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STRONGLY DEPENDENT THEORIES

BY

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ABSTRACT

We further investigate the class of models of a strongly dependent (first order complete) theory T, continuing [Sh:715], [Sh:783] and related works. Those are properties (= classes) somewhat parallel to superstability among stable theory, though are different from it even for stable theories. We show equivalence of some of their definitions, investigate relevant ranks and give some examples, e.g., the first order theory of the p-adics is strongly dependent. The most notable result is: if $|A| + |T| \leq \mu, \mathbf{I} \subseteq \mathfrak{C}$ and $|\mathbf{I}| \geq \beth_{|T|^+}(\mu)$, then some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality μ^+ is an indiscernible sequence over A.

Annotated contents

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witness $\kappa < \kappa_{\rm ict}(T)$" (1.5), and we deduce that without loss of generality $m = 1$ in (1.7). An observation (1.10) will help to prove the equivalence of some variants. To some extent, indiscernible sequences can replace an element and this is noted in 1.8, 1.9 dealing with the variant $\kappa_{\rm icu}(T)$. We end with some examples, in particular (as promised in [Sh:783]) the first order theory of the p-adic is strongly dependent and this holds for similar fields and for some ordered abelian groups expanded by subgroups. Also, there is a (natural) strongly stable not strongly^2 stable T.

We give equivalent conditions to strongly dependent by cutting indiscernibles (2.1) and recall the parallel result for T dependent. Then we define $\kappa_{\rm ict,2}(T)$ (in 2.3) and show that it always almost is equal to $\kappa_{\rm ict}(T)$ in 2.8. The exceptional case is "T is strongly dependent but not strongly^2 dependent" for which we give equivalent conditions (2.3 and 2.10).

We define $M_0 \le_A M_1, M_0 \le_{A,p} M_2$ (in 3.2) and observe some basic properties in 3.3. Then in 3.5 for most $\ell = 1,\dots,12$ we define $<_\ell$ $, <^\ell_{\rm at}, <^\ell_{\rm pr}, \le^\ell$, explicit $\bar\Delta$-splitting, and last but not least the ranks dp-rk$_{\bar\Delta,\ell}(\mathfrak{x})$. Easy properties are in 3.7, the equivalence of "rank is infinite" is $\ge |T|^+, T$ is strongly dependent in 3.7 and more basic properties in 3.9. We then add more cases ($\ell > 12$) to the main definition in order to deal with (a version of) strong dependency and then have parallel claims.

We prove that if $\mu \ge |A| + |T|$ and $a_\alpha \in {}^m\mathfrak{C}$ for $\alpha < \beth_{\mu^+}$ *then* for some $u \subseteq \beth_{\mu^+}$ of cardinality $\mu^+, \langle a_\alpha : \alpha \in u\rangle$ is indiscernible over A.

We consider shortly several further relatives of strongly dependent.

We redefine explicitly $\bar\Delta$-splitting and dp-rk$_{\bar\Delta,\ell}$ for more cases, i.e. more ℓ's and for the case of finite Δ_ℓ's in a way fitting dependent T (in 5.9), point out the basic equivalence (in 5.9), consider a variant (5.11) and questions (5.10, 5.12).

0. Introduction

Our motivation is trying to solve the equations

"x/dependent = superstable/ stable"

(e.g., among complete first order theories). In [Sh:783, §3] mainly two approximate solutions are suggested: strongly$^\ell$ dependent for $\ell = 1, 2$; here we try to investigate them not relying on [Sh:783, §3]. We define $\kappa_{\mathrm{ict}}(T)$ generalizing $\kappa(T)$; the definition has the form "$\kappa < \kappa_{\mathrm{ict}}(T)$ iff there is a sequence $\langle \varphi_i(\bar{x}, \bar{y}_i) : i < \kappa\rangle$ of formulas such that ...".

Now T is strongly dependent (= strongly^1 dependent) iff $\kappa_{\mathrm{ict}}(T) = \aleph_0$; prototypical examples are: the theory of dense linear orders, the theory of real closed fields, the model completion of the theory of trees (or trees with levels), and the theory of the p-adic fields (and related fields with valuations). (The last one is strongly^1 not strongly^2 dependent, see 1.17.)

For T superstable, if $\langle \bar{a}_t : t \in I \rangle$ is an indiscernible set over A and C is finite, then for some finite $I^* \subseteq I, \langle \bar{a}_t : t \in I \backslash I^* \rangle$ is indiscernible over $A \cup C$, moreover over $A \cup C \cup \{\bar{a}_t : t \in I^*\}$. In §2 we investigate the parallel here, when I is a linear order, complete for simplicity (see more and history in [Sh:950, §1C, 1.37]). But we get two versions: strongly$^\ell$ dependent $\ell = 1, 2$ according to whether we like to generalize the first version of the statement above or the "moreover".

Next, in §3, we define and investigate rank, not of types but of related objects $\mathfrak{x} = (p, M, A)$ where, e.g., $p \in \mathbf{S}^m(M \cup A)$; but there are several variants. For some of them we prove "T is strongly dependent iff the rank is always $< \infty$ iff the rank is bounded by some $\gamma < |T|^+$". We first deal with the ranks related to "strongly^1 dependent" and then for the ones related to "strongly^2 dependent".

Further serious evidence for those ranks being of interest is in §4, where we use them to get indiscernibles. Recall that if T is stable, $|A| \leq \lambda = \lambda^{|T|}, a_\alpha \in \mathfrak{C}$ for $\alpha < \mu := \lambda^+$, then for some stationary $S \subseteq \mu, \langle a_\alpha : \alpha \in S \rangle$ is indiscernible over $A, |S| = \mu$; we can write this as $\lambda \to (\lambda)^{<\omega}_{T,\mu}$; We can get similar theorems from set theoretic assumptions: e.g., μ a measurable cardinal, very interesting and important but not for the present model theoretic investigation.

We may wonder: Can we classify first order theories by $\lambda \to_T (\mu)_\kappa$, as was asked by Grossberg and the author (see on this question [Sh:702, 2.9–2.20]). A positive answer appears in [Sh:197], *but* under a very strong assumption on T: not only T is dependent but for every subset $P_1, \ldots, P_n$ of $|M|$ the theory $\mathrm{Th}(M, P_1, \ldots, P_n)$ is dependent, i.e., being dependent is preserved by monadic expansions.

Here we prove that if T is strongly stable and $\mu \geq |T|$, then $\beth_{\mu^+} \to_T (\mu^+)^{<\omega}_{\mu^+}$. We certainly hope for a better result (using $\beth_n(|T|)$ for some fixed n or even $(2^\mu)^+$ instead of $\beth_{\mu^+}$) and weaker assumptions, say "T is dependent" (or less) instead of "T is strongly dependent". But still it seems worthwhile to prove 4.1, particularly having waited so long for something.

Let strongly$^\ell$ stable mean strongly$^\ell$ dependent and stable. As it happens (for T), being superstable implies strong^2 stable implies strong^1 stable, but the inverses fail. So strongly$^\ell$ dependent does not really solve the equation we have started with. However, this is not necessarily bad; the notion "strongly$^\ell$ stable" seems interesting in its own right. This applies to the further variants.

We give a "simplest" example of a theory T which is strongly^1 stable and not strongly^2 stable at the end of §1 as well as prove that the (theories of the) p-adic field is strongly stable (for any prime p) as well as similar enough fields.

In §5 we comment on some further properties and ranks. Such further properties hopefully will be crucial in [Sh:F705], if it materializes; it tries to deal mainly with $K^{\rm or}$-representable theories and contains other beginnings as well. We comment on ranks parallel to those in §3 suitable for all dependent theories.

We further try to look at theories of fields. Also, we deal with the search for families of dependent theories T which are unstable but "minimal", much more well behaved. For many years it seems quite bothering that we do not know how to define o-minimality as naturally arising from a parallel to stability theory rather than as an analog to minimal theories or to generalize examples related to the theory of the field of the reals and its expansions. Of course, the answer may be a somewhat larger class. This motivates Firstenberg–Shelah [FiSh:E50] (on Th($\mathbb{R}$), specifically on "perpendicularly is simple"), and some definitions in §5. Another approach to this question is of Onshuus in his very illuminating works on th-forking [On0x1] and [On0x2].

A result from [Sh:783, §3,§4] used in [FiSh:E50] says that

0.1. Claim: *Assume T is strongly^2 dependent.*

(a) *If G is a definable group in $\mathfrak{C}_T$ and h is a definable endomorphism of G with finite kernels then h is almost onto G, i.e., the index $(G : \mathrm{Rang}(h))$ is finite.*
(b) *It is not the case that: there are a definable (with parameters) subset $\varphi(\mathfrak{C}, \bar a_1)$ of $\mathfrak{C}$, an equivalence relation $E_{\bar a_2} = E(x, y, \bar a_2)$ on $\varphi(\mathfrak{C}, \bar a_1)$ with infinitely many equivalence classes and $\vartheta(x, y, z, \bar a_3)$ such that $E(c, c, \bar a_2) \Rightarrow \vartheta(x, y, c, \bar a_3)$ is a one-to-one function from (a co-finite subset of) $\varphi(\mathfrak{C}, \bar a_1)$ into $c/E_{\bar a_2}$.*

We continue investigating dependent theories in [Sh:900], [Sh:877], [Sh:906], more recently [Sh:950] and Kaplan–Shelah [KpSh:946], [**?**] and concerning definable groups in [Sh:876], [Sh:886] and [KpSh:993].

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0.2. *Notation:* (1) Let $\varphi^{\mathbf{t}}$ be φ if $\mathbf{t} = 1$ or $\mathbf{t} =$ true *and* $\neg\varphi$ if $\mathbf{t} = 0$ or $\mathbf{t} =$ false.
(2) $\mathbf{S}^\alpha(A, M)$ is the set of complete types over A in M (i.e., finitely satisfiable in M) in the free variables $\langle x_i : i < \alpha\rangle$.

