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Δ-LOGICS AND GENERALIZED QUANTIFIERS

J.A. MAKOWSKY *
II. Math. Institut, Freie Universität Berlin, 1 Berlin 33, West Berlin

Saharon SHELAH
The Hebrew University, Jerusalem Israel

Jonathan STAVI
Department of Mathematics, Bar Ilan University, Ramat Gan, Israel

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0. Introduction

The basic motivation for the study of abstract model theory is the search for languages ("abstract logics") which have a stronger expressive power than ordinary first-order logic $L_{\omega\omega}$ and yet have a workable model theory. Previous work in the subject has been devoted mainly to characterizations of known logics ($L_{\omega\omega}$ by Lindstrom [33,34], $L_{\infty\omega}$ and its sublogics in Barwise [3,4]) as maximal with respect to some of their model theoretic properties. A general discussion of desirable properties of (model-theoretic) languages can be found in Feferman [10–12] and Kreisel [31].

During the years in which this abstract point of view has evolved there have also been intensive studies of particular languages, notably the language $L_{\omega\omega}[Q_1]$ obtained from $L_{\omega\omega}$ by adding the quantifier "there exist uncountably many" (cf. Fuhrken [17] and Keisler [29]) and other languages based upon generalized quantifiers. Some of them are treated in Bell and Slomson [7] and the present work will give an up-to-date survey in the examples. In [53, §4] (see [47]) Shelah proved the compactness of the languages $L_{\omega\omega}[Q^C]$ obtained from $L_{\omega\omega}$ by adding to $L_{\omega\omega}$ a quantifier saying that an ordering has cofinality ω. $L_{\omega\omega}[Q_1]$, $L_{\omega\omega}[Q^C]$ and

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various related logics are countably or fully compact and satisfy, as will be shown in the present work, a downward Lowenheim–Skolem Theorem to $\aleph_1$ and are axiomatizable (i.e., have a recursively enumerable set of valid sentences). But Craig's interpolation theorem fails for them as was noted for $L_{\omega\omega}[Q_1]$ by Keisler (cf. Feferman [10, p. 216, footnote (3)]).

This paper grew out of the search (suggested and motivated by Feferman in [10–12] and in private conversations, but conceived independently by other people, in particular Barwise, Friedman and Keisler) for a manageable extension of $L_{\omega\omega}[Q_1]$ which will satisfy the interpolation theorem in addition to having the above-mentioned nice properties. At present no such extension is known but the search for it has led to a wide class of compact and axiomatizable logics based on quantifiers, which are studied in Section 3 of this paper as well as in Shelah [47] or Hutchinson [24].

It was realized by the people mentioned above and others that with every logic L one can associate a smallest extension $\Delta(L)$ haivng a weakened interpolation property known as Souslin Kleene interpolation. The operator Δ preserves compactness and other, but not all, nice model theoretic properties of logics. The systematic study of properties preserved by the Δ-operation in Section 2 is largely taken from Makowsky [37], though the simpler facts proved there are not claimed to be new (cf. also Paulos [44–46]). It provides a basis for the further study of the quantifiers introduced in Section 3. This approach demonstrates the fruitfulness of the abstract point of view in discovering and proving properties of "concrete" generalized quantifiers.

But the Δ-closure is more than just a technical tool to construct logics. It is a closure operator motivated by Beth's Theorem (or variations of it), which adds to a logic L everything which is, in some sense, implicit in it. The Δ-closure also provides a means of evaluating the choice of generalized quantifiers, a problem which seems even more delicate than the "choice of infinitary languages" (cf. Kreisel [31]) since we seem to lack not only a programme à la Kreisel but also experience and intuition. But one may say that the expressive power of a quantifier Q is better shown by the Δ-closure of the logic generated by it. For instance, the difficulties of finding a reasonable description of $\Delta(L_{\omega\omega}[Q_1])$ might indicate that pure cardinality quantifiers are not the right choice.

In Section 4 we study sublogics of $L_{\omega_1\omega}$. Our main task is to identify $\Delta(L)$ for certain logics L (containing generalized quantifiers) with admissible fragments L_A of $L_{\omega_1\omega}$. The simplest case was treated by Barwise [4]

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$\mathfrak{B}$ in which χ is interpreted may be of arbitrary type though the induced structure $\mathfrak{A}$ is automatically of type τ. For example, if $\tau = \langle\ \rangle$, $K = \{\langle A\rangle \mid A$ is infinite$\}$ then a quantifier Q of type τ binds one variable and $Qx\varphi(x, \bar{w})$ is interpreted as "there exist infinitely many x such that φ".

Note the role of the formula φ_0 (in the above χ) – it serves to define the domain A of the structure which is claimed in χ to belong to K. Thus "relativization" is built in the formation of formulas involving the quantifier (in this we deviate from Lindstrom [33] and follow Barwise [4]).

There is no difficulty in introducing quantifiers of certain many sorted types. Call a type $\tau = \langle I_0, I_1, I_2, I_3, \rho_1, \rho_2, p_3\rangle$ semi-simple when for some $h, k, l \geqslant 0$:

$$I_0 = \{1, ..., h\}, I_1 = \{1, ..., k\}, I_2 = \emptyset, I_3 = \{1, ..., l\}$$

(when $h = 1$ τ is a simple type). A quantifier of type τ (where τ is semi-simple as above) would produce formulas χ of the form

$$Qx_1 \dots x_h x_{11} \dots x_{1n_1} \dots x_{k1} \dots x_{kn_k}[\varphi_1, ..., \varphi_h, \psi_1, ..., \psi_k, t_1, ..., t_l]$$

where $\varphi_1, ..., \varphi_h$ take the role of φ_0 above so that $\mathfrak{A}$ will be an (h-sorted) structure of type τ. The definition of satisfaction of the formula χ is left to the reader. When $h = 0$ (hence $l = 0$ and each relation is 0-ary) structures of type τ consist simply of a sequence of k truth-values and a quantifier of type τ is just a propositional connective.

It is also possible to consider infinitary connectives (see Friedman [16] and Harrington [54]) and more generally quantifiers which bind infinitely many variables and/or operate on infinitely many formulas and terms, but we shall not do this here.

1.4. Logics (model theoretic languages): Instead of defining abstract logics axiomatically as in Barwise [3,4] we introduce them in a concrete restricted way. A logic L is given by a family $\langle Q^i \mid i \in I\rangle$ of quantifier symbols, a family $\langle \tau^i \mid i \in I\rangle$ of semi-simple types and a family $\langle K^i \mid i \in I\rangle$, K^i closed under isomorphisms and included in $S(\tau^i)$ for each i. I may be a proper class. K^i will serve as the interpretation of Q^i. For an arbitrary type τ we construct atomic formulas of $L(\tau)$ as in the ordinary language $L_{\omega\omega}(\tau)$ (using infinitely many variables of each sort). Arbitrary formulas of $L(\tau)$ are now obtained from atomic formulas by the usual logical operations of $L_{\omega\omega}(\neg, \wedge, \forall$, etc.) and the quantifiers Q^i. We shall write $L = L_{\omega\omega}[Q^i]_{i \in I}$ though L really depends (for its semantics) also on the family $\langle K^i | i \in I \rangle$. The precise definitions of the basic syntactical and semantical notions for a logic L are left to the reader. A logic L is called finitely generated when it is of the form $L = L_{\omega\omega}[Q^1, ..., Q^n]$ for some $n < \omega$.

Although this notion of a logic is narrower than the abstract notions considered in Barwise [3] or [4], any abstract logic which satisfies some reasonable closure conditions and in which only finitely many non-logical symbols "occur" (cf. Barwise [4, I, §7]) in each sentence can be put in this form: Assigning a quantifier Q^K (i.e., a quantifier Q interpreted by the class K) to each class K of structures (of any semi-simple type) which is elementary (EC) in the given abstract logic. Thus, for example, the part of $L_{\infty\omega}$ consisting of sentences in which only finitely many non-logical symbols occur is equivalent (in expressive power) to some logic in our sense, though the definition of satisfaction for that logic would presuppose the ordinary semantics of $L_{\infty\omega}$.

Examples. (1) K_1 of type $\langle\ \rangle$ with $K_1 = \{\langle A\rangle | A \neq \emptyset\}$: Q^{K_1} can be identified with $\exists$.

(2) K_2^α of type $\langle\ \rangle$ with $K_2^\alpha = \{\langle A\rangle | \bar{\bar{A}} \geqslant \aleph_\alpha\}$: $Q^{K_2^\alpha}$ can be identified with Q_α ("there exist at least $\aleph_\alpha$").

(3) K_3 of type $\langle 2\rangle$ with $K_3 = \{\mathfrak{A} \mid \mathfrak{A} = \langle A, R\rangle$ and R is a well-ordering of $A\}$; Q^{K_3} will be dentoed by Q^{wo}.

(4) $K_4^{n,\alpha}$ of type $\langle n\rangle$ with $K_4^{n,\alpha} = \{\mathfrak{A} \mid \mathfrak{A} = \langle A, R\rangle$ such that there is an $S \subseteq A$ with $S^n \subseteq R$ and $\bar{\bar{S}} \geqslant \aleph_\alpha\}$: $Q^{K_4^{n,\alpha}}$ will be denoted by $Q_\alpha^{\mathrm{MM}(n)}$ and was first discussed by Magidor and Malitz [36]. For $\alpha = 0$ this quantifier is sometimes called the Ramsey quantifier.

(5) K_5 of type $\langle 1\rangle$ with $K_5 = \{\langle A, R\rangle | \bar{\bar{R}} = \bar{\bar{A}}\}$. If we restrict ourselves to single-sorted structures and put for φ_0 in $Q^{K_5}\, x_0 = x_0$ then the resulting logic corresponds to Chang's quantifier in Bell–Slomson [7, Ch. 13]. Chang's quantifier will be denoted Q_{ccc}. The more general form was introduced by Härtig [22].

Given a logic L and a class of structures K of a type τ we say K is *L-elementary* ($K \in \mathrm{EC}_{\mathrm{L}}^\tau$) if there is a sentence $\varphi \in \mathrm{L}(\tau)$ such that $K = \mathrm{Mod}(\varphi)$. K is a *L-projective* class ($K \in \mathrm{PC}_{\mathrm{L}}^\tau$) if there is a type $\tau' \supseteq \tau$ and a sentence φ in $\mathrm{L}(\tau')$ with $K = \mathrm{Mod}(\varphi){\restriction}_\tau$.

If L_1 and L_2 are two logics, the logic $L_1 \cap L_2$ is defined by $L_1 \cap L_2 =$

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and similar work was done by Makowsky [39]. However, in some cases this leads to L_A with A a non-transitive set. Therefore we also introduce a new closure operation $\tilde{\Delta}$ on sublogics of $L_{\omega_1\omega}$ which is, in some cases, at least for identifications of that type, better behaved than Δ. Related work was done recently by Paulos [45,46] and Swett [49].

Each section contains numerous examples, and open problems are stated at the end of Sections 2, 3 and 4. We consider these examples (and counter-examples) as an important part of this paper. Abstract model theory gives us only an approach to general questions; the intuition and experience for it can only be found by dealing with concrete problems. Several of the basic ideas of Section 2 have been essentially known to Keisler, Barwise, Friedman, Shelah, Paulos and possibly others. The detailed study has been done by Makowsky (cf. [37,39]). Section 3 is mainly due to Shelah and Stavi and Section 4 to Makowsky and Stavi. We wish to thank S. Feferman for many challenging questions and helpful discussions which greatly encouraged us to pursue the subject.

1. Preliminaries

1.1. Unexplained notation is standard. For model theory the books of Chang–Keisler [8], Keisler [28], Shoenfield [48] or Bell–Slomson [7] will do. For admissible sets wer refer to Barwise [5] although admissible sets are only used in Section 4.

1.2. A many sorted similarity type is a 7-tuple $\tau = \langle I_0, I_1, I_2, I_3, \rho_1, \rho_2, \rho_3\rangle$ where I_0 is a set indexing the sorts, I_1 indexes the relations, I_2 the operations and I_3 the distinguished elements (of any structure of type τ). ρ_1, ρ_2, ρ_3 are functions defined on I_1, I_2, I_3, respectively, and showing the number of places and sorts of arguments and value for each relation, operation and distinguished element. A simple type is a type τ for which $I_0 = \{1\}$, $I_1 = \{1, ..., k\}$ for some k, $I_2 = \emptyset, I_3 = \{1, ..., l\}$ for some l $(k, l \geqslant 0)$. Structures of type τ are then single-sorted, with finitely many relations and distinguished elements (no operations). We denote such a type τ by $\langle n_1, ..., n_k; l\rangle$ where n_i is the number of places of the i^{th} relation (given by $\rho_1(i)$). When $l = 0$ we simply write $\tau = \langle n_1, ..., n_k\rangle$. For any type τ, $S(\tau)$ $(S(\tau)_\kappa, S_\kappa)$ is the class of all structures of this type (of cardinality κ).

A set of relation symbols, operation symbols and constants is called a vocabulary. With each type τ we associate in some standard way a vocabulary, which is used to construct (atomic) formulas of $L_{\omega\omega}(\tau)$. For definiteness let us agree that equations $t = t'$ are allowed as atomic formulas (for any τ) just in case t and t' are terms of the same sort.

1.3. Generalized quantifiers have been introduced by Mostowski [52] and in greater generality by Lindstrom [33] (see also Kalish–Montague [27]).

Let τ be a simple type. A quantifier Q of type τ is a variable-binding operator that makes a formula out of formulas and terms. If $\tau = \langle n_1, \ldots, n_k; l\rangle$ then a typical formula $\chi(\bar{w})$ beginning with Q (here $\bar{w} = (w_1, w_2, \ldots)$ is a list containing the free variables of χ) has the form

$$Qx_0, \langle x_{ij}\rangle_{1\leqslant i\leqslant k,\, 1\leqslant j\leqslant n_i} [\varphi_0(x_0, \bar{w}), \psi_1(x_{11}, \ldots, x_{1n_1}, \bar{w}), \ldots, \psi_k(x_{k1}, \ldots, x_{kn_k}, \bar{w}), t_1(\bar{w}), \ldots, t_l(\bar{w})],$$

where the free variables of each formula or term are among those displayed, the variables x_0, x_{ij}, w_1, w_2, ..., are distinct and all these variables and the terms $t_1, \ldots, t_l$ are of the same sort.

