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HEREDITARILY SEPARABLE GROUPS AND MONOCHROMATIC UNIFORMIZATION

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ABSTRACT

We give a combinatorial equivalent to the existence of a non-free hereditarily separable group of cardinality $\aleph_1$. This can be used, together with a known combinatorial equivalent of the existence of a non-free Whitehead group, to prove that it is consistent that every Whitehead group is free but not every hereditarily separable group is free. We also show that the fact that $\mathbb{Z}$ is a p.i.d. with infinitely many primes is essential for this result.

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Introduction

An abelian group G is said to be **separable** if every finite rank pure subgroup is a free direct summand of G; G is **hereditarily separable** if every subgroup of G is separable. It is well-known that a Whitehead group is hereditarily separable. In fact, we have the following implications:

free $\Rightarrow$ W-group $\Rightarrow$ hereditarily separable.

Whitehead's Problem asks if the first arrow is reversible. For each of the two arrows, it has been proved by the third author that it is independent of ordinary set theory whether the arrow reverses. (See [7], [9] and [10], or the account in [3].)

Now if we consider the two arrows together, there are four possible cases, three of which have already been shown to be consistent:

1. *Both arrows reverse.* That is, every hereditarily separable group is free. This is true in a model of V = L. (See [3, VII.4.9].)

2. *Neither arrow reverses.* That is, there are Whitehead groups which are not free, and hereditarily separable groups which are not Whitehead. This is true in a model of MA + $\neg$CH. (See [3, VII.4.5, VII.4.6 and XII.1.11].)

3. *The second arrow reverses but not the first.* That is, every hereditarily separable group is Whitehead and there are Whitehead groups which are not free. This is true in a model of Ax(S) + $\diamondsuit^*(\omega_1 \setminus S)$ plus $\diamondsuit_\kappa(E)$ for every regular $\kappa > \aleph_1$ and every stationary subset E of κ. (See [3, Exer. XII.16].)

4. *The first arrow reverses but not the second.* That, is every Whitehead group is free, but there are non-free hereditarily separable groups. It is an application of the main theorem of this paper that this case is consistent. (See Section 3.)

We also give additional information about the circumstances under which Cases 2 and 3 can occur. (See Section 4.) Finally, we show that Case 4 is impossible for modules over a p.i.d. with only finitely many (but at least two) primes. (See Section 5.)

Our methods involve the use of notions of uniformization, which have played an important role in this subject since [8]. (See, for example, [3] and the recent [4].) It has been proved that there exists a non-free Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which satisfies 2-uniformization. (Definitions are given in detail in the next section.) Our main theorem here is the following:

THEOREM 1: *A necessary and sufficient condition for the existence of a non-free hereditarily separable group of cardinality $\aleph_1$ is the existence of a ladder system on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours.*

A ladder system $\eta = \{\eta_\delta\colon \delta \in S\}$ on S is said to satisfy monochromatic uniformization for ω colours if for every function $c\colon S \to \omega$, there is a function $f\colon \omega_1 \to \omega$ such that for every $\delta \in S$, $f(\eta_\delta(n)) = c(\delta)$ for all but finitely many $n \in \omega$.

We believe the main theorem is of independent interest aside from its use in Case 4. We will prove sufficiency in Section 1 and necessity in Section 2. We will then derive the consistency of Case 4 by standard forcing techniques like those used in [10]. (Actually, we need only the sufficiency part of the main theorem for this.) A knowledge of forcing is required only for Sections 3 and 4.

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Preliminaries

We will always be dealing with abelian groups or $\mathbb{Z}$-modules; we shall simply say "group". A group G is said to be **$\aleph_1$-free** if every countable subgroup of G is free, or equivalently, every finite rank subgroup is free. (See [3, IV.2.3]; throughout the paper we will usually cite [3] for results we need, rather than the original source.) An $\aleph_1$-free group G is separable if and only if every pure subgroup H of finite rank is a direct summand of G, i.e., there is a projection $h\colon G \to H$ (a homomorphism such that $h \restriction H$ is the identity on H). The following are two useful facts (cf. [3, IV.2.7 and VII.4.2]):

LEMMA 2:

(i) *An $\aleph_1$-free group G is separable if every pure cyclic subgroup of G is a direct summand of G.*

(ii) *An $\aleph_1$-free group G is hereditarily separable if B is separable whenever B is a subgroup of G such that G/B is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$ and there is a finite set P of primes such that the order of every element of G/B is divisible only by primes in P.*

A group G is said to be a **Whitehead group** if $\mathrm{Ext}(G,\mathbb{Z}) = 0$. Every Whitehead group is separable ([3, XII.1.3]). Since a subgroup of a Whitehead group is also a Whitehead group, every Whitehead group is hereditarily separable.

A group is said to be a **Shelah group** if it has cardinality $\aleph_1$, is $\aleph_1$-free, and for every countable subgroup B there is a countable subgroup $B' \supseteq B$ such that for every countable subgroup C satisfying $C \cap B' = B$, C/B is free. In that case, we say that B' has the **Shelah property** over B. In [7] and [9] it is proved a consequence of Martin's Axiom plus $\neg$CH that the Whitehead groups of cardinality $\aleph_1$ are precisely the Shelah groups.

Notions of uniformization (in our sense) were first defined in [2] and [8]. Let S be a subset of $\lim(\omega_1)$. If $\delta \in S$, a **ladder on** δ is a function $\eta_\delta\colon \omega \to \delta$ which is strictly increasing and has range cofinal in δ. A **ladder system on** S is an indexed family $\eta = \{\eta_\delta\colon \delta \in S\}$ such that each η_δ is a ladder on δ. The ladder system η is **tree-like** if whenever $\eta_\delta(n) = \eta_\tau(m)$, then $n = m$ and $\eta_\delta(k) = \eta_\tau(k)$ for all $k < n$.

For a cardinal $\lambda \geq 2$, a λ**-coloring** of a ladder system η on S is an indexed family $c = \{c_\delta\colon \delta \in S\}$ such that $c_\delta\colon \omega \to \lambda$. A **uniformization** of a coloring c of a ladder system η on S is a pair $\langle g, g^*\rangle$ where $g\colon \omega_1 \to \lambda$, $g^*\colon S \to \omega$ and for all $\delta \in S$ and all $n \geq g^*(\delta)$, $g(\eta_\delta(n)) = c_\delta(n)$. If such a pair exists, we say that c can be uniformized. We say that (η, λ)**-uniformization holds** or that η **satisfies** λ**-uniformization** if every λ-coloring of η can be uniformized.

A **monochromatic** colouring c of a ladder system η is one such that for each $\delta \in S$, c_δ is a constant function. We shall, from now on, consider a monochromatic colouring with λ colours to be a function $c\colon S \to \lambda$ (which gives the constant value, $c(\delta)$, of the colouring of η_δ). Then a uniformization of a monochromatic colouring c is a pair $\langle f, f^*\rangle$ where for all $\delta \in S$ and all $n \geq f^*(\delta)$, $f(\eta_\delta(n)) = c(\delta)$. If every monochromatic λ-colouring of η can be uniformized we say η **satisfies monochromatic uniformization for** λ **colours**.

Define a **ladder system based on a countable set** to be an indexed family $\eta = \{\eta_\delta\colon \delta \in S\}$ such that each η_δ is a function from ω to a fixed countable set I. We can define notions of colouring and uniformization analogous to those above.

(See [3, pp. 367–369].) The following two results, though stated and proved for ladder systems on a stationary subset of ω_1, are true also for ladder systems based on a countable set.

For ease of reference, we include the following result (compare [3, XII.3.3]).