1. Strongly dependent: Basic variant

1.1. *Convention:* (1) T is complete first order fixed.
(2) $\mathfrak{C} = \mathfrak{C}_T$ a monster model for T.

Recall, see [Sh:783]:

1.2. *Definition:* (1) $\kappa_{\rm ict}(T) = \kappa_{{\rm ict},1}(T)$ is the minimal κ such that for no $\bar\varphi = \langle \varphi_i(\bar x, \bar y_i) : i < \kappa\rangle$ is $\Gamma_\lambda = \Gamma^{\bar\varphi}_\lambda$ consistent with T for some ($\equiv$ every) λ, where $\ell g(\bar x) = m, \ell g(\bar y^i_m) = \ell g(\bar y_i)$ and

$$\Gamma_\lambda = \{\varphi_i(\bar x_\eta, \bar y^i_\alpha)^{{\rm if}(\eta(i)=\alpha)} : \eta \in {}^\kappa\lambda, \alpha < \lambda \text{ and } i < \kappa\}.$$

(1A) We say that $\bar\varphi = \langle\varphi_i(\bar x, \bar y_i) : i < \kappa\rangle$ witness $\kappa < \kappa_{\rm ict}(T)$ (with $m = \ell g(\bar x)$) *when* it is as in part (1).
(2) T is strongly dependent (or strongly[1] dependent) *when* $\kappa_{\rm ict}(T) = \aleph_0$.

Easy (or see [Sh:783]):

1.3. *Observation:* If T is strongly dependent *then* T is dependent.

1.4. *Observation:* (1) $\kappa_{\rm ict}(T^{\rm eq}) = \kappa_{\rm ict}(T)$.
(2) If $T_\ell = {\rm Th}(M_\ell)$ for $\ell = 1, 2$, then $\kappa_{\rm ict}(T_1) \le \kappa_{\rm ict}(T_2)$ when:
 (∗) M_1 is (first order) interpretable in M_2.
(3) If $T' = {\rm Th}(\mathfrak{C}, c)_{c\in A}$, then $\kappa_{\rm ict}(T') = \kappa_{\rm ict}(T)$.
(4) If M is the disjoint sum of M_1, M_2 (or the product) and ${\rm Th}(M_1)$, ${\rm Th}(M_2)$ are dependent, then so is ${\rm Th}(M)$; so M_1, M_2, M has the same vocabulary.
(5) In Definition 1.2, for some $\lambda, \Gamma^{\bar\varphi}_\lambda$ is consistent with T iff for every λ, $\Gamma^{\bar\varphi}_\lambda$ is consistent with T.

Remark: Concerning Part (4) for "strongly dependent", see Cohen–Shelah [CoSh:E65, Th.24].

Proof. Easy. $\blacksquare_{1.4}$

1.5. *Observation:* Let $\ell g(\bar{x})=m$, $\bar{\varphi}=\langle \varphi_i(\bar{x},\bar{y}_i):i<\kappa\rangle$ and let $\bar{\varphi}'=\langle \bar{\varphi}'_i(\bar{x},\bar{y}'_i):i<\kappa\rangle$ where $\varphi'_i(\bar{x},\bar{y}'_i) = [\varphi_i(\bar{x},\bar{y}^1_i) \wedge \neg\varphi_i(\bar{x},\bar{y}^2_i)]$, and let $\bar{\varphi}'' = \langle \varphi''_i(\bar{x},\bar{y}''_i) : i < \kappa\rangle$ where $\bar{\varphi}''_i(\bar{x},\bar{y}''_i) = [\bar{\varphi}_i(\bar{x},\bar{y}^1_i) \equiv \neg\varphi_i(\bar{x},\bar{y}^2_i)]$. *Then* $\circledast^1_{\bar{\varphi}} \Rightarrow \circledast^2_{\bar{\varphi}} \Leftrightarrow \circledast^3_{\bar{\varphi}} \Leftrightarrow (\exists\eta \in {}^\kappa 2)\circledast^2_{\bar{\varphi}^{[\eta]}} \Leftrightarrow (\exists\eta \in {}^\kappa 2)\circledast^3_{\bar{\varphi}^{[\eta]}}$ and $\circledast^\ell_{\bar{\varphi}} \Leftrightarrow \circledast^\ell_{\bar{\varphi}'} \Leftrightarrow \circledast^\ell_{\bar{\varphi}''}$ for $\ell = 2, 3$ and $\circledast^3_{\bar{\varphi}} \Leftrightarrow \circledast^1_{\bar{\varphi}'} \Leftrightarrow \circledast^1_{\varphi''}$ where $\bar{\varphi}^{[\eta]} = \langle \varphi_i(\bar{x},\bar{y}_i)^{\eta(i)} : i < \kappa\rangle$ and

- $\circledast^1_{\bar{\varphi}}$ $\bar{\varphi}$ witness $\kappa < \kappa_{\rm ict}(T)$,
- $\circledast^2_{\bar{\varphi}}$ we can find $\langle \bar{a}^i_k : k<\omega, i<\kappa\rangle$ in $\mathfrak{C}$ such that $\ell g(\bar{a}^i_k)=\ell g(\bar{y}_i)$, $\langle \bar{a}^i_k : k<\omega\rangle$ is indiscernible over $\cup\{\bar{a}^j_k : j < \kappa, j \neq i, k < \omega\}$ for each $i < \kappa$ and $\{\varphi_i(\bar{x},\bar{a}^i_0) \wedge \neg\varphi_i(\bar{x},\bar{a}^i_1) : i < \kappa\}$ is consistent, i.e., finitely satisfiable in $\mathfrak{C}$,
- $\circledast^3_{\bar{\varphi}}$ like $\circledast^2_{\bar{\varphi}}$ but in the end $\{\varphi_i(\bar{x},\bar{a}^i_0) \equiv \neg\varphi_i(\bar{x},\bar{a}^i_1) : i < \kappa\}$ is consistent.

1.6. *Remark:* (1) We could have added the indiscernibility condition to $\circledast^1_{\bar{\varphi}}$, i.e., to 1.2(1), as this variant is equivalent to $\circledast^1_{\bar{\varphi}}$.

(2) Similarly we could have omitted the indiscernibility condition in $\circledast^2_{\bar{\varphi}}$ but demand in the end: "if $k_i < \ell_i < \omega$ for $i < \kappa$ then $\{\varphi_i(\bar{x},\bar{a}^i_{k_i}) \wedge \neg\varphi_i(\bar{x},a^i_{\ell_i}) : i < \kappa\}$ is consistent" and get an equivalent condition.

(3) Similarly we could have omitted the indiscernibility condition in $\circledast^3_{\bar{\varphi}}$ but demand in the end "if $k_i < \ell_i < \omega$ for $i < \kappa$ then $\{\varphi_i(\bar{x},\bar{a}^i_{k_i}) \equiv \neg\varphi_i(\bar{x},\bar{a}^1_{\ell_i}) : i < \kappa\}$ is consistent" and get an equivalent condition.

(4) We could add $\circledast^3_{\bar{\varphi}} \Leftrightarrow \circledast^1_{\bar{\varphi}'}$.

(5) In $\circledast^2_{\bar{\varphi}}, \circledast^3_{\bar{\varphi}}$ (and the variants above) we can replace ω by any λ (see 1.7).

(6) What about $\circledast^2_{\bar{\varphi}} \Rightarrow \circledast^1_{\bar{\varphi}}$? We shall now describe a model whose theory is a counterexample to this implication. We define a model M with $\tau_M = \{P, P_i, R_i : i < \kappa\}, P$ a unary predicate, P_i a unary predicate, R_i a binary predicate, as follows:

   (a) $|M|$ the universe of M is $(\kappa \times \mathbb{Q}) \cup {}^\kappa\mathbb{Q}$,
   (b) $P^M = {}^\kappa\mathbb{Q}$,
   (c) $P^M_i = \{i\} \times \mathbb{Q}$,
   (d) $R^M_i = \{(\eta,(i,q)) : \eta \in {}^\kappa\mathbb{Q}, q \in \mathbb{Q}$ and $\mathbb{Q} \models \eta(i) \geq q\}$,
   (e) $\varphi_i(x,y) = P(x) \wedge P_i(y) \wedge R_i(x,y)$ for $i < \kappa$.

   Now

   (α) Why (for $\mathrm{Th}(\bar{M})$) do we have $\circledast^2_{\bar{\varphi}}$?

For $i < \kappa, k < \omega$ let $a^i_k = (i,k) \in P^M_i$ recalling $\omega \subseteq \mathbb{Q}$.

Easily $\langle a^i_k : k < \omega, i < \kappa\rangle$ are as required in $\circledast^2_{\bar\varphi}$; e.g., the unique $\eta \in {}^\kappa\mathbb{Q}$ realizing the type. Also, for each $i < \kappa$, the sequence $\langle a^i_k : k < \omega\rangle$ is indiscernible over $\{a^j_m : j < \kappa, j \neq i$ and $m < \omega\}$.

Why? Because for every automorphism π of the rational order $(\mathbb{Q}, <)$, for the given $i < \kappa$ we can define a function $\hat\pi_i$ with domain M by

$(*)_1$ for $j < \kappa$ and $q \in \mathbb{Q}$ we let $\hat\pi_i((j,q))$ be (j,q) if $j \neq i$ and $(j,\pi(q))$ if $j = i$,

$(*)_2$ for $\eta,\nu \in {}^\chi\mathbb{Q}$ we have $\hat\pi_i(\eta) = \nu$ iff $(\forall j < \kappa)(j \neq i \Rightarrow \eta(j) = \nu(j))$ and $\nu(i) = \pi_i(\eta(i))$.

