i^\beta\rangle_{i\in I}/U$. By Łos's theorem on ultraproducts, we have $\vDash \varphi[\langle a_i^\beta\rangle_{\beta<\alpha}, \boldsymbol{b}]$ for all $i \in P$, with some $P \in U$, hence for at least one i. This shows what we want since $a_i^\beta \in A$ for all $\beta < \alpha$.

Turning to the data given in the proposition, let $M = \bigcup_{i<\alpha} M_i$, and let U be a κ-complete ultrafilter on some set I such that p is realized in M^U by $d \in M^U$, say. [Such U exists, since κ is compact, by the familiar argument: we let I be the set of all subsets of p of cardinality $<\kappa$, and U a κ-complete ultrafilter on I such that for every $i \in I$, $[i] \overset{\text{def}}{=} \{j \in I : i \subseteq j\}$ belongs to U; since $\bigcap_{\beta<\alpha}[i_\beta] = [\bigcup_{\beta<\alpha} i_\beta]$ $(\alpha < \kappa)$, and thus the intersection of $<\kappa$ many $[i]$'s is non-empty, such U exists by κ being compact (see [4]). For any $i \in \mathscr{P}_{<\kappa}(p) = I$, let $a_i \in |M|$ be such that $\vDash (\bigwedge i)[a_i]$. Then $d = \langle a_i\rangle_{i\in I}/U$ will realize p.] Let M_i^U mean the canonical copy of M_i^U inside M^U; then, as it is easily seen, $M^U = \bigcup_{i<\alpha} M_i^U$, since $M = \bigcup_{i<\alpha} M_i$, $\alpha < \kappa$, and I is κ-complete. Thus, since d is a *finite* tuple, $d \in M_i^U$ for some $i \in I$. Applying our above general observation, we have $M_i^U \perp_{M_i} M$, hence $d \perp_{N_i} M$, hence $p = \text{tp}(d/M)$ dnf over M_i, as desired. □

Proposition 4.10. *With α an arbitrary ordinal >0, assume $p \in S^{<\omega}(\bigcup_{i<\alpha}^{\uparrow} M_i)$, $C \subset M_0$, and $p \mid M_i$ dnf over C for every $i < \alpha$. Then p dnf over C.*

Proof. If $\text{cf}\,\alpha \geq \kappa$, then every $B \in \mathscr{P}_{<\kappa}(\bigcup_{i<\alpha}^{\uparrow} M_i)$ is contained in some M_i, $i < \alpha$, hence the assertion follows from $(C)_{<\kappa}$. Otherwise, if $\text{cf}\,\alpha < \kappa$, then we may assume without loss of generality that $\alpha = \operatorname{cf}\alpha < \kappa$, and by 4.9, we have that p dnf over C. □

All along, we have been interested in the case when κ is a compact cardinal greater than $\aleph_0$, although so far, the assumption $\kappa > \aleph_0$ was not essential. At this point, we can make an essential use of that assumption, and we formally make the

Assumption 4.11. $\kappa > \aleph_0$ for the rest of this section.

Under 4.11, 'superstability' is a consequence of (1-) stability unlike in the finitary case.

Proposition 4.12. *Suppose α is an arbitrary ordinal >0, and $p \in S^{<\omega}(\bigcup^{\uparrow}_{i<\alpha} M_i)$. Then p dnf over M_i for some $i < \alpha$.*

Proof. For $\alpha < \kappa$, this is 4.9. The case $\operatorname{cf}\alpha < \kappa$ is now an immediate consequence. Assume $\operatorname{cf}\alpha \geq \kappa$, and assume, contrary to the assertion, that p forks over M_i for all $i < \alpha$. By induction on $n < \omega$, we define the increasing sequence $\langle i_n \rangle_{n<\omega}$ of ordinals $<\alpha_0$. i_0 is arbitrary. Having defined i_n, by $(\mathrm{C})_{<\kappa}$, there is $B \subset \bigcup^{\uparrow}_{i<\alpha} M_i$ of cardinality $<\kappa$ such that $p \mid (B \cup M_{i_n})$ forks over M_{i_n}. Since $\operatorname{cf}\alpha \geq \kappa$, there is $i_{n+1} \geq i_n$ with $B \subset M_{i_{n+1}}$; we have that $p \mid M_{i_{n+1}}$ forks over M_{i_n} (by (M)). Having defined the i_n, we let $M^* = \bigcup^{\uparrow}_{n<\omega} M_{i_n} < M$. By (M), $q = p \mid M^*$ forks over each M_{i_n}, $n < \omega$; i.e., we have a situation when the assertion of 4.9 fails to hold, with $\alpha = \omega$. But $\omega < \kappa$; contradiction to 4.9. □

Corollary 4.13. (i) *For $M = \bigcup^{\uparrow}_{i<\alpha} M_i$, and finite tuples a and b, if $a \underset{M_i}{\sim} b$ for all $i < a$, then $a \underset{M}{\sim} b$.*

(ii) *For $M = \bigcup^{\uparrow}_{i<\alpha} M_i$ and $p \in S^{<\omega}(M)$, if $p \mid M_i$ dnf over M_0 for all $i < \alpha$, then p dnf over M_0.*

(iii) $(B)^{<\omega}_{\kappa'}$ *For any type $p \in S^{<\omega}(M)$, there is $N < M$ of power κ' such that p dnf over N.*

Proof. (i) By 4.12 there is $i < \alpha$ such that $a \perp_{M_i} M$, $b \perp_{M_i} M$. By $a \underset{M_i}{\sim} b$ and (U), $a \underset{M}{\sim} b$ follows.

(ii) By 4.12, p dnf over M_i for some $i < \alpha$. Since $p \mid M_i$ dnf over M_0, by (T), p dnf over M_0.

(iii) By $(\mathrm{B})_{(\kappa')^+}$, there is $N < M$ with $\|N\| \leq (\kappa')^+$ such that p dnf over N. Write $N = \bigcup^{c}_{i<(\kappa')^+} N_i$ such that $\|N_i\| = \kappa'$. By 4.12, there is $i < (\kappa')^+$ such that $p \mid N$ dnf over N_i, hence, by (T), p dnf over N_i. □

Proposition 4.14. *Let $\kappa_1 = \sup\{|S^{<\omega}(M)| : \|M\| = \kappa'\}$. Then*

$$|S^{<\omega}(M)| \leq \max(|M|, \kappa_1) \quad \textit{for all } M \in K.$$

Proof. By induction on $\|M\|$ $(\geq \kappa')$. For $\|M\| = \kappa'$, the assertion is obvious. Suppose $\|M\| > \kappa'$. Then $M = \bigcup^{\uparrow}_{i<\alpha} M_i$, with $\alpha \leq \|M\|$, $\|M_i\| < \|M\|$ for every

$i<\alpha$. Any $p \in S^{<\omega}(M)$ dnf over M_{i_p} for some $i_p < \alpha$; by (U), p is the unique nf extension of $p \mid M_{i_p}$ to M. It follows that the mapping

$$S^{<\omega}(M) \to \coprod_{i<\alpha} S^{<\omega}(M_i) \quad \text{(disjoint sum)}$$

$$p \mapsto p \mid M_{i_p}$$

is an injective mapping. Since, by the induction hypothesis, $|S^{<\omega}(M_i)|| \leqslant \max(\|M_i\|, \kappa_1)$, we conclude that

$$|S^{<\omega}(M)| \leqslant |\alpha| \times \max(\|M\|, \kappa_1) = \max(\|M\|, \kappa_1). \quad \square$$

Corollary 4.15. *If K is stable in κ', then it is stable in all $\mu \geqslant \kappa'$.* $\square$

Assumption 4.16. K is stable in κ'.