LEMMA 3: *If there is a ladder system on a stationary subset of ω_1 which satisfies λ-uniformization (resp. monochromatic uniformization for λ colours), then there is a tree-like ladder system on a stationary subset of ω_1 which satisfies λ-uniformization (resp. monochromatic uniformization for λ colours).*

Proof: Suppose $\{\eta_\delta\colon \delta \in S\}$ satisfies λ-uniformization (resp. monochromatic uniformization for λ colours). Choose a one-one map θ from ${}^{<\omega}\omega_1$ to ω_1, such that $\theta(\sigma) \leq \theta(\sigma')$ if σ' is a sequence extending σ and such that for any $\tau \in {}^{<\omega}\omega_1$, $\theta(\tau) \geq \tau(n)$ for all $n \in \mathrm{dom}(\tau)$. Let C be a closed unbounded subset of ω_1 consisting of limit ordinals such that for every $\alpha \in C$, $\theta[{}^{<\omega}\alpha] \subseteq \alpha$. Let $S' = S \cap C$. For $\alpha \in S'$, define $\zeta_\alpha(n) = \theta(\langle \eta_\alpha(0), \ldots, \eta_\alpha(n)\rangle)$. Then $\{\zeta_\alpha\colon \alpha \in S'\}$ is tree-like and satisfies λ-uniformization (resp. monochromatic uniformization for λ colours). ∎

Remark: With a little more care we can prove that if there is a ladder system on S which satisfies λ-uniformization, then there is a tree-like ladder system on the same set S which satisfies λ-uniformization. (Compare [3, Exer. XII.17].)

The third author has proved that there is a non-free Whitehead group of cardinality $\aleph_1$ if and only if there is a ladder system on a stationary subset of ω_1 which satisfies 2-uniformization. (See [3, §XII.3].) The main theorem of this paper is an analogous necessary and sufficient condition for the existence of a non-free hereditarily separable group. Since every Whitehead group is hereditarily separable, we can conclude that if there is a ladder system on a stationary subset of ω_1 which satisfies 2-uniformization, then there is a ladder system on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours. It is perhaps reassuring to know that there is a simple direct proof of this consequence:

PROPOSITION 4: *If there is a ladder system η on a stationary subset of ω_1 which satisfies 2-uniformization, then there is a ladder system on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours.*

Proof: By Lemma 3, we can assume that $\eta = \{\eta_\delta\colon \delta \in S\}$ is tree-like. Fix a monochromatic colouring c of η with ω colours. Let c' be a 2-colouring of η such that for $\delta \in S$, and $k \in \omega$ such that $2^k > c(\delta)$,

$$c'_\delta(2^k+1), c'_\delta(2^k+2), \ldots, c'_\delta(2^{k+1})$$

is the sequence

$$0^{c(\delta)}, 1, 1, 1, \ldots, 1,$$

i.e., $c(\delta)$ zeroes followed by $2^k - c(\delta)$ ones. Let $\langle g, g^* \rangle$ be a uniformization of c'. To define $\langle f, f^* \rangle$, let $f^*(\delta)$ be the least n so that $n = 2^{k+1}$ where $2^k > \max\{c(\delta), g^*(\delta)\}$. We define f so that for all $\delta \in S$ and $m \geq f^*(\delta)$, $f(\eta_\delta(m)) = c(\delta)$. To see that f is well-defined, consider the case when $\eta_\tau(m) = \eta_\delta(m)$ and $m \geq f^*(\tau), f^*(\delta)$. Since η is tree-like, $\eta_\delta(j) = \eta_\tau(j)$ for all $j \leq m$. By definition of f^*, there is a k such that $2^{k+1} \leq m$, and $2^k > \max\{c(\delta), c(\tau), g^*(\delta), g^*(\tau)\}$. But then the values of $g(\eta_\delta(j))$ for $j = 2^k+1, \ldots, 2^{k+1}$ code $c(\delta)$ and also $c(\tau)$, so $c(\delta) = c(\tau)$. ∎

Remark: The proof actually shows that if η is tree-like and satisfies 2-uniformization, then η satisfies monochromatic uniformization for ω colours.

1. Sufficiency

THEOREM 5: *If there is a ladder system on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours, then there is a non-free group of cardinality $\aleph_1$ which is hereditarily separable.*

Proof: By hypothesis there is a stationary subset S of ω_1 and a ladder system $\eta = \{\eta_\delta : \delta \in S\}$ such that every monochromatic colouring with ω colours can be uniformized. By Lemma 3, without loss of generality we can assume that η is tree-like.

We begin by defining the group. Let p_n $(n < \omega)$ be an enumeration of the primes. The group G will be generated by $\{x_\alpha : \alpha < \omega_1\} \cup \{y_{\delta,n} : \delta \in S, n < \omega\}$, subject to the relations

$$p_n y_{\delta,n+1} = y_{\delta,n} + x_{\eta_\delta(n)}.$$

For any α, we let G_α denote the subgroup of G generated by $\{x_\beta, y_{\delta n} : \beta < \alpha, \delta \in S \cap \alpha\}$. It is standard that G is $\aleph_1$-free (in fact $\aleph_1$-separable) but not free. The rest of the proof will be devoted to proving that G is hereditarily separable.

Assume B is a subgroup of G such that G/B is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$ and there is a finite set P of primes such that the order of every element of G/B is divisible only by primes in P; we need to prove that if Z is a rank 1 pure subgroup of B, then there is a projection of B onto Z. (See Lemma 2.) Since G/B is countable there is α so that $G_\alpha + B = G$ and $Z \subseteq G_\alpha$. Fix such an α and call it α^*. Next, choose in G_{α^*} a system of representatives for G/B and let $g\colon G \to G_{\alpha^*}$ be the function which assigns to an element of G its coset representative. Finally choose n^* so that for all $n \geq n^*$, G/B is uniquely p_n-divisible.

Let h_1 be a projection of $B \cap G_{\alpha^*}$ onto Z. Such a projection exists since $B \cap G_{\alpha^*}$ is free. We will extend h_1 to a projection h from B to Z by defining h on $\{x_\beta - g(x_\beta),\ y_{\delta,n} - g(y_{\delta,n})\colon \beta \geq \alpha^*,\ n \in \omega,\ \delta \in S,\ \delta \geq \alpha^*\}$. Such a definition suffices (provided it works), since B is generated by this set together with $B \cap G_{\alpha^*}$.

Define the colouring $c\colon (S \setminus \alpha^*) \to G_{\alpha^*}$ so that $c(\delta) = g(y_{\delta,n^*})$. (Since the values are taken in G_{α^*}, which is a countable set, this is an allowable colouring.) Let the pair $\langle f, f^* \rangle$ uniformize c. We can assume that $f^*(\delta) \geq n^*$ and that $\eta_\delta(f^*(\delta)) \geq \alpha^*$ for all $\delta \in S \setminus \alpha^*$. We define the function h in three stages. First for each $\delta \geq \alpha^*$ and $n \geq f^*(\delta)$, $h(y_{\delta,n} - g(y_{\delta,n})) = 0$ and

$$h(x_{\eta_\delta(n)} - g(x_{\eta_\delta(n)})) = h_1(p_n g(y_{\delta,n+1}) - g(y_{\delta,n}) - g(x_{\eta_\delta(n)})).$$

There are two potential problems with the second definition: namely, why is the right-hand side of the equation defined; and why is the definition independent of δ? (Note that $x_{\eta_\delta(n)}$ may equal $x_{\eta_\tau(n)}$ for some τ). For the first problem, note

$$p_n g(y_{\delta,n+1}) - g(y_{\delta,n}) - g(x_{\eta_\delta(n)}) \equiv p_n y_{\delta,n+1} - y_{\delta,n} - x_{\eta_\delta(n)} \equiv 0 \pmod{B},$$

hence $p_n g(y_{\delta,n+1}) - g(y_{\delta,n}) - g(x_{\eta_\delta(n)}) \in B \cap G_{\alpha^*}$. The independence of the definition from δ is a consequence of the following lemma, after noting that $n \geq f^*(\delta)$ implies $g(y_{\delta,n^*}) = g(y_{\tau,n^*})$ when $\eta_\delta(n) = \eta_\tau(n)$.