So $\hat\pi_i$ is an automorphism of M over $\bigcup_{j\neq i} P^M_j$ which includes the function $\{(a^i_q, a^i_{\pi(q)}) : q \in \mathbb{Q}\}$

(β) Why (for $\mathrm{Th}(M)$) do we not have $\circledast^1_{\bar\varphi}$?

Because $M \models (\forall y_1,y_2)[P_i(y_1) \wedge P_i(y_2) \wedge y_1 \neq y_2 \rightarrow \bigvee^2_{\ell=1}(\forall x)(\varphi_i(x,y_\ell) \wedge P(x) \rightarrow \varphi_i(x, y_{3-\ell}))]$.

(γ) T is dependent. Why? Left to the reader (use restriction to any finite $\tau \subseteq \tau_M$).

Proof. The following series of implications clearly suffices.

$\circledast^1_{\bar\varphi}$ implies $\circledast^2_{\bar\varphi}$

Why? As $\circledast^1_{\bar\varphi}$, clearly for any $\lambda \geq \aleph_0$ we can find $\bar a^i_\alpha \in {}^{\ell g(\bar y_i)}\mathfrak{C}$ for $i < \kappa, \alpha < \lambda$ and $\langle \bar c_\eta : \eta \in {}^\omega\lambda\rangle, \bar c_\eta \in {}^{\ell g(\bar x)}\mathfrak{C}$ such that $\models \varphi_i[\bar c_\eta, \bar a^i_\alpha]$ iff $\eta(i) = \alpha$. By some applications of the Ramsey theorem (or polarized partition relations), without loss of generality $\langle \bar a^i_\alpha : \alpha < \lambda\rangle$ is indiscernible over $\bigcup\{\bar a^j_\beta : j < \kappa, j \neq i, \beta < \lambda\}$ for each $i < \omega$. Now those $\bar a^i_\alpha$'s witness $\circledast^2_{\bar\varphi}$ as $\bar c_\eta$ witness the consistency of the required type when $\eta \in {}^\kappa\{0\}$.

$\circledast^2_{\bar\varphi} \Rightarrow \circledast^3_{\bar\varphi}$ (hence in particular $\circledast^2_{\bar\varphi'} \Rightarrow \circledast^3_{\bar\varphi'}$ and $\circledast^2_{\bar\varphi''} \Rightarrow \circledast^3_{\bar\varphi''}$).

Trivial; read the definitions.

$\circledast^3_{\bar\varphi} \Rightarrow \circledast^2_{\bar\varphi}$ (hence in particular $\circledast^3_{\bar\varphi'} \Rightarrow \circledast^3_{\bar\varphi'}$ and $\circledast^2_{\bar\varphi''} \Rightarrow \circledast^3_{\bar\varphi''}$).

By compactness, for the dense linear order $\mathbb{R}$ we can find $\bar a^i_t$ for $i < \kappa, t \in \mathbb{R}$ such that for each $i < \kappa$ the sequence $\langle \bar a^i_t : t \in \mathbb{R}\rangle$ is indiscernible over $\bigcup\{\bar a^j_s : j \neq i, j < \kappa, s \in \mathbb{R}\}$ and for any $s_0 <_{\mathbb{R}} s_1$ the set $\{\varphi_i(\bar x, \bar a^i_{s_0}) \equiv \neg\varphi_i(\bar x, \bar a^i_{s_1}) : i < \kappa\}$ is consistent, say realized by $\bar c = \bar c_{s_0,s_1}$. Now let $u = \{i < \kappa : \mathfrak{C} \models \varphi_i[\bar c, \bar a^i_{s_0}]\}$, and for $n < \omega$ define $\bar b^i_n$ as $\bar a^i_{s_0+n(s_1-s_0)}$ if $i \in u$ and as $\bar a^i_{s_1-n(s_1-s_0)}$ if $i \in \kappa\backslash u$. Now $\langle \bar b^i_n : n < \omega, i < \kappa\rangle$ exemplifies $\circledast^2_{\bar\varphi}$.

$\circledast^2_{\bar\varphi}$ implies $\circledast^1_{\bar\varphi'}$ (hence by the above $\circledast^2_{\bar\varphi} \Rightarrow \circledast^2_{\bar\varphi'}$ and $\circledast^3_{\bar\varphi} \Rightarrow \circledast^3_{\bar\varphi'}$).

Let $\langle \bar a^i_\alpha : \alpha<\omega, i<\kappa\rangle$ witness $\circledast^2_{\bar\varphi}$ and $\bar c$ realizes $\{\varphi_i(\bar x, a^i_0)\wedge\neg\varphi_i(\bar x,\bar a^i_1) : i<\kappa\}$. Without loss of generality a^i_t is well defined for every $t\in\mathbb{Z}$ not just $t\in\omega$ (and $i<\kappa$), and $\langle a^i_t : t\in\mathbb{Z}\rangle$ is an indiscernible sequence over $\{a^j_s : j\in\kappa\backslash\{i\}$ and $s\in\mathbb{Z}\}$. Also, without loss of generality for each $i<\kappa, \langle \bar a^i_\alpha : \alpha\in[2,\omega)\rangle$ as well as $\langle a^i_{-1-n} : n\in\omega\rangle$ are indiscernible sequences over

$$\bigcup\{\bar a^j_t : j<\kappa, j\neq i \text{ and } t\in\mathbb{Z}\}\cup\{\bar c\}.$$

For $t\in\mathbb{Z}, i<\kappa$ let $\bar b^i_t=\bar a^i_{2t}{}^\frown\bar a^i_{2t+1}$, so $\mathfrak{C}\models\varphi'_i[\bar c,\bar b^i_0]$ (as this just means $\mathfrak{C}\models\varphi_i(\bar c,\bar a^i_0)\wedge\neg\varphi_i[\bar c,\bar a^i_1]$) and $\mathfrak{C}\models\neg\varphi'_i[\bar c,\bar b^i_s]$ when $s\in\mathbb{Z}\backslash\{0\}$ (as the sequences $\bar c{}^\frown\bar a^i_{2s}$ and $\bar c{}^\frown\bar a^i_{2s+1}$ realize the same type). So $\langle \bar b^i_\alpha : \alpha<\omega, i<\kappa\rangle$ witness $\circledast^1_{\bar\varphi'}$.

$\circledast^3_{\bar\varphi'}$ implies $\circledast^3_{\bar\varphi''}$.

Read the definitions.

$\circledast^3_{\bar\varphi''}$ implies that for some $\eta\in{}^\kappa 2$ we have $\circledast^1_{\bar\varphi^{[\eta]}}$.

As in the proof of $\circledast^2_{\varphi}\Rightarrow\circledast^1_{\bar\varphi'}$; but we elaborate: let $\langle\langle \bar a^i_\alpha{}^\frown\bar b^i_\alpha : \alpha<\omega\rangle : i<\kappa\rangle$ witness $\circledast^3_{\bar\varphi''}$ noting $\bar\varphi''=\langle\varphi''_i(\bar x,\bar y^i_1,\bar y^i_2) : i<\kappa\rangle$ where $\ell g(\bar y^i_1)=\ell g(\bar y_i)=\ell g(\bar y^i_2)$. Let $\bar c$ realize $\{\varphi''_i(\bar x,\bar a^i_0,\bar b^i_0)\equiv\neg\varphi''_i(\bar x,\bar a^i_1,\bar b^i_1) : i<\kappa\}$. Without loss of generality, for each $i<\kappa$ the sequence $\langle \bar a^i_\alpha{}^\frown\bar b^i_\alpha : 2\leq\alpha<\omega\rangle$ is indiscernible over $\bigcup\{\bar a^j_\alpha{}^\frown\bar b^j_\alpha : j\in\kappa\backslash\{i\}$ and $\alpha<\omega\}\cup\bar c$.

By this extra indiscernibility assumption, for each $i<\kappa$ we can find $\ell_0(i),\ell_1(i)\in\{0,1\}$ such that $n\geq 2\Rightarrow\mathfrak{C}\models\varphi_i[\bar c,\bar a^i_n]^{\ell_0(i)}\wedge\varphi_i[\bar c,\bar b^i_n]^{\ell_1(i)}$. By the choice of $\bar c$ we have $\mathfrak{C}\models\varphi''_i(\bar c,\bar a^i_0,\bar b^i_0)\equiv\varphi''_i(\bar c,\bar a^i_1,\bar b^i_1)$, hence by the choice of φ''_i we *cannot* have $\mathfrak{C}\models\varphi_i[\bar c,\bar a^i_0]^{\ell_0(i)}\wedge\varphi_i[\bar c,\bar a,\bar b^i_0]^{\ell_1(i)}\wedge\varphi_i[\bar c,\bar a^i_1]^{\ell_0(i)}\wedge\varphi_i[\bar c,\bar b^i_1]^{\ell_1(i)}$.

Hence there are $\ell_3(i),\ell_4(i)\in\{0,1\}$ such that

- $\ell_4(i)=0\Rightarrow\mathfrak{C}\models\varphi_i[\bar c,\bar a^i_{\ell_3(i)}]^{1-\ell_0(i)}$,
- $\ell_4(i)=1\Rightarrow\mathfrak{C}\models\varphi_i[\bar c,\bar b^i_{\ell_3(i)}]^{1-\ell_1(i)}$.

Lastly, choose $\eta=\langle 1-\ell_{\ell_4(i)}(i) : i<\kappa\rangle$ and we choose $\langle \bar d^i_\alpha : \alpha<\omega, i<\kappa\rangle$ as follows:

- if $\ell_4(i)=0$ and $n=0$ then $\bar d^i_n=\bar a^i_{\ell_3(i)}$,
- if $\ell_4(i)=0$ and $n>0$ then $\bar d^i_n=\bar a^i_{1+n}$,
- if $\ell_4(i)=1$ and $n=0$ then $\bar d^i_n=\bar b^i_{\ell_3(i)}$,
- if $\ell_4(i)=1$ and $n>0$ then $\bar d^i_n=\bar b^i_{1+n}$.