Remark. In view of 3.5, and the fact that we are interested in the case when K is categorical in some $\lambda > \kappa'$, the last assumption is reasonable.

Lemma 4.17. *Let us write $A \perp_c [B]^{<\omega}$ to mean that $A \perp_c b$ for all* **finite** *tuples b of elements of B. Let $\kappa'' = (\kappa')^{<\kappa}$. Then, for any A, M and $B \subset M$ with $|B| \leqslant \mu$, there is $N < M$ such that*

$$B \subset N, \qquad \|N\| \leqslant \max(\|A\|, \kappa'') \quad \text{and} \quad A \underset{N}{\perp} [M]^{<\omega}.$$

Proof. Let $\mu = \max(|A|, \kappa'')$. If $\mu^{<\kappa} = \mu$, then, by $(B)_\mu$, we can find N with $\|N\| \leqslant \mu$ satisfying the stronger condition $A \perp_N M$. Otherwise, $\mu > \kappa''$ (since κ'' satisfies $(\kappa'')^{<\kappa} = \kappa''$), and μ is a limit cardinal. Let $\mu = \lim_{i<\alpha} \mu_i$, with the μ_i increasing successor cardinals, each $\geqslant \kappa''$. Let us write B as $B = \bigcup^{\uparrow}_{i<\alpha} B_i$ with $|B_i| \leqslant \mu_i$. Let P be a model, $P < \mathscr{C}$, containing the set A, of power μ, and let $P = \bigcup^{\uparrow}_{i<\alpha} P_i$ with models P_i, each of cardinality μ_i. By induction on $i < \alpha$, define $N_i < M$ such that $B_i \subset N_i$, $\|N_i\| \leqslant \mu_i$, $N_j < N_i$ for $j < i$, and $P_i \perp_{N_i} M$: this is easily done by $(B)_{\mu_i}$. Let $N = \bigcup_{i<\alpha} N_i$. Then $B \subset N$. By (M), we have $P_i \perp_N M$ for all $i < \alpha$. Now, let a be any finite tuple of elements of M. Then (by (S)), $a \perp_N P_i$; hence, by 4.13(ii), $a \perp_N P$ (since $P = \bigcup^{\uparrow}_{i<\alpha} P_i$), hence $P \perp_N a$. Since a was an arbitrary finite tuple from M, we have $P \perp_N [M]^{<\omega}$, and a fortiori $A \perp_N [M]^{<\omega}$. Note that $\|N\| \leqslant \mu$. $\square$

Proposition 4.18. *Let λ be a cardinal $> \kappa'' = (\kappa')^{<\kappa}$. Suppose that $M = \bigcup^{\uparrow}_{i<\alpha} M_i$, and that each M_i is λ-saturated. Then M is λ-saturated.*

Proof. If cf $\alpha \geqslant \lambda$, then every subset of M of cardinality $<\lambda$ is included in some M_i, thus the assertion is clearly true in this case. We thus may assume that cf $\alpha < \lambda$; then, by taking a suitable subsequence of $\langle M_i \rangle_{i<\alpha}$, we may assume that $\alpha = \text{cf}\, \alpha < \lambda$. Let $A \subset M$, $|A| < \lambda$. Let $\mu = \max(\|A\|, \kappa'', |\alpha|)$; we have $\mu < \lambda$. We construct, by induction on $\beta<\kappa$, for each $i<\alpha$ a $<$-continuous chain $\langle N_i^\beta\rangle_{\beta<\kappa}$ of models $N_i^\beta<M_i$ of cardinality $\leqslant\mu$ such that $A\subset\bigcup_{i<\alpha}N_i^0$,

$$\bigcup_{j<\alpha}N_j^\beta \underset{N_i^{\beta+1}}{\perp} [M_i]^{<\omega} \tag{1}$$

(see 4.17) and

$$N_j^\beta<N_i^\beta \qquad (\beta<\kappa,\ j<i,\ i<\alpha),$$

as follows. We choose $N_i^0<M_i$ of power $\leqslant\mu$ to contain $A\cap M_i$; at limit ordinals β, we take unions. Suppose $\beta<\kappa$ and the N_i^β have been defined ($i<\alpha$). We define $N_i^{\beta+1}$ by induction on $i<\alpha$. Assuming $i<\alpha$ and that $N_j^{\beta+1}$ has been defined for every $j<i$, we let $N_i^{\beta+1}$ be an $\mathscr{F}$-submodel of M_i of power $\leqslant\mu$ such that (1) holds and such that $\bigcup_{j<i}N_j^{\beta+1}<N_i^{\beta+1}$: $N_i^{\beta+1}$ exists by 4.17, the choice of μ, and $|i|\leqslant\mu$, $\|N_j^{\beta+1}\|\leqslant\mu$. This completes the definition of the N_i^β ($\beta<\kappa$, $i<\alpha$).

Let $N_i=\bigcup_{\beta<\kappa}N_i^\beta$ and $N=\bigcup_{i<\alpha}N_i$. Then $A\subseteq N$, $N_i<N<M$, $\|N\|\leqslant\mu$, and by $(\mathrm{C})_{<\kappa}$, from (1) we obtain

$$N \underset{N_i}{\perp} [M_i]^{<\omega}. \tag{2}$$

Now, consider any $p\in S^{<\omega}(A)$. Extend p to some $q\in S(N)$, and choose, by 4.12, some $i<\alpha$ such that q dnf over N_i. Since $\|N_i\|<\lambda$, and M_i is λ-saturated, there is $a\in M_i$ realizing $q\restriction N_i$. From (2), we obtain

$$N \underset{N_i}{\perp} a$$

hence (by (S)), a realizes the unique non-forking extension of $q\restriction N_i$ to N, which is q. Thus, a realizes p, and the proof is complete. □

Proposition 4.19. *For every $\lambda>\kappa'$ there is a saturated model of cardinality λ.*