Lemma 6: *Let $n \geq n^*$. Suppose that $\tau, \delta \in S$, $\eta_\tau(n) = \eta_\delta(n)$ and $g(y_{\delta,n^*}) = g(y_{\tau,n^*})$. Then $g(y_{\delta,n+1}) = g(y_{\tau,n+1})$.*

Proof: The proof is by induction on $n \geq n^*$. Since the ladder system is tree-like, we can assume by induction that $g(y_{\delta,n}) = g(y_{\tau,n})$. Now

$$p_n g(y_{\delta,n+1}) \equiv g(y_{\delta,n}) + g(x_{\eta_\delta(n)}) = g(y_{\tau,n}) + g(x_{\eta_\tau(n)}) \equiv p_n g(y_{\tau,n+1}) \pmod{B}.$$

Since G/B is uniquely p_n-divisible (by choice of n^*), $g(y_{\delta,n+1}) \equiv g(y_{\tau,n+1})$ (mod B) and hence $g(y_{\delta,n+1}) = g(y_{\tau,n+1})$ by definition of g. ∎

To complete the definition, the second step is to define $h(x_\beta - g(x_\beta))$ arbitrarily (say 0) for any β not covered in the first step (i.e., $\beta \geq \alpha^*$ and $\beta \neq \eta_\delta(n)$ for any δ and any $n \geq f^*(\delta)$). Finally, for all δ and $n < f^*(\delta)$ define $h(y_{\delta,n} - g(y_{\delta,n}))$ as required by the equation

$$(y_{\delta,n} - g(y_{\delta,n})) + (x_{\eta_\delta(n)} - g(x_{\eta_\delta(n)})) - p_n(y_{\delta,n+1} - g(y_{\delta,n+1})) \\ + g(y_{\delta,n}) + g(x_{\eta_\delta(n)}) - p_n g(y_{\delta,n+1}) = 0.$$

(Do this by "downward induction".)

It remains to see that h induces a homomorphism. Consider the free group $F = L \oplus (B \cap G_{\alpha^*})$ where L is the group freely generated by $\{u_\beta, w_{\delta,n}\colon \delta \in S, n \in \omega, \beta > \alpha^*, \eta_\delta(n) > \alpha^*\}$. There is a surjective map $\varphi\colon F \to B$ which is the identity on $B \cap G_{\alpha^*}$ and such that $\varphi(u_\beta) = x_\beta - g(x_\beta)$ and $\varphi(w_{\delta,n}) = y_{\delta,n} - g(y_{\delta,n})$. The kernel K of φ is generated by elements of the form $(w_{\delta,n} + u_{\eta_\delta(n)} - p_n w_{\delta,n+1}) + (g(y_{\delta,n}) + g(x_{\eta_\delta(n)}) - p_n g(y_{\delta,n+1}))$. Let $\hat{h}\colon F \to Z$ be defined so that $\hat{h} \restriction B \cap G_{\alpha^*} = h_1$, $\hat{h}(u_\beta) = h(x_\beta - g(x_\beta))$ and $\hat{h}(w_{\delta,n}) = h(y_{\delta,n} - g(y_{\delta,n}))$. Since $\hat{h}$ is constantly 0 on K, it induces a homomorphism from B to Z which agrees with h on the generators of B. ∎

Remark: The same proof works with any tree-like ladder system based on a countable set. (The assumption that the ladder system is tree-like is necessary, as witnessed by Hausdorff gaps). In particular, if there is a set of $\aleph_1$ branches through the binary tree of height ω which satisfies monochromatic uniformization for ω colours, then the group built from these branches is hereditarily separable. This group is just the group constructed in [3, VII.4.3]; there it is shown that MA + $\neg$CH implies this group is hereditarily separable. Given these comments, one might expect that it is possible to show that MA + $\neg$CH implies that any system of $\aleph_1$ branches through the binary tree satisfies monochromatic uniformization for ω colours. Indeed, this is the case: given a set of $\aleph_1$ branches and a monochromatic colouring c by ω colours, let the poset, $\mathbb{P}$, consist of pairs (s, B) where s is a function from ${}^n2 \to \omega$ and B is a finite subset of the branches such that for all $b \in B$, $s(b \restriction n) = c(b)$. If $(t, c) \in \mathbb{P}$ and $\mathrm{dom}(t) = {}^m2$, we define $(t, C) \geq (s, B)$ iff $s \subseteq t$, $B \subseteq C$ and for all $b \in B$ and $n \leq k \leq m$, $t(b \restriction k) = s(b \restriction n) = c(b)$. The proof that for each n, $\{(s, B)\colon {}^n2 \subseteq \mathrm{dom}(s)\}$ is dense uses the fact that the colouring is monochromatic. On the other hand the poset is c.c.c., since any two conditions with the same first element are compatible.

2. Necessity

The following lemma can be derived as a consequence of the fact that the Richman type of a finite rank torsion free group is well-defined (see [6] or [5]); but for the convenience of the reader we give a self-contained proof.

LEMMA 7: *Suppose A is a torsion free group of rank $r+1$ and every rank r subgroup is free. If B and C are pure rank r subgroups, then the type of A/B is the same as the type of A/C.*

Proof: The proof is by induction on r. We can assume that $r \geq 1$ and $B \neq C$. Consider first the case $r = 1$; then $B \cap C = 0$. If $b \in B$ and $c \in C$ are generating elements, then $A \subseteq \mathbb{Q}b \oplus \mathbb{Q}c$ and it is enough to prove that m divides $b \pmod{C}$ if and only if m divides $c \pmod{B}$. Now if m divides $b \pmod{C}$, then $b = ma + nc$ for some $a \in A$ and $n \in \mathbb{Z}$. Since $B = \langle b \rangle$ is pure in A, m and n must be relatively prime. Hence there exist $s, t \in \mathbb{Z}$ such that $ns + mt = 1$. But then $c = m(tc - sa) + sb$, so m divides $c \pmod{B}$.

Now suppose $r > 1$. Consider $B \cap C$; since $r + 1 = \mathrm{rk}(B + C) = \mathrm{rk}(B) + \mathrm{rk}(C) - \mathrm{rk}(B \cap C)$ and $2r > r + 1$, we have that $\mathrm{rk}(B \cap C) \geq 1$. Since $B \cap C$ is a pure free subgroup of A we can find $\langle x \rangle \subseteq B \cap C$ which is a pure subgroup of A. Note that $A/\langle x \rangle$ has the property that every subgroup of rank $r - 1$ is free. Now apply the induction hypothesis to $A/\langle x \rangle$, $B/\langle x \rangle$ and $C/\langle x \rangle$. ∎

THEOREM 8: *If there is a non-free hereditarily separable group G of cardinality $\aleph_1$, then there is a ladder system on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours.*

Proof: We can write $G = \bigcup_{\alpha<\omega_1} G_\alpha$, a union of a continuous chain of countable free pure subgroups where, without loss of generality, we can assume that there is a stationary subset S of ω_1, consisting of limit ordinals, and an integer $r \geq 0$ such that for all $\delta \in S$, $G_{\delta+1}/G_\delta$ is non-free of rank $r + 1$ and every subgroup of $G_{\delta+1}/G_\delta$ of rank r is free. (We use the fact that if a stationary subset of ω_1 is partitioned into countably many pieces, then one of the pieces must be stationary: cf. [3, II.4.5].) Thus for each $\delta \in S$ there is a pure free subgroup F_δ/G_δ of $G_{\delta+1}/G_\delta$ of rank r such that $M_\delta \stackrel{\text{def}}{=} G_{\delta+1}/F_\delta$ is rank 1 and non-free.

Moreover either M_δ is divisible or there is a prime p_δ and an element $y^\delta + F_\delta$ of M_δ which is not divisible by p_δ. Without loss of generality (again using [3, II.4.5]), we can assume that there is a prime p such that $p_\delta = p$ for all $\delta \in S$ such that M_δ is not divisible.

For each $\delta \in S$, let $\{y^\delta_\ell : 0 \le \ell \le r\} \subseteq G_{\delta+1}$ be such that $\{y^\delta_\ell + G_\delta : 0 \le \ell \le r-1\}$ is a basis of F_δ/G_δ, $y^\delta_r \notin F_\delta$ and $y^\delta_r + F_\delta$ is not divisible by p in M_δ if M_δ is not divisible. Then $G_{\delta+1}$ is generated by $G_\delta \cup \{y^\delta_\ell : \ell \le r\} \cup \{z^\delta_n : n \in \omega\}$, where the z^δ_n satisfy equations

$$r^\delta_n z^\delta_n = \sum_{\ell \le r} s^{\delta,n}_\ell y^\delta_\ell + g^\delta_n$$

where $g^\delta_n \in G_\delta$ and r^δ_n, $s^{\delta,n}_\ell \in \mathbb{Z}$.