Now let K be a class of structrues of type τ, closed under isomorphism: K gives rise to an interpretation of Q which we describe below in the special case $\tau = \langle 1,2;1\rangle$ to simplify notation. The formula $\chi(\bar{w})$ is now of the form

$$Qxyz_1z_2[\varphi_0(x,\bar{w}), \psi_1(y,\bar{w}), \psi_2(z_1,z_2,\bar{w}), t(\bar{w})].$$

Let $\mathfrak{B}$ be a structure of any type and let $\bar{b}$ be elements of $\mathfrak{B}$ assigned as values to the variables $\bar{w}$. Let B_0 be the basic domain of $\mathfrak{B}$ corresponding to the (common) sort of the variables x, y, z_1, z_2. We define: $\mathfrak{B} \models \chi[\bar{b}]$ iff $\langle A, R_1, R_2, c\rangle$ is a structure in K, where

$$A = \{a \in B_0 \mid \mathfrak{B} \models \varphi_0[a,\bar{b}]\},$$

$$R_1 = \{a \in B_0 \mid \mathfrak{B} \models \psi_1[a,\bar{b}]\},$$

$$R_2 = \{(a_1,a_2) \in B_0^2 \mid \mathfrak{B} \models \psi_2[a_1,a_2,\bar{b}]\},$$

$$c = t[\bar{b}] \text{ (evaluated in } \mathfrak{B}\text{)}.$$

Thus $\mathfrak{B} \models \chi[\bar{b}]$ iff $(A \neq \emptyset)$ $R_1 \subseteq A$, $R_2 \subseteq A^2$, $c \in A$ and the structured $\mathfrak{A} = \langle A, R_1, R_2, c\rangle$ is in K. A short suggestive notation is: $A = \varphi_0(\mathfrak{B},\bar{b})$, $R_1 = \psi_1(\mathfrak{B},\bar{b})$, $R_2 = \psi_2(\mathfrak{B}^2,\bar{b})$, $c = t(\bar{b})$. We emphasize that the structure

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$= L_{\omega\omega}[Q^K]_{K \in EC_{L_1} \cap EC_{L_2}}$. L_1 is a *sublogic* of L_2, $L_1 \leqslant L_2$, if $EC^\tau_{L_1} \subseteq EC^\tau_{L_2}$ for all τ. L_1 is a *PC-sublogic* of L_2, $L_1 \leqslant PC\ L_2$, if $PC^\tau_{L_1} \subseteq PC^\tau_{L_2}$ for all τ. L_1 is *equivalent* to L_2 $L_1 \sim L_2$, if $EC^\tau_{L_1} = EC^\tau_{L_2}$ and $L_1 \sim_{PC} L_2$ if $PC^\tau_{L_1} = PC^\tau_{L_2}$ (for all τ). $\mathrm{Th}_L(\mathfrak{A}) = \{\varphi \in L(\tau) \mid \mathfrak{A} \models \varphi\}$ with $\tau = \tau(\mathfrak{A})$. $\mathfrak{A} \equiv_L \mathfrak{B}$ if $\mathrm{Th}_L(\mathfrak{A}) = \mathrm{Th}_L(\mathfrak{B})$; $\mathfrak{A}$ and $\mathfrak{B}$ are then called *L-elementarily equivalent.* If L is a logic, $L = L_{\omega\omega}[Q^\alpha]_{\alpha \in A}$ we write $L[Q^\beta]_{\beta \in B}$ for $L_{\omega\omega}[Q^\alpha]_{\alpha \in A \cup B}$.

Examples. (6) $L \leqslant L[Q]$ for any quantifier Q.

(7) $L \sim L[Q]$ iff $K^Q \in EC_L$.

(8) $L_{\omega\omega}[Q_0] \leqslant L_{\omega_1\omega}$.

(9) $L_{\omega\omega}[Q^{wo}] \leqslant L_{\omega_1\omega_1}$.

(10) $L_{\omega_1 G}$ is the logic obtained from $L_{\omega_1\omega}$ by adding the following formation rule: If $\varphi_i(x_1, ..., x_{p_i}, y_1, ..., y_k)$ $(i < \omega)$ are formulas having only the displayed free variables then

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots \bigwedge \varphi_i \quad \text{and}$$

$$\exists x_1 \forall x_2 \ldots \bigvee \varphi_i \text{ are formulas}$$

of $L_{\omega_1 G}$. The semantics of $L_{\omega_1 G}$ may be explained via two-person games. $L_{\omega_1\omega_1} \not\leqslant L_{\omega_1 G}$ and $L_{\omega_1 G} \not\leqslant L_{\infty\infty}$ (see Barwise [3] and references there). Strictly speaking, $L_{\omega_1 G}$ as described here is not a logic in our sense but see remarks preceding (1).

(11) $L_{\omega\omega}[Q^E_\alpha]$ with defining class K of type $\langle 2 \rangle$. $K_\alpha = \{\mathfrak{A} \mid \mathfrak{A} = \langle A, R\rangle$ and R is an equivalence relation with at least $\aleph_\alpha$-many equivalence classes$\}$. One easily verifies that K and its complement are PC_L with $L = L_{\omega\omega}[Q_\alpha]$ but not EC_L. Furthermore $L_{\omega\omega}[Q_\alpha] \leqslant L_{\omega\omega}[Q^E_\alpha]$ and hence $L_{\omega\omega}[Q_\alpha] \sim_{PC} L_{\omega\omega}[Q^E_\alpha]$. This logic was considered by Feferman [12].

1.5. Let L be a logic, $\mathfrak{A}$ a structure of type τ and $\mathfrak{B}$ a substructure of $\mathfrak{A}$. $\mathfrak{B} <_L \mathfrak{A}$, $\mathfrak{B}$ is an *L-elementary substructure of* $\mathfrak{A}$, if $\mathrm{Th}_L(\mathfrak{B}, \bar{b}) = \mathrm{Th}_L(\mathfrak{A}, \bar{b})$ for every finite sequence $\bar{b}$ in $\mathfrak{B}$.

A logic L satisfies the *Löwenheim–Skolem Theorem* for κ, κ a cardinal, if every sentence ϕ of L, which has a model, has a model of cardinality $\leqslant (<)\kappa$. We denote this property by $LS(\kappa)$ ($LS(<\kappa)$). A logic L satisfies the *Löwenheim–Skolem–Tarski Theorem for* κ if every τ-structure $\mathfrak{A}$ has a L-elementary substructure of cardinality $\leqslant \kappa (<\kappa)$, provided $\bar{\bar{\tau}} \leqslant \kappa$ ($\bar{\bar{\tau}} < \kappa$ respectively). We denote this property by $LST(\kappa)$ ($LST(<\kappa)$). A logic is (κ, λ)*-compact*, for infinite cardinals κ, λ, $\kappa \geqslant \lambda$, if for every set of sentences Σ of L, $\bar{\bar{\Sigma}} \leqslant \kappa$, such that every $\Sigma_0 \subseteq \Sigma$, $\bar{\bar{\Sigma}}_0 < \lambda$, has a model, Σ has a model.

A logic L is *axiomatizable* if $L = L_{\omega\omega}[Q^1, \ldots, Q^n]$ ($n < \omega$) and the set of valid sentences of L is recursively enumerable.

A logic L has the *Tarski property* for κ, κ a regular cardinal, if the union of an L-elementary proper chain of cofinality $\geqslant \kappa$ is an L-elementary extension of all the members of the chain. We denote this property $T(\kappa)$. L has the Tarski property if it satisfies $T(\omega)$.

The *Löwenheim number* of a logic L is the smallest cardinal κ such that $LS(\kappa)$ holds for L. The *Hanf number* of a logic L is the smallest cardinal κ such that whenever a sentence of L has a model of cardinality κ then it has arbitrarily large models. Both numbers exist if $L = L_{\omega\omega}[Q^i]_{i \in I}$ where I is a set.

We say that an ordinal α is *L-accessible* if there is a class K of type $\langle 2 \rangle$ which is PC_L, all its members are well ordered and there is a $\mathfrak{A} \in K$ such that $\langle \alpha, < \rangle$ is embeddable in $\mathfrak{A}$. The *well-ordering number* of L is the least ordinal which is not L-accessible (if it exists). We shall abbreviate it by wo-number.

A logic L is *bounded* if and only if the class of well-orderings is not PC_L This is equivalent to Barwise's definition [3,4].)

A logic has the *Karp property* if for all $\mathfrak{A}$, $\mathfrak{B}$ $\mathfrak{A} \cong_p \mathfrak{B}$ (cf. [4]) implies $\mathfrak{A} \equiv_L \mathfrak{B}$. Various definability properties (Craig, Beth) will be studied in Section 2.

1.6. Logics may be characterized in terms of their model-theoretic properties. As an illustration we give two theorems.

Theorem 1.1 (Lindstrom [34,35]). *If* L *satisfies one of the following* (i)–(iv) *then* $L \sim L_{\omega\omega}$

(i) $LS(\omega)$ *and* (ω, ω)*-compact*;

(ii) $LS(\omega)$ *and the Hanf number of* L *is* ω;

(iii) $LS(\omega)$ *and* L *is axiomatizable*;

(iv) L *is* (κ, ω)*-compact and satisfies* $T(\kappa)$ *and* $LST(<\kappa)$ (*for some* $\kappa > \omega$).

Theorem 1.2 (Barwise [4]).

(i) *If* L *satisfies* $LS(\omega)$ *then* L *has the Karp property*.

(ii) *If* L *satisfies Craig's Theorem and has the Karp property then* L *satisfies* $LS(\omega)$.

(iii) *Let* $\kappa = \beth_\kappa$ *or* $\kappa = \omega$. *If* L *has the Karp property and the well-ordering number of* L *is* $\leqslant \kappa$ *then* $L \leqslant L_{\kappa\omega}$.

Since $\kappa = \omega$ is allowed in (iii) parts (i), (ii) of 1.1 follow easily. There are also theorems characterizing certain logics as the minimal logic with respect to certain properties. Examples of this sort will be discussed in Section 4.

2. The Δ-closure

Consider the following interpolation and definability properties of a logic L, for a given similarity type τ. $\mathfrak{A}$ varies over structures of type τ. $\bar{K}$ is the complement of K.

$(\mathrm{I})_{\mathrm{L},\tau}$: Whenever K_1, K_2 are disjoint $\mathrm{PC}_{\mathrm{L}}(\tau)$ classes there is some $K_3 \in \mathrm{EC}_{\mathrm{L}}(\tau)$ such that $K_1 \subseteq K_3$, $K_2 \subseteq \bar{K}_3$.

$(\Delta)_{\mathrm{L},\tau}$: If $K, \bar{K} \in \mathrm{PC}_{\mathrm{L}}(\tau)$ then $K \in \mathrm{EC}_{\mathrm{L}}(\tau)$.

$(\mathrm{B})_{\mathrm{L},\tau}$: If $K \in \mathrm{EC}_{\mathrm{L}}(\tau')$, where τ' is obtained from τ by adding (an index for) one n-ary relations, and $\forall \mathfrak{A}\, \exists^{\leq 1} R(\langle \mathfrak{A}, R\rangle \in K($ then $\{\langle \mathfrak{A}, a_1, \ldots, a_n\rangle \mid \exists R(\langle \mathfrak{A}, R\rangle \in K$ and $\langle a_1, \ldots, a_n\rangle \in R\} \in \mathrm{EC}_{\mathrm{L}}$.

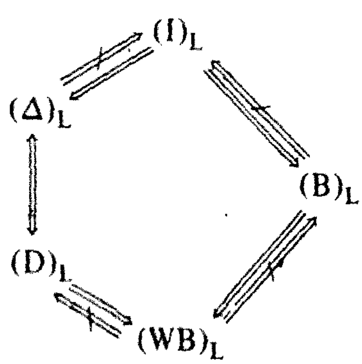
$(\mathrm{WB})_{\mathrm{L},\tau}$: As $(B)_{\mathrm{L},\upsilon}$ with "$\forall \mathfrak{A}\, \exists^{\leq 1} R$" replaced by "$\forall \mathfrak{A}\, \exists!\, R$".

$(\mathrm{D})_{\mathrm{L},\tau}$: Same as $(\mathrm{WB})_{\mathrm{L},\tau}$ except that only $K \in \mathrm{PC}_{\mathrm{L}}(\tau)$ is assumed (rather than $K \in \mathrm{EC}_{\mathrm{L}}(\tau)$).

When $(\mathrm{I})_{\mathrm{L},\tau}$ holds for all types τ we write $(\mathrm{I})_{\mathrm{L}}$ and say that L has the *interpolation property* (or the *Craig property*). Define $(\Delta)_{\mathrm{L}}$, $(\mathrm{B})_{\mathrm{L}}$, $(\mathrm{WB})_{\mathrm{L}}$, $(\mathrm{D})_{\mathrm{L}}$ similarly. $(\Delta)_{\mathrm{L}}$ is called the *Δ-interpolation* (sometimes *Souslin–Kleene interpolation*) *property*, $(\mathrm{B})_{\mathrm{L}}$ - the *Beth property*, $(\mathrm{WB})_{\mathrm{L}}$ - the *weak Beth property*. $(\mathrm{D})_{\mathrm{L}}$ is equivalent to $(\Delta)_{\mathrm{L}}$ (by Feferman [10]). When $(\Delta)_{\mathrm{L}}$ holds we sometimes say that L is *Δ-closed*.

It is easy to see that $(\mathrm{I})_{\mathrm{L}}$ holds iff $(\mathrm{I})_{\mathrm{L},\tau}$ holds for all semi-simple types τ, and similarly for the other properties. [It is, apparently, not enough that $(\mathrm{I})_{\mathrm{L},\tau}$ holds for all single-sorted τ.]

Theorem 2.1.



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For proofs of the implications see Feferman [10] and Jensen [26]; for the counterexamples see Proposition 2.21 ($\Delta \nRightarrow \mathrm{I}$), Corollary 2.23 ($\mathrm{WB} \nRightarrow \Delta$) and Makowsky–Shelah [40].

For $\mathrm{L} = \mathrm{L}_{\omega\omega}$, $(\mathrm{I})_{\mathrm{L}}$ and $(\mathrm{B})_{\mathrm{L}}$ are the well-known theorems of Craig and Beth respectively.

Examples. (1) If L is L_A, A a countable admissible set, then $(\mathrm{I})_{\mathrm{L}}$ holds and hence also $(\Delta)_{\mathrm{L}}$ and $(\mathrm{B})_{\mathrm{L}}$ and $(\mathrm{WB})_{\mathrm{L}}$ (Barwise [2]).

(2) If L is L_A and A is a union of countable admissible sets, then $(\mathrm{I})_{\mathrm{L}}$ holds. In particular, for $\mathrm{L}_{\omega_1\omega}$. In Section 4 a converse of this is proved (Theorem 4.15).

(3) If L is $\mathrm{L}_{\omega\omega}[\mathrm{Q}_1]$ $(\Delta)_{\mathrm{L}}$ does not hold (cf. Feferman [10], p. 216, footnote 3) nor does $(\mathrm{B})_{\mathrm{L}}$ (cf. Friedman [13] and Makowsky–Shelah [40]).

(4) If L is $\mathrm{L}_{\omega\omega}[\mathrm{Q}_1^{\mathrm{MM}(n)}]$ $(\Delta)_{\mathrm{L}}$ does not hold (Magidor and Makowsky), since the irrationals and ω_1-many copies of them are L-elementarily equivalent as dense orderings yet by Theorem 2.15 they can be distinguished by complementary PC_{L}-classes. Badger [1] showed that $(\mathrm{B})_{\mathrm{L}}$ does not hold either.

(5) $\mathrm{L}_{\omega_1\mathrm{G}}$ satisfies the following approximation theorem due to Harnik [21]. Let $\boldsymbol{R}, \boldsymbol{Q}$ be disjoint sequences of relation symbols, $\varphi(\boldsymbol{R}, \boldsymbol{Q})$ be a $\mathrm{L}_{\omega_1\mathrm{G}}$ sentence. Then there is a $\mathrm{L}_{\omega_1\mathrm{G}}$ sentence $\varphi^*(\boldsymbol{Q})$ such that for any sentence $\delta(\boldsymbol{Q})$ in $\mathrm{L}_{\omega_1\mathrm{G}}$ we have: (a) If $\delta(\mathrm{Q}) \Rightarrow \varphi(\boldsymbol{R}, \boldsymbol{Q})$ is valid so is $\delta(\boldsymbol{Q}) \Rightarrow \varphi^*(\boldsymbol{Q})$ and (b) if $\varphi(\boldsymbol{R}, \boldsymbol{Q}) \Rightarrow \delta(\mathrm{Q})$ is valid so is $\varphi^*(\boldsymbol{Q}) \Rightarrow \delta(\boldsymbol{Q})$. Barwise [3] showed that $(\mathrm{I})_{\mathrm{L}_{\omega_1\mathrm{G}}}$ does not hold, and J. Burgess showed that even Δ-interpolation fails. In fact he proved a more general result about absolute logics as defined in [3]. An absolute logic is a logic L (in the sense of [4], say) such that the relations

$$\{(\varphi, \tau) \mid \varphi \text{ is a sentence of } \mathrm{L}(\tau)\},$$

$$\{\mathfrak{U}, \varphi, \tau) \mid \mathfrak{U} \text{ is a structure of type } \tau, \varphi \text{ a sentence of } \mathrm{L}(\tau) \text{ and } \mathfrak{U} \models_{\mathrm{L}} \varphi\},$$

are respectively Σ_1, and Δ_1-definable over the universe. In the next proof the first relation is also assumed to be Δ_1; we could even assume it to be Δ_0 with no loss of generality.