Proof. By 4.15 and 4.16, K is stable in all $\mu\geqslant\kappa'$. For λ a successor cardinal, the assertion follows from 2.6 (put $\mu=\lambda$ in 2.6). Assume λ is a limit cardinal: $\lambda=\lim_{i<\alpha}\lambda_i$ with strictly increasing successor cardinals $\lambda_i>\kappa'$. Let, by induction on $i<\alpha$, M_i be saturated of power λ_i such that $\bigcup_{j<i}^{\uparrow}M_j<M_i$ (see 2.2(ii)); M_i exists by 2.6 since λ_i is a successor cardinal. Consider $M=\bigcup_{i<\alpha}^{\uparrow}M_i$. For any $i_0<\alpha$, M_i is λ_{i_0}-saturated for all i, $i_0\leqslant i<\alpha$, hence, by 4.18, M is λ_{i_0}-saturated. Since this is true for all $i_0<\alpha$, and $\lambda=\lim_{i<\alpha}\lambda_i$, M is λ-saturated; clearly, $\|M\|=\lambda$. □

Lemma 4.20. *Let λ be an infinite cardinal such that $\lambda^{<\kappa}=\lambda$, and $\lambda\geqslant\kappa'$. Suppose $\langle N_i\rangle_{i<\lambda^+}$ is a $<$-continuous chain of models N_i of power $\leqslant\lambda$, $\langle M_i\rangle_{i<\lambda^+}$ is an increasing chain of models such that $M_i<N_i$ for all $i<\lambda^+$. Let*

$$N_{\lambda^+}=\bigcup_{i<\lambda^+}N_i, \qquad M_{\lambda^+}=\bigcup_{i<\lambda}M_i.$$

Then there is $i<\lambda^+$ such that $M_{\lambda^+}\perp_{M_i}N_i$.

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Proof. For each $A \subset N_{\lambda^+}$ of cardinality $<\lambda$, let $i(A) < \lambda^+$ be chosen so that $M_{\lambda^+} \perp_{M_{i(A)}} A$: such $i(A)$ exists, by $(B)_\lambda$ and (M). Given any $i < \lambda^+$, the number of subsets of N_i of cardinality $<\kappa$ is $\leqslant \lambda^{<\kappa} = \lambda$, hence

$$i^* \stackrel{\text{def}}{=} \sup\{i(A) : A \in \mathcal{P}_{<\kappa}(N_i)\} < \lambda^+.$$

Define the sequence $\langle i_\alpha \rangle_{\alpha \leqslant \kappa}$ by $i_0 = 0$, $i_{\alpha+1} = (i_\alpha)^*$, $i_\delta = \lim_{\alpha<\delta} i_\alpha$ for δ limit. Since $\kappa \leqslant \lambda$, $i_\kappa < \lambda^+$. Any $A \subset N_{i_\kappa} = \bigcup_{\alpha<\kappa} N_{i_\alpha}$ with $|A| < \kappa$ is in some N_{i_α}, $\alpha < \kappa$, and thus

$$M_{\lambda^+} \underset{M_{(i_\alpha)^*}}{\perp} A, \quad \text{and} \quad M_{\lambda^+} \underset{M_{i_\kappa}}{\perp} A \qquad \text{by (M) and (S).}$$

It follows (by $(C)_{<\kappa}$) that $M_{\lambda^+} \perp_{M_{i_\kappa}} N_{i_\kappa}$. □

Definition 4.21. Let $A, B \subset \mathscr{C}$, $M < \mathscr{C}$. *A dominates B over M* if for any C, $A \perp_M C$ implies $A \cup B \perp_M C$.

Proposition 4.22. (i) *Let M be saturated, of power λ, such that $\lambda^{<\kappa} = \lambda$ and $\lambda > \kappa''$, and let c be any finite tuple of elements in $\mathscr{C}$. Then there is a saturated model $N \supset M \cup \{c\}$ of power λ, dominated by c over M.*

(ii) *Let, in addition to the data of* (i), *M^* be an $\mathscr{F}$-substructure of M of power $<\lambda$ such that $C \perp_{M^*} M$, and r a type, $r \in S^\alpha(M^* \cup \{c\})$, with $\alpha \leqslant \lambda$. Then N can be found as in* (i), *and also such that it realizes r.*

Proof. It suffices to show (ii): by 4.13(iii) (or even $(B)_{\kappa''}$), we can find M^* with $\|M^*\| < \lambda$ and $c \perp_{M^*} M$; take r to be any 1-type, say.

Let us assume the hypotheses of (ii). We will exploit the obvious fact that if f is a Σ_1-elementary map with a domain including M and c, then $f(c)$, $f''M$, $f''M^*$ and $f(r)$ satisfy the conclusion iff c, M, M^* and r do.

Let $M < M'$, M' saturated of power λ, and $c \perp_M M'$. Then there is a Σ_1-elementary map with domain including M and c for which $f''M = M'$, $f(c) = c$ and $f \upharpoonright M^* =$ identity: by (T), $c \perp_{M^*} M'$; since M, M' are both saturated of power λ, and $\|M^*\| < \lambda$, there is an automorphism g of $\mathscr{C}$ such that $g''M = M'$, $g \upharpoonright M^* =$ identity (see 2.2(iii)); we have $g(c) \perp_{M^*} M'$, hence by (U), $c \underset{M_i}{\sim} f(c)$; it follows that the function f with domain $M \cup \{c\}$ for which $f \upharpoonright M = g \upharpoonright M$ and $f(c) = c$ is Σ_1-elementary.

Assume that the assertion fails for the given M, c, M^* and r. Then, by the above, it also fails for M' and the same c, M^* and r, whenever $M < M'$, M' is saturated of power λ, and $c \perp_M M'$.

We construct $<$-continuous chains $\langle M_i \rangle_{i<\lambda^+}$, $\langle N_i \rangle_{i<\lambda^+}$ of saturated models M_i, N_i of power λ such that

$$M_i < N_i, \qquad M_0 = M, \qquad c \underset{M_0}{\perp} M_i, \qquad N_i \underset{M_i}{\perp} M_{i+1}$$

for all $i < \lambda^+$. We put $M_0 = M$, N_0 any saturated model of power λ containing M_0

and c, and realizing r. Suppose we have constructed all items with indices up to and including i. Since the assertion of the proposition fails for M_i, c, M^* and r, N_i realizes r, (since N_0 does), we have that N_i fails to be dominated by c over M_i. It follows that there is B with $c \mathrel{\smile\!\!\!\!\!\!|}_{M_i} B$, but $N_i \mathrel{\not\smile\!\!\!\!\!\!|}_{M_i} B$. By $(C)_{<\kappa}$, B may be chosen to be of power $<\kappa$; we choose a saturated model M_{i+1} of power λ containing $B \cup M_i$ such that $c \mathrel{\smile\!\!\!\!\!\!|}_{M_i} M_{i+1}$ by (E) and (U); we will have $N_i \mathrel{\smile\!\!\!\!\!\!|}_{M_i} N_{i+1}$. We define N_{i+1} to be any saturated model of power λ containing N_i and M_{i+1} (see 2.2(ii)).