Define functions φ_δ on ω for each $\delta \in S$ by:

$$\varphi_\delta(n) = \langle g^\delta_m, s^{\delta,m}_\ell, r^\delta_m : \ell \le r, m \le n \rangle.$$

Notice that $\varphi_\delta(n)$ determines the isomorphism type of the finitely generated subgroup of $G_{\delta+1}/G_\delta$ generated by (the cosets of) $\{y^\delta_\ell : \ell \le r\} \cup \{z^\delta_m : m \le n\}$.

As in [3, XII.3] — see especially Theorem XII.3.3 and the beginning of the proof of XII.3.1 (p. 381) — if we show that $\Phi = \{\varphi_\delta : \delta \in S\}$ satisfies monochromatic uniformization for ω colours, then there is a ladder system on a stationary subset of ω_1 with the same property.

So fix a monochromatic colouring $c: S \to \omega$ of Φ. We are going to use c to define a subgroup B of G with a pure cyclic subgroup Z. By the hypothesis on G, there will be a projection $h: B \to Z$. Because of the way we define B we will be able to use h to define $f: \{\varphi_\delta(n) : \delta \in S, n \in \omega\} \to \omega$ such that for each $\delta \in S$, $f(\varphi_\delta(n)) = c(\delta)$ for all but finitely many $n \in \omega$.

We will define a continuous chain of subgroups B_α of G_α by induction on α and let $B = \bigcup_{\alpha \in \omega_1} B_\alpha$. To begin, let $\{x_n : n \in \omega\}$ be a basis of G_0, and let B_0 be the subgroup of G_0 generated by $\{px_0\} \cup \{px_{n+1} - x_n : n \in \omega\}$. Thus $G_0/B_0 \cong Z(p^\infty)$ and $Z \stackrel{\text{def}}{=} \mathbb{Z}px_0$ is a pure subgroup of B_0.

Let $A = \{t_n : n \in \omega\} \subseteq G_0$ be a complete set of representatives of G_0/B_0 such that $t_0 = 0$. For each pair (d, a) where $d > 0$ and $a \in A$, fix an element $[d, a] \in \omega$ such that $dt_{[d,a]} + B_0 = a + B_0$.

We will define the B_α so that for all α

1. $B_\alpha + G_0 = G_\alpha$ and

2. for all $\beta < \alpha$, $B_\alpha \cap G_\beta = B_\beta$.

Notice then that $G_\alpha/B_\alpha \cong G_0/B_0$, and Z is pure in each B_α.

The crucial case is when we have defined B_δ already and $\delta \in S$. We will define $B_{\delta,m}$ by induction on m and then let $B_{\delta+1} = \bigcup_{m\in\omega} B_{\delta,m}$. Let

$$B_{\delta,0} = \langle B_\delta \cup \{y_\ell^\delta \colon \ell < r\} \cup \{y_r^\delta - t_{c(\delta)}\}\rangle.$$

Then $B_{\delta,0} \cap G_\delta = B_\delta$ since $\{y_\ell^\delta \colon \ell \le r\}$ is independent mod G_δ. Suppose $B_{\delta,m}$ has been defined so that $B_{\delta,m} \cap G_\delta = B_\delta$. Thus $(B_{\delta,m} + G_\delta)/B_{\delta,m} \cong G_0/B_0$. Let $d_m > 0$ be minimal such that $d_m z_m^\delta \in B_{\delta,m} + G_\delta$. If $d_m z_m^\delta \equiv a_m \in A$ (mod $B_{\delta,m}$), let

$$B_{\delta,m+1} = B_{\delta,m} + \mathbb{Z}(z_m^\delta - t_{[d_m,a_m]}).$$

Then we will have $B_{\delta,m+1} \cap G_\delta = B_\delta$. So, in the end, $B_{\delta+1} \cap G_\delta = B_\delta$. Moreover, $B_{\delta+1} + G_0 = G_{\delta+1}$, because, by construction, every generator of $G_{\delta+1}$ belongs to $B_{\delta+1} + G_0$.

If $\delta \notin S$, the construction of $B_{\delta+1}$ is essentially the same, except that the colouring c plays no role; we begin with a set $Y \subseteq G_{\delta+1}$ which is maximal independent mod G_δ and let $B_{\delta,0} = \langle B_\delta \cup Y\rangle$; then define $B_{\delta,m}$ by induction as before (using a well-ordering of type ω of a set of generators of $G_{\delta+1}$ mod G_δ). This completes the description of the construction of B.

Now fix a projection $h \colon B \to Z$ and fix a well-ordering, $\prec$, of $Z^{r+1} \times \omega$ of order type ω. We are going to define the uniformizing function f. We must define $f(\nu)$ for each ν of the form $\varphi_\delta(n)$. (Note that there may be many δ such that $\nu = \varphi_\delta(n)$.) Suppose

$$\nu = \langle g_m, s_\ell^m, r_m \colon \ell \le r, m \le n\rangle.$$

Let $\sigma = \sigma(\nu)$ be minimal such that $g_m \in G_\sigma$ for all $m \le n$. For each $k \in \omega$ we can construct a group $B_\nu^{(k)}$ just as in the construction of $B_{\delta+1}$, which is generated by $B_\sigma \cup \{y_\ell \colon \ell < r\} \cup \{y_r - t_k\}$ together with elements of the form $z_m - t_{[d_m,a_m]}$ $(m \le n)$ where the z_m satisfy the relations

$$(*) \qquad r_m z_m = \sum_{\ell \le r} s_\ell^m y_\ell + g_m.$$

This is an abstract group, which can be regarded as a subgroup of the free group on $G_\sigma \cup \{y_\ell \colon \ell \le r\} \cup \{z_m \colon m \le n\}$ modulo the relations in G_σ and the relations given by ν (i.e., the equations (*)). If δ is such that $\varphi_\delta(n) = \nu$ and $c(\delta) = k$, then there is an embedding of $B_\nu^{(k)}$ into $B_{\delta,n+1}$ which fixes B_σ (and is an isomorphism if $\sigma = \delta$). As before, $Z = \langle px_0\rangle$ is a pure subgroup of $B_\nu^{(k)}$. Since $B_\nu^{(k)}$ is isomorphic to a subgroup of G, it is separable.

Since h exists, there is a $\prec$-least tuple $\langle w_\ell : \ell \le r\rangle\frown\langle k\rangle$ in $Z^{r+1}\times\omega$ for which there is a projection $h' : B_\nu^{(k)} \to Z$ with $h'\restriction B_\sigma = h\restriction B_\sigma$, $h'(y_\ell) = w_\ell$ for $\ell < r$ and $h'(y_r - t_k) = w_r$. Note that this tuple determines h' on $B_\nu^{(k)}$. Define $f(\nu) = k$.

We have to show that this definition works, that is, for each $\delta \in S$, $f(\varphi_\delta(n))$ equals $c(\delta)$ for sufficiently large $n \in \omega$. Fix $\delta \in S$. With respect to the well-ordering $\prec$, there are only finitely many "wrong guesses" which come before the "right answer" $\langle h(y_\ell^\delta) : \ell < r\rangle\frown\langle h(y_r^\delta - t_{c(\delta)})\rangle\frown\langle c(\delta)\rangle$. So we just have to show that no wrong guess can work for all n if it involves a $k \ne c(\delta)$. If there were a wrong guess that worked for all n for some $k \ne c(\delta)$, then there would be a projection h' onto Z whose domain, B', contains $\{y_\ell^\delta : \ell < r\}\cup\{y_r^\delta - t_k\}$, elements of the form $z_n^\delta - a_{\delta,n}$ for all $n \in \omega$ (with $a_{\delta,n} \in A$), and B_σ where σ is minimal such that $g_n^\delta \in G_\sigma$ for all $n \in \omega$.