Theorem 2.2 (Burgess): *Let* L *be an absolute logic and assume that the class of well-founded binary relations is* PC_{L}. *Then* L *is not* Δ*-closed.*

Proof. Let K be the following class of structures of type $\langle 2\rangle$. $K = \{\mathcal{A} = \langle A, E\rangle \mid$ for some sentence φ of $\mathrm{L}(\langle 2\rangle)$, $\mathcal{A}$ is isomorphic to $\langle \mathrm{TC}\{\varphi\}, \epsilon\rangle$ and $\mathcal{A} \models_{\mathrm{L}} \neg\varphi\}$. Using Σ_1 formulas of the language of set theory which express: "x is (is not) a sentence of $\mathrm{L}\langle 2\rangle$", "$x$ is a sentence of $\mathrm{L}\langle 2\rangle$ and is true (false) in the structure $\langle y, \epsilon\rangle$", it is easy to see that both K and $\bar{K}$ are PC_{L}. (One uses an extra sort of elements and an extra predicate ϵ to embed all objects involved in a well founded model of ZFC^-.) However, K is not EC_{L}, for suppose $K = \mathrm{Mod}(\chi)$ and let $\mathcal{A} = \langle \mathrm{TC}\{\chi\}, \epsilon\rangle$. Then $\mathcal{A} \models \chi \Leftrightarrow \mathcal{A} \in K \Leftrightarrow \mathcal{A} \models \neg\chi$, contradiction. □

Remark. If one defines PC_{L} allowing only extra predicates, not extra sorts of elements, the definition of K must be modified to ensure all structures in K are infinite. In this case the theorem still goes through if we add the hypothesis that L contains the quantifier "there exist infinitely many".

Corollary 2.3. $\mathrm{L}_{\omega_1 G}$ *is not Δ-closed.*

Proof. $\mathrm{L}_{\omega_1 G}$ is absolute (cf. [3]) and can express well foundedness. □

Lindstrom showed the following general result. (cf. [34])

Theorem 2.4. *If* $\mathrm{L} = \mathrm{L}_{\omega\omega}[\mathrm{Q}^1, \ldots, \mathrm{Q}^n]$, $\mathrm{L} > \mathrm{L}_{\omega\omega}$ *and* L *satisfies* $\mathrm{LS}(\aleph_0)$ *then* L *does not satisfy* $(\mathrm{WB})_{\mathrm{L}}$.

The proof uses the fact that $(\mathrm{WB})_{\mathrm{L}}$ and $\mathrm{LS}(\aleph_0)$ (together with Gödel's incompleteness theorem) are sufficient to show the existence of non-standard models of arithmetic.

(6) $\mathrm{L}_{\omega\omega}[\mathrm{Q}_0]$ does not have the weak Beth property.

(7) Malitz had shown that $\mathrm{L}_{\omega_1\omega_1}$ does not have the weak Beth property (cf. Makowsky–Shelah [40]; or analyze the proof of [42, Theorem 4.2]).

(8) Second-order logic: $(\Delta)_{\mathrm{L}}$ fails, as is not hard to see, but $(\mathrm{I})_{\mathrm{L}}$ is true for the single-sorted part of second-order logic.

(9) $\mathrm{L}_{\infty\omega}$ is not Δ-closed (cf. the proof of Proposition 2.19) nor does it have the Beth property (cf. Gregory [20] and Makowsky–Shelah [40]). Shelah [cf. 40] showed that even weak Beth fails for $\mathrm{L}_{\infty\omega}$. Malitz [42] furthermore showed that $\mathrm{L}_{\kappa\omega}$ has interpolants in $\mathrm{L}_{\mu\kappa}$ with $\mu = (2^{\kappa})^{\kappa}$ and κ regular. For a semantical proof of this see Green [19].

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Feferman [10, p. 211] has characterized logics satisfying $(\Delta)_L$ in terms of truth maximality and truth adequacy. The definitions are quite complicated, so we omit them.

If now a logic does not satisfy one of these properties $(I)_L$, $(\Delta)_L$, $(D)_L$, $(B)_L$ or $(WB)_L$ one might ask if any extension L′ of L does. All the properties but $(I)_L$ speak of a uniquely defined class which must be in EC_L. If it is not, one might add it to L using additional quantifiers. In the following, we investigate this possibility for $(\Delta)_L$, $(WB)_L$ and $(B)_L$, the latter two only in outline.

Definition. Let L be a logic and let $\{K_\alpha\}_{\alpha\in A}$ be a list of all classes K such that $K, \bar{K} \in EC_L^\tau$ for some semi-simple type τ. Now put $\Delta(L) = L_{\omega\omega}[Q^\alpha]_{\alpha\in A}$ where the generalized quantifier Q^α has K^α as its defining class.

Δ is in fact a closure operator on logics. To prove this, and other nice properties of Δ, we need a crucial lemma.

Lemma 2.5. *Let* L *be a logic and* $L' = L[Q^\alpha]_{\alpha\in A}$ *where each generalized quantifier* Q^α *corresponds to a* PC_L *class* K_α *whose complemnet is* PC_L *too. Then* $L' \leqslant_{PC} L$.

Remark. The proof will implicitly give an effective way of associating with each type τ and sentence φ of L′ a sentence $\tilde{\varphi}$ of L such that $\{\mathfrak{A}\restriction\tau \mid \mathfrak{A}\models\varphi\} = \{\mathfrak{B}\restriction\tau\mid\mathfrak{B}\models\tilde{\varphi}\}$, assuming that the syntax of L, L′ and the type τ are recursively presented and we can find, as a recursive function of $\alpha\in A$, sentences of L defining K_α and $\bar{K}_\alpha$ as projective classes.

Proof. Let K be a class in $PC_{L'}^\tau$: $K = \{\mathfrak{A}\restriction\tau\mid\mathfrak{A}\models\varphi\}$ where φ is a sentence of $L'(\tau')$ $(\tau'\supseteq\tau)$. For each subformula ψ of φ introduce a new predicate P_ψ whose arity corresponds to the number and sorts of the free variables of ψ (in particular P_φ is a propositional constant). Define the sentence σ_ψ ($\psi = \psi(\bar{x})$ a subformula of φ) as follows:

If ψ is atomic, σ_ψ is $\forall\bar{x}(P_\psi(\bar{x}) \leftrightarrow \psi(\bar{x}))$;

if ψ is $\neg\psi_1$, σ_ψ is $\forall\bar{x}(P_\psi(\bar{x}) \leftrightarrow \neg P_{\psi_1}(\bar{x}))$

and similarly for other connectives;

If ψ is $\forall y\,\psi_1$, σ_ψ is $\forall\bar{x}(P_\psi(\bar{x}) \leftrightarrow \forall y P_{\psi_1}(\bar{x}, y))$

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and similarly for $\exists$;

$$\text{if } \psi(\bar{x}) \text{ is (say) } \mathrm{Q}uvw[\psi_0(u,\bar{x}), \psi_1(v,w,\bar{x})] \text{ then } \sigma_\psi \text{ is}$$

$$\forall \bar{x}(P_\psi(\bar{x}) \leftrightarrow \mathrm{Q}uvw[P_{\psi_0}(u,\bar{x}), P_{\psi_1}(v,w,\bar{x})]);$$

similarly for generalized quantifier of L′ of any type.

Let $\psi_1, \ldots, \psi_k$ be a list of all subformulas of φ, and let $\varphi_1 = P_\varphi \wedge \sigma_{\psi_1} \wedge \ldots \wedge \sigma_{\psi_k}$. It is clear that $K = \{\mathfrak{B} \restriction \tau \mid \mathfrak{B} \models \varphi_1\}$ (in fact $\varphi \equiv \exists P_{\psi_1} \ldots P_{\psi_k} \varphi_1$).

Thus it will suffice to prove that the class of models of σ_ψ is in PC_L for each subformula ψ of φ (since PC_L is closed under finite intersections and projections). The only case in which σ_ψ is not a sentence of L is when ψ begins with Q^α for some $\alpha \in A$. For definiteness say that Q^α is of type $\langle 2 \rangle$ so that σ_ψ is of the form:

$$\forall x(P(\bar{x}) \leftrightarrow \mathrm{Q}^\alpha uvw[P_0(u,\bar{x}), P_1(v,w,\bar{x})]),$$

which is equivalent to the conjunction of

$$\forall x(P(\bar{x}) \rightarrow \mathrm{Q}^\alpha uvw[P_0(u,\bar{x}), P_1(v,w,\bar{x})]) \tag{1}$$

and

$$\forall x(\neg P(\bar{x}) \rightarrow \neg \mathrm{Q}^\alpha uvw[P_0(u,\bar{x}), P_1(v,w,\bar{x})]). \tag{2}$$

Given that $K_\alpha \in \mathrm{PC}_\mathrm{L}$ it is easy to see that the class of models of (1) is in PC_L. Similarly the fact that $\bar{K}_\alpha \in \mathrm{PC}_\mathrm{L}$ implies the same for (2). We leave the details to the reader. □

Corollary 2.6. *$K \in \mathrm{PC}_{\Delta(\mathrm{L})}$ iff $K \in \mathrm{PC}_\mathrm{L}$ (i.e. $\mathrm{L} \sim_{\mathrm{PC}} \Delta(\mathrm{L})$).*

This is immediate from 2.5.

Lemma 2.7. (i) *$\Delta(\mathrm{L})$ satisfies Δ-interpolation.*

(ii) *L satisfies Δ-interpolation iff $\mathrm{L} \sim \Delta(\mathrm{L})$.*

(iii) *$\Delta(\mathrm{L})$ is the largest extension L′ of L such that $\mathrm{L}' \sim_{\mathrm{PC}} \mathrm{L}$.*

Proof. (i) First suppose $K, \bar{K} \in \mathrm{PC}^\tau_{\Delta(\mathrm{L})}$ where τ is a semi-simple. By 2.6 $K, \bar{K} \in \mathrm{PC}^\tau_\mathrm{L}$ hence, by definition of $\Delta(\mathrm{L})$, $K \in \mathrm{EC}^\tau_{\Delta(\mathrm{L})}$. Next note that if a logic satisfies Δ-interpolation for semi-simple types then it satisfies Δ-interpolation for all types (proof easy).

(ii) By 2.6 and (i).

(iii) By 2.6 $\Delta(\mathrm{L}) \sim_{\mathrm{PC}} \mathrm{L}$. If $\mathrm{L}' \sim_{\mathrm{PC}} \mathrm{L}$ then $K \in \mathrm{EC}_{\mathrm{L}'} \Rightarrow K, \bar{K} \in \mathrm{PC}_{\mathrm{L}'} \Rightarrow K, \bar{K} \in \mathrm{PC}_{\mathrm{L}} \Rightarrow K \in \mathrm{EC}_{\Delta(\mathrm{L})}$.
Thus $\mathrm{L}' \leqslant \Delta(\mathrm{L})$. □

Theorem 2.8. Δ *is a closure operator, i.e.,* –
(i) $\mathrm{L} \leqslant \Delta(\mathrm{L})$;
(ii) $\Delta(\Delta(\mathrm{L})) \sim \Delta(\mathrm{L})$;
(iii) $\mathrm{L}_1 \leqslant \mathrm{L}_2 \Rightarrow \Delta(\mathrm{L}_1) \leqslant \Delta(\mathrm{L}_2)$.

Proof. (i) is obvious by the definition of Δ. (ii) follows from 2.7 (i), (ii). To prove (iii) note that $K \in \mathrm{EC}_{\Delta(\mathrm{L}_1)} \Rightarrow K, \bar{K} \in \mathrm{PC}_{\mathrm{L}_1} \Rightarrow K, \bar{K} \in \mathrm{PC}_{\mathrm{L}_2} \Rightarrow K \in \mathrm{EC}_{\Delta(\mathrm{L}_2}$

Remark. The proof of 2.7 (i) explains why in the definition of $\Delta(\mathrm{L})$ we consider classes K such that $K, \bar{K} \in \mathrm{PC}^{\tau}_{\mathrm{L}}$ for some semi-simple (not only simple) type τ.

Example. (10) $\mathrm{L}_{\omega\omega}[\mathrm{Q}_1] \leqslant \Delta(\mathrm{L}_{\omega_2\omega})$.

Proof. It is enough to show that $K_3 = \{\langle A\rangle \mid \bar{\bar{A}} \geqslant \aleph_1\}$ and $\bar{K}_3$ are $\mathrm{PC}_{\mathrm{L}_{\omega_2\omega}}$, which is left to the reader (or see proof of 2.19). □

For the sake of comparison we now define two other closure operations for logics, connected with $(\mathrm{B})_{\mathrm{L}}$ and $(\mathrm{WB})_{\mathrm{L}}$. The way $(\mathrm{B})_{\mathrm{L}}$ and $(\mathrm{WB})_{\mathrm{L}}$ are formulated, the class K which is supposed to be elementary in L is a class involving a simple type $\langle n_1, \ldots, n_k; l\rangle$, $l > 0$, i.e., involving distinguished elements.

Definitions. Let K_β, $\beta \in \mathrm{B}$ be a list of the counterexamples to $(\mathrm{WB})_{\mathrm{L}}((\mathrm{B})_{\mathrm{L}})$. Then $\mathrm{WB}^1(\mathrm{L})(\mathrm{B}^1(\mathrm{L}))$ is the logic $\mathrm{L}[\;^{K_\beta}]_{\beta\in\mathrm{B}}$. Now we proceed by induction:

$$\mathrm{WB}^{n+1}(\mathrm{L}) = \mathrm{WB}^1(\mathrm{WB}^n(\mathrm{L})),\ \mathrm{B}^{n+1}(\mathrm{L}) = \mathrm{B}^1(\mathrm{B}^n(\mathrm{L}))\,.$$

Finally let $\mathrm{WB}(\mathrm{L}) = \bigcup_{n<\omega} \mathrm{WB}^n(\mathrm{L})$, $\mathrm{B}(\mathrm{L}) = \bigcup_{n<\omega} \mathrm{B}^n(\mathrm{L})$.

Proposition 2.9. WB(L) (B(L)) *is the smallest extension of* L *having the weak Beth (resp. Beth) property.*

Proof. That WB(L) satisfies (WB) follows directly from the definition. Now assume L′ satisfies (WB) and $\mathrm{L} \leqslant \mathrm{L}'$. We proceed to show that

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$\mathrm{WB}(\mathrm{L}) \leqslant \mathrm{L}'$. $\mathrm{WB}^1(\mathrm{L}) \leqslant \mathrm{L}'$ since L' satisfies (WB). Now $\mathrm{WB}^{n+1}(\mathrm{L}) = \mathrm{WB}^1(\mathrm{WB}^n(\mathrm{L}))$ hence $\mathrm{WB}^n(\mathrm{L}) \leqslant \mathrm{L}'$ for all n. Similarly for B(L). □

Corollary 2.10. WB(L) *and* B(L) *are closure operations.* □

Remark. By Theorem 2.4 WB(L) is not finitely generated if L satisfies LS(ω) and extends properly $\mathrm{L}_{\omega\omega}$. The same is true for B(L) and Δ(L).

For WB(L) we also have an analogue of Lemma 2.5 since the condition of $(\mathrm{WB})_\mathrm{L}$ says that every structure has a required expansion.