Finally, let $i < \lambda^+$ be a limit ordinal, and assume the construction done below i. Let $M_i = \bigcup_{j<i} M_j$, $N_i = \bigcup_{j<i} N_j$. M_i and N_i are saturated of power λ by 4.18. The property $c \mathrel{\smile\!\!\!\!\!\!|}_{M_0} M_i$ follows from the induction hypothesis and 4.13(ii).

We have constructed items making up a counterexample to 4.20; this contradiction proves the assertion. □

Definition 4.23. (i) $p, q \in S(M)$ are *weakly orthogonal*, $p \perp_w q$, if $a \mathrel{\smile\!\!\!\!\!\!|}_M b$ whenever a realizes p and b realizes q.

(ii) $p, q \in S(M)$ are *orthogonal*, $p \perp q$, if for all $N \supseteq M$, and any non-forking extensions p', q' of p, q, respectively, to N, we have $p' \perp_w q'$.

Proposition 4.24. (i) *If $M \subseteq N$, $p, q \in S(N)$ both do not fork over M, and $p \perp_w q$, then $p \mid M \perp_w q \mid M$.*

(ii) *If $p, q \in S^{<\omega}(M)$, M is $(\kappa')^+$-saturated, then $p \perp_w q$ implies that $p \perp q$.*

Proof (compare Theorem V.12 in [9]). (i) Assume the hypothesis in (i), and let a, b realize $p \mid M$, $q \mid M$, respectively. Let a', b' be such that

$$a'{}^\wedge b' \mathrel{\smile\!\!\!\!\!\!|}_M N \quad \text{and} \quad a'{}^\wedge b' \underset{M}{\approx} a^\wedge b.$$

Then, since a' and b' realize p and q, respectively, we have $a' \mathrel{\smile\!\!\!\!\!\!|}_N b'$. Since $a' \mathrel{\smile\!\!\!\!\!\!|}_M N$, by (T) we conclude $a' \mathrel{\smile\!\!\!\!\!\!|}_M b'$; since $a'{}^\wedge b' \underset{M}{\approx} a^\wedge b$, $a \mathrel{\smile\!\!\!\!\!\!|}_M b$ follows.

(ii) Note, first of all, that for types p, q over a model M, $p \in S_x(M)$, $q \in S_y(M)$, x and y disjoint tuples of variables, $p \perp_w q$ means that $p(x) \cup q(y)$ is complete, that is, there is a unique $r \in S_{x^\wedge y}(M)$ with $p(x) \cup q(y) \subset r(x, y)$. Assume the hypotheses of (ii), and contrary to the assertion, $p \not\perp q$. There is a model N extending M such that for the nonforking extensions p', q' of p, q, respectively, to N, we have $p' \not\perp_w q'$. Since by (i), for any $\mathcal{F}$-extension of N, the non-forking extensions of p', q' remain non-weakly-orthogonal, we may assume that N is $(\kappa')^+$-saturated.

Note that the definition of non-forking implies that if a type $p \in S(N)$ does not fork over M, then p does not split over M: for any tuples a, b from N and for c realizing p, if $a \underset{M}{\approx} b$, then $a \underset{c}{\approx} b$: if we had a formula φ such that $\models \varphi(c, a) \wedge \neg\varphi(c, b)$, then by $c \mathrel{\smile\!\!\!\!\!\!|}_M N$, there would be $c' \in M$ with $\models \varphi(c', a) \wedge \neg\varphi(c', b)$, contradicting $a \underset{M}{\approx} b$.

The fact that $p' \not\perp_w q'$ means that there is a formula $\varphi(x, y, z)$ and a tuple $c \in N$ such that both $p'(x) \cup q'(y) \cup \{\varphi(x, y, c)\}$ and $p'(x) \cup q'(y) \cup \{\neg\varphi(x, y, c)\}$ are

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consistent. By 4.13(iii) and (M), there is $M_0 < M$ of power κ' such that p and q do not fork over M_0. Since M is $(\kappa')^+$-saturated, there is $d \in M$ such that $c \underset{M_0}{\sim} d$.

We claim that $p'(x) \cup q'(y) \cup \{\varphi(x, y, d)\}$ and $p'(x) \cup q'(y) \cup \{\neg\varphi(x, y, d)\}$ are both consistent. Let P, Q be subsets of p' and q', respectively, of power $<\kappa$. Let a be a $<\kappa$-tuple of elements of N containing all the parameters mentioned in P or Q. By N being $(\kappa')^+$-saturated, there is b in N such that $d\frown a \underset{M_0}{\sim} c\frown b$. Let $\hat{P}$, $\hat{Q}$ be obtained from P and Q, respectively, by replacing each parameter from a in any formula by the corresponding parameter in b. Since $a \underset{M_0}{\sim} b$, and p', q' do not split over M_0, $\hat{P} \subset p'$, $\hat{Q} \subset q'$. It follows that $\hat{P}(x) \cup \hat{Q}(y) \cup \{\varphi(x, y, c)\}$, $\hat{P}(x) \cup \hat{Q}(y) \cup \{\neg\varphi(x, y, c)\}$ are both consistent; since $d\frown a \underset{\emptyset}{\sim} c\frown b$, we conclude that $P(x) \cup Q(y) \cup \{\varphi(x, y, d)\}$ and $P(x) \cup Q(y) \cup \{\neg\varphi(x, y, d)\}$ are both consistent, which was to be shown.

Of course, the claim implies that $p(x) \cup q(y) \cup \{\varphi(x, y, d)\}$, $p(x) \cup q(y) \cup \{\neg\varphi(x, y, d)\}$ are both consistent; since $d \in M$, this contradicts the assumption $p \perp_{\mathrm{w}} q$. □

Proposition 4.25. *Suppose there is a model which is $(\kappa')^+$-saturated, but not saturated. Then, for M^* the saturated model of power $(\kappa')^+$, there are types $p^*, q^* \in S^1(M^*)$, neither realized in M^*, such that $p^* \perp q^*$.*

Proof. Suppose M is not saturated, but $(\kappa')^+$-saturated. Let $M_0 < M$, $p_0 \in S^1(M_0)$ such that $\mu \overset{\text{def}}{=} \|M_0\| < \|M\|$ and p_0 is not realized in M; $\mu \geq (\kappa')^+$.