Let $\tilde{G}$ denote

$$G_0 + B' = G_\sigma + \langle\{y_\ell^\delta : \ell \le r\}\cup\{z_n^\delta : n \in \omega\}\rangle.$$

Notice that for each $g \in G_0$, there is a j such that $p^j g \in B_0 \subseteq B_\sigma$, which is a subset of $\mathrm{dom}(h)$ and $\mathrm{dom}(h')$. Hence we can extend h and h' uniquely to homomorphisms from $\tilde{G}$ into $\mathbb{Q}^{(p)}\otimes Z$. (Here $\mathbb{Q}^{(p)}$ is the group of rationals whose denominators are powers of p.) Denote the extension by $\tilde{h}$ (resp. $\tilde{h}'$). We claim that $\tilde{h} = \tilde{h}'$.

Assume for the moment that this is true. Then $\tilde{h}(y_r^\delta) = \tilde{h}'(y_r^\delta)$. Now $\tilde{h}(y_r^\delta - t_{c(\delta)}) \in Z$ and $\tilde{h}'(y_r^\delta - t_k) \in Z$. So

$$\tilde{h}(t_{c(\delta)} - t_k) = \tilde{h}(y_r^\delta - t_{c(\delta)}) - \tilde{h}'(y_r^\delta - t_k) \in Z.$$

Since $k \ne c(\delta)$, there is an $s \in \mathbb{Z}$ so that $s(t_{c(\delta)} - t_k) \equiv x_0 \pmod{B_0}$, so $\tilde{h}(x_0) \in Z$. But $p(\tilde{h}(x_0)) = \tilde{h}(px_0) = px_0$; this contradicts the fact that px_0 generates Z.

It remains to prove the claim. Let $H = \tilde{G}/G_\sigma$, which is isomorphic to $G_{\delta+1}/G_\delta$. Now $\tilde{h} - \tilde{h}'$ induces a homomorphism from H into $\mathbb{Q}^{(p)}$ since $\tilde{h}$ and $\tilde{h}'$ agree on B_σ, hence on G_0 (since $G_0/B_0 \cong Z(p^\infty)$) and so on G_σ. So it suffices to prove that $\mathrm{Hom}(H, \mathbb{Q}^{(p)}) = 0$. Assume, to the contrary, that there is a non-zero

$\psi: H \to \mathbb{Q}^{(p)}$. Let $K = \ker(\psi)$; then the rank of K is r. Now H/K is isomorphic to a subgroup of $\mathbb{Q}^{(p)}$ and hence is not divisible. By Lemma 7, H/K is isomorphic to $M_\delta = G_{\delta+1}/F_\delta$. So by the choice of S and p, H/K is not p-divisible; since H/K is isomorphic to a subgroup of $\mathbb{Q}^{(p)}$, this implies H/K is free. But this is impossible, since H is not free and K is a subgroup of rank r, and hence free. ∎

COROLLARY 9: *If there is an hereditarily separable group of cardinality $\aleph_1$ which is not free, then there exist $2^{\aleph_1}$ different $\aleph_1$-separable groups of cardinality $\aleph_1$ which are hereditarily separable.*

Proof: By the theorem, the given hypothesis implies that there is a ladder system η on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours. Using this ladder system, we can construct an $\aleph_1$-separable group which is hereditarily separable as in the proof of Theorem 5. By a standard trick we can, in fact, construct such groups with $2^{\aleph_1}$ different Γ-invariants. (Compare [3, VII.1.5].) ∎

Similarly to the proof of Theorem 8 we can prove the following:

THEOREM 10: *If there is an hereditarily separable group G of cardinality $\aleph_1$ which is not a Shelah group, then there is a ladder system based on a countable set which satisfies monochromatic uniformization for ω colours.*

3. Consistency of Case 4

The consistency of Case 4 in the Introduction will now follow from Theorem 5 and the following set-theoretic result.

THEOREM 11:

It is consistent with ZFC + GCH that the following all hold:

(i) *there is a ladder system on a stationary subset of ω_1 which satisfies monochromatic uniformization for ω colours;*
(ii) *there is no ladder system on a stationary subset of ω_1 which satisfies 2-uniformization;*
(iii) *$\diamondsuit_\kappa(E)$ holds for every stationary subset, E, of every regular cardinal $\kappa > \aleph_1$.*

Proof: We assume familiarity with the methods of [10]. For simplicity let our ground model be L; fix a stationary, co-stationary subset S of ω_1 and a ladder system η on S. Our forcing $\mathbb{P}$ will be an iterated forcing with countable support using two types of posets: R, which adds a Cohen subset of ω_1, and $Q(c)$ which is the poset uniformizing a monochromatic colouring $c\colon S \to \omega$ of η with countable onditions, i.e.,

$$Q(c) = \{f\colon f\colon \alpha \to \omega \text{ for some successor } \alpha < \omega_1 \text{ and for all } \delta \in S \cap \alpha,$$
$$f(\eta_\delta(n)) = c(\delta) \text{ for almost all } n \in \omega\}.$$

In the iteration $\mathbb{P}$ we force with $\tilde{R}$ at successors of even ordinal stages and force with $\tilde{Q}(\tilde{c})$ at the successors of odd ordinal stages, where, as usual, the names $\tilde{c}$ are chosen so that all possibilities occur. The posets R and $Q(c)$ are proper, so stationary sets are preserved by $\mathbb{P}$. Also, $\mathbb{P}$ is $(\omega_1 \setminus S)$-closed and of cardinality $\aleph_2$, so GCH holds in the generic extension as well as $\diamondsuit_\kappa(E)$ for every stationary subset of every regular cardinal $\kappa > \aleph_1$.

It remains to show that in the generic extension 2-uniformization fails for every stationary subset E of ω_1 and every ladder system $\zeta = \{\zeta_\delta\colon \delta \in E\}$. By doing an initial segment of the forcing we can assume that E and ζ are both in the ground model. Let X be the generic set for the first copy of R in the iteration of $\mathbb{P}$. Consider the 2-colouring $\{c_\delta\colon \delta \in E\}$ of ζ defined as follows: $c_\delta(n) = 0$ if and only if $\delta + n \in X$. The proof that this colouring is not uniformized now follows along the same lines as that in [10]. ∎

COROLLARY 12: *It is consistent with ZFC + GCH that there is an hereditarily separable group of cardinality $\aleph_1$ which is not free, and every Whitehead group (of arbitrary cardinality) is free.*

Proof: We use the model of ZFC + GCH constructed in Theorem 11. Clause (i) in Theorem 11 together with Theorem 5 imply that there is a non-free hereditarily separable group of cardinality $\aleph_1$. Clause (ii) implies that there is no non-free Whitehead group of cardinality $\aleph_1$. (See [3, XII.3.1(i)].) Finally clause (iii) enables one to do an inductive proof that there is no non-free Whitehead group of any cardinality (as, for example, in [3, XII.1.6]). ∎

4. Cases 2 and 3

In Cases 2 and 3 of the Introduction we are in the situation where there is a Whitehead group which is not free; here we shall consider two hypotheses which are stronger than this hypothesis: first, that there is a Whitehead group which is not a Shelah group; and, second, that every Shelah group is a Whitehead group.

The following theorem says that the hypothesis that there is a Whitehead group which is not a Shelah group is not consistent with Case 3. It also gives another consistency proof for Case 2 since it is known that it is consistent that there are Whitehead groups which are not Shelah groups (see [3, XII.3.11]).

THEOREM 13: *If there is a Whitehead group of cardinality $\aleph_1$ which is not a Shelah group, then there is an hereditarily separable group which is not a Whitehead group.*

Proof: By [3, XII.3.19] there is a ladder system $\eta = \{\eta_\delta : \delta \in \omega_1\}$ based on a countable set I which satisfies 2-uniformization. Without loss of generality we can assume that $I = \omega$ and each $\eta_\delta : \omega \to \omega$ is strictly increasing. Moreover, as in the proof of Lemma 3, we can assume that η is tree-like, and hence, as in the proof of Lemma 4, η satisfies monochromatic uniformization for ω colours.