Lemma 2.11. *For every sentence (formula)* φ *of* WB(L) *there is a sentence (formula)* $\tilde{\varphi}$ *of* L *having additional predicates such that for all structures* $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ *iff there is an expansion* $\mathfrak{A}^*$ *iff* $\mathfrak{A}$ *with* $\mathfrak{A}^* \models \tilde{\varphi}$. *In particular* $\mathrm{WB}(\mathrm{L}) \leqslant \Delta(\mathrm{L})$.

Proof. For every quantifier in φ which is not a quantifier of L we add a new predicate T which is interpreted by the implicitly definable relation, where it comes from. The details are analogous to the proof of Lemma 2.5. □

In the case of B(L) we run into troubles since not every structure need have an expansion of the required type.

Definition. Let $\mathfrak{A}$ be a structure. We define, analogously to Bell and Slomson [7, Ch. 10, §4], the L-full expansion of $\mathfrak{A}$ by adding for every formula $\varphi(\bar{x})$ a new predicate $R^\varphi(\bar{x})$ with the obvious interpretation. Let the resulting structure be denoted by $\mathfrak{A}^*$.

Lemma 2.12. *For each logic* L *and similarity type* τ, *denoting by* τ^* *the type of the* WB(L)*-full expansions of models of type* τ, *there is a set* Γ *of* $\mathrm{L}(\tau^*)$ *sentences such that the following holds:*

If $\mathfrak{U}$ *is any structure of type* τ *and* $\mathfrak{U}^*$ *is its* WB(L)*-full expansion, then* $\mathfrak{U}^* \models \Gamma$ *and* $\mathfrak{U}^*$ *is the unique expansion of* $\mathfrak{U}$ *to type* τ^* *which satisfies* Γ.

Proof. If φ is a formula of WB(L)(τ) and $\psi_0, \psi_1, \ldots$ are its immediate subformulas then Γ will contain an axiom describing how R^φ is related to $R^{\psi_0}, R^{\psi_1}, \ldots$. The only interesting case is when φ begins with a quantifier Q of WB(L) which is not in L. Then the corresponding axiom will essen-

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tially say that R^{φ} satisfies the implicit definition which gave rise to the quantifier Q in the structure defined by $R^{\psi_0}, R^{\psi_1}, \ldots$. Actually we are oversimplifying a bit but the details can be left to the reader, who should notice that the proof applies to WB(L) but not to B(L). □

The operators Δ and WB preserve some of the "nice" model-theoretic properties of a logic L, while to prove similar results for B seems more difficult.

Theorem 2.13. (i) *If* L *is* (κ, λ)*-compact so are* Δ(L) *and* WB(L).

(ii) *If* L *is bounded, so are* Δ(L) *and* WB(L). *In fact,* L, Δ(L) *and* WB(L) *have the same* wo*-number.*

(iii) L, Δ(L) *and* WB(L) *have the same Hanf- and Löwenheim-numbers.*

Proof. L, WB(L) and Δ(L) have the same projective classes. The properties of compactness, boundedness, well-ordering number, Lowenheim number and Hanf number can all be defined by reference to projective classes only, hence the theorem holds. □

Applications.

(1) Feferman's quantifier Q_α^F, (cf. ex. 11, Section 1). The logic $L_{\omega\omega}[Q_\alpha^F]$ is (ω, ω)-compact (for many α's) and satisfies $LS(\omega_\alpha)$ since it is included in $\Delta(L_{\omega\omega}[Q_\alpha])$.

(2) Lindstrom's Theorem gives new proofs for the fact that $L_{\omega\omega}$ satisfies $(WB)_L$ and $(\Delta)_L$: $L_{\omega\omega}$ is maximal with respect to compactness and $LS(\omega)$ or with respect to Hanf-number = Löwenheim-number = ω, all properties preserved under the Δ-operation. Thus $\Delta(L_{\omega\omega}) \sim L_{\omega\omega}$.

(3) Another application of Lindstrom's Theorem gives us information about $L_{\omega\omega}[Q_1]$ and its Δ-closure: (This statement for $L_{\omega\omega}[Q_{ccc}]$ only makes sense for unrelativized logics; here Q_{ccc} is Chang's quantifier §1, ex. 5.)

Proposition 2.14. $\Delta(L_{\omega\omega}[Q_\alpha]) \cap L_{\omega_1\omega} = L_{\omega\omega}$ *for* $\alpha = 1$, *or any* α *such that* $L_{\omega\omega}[Q_\alpha]$ *is countably compact. Also, if* $L_{\omega\omega}[Q_{ccc}]$ *is* (ω, ω)*-compact then* $\Delta(L_{\omega\omega}[Q_{ccc}]) \cap L_{\omega_1\omega} = L_{\omega\omega}$.

Note: The GCH implies (ω, ω)-compactness of $L_{\omega\omega}[Q_{ccc}]$ and of $L_{\omega\omega}[Q_\alpha]$ for most α. (See Bell and Slomson [7].)

Proof. $L_{\omega\omega}[Q_{ccc}]$ and $L_{\omega\omega}[Q_\alpha]$ ($\alpha = 1$, say), are both (ω, ω)-compact and we assume $L_{\omega_1\omega}$ satisfies $LS(\omega)$. Both these properties are preserved under the formation of sublogics. Hence in both cases the intersection satisfies $LS(\omega)$ and is (ω, ω)-compact. But by Lindstrom's Theorem (Theorem 0.1) this must be $L_{\omega\omega}$. Note that we are applying Lindstrom's theorem to a logic which does not allow relativization. □

(4) Consider the quantifier Q^D with $K^{Q^D} = \{\mathfrak{A} \mid \mathfrak{A}$ of type $\langle 2\rangle$ and is a dense linear ordering with a countable dense subset$\}$ $(= K^D)$.

Theorem 2.15. $L_{\omega\omega}[Q^D] \leqslant \Delta(L_{\omega\omega}[Q_1])$ *and hence* $L_{\omega\omega}[Q^D]$ *is* (ω, ω)-*compact and satisfies* $LS(\omega_1)$.

Proof. We show that K^D and $(K^D)^-$ are $PC_{L_{\omega\omega}[Q_1]}$. For K^D this is straightforward. For $(K^D)^-$ we observe that if a dense linear order $\langle A, <\rangle$ has no countable dense subset then there are at least ω_1 many disjoint rectangles (= cartesian products of intervals) in A^2. For assume there is a maximal countable set of disjoint rectangles avoiding the diagonal, then the projection of their endpoints into A is a dense sebset of A. With this observation we easily see that $(K^D)^-$ is $PC_{L_{\omega\omega}[Q_1]}$. This idea goes back to Kurepa [51].□

(5) Theorem 1.2 (iii) as proven in Barwise [4] gives slightly more:

Theorem 2.16. *Let* $\kappa = \beth_\kappa$ *or* $\kappa = \omega$ *and* L *be a logic with* wo-*number* κ. *If* K *is closed under partial isomorphisms and* $K \in EC_L$ *then* $K \in EC_{L_{\kappa\omega}}$.

Since Δ preserves wo-numbers we obtain the following interpolation theorem for $L_{\kappa\omega}$. Note that the wo-number of $L_{\kappa\omega}$ is κ if $\kappa = \omega$ or $\kappa = \beth_\kappa$ (this follows easily from [6]).

Theorem 2.17. *If* K *is closed under partial isomorphism and* K *is* $EC_{\Delta(L_{\kappa\omega})}$ ($\kappa = \beth_\kappa$ *or* $\kappa = \omega$ *then* K *is* $EC_{L_{\kappa\omega}}$.

Corollary 2.18. *If* K *is closed under partial isomorphism and* K *and* $\bar{K}$ *are* $PC_{L_{\infty\omega}}$ *then* K *is* $EC_{L_{\infty\omega}}$.

Not all model-theoretic properties are preserved by Δ.

Proposition 2.19. Δ *does not preserve the Karp-property.*

Proof. Let L be $L_{\omega_2\omega}$. L has the Karp property. There are sentences ϕ_0 and ϕ_1 in $L_{\omega_2\omega}$ characterizing the structures $\langle\omega,<\rangle$ and $\langle\omega_1,<\rangle$, respectively. Now put $K_1 = \{\mathfrak{A} \mid \mathfrak{A} = \langle A\rangle$ and A can be mapped 1 — 1 into $\omega\}$ and $K_2 = \{\mathfrak{A} \mid \mathfrak{A} = \langle A\rangle$ and $\mathfrak{A}$ can be expanded to $\mathfrak{A}' = \langle A_1, A_2, f, <\rangle$ where $\langle A_2,<\rangle \simeq \langle\omega_1,<\rangle$ and f is an injection of A_2 into $A_1\}$. Obviously, K_1, K_2 are PC_L and K_2 is the complement of K_1. Hence K_1 and K_2 are $EC_{\Delta(L)}$. But K_1 contains only countable structures, K_2 only uncountable structures. In view of the fact that all infinite sets are partially isomorphic, it follows that $\Delta(L)$ does not have the Karp property. □

Remarks. (i) A similar argument shows that in $\Delta(L_{\kappa\omega})$ all the quantifiers Q_α for $\aleph_\alpha < \kappa$ are definable. In particular, for $\kappa > \omega_1$ $L_{\kappa\omega}$ is not Δ-closed.

(ii) If L has LS(ω) then it has, by a result of Barwise [4], the Karp property and so has Δ(L).

(iii) Kueker [50] studies logics with LS(ω) in a general context and found that the logic generated by his closed and co-closed classes is Δ-closed.

Proposition 2.20. *Δ does not preserve the Tarski property.*

Proof. Again let $L = L_{\omega_2\omega}$. L has the Tarski property (in fact, all $L_{\kappa\omega}$ have it). Now let $\{\mathfrak{A}_i\}_{i<\omega_1}$ be countable structures with equaltiy only, where every $\mathfrak{A}_{i+1}$ is a proper extension of $\mathfrak{A}_i$. $\{\mathfrak{A}_i\}_{i<\omega_1}$ is an elementary chain of $\Delta(L_{\omega_2\omega})$ since for every finite subset A of A_i, $\langle A_i, a \in A\rangle \simeq \langle A_{i+1}, a \in A\rangle$. But $\bigcup_{i<\omega_1} \mathfrak{A}_i$ cannot be an Δ(L)-elementary extension of any of the $\mathfrak{A}_i$'s since $\bigcup_{i<\omega_1} \mathfrak{A}_i$ is uncountable and $L_{\omega\omega}[Q_1] \leqslant \Delta(L_{\omega_2\omega})$ by ex. 10. □

$\Delta(L_{\omega_2\omega})$ also gives an example of a Δ-closed logic which does not satisfy $(I)_L$. In fact, Friedman (unpublished) proved the following theorem. The proof below is essentially due to Hutchinson (cf. [24]):

Proposition 2.21. *Let* L *be such that* $L_{\omega\omega}[Q_1] \leqslant L \leqslant \Delta(L_{\omega_2\omega})$. *Then there are disjoint* PC_L*-classes* K_1, K_2 *which cannot be separated in* $\Delta(L_{\omega_2\omega})$.

Proof. Let $K_1 = \{\mathfrak{A} \mid \mathfrak{A} = \langle A,<\rangle$ where $<$ is an ordering of A of cofinality $\omega\}$ and $K_2 = \{\mathfrak{A} \mid \mathfrak{A} = \langle A,<\rangle$ where $<$ is an ordering of cofinality $\omega_1\}$. Clearly, K_1, K_2 are disjoint PC-classes in $L_{\omega\omega}[Q_1]$ (using $\aleph_1$-like orderings).

Now assume, for contradiction, that ϕ is a formula of $\Delta(L_{\omega_2\omega})$ which separates K_1 from K_2. Using Lemma 2.5 ϕ is equivalent, using additional sorts and predicates, to some formula $\tilde{\phi}$ of $L_{\omega_2\omega}$. Assume further that $\langle\omega_2,<\rangle = \mathfrak{B}$ and $\mathfrak{B} \models \phi$ (if not take $\neg\phi$). Expand $\mathfrak{B}$ to a model $\tilde{\mathfrak{B}}$ of $\tilde{\phi}$. Using the Löwenheim–Skolem Theorem for $L_{\omega_2\omega}$ we can find a $\mathfrak{B}_0 \subset \mathfrak{B}$ such that $\mathfrak{B}_0 \models \tilde{\phi}$ and $|(\mathfrak{B})^{=}| \leqslant \omega_1$. Let $b \in |\mathfrak{B}| - |\mathfrak{B}_0|$ and for all $b_0 \in |\mathfrak{B}_0|$, $\mathfrak{B} \models b > b_0$. Let $\mathfrak{B}_1$ be such that $\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \mathfrak{B}$, $\mathfrak{B}_1 \models \phi$, $b \in |\mathfrak{B}_1|$ and that $|(\mathfrak{B}_1)^{=}| \leqslant \omega_1$ (such a $\mathfrak{B}_1$ exists using the Löwenheim–Skolem Theorem once more). Now iterating this process ω-many times gives us a $\mathfrak{B}_\omega$ such that $\langle|\mathfrak{B}_\omega|,<\rangle = \mathfrak{B}_\omega$ has cofinality ω, iterating ω_1-many times gives us $\mathfrak{B}_{\omega_1}$ of cofinality ω_1, which contradicts our assumptions (as $\mathfrak{B}_\omega \models \tilde{\varphi}$, $\mathfrak{B}_{\omega_1} \models \tilde{\varphi}$ hence $\mathfrak{B}_\omega \models \varphi$, $\mathfrak{B}_{\omega_1} \models \varphi$, yet φ separates K_1 and K_2). □

In contrast to this we have:

Theorem 2.22. *If* L *has the Tarski property, so has* (WB(L).

Proof. Let $\mathfrak{A}_\alpha$, $\alpha < \delta$ be a WB(L)-elementary chain of structures of type τ and $\mathfrak{A} = \bigcup_{\alpha<\delta}\mathfrak{A}_\alpha$. Consider the WB(L)-full expansions $\mathfrak{A}^*_\alpha$ of each $\mathfrak{A}_\alpha$ and $\mathfrak{A}^*$ of $\mathfrak{A}$ (let τ^* be their similarity type). For $\alpha < \beta$, $\mathfrak{A}_\alpha <_{\mathrm{WB(L)}} \mathfrak{A}_\beta$ hence $\mathfrak{A}^*_\alpha <_{\mathrm{L}} \mathfrak{A}^*_\beta$ (in fact $\mathfrak{A}^*_\alpha <_{\mathrm{WB(L)}} \mathfrak{A}^*_\beta$ since every WB(L)(τ^*) formula can be "translated" to a WB(L)(τ) formula). Let $\mathfrak{A}' = \bigcup_{\alpha<\delta}\mathfrak{A}^*_\alpha$. Since L has the Tarski property $\mathfrak{A}^*_\alpha <_{\mathrm{L}} \mathfrak{A}'$ for each α. Let Γ be the set of sentences in Lemma 2.12. Then $\mathfrak{A}^*_0 \models \Gamma$ hence $\mathfrak{A}' \models \Gamma$, and since $\mathfrak{A}'$ is an expansion of $\mathfrak{A}$ it follows from Lemma 2.12 that $\mathfrak{A}' = \mathfrak{A}^*$. Thus $\mathfrak{A}^*_\alpha <_{\mathrm{L}} \mathfrak{A}^*$ and so $\mathfrak{A}_\alpha <_{\mathrm{WB(L)}} \mathfrak{A}$ for each α. (We are using the trivial fact that if L$'$ is any logic and $\mathfrak{A}^*$, $\mathfrak{B}^*$ are the L$'$-full expansions of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{A}^* \subseteq \mathfrak{B}^*$ then $\mathfrak{A}' <_{\mathrm{L}'} \mathfrak{B}'$.) □

Corollary 2.23. $(\mathrm{WB})_{\mathrm{L}} \not\Rightarrow (\Delta)_{\mathrm{L}}$.