Now check that $\langle \bar d^i_\alpha : \alpha<\omega$ and $i<\kappa\rangle$ witness $\circledast^1_{\bar\varphi^{[\eta]}}$.

$\circledast^3_{\bar\varphi^{[\eta]}},\circledast^3_{\bar\varphi}$ are equivalent where $\eta\in{}^\kappa 2$.

Why? Because the formula $(\varphi_i(x,\bar a^i_0)\equiv\neg\varphi_i(x,\bar a^i_1))$ is equivalent to $(\varphi_i(x,a^i_0)^{\eta(i)}\equiv\neg\varphi_i(x,\bar a^i_1)^{\eta(i)})$. $\blacksquare_{1.5}$

1.7. *Observation:* (1) In Definition 1.2, without loss of generality $m(=\ell g(\bar x))$ is 1.

(2) For any κ we have: $\kappa<\kappa_{\rm ict}(T)$ *iff* for some infinite linear order I_i (for $i<\kappa$) and $\langle\bar a^i_t:t\in I_i,i<\kappa\rangle$ such that $\langle\bar a^i_t:t\in I_i\rangle$ is indiscernible over $\bigcup\{\bar a^j_s:s\in I_j$ and $j\neq i,j<\kappa\}\cup A$ and finite C, for κ ordinals $i<\kappa$, the sequence $\langle\bar a^i_t:t\in I_i\rangle$ is not indiscernible over $A\cup C$.

(3) In 1.5, for any $\lambda(\geq\aleph_0)$, from the statement $\circledast^2_{\bar\varphi}$ we get an equivalent one if we replace ω by λ; similarly for $\circledast^3_{\bar\varphi}$.

Proof. (1) For some m, there is $\bar\varphi=\langle\varphi_i(\bar x,\bar y_i):i<\kappa\rangle,\ell g(\bar x)=m$ witnessing $\kappa<\kappa_{\rm ict}(T)$; without loss of generality m is minimal. Fixing $\bar\varphi$ by 1.5 we know that $\circledast^2_{\bar\varphi}$ from observation 1.5 holds. Let $\langle\bar a^i_\alpha:i<\kappa,\alpha<\lambda\rangle$ exemplify $\circledast^2_{\bar\varphi}$ with λ instead ω and let $\bar c=\langle c_i:i<m\rangle$ realize $\{\varphi_i(\bar x,\bar a^i_0)\wedge\neg\varphi_i(\bar x,\bar a^i_1):i<\kappa\}$.

Case 1: For some $u\subseteq\kappa,|u|<\kappa$ for every $i\in\kappa\backslash u$ the sequence $\langle\bar a^i_\alpha:\alpha<\lambda\rangle$ is an indiscernible sequence over $\bigcup\{\bar a^j_\beta:j\in\kappa\backslash u\backslash\{i\}\}\cup\{c_{m-1}\}$.

In this case for $i\in\kappa\backslash u$ let $\psi_i(\bar x',\bar y'_i):=\varphi_i(\bar x\restriction(m-1),\langle x_{m-1}\rangle{}^\frown\bar y_i)$ and $\bar\psi=\langle\psi_i(\bar x',\bar y'_i):i\in\kappa\backslash u\rangle$ and $\bar b^i_\alpha=\langle c_{m-1}\rangle{}^\frown\bar a^i_\alpha$ for $\alpha<\lambda,i\in\kappa\backslash u$ and $\bar\varphi=\langle\psi_i(\bar x',\bar y'_i):i\in\kappa\backslash u\rangle$. Now $\langle\bar b^i_\alpha:\alpha<\lambda,i\in\kappa\backslash u\rangle$ witness that (abusing our notation) $\circledast^2_{\bar\psi}$ holds (the consistency is exemplified by $\bar c\restriction(m-1)$), hence (in the notation of 1.5) $\circledast^1_{\bar\psi[\eta]}$ holds for some $\eta\in{}^{\kappa\backslash u}2$, contradiction to the minimality of m.

Case 2: Not Case 1.

We choose v_ζ by induction on $\zeta<\kappa$ such that

$\bigotimes_\zeta$ (a) $v_\zeta\subseteq\kappa\backslash\bigcup\{v_\varepsilon:\varepsilon<\zeta\}$,

(b) v_ζ is finite,

(c) for some $i\in v_\zeta,\langle\bar a^i_\alpha:\alpha<\lambda\rangle$ is not indiscernible over

$$\bigcup\{\bar a^j_\beta:j\in v_\zeta\backslash\{i\},\beta<\lambda\}\cup\{c_{m-1}\},$$

(d) under (a)+(b)+(c),$|v_\zeta|$ is minimal.

In the induction step, the set $u_\zeta=\cup\{v_\varepsilon:\varepsilon<\zeta\}$ cannot exemplify Case 1, so for some ordinal $i(\zeta)\in\kappa\backslash u_\zeta$ the sequence $\langle\bar a^{i(\zeta)}_\alpha:\alpha<\lambda\rangle$ is not indiscernible over $\bigcup\{\bar a^j_\beta:j\in\kappa\backslash u_\zeta\backslash\{i(\zeta)\}$ and $\beta<\lambda\}\cup\{c_{m-1}\}$, so by the finite character of indiscernibility, there is a finite $v\subseteq\kappa\backslash u_\zeta\backslash\{i(\zeta)\}$ such that $\langle\bar a^{i(\zeta)}_\alpha:\alpha<\lambda\rangle$ is not indiscernible over $\bigcup\{\bar{a}^j_\beta : j \in v, \beta < \lambda\} \cup \{c_{m-1}\}$. So $v' = \{i(\zeta)\} \cup v$ satisfies (a)+(b)+(c), hence some finite $v_\zeta \subseteq \kappa \backslash u_\zeta$ satisfies clauses (a),(b),(c) and (d).

Having carried the induction let $i_*(\zeta) \in v_\zeta$ exemplify clause (c). We can find a sequence $\bar{d}_\zeta$ from $\bigcup\{\bar{a}^j_\beta : j \in v_\zeta \backslash \{i_*(\zeta)\}$ and $\beta < \lambda\}$ such that $\langle \bar{a}^{i_*(\zeta)}_\alpha : \alpha < \lambda\rangle$ is not indiscernible over $\langle c_{m-1}\rangle{}^\frown \bar{d}_\zeta$.

Also, we can find $n(\zeta) < \omega$ and ordinals $\beta_{\zeta,\ell,0} < \beta_{\zeta,\ell,1} < \cdots < \beta_{\zeta,\ell,n(\zeta)-1} < \lambda$ for $\ell = 1, 2$ such that the sequences

$$\bar{d}{}^\frown \bar{a}^{i_*(\zeta)}_{\beta_\zeta,1,0}{}^\frown \cdots {}^\frown \bar{a}^{i_*(\zeta)}_{\beta_\zeta,1,n(\zeta)-1} \text{ and } \bar{d}{}^\frown \bar{a}^{i_*(\zeta)}_{\beta_\zeta,2,0}{}^\frown \cdots {}^\frown \bar{a}^{i_*(\zeta)}_{\beta_\zeta,2,n(\zeta)-1}$$

realize different types over c_{m-1}.

Now we consider $\bar{a}^{i_*(\zeta)}_\beta{}^\frown \cdots {}^\frown \bar{a}^{i_*(\zeta)}_{\beta+n(\zeta)-1}$ where

$$\beta := \max\{\beta_{\zeta,1,n(\zeta)-1} + 1, \beta_{\zeta,2,n(\zeta)-1} + 1\},$$

so renaming, without loss of generality $\beta_{\zeta,1,n(\zeta)-1} < \beta_{\zeta,2,0}$. Omitting some $a^{i_*(\zeta)}_\beta$'s, without loss of generality $\beta_{\beta_\zeta,1,m} = m, \beta_{\zeta,2,m} = n(\zeta)+m$ for $m < n(\zeta)$. Now we define $\bar{b}^\zeta_\beta := \bar{d}_\zeta{}^\frown \bar{a}^{i_*(\zeta)}_{n(\zeta)\beta}{}^\frown \cdots {}^\frown \bar{a}^{i_*(\zeta)}_{n(\zeta)\beta+n(\zeta)-1}$ for $\beta < \lambda, \zeta < \kappa$.

By the indiscernibility of $\langle \bar{a}^{i_\zeta(*)}_\gamma : \gamma < \lambda\rangle$ over $\bar{d}_\zeta \cup \bigcup\{\bar{a}^j_\beta : j \in \kappa\backslash v_\zeta, \beta < \lambda\} \subseteq \bigcup\{a^j_\beta : j \in \kappa\backslash\{i_\zeta(*)\}, \beta < \lambda\}$ we can deduce that $\langle \bar{b}^\zeta_\beta : \beta < \lambda\rangle$ is an indiscernible sequence over $\bigcup\{\bar{b}^\varepsilon_\beta : \varepsilon \in \kappa\backslash\{\zeta\}, \alpha < \gamma$ and $\beta < \lambda\}$. But by an earlier sentence $\bar{b}^\zeta_0, \bar{b}^\zeta_1$ realizes different types over c_{m-1}, so we can choose $\varphi'_\zeta(x, \bar{y}_\zeta)$ such that $\mathfrak{C} \models \varphi'_\zeta(c_{m-1}, \bar{b}^\zeta_0) \wedge \neg\varphi'_i(c_{m-1}, \bar{b}^i_1)$ for $i < \kappa$.