We claim that there is M_1, $M_0 < M_1 < M$, $\|M_1\| = \mu$, and an extension p_1 of p_0 to M_1 such that all extensions of p_1 to M do not fork over M_1. Suppose not, and define by induction on $n < \omega$, $M_0 = M_0^0 < M_0^1 < \cdots < M_0^n < \cdots$, $p^0 = p_0$, $p^n \in S^1(M_0^n)$, $p^n \subset p^{n+1}$, such that $\|M_0^n\| = \mu$ and p^{n+1} forks over M_0^n; by the indirect supposition, this is clearly possible; let $M^* = \bigcup_{n<\omega} M_0^n$, and by 2.14, let $p^* \in S^1(M^*)$ be such that $p^* \mid M_0^n = p^n$; then, by (M), p^* forks over each M_0^n $(n < \omega)$, in contradiction to 4.12; this shows our claim.

Note that saying that all extensions of p_1 to M do not fork over M_1 means that p_1 has a unique extension to M, and to any set A with $M_1 \subset A \subset M$.

We can easily construct M_2 such that $M_1 < M_2 < M$, $\|M_2\| = \mu$, and M_2 is $(\kappa')^+$-saturated: let $M_2 = \bigcup^{\mathrm{c}}_{i<(\kappa')^+} M_2^i$, where $M_2^0 = M_1$, each M_2^i has power μ, and M_2^{i+1} realizes all types over M_2^i (we have stability in μ). Let p_2 be the unique extension of p_1 to M_2. Since $\|M_2\| = \mu < \|M\|$, there is $b \in M - M_2$; let $q_2 = \mathrm{tp}(b/M_2)$. We claim that $p_2 \perp_{\mathrm{w}} q_2$: indeed, if a realizes p_2, then $\mathrm{tp}(a/M)$ must be the unique, hence the non-forking, extension of p_2, i.e. $a \downarrow_{M_2} M$, and thus $a \downarrow_{M_2} b$, as desired.

Let $N < M_2$ be of power κ' such that p_2 and q_2 dnf over N (by 4.13(iii)). By 2.2(ii) applied to $\mathrm{Diag}_{\mathscr{F}}(N)$ and 4.19, let M^* be saturated of power $(\kappa')^+$ such that $N < M^* < M_2$, and let $p^* = p_2 \mid M^*$, $q^* = q_2 \mid M^*$. Since p_2, q_2 dnf over M^*, by 4.24(i), $p^* \perp_{\mathrm{w}} q^*$, and by 4.24(ii), $p^* \perp q^*$. It is clear by the choice of p^* and q^* that they are not realized in M^*. □

Lemma 4.26. *Suppose there is a model which is $(\kappa')^+$-saturated, but not saturated. Then, for any $\mu > \kappa''$ ($= (\kappa')^{<\kappa}$) there is a pair M, N of saturated models of power μ such that $M \precneqq N$, and there is a type $q \in S^1(M)$ not realized in M which has a unique extension to N.*

Proof. First, let μ be a successor cardinal. Let M be any saturated extension of M^*, the saturated model of power $(\kappa')^+$, with $\|M\| = \mu$; note $\mu \geqslant (\kappa')^+$. Let p, q be the non-forking extensions of p^* and q^*, respectively, to M, where p^* and q^* are from 4.25. We have that p and q are orthogonal to each other. Let $a \in \mathscr{C}$ realize p, and, by 4.22(i), let $N \supset M \cup \{a\}$ be a saturated model of power μ, dominated by a over M. Let b be any realization of q; since $p \perp q$, we have $a \perp\!\!\!\perp_M b$, and since N is dominated by a over M, $N \perp\!\!\!\perp_M b$; this shows that any extension of q to N dnf over M, that is, q has a unique extension to N, as desired.

Secondly, let μ be a limit cardinal. Since $\mu > \kappa''$, and either $\kappa'' = \kappa'$, or $\kappa'' = (\kappa')^+$, necessarily $\mu > (\kappa')^+$. Let $\langle \mu_i \rangle_{i<\mathrm{cf}\,\mu}$ be a strictly increasing sequence of cardinals such that $\mu = \sup_{i<\mathrm{cf}\,\mu} \mu_i$, $\mu_0 = (\kappa')^+$, μ_{j+1} is a successor cardinal for all $j < \mathrm{cf}\,\mu$, and $\mu_i = \sup_{j<i} \mu_j$ whenever $i < \mathrm{cf}\,\mu$ is a limit ordinal. Let $M_0 = M^*$ be the saturated model of power $(\kappa')^+$, a an element realizing p^* (from 4.25), N_0 any model of power $(\kappa')^+$ containing $M_0 \cup \{a\}$.

We construct, by induction on i, $0 < i < \mathrm{cf}\,\mu$, models M_i, N_i such that

(i) $\langle M_i \rangle_{i<\mathrm{cf}\,\mu}$, $\langle N_i \rangle_{i<\mathrm{cf}\,\mu}$ are continuous $<$-chains; and, for all $i, j < \mathrm{cf}\,\mu$,

(ii) M_{j+1}, N_{j+1} are saturated of power μ_{j+1};

(iii) $M_i < N_i$;

(iv) $a \perp\!\!\!\perp_{M_0} M_i$;

(v) N_{j+1} is dominated by a over M_{j+1}.

Suppose $0 < i < \mathrm{cf}\,\mu$ and M_j, N_j have been defined for all $j < i$ with the required properties. Let first i be a limit ordinal. Put $M_i = \bigcup_{j<i} M_j$, $N_i = \bigcup_{j<i} N_j$. Property (iii) for i follows from the same for $j < i$, and (iv) for i follows from the same for $j < i$, and 4.13(ii).

Next, let $i = j + 1$. Let $\hat{M}$ be any saturated model of power μ_{j+1} extending M_j such that $a \perp\!\!\!\perp_{M_j} \hat{M}$, and let $r = \mathrm{tp}(N_j / M_j \cup \{a\})$; we will use 4.22(ii) with μ_{j+1} as λ, $\hat{M}$ as M, a as c, M_j as M^*, and r as r. 4.22(ii) is applicable, and we get the saturated model $\hat{N}$ of power μ_{j+1} such that $\hat{M} < \hat{N}$, $a \in \hat{N}$, $\hat{N}$ is dominated by a over $\hat{M}$, and $\hat{N}$ realizes r. Let N_j' be a realization of r in $\hat{N}$. There is an automorphism g of $\mathscr{C}$ which is the identity on $M_j \cup \{a\}$, and for which $g(N_j') = N_j$. Let $M_{j+1} = g''\hat{M}$, $N_{j+1} = g''\hat{N}$. Then, since $a \perp\!\!\!\perp_{M_j} \hat{M}$, and (iv) holds at j, we have $a \perp\!\!\!\perp_{M_0} \hat{M}$; hence $a \perp\!\!\!\perp_{M_0} M_{j+1}$ as required for (iv). Since $\hat{N}$ is dominated by a over $\hat{M}$, and $g(a) = a$, (v) holds. The construction also ensures that $M_j < M_{j+1}$, $N_j < N_{j+1}$ (for (i)) and $M_{j+1} < N_{j+1}$ (for (iii)). This completes the construction.