Proof. $\Delta(L_{\omega_2\omega})$ does not have the Tarski property (by 2.20) but $\mathrm{WB}(L_{\omega_2\omega})$ does, by 2.22. □

Example. (11) WB does not preserve the Karp property. To see this we use Shelah's theorem (cf. ex. 9 above) that $\mathrm{WB}(L_{\infty\omega}) > L_{\infty\omega}$. By Theorem 2.13 (ii) $\mathrm{WB}(L_{\infty\omega})$ is a bounded logic. By [4, Corollary 3.3] every bounded logic having the Karp property is $\leqslant L_{\infty\omega}$. Hence $\mathrm{WB}(L_{\infty\omega})$ does not have the Karp property.

If L is axiomatizable, one would like to know if $\Delta(\mathrm{L})$ also is axiomarizable. The construction of Δ does not indicate any solution to this problem, but the following is true:

Theorem 2.24. *Let* L *be an axiomatizable logic. Let* $\mathrm{L}' = \mathrm{L}[\mathrm{Q}^1, ..., \mathrm{Q}^n]$ *be such that the defining classes* $K_1, ..., K_n$ *of* $\mathrm{Q}^1, ..., \mathrm{Q}^n$, *respectively, are in* $\mathrm{EC}_{\Delta(\mathrm{L})}$. *Then* L′ *is axiomatizable.*

Proof. We have to show that the set of valid sentences V' of L′ is recursively enumerable. By our assumption, the set of valid sentences V of L is r.e. Now let ϕ be a sentence of L′, hence of $\Delta(\mathrm{L})$. By the remark following Lemma 2.5 there is an effective translation of ϕ into a sentence $\tilde{\phi}$ of L. The effectivity of the translation is guaranteed by restriction to finitely many quantifiers in L′. Now $\varphi \in V'$ iff $\neg\varphi$ has no model iff $(\neg\varphi)^\sim$ has no model iff $\neg((\neg\varphi)^\sim) \in V$, hence the result. □

Corollary 2.25. *The same is true if we replace* $\mathrm{EC}_{\Delta(\mathrm{L})}$ *by* $\mathrm{EC}_{\mathrm{WB}(\mathrm{L})}$.

Applications.

(1) $\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathrm{E}}_\alpha]$ is axiomatizable for $\alpha = 1$ (and many $\alpha \geqslant 1$). This solves a problem posed by Feferman [10]. Stavi had given an explicit axiomatization (by schemes) for $\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathrm{E}}]$.

(2) $\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathrm{D}}]$ is axiomatizable. Both these results follow from the fact that $\mathrm{L}_{\omega\omega}[\mathrm{Q}_\alpha]$ for $\alpha = 1$ and many other α is axiomatizable.

(3) If $\mathrm{L}_{\omega\omega}[\mathrm{Q}^1, ..., \mathrm{Q}^n]$ satisfies LS(ω) and properly extends L then $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^1, ..., \mathrm{Q}^n])$ and $\mathrm{WB}(\mathrm{L}_{\omega\omega}[\mathrm{Q}^1, \mathrm{Q}^n])$, cannot be obtained by adding only finitely many new generalized quantifiers, by Theorem 2.4.

(4) Theorem 2.24 gives a new proof for $(\Delta)_{\mathrm{L}}$ in $\mathrm{L}_{\omega\omega}$, since $\mathrm{L}_{\omega\omega}$ is maximal with respect to axiomatizability and LS(ω).

Problems.

We concentrated mainly on the Δ-closure and the WB-closure, both being very smooth operations on logics.

Problem 2.1. Does the B-closure have similar features?

Problem 2.2. Is there a way of defining reasonable an I-closure?

Trivially, one could add all PC_{L}-classes to a logic L extending it in ω-many steps to L′ such that $\mathrm{EC}_{\mathrm{L}'} = \mathrm{PC}_{\mathrm{L}'}$ but this seems to strong. For $\mathrm{L}_{\omega\omega}$

for example, this construction gives all the classes of structures definable in set theory, i.e., goes beyond second-order logic.

We were operating on logics defining intersections and closures. Unions can be defined similarly.

Problem 2.3. Investigate the model-theoretic properties of logics and their behaviour under these constructions.

3. Cofinally invariant classes of structures and $\Delta(L_{\omega\omega}[Q_1]$

Malitz and Magidor in [36] asked whether one can characterize $L_{\omega\omega}[Q_1]$ in model-theoretic terms such as axiomatizability, compactness or Löwenheim- and Hanf-numbers. With the exception of axiomatizability the preservation Theorem 2.13 tells us that this is not the case. The modified question of course would be to characterize $\Delta(L_{\omega\omega}[Q_1])$. This chapter is a result of attempts to do so, but does not give a solution to this problem.

Now Theorem 2.24 gives us even axiomatizable extensions of $L_{\omega\omega}[Q_1]$. In this chapter we shall construct more logics which are (ω, ω)-compact, satisfy $LS(\omega_1)$ or even $LST(\omega_1)$ and are axiomatizable. Our starting point is the following observation:

Theorem 3.1. *Let* L *be a logic, K a class of structures closed under isomorphism such that for some κ, K and its complement are both "PC_L on structures of cardinality $\leqslant \kappa$" (i.e. $K \cap S_\kappa = K_1 \cap S_\kappa$, $\bar{K} \cap S_\kappa = K_2 \cap S_\kappa$ for some $K_1, K_2 \in PC_L$). Let $L' = L[Q^K]$ (or, more generally, let L' be obtained from* L *by adding finitely many quantifiers of this kind).*

(1) *If L' satisfies the Löwenheim–Skolem theorem for κ for single sentences ($LS(\kappa)$) and* L *is axiomatizable then L' is axiomatizable.*

(2) *If L' satisfies the Löwenheim–Skolem theorem for κ for sets of sentences of cardinality $\leqslant \kappa$ (and, in particular, if L' satisfies $LST(\kappa)$) and* L *is (λ, μ)-compact, $\mu \leqslant \lambda \leqslant \kappa$, then L' is (λ, μ)-compact.*

[This generalizes results of §2 on preservation of axiomatizability and compactness by the Δ-operation.]

Proof. (1) As in the proof of Lemma 2.5 we can effectively associate with each sentence φ of L' a sentence $\tilde{\varphi}$ of L such that a structure $\mathfrak{A} \in S_\kappa$ is a model of φ iff $\mathfrak{A}$ has an expansion to a model of $\tilde{\varphi}$. Since L' (hence L) satisfies $LS(\kappa)$ it is clear that φ has a model iff $\tilde{\varphi}$ has a model. Thus the set of valid sentences of L′ is effectively reducible to the set of valid sentences of L.

(2) Similarly, with each set Φ of sentences of L′ we can associate a set $\tilde{\Phi}$ of sentences of L such that for $\mathfrak{A} \in S_\kappa : \mathfrak{A} \models \Phi$ iff $\mathfrak{A}$ can be expanded to a model of $\tilde{\Phi}$. $\tilde{\Phi}$ stands in a 1–1 correspondence with Φ and the statement of (λ, μ) compactness applied to $\Phi(\bar{\bar{\Phi}} \leqslant \lambda)$ easily reduces to the statement applied to $\tilde{\Phi}$, bearing in mind that $\lambda \leqslant \kappa$ so LS is applicable to Φ and $\tilde{\Phi}$. □

Examples. (1) The quantifier Q^C defined by $K^C = \{\mathfrak{A} \mid \mathfrak{A} = \langle A, < \rangle$ where $<$ is a linear ordering of cofinality $\omega\}$. Shelah [47,53, §4]) proved that $L_{\omega\omega}[Q^C]$ is (κ, ω)-compact for every κ. It will follow from the results in this chapter that $L_{\omega\omega}[Q^C]$ satisfies LST(ω_1). Taking L in Theorem 3.1 to be $L_{\omega\omega}[Q_1]$ we see that $L_{\omega\omega}[Q^C]$ is axiomatizable (to verify the hypothesis with $\kappa = \aleph_1$ one uses $\aleph_1$-like orderings) and (ω, ω)-compact.

The main difficulty in the applications of Theorem 3.1 is to verify that L[Q] has the same Löwenheim-number as L.

The following work developed of course from the special to the general: Example (3) was suggested by S. Feferman [10], ex. (4) was studied previously by Shelah [47] who also first defined ex. (5) and ex. (6).

The cofinally invariant (c.i.) quantifiers we are going to study in this chapter arose on the way of trying to find a broad class of applications for Theorem 3.1 using, as in ex. (1), the nice properties of $L_{\omega\omega}[Q_1]$ as a point of departure.

Let us point out, though, that recently J. Hutchinson [23] [24] has found another approach to construct extensions of $L_{\omega\omega}[Q_1]$ using, following an idea due to H. Friedman, nonstandard models of set theory.

Let τ be a similarity type. The syntax of the (monadic) second order language $L^{(2)}_{\omega\omega}(\tau)$ can be described as follows: we add a new sort of variables $X, Y, \ldots$ (called set variables) and a new predicate symbol $\in$ for each sort of τ, and then build formulas of $L_{\omega\omega}(\tau^{(2)})$ in the usual way, where $\tau^{(2)}$ is the type obtained from τ by the above additions (if τ is single sorted, $\tau^{(2)}$ is two sorted). For each structure $\mathfrak{A}$ of type τ let $\tilde{\mathfrak{A}}$ be the structure of type $\tau^{(2)}$ obtained by letting the set variables range over *countable* sets of elements (of the corresponding sort) and $\in$ denote membership. Thus if $\mathfrak{A} = \langle A, \ldots \rangle$ is single sorted, $\tilde{\mathfrak{A}} = \langle \mathfrak{A}, P_{<\aleph_1}(A), \in \rangle$ where $P_{<\aleph_1}(A)$ is the set of countable subsets of $\mathfrak{A}$. If φ is a sentence of $L^{(2)}_{\omega\omega}(\tau) (= L_{\omega\omega}(\tau^{(2)}))$ we let $\mathfrak{A} \models^{(2)} \varphi$ mean $\tilde{\mathfrak{A}} \models \varphi$. Thus $\models^{(2)}$ is the ordinary satisfaction relation of "weak" second order logic.

For any logic L and type τ let $L^{(2)}(\tau)$ be the set of formulas of $L(\tau^{(2)})$ in which the quantifiers of L (other than $\forall$, $\exists$) bind only "individual variables" not "set variables". Naturally we define: $\mathfrak{A} \models_{L^{(2)}} \varphi$ iff $\bar{\mathfrak{A}} \models_L \varphi$. (Incidentally, $L^{(2)}$ or even $L^{(2)}_{\omega\omega}$ is an abstract logic in the sense of [3], [4], say, but not a logic in the technical sense of this paper.)

Returning now to $L_{\omega\omega}(\tau^{(2)})$, suppose τ is single sorted. We shall be interested in structure $\mathfrak{A}_S = \langle \mathfrak{A}, S, \epsilon \rangle$ of type $\tau^{(2)}$ ($\mathfrak{A}$ of type τ), in which S is a subset of $P_{<\aleph_1}(A)$ which is cofinal in the partial ordering $\subsetneq$ of $P_{<\aleph_1}(A)$. Thus the set variables, rather than ranging over all countable sets (as in $\bar{\mathfrak{A}}$) range only over a cofinal collection S of such sets. Such structures $\mathfrak{A}_S$ will be called *cofinal structures* over $\mathfrak{A}$ while $\bar{\mathfrak{A}}$ itself is the *full structure* over $\mathfrak{A}$.

A formula $\varphi(x, y, \ldots; X, Y, \ldots)$ of $L_{\omega\omega}(\tau^{(2)})$ will be called *cofinally invariant* (*c.i.*) if for every $\mathfrak{A}$ of type τ and cofinal structure $\mathfrak{A}_S$ over $\mathfrak{A}$ and elements $a, b, \ldots \in |\mathfrak{A}|$, $s, t, \ldots \in S$ we have

$$\mathfrak{A}_S \models \varphi[a, b, \ldots; s, t, \ldots] \text{ iff } \bar{\mathfrak{A}} \models \varphi(a, b, \ldots; s, t, \ldots).$$

By a *c.i. class* of type τ in the wider sense we mean the class of models of some c.i. sentence $\varphi \in L_{\omega\omega}(\tau^{(2)})$. In order to take care of the relativization built into our quantifiers we shall define a c.i. class of type τ (in the strict sense) as a class K of models of type τ for which there exists a c.i. sentence φ in $L_{\omega\omega}([\tau, 1]^{(2)})$ ($[\tau, 1]$ is obtained from τ by adding one unary predicate) such that for every structure $\mathfrak{A}$ of type τ and set $B \supseteq |\mathfrak{A}|$:

$$\mathfrak{A} \in K \text{ iff } (B, \mathfrak{A}) \models \varphi.$$

If τ is a simple type and K a c.i. class of type τ then the quantifier Q^K will be called a *c.i. quantifier*. A *c.i. logic* is a logic $L = L_{\omega\omega}[Q_i]_{i\in I}$ where each Q_i is a c.i. quantifier (of some simple type τ_i). Note that a c.i. logic has only countably many quantifiers.

Examples (in each example the sentence φ shows the defining class of the quantifier to be c.i.: P is the unary predicate added in passing from τ to $[\tau, 1]$).

(2) Q_1 of example (2), Sect. 1 is c.i. with

$$\varphi = \neg\exists X \forall y \exists x (P(x) \Rightarrow x \in X).$$

(3) Q^E from example (11), Sect. 1 is c.i. with

$$\varphi = \neg\exists X\, \forall y [P(y) \to \exists x (x \in X \wedge x E y)] \wedge (\text{E is an equivalent relation}).$$

(4) Q^C from example (1) is c.i. with

$$\varphi = \exists X \forall y [\exists x (y<x) \to \exists x (x \in X \wedge y < x)] \wedge (< \text{ is a linear ordering})$$

(5) Q^D from Th. 2.15 is c.i. with

$$\varphi = \exists X \forall y \forall z [y<z \Rightarrow \exists x (x \in X \wedge y < x < z] \wedge (< \text{ is a linear ordering})$$

(6) Q^B is of type $\langle 2 \rangle$ with

$K^B = \{ \mathfrak{A} \mid \mathfrak{A} = \langle A, R \rangle$ s.t. there is a countable $Y \subset |\mathfrak{A}|$ with $\forall x [\exists y R(xy) \Rightarrow (\exists y \in Y) R(xy)]$ Q^B is c.i.$\}$

(the reader will easily find the sentence φ showing this).

Remark. $L_{\omega\omega}[Q^B]$ might have applications to separable metric spaces and similar structures where separability is needed as a basic concept. (cf. Makowsky [56]).

Remark. Q^E can be generalized by looking at an equivalence relation between n-tuples. This gives a quantifier Q^{E_n} of type $\langle 2n \rangle$. Similarly one can define Q^{C_n}, Q^{D_n} by looking at an ordering of n-tuples, and $Q^{B_{m,n}}$ (of type $\langle m+n \rangle$) by considering a relation between m-tuples and n-tuples. These quantifiers would be c.i. if we made the natural generalization of allowing variables over countable n-ary relations (for all n) in $L^{(2)}_{\omega\omega}$ thus replacing the monadic second order language by the full language. We could then generalize the notion of cofinal structure, c.i. formula etc. and extend all the results of this section to the wider class of c.i. logic, thus defined.

(7) Monotone classes: A class of structures K of type τ is *monotone* if there exists a first-order formula with one additional unary predicate P $\varphi(P)$ such that whenever $\langle \mathfrak{A}, P \rangle \models \varphi(P)$ and $P \subseteq R \subseteq |\mathfrak{A}|$ then $\langle \mathfrak{A}, R \rangle \models \varphi(R)$ and $K = \{ \mathfrak{A} \mid$ there exists a countable P with $\langle \mathfrak{A}, P \rangle \models \varphi(P)\}$. Monotone classes are, by similar arguments as above, c.i. at least in the wider sense. Furthermore, all the examples (1)–(6) are monotone.