So $\langle \bar{b}^\zeta_\alpha : \alpha < \omega, \zeta < \kappa\rangle$ and $\bar{\varphi}' = \langle \varphi'_\zeta(x, \bar{y}_\zeta) : \zeta < \kappa\rangle$ satisfy the demands on $\langle \bar{a}^i_k : k < \omega, i < \kappa\rangle, \langle \varphi_i(x, \bar{y}_i) : i < \kappa\rangle$ in $\circledast^2_{\bar{\varphi}}$ *for* $m = 1$ (by 1.5's notation), so by 1.5 also $\circledast^1_{\bar{\varphi}[\eta]}$ holds for some $\eta \in {}^\kappa 2$, so we are done.

(2) Implicit in the proof of part (1) (and see Case 1 in the proof of 2.1).

(3) Trivial. $\blacksquare_{1.7}$

A relative of $\kappa_{\rm ict}(T)$ is

1.8. *Definition:* (1) $\kappa_{\rm icu}(T) = \kappa_{{\rm icu},1}(T)$ is the minimal κ such that for no $m < \omega$ and $\bar{\varphi} = \langle \varphi_i(\bar{x}_i, \bar{y}_i) : i < \kappa\rangle$ with $\ell g(\bar{x}^i) = m \times n_i$ can we find $\bar{a}^i_\alpha \in {}^{\ell g(\bar{y}_i)}\mathfrak{C}$ for $\alpha < \lambda, i < \kappa$ and $\bar{c}_{\eta,n} \in {}^m\mathfrak{C}$ for $\eta \in {}^\kappa\lambda$ such that:

(a) $\langle \bar{c}_{\eta,n} : n < \omega\rangle$ is an indiscernible sequence over $\bigcup\{\bar{a}^i_\alpha : \alpha < \lambda, i < \kappa\}$,

(b) for each $\eta \in {}^\kappa\lambda$, $\alpha < \lambda$ and $i < \kappa$ we have $\mathfrak{C} \models \varphi_i(\bar{c}_{\eta,0}{}^\frown \cdots {}^\frown \bar{c}_{\eta,n_i-1}, \bar{a}^i_\alpha)^{{\rm if}(\alpha=\eta(i))}$.

(2) If $\bar{\varphi}$ is as in (1), then we say that it witnesses $\kappa < \kappa_{\rm icu}(T)$.

(3) T is strongly^{1,*} dependent if $\kappa_{\rm icu}(T) = \aleph_0$.

1.9. Claim: (1) $\kappa_{icu}(T) \geq \kappa_{ict}(T)$.
(2) *If* $\mathrm{cf}(\kappa) > \aleph_0$ *then* $\kappa_{icu}(T) > \kappa \Leftrightarrow \kappa_{ict}(T) > \kappa$.
(3) *The parallels of 1.4, 1.5, 1.7(2) hold.*[1]

Proof. (1) Trivial.
(2) As in the proof of 1.7.
(3) Similar. $\blacksquare_{1.9}$

* * *

To translate a statement on several indiscernible sequences to one (e.g., in 2.1), one notes:

1.10. *Observation:* Assume that for each $\alpha < \kappa, I_\alpha$ is an infinite linear order, the sequence $\langle \bar{a}_t : t \in I_\alpha \rangle$ is indiscernible over $A \cup \cup\{\bar{a}_t : t \in I_\beta$ and $\beta \in \kappa \backslash \{\alpha\}\}$ (and for notational simplicity $\langle I_\alpha : \alpha < \kappa \rangle$ are pairwise disjoint) and let $I = \Sigma\{I_\alpha : \alpha < \kappa\}, t \in I_\alpha \Rightarrow \ell g(\bar{a}_t) = \zeta(\alpha)$, and lastly for $\alpha \leq \kappa$ we let $\xi(\alpha) = \Sigma\{\zeta(\beta) : \beta < \alpha\}$.

Then there is $\langle \bar{b}_t : t \in I \rangle$ such that

(a) $\ell g(\bar{b}_t) = \xi(\kappa)$,
(b) $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence over A,
(c) $t \in I_\alpha \Rightarrow \bar{a}_t = \bar{b}_t \restriction [\xi_\alpha, \xi_\alpha + \zeta_\alpha)$,
(d) if $C \subseteq \mathfrak{C}$ and $\mathscr{P}$ is a set of cuts of I such that [J is a convex subset of I not divided by any member of $\mathscr{P} \Rightarrow \langle \bar{b}_t : t \in J \rangle$ is indiscernible over $A \cup C$] *then* we can find $\langle \mathscr{P}_\alpha : \alpha < \kappa \rangle, \mathscr{P}_\alpha$ is a set of cuts of I_α such that $\Sigma\{|\mathscr{P}_\alpha| : \alpha < \kappa\} = |\mathscr{P}|$ and, if $\alpha < \kappa, J$ is a convex subset of I_α not divided by any member of $\mathscr{P}_\alpha$, then $\langle \bar{a}_t : t \in J \rangle$ is indiscernible over $A \cup C$,
(e) if $C \subseteq \mathfrak{C}$ and $\mathscr{P}$ is a set of cuts of I such that [J is a convex subset of I not divided by any member of $\mathscr{P} \Rightarrow \langle \bar{b}_t : t \in J \rangle$ is indiscernible over $A \cup C \cup \{b_s : s \in I \backslash J\}$] *then* we can find $\langle \mathscr{P}_\alpha : \alpha < \kappa \rangle, \mathscr{P}_\alpha$ is a set of cuts of I_α such that $\Sigma\{|\mathscr{P}_\alpha| : \alpha < \kappa\} = |\mathscr{P}|$ and, if $\alpha < \kappa, J$ is a convex subset of I_α not divided by any member of $\mathscr{P}_\alpha$, then $\langle \bar{a}_t : t \in J \rangle$ is indiscernible over $A \cup C \cup \{\bar{a}_t : t \in I \backslash J\}$,
(f) moreover, in clauses (d), (e) we can choose $\mathscr{P}_\alpha$ as the set of non-trivial cuts of I_α induced by $\mathscr{P}$, i.e., $\{(J' \cap I_\alpha, J'' \cap I_\alpha) : (J', J'') \in \mathscr{P}\} \backslash \{(I_\alpha, \emptyset), (\emptyset, I_\alpha)\}$.

[1] And of course more than 1.7(2), using an indiscernible sequence of m_*-tuples, for any $m_* < \omega$.

Proof. Straightforward; e.g.:

Without loss of generality $\langle I_\alpha : \alpha < \kappa\rangle$ are pairwise disjoint and let $I = \Sigma\{I_\alpha : \alpha < \kappa\}$. We can find $\bar{b}^\alpha_t \in {}^{\zeta(\alpha)}\mathfrak{C}$ for $t \in I, \alpha < \kappa$ such that: if $n < \omega$, $\alpha_0 < \cdots < \alpha_{n-1} < \kappa, t^\ell_0 <_I \cdots <_I t^\ell_{k_\ell - 1}$ and $s^\ell_0 <_{I_{\alpha_\ell}} \cdots <_{I_{\alpha_\ell}} s^\ell_{k_\ell - 1}$ for $\ell < n$, then the sequence $(\bar{b}^{\alpha_0}_{t^0_0}{}^\frown \cdots {}^\frown \bar{b}^{\alpha_0}_{t^0_{k_0-1}})^\frown \cdots {}^\frown(\bar{b}^{\alpha_{n-1}}_{t^{n-1}_0}{}^\frown \cdots {}^\frown \bar{b}^{\alpha_{n-1}}_{t^{n-1}_{k_{n-1}-1}})$ realizes the same type as the sequence $(\bar{a}^\alpha_{s^0_0}{}^\frown \cdots {}^\frown \bar{a}^{\alpha_0}_{s^0_{k_n-1}})^\frown \cdots {}^\frown(\bar{a}^{\alpha_{n-1}}_{s^{n-1}_0}{}^\frown \cdots {}^\frown \bar{a}^{\alpha_{n-1}}_{s^{n-1}_{k_{n-1}-1}})$; this is possible by compactness. Using an automorphism of $\mathfrak{C}$, without loss of generality $t \in I_\alpha \Rightarrow \bar{b}^\alpha_t = \bar{a}^\alpha_t$. Now for $t \in I$ let $\bar{a}^*_t$ be $(\bar{a}^0_t{}^\frown \bar{a}^1_t{}^\frown \cdots {}^\frown \bar{a}^1_\alpha \cdots)_{\alpha<\kappa}$.

Clauses (a)+(b)+(c) hold trivially and clauses (d), (e), (f) follow. $\blacksquare_{1.10}$

* * *

In the following we consider "natural" examples which are strongly dependent; see more in 2.5.

1.11. Claim: (1) *Assume T is a complete first order theory of an ordered abelian group expanded by some individual constants and some unary predicates $P_i (i < i(*))$ which are subgroups and T has elimination of quantifiers.*

T is strongly dependent iff *we cannot find $i_n < i(*)$ and $\iota_n \in \mathbb{Z}\backslash\{0\}$ for $n < \omega$ such that:*

- $(*)$ *we can find $b_{n,\ell} \in \mathfrak{C}$ for $n, \ell < \omega$ such that*
  - (a) $\ell_1 < \ell_2 \Rightarrow \iota_n(b_{n,\ell_2} - b_{n,\ell_1}) \notin P^{\mathfrak{C}}_{i_n}$,
  - (b) *for every $\eta \in {}^\omega\omega$ there is c_η such that $c_\eta - b_{n,\eta(n)} \in P^{\mathfrak{C}}_{i_n}$ for $n < \omega$.*

(2) *Let M be $(\mathbb{Z}, +, -, 0, 1, <, P_n)$ where $P_n = \{na : a \in \mathbb{Z}\}$, so we know that $T = \mathrm{Th}(M)$ has elimination of quantifiers.* Then *T is strongly dependent, hence* $\mathrm{Th}(\mathbb{Z}, +, -, 0, <)$ *is strongly dependent.*

1.12. *Remark:* (1) This generalizes the parallel theorem for stable abelian groups.