For each $\delta \in \omega_1$ and $n \in \omega$ let

$$k_{\delta,n} = (\eta_\delta(n) + 1)!$$

and

$$k'_{\delta,n} = \eta_\delta(n)!.$$

Let G be the group generated by $\{x_n : n < \omega\} \cup \{y_{\delta,n} : \delta \in \omega_1, n < \omega\}$, subject to the relations

$$(1) \qquad k_{\delta,n+1} y_{\delta,n+1} = y_{\delta,n} + x_{\eta_\delta(n)}.$$

As in the proof of Theorem 5, G is hereditarily separable.

It remains to show that G is not a Whitehead group. For this we shall define an epimorphism $\pi: H \to G$ with kernel $\mathbb{Z}$ which does not split. Let H be the group generated by $\{x'_n : n < \omega\} \cup \{y'_{\delta,n} : \delta \in \omega_1, n < \omega\} \cup \{z\}$, subject to the relations

$$(2) \qquad k_{\delta,n+1} y'_{\delta,n+1} = y'_{\delta,n} + x'_{\eta_\delta(n)} + k'_{\delta,n+1} z.$$

There is an epimorphism π taking $y'_{\delta,n}$ to $y_{\delta,n}$, x'_m to x_m, and z to 0; the kernel of π is the pure subgroup of H generated by z. Aiming for a contradiction, assume

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there is a splitting of π, i.e., a homomorphism $\varphi\colon G \to H$ such that $\pi\circ\varphi = 1_G$. So $\varphi(y_{\delta,n}) - y'_{\delta,n} \in \ker(\pi)$ for all $\delta < \omega_1$, $n \in \omega$. Since a countable union of countable sets is countable, there exists $\delta \neq \tau$ such that $\eta_\delta(0) = \eta_\tau(0)$, $\eta_\delta(1) = \eta_\tau(1)$ and $\varphi(y_{\delta,0}) - y'_{\delta,0} = \varphi(y_{\tau,0}) - y'_{\tau,0}$. Let m (≥ 2) be minimal such that $\eta_\delta(m) \neq \eta_\tau(m)$.

We claim that $\varphi(y_{\delta,n}) - y'_{\delta,n} = \varphi(y_{\tau,n}) - y'_{\tau,n}$ if $n < m$. The proof is by induction on $n < m$; the initial case $n = 0$ is by choice of δ and τ. So supposing the result is true for $n < m-1$, we will prove it for $n+1$. Applying the homomorphism φ to equation (1) for τ as well as δ and subtracting we get that (in H)

$$k_{\delta,n+1}\varphi(y_{\delta,n+1}) - k_{\tau,n+1}\varphi(y_{\tau,n+1}) = \varphi(y_{\delta,n}) - \varphi(y_{\tau,n}) \tag{3}$$

since $x_{\eta_\delta(n)} = x_{\eta_\tau(n)}$ because $n < m$. But then by induction

$$k_{\delta,n+1}\varphi(y_{\delta,n+1}) - k_{\tau,n+1}\varphi(y_{\tau,n+1}) = y'_{\delta,n} - y'_{\tau,n}. \tag{4}$$

Now by equation (2), since $x_{\eta_\delta(n)} = x_{\eta_\tau(n)}$ and $k'_{\delta,n+1} = k'_{\tau,n+1}$ (the latter because $n < m-1$), we have

$$k_{\delta,n+1}y'_{\delta,n+1} - k_{\tau,n+1}y'_{\tau,n+1} = y'_{\delta,n} - y'_{\tau,n}, \tag{5}$$

so by equations (4) and (5) we have

$$k_{\delta,n+1}(\varphi(y_{\delta,n+1}) - y'_{\delta,n+1}) = k_{\tau,n+1}(\varphi(y_{\tau,n+1}) - y'_{\tau,n+1}). \tag{6}$$

Since $n < m-1$, $k_{\delta,n+1} = k_{\tau,n+1}$, so cancelling $k_{\delta,n+1}$ from equation (6), we obtain the desired result, and the claim is proved.

Now equation (4) holds for $n = m-1$ so

$$k_{\delta,m}\varphi(y_{\delta,m}) - k_{\tau,m}\varphi(y_{\tau,m}) = y'_{\delta,m-1} - y'_{\tau,m-1}. \tag{7}$$

In this case, instead of (5) we have

$$k_{\delta,m}y'_{\delta,m} - k_{\tau,m}y'_{\tau,m} - (k'_{\delta,m} - k'_{\tau,m})z = y'_{\delta,m-1} - y'_{\tau,m-1}, \tag{8}$$

so combining (7) and (8) we have

$$k_{\delta,m}\varphi(y_{\delta,m}) - k_{\tau,m}\varphi(y_{\tau,m}) = k_{\delta,m}y'_{\delta,m} - k_{\tau,m}y'_{\tau,m} - (k'_{\delta,m} - k'_{\tau,m})z. \tag{9}$$

Say $\eta_\delta(m) < \eta_\tau(m)$. Then $k_{\delta,m}$, $k'_{\tau,m}$ and $k_{\tau,m}$ are all divisible by $k_{\delta,m} = (\eta_\delta(m)+1)!$ so equation (9) implies that $(\eta_\delta(m)+1)!$ divides $k'_{\delta,m}z = \eta_\delta(m)!z$ in H which is a contradiction, since z generates a pure subgroup of H. ∎

Now we consider the hypothesis that every Shelah group is a Whitehead group. This is true in a model of Martin's Axiom, in which case there are hereditarily separable groups which are not Whitehead groups, i.e., Case 2 holds. Here we show that it is consistent that every Shelah group is a Whitehead group but every hereditarily separable group is a Whitehead group, i.e., there is a model for Case 3 in which every Shelah group is a Whitehead group. For this purpose we use the notion of stable forcing. A poset, $\mathbb{P}$, is **stable** if for every countable subset P_0 there is a countable subset P_1 so that for every $p \in \mathbb{P}$ there is an extension p' of p and an element $p^* \in P_1$ so that p' and p^* are compatible with exactly the same elements of P_0. In [1] the basic facts about c.c.c. stable forcings are proved. There are a few basic facts that we will use:

PROPOSITION 14:

1. [1] *Any iteration of c.c.c. stable forcings with finite support is c.c.c. and stable.*
2. *The forcing adding any number of Cohen reals is stable.*
3. *If A is a Shelah group and*

$$0 \to \mathbb{Z} \longrightarrow B \xrightarrow{\pi} A \to 0$$

is a short exact sequence, then the finite forcing, $Q(\pi)$, constructing the splitting of π is (c.c.c. and) stable.

Proof: We will prove only the last of the statements. Write A as $\bigcup_{\alpha<\omega} A_\alpha$ (an ω_1-filtration) where each A_α is pure in A and $A_{\alpha+1}$ has the Shelah property over A_α. The forcing $Q(\pi)$ is the set of partial splittings of π whose domains are finite rank pure subgroups of A. (This forcing is c.c.c. — see, e.g., [3, XII.1.11].) Given P_0, choose α so that every element of P_0 has domain contained in A_α. Let P_1 be the set of elements of $Q(\pi)$ whose domains are contained in $A_{\alpha+\omega}$. Given $p \in Q(\pi)$, let G be the pure subgroup of A generated by $A_\alpha \cup \mathrm{dom}(p)$. There exists $n \in \omega$ such that $G \cap A_{\alpha+\omega} = G \cap A_{\alpha+n}$ (since G has finite rank over A_α). Then

$$G = (G \cap A_{\alpha+n}) \oplus \langle y_0, \dots, y_m \rangle$$

for some $y_0, \dots, y_m$ since $G \cap A_{\alpha+n+1} = G \cap A_{\alpha+n}$ and $A_{\alpha+n+1}$ has the Shelah property over $A_{\alpha+n}$. Extend p to $p' \in Q(\pi)$ such that

$$\mathrm{dom}(p') = M \oplus \langle y_0, \dots, y_m \rangle$$

where $M \subseteq G \cap A_{\alpha+n}$ is a finite rank pure subgroup of $G \cap A_{\alpha+n}$ such that $\mathrm{dom}(p) \subseteq M \oplus \langle y_0, \ldots, y_m \rangle$. Let $p^* = p' \restriction M \in P_1$.