Theorem 3.2. *Let* L *be a c.i. logic, τ any single sorted type. Every formula φ of $L^{(2)}(\tau)$ can be translated into a formula $\tilde{\varphi}$ of $L^{(2)}_{\omega\omega}(\tau)$ with the same free variables such that for all cofinal structures $\mathcal{U}_S$, elements $\bar{a}$ of $|\mathcal{U}|$ and elements $\bar{B}$ of S we have*:

$$\oplus \qquad \mathcal{U}_S \models \varphi[\bar{a}, \bar{B}] \text{ iff } \mathcal{U}_S \models \tilde{\varphi}[\bar{a}, \bar{B}].$$

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Proof. We define $\tilde{\varphi}$ by induction on φ. If φ is atomic $\tilde{\varphi}$ is φ. $\tilde{\varphi}$ commutes with $\neg, \wedge, \vee, \forall, \exists$. If φ is $Q_i \bar{x}_0 \bar{x}_1 \dots \bar{x}_k [\psi_0(\bar{x}_0), \psi_1(\bar{x}_1), \dots \psi_k(\bar{x}_k), t_1, \dots, t_l]$ and the defining class K_i of Q_i is seen to be c.i. by the sentence $\chi_i(P, R_1, \dots, R_k, c_1, \dots, c_l)$ then $\tilde{\varphi}$ is $\chi_i(\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_k, t_1, \dots, t_l)$. The verification of ⊕ is straightforward using the absoluteness of χ_i between $\mathfrak{U}_S$ and $\mathfrak{U}$. □

It follows from Theorem 3.2 that if φ is a c.i. formula of $L^{(2)}(\tau)$ (where cofinal invariance is defined for $L^{(2)}(\tau)$ formulas just as for $L^{(2)}_{\omega\omega}(\tau)$ formulas) the $\tilde{\varphi}$ is c.i. too. From this it is easy to deduce:

Corollary 3.3. *If $K \in EC_L$ for some c.i. logic L then K is a c.i. class.*

The main results for c.i. logic are:

Theorem 3.4. *If K is a c.i. class in the wider sense then there are PC-classes K_1, K_2 in $L_{\omega\omega}[Q_1]$ such that $K \cap S_{\aleph_1} = K_1 \cap S_{\aleph_1}$ and $K_2 \cap S_{\aleph_1} = \bar{K} \cap S_{\aleph_1}$* and

Theorem 3.5. *If L is a c.i. logic then L satisfies* LST(ω_1).

Corollary 3.6. *Let L be a finitely generated c.i. logic. Then L satisfies* LST(ω_1), *is (ω, ω)-compact and axiomatizable.*

Proof. This follows from Theorem 3.5 and Theorem 3.1. □

Proof of Theorem 3.4. It suffices to show that for each c.i. sentence φ there is a class $K_1 \in PC_{L_{\omega\omega}[Q_1]}$ such that $K_1 \cap S_{\aleph_1} = K \cap S_{\aleph_1}$ where K is the class of models of φ. So let φ be a c.i. sentence and $|\mathfrak{U}| = \leqslant \aleph_1$.

$\mathfrak{A} \in K$ iff either $\mathfrak{A}$ is countable and $\mathfrak{A}_{S_0} \models \varphi$, with $S_0 = \{|\mathfrak{A}|\}$, or there is an ω_1-like ordering $<$ of $|\mathfrak{A}|$ and $\mathfrak{A}_S \models \varphi$ where S is the set of initial segments of $<$. Clearly, this can be translated into $L_{\omega\omega}[Q_1]$ using additional predicates. □

To prove Theorem 3.5 we prove something a little bit stronger:

Definition. Let κ be a cardinal. *$P(\kappa)$ holds* if the set of countable subsets of κ, $P_{<\aleph_1}(\kappa)$, partially ordered under inclusion, has a cofinal subset of power $\leqslant \kappa$.

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Theorem 3.7. *If $P(\kappa)$ holds and $\kappa \geqslant \aleph_1$ and $\bar{\bar{\tau}} \leqslant \kappa$ then for every τ-structure $\mathfrak{A}$ and $C \subseteq |\mathfrak{A}|$, $\bar{\bar{C}} \leqslant \kappa$ there is an* L*-elementary substructure $\mathfrak{A}_0$ of $\mathfrak{A}$ with $C \subseteq |\mathfrak{A}_0|$ and $|\mathfrak{A}_0| \leqslant \kappa$.*

Proof. Let $\mathfrak{A}$ and C be given. Look at $\mathfrak{A}$ as a many sorted first-order structure and select a family of Skolem functions for $\mathfrak{A}$ (since $\bar{\bar{\tau}} \leqslant \kappa$ there are κ such functions). Now if $B \subseteq |\mathfrak{A}|$ and $T \leqslant P_{<\aleph_1}(\mathfrak{A})$ let $\mathrm{cl}(B, T)$ be the smallest set $(B', T') \supseteq (B, T)$ which is closed under the Skolem functions and is transitive (i.e., $\cup T' \subseteq A'$). This can be obtained in ω-many steps; hence, if $(B \cup T)^{=} \leqslant \kappa$ then $(B' \cup T')^{=} \leqslant \kappa$.

We now define by induction on $\alpha \leqslant \omega_1$

$$(A_0, S_0) = \mathrm{cl}(C, \emptyset)$$

$$(A_\delta, S_\delta) = \bigcup_{\alpha < \delta} (A_\alpha, S_\alpha) \text{ for } \delta \text{ a limit ordinal}$$

$$(A_{\alpha+1}, S'_{\alpha+1}) = \mathrm{cl}(A_\alpha, S_\alpha) \text{ and}$$

$$S_{\alpha+1} = S'_{\alpha+1} \cup \text{"a cofinal subset of } P_{\aleph_1}(A_{\alpha+1}) \text{ of power} \leqslant \kappa\text{"}.$$

Let $\mathcal{U}_{\omega_1} = \mathcal{U} | A_{\omega_1}$. Clearly the structure $(\mathcal{U}_{\omega_1}, S_{\omega_1}, \epsilon)$ is an $\mathrm{L}_{\omega\omega}$-elementary substructure of $\mathcal{U}$. Also, every countable subset of A_{ω_1} is included in some $A_{\alpha+1} (\alpha < \omega_1)$ hence in some member of $S_{\alpha+1} \subseteq S_{\omega_1}$. Thus $(\mathcal{U}_{\omega_1}, S_{\omega_1}, \epsilon)$ is a cofinal structure. For any formula φ of $\mathrm{L}(\tau)$ with parameters from A_{ω_1} we have ($\tilde{\varphi}$ being the translation given in Theorem 3.2):

$$\mathcal{U} \models \varphi \text{ iff } \mathcal{U} \models \tilde{\varphi} \text{ iff } (\mathcal{U}_{\omega_1}, S_{\omega_1}, \epsilon) \models \tilde{\varphi} \text{ iff } \mathcal{U}_{\omega_1} \models \varphi.$$

Thus $\mathcal{U}_{\omega_1} <_{\mathrm{L}} \mathcal{U}$. Clearly $\bar{\bar{A}}_{\omega_1} \leqslant \kappa$ so $\mathcal{U}_{\omega_1}$ has all the required properties. □

Remark. It is easily verified that $P(\kappa) \Rightarrow P(\kappa^+)$, in particular $P(\aleph_n)$ $n < \omega$. Hence Theorem 3.5 follows. But Theorem 3.7 gives us also information about the possible other cardinalities of elementary submodels for c.i. logic L. To exploit this even more we investigate the property $P(\kappa)$ further:

Theorem 3.8. (i) *If $P(\kappa)$ and $\kappa \geqslant \aleph_1$ then $\mathrm{cf}(\kappa) > \omega$.*

(ii) *$\kappa \geqslant 2^{\aleph_0}$ implies $P(\kappa) \Leftrightarrow \kappa^{\aleph_0} = \kappa$.*

Proof. (i) Let S be cofinal in $(P_{<\aleph_1}(\kappa), \subseteq)$, $\bar{\bar{S}} \leqslant \kappa$ and assume, for contradiction, that $\mathrm{cf}(\kappa) = \omega$. Then $S = \bigcup_{n<\omega} S_n$ with $\bar{\bar{S}}_n = \lambda_n < \kappa$ for suitable

S_n's. Since $(\cup S_n)^{=} \leqslant \lambda_n \cdot \aleph_0 < \kappa$ there is some $\beta_n < \kappa$ such that there is no A with $\beta_n \in A \in S_n$. Let $B = \{\beta_n \mid n \in \omega\}$. Then for all n and all $A (A \in S_n) \Rightarrow B \nsubseteq A$. Thus S is not cofinal, a contradiction.

(ii) $\kappa^{\aleph_0} = \kappa \Rightarrow P(\kappa)$ is clear. So let $\kappa \geqslant 2^{\aleph_0}$ and $S \subseteq P_{<\aleph_1}(\kappa)$, S cofinal and $\bar{\bar{S}} \leqslant \kappa$. Let T be a family of $\kappa^{\aleph_0}$ almost disjoint countable subsets of κ. For $s \in S$ let $T_s = \{t \in T \mid t \subseteq s\}$. By the cofinality of S we have $T = \bigcup_{s \in S} T_s$ thus $\kappa^{\aleph_0} = \bar{\bar{T}} \leqslant \Sigma_{s \in S} \bar{\bar{T}}_s \leqslant S \cdot 2^{\aleph_0} \leqslant \kappa \cdot 2^{\aleph_0} = \kappa$. □

Theorem 3.6 is best possible even for the quantifier Q^{B} as shows:

Theorem 3.9. *There is a sentence φ of $\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathrm{B}}]$ such that, for all κ, φ has a model of power κ iff $\kappa \geqslant \aleph_1$ and $P(\kappa)$.*

Proof. Let φ_1 be $\forall x \exists^{\leqslant \aleph_0} y R(y, x)$ and φ_2 be $\neg \mathrm{Q}^{\mathrm{B}} x, y(\neg R(y, x))$, where R is a binary predicate. $\langle A, R\rangle \models \varphi_2$ iff there is no countable $Y \subseteq A$ with $\forall x[\exists y \neg R(y, x) \Rightarrow (\exists y \in Y)(\neg R(y, x))]$ iff for every countable $Y \subseteq A \exists x[\exists y \neg R(y, x) \wedge (\forall y \in Y) R(y, x))]$. Let φ be $\varphi_1 \wedge \varphi_2$. Now $\langle A, R\rangle \models \varphi$ and $S = \{\{y \mid R(y, x)\} \mid x \in A\}$ implies $\bar{\bar{A}} \geqslant \aleph_1 \bar{\bar{S}} \leqslant \bar{\bar{A}}$, S is cofinal in $P_{<\aleph_1}(A)$. Thus if $\bar{\bar{A}} = \kappa$ then $\kappa \geqslant \aleph_1$ and $P(\kappa)$. Conversely, if $\kappa \geqslant \aleph_1$ and $P(\kappa)$ let $A = \{a_i \mid i < \kappa\}$ $a_i \neq a_j$ for $i \neq j$ and $S = \{b_j \mid j < \kappa\}$, S cofinal in $P_{<\aleph_1}(A)$. Then $\langle \kappa, R\rangle \models \varphi$ with $R(i, j)$ iff $a_i \in b_j$. □

For $\mathrm{L} = \mathrm{L}_{\omega\omega}[\mathrm{Q}_1, \mathrm{Q}^{\mathrm{C}}]$ we have a better result:

Theorem 3.10. $\mathrm{L} = \mathrm{L}_{\omega\omega}[\mathrm{Q}_1, \mathrm{Q}^{\mathrm{C}}]$ *satisfies* $\mathrm{LST}(\kappa)$ *for every* $\kappa \geqslant \aleph_1$.

Proof. Let $\mathfrak{B}$ be a structure of type τ $\kappa > \omega_0$ and $\bar{\bar{\tau}} \leqslant \kappa$, $C \subseteq |\mathfrak{B}|$ and $\bar{\bar{C}} = \kappa$. Define $\langle \mathfrak{A}_\alpha \mid \alpha \leqslant \omega_1 \rangle$ such that:

(i) $|\mathfrak{A}_\alpha| \subset |\mathfrak{B}|$, $\overline{\overline{|\mathfrak{A}_\alpha|}} = \kappa$.

(ii) $C \subseteq |\mathfrak{A}_0|$, $|\mathfrak{A}_\delta| = \bigcup_{\beta < \delta} |\mathfrak{A}_\beta|$ for a δ limit ordinal and $|\mathfrak{A}_\alpha| \subseteq |\mathfrak{A}_{\alpha+1}|$.

(iii) $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{B}$.

(iv) Let $\alpha < \omega_1$ and $\bar{a} \in |\mathfrak{A}_\alpha|$. Then for every $\varphi \in \mathrm{L}$: (a) If $\mathfrak{B} \models \exists x \varphi(x, \bar{a})$ then for some $b \in |\mathfrak{A}_{\alpha+1}|$ $\mathfrak{B} \models \varphi(b, \bar{a})$. (b) If $\mathfrak{B} \models \mathrm{Q}_1 x \varphi(x, \bar{a})$ then there exist at least $\aleph_1$ many elements $b \in |\mathfrak{A}_{\alpha+1}|$ with $\mathfrak{B} \models \varphi(b, \bar{a})$. (c) If $\mathfrak{B} \models Cxy\varphi(x, y, \bar{a})$ then there exist $b_0, b_1, b_2, \ldots \in |\mathfrak{A}_{\alpha+1}|$ such that $\{b_n \mid n \in \omega\}$ is cofinal in $\{(x, y) \mid \mathfrak{B} \models \varphi(x, y, \bar{a})\}$. (d) If $\{(x, y) \mid \mathfrak{B} \models \varphi(x, y, \bar{a})$ is an ordering of cofinality $> \omega$ (call it $<_{\varphi,a}$ and let $\mathrm{D}_{\varphi,a}$ be its field) and $|\mathfrak{A}_\alpha| \cap \mathrm{D}_{\varphi,a}$ is not cofinal in $<_{\varphi,a}$ then there is some $b \in |\mathfrak{A}_{\alpha+1}|$ which is greater in $<_{\varphi,a}$ than every element of $|\mathfrak{A}_\alpha| \cap \mathrm{D}_{\varphi,a}$. Clearly, $\mathfrak{A}_{\alpha+1}$ can be obtained from $\mathfrak{A}_\alpha$ by adding $\leqslant \kappa$ elements, so the construction is possible and $|\mathfrak{A}_{\omega_1}| = \kappa$ since $\kappa \geqslant \aleph_1$. Also, each $\mathfrak{A}_\delta$, δ a limit ordinal, is closed under the operations of $\mathfrak{B}$ and $\mathfrak{A}_\delta <_{L_{\omega\omega}} \mathfrak{B}$ (by the Tarski–Vaught Theorem for $L_{\omega\omega}$). In fact, $\mathfrak{A}_\delta <_{L_{\omega\omega}[Q_1]} \mathfrak{B}$ as one easily verifies.