(2) Note that if G is the ordered abelian group with sets of elements $\mathbb{Z}[x]$, addition of $\mathbb{Z}[x]$ and $p(x) > 0$ iff the leading coefficient is > 0, in $\mathbb{Z}, P_n$ as above (so definable), then $\mathrm{Th}(G)$ is not strongly dependent using P_n for n prime.

(3) On elimination of quantifiers for ordered abelian groups, see Gurevich [Gu77].

Proof. (1) The main point is the if direction. We use the criterion from 2.1(2),(4) below. So let $\langle \bar{a}_t : t \in I\rangle$ be an infinite indiscernible sequence and $c \in \mathfrak{C}$

(with $\bar{a}_t$ not necessarily finite). Without loss of generality $\mathfrak{C} \models$ "$c > 0$" and $\bar{a}_t = \langle a_{t,\alpha} : \alpha < \alpha^* \rangle$ list the members of M_t, a model and even a $|T|^+$-saturated model (see 2.1(4)), and let $p_t = \operatorname{tp}(c, M_t)$.

Note that

$(*)_1$ if $a_{s,i} = a_{t,j}$ and $s \neq t$, then $\langle a_{r,i} : r \in I \rangle$ is constant.

Obviously, without loss of generality, $c \notin \bigcup\{M_t : t \in I\}$ but $\mathfrak{C}$ is torsion free (as an abelian group because it is ordered), hence

$(*)_2$ $\iota \in \mathbb{Z}\backslash\{0\} \Rightarrow \iota c \notin \bigcup\{M_t : t \in I\}$,

$(*)_3$ for $t \in I, a \in M_t$ and $\iota \in \mathbb{Z}\backslash\{0\}$, let $\eta^\iota_a \in {}^{i(*)+1}2$ be such that $[\eta^\iota_a(i(*)) = 1 \Leftrightarrow \iota c > a]$ and, for $i < i(*), [\eta^\iota_a(i) = 1 \Leftrightarrow \iota c - a \in P^{\mathfrak{C}}_i]$,

$(*)_4$ for $t \in I$ and $a \in M_t$ let $p_a := \bigcup_{\iota \in \mathbb{Z}\backslash\{0\}}(p^\iota_a \cup q^\iota_a)$ where [2] $p^\iota_a(x) := \{\iota x \neq a, (\iota x > a)^{\eta^\iota_a(i(*))}\}$ and $q^\iota_a(x) := \{P_i(\iota x - a)^{\eta^\iota_a(i)} : i < i(*)\}$.

Now

$\boxdot_0$ for $\iota \in \mathbb{Z}\backslash\{0\}$ and $\alpha < \alpha^*$ let $I^\iota_\alpha = \{t \in I : a_{t,\alpha} < \iota c\}$;

$\boxdot_1$ $\langle u_{-1}, u_0, u_1 \rangle$ is a partition of α^*, where

(a) $u_{-1} = \{\alpha < \alpha^*$: for every $s <_I t$ we have $\mathfrak{C} \models a_{t,\alpha} < a_{s,\alpha}\}$,

(b) $u_0 = \{\alpha < \alpha^*$: for every $s <_I t$ we have $\mathfrak{C} \models a_{s,\alpha} = a_{t,\alpha}\}$,

(c) $u_1 = \{\alpha < \alpha^*$: for every $s <_I t$ we have $\mathfrak{C} \models a_{s,\alpha} < a_{t,\alpha}\}$;

$\boxdot_2$ if $\iota \in \mathbb{Z}\backslash\{0\}$ *then*

(a) I^ι_α is an initial segment of I when $\alpha \in u_1$,

(b) I^ι_α is an end segment of I when $\alpha \in u_{-1}$,

(c) $I^\iota_\alpha \in \{\emptyset, I\}$ when $\alpha \in u_0$,

(d) $\{I^\iota_\alpha : \alpha \in u_1\}\backslash\{\emptyset, I\}$ has at most 2 members.

[Why? Recall $<^{\mathfrak{C}}$ is a linear order. So for each $\iota \in \mathbb{Z}\backslash\{0\}, \alpha \in u_1$, by the definition of u_1 the set $I^\iota_\alpha := \{t \in I : a_{t,\alpha} < \iota c\}$ is an initial segment of I, also $t \in I\backslash I^\iota_\alpha \Rightarrow \iota c <^{\mathfrak{C}} a_{t,\alpha}$ as $c \notin \bigcup\{M_s : s \in I\}$ by $(*)_2$.

Now suppose $\alpha, \beta \in u_1$ and $|I^\iota_\beta \backslash I^\iota_\alpha| > 1$ and $I^\iota_\alpha, I^\iota_\beta \notin \{\emptyset, I\}$; then choose $t_1 <_I t_2$ from $I^\iota_\beta \backslash I^\iota_\alpha$ and $t_0 \in I^\iota_\alpha, t_3 \in I\backslash I^\iota_\beta$. As $I^\iota_\alpha, I^\iota_\beta$ are initial segments and $t_0 <_I t_1 <_I t_2 <_I t_3$, necessarily $\mathfrak{C} \models$ "$a_{t_0,\alpha} < \iota c < a_{t_1,\alpha} \wedge a_{t_2,\beta} < \iota c < a_{t_3,\beta}$". If $a_{t_1,\alpha} \leq^{\mathfrak{C}} a_{t_2,\beta}$ we can deduce a contradiction ($\mathfrak{C} \models$ "$\iota c < a_{t_1,\alpha} \leq a_{t_2,\beta} < \iota c$"). Otherwise, by the indiscernibility of the sequence $\langle (a_{t,\alpha}, a_{t,\beta}) : t \in I \rangle$ we get $\mathfrak{C} \models a_{t_3,\beta} < a_{t_0,\alpha}$ and a similar contradiction. So $|I^\iota_\beta \backslash I^\iota_\alpha| \leq 1$.

[2] Recall that $\varphi^1 = \varphi, \varphi^0 = \neg\varphi$.

So $I^\iota_\alpha, I^\iota_\beta \notin \{\emptyset, I\} \Rightarrow |I^\iota_\beta \backslash I^\iota_\alpha| \le 1$ and by symmetry $|I^\iota_\alpha \backslash I^\iota_\beta| \le 1$. So $|\{I^\iota_\alpha : \alpha \in u_1\} \backslash \{\emptyset, I\}| \le 2$, i.e., clause (d) of $\boxdot_2$ holds; the other clauses should be clear.]

Now clearly

$\boxdot_3$ if $\alpha, \beta < \alpha(*), \iota \in \mathbb{Z}\backslash\{0\}$ and $a_{t,\alpha} = -a_{t,\beta}$ (for some equivalently for every $t \in I$) *then*:

(a) $(\alpha \in u_1) \equiv (\beta \in u_{-1})$,

(b) $((\iota c) < a_{t,\alpha}) \equiv (a_{t,\beta} < ((-\iota)c))$ recalling $\iota c, (-\iota)c \notin \bigcup_{t \in I} M_t$,

(c) $I^\iota_\alpha = I \backslash I^\iota_\beta$.

Also

$\boxdot_4$ if ι_1, ι_2 are from $\{1, 2, \dots\}$ and $\iota_1 a_{t,\alpha} = \iota_2 a_{t,\beta}$ *then*

(a) $[\alpha \in u_{-1} \equiv \beta \in u_{-1}], [\alpha \in u_0 \equiv \beta \in u_0]$ and $[\alpha \in u_1 \equiv \beta \in u_1]$,

(b) $(t \in I^{\iota_2}_\alpha) \Leftrightarrow (t \in I^{\iota_1}_\beta)$, hence $I^{\iota_2}_\alpha = I^{\iota_1}_\beta$.

[Why? Clause (a) is obvious. For clause (b) note that $t \in I^{\iota_2}_\alpha \Leftrightarrow a_{t,\alpha} < \iota_2 c \Leftrightarrow \iota_1 a_{t,\alpha} < \iota_1(\iota_2 c) \Leftrightarrow \iota_2 a_{t,\beta} < \iota_2(\iota_1 c) \Leftrightarrow a_{t,\beta} < \iota_1 c \Leftrightarrow t \in I^{\iota_1}_\beta$.]

By symmetry, i.e., by $\boxdot_3$, clearly

$\boxdot_5$ the statement (d) in $\boxdot_2$ holds for $\alpha \in u_{-1}$.

Obviously

$\boxdot_6$ if $\alpha \in u_0$ then $I^\iota_\alpha \in \{\emptyset, I\}$.

Together

$\boxdot_7$ $\{I^\iota_\alpha : \alpha < \alpha^*$ and $\iota \in \mathbb{Z}\backslash\{0\}\}\backslash\{\emptyset, I\}$, hence has ≤ 4 members.

Hence

$\circledast_0$ There are initial segments J_ℓ of I for $\ell < \ell(*) \le 4$ such that: if s, t belongs to I and $\ell < \ell(*) \Rightarrow [s \in J_\ell \equiv t \in J_\ell]$ *then* $\eta^\iota_{a_{t,\alpha}}(i(*)) = \eta^\iota_{a_{s,\alpha}}(i(*))$.

[Why? By the above and the definition of $\eta^\iota_{a_{t,\alpha}}(i(*))$ we are done.]

$\circledast_1$ For each $t \in I$ we have $\bigcup\{p_a(x) : a \in M_t\} \vdash p_t(x)$.