It suffices to prove that if $q \in P_0$ is compatible with p^*, then q is compatible with p'. So suppose that $r \in Q(\pi)$ such that $r \geq q, p^*$. Without loss of generality $\mathrm{dom}(r) \subseteq G \cap A_{\alpha+n}$. Define r' with

$$\mathrm{dom}(r') = \mathrm{dom}(r) \oplus \langle y_0, \ldots, y_m \rangle$$

by: $r' \restriction \mathrm{dom}(r) = r$ and $r' \restriction \langle y_0, \ldots, y_m \rangle = p' \restriction \langle y_0, \ldots, y_m \rangle$. Clearly $r' \geq q, p'$. Moreover, $\mathrm{dom}(r')$ is pure in A since G is pure in A and $\mathrm{dom}(r)$ is pure in $G \cap A_{\alpha+n}$; so $r' \in Q(\pi)$. ∎

THEOREM 15: *It is consistent that every Shelah group is a Whitehead group and every hereditarily separable group is a Whitehead group.*

Proof: We do our forcing over L by iteratively adding subsets of ω_1 by finite conditions and adding splittings for Shelah groups. More precisely, our forcing $\mathbb{P}$ will be an iterated forcing with finite support and of length ω_2 using two types of posets: R, the finite functions from ω_1 to 2, and $Q(\pi)$ which is the finite forcing splitting π as in Proposition 14(3). If we choose the iterants correctly, then in the generic extension every Shelah group of cardinality $\aleph_1$ will be a Whitehead group and $\diamondsuit(E)$ will hold for every stationary subset of every regular cardinal greater than $\aleph_1$. It will suffice then to show that every hereditarily separable group of cardinality $\aleph_1$ is a Shelah group (because we have all instances of diamond above $\aleph_1$: cf. [3, Exer. XII.16(ii)]).

By Theorem 10 it is enough to show that, in the generic extension, if $\Phi = \{\varphi_\alpha \colon \alpha < \omega_1\}$ is a ladder system based on ω, then Φ does not satisfy monochrome uniformization for ω colours. In fact, we will show that Φ does not satisfy monochrome uniformization for 2 colours. By absorbing an initial segment of the forcing into the ground model we can assume that Φ is in the ground model and the forcing $\mathbb{P}$ is first R, the finite functions from ω_1 to 2, followed by a name T for a c.c.c. stable forcing. We define the colouring $c \colon \omega_1 \to 2$ to be the generic set for R; let $\tilde{c}$ be a name for c.

In order to obtain a contradiction, assume that this colouring can be uniformized. Then there is a pair $\langle \tilde{f}, \tilde{f}^* \rangle$ of names for functions and there is a $p' \in \mathbb{P}$ such that $p' \Vdash$ "$\langle \tilde{f}, \tilde{f}^* \rangle$ uniformizes $\tilde{c}$". Now let P_0 be a countable subset of $\mathbb{P}$ containing p' as well as for every $n < \omega$, a maximal antichain which determines the value of $\tilde{f}(n)$. Let P_1 be as given by the definition of a stable poset for this P_0. For each $\alpha \in \omega_1$ choose $p_\alpha \geq p'$ so that p_α determines the values of $\tilde{f}^*(\alpha)$ and $\tilde{c}(\alpha)$ and there exists $p^*_\alpha \in P_1$ so that p_α and p^*_α are compatible with exactly the same elements of P_0. Say

$$p_\alpha \Vdash \tilde{f}^*(\alpha) = m_\alpha \wedge \tilde{c}(\alpha) = e_\alpha.$$

By the pigeon-hole principle, there exists an uncountable set $E \subseteq \omega_1$ and $p^* \in P_1$ so that for all $\alpha \in E$, $p^*_\alpha = p^*$. Since p^* is compatible with p', there exists $q_1 \geq p^*, p'$. By the definition of R, there exists $\alpha_0 \in E$ and $q_2 \in \mathbb{P}$ such that $q_1 \leq q_2$ and $q_2 \Vdash \tilde{c}(\alpha_0) \neq e_{\alpha_0}$. So $q_2 \Vdash$ "$\exists k > m_{\alpha_0}$ s.t. $\tilde{f}(\varphi_\alpha(k)) \neq e_{\alpha_0}$". Thus there exists $q_3 \geq q_2$ and $k_0 > m_{\alpha_0}$ such that $q_3 \Vdash \tilde{f}(\varphi_\alpha(k_0)) \neq e_{\alpha_0}$. But there is a maximal antichain in P_0 of conditions forcing the value of $\tilde{f}(\varphi_\alpha(k_0))$. Hence there exists $r \in P_0$ and q_4 such that $r \leq q_4$, $q_3 \leq q_4$ and $r \Vdash \tilde{f}(\varphi_\alpha(k_0)) = 1 - e_{\alpha_0}$. Then r is compatible with $p^* = p^*_{\alpha_0}$ and hence with p_{α_0}. But this is a contradiction since $p_{\alpha_0} \Vdash \tilde{c}(\alpha) = e_{\alpha_0} \wedge \tilde{f}(\varphi_\alpha(k_0)) = \tilde{c}(\alpha)$ since $k_0 > m_{\alpha_0}$. ∎

5. Finitely many primes

The proof of Theorem 5 uses infinitely many primes. Otherwise said, the type of the (torsion-free rank one) non-free quotients $G_{\delta+1}/G_\delta$ in that construction is $(1, 1, 1, \ldots)$. We may ask what happens if we are allowed only finitely many primes. For example, we may consider modules over $\mathbb{Z}_{(P)}$ (where P is a set of primes and $\mathbb{Z}_{(P)}$ denotes the rationals whose denominators in reduced form are not divisible by an element of P) and ask whether the main theorem, Theorem 1, holds. If P is infinite, i.e., $\mathbb{Z}_{(P)}$ has infinitely many primes, then our proofs apply and there is a non-free hereditarily separable $\mathbb{Z}_{(P)}$-module of cardinality $\aleph_1$ if and only if there is a ladder system on ω_1 which satisfies monochrome uniformization for ω colours. On the other hand if the cardinality of P is finite but at least two, we can show that Theorem 1 does not hold, and Case 4 in the Introduction is impossible. In fact, this section is devoted to proving the following result:

THEOREM 16: *Suppose R is a countable p.i.d. with only finitely many but at least 2 primes. If there is an hereditarily separable R-module of cardinality $\aleph_1$ which is not free, then there is a Whitehead R-module of cardinality $\aleph_1$ which is not free.*

Proof: The method of proof is to show that if there is an hereditarily separable R-module of cardinality $\aleph_1$ which is not free, then there is a ladder system on a stationary subset of ω_1 which satisfies 2-uniformization. We first prove that

(†) there is a tree-like ladder system $\eta = \{\eta_\delta\colon \delta \in S\}$ on a stationary subset S of $\lim(\omega_1)$ such that for every 2-colouring $c = \{c_\delta\colon \delta \in S\}$ of η, there is a function $f\colon \omega_1 \times \omega \to 2$ such that for all $\delta \in S$ there exists $m_\delta \in \omega$ such that $f(\eta_\delta(n), m_\delta) = c_\delta(n)$ for all $n \in \omega$.

Let N be an hereditarily separable R-module of cardinality $\aleph_1$. As in the proof of Theorem 8, we write $N = \cup_{\alpha<\omega_1} N_\alpha$ as a union of a continuous chain of countable free pure submodules where there is a stationary subset S of ω_1, consisting of limit ordinals, and an integer $r \geq 0$ such that for all $\delta \in S$, $N_{\delta+1}/N_\delta$ is non-free of rank $r+1$ and every subgroup of $N_{\delta+1}/N_\delta$ of rank r is free. There is a pure free subgroup F_δ/N_δ of $N_{\delta+1}/N_\delta$ of rank r such that $N_{\delta+1}/F_\delta$ is rank 1 and non-free.