We now show that $\mathfrak{A} = \mathfrak{A}_{\omega_1} <_L \mathfrak{B}$ by proving that $\langle \mathfrak{A}, a\rangle \models \psi(\bar{a})$ iff $\langle \mathfrak{B}, \bar{a}\rangle \models \psi(\bar{a})$ for all $\bar{a} \in |\mathfrak{A}|$, by induction on C-quantifier rank of ψ. The only non-trivial case is $\psi(\bar{a})$ of the form $Cxy\varphi(\bar{a}, xy)$. By induction hypothesis $\varphi(x, y, \bar{a})$ linearly orders $\mathfrak{A}$ iff it does it for $\mathfrak{B}$. Thus we may assume that $<^{\mathfrak{B}}_{\varphi,a}$ is a linear ordering of $D^{\mathfrak{B}}_{\varphi,a}$ and that $<^{\mathfrak{A}}_{\varphi,a} = <^{\mathfrak{B}}_{\varphi,a} \cap |\mathfrak{A}|^2$ and $D^{\mathfrak{A}}_{\varphi,a} = D^{\mathfrak{B}}_{\varphi,a} \cap A$ (again by induction hypothesis). If $<^{\mathfrak{B}}_{\varphi,a}$ is of cofinality ω then $\mathfrak{A}_{\alpha+1}$ and hence $\mathfrak{A}$ contains a countable sequence for $<^{\mathfrak{B}}_{\varphi,a}$ and hence for $<^{\mathfrak{A}}_{\varphi,a}$. If $<^{\mathfrak{B}}_{\varphi,a}$ has cofinality bigger than ω then either $|\mathfrak{A}| \cap D^{\mathfrak{B}}_{\varphi,a}$ is cofinal in $<^{\mathfrak{B}}_{\varphi,a}$ and then the cofinality of $<^{\mathfrak{B}}_{\varphi,a}$ is the same as of $<^{\mathfrak{A}}_{\varphi,a}$, or $A \cap D^{\mathfrak{B}}_{\varphi,a}$ is not cofinal in $<^{\mathfrak{B}}_{\varphi,a}$ and then for each $\beta \leqslant \alpha$ $\mathfrak{A}_{\beta+1}$ contains an element bigger than all of $|\mathfrak{A}_\beta| \cap D^{\mathfrak{B}}_{\varphi,a}$ so the cofinality of $<^{\mathfrak{A}}_{\varphi,a}$ is ω_1. □

Problems. Ebbinghaus [9] recently proved that $L_{\omega\omega}[Q^B]$ does not satisfy interpolation.

Problem 3.1. (i) Is $\Delta(L_{\omega\omega}[Q^B]$ finitely generated?
(ii) Is $\Delta(L_{\omega\omega}[Q_1])$ finitely generated?

Problem 3.2. (Feferman) Is there an extension L of $L_{\omega\omega}[Q_1]$ which is (ω, ω)-compact, axiomatizable and satisfies $(I)_L$ or $(\Delta)_L$?

Problem 3.3. Are the c.i. logics Δ-closed? Do they satisfy $(B)_L$? Find (natural) examples of c.i. logics which require non-monotone quantifiers (see ex. (7)).

Modifying a question by H. Friedman [15] one might ask:

Problem 3.4. (i) Is there any L properly extending $L_{\omega\omega}$ which satisfies $(B)_L$, $(I)_L$ or $(WB)_L$, and is axiomatizable?
(ii) Is there any L properly extending $L_{\omega\omega}$ which satisfies $(B)_L$ or $(WB)_L$ and is (κ, ω)-compact for some $\kappa \geqslant \omega$?

4. Δ-sublogics of $L_{\omega_1\omega}$ and Scott sentences

Let $\mathfrak{A}$ be a structure of semi-simple similarity type. We define

$$I(\mathfrak{A}) = \{\mathfrak{A}' \mid \mathfrak{A} \cong \mathfrak{A}'\}$$

and

$$\mathrm{PI}(\mathfrak{A}) = \{\mathfrak{A}' \mid \mathfrak{A} \cong_p \mathfrak{A}'\}.$$

If $\mathfrak{A}$ is countable $\mathrm{PI}(\mathfrak{A}) = \mathrm{Mod}(\sigma^{\mathfrak{A}})$ where $\sigma^{\mathfrak{A}}$ is the canonical Scott sentence of $\mathfrak{A}$ in $\mathrm{L}_{\omega_1\omega}$. If $\mathfrak{A}$ is a term model then $\mathrm{I}(\mathfrak{A}) = \mathrm{PI}(\mathfrak{A})$. (See [4, I, §10] for definitions of $\cong_p$, $\sigma^{\mathfrak{A}}$.)

In this chapter we study the Δ-clsoure of logics obtained by adding $\mathrm{I}(\mathfrak{A})$ or $\mathrm{PI}(\mathfrak{A})$ for some fixed $\mathfrak{A}$ (or a family of $\mathfrak{A}$'s) as a quantifier to $\mathrm{L}_{\omega\omega}$. We shall mainly concentrate on sublogics of $\mathrm{L}_{\omega_1\omega}$, but our considerations go a little bit further.

There are two ways of looking at sublogics of $\mathrm{L}_{\omega_1\omega}$ (or $\mathrm{L}_{\kappa\lambda}$ in general): Either we look at logics of the form $\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha]_{\alpha\in A}$ or at logics $\mathrm{L}_A = \mathrm{L}_{\omega_1\omega} \cap A$ for some transitive set A closed under some set-theoretic operations. We shall consider both approaches here. We assume general acquaintance with admissible sets (cf. [5]) but unlike [5] consider only sets without urelements in this section.

The basic relation between the two approaches was discovered by Barwise, who showed (cf. [4, II, 4.1]), that the Δ-closure of ω-logic or of the logic $\mathrm{L}_{\omega\omega}[\mathrm{Q}_0]$, is the logic L_A where $A = \omega^+$ the least admissible set containing ω. This was generalised by Barwise [4, II, 4.4] and Makowsky [37,39] to get the following theorem (see [4] for the proof or compare the proof of 4.4 below).

Theorem 4.1. *Let $p \subseteq \omega$ and let* L *be a Δ-closed logic in which the structure $\mathcal{U} = \langle\omega, <, p\rangle$ is characterizable (that is $\mathrm{I}(\mathcal{U})(=\mathrm{PI}(\mathcal{U})) \in \mathrm{EC_L}$). Then $\mathrm{L}_{(\omega,p)^+} \leqslant \mathrm{L}$ where $(\omega, p)^+$ is the smallest admissible set containing ω and p as elements. In particular $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathrm{I}(\mathcal{U})}]) \sim \mathrm{L}_{(\omega,p)^+}$.*

[Note that by regarding L_A (admissible A) as a logic in the sense of Section 1 we are effectively restricting attention to the sentences of L_A in which only finitely many non-logical symbols occur.]

Corollary 4.2. *If* L *is a Δ-closed logic in which each structure $\langle\omega, <, p\rangle$ ($p \subseteq \omega$ is characterizable then $\mathrm{L}_{\omega_1\omega} \leqslant \mathrm{L}$.*

Thus $\mathrm{L}_{\omega_1\omega}$ is the least logic which is Δ-closed and satisfies Scott's theorem (every countable structure is characterizable).

The main aim of this section is to try to characterize $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathcal{U}}])$ where $\mathrm{Q}^{\mathcal{U}}$ abbreviates $\mathrm{Q}^{\mathrm{PI}(\mathcal{U})}$ and $\mathcal{U}$ is not of the form $(\omega, <, \ldots)$. It was conjectured by H. Friedman [14] and announced by Makowsky [39] that:

Statement 4.3. If $\mathfrak{U}$ is an arbitrary countable structure of finite similarity type and $\sigma^{\mathfrak{U}}$ is its canonical Scott Sentence then $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathfrak{U}}] = \mathrm{L}_A$ with $\mathrm{Q}^{\mathfrak{U}}$ defined by $\mathrm{PI}(\mathfrak{U})$ and $A = (\sigma^{\mathfrak{U}})^+$.

It turns out that Statement 4.3 is false. To see this we first consider the case $\mathfrak{U} = \langle \alpha, < \rangle$ where α is an ordinal and the quantifier Q^α of type $\langle 2 \rangle$ defined by $\mathrm{I}(\mathfrak{U}) = \mathrm{PI}(\mathfrak{U})$).

Definition. Let a be a set. Then the *Σ_1-definable part of the next admissible set a^+* is the set $S_a = \{b \in a^+ \mid$ there is Σ_1-formula $\varphi(x, y)$ such that $\langle a^+, \in \rangle \models \exists! y \varphi(a, y) \wedge \varphi(a, b)\}$. That is to say, $b \in S_a$ iff b is Σ_1-definable in a^+ from the parameter a.

Theorem 4.4. $\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha] \sim_{PC} \mathrm{L}_{S_\alpha} = \mathrm{L}_{\infty\omega} \cap S_\alpha$. *Furthermore,* $\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha] \leqslant_{EC} \mathrm{L}_{S_\alpha}$.

Proof. We first show $\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha] \leqslant_{EC} \mathrm{L}_{S_\alpha}$. Since $\alpha \in S_\alpha$, $\langle \alpha, \in \rangle$ can be characterized up to isomorphism in L_{S_α}, hence Q^α is definable in L_{S_α}.

It remains to show that every $K \in \mathrm{EC}_{\mathrm{L}_{S_\alpha}}$ is $\mathrm{PC}_{\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha]}$. Let $\varphi \in S_\alpha$ be a sentence and let σ_φ be a Σ_1-formula that defines φ in α^+ using α as the only parameter. Consider the conjunction ψ of the following sentences in $\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha]$ (for a two-sorted structure). For the first sort we have:

(i) the axioms for KP or a strong enough finite set of them.

(ii) c_1 (a constant) is an ordinal and $\langle c_1, \in \rangle \cong \langle \alpha, < \rangle$ (using Q^α and sentences in $\mathrm{L}_{\omega\omega}$).

(iii) $\exists! x \sigma_\varphi(c_1, x) \wedge \sigma_\varphi(c_1, c_2)$. (This insures that the constant c_2 "is" φ.)

(iv) c_3 (another constant) is a structure satisfying c_2 (which is expressible in $\mathrm{L}_{\omega\omega}$).

For the second sort we have:

(v) The universe of the second sort is isomorphic to c_3 (which is again expressible in $\mathrm{L}_{\omega\omega}$, due to the fact that c_3 has a finite similarity type).

The Σ_1-definition of c_2 and its uniqueness guarantee that the models of ψ do not admit nonstandard elements for c_2; so, clearly, all the models of ψ restricted to the second sort are models of φ and vice versa. □

Corollary 4.5. $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha]) \sim \Delta(\mathrm{L}_{S_\alpha})$. □

For *countable* ordinals α we get more:

Theorem 4.6. *If $\alpha \in \mathrm{HC}$ then L_{S_α} satisfies interpolation.*

Proof. Let φ and ψ be sentences of L_{S_α} such that $\varphi \Rightarrow \psi$. Since $\alpha \in \mathrm{HC}$, α^+ is a countable admissible set and L_{α^+} satisfies $(\mathrm{I})_{\mathrm{L}}$. Hence there is an interpolant θ, i.e., $\theta \in \alpha^+$ and $\varphi \Rightarrow \theta$ and $\theta \Rightarrow \psi$ where the interpolant contains only extralogical symbols occurring in φ and ψ. All we have to show is that we can find a θ which is in S_α. Let $R(\varphi, \psi, x)$ be an abbreviation for: x is a triple $\langle \theta_0, \pi_1, \pi_2 \rangle$ where θ_0 is a sentence containing only symbols occurring in both φ and ψ, π_1 is a derivation of $\varphi \Rightarrow \theta_0$ and π_2 is a derivation of $\theta_0 \Rightarrow \psi$ (within L_{α^+}). Let $R'(\varphi, \psi, \alpha_0)$ be an abbreviation for: α_0 is an ordinal and $(\exists x \in \mathrm{L}_{\alpha_0}\{\varphi, \psi\})\, R(\varphi, \psi, x) \wedge (\forall \beta < \alpha_0)\, \neg\, (\exists x \in \mathrm{L}_\beta\{\varphi, \psi\})\, (R(\varphi, \psi, x))$ where $\mathrm{L}_\alpha\{a, b\}$ is the α-th stage of sets relatively constructible (over $\{a, b\}$).

Finally, let $R''(\varphi, \psi, \alpha_0, y)$ be an abbreviation for: $R'(\varphi, \psi, \alpha_0)$ and $y = \bigwedge \{\theta \mid \mathrm{L}_{\alpha_0}\{\varphi, \psi\}$ contains a triple $x = \langle \theta, \pi_1, \pi_2 \rangle$ s.t. $R(\varphi, \psi, x)\}$. Note that R, R' and R'' are primitive recursive relations and that α in R' and α and y in R'' are uniquely determined by φ_x and ψ. Note, further, that if $R''(\varphi, \psi, \alpha_0, y)$ then y is a non-empty conjunction of interpolants for φ and ψ; hence, y itself is an interpolant for φ and ψ. Since φ and ψ are in S_α there exist Σ_1-formulas $\sigma_1(x, y)$ and $\sigma_2(x, y)$ such that

$$\langle \alpha^+, \in \rangle \models \exists! y\, \sigma_1(\alpha, y) \wedge \exists! y\, \sigma_2(\alpha, y) \wedge \sigma_1(\alpha, \varphi) \wedge \sigma_2(\alpha, \psi).$$

Let $\sigma_0(\alpha, y)$ be the Σ_1-formula expressing $\exists u, v, \alpha_0 [\sigma_1(\alpha, u) \wedge \sigma_2(\alpha, v) \wedge R''(u, v, \alpha_0, y)]$. Clearly, $\langle \alpha^+, \in \rangle \models \sigma_0(\alpha, y)$ holds for a unique y, say θ; hence, $\theta \in S_{\alpha^+}$ and θ is an interpolant for φ and ψ. □

Corollary 4.7. *If $\alpha < \omega_1$, $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha]) \sim \mathrm{L}_{S_\alpha}$.* □

The proof of the next result is easy and left to the reader.

Theorem 4.8. *If β is an ordinal, $\beta < \mathrm{O}(\alpha^+)$. Then $\mathrm{I}(\langle \beta, < \rangle) \in \mathrm{EC}$ in L_{S_α} iff $\beta \in S_\alpha$.*

Thus for countable α we have:

Corollary 4.9. *If $\alpha < \omega_1$ then $\mathrm{I}(\langle \beta, < \rangle)$ is EC in $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha])$ iff $\beta \in S_{\alpha^+}$.*

Proof. For $\beta < \mathrm{O}(\alpha^+)$ this follows from 4.7 and 4.8. For $\beta \geqslant \mathrm{O}(\alpha^+)$ this follows from a result due to Barwise and Kunen [6] to the effect that β cannot be characterized in L_{α^+} even as a projective class. □

There are also generalizations of Corollary 4.7 if we replace the ordinal by an arbitrary hereditarily countable set a and consider $\mathfrak{U}$ for the structure $\mathfrak{U} = \langle \mathrm{TC}\{a\}, \epsilon, a\rangle$, $a \in \mathrm{HC}$.

Furthermore, if $\mathfrak{U}$ is an arbitrary countable structure (of semi-simple type) whose universe is a set of urelements and $\sigma^{\mathfrak{U}}$ is its canonical Scott sentence, then we have:

Theorem 4.10. *If* $\sigma^{\mathfrak{U}} \in (\mathfrak{U})^+$, *then* $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\mathfrak{U}}] \sim \mathrm{L}_{S_{\sigma^{\mathfrak{U}}}}$

Which looks very much like statement 4.3, only that there is an additional hypothesis σ^+ in the conclusion is replaced by $S_{\sigma^{\mathfrak{U}}}$.

We now proceed to show that statement 4.3 is false, the main reason being that S_σ in general is not transitive. It is enough by 4.9 to show that for some countable ordinal α $S_\alpha \not\supseteq 0(\alpha^+)$.

Example. Let $\mathrm{L}_\gamma (\gamma < \omega_1)$ be a model of (enough axioms of) ZF and let $\alpha = \omega_1^{\mathrm{L}\gamma}$. Thus α is uncountable in $\mathrm{L}\gamma$. The definitions of α^+ and S_α are absolute; hence, S_α of L_γ is the real S_α. But, by definition of S_α we have $\mathrm{L}_\gamma \models$ "S_α is countable and α is uncountable"; hence, $\alpha \not\subseteq S_\alpha$.