[Why? Use the elimination of quantifiers and the closure properties of M_t. That is, every formula in $p_t(x)$ is equivalent to a Boolean combination of quantifier free formulas. So it suffices to deal with the cases $\varphi(x, \bar{a}) \in p_t(x)$ which is atomic or negation of atomic and x appear. As for $b_1, b_2 \in \mathfrak{C}$, exactly one of the possibilities $b_1 < b_2, b_1 = b_2, b_2 < b_1$ holds, and, by symmetry, it suffices to deal with $\sigma_1(x, \bar{a}) > \sigma_2(x, \bar{a}), \sigma_1(x, \bar{a}) = \sigma_2(x, \bar{a}), P_i(\sigma(x, \bar{a})), \neg P_i(\sigma(x, \bar{a}))$ where $\sigma(x, \bar{y}), \sigma_1(x, \bar{y}), \sigma_2(x, \bar{y})$ are terms in $\mathbb{L}(\tau_T)$. As we can substract, it suffices to deal with $\sigma(x, \bar{a}) > 0, \sigma(x, \bar{a}) = 0, P_i(\sigma(x, \bar{a})), \neg P_i(\sigma(x, \bar{a}))$. By linear algebra, as M_t is closed under the operations, without loss of generality $\sigma(x,\bar a) = \iota x - a_{t,\alpha}$ for some $\iota \in \mathbb{Z}$ and $\alpha < \alpha^*$, and without loss of generality $\iota \neq 0$. The case $\varphi(x) = (\iota x - a_{t,\alpha} = 0) \in p(x)$ implies $c \in M_t$ (as M is torsion free), which we assume does not hold. In the case $\varphi(x,\bar a) = (\iota x - a_{t,\alpha} > 0)$ use $p^\iota_{a_{t,\alpha}}(x)$, in the case $\varphi(x,\bar a) = P_i(\iota x - a_{t,\alpha})$ or $\varphi(x,\bar a) = \neg P_i(\iota x - a_{t,\alpha})$ use $q^\iota_{a_{t,\alpha}}(x)$ for $\eta^\iota_{a_{t,\alpha}}(i)$.]

$\circledast_2$ If $\iota \in \mathbb{Z}\backslash\{0\}, n < \omega$ and $a_0,\dots,a_{n-1} \in M_t$, then for some $a \in M_t$ we have $\ell < n \wedge i < i(*) \wedge \eta^\iota_{a_\ell}(i) = 1 \Rightarrow \eta^\iota_a(i) = 1$.

[Why? Let $a' \in M_t$ realize $p_t \restriction \{a_0,\dots,a_{n-1}\}$, exist as M_t was chosen as $|T|^+$-saturated; less is necessary. Now $\iota c - a_\ell \in P^{\mathfrak{C}}_i \Rightarrow \iota a' - a_\ell \in P^{\mathfrak{C}}_i \Rightarrow (\iota c - \iota a') = ((\iota c - a_\ell) - (\iota a' - a_\ell)) \in P^{\mathfrak{C}}_i$ and let $a := \iota a'$.]

$\circledast_3$ Assume $\iota \in \mathbb{Z}\backslash\{0\}, i < i(*), \alpha < \alpha^*, s_1 <_I s_2$ and $t \in I\backslash\{s_1,s_2\}$; then:

(a) if $\eta^\iota_{a_{s_1,\alpha}}(i) = 1$ and $\eta^\iota_{a_{s_2,\alpha}}(i) = 0$, then $\eta^\iota_{a_{t,\alpha}}(i) = 0$,

(b) if $\eta^\iota_{a_{s_1,\alpha}}(i) = 0$ and $\eta^\iota_{a_{s_2,\alpha}}(i) = 1$, then $\eta^\iota_{a_{t,\alpha}}(i) = 0$.

[Why? As we can invert the order of I it is enough to prove clause (a). By the choice of $a \mapsto \eta^\iota_a$ we have $\iota c - a_{s_1,\alpha} \in P^{\mathfrak{C}}_i, \iota c - a_{s_2,\alpha} \notin P^{\mathfrak{C}}_i$, hence $a_{s_1,\alpha} - a_{s_2,\alpha} \notin P^{\mathfrak{C}}_i$, hence also $a_{s_2,\alpha} - a_{s_1,\alpha} \notin P^{\mathfrak{C}}_i$.

By the indiscernibility we have $a_{t,\alpha} - a_{s_1,\alpha} \notin P^{\mathfrak{C}}_i$ and as $\iota c - a_{s_1,\alpha} \in P^{\mathfrak{C}}_i$ we can deduce $\iota c - a_{t,\alpha} \notin P^{\mathfrak{C}}_i$, hence $\eta^\iota_{a_{t,\alpha}}(i) = 0$. So we are done.]

$\circledast_4$ For each $\iota \in \mathbb{Z}\backslash\{0\}, i < i(*)$ and $\alpha < \alpha^*$, the set $I^\iota_{i,\alpha} := \{t : \eta^\iota_{a_{t,\alpha}}(i) = 1\}$ is $\emptyset, I$ or a singleton.

[Why? By $\circledast_3$.]

$\circledast_5$ if $I_* = \bigcup\{I^\iota_{i,\alpha} : \iota \in \mathbb{Z}\backslash\{0\}, i < i(*), \alpha < \alpha^*$ and $I^\iota_{i,\alpha}$ is a singleton$\}$ is infinite, then (possibly inverting I) we can find $t_n \in I$ and $\beta_n < \alpha^*, \iota_n \in \mathbb{Z}\backslash\{0\}$ and $i_n < i(*)$ for $n < \omega$ such that

(a) $t \in I$, then $[\iota_n c - a_{t,\beta_n} \in P^{\mathfrak{C}}_{i_n}] \Leftrightarrow t = t_n$ for every $n < \omega$,

(b) $\langle a_{t,\beta_n} - a_{s,\beta_n} : s \neq t \in I\rangle$ are pairwise not equal mod $P^{\mathfrak{C}}_{i_n}$,

(c) $t_n < t_{n+1}$ for $n < \omega$.

[Why? Should be clear.]

$\circledast_6$ If $I_* = \bigcup\{I^\iota_{i,\alpha} : \iota \in \mathbb{Z}\backslash\{0\}, \alpha < \alpha^*, i < i(*)$ and $I^\iota_{i,\alpha}$ is a singleton$\}$ is finite and $J_\ell (\ell < \ell(*) \leq 4)$ are as in $\circledast_0$, *then* $\mathrm{tp}(\bar a_s, \{c\}) = \mathrm{tp}(\bar a_t, \{c\})$ whenever $(s,t \in I\backslash I_*) \wedge \bigwedge_{\ell<\ell(*)} (s \in J_\ell \equiv t \in J_\ell)$ recalling $\bar a_t$ list the elements of M_t.

[Why? By $\circledast_4$ and $\circledast_1$ (and $\circledast_0$) recalling the choice of p_a in $(*)_4$.]

Assume $c, \langle \bar a_t : t \in I\rangle$ exemplify T is not strongly dependent; then I_* cannot be finite (by $\circledast_6$) hence I_* is infinite, so by $\circledast_5$ we can find $\langle (t_n,\beta_n,\iota_n,i_n) : n < \omega\rangle$ as there.

That is, for $n < \omega, \ell < \omega$ let $b_{n,\ell} := a_{t_\ell,\beta_n}$. So

$\circledast_7$ $\iota_n c - b_{n,\ell} \in P^{\mathfrak{C}}_{i_n}$ *iff* $\iota_n c - a_{t_\ell,\beta_n} \in P^{\mathfrak{C}}_{i_n}$ iff $t_\ell = t_n$ iff $\ell = n$,

$\circledast_8$ if $\ell_1 < \ell_2$ then $b_{n,\ell_2} - b_{n,\ell_2} \notin P^{\mathfrak{C}}_{i_n}$.

[Why? By clause (b) of $\circledast_5$.]

Now

$\circledast_9$ if $\eta \in {}^\omega\omega$ is increasing, then there is $c_\eta \in \mathfrak{C}$ such that $n < \omega \Rightarrow \iota_n c_\eta - b_{n,\eta(n)} \in P^{\mathfrak{C}}_{i_n}$.

[Why? As $\langle \bar{a}_t : t \in I\rangle$ is an indiscernible sequence, there is an automorphism $f = f_\eta$ of $\mathfrak{C}$ which maps $\bar{a}_{t_n}$ to $\bar{a}_{t_{\eta(n)}}$ for $t \in I$, so $f_\eta(b_{\eta,n}) = b_{n,\eta(n)}$. Hence $c_\eta = f_\eta(c)$ satisfies $n < \omega \Rightarrow \iota_n f(c) - b_{\eta,\eta(n)} \in P^{\mathfrak{C}}_{i_n}$.]

Now $\langle b_{n,\ell} : n, \ell < \omega\rangle$ almost satisfies $(*)$ of 1.11. Clause (a) holds by $\circledast_8$ and clause (b) holds for all increasing $\eta \in {}^\omega\omega$. By compactness we can find $\langle \bar{b}'_{n,\ell} : n, \ell < \omega\rangle$ satisfying (a) + (b) of $(*)$ of 1.11.

[Why? Let $\Gamma = \{P_{i_n}(\iota_n x_\eta - y_{n,\eta(n)}) : \eta \in {}^\omega\omega, n < \omega\} \cup \{\neg P_{i_n}(\iota_n x_{n,\ell_1} - \iota_n x_{n,\ell_2}) : n < \omega, \ell_1 < \ell_2 < \omega\}$. If Γ is satisfied in $\mathfrak{C}$ we are done, otherwise there is a finite inconsistent $\Gamma' \subseteq \Gamma$. Let n_* be such that: if $y_{n,\ell}$ appear in Γ' then $n, \ell < n_*$. But the assignment $y_{n,\ell} \mapsto b_{nn_*+\ell}$ for $n < n_*, \ell < n_*$ exemplified that Γ' is realized, so we have proved half of the claim. The other direction should be clear, too.]