Having the M_i, N_i with (i) to (v), we put $M = \bigcup_{i<\mathrm{cf}\,\mu} M_i$, $N = \bigcup_{i<\mathrm{cf}\,\mu} N_i$. As $a \perp\!\!\!\perp_{M_0} M$ (by 4.13(ii)), and $a \notin M_0$, we have $a \notin M$ [if $a \in M$, the formula $x = a$ belongs to $\mathrm{tp}(a/M)$, hence there is $a' \in M_0$ with $a' = a$]. Thus, $M \precneqq N$. Now, with $q^* \in S^1(M^*)$ ($M^* = M_0$), the type from 4.25, let q be the non-forking extension of

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q^* to M. 4.13(i) says that any type $q' \in S^1(N) = S^1(\bigcup_{i<\mathrm{cf}\,\mu} N_{i+1})$ is determined by its restrictions $q' \mid N_i$ for $i < \mathrm{cf}\,\mu$. But, since N_{i+1} is dominated by a over M_{i+1}, and $\mathrm{tp}(a/M_{i+1})$, the non-forking extension of p^* to M_{i+1} is orthogonal to $q \mid M_{i+1}$, $q \mid M_{i+1}$ has a unique extension to N_{i+1}, by the argument at the beginning of this proof. It follows that q has a unique extension to N as promised. Since q^* is not realized in M^*, its non-forking extension q to M is not realized in M (for the same reason as $a \notin M$). □

Proposition 4.27. *Suppose there is a model which is $(\kappa')^+$-saturated, but not saturated. Then for every successor cardinal $\mu^+ > (\kappa'')^+$ $(\kappa'' = (\kappa')^{<\kappa})$, there is a non-saturated model of power μ^+.*

Proof. By 4.26, there is a pair $M \lneq N$ of saturated models of power μ with a type $q \in S^1(M)$, not realized in M, such that q has a unique extension to N. Let $M^0 < M$ be a model of power κ' such that q dnf over M^0. By induction on $i < \mu^+$, we define M_i, a saturated model of power μ, such that $M_0 = M$, $\langle M_i \rangle_{i<\mu^+}$ is $<$-continuous, as follows. We put $M_1 = N$. If i is a limit ordinal $<\mu^+$, we put $M_i = \bigcup_{j<i} M_j$; by 4.18, M_i is saturated of power μ. For $i = j+1$, having constructed M_j, note that, by 2.2(iii), there is an isomorphism $g: M_0 \xrightarrow{\cong} M_j$ which is the identity on M^0; with h any automorphism of $\mathscr{C}$ extending g, we let $M_{j+1} = h''N$; in other words, there is an automorphism g_j leaving M_0 fixed, taking M to M_j, N to M_{j+1}.

We claim that q has a unique extension to $M_{\mu^+} \stackrel{\mathrm{def}}{=} \bigcup_{i<\mu^+} M_i$; in fact, by induction on $i \leq \mu^+$, we show that q has a unique extension to M_i. For i a limit ordinal, the induction step follows by 4.13. Let $i = j+1$; let the unique extension of q to M_j be q_j. q_j dnf over M_0. With g_j the isomorphism mentioned in the construction, $g_j(q) \in S^1(g_j''M)$ has a unqiue extension to $g_j''N$. But also, $g_j(q)$ dnf over $q_j''M^0 = M^0$. Since $g_j''M = M_j$, $g_j''N = M_{j+1}$, we have $q_j = g_j(q)$ (since they are both nf extensions of q), and q_j has a unique extension to M_{j+1}. It follows that q has a unique extension to M_{j+1}. This proves our claim.

Since q is not realized in M, its unique nf extension to M_{μ^+} is not realized in M_{μ^+}. $\|M\| = \mu < \|M_{\mu^+}\| = \mu^+$ (since $M_J \lneq M_{j+1}$ for all $j < \mu^+$); M_{μ^+} is a non-saturated model of power μ^+. □

5. Summary

Conclusion 5.1. = *Theorem* of the Introduction.

Proof. (i) Assume T is a theory in a fragment $\mathscr{F}$ of $L_{\kappa\omega}$, κ is a compact cardinal $>\omega$, and T is categorical in the successor cardinal $\mu^+ > (\kappa')^{<\kappa^+}$, $\kappa' = \max(\kappa, |\mathscr{F}|)$. By 1.9, every $M \in K$ is existentially closed, hence (by 1.6), K has the amalgamation property. By 1.8, K has the joint embedding property. This means that the assumption stated at the beginning of Section 2, and Assumption 4.0 are true. By 4.3, K does not have long definable order (K is 1-stable), i.e., Assumption 4.4 holds true. K is stable in κ' by 3.5; Assumption 4.16 holds true. By 4.19, for every $\lambda > \kappa'$, there is a saturated model of cardinality λ; it is unique up to isomorphism by 2.2(iv) and 2.4. It now suffices to show that, for every $\lambda \geq \min(\mu^+, \beth_{(2^{\kappa'})^+})$, every model of power λ is saturated. Since T is categorical in μ^+, the only model of T of power μ^+ is saturated.

Assume $\lambda \geq \min(\mu^+, \beth_{(2^{\kappa'})^+})$, $\|M\| = \lambda$. Either $\lambda \geq \mu^+$, or $\lambda \geq \beth_{(2^{\kappa'})^+}$. In the first case, M is μ^+-saturated: if it were not, by an obvious downward Löwenheim–Skolem argument, there would be a non-saturated model of power μ^+, in contradiction to the categoricity of T in μ^+. In the second case, M is $(\kappa')^+$-saturated: otherwise, there are $M_0 < M$, $\|M_0\| = \kappa'$, and a type $p \in S^1(M_0)$ omitted in M; by 3.2, p is omitted in some model of power μ^+, which contradicts the categoricity of T in μ^+. Thus, in either case, M is $(\kappa')^+$-saturated ($\mu > \kappa'$). If M were not saturated, then by 4.27, there would be a non-saturated model of power μ^+, again in contradiction to the categoricity of T in μ^+.

(ii) The proof of this part does not use Section 4, however, unlike part (i), it uses the more difficult arguments of Sections 1 and 3. Assume T, L, $\mathscr{F}$ and κ' are as before, and T is categorical in λ where $\lambda > \beth_{\kappa+1}(\kappa')$. In the proof of Conclusion of 3.7, we concluded that, under the given hypotheses, the unique model of T of power λ is saturated. Suppose $\mu = \beth_\delta$ with δ divisible by $(2^{\kappa'})^+$, and assume, for reaching a contradiction, that M is a non-saturated model of T of power μ. Since clearly, for any $\alpha < \delta$ we have $\alpha + (2^{\kappa'})^+ \leq \delta$, for any cardinal $\sigma < \mu$, we have $\beth_{(2^{\kappa'})^+}(\sigma) \leq \mu$.