Remark. The quantifier Q^{ω_1} is quite strong: In $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\omega_1}])[\mathrm{Q}]$ is definable and also the quantifier of wellfoundedness on countable domains. It follows from a result of Stavi (unpublished) that the set of valid sentences of $\mathrm{L}_{\omega\omega}[\mathrm{Q}^{\omega_1}]$ is extremely complicated, in fact not Σ_1 over the universe using only the parameter ω_1.

Statement 4.3 fails mainly because $\Delta(\mathrm{L}_{\omega\omega}[\mathrm{Q}^\alpha])$ need not be of the form L_A with A transitive. For the rest of this section we shall study fragments of $\mathrm{L}_{\omega_1\omega}$ with A transitive and primitive recursively closed. Another way to look at Δ-logics is the following: $\Delta(\mathrm{L})$ was defined to be the smallest logic of the form $\mathrm{L}_{\omega\omega}[\mathrm{Q}_i]_{i\in I}$ which is Δ-closed and has all the L-elementary classes as quantifiers in it, i.e. $\Delta(\mathrm{L})$ has a special syntactical form. This suggests for $\mathrm{L} \leqslant \mathrm{L}_{\omega_1\omega}$ to replace $\Delta(\mathrm{L})$ by $\tilde{\Delta}(\mathrm{L})$, which is the smallest logic of the form L_A, where A is primitive recursively closed and transitive, Δ-closed and contains L. To discuss the operation $\tilde{\Delta}$ we need a theorem due to Friedman [14] and independently proved by Stavi.

Theorem 4.11. *Let A be a transitive primitive recursively closed set. If L_A is Δ-closed then A is the union of admissible sets.*

Proof. We first recall a fact about the next admissible set (cf. Barwise [5, Ch. II]).

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Lemma 4.12. *If $b \in a^+$ then there exists a Σ_1-formula σ and elements $a_1, \ldots, a_n \in \mathrm{TC}\{a\}$ such that whenever $\mathfrak{M} = \langle M, \in\rangle$ is a model of* KP *and an end extension of $\langle a^+, \epsilon\rangle$ then $\mathfrak{M} \models \exists ! x\sigma(a_1, \ldots, a_n, x) \wedge \sigma(a_1, \ldots, a_n, b)$.*

We now want to show that under the hypothesis of Theorem 4.11 $A = \bigcup_{a \in A} a^+$. By a result in [5] $a^+ = \bigcup_{\alpha < 0(a^+)} \mathrm{L}_\alpha\{a\}$ so it suffices to show that if $a \in A$ and $\alpha < 0(a^+)$ then $\alpha \in A$ since A is primitive recursive closed. Without loss of generality (since A is transitive) we can assume that $\alpha = \omega^\alpha$ and that $0(a^+) > \omega$.

Lemma 4.13. *If $a \in A$, $\alpha < 0(a^+)$ then $\mathrm{I}(\langle\alpha, <\rangle)$ is* EC *in* L_A.

Proof. (a) $\mathrm{I}(\langle\alpha, <\rangle)$ is PC in L_A. Let φ be the conjunction of the following sentences in a two-sorted language, $\in$ a binary relation over the first sort, $<$ a binary relation over the second sort and I a binary relation over both sorts, unprimed variables for the first and primed variables for the second sort.

(i) The universe V_1 of the first sort together with $\in$ satisfies KP (or a strong enough finite subset of KP).

(ii) $\bigwedge \{\forall x(x \in \hat{c}_1 \Leftrightarrow \bigvee_{c_2 \in c_1} x = \hat{c}_2) | c_1 \in \mathrm{TC}\{a\}\}$ where $\hat{c}_1, \hat{c}_2$ are names of elements in V_1,

(iii) $\forall x' \exists ! x\, \mathrm{I}(x, x')$,

(iv) $\forall x y x' y' [\mathrm{I}(x, x') \wedge \mathrm{I}(y, y') \Rightarrow (x' < y' \Leftrightarrow x \in y)]$,

(v) $\exists z [\sigma(\hat{a}_1, \ldots, \hat{a}_n, z) \wedge \forall x \exists x' [\mathrm{I}(x, x') \Leftrightarrow x \in z]$,

where σ is the Σ_1-formula from Lemma 4.12 and $\hat{a}_1, \ldots, \hat{a}_n$ are names for the parameters in Lemma 4.12 designing elements of V_1. Clearly, whenever $\mathfrak{M} \models \varphi$ the restriction of $\mathfrak{M}$ to the second sort is isomorphic to $\langle\alpha, <\rangle$ and φ is consistent, hence $\mathrm{I}(\langle\alpha, <\rangle)$ is PC in L_A. (Note only clause (ii) is infinitary.)

(b) $I(\langle\alpha, <\rangle)$ is PC in L_A. To see that we note that $\langle B, <\rangle \not\cong \langle\alpha, <\rangle$ iff either $\langle B, <\rangle$ is not well founded (which is PC in $\mathrm{L}_{\omega\omega}$) or $\langle B, <\rangle$ is isomorphic to an initial segment of $\langle\alpha, <\rangle$ (which is PC in L_A using (a)) or $\langle\alpha, <\rangle$ is isomorphic to an initial segment of $\langle B, <\rangle$ (which is PC in L_A again by (a)). So by our assumption on L_A $\mathrm{I}(\langle\alpha, <\rangle)$ is EC in L_A. □

Continuation of proof of Theorem 4.11. By Lemma 4.13 there is a sentence ψ in L_A characterizing $\langle\alpha, <\rangle$ up to isomorphism. ψ is also a sentence of $\mathrm{L}_{\infty\omega}$, so, by a result due to C. Karp [55] the quantifier rank $\mathrm{qr}(\psi)$ is bigger or equal to α if $\alpha = \omega^\alpha$. Therefore, since ψ is in L_A $\alpha \leqslant \mathrm{qr}(\psi) < 0(A)$ and $\alpha \in A$. □

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Remarks. Clause (ii) in Lemma 4.13 contains possibly countably many constants $\hat{c}$ (for every $c \in \mathrm{TC}(a)$). Now we only work with finite similarity type. But this difficulty can easily be overcome. We replace the constants for c by a formula $\pi_c(x)$ which defines c using its Σ-structure:

$$\pi_c(x) \text{ is } (\forall u \in v) \bigvee_{b \in c} \pi_b(u) \wedge \bigwedge_{b \in c} (\exists u \in v) \pi_b(u).$$

Clause (ii) then reads $\exists x \pi_a(x)$ and in clause (v) we replaces all the a_i's by their defining formulas $\pi_{a_i}(x_i)$ and bind x_i by an existential quantifier.

The proof also shows the following:

Theorem 4.14. *Let a be a set and B any transitive primitive recursive closed set containing a. Then every K which is* PC *in* L_{a^+} *is already* PC *in* L_B.

A converse of Theorem 4.11 is the following:

Theorem 4.15 *Let A be transitive and primitive recursive closed. Then* L_A *satisfies interpolation iff A is a union of admissible sets and* $A \subsetneq \mathrm{HC}$.

Proof. If $A \subsetneq \mathrm{HC}$ and satisfies the hypothesis then L_A satisfies $(\mathrm{I})_\mathrm{L}$ by Barwise [2]. If $A \subsetneq \mathrm{HC}$ and L_A satisfies $(\mathrm{I})_\mathrm{L}$ then L_A satisfies $(\Delta)_\mathrm{L}$ and hence A is a union of admissible sets by Theorem 4.11. So suppose L_A satisfies $(\mathrm{I})_\mathrm{L}$ and A is transitive and primitive recursive closed. It remains to show that $A \subsetneq \mathrm{HC}$. If not, there is $a \in A - \mathrm{HC}$. Let a be in $A - \mathrm{HC}$ of minimal rank. Then $a \subsetneq \mathrm{HC}$ and since $a \notin \mathrm{HC}$, $\bar{\bar{a}} \geqslant \aleph_1$. Also $\omega \in A$ since we assume A has no urelements. Now $K_1 = \{\mathfrak{A} \mid |\bar{\bar{\mathfrak{A}}}| \leqslant \aleph_0\}$ and $K_2 = \{\mathfrak{A} \mid |\bar{\bar{\mathfrak{A}}}| \geqslant \bar{\bar{a}}\}$ are easily seen to be PC in L_A and are disjoint. But since L_A satisfies the Karp property K_1 and K_2 cannot be separated by an EC class in L_A. □

Remark. The set a in the proof of Theorem 4.15 has cardinality $\leqslant 2^{\aleph_0}$ since $a \subsetneq \mathrm{HC}$. Now if $\aleph_1 = 2^{\aleph_0}$ K_1 and K_2 are disjoint and complementary and we have:

Theorem 4.16. *If* $\aleph_1 = 2^{\aleph_0}$ *and A is primitive recursive closed and transitive then* L_A *is* Δ*-closed iff A is a union of admissible sets and* $A \subsetneq \mathrm{HC}$.

We can now give a precise definition of $\tilde{\Delta}(\mathrm{L})$.

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Definition. Let $L \leqslant L_{\omega_1\omega}$. $\tilde{\Delta}(L) = L_A$ with $A = \cap\{B \mid L \leqslant L_B, L_B$ is Δ-closed and B transitive and primitive recursive closed$\}$.

Note that it is not obvious that $L \leqslant \tilde{\Delta}(L)$.

Proposition 4.17. *$\tilde{\Delta}(L)$ is Δ-closed.*

Proof. Let $\tilde{\Delta}(L) = L_A$. Since $A \subsetneq HC$ it is enough to show that A is a union of admissible sets. We shall first show that for every $a \in A$, $a^+ \subsetneq A$. If $a \in A$ then $a \in B$ for some B which by Theorem 4.11 is a union of admissible sets C_i (by the definition of A). So $a \in C_i$ for some C_i. But, then $a^+ \subsetneq C_i$ and $a^+ \subsetneq A$. Now $A = \bigcup_{a\in A}\{a\}^+ = \bigcup_{a\in A} a^+$ which proves the proposition. □

To conclude this section we want to prove an analogue of Conjecture 4.3 for $\tilde{\Delta}(L)$. For this we need a theorem due to Nadel [43].

Let $\mathfrak{A}$ be a countable structure, $\sigma^{\mathfrak{A}}$ its canonical Scott sentence. We call a sentence φ a Scott sentence of $\mathfrak{A}$ if φ is logically equivalent to $\sigma^{\mathfrak{A}}$.

Theorem 4.18. *Let A be a countable admissible set, $\omega \in A$, and φ a Scott sentence in A. Then the canonical Scott sentence σ equivalent to φ is also in A.* (Proof in [43].)

Theorem 4.19. *Let $\mathcal{U}$ be a countable structure (of finite (semi-simple) similarity type) and let $Q^{\mathcal{U}}$ be the quantifier defined by* $PI(\mathcal{U})$, $L = L_{\omega\omega}[Q^{\mathcal{U}}]$. *Then:*

(1) *If* $PI(\mathcal{U}) \in EC_{L_{\omega\omega}}$ *then* $\tilde{\Delta}(L) \sim L_{\omega\omega}(\sim \Delta(L))$.

(2) *If* $PI(\mathcal{U}) \notin EC_{L_{\omega\omega}}$ *then* $\tilde{\Delta}(L) \sim L_{(\sigma^{\mathcal{U}})^+}$ *where $\sigma^{\mathcal{U}}$ is the canonical Scott sentence of $\mathcal{U}$.*

In both cases $\tilde{\Delta}(L) \geqslant \Delta(L) \geqslant L$.

Proof. (1) If $PI(\mathcal{U}) \in EC_{L_{\omega\omega}}$ then $L \leqslant L_{\omega\omega} = L_{HF}$ and the result follows immediately from the definition of $\tilde{\Delta}$.

(2) Suppose $PI(\mathcal{U}) \notin EC_{L_{\omega\omega}}$. Suppose $L \leqslant L_B$ where L_B is as in the definition of $\tilde{\Delta}$. Then B is a union of admissible sets (by Theorem 4.11) and $PI(\mathcal{U}) \in EC_L \subsetneq EC_{L_B}$. Thus L_B contains a Scott sentence for $\mathcal{U}$ and $L_B \not\leqslant L_{\omega\omega}$ hence $\omega \in B$. By Theorem 4.18 $\sigma^{\mathcal{U}} \in B$. Taking the intersection over all B we get $\sigma^{\mathcal{U}} \in A$ where $\tilde{\Delta}(L) = L_A$. Therefore $A \supsetneq (\sigma^{\mathcal{U}})^+$. On the other hand the admissible set $B = (\sigma^{\mathcal{U}})^+$ clearly satisfies $L \leqslant L_B$. Therefore $A \subsetneq B$, hence $A = B$. □

By Theorem 4.10 $\tilde{\Delta}(L)$ and $\Delta(L)$ coincide for $L = L_{\omega\omega}[Q^{\mathcal{U}}]$ if $S_\sigma = \sigma^+$ and $\sigma \in (\mathcal{U})^+$. By Theorem 4.1 this is the case for $\mathcal{U} = \langle \omega, <, P\rangle$. In

Makowsky [38] a family of generalized quantifiers, the ω-securable quantifiers, is studied and it is shown that for logics $L = L_{\omega\omega}[Q]$ with Q ω-securable $\tilde{\Delta}(L)$ and $\Delta(L)$ coincide, too. But obviously $\tilde{\Delta}(L)$ was introduced only to make precise how statement 4.3 fails and no preservation theorems of the type discussed in Section 2 were discussed for $\tilde{\Delta}(L)$ mainly because $\tilde{\Delta}(L)$ only applies to sublogics of $L_{\omega_1\omega}$.

Problems

Problem 4.1. Characterize $\Delta(L_{\omega\omega}[Q\ \ ])$ for arbitrary countable structures

Problem 4.2. Let $L \leqslant L_{\omega_1\omega}$. Is $\tilde{\Delta}(L) \geqslant L$? A simpler auxiliary question: Let A, B be countable admissible sets and let $K \in EC_{L_A} \cap EC_{L_B}$. Does $K \in EC_{L_{A \cap B}}$? If this is not always true then $L_{\omega\omega}[Q^K]$ will be an example of a logic $L(\leqslant L_A)$ such that $\tilde{\Delta}(L) \geqslant L$.

Problem 4.3. Characterize $L_{\omega_1\omega}$ or fragments of it (other than $L_{\omega\omega}$) as maximal logics for some model theoretic properties.

Barwise [3, §3] gives such a characterization using the notion of an absolute logic which is not purely model theoretic. For some time we thoguht that $L_{\omega_1\omega}$ might be the maximum of logic satisfying $LS(\omega)$ and having well-ordering number $\leqslant \omega_1$ (similar characterizations could be proposed for certain fragments). This conjecture was rejected in a very strong sense by Harrington [54] and, independently, by Kunen [57].

Added in proof

Let L^p be the fragment of weak second-order logic allowing existential quantification over *countable* sets and relations provided *they do no not occur negatively in the scope of the existential quantifier*. L^p was introduced by Makowsky and further studied in [40,56]. L^p is (equivalent to) a c.i. logic and contains $L(Q^B)$; it is countably compact and, as well as a c.i. logic, has a completeness theorem with a natural finite list of axioms and schemata provided by Stavi (cf. [40]).

We do not know whether L^p is the strongest c.i. logic.

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Added in proof

Let L^P be the fragment of weak second-order logic allowing existential quantification over *countable* sets and relations provided *they do no not occur negatively in the scope of the existential quantifier*. L^P was introduced by Makowsky and further studied in [40,56]. L^P is (equivalent to) a c.i. logic and contains $L(Q^B)$; it is countably compact and, as well as a c.i. logic, has a completeness theorem with a natural finite list of axioms and schemata provided by Stavi (cf. [40]).

We do not know whether L^P is the strongest c.i. logic.

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