It follows from the fact that there are only finitely many primes that the type of $N_{\delta+1}/F_\delta$ is $(t_1, t_2, \ldots, t_n)$ where each t_i is either 0 or ∞ and at least one $t_i = \infty$. Thus without loss of generality we may assume that there is a fixed prime $p \in R$ such that for all $\delta \in S$ there exists $\{y_\ell^\delta\colon 0 \leq \ell \leq r\} \subseteq N_{\delta+1}$ such that $\{y_\ell^\delta + N_\delta\colon 0 \leq \ell \leq r-1\}$ is a basis of F_δ/N_δ, $y_r^\delta \notin F_\delta$ and $y_r^\delta + F_\delta$ is p-divisible in $N_{\delta+1}/F_\delta$. Then $N_{\delta+1}$ contains elements z_n^δ $(n \in \omega)$, where $z_0^\delta = y_r^\delta$ and the z_n^δ $(n \geq 1)$ satisfy equations

$$(\blacktriangle_n) \qquad pz_n^\delta = \textstyle\sum_{\ell\leq r} s_\ell^{\delta,n} y_\ell^\delta + g_n^\delta + \sum_{j<n} r_j^{\delta,n} z_j^\delta$$

where $g_n^\delta \in N_\delta$, $r_\ell^{\delta,n}$, $s_\ell^{\delta,n} \in R$, and no element of $N_{\delta+1}/\langle F_\delta \cup \{z_n^\delta\colon n \in \omega\}\rangle$ has order p.

Define functions φ_δ on ω for each $\delta \in S$ by:

$$\varphi_\delta(n) = \langle g_m^\delta, s_\ell^{\delta,m}, r_j^{\delta,m}\colon \ell \leq r, m \leq n, j < m\rangle.$$

Let c be a 2-colouring of $\Phi = \{\varphi_\delta\colon \delta \in S\}$. Following the pattern of the proof of Theorem 8, we will use c to define a subgroup B of N. We begin by letting $\{x_n\colon n \in \omega\}$ be a basis of N_0, and letting B_0 be the subgroup of N_0 generated by $\{px_0\} \cup \{px_{n+1} - x_n\colon n \in \omega\}$. Also, let $A = \{t_n\colon n \in \omega\} \subseteq N_0$ be a complete set of representatives of N_0/B_0 such that $t_0 = 0$ and for each $a \in A$, fix an element $[p, a] \in \omega$ such that $pt_{[p,a]} + B_0 = a + B_0$.

Assume we have defined B_δ so that $B_\delta + N_0 = N_\delta$ and for all $\beta < \delta$, $B_\delta \cap N_\beta = B_\beta$. We now define $B_{\delta,m}$ inductively so that $z_m^\delta \in B_{\delta,m} + N_0$. Let

$B_{\delta,0}$ be generated by $B_\delta \cup \{y_0^\delta, \ldots, y_r^\delta\}$. If $B_{\delta,m-1}$ has been defined, we have $pz_m^\delta \in B_{\delta,m-1} + N_0$, so $pz_m^\delta \equiv a_m^\delta \pmod{B_{\delta,m-1}}$ for some $a_m^\delta \in A$. Let

$$B_{\delta,m} = B_{\delta,m-1} + R(z_m^\delta - t_{[p,a_m^\delta]} - c_\delta(m)x_0).$$

Having defined $B_{\delta,m}$ for all m, we can extend $\bigcup_{m\in\omega} B_{\delta,m}$ to $B_{\delta+1}$ such that $B_{\delta+1} + N_0 = N_{\delta+1}$ and $B_{\delta+1} \cap N_\delta = B_\delta$.

Finally, let $B = \bigcup_{\alpha<\omega_1} B_\alpha$ and fix a projection h: $B \to Rpx_0$ and a well-ordering, $\prec$, of R^{r+1} of order type ω. Extend h to a homomorphism, also denoted h, from N into Qpx_0, where Q is the quotient field of R. Given ν of the form $\varphi_\delta(n)$ and $m \in \omega$, we are going to define $f(\nu, m)$. Let $\langle w_\ell^m : \ell \le r\rangle$ be the mth tuple in R^{r+1} according to $\prec$. We shall suppose that

$$(\#_m) \qquad h(y_\ell^\delta) = w_\ell^m px_0$$

for $\ell \le r$ for some δ such that $\nu = \varphi_\delta(n)$, and show that under this supposition (and with the information given by ν) we can compute $c_\delta(n)$; we will then define this value of $c_\delta(n)$ to be $f(\nu, m)$. Since one of our suppositions $(\#_m)$ about the values of $h(y_\ell^\delta)$ must be right, (†) will be proved.

The proof is by induction on $k \le n$ that we can compute $h(z_k^\delta)$, a_k^δ, and $c_\delta(k)$. In fact, for $0 < k \le n$ we have an equation

$$(\blacktriangle_k) \qquad pz_k^\delta = \textstyle\sum_{\ell\le r} s_\ell^{\delta,k} y_\ell^\delta + g_k^\delta + \sum_{j<k} r_j^{\delta,k} z_j^\delta$$

satisfied by z_k^δ. Since by induction and our supposition we know the value of h for all the elements on the right-hand side, we can compute $h(z_k^\delta)$. Since by induction we also know $c_\delta \restriction k$, we know $B_{\delta,k-1}$, so we can calculate a_k^δ ($\equiv pz_k^\delta \pmod{B_{\delta,k-1}}$). Finally, we know that

$$h(z_k^\delta - t_{[p,a_k^\delta]} - c_\delta(k)x_0) = h(z_k^\delta - t_{[p,a_k^\delta]}) - c_\delta(k)x_0$$

belongs to Rpx_0, and we know $h(z_k^\delta - t_{[p,a_k^\delta]})$ by induction (because we know $h\restriction N_0$). Now $h(z_k^\delta - t_{[p,a_k^\delta]}) - x_0$ and $h(z_k^\delta - t_{[p,a_k^\delta]})$ cannot both belong to Rpx_0. If $h(z_k^\delta - t_{[p,a_k^\delta]})$ belongs to Rpx_0, $c_\delta(k)$ must equal 0; otherwise let $c_\delta(k) = 1$. (If the latter value does not make $h(z_k^\delta - t_{[p,a_k^\delta]} - c_\delta(k)x_0)$ belong to Rpx_0, then our supposition must have been wrong, and we can let $f(\nu, m)$ be arbitrary.)

This completes the proof of (†). At this point we use the assumption that there are at least two primes. Then the proof of necessity, i.e. of Theorem 8, is still valid. (Referring to the last paragraph of that proof, we use the fact that there are two primes when we assert that H/K is not divisible.) Moreover, there is a single ladder system η which satisfies the property of (†) as well as monochromatic uniformization for ω colours. (Indeed, by reducing to a smaller set we can assume that the same set S is used in both the proof of (†) and the proof of Theorem 8; then we can let η be a ladder system derived from functions φ_δ which give combined information about the equations used in the proof of (†) and the equations used in the proof of Theorem 8.)

Given a 2-colouring c of η, let f be as in (†). Define a monochromatic colouring c' of η by: $c'(\delta) = m_\delta$ where m_δ is such that $f(\eta_\delta(n), m_\delta) = c_\delta(n)$ for all $n \in \omega$. There is a uniformization $\langle g, g^*\rangle$ of c'. Define $h\colon \omega_1 \to 2$ by: $h(\nu) = f(\nu, g(\nu))$. Then for all $\delta \in S$ for sufficiently large n,

$$h(\eta_\delta(n)) = f(\eta_\delta(n), g(\eta_\delta(n))) = f(\eta_\delta(n), m_\delta) = c_\delta(n). \qquad \blacksquare$$

The third author, in [9, Thm. 3.6], claimed to prove that if the non-freeness of G involves only finitely many primes, then G is hereditarily separable if and only if G is Whitehead. However, the proof given seems to be irredeemably defective. We do not know if the result claimed is true. Thus we still have the following open questions:

> If R is a countable p.i.d. with exactly one prime, does Theorem 16 hold? If R has finitely many primes, is every hereditarily separable R-module of cardinality $\aleph_1$ a Whitehead module? If not, find a combinatorial equivalent, analogous to Theorem 1, to the existence of a hereritarily-separable R-module which is not a Whitehead module.

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