There is $M_0 < M$ with $\sigma \stackrel{\text{def}}{=} \|M_0\| < \mu$ and there is $p \in S^{<\omega}(M_0)$ omitted by M. By 3.3 (with σ for μ of 3.3), in every power $>\kappa'$ there is a non-$(\kappa')^+$-saturated model; this applied to λ, we get a contradiction.

We conclude that for any μ of the kind we are considering, all models of μ have to be saturated; this says (by 2.2(iv) and 2.4) that T is categorical in μ. □

Appendix: Squares on stationary sets

This section contains set-theoretic material used in Section 1.

For a set X of ordinals, X' denotes the set of *limit points of* X: $\alpha \in X'$ iff $\alpha > 0$ and $\sup(\alpha \cap X) = \alpha$. A subset X of λ (λ any ordinal) is *closed in* λ if $X' - \{\lambda\} \subset X$; it is *unbounded in* λ if $\lambda \in X'$; it is a *club in* λ if it is both the above; it is *stationary in* λ if it meets every club in λ.

A *square-system on* λ (in the 'limit formulation') is a system $\boldsymbol{C} = \langle C_\alpha : \alpha \in S\rangle$ of sets C_α such that $S \subset \lim(\lambda)$ (= the set of limit ordinals $<\dot{\lambda}$), C_α is a club in α and $(C_\alpha)' \subseteq S$ for all $\alpha \in S$, and finally, $C_\beta = C_\alpha \cap \beta$ whenever $\alpha \in S$ and $\beta \in (C_\alpha)'$ (and hence $\beta \in S$). When S is to be mentioned with the square-system, we call the square-system S-*indexed*. Of course, the square-system $\boldsymbol{C}$ is of interest only if S is

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large enough (e.g., stationary in λ), and if the C_α are small enough (e.g., have small ordertypes). We call $\boldsymbol{C}$ a $<\chi$-square-system if each C_α is of ordertype $<\chi$.

A simple argument in [14] (2, Lemma 1), using a lemma due to Engelking and Karlowich, provides square-systems via the following

Proposition A.1. *Suppose μ, χ are infinite cardinals with $\mu^{<\chi}=\mu$. Then we can find a family $\langle S_\xi\rangle_{\xi<\mu}$ of subsets S_ξ of μ^+ such that*

$$\bigcup_{\xi<\mu} S_\xi = \{\delta<\mu^+ : \mathrm{cf}\,\delta<\chi\}$$

and for each $\xi<\mu$, there exists an S_ξ-indexed $<\chi$-square-system on λ.

For the application in Section 1, we need a modified kind of square-system, also incorporating a controlled role for successor ordinals.

Definition A.2. A *modified square-system on λ* is a system $\langle C_\alpha : \alpha\in S\rangle$ of sets such that $S\subset\lambda$, and, for all $\alpha\in S$,

(i) C_α is a subset of α which is closed in α, and if α is a limit ordinal, C_α is a club in α.

(ii) $C_\alpha\subseteq S$.

(iii) $\beta\in C_\alpha \Rightarrow C_\beta = C_\alpha\cap\beta$.

(iv) If C_α has a last element β which is a limit ordinal, then $\alpha=\beta+1$.

The main difference is that (ii) and (iii) now apply to all elements of C_α rather than to its limit points only as before.

Proposition A.3. *Suppose μ, χ, σ are infinite cardinals such that $\mu^{<\chi}=\mu$, σ is regular, $\sigma<\chi$. Then there is a modified square-system $\langle C_\alpha:\alpha\in S\rangle$ on μ^+ such that $\{\alpha\in S:\mathrm{otp}(C_\alpha)=\sigma\}$ is stationary in μ^+.*

Proof. We start with the square-systems $\langle C^\xi_\alpha : \alpha\in S_\xi\rangle$ given by A.1 ($\xi<\mu$); in particular, $\mathrm{otp}(C^\xi_\alpha)<\chi$ for all $\alpha\in S_\xi$; also, $\bigcup_{\xi<\mu}S_\xi=\{\delta<\mu^+:\mathrm{cf}\,\delta<\chi\}$. The set $\{\alpha<\mu^+:\mathrm{cf}\,\alpha=\sigma\}$ is a stationary set contained in $\bigcup_{\xi<\mu}S_\xi$; therefore, there is ξ such that $\hat{S}_\xi \stackrel{\mathrm{def}}{=} \{\alpha\in S_\xi:\mathrm{cf}\,\alpha=\sigma\}$ is stationary. For notational simplicity, assume $\xi=0$.

Let us assume that $\sigma>\omega$. (The case $\sigma=\omega$ is essentially trivial.) The stationary set $\hat{S}_0$ is the union of the less than μ^+ many sets $S^\gamma=\{\alpha\in\hat{S}_0:\mathrm{otp}(C^0_\alpha)=\gamma\}$ ($\gamma<\chi$); hence there is $\gamma<\chi$ such that S^γ is stationary; let us fix such a γ. Since for any $\alpha\in S^\gamma$, $\mathrm{cf}\,\alpha=\sigma$ and C^0_α is cofinal in α, we have that $\mathrm{cf}\,\gamma=\sigma$. Let $D_0\subset\gamma$ be a club in γ of order-type $\mathrm{otp}(D_0)=\sigma$ such that every element of D_0 is a limit ordinal (D_0 exists since σ is a regular cardinal $>\omega$). Put $D=D_0\cup\{\gamma\}$. Now, let

$$S^* \stackrel{\mathrm{def}}{=} \{\alpha\in S_0:\mathrm{otp}(C^0_\alpha)\in D\},$$
$$S^{**} \stackrel{\mathrm{def}}{=} \{\alpha\in S_0:\mathrm{otp}(C^0_\alpha)\in D'\},$$
$$C^*_\alpha \stackrel{\mathrm{def}}{=} ((C^0_\alpha)'-\{\alpha\})\cap S^* \quad \text{for } \alpha\in S^*.$$

Claim 1. *If $\alpha\in S^*$, then C^*_α is closed in α, and if $\alpha\in S^{**}$, C^*_α is a club in α.*

Proof. Indeed, let $\alpha \in S^*$. $D \cap \text{otp}(C^0_\alpha)$ is closed in $\text{otp}(C^0_\alpha)$; and if $\alpha \in S^{**}$, it is a club in $\text{otp}(C^0_\alpha)$. Since C^0_α is a club in α, it follows that the set

$$C^+_\alpha \stackrel{\text{def}}{=} \{\beta \in C^0_\alpha : \text{otp}(C^0_\alpha \cap \beta) \in D\}$$

is closed in α, and it is a club in α in case $\alpha \in S^{**}$. As we now show,

$$C^+_\alpha = C^*_\alpha;$$

this will show the claim.

Suppose first $\beta \in C^+_\alpha$. Since $\text{otp}(C^0_\alpha \cap \beta)$, being in D, is a limit ordinal, and C^0_α is closed in α, $\sup(C^0_\alpha \cap \beta) = \beta$. This means that $\beta \in (C^0_\alpha)'$, hence $\beta \in S^0$ and $C^0_\beta = C^0_\alpha \cap \beta$, and thus $\text{otp}(C^0_\beta) = \text{otp}(C^0_\alpha \cap \beta) \in D$. We have shown that $\beta \in S^*$ and $\beta \in C^*_\alpha$.

Secondly, assume $\beta \in C^*_\alpha$. Then $\beta \in (C^0_\alpha)'$, hence $\beta \in S^0$ and $C^0_\beta = C^0_\alpha \cap \beta$. Since also $\beta \in S^*$, $\text{otp}(C^0_\beta) \in D$; it follows that $\text{otp}(C^0_\alpha \cap \beta) \in D$; hence $\beta \in C^+_\alpha$ as desired. □ (Claim 1)

Claim 2. *If $\alpha \in S^*$, then $(C^*_\alpha)' - \{\alpha\} = C^*_\alpha \cap S^{**}$.*

Proof. Indeed, by the above, $C^+_\alpha = C^*_\alpha$. Assume that $\beta \in (C^*_\alpha)' = (C^+_\alpha)'$, $\beta < \alpha$. Then $\text{otp}(C^0_\alpha \cap \beta)$ is a limit point of D and $\beta \in (C^0_\alpha)'$; hence $C^0_\alpha \cap \beta = C^0_\beta$, and $\text{otp}(C^0_\beta) \in D'$, and thus $\beta \in S^{**}$. This shows that the left-hand side is contained in the right-hand side.

Conversely, if $\beta \in C^*_\alpha \cap S^{**}$, then $\beta \in (C^0_\alpha)'$, hence $C^0_\beta = C^0_\alpha \cap \beta$; $\text{otp}(C^0_\alpha \cap \beta) \in D'$, and β is a limit point of C^+_α, which suffices. □ (Claim 2)

We also have

$$C^*_\alpha \subset S^* \quad \text{for all } \alpha \in S^*, \tag{3}$$

$$C^*_\beta = C^*_\alpha \cap \beta \quad \text{whenever } \beta \in C^*_\alpha,\ \alpha \in S^*; \tag{4}$$

(3) is clear, and (4) follows from $\beta \in (C^0_\alpha)' \Rightarrow C^0_\beta = C^0_\alpha \cap \beta$.

In Claim 1, for (3) and (4) we have analogs of (i), (ii), (iii), respectively, of A2, with S^* for S, C^*_α for C_α, and in (i), with "$\alpha \in S^{**}$" for "α is a limit ordinal". To arrive at the final set S, we replace in S^* every $\alpha \in S^* - S^{**}$ by $\alpha + 1$ (thereby forcing every limit ordinal in S to be in S^{**}), and to make (iv) hold, we also add $\alpha + 1$ to S for all $\alpha \in S^{**}$; we define C_α accordingly. In more detail:

$$S \stackrel{\text{def}}{=} S^{**} \cup \{\alpha + 1 : \alpha \in S^* - S^{**}\} \cup \{\alpha + 1 : \alpha \in S^{**}\},$$

$$\hat{C}_\alpha \stackrel{\text{def}}{=} (C^*_\alpha \cap S^{**}) \cup \{\beta + 1 : \beta \in C^*_\alpha \cap (S^* - S^{**})\} \cup \{\beta + 1 : \beta \in C^*_\alpha \cap S^{**}\} \quad \text{for } \alpha \in S^*,$$

$$C_\alpha \stackrel{\text{def}}{=} \hat{C}_\alpha \quad \text{for } \alpha \in S^{**},$$

$$C_{\alpha+1} \stackrel{\text{def}}{=} \hat{C}_\alpha \quad \text{for } \alpha \in S^* - S^{**},$$

$$C_{\alpha+1} \stackrel{\text{def}}{=} \hat{C}_\alpha \cup \{\alpha\} \quad \text{for } \alpha \in S^{**};$$

the last three definitions specify C_α for every $\alpha \in S$.

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Since $\hat{C}_\alpha$ is obtained from C^*_α by replacing some elements by their successors, and adding successors of some other elements, we have $(\hat{C}_\alpha)' = (C^*_\alpha)'$. Since, by Claim 2, $(C^*_\alpha)' - \{\alpha\} \subset S^{**}$, and by Claim 1, $C^*_\alpha \cap S^{**} \subseteq \hat{C}_\alpha$, we see that $\hat{C}_\alpha$ is closed in α. It is also clear that $\sup \hat{C}_\alpha = \sup C^*_\alpha$ in case C^*_α has no largest element; by Claim 1, it follows that $\sup \hat{C}_\alpha = \alpha$, hence $\hat{C}_\alpha$ is a club on α, in case $\alpha \in S^{**}$. If $\alpha \in S$ and α is a limit ordinal, then necessarily $\alpha \in S^{**}$. Thus, we see that A.2(i) holds as desired. (ii) is clear from the definitions and (3), (iii) from the definitions and (4).

Finally, for (iv), assume that $\delta \in S$, β is the largest element of C_δ, and β is a limit ordinal, to show that $\delta = \beta + 1$. $\delta \in S^{**}$ is impossible since then C_δ has no largest element. If $\delta = \alpha + 1$, $\alpha \in S^{**}$, then $\beta = \alpha$, thus $\delta = \beta + 1$ as desired. However, if $\delta = \alpha + 1$ and $\alpha \in S^* - S^{**}$, then α is a limit ordinal and $\beta < \alpha$; since all limit ordinals in C_δ are in S^{**}, $\beta \in S^{**}$; but then $\beta + 1 < \alpha$ and thus $\beta + 1 \in C_\delta$, by the presence of the third term in the union defining $C_\delta = \hat{C}_\alpha$; this contradicts the assumption that β is maximal in C_δ. □

Let us quote

Proposition A.4. *If μ is a strong limit cardinal with $2^\mu = \mu^+$, then $\diamondsuit_S$ holds for every stationary $S \subseteq \{\delta < \mu^+ : \operatorname{cf} \delta \neq \mu\}$.*

Proof. See the end of [12]. □

Conclusion A.5. *Let κ be compact $>\aleph_0$, κ' any cardinal, $\mu = \beth_\kappa(\kappa')$, σ a regular cardinal $<\kappa$. Then there is $S \subset \mu^+$ and an S-indexed modified square-system $\langle C_\alpha : \alpha \in S \rangle$ on μ^+ such that $S^* \overset{\text{def}}{=} \{\alpha \in S : \operatorname{otp}(C_\alpha) = \sigma\}$ is stationary; also, $\diamondsuit_{S^*}$ holds.*

Proof. By A.3 and A.4, since $\mu^{<\kappa} = \mu$ is clear, and $2^\mu = \mu^+$ is true by [16]. □

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