(2) The first assertion (on T) holds by part (1); the second holds as the set of terms $\{0, 1, 2, \ldots, n-1\}$ is provably a set of representatives for $\mathbb{Z}/P_n$ which is finite. $\blacksquare_{1.11}$

1.13. *Example:* $\mathrm{Th}(M)$ is not strongly stable when M satisfies the following:

(a) it has universe ${}^\omega\mathbb{Q}$

(b) it is an abelian group as a power of $(\mathbb{Q}, +)$,

(c) it $P^M_n = \{f \in M : f(n) = 0\}$, a subgroup.

We now consider the p-adic fields and more generally valued fields.

1.14. *Definition:* (1) We define a valued field M as one in the Denef–Pas language, i.e., a model M such that:

(a) the elements of M are of three sorts:

  (α) the field P^M_0 which (as usual) we call K^M, so $K = K^M$ is the field of M and has universe P^M_0, so we have appropriate individual constants (for $0, 1$), and the field operations (including the inverse which is partial),

(β) the residue field P_1^M which (as usual) is called k^M, so $k = k^M$ is a field with universe P_1^M, so with the appropriate $0, 1$ and field operations,

(γ) the valuation ordered abelian group P_2^M which (as usual) we call Γ^M, so $\Gamma = \Gamma^M$ is an ordered abelian group with universe P_2^M, so with 0, addition, subtraction and the order;

(b) the functions (and individual constants) of K^M, k^M, Γ^M and the order of Γ^N (actually mentioned in clause (a));

(c) $\mathrm{val}^M : K^M \to \Gamma^M$, the valuation;

(d) $\mathrm{ac}^M : K^M \to k^M$, the function giving the "leading coefficient" (when, as in natural cases, the members of K are power series);

(e) of course, satisfying the sentences saying that the following hold:

(α) Γ^M is an ordered abelian group,

(β) k is a field,

(γ) K is a field,

(δ) val,ac satisfies the natural demands.

(1A) Above we replace "language" by ω-language *when*: in clause (b), i.e., (a)(γ), Γ^M has 1_Γ (the minimal positive elements) and we replace (d) by

$(d)^-_\omega$ $\mathrm{ac}^M_n : K^M \to k^M$ satisfies: $\bigwedge_{\ell<n} \mathrm{ac}^M_\ell(x) = \mathrm{ac}^M_k(y) \Rightarrow \mathrm{val}^M(x-y) > \mathrm{val}^m(x) + n$.

(2) We say that such M (or $\mathrm{Th}(M)$) has elimination of the field quantifier *when*: every first order formula (in the language of $\mathrm{Th}(M)$) is equivalent to a Boolean combination of atomic formulas, formulas about k^M (i.e., all variable, free and bounded vary on P_1^M) and formulas about Γ^M; note this definition requires clause (d) in part (1).

The following is well known (on 1.15 and 1.16 see, e.g., [Pa90], [CLR06]).

1.15. Claim: (1) *Assume* Γ *is a divisible ordered abelian group and* k *is a perfect field of characteristic zero. Let* K *be the field of power series for* (Γ, k), *i.e.,* $\{f : f \in {}^\Gamma k$ *and* $\mathrm{supp}(f)$ *is well ordered*$\}$ *where* $\mathrm{supp}(f) = \{s \in \Gamma : f(s) \neq 0_k\}$. Then *the model defined by* (K, Γ, k) *has elimination of the field quantifiers.*

(2) *For p prime, we can consider the p-adic field as a valued field in the Denef–Pas ω-language and its first order theory has elmination of the field quantifiers (this version of the p-adics and the original one are (first-order) bi-interpretable; note that the field k here is finite and formulas speaking on Γ which is the ordered abelian group $\mathbb{Z}$ are well understood).*

We will actually be interested only in valuation fields M with elimination of the field quantifiers. The following is well known.

1.16. Claim: *Assume $\mathfrak{C} = \mathfrak{C}_T$ is a (monster, i.e., quite saturated) valued field in the Denef–Pas language (or in the ω-language) with elimination of the field quantifiers. If $M \prec \mathfrak{C}$ then:*

(a) *it satisfies the cellular decomposition of Denef which implies* [3]*: if $p \in \mathbf{S}^1(M)$ and $P_0(x) \in p$ then p is equivalent to $p^{[*]} := \bigcup\{p_c^{[*]} : c \in P_0^M\}$ where $p_c^{[*]} = p_c^{[*,1]} \cup p_c^{[*,2]}$ and $p_c^{[*,1]} = \{\varphi(\mathrm{val}(x-c), \bar{d}) \in p : \varphi(x, \bar{y})$ is a formula speaking on Γ^M only so $\bar{d} \subseteq \Gamma^M, c \in P_0^M\}$ and $p_c^{[*,2]} = \{\varphi(\mathrm{ac}(x-c), \bar{d}) \in p : \varphi$ speaks on k^M only$\}$, but for the ω-language we should allow $\varphi(\mathrm{ac}_0(x-c), \dots, \mathrm{ac}_n(x-c), \bar{d})$ for some $n < \omega$;*

(b) *if $p \in \mathbf{S}^1(M), P_0(x) \in p$ and $c_1, c_2 \in P_0^M$ and $\mathrm{val}^M(x-c_1) <^{\Gamma^M} \mathrm{val}^M(x-c_2)$ belongs to $p(x)$ then $p_{c_2}^{[*]}(x) \vdash p_{c_1}^{[*]}(x)$ and even $\{\mathrm{val}(x-c_1) < \mathrm{val}(x-c_2)\} \vdash p_{c_1}^{[*]}(x)$;*

(c) *for $\bar{c} \in {}^{\omega>}(k^M)$, the type $\mathrm{tp}(\bar{c}, \emptyset, k^M)$ determines $\mathrm{tp}(\bar{c}, \emptyset, M)$, and similarly for Γ^M.*

1.17. Claim: (1) *The first order theory T of the p-adic field is strongly dependent.*

(2) *For the theory T of a valued field $\mathbb{F}$ which has elimination of the field quantifier we have: T is strongly dependent iff the theory of the valued ordered group and the theory of the residue fields of $\mathbb{F}$ are strongly dependent.*

(3) *Like (2), when we use the ω-language and we assume k^M is finite.*

1.18. *Remark:* (1) In 1.17 we really get that T is strongly dependent over the residue field + the valuation ordered abelian group.

(2) We had asked in a preliminary version of [Sh:783, §3]: show that the theory of the p-adic field is strongly dependent. Udi Hrushovski has noted that the criterion $(\mathrm{St})_2$ presented there (and repeated in 0.1 here from [Sh:783, 3.10=ss.6]) apply, so T is not strongly^2 dependent. Namely, take the following equivalence relation E on $\mathbb{Z}_p$:$\mathrm{val}(x-y) \geq \mathrm{val}(c)$, where c is some fixed element with infinite valuation. Given x, the map $y \mapsto (x+cy)$ is a bijection between $\mathbb{Z}_p$ and the class x/E.

(3) By [Sh:783, §3], the theory of real closed fields, i.e., $\mathrm{Th}(\mathbb{R})$ is strongly dependent. Onshuus shows that also the theory of the field of the reals is not

[3] Note: $p \in \mathbf{S}^1(A, M), A \subseteq M$ is a little more complicated.

strongly[2] dependent (e.g., though Claim [Sh:783, 3.10=ss.6] does not apply but its proof works using pairwise, not too near $\bar{b}$'s, in general just an uncountable set of $\bar{b}$'s).

(4) See more in §5.

Of course,

1.19. *Observation:* (1) For a field K, $\mathrm{Th}(K)$ being strongly dependent is preserved by finite extensions in the field theoretic sense by 1.4(2).

(2) In 1.17, if we use the ω-language and k^N is infinite, the theory is not strongly dependent.

Proof. (1) Recall that by 1.11(2), the theory of the valued group (which is an ordered abelian group) is strongly dependent, and this holds trivially for the residue field being finite. So by 1.15(2) we can apply part (3).

(2) We consider the models of T as having three sorts: P_0^M the field, P_1^M the ordered abelian group (like value of valuations) and P_2^M the residue field.

Let

$\boxdot_1$ (a) I be an infinite linear order, without loss of generality complete and dense (and with no extremal members),

(b) $\langle \bar{a}_t : t \in I\rangle$ be an indiscernible sequence, $\bar{a}_t \in {}^\alpha\mathfrak{C}$ and let $c \in \mathfrak{C}$ (a singleton!),

and we shall prove

$\boxdot_2$ for some finite $J \subseteq I$ we have: if $s, t \in I \backslash J$ and $(\forall r \in J)(r <_I s \equiv r <_I t)$, then $\bar{a}_s, \bar{a}_t$ realizes the same type over $\{c\}$.

This suffices by 2.1 and, as there, by 2.1(4) without loss of generality

$\boxdot_3$ $\bar{a}_t = \langle a_{t,i} : i < \alpha\rangle$ list the elements of an elementary submodel M_t of $\mathfrak{C} = \mathfrak{C}_T$ (we may assume M_t is $\aleph_1$-saturated; alternatively we could have assumed that it is quite complete).

It easily follows that it suffices to prove (by the L.S.T. argument, but not used)

$\boxdot_2'$ for every countable $u \subseteq \alpha$ there is a finite $J \subseteq I$ which is O.K. for $\langle \bar{a}_t \restriction u : t \in I\rangle$.

Let $\mathbf{f}_{t,s}$ be the mapping $a_{s,i} \mapsto a_{t,i}$ for $i < \alpha$; clearly it is an isomorphism from M_s onto M_s.

Now

$\boxdot_4$ $p_t = \mathrm{tp}(c, M_t)$, so $(p_t)_a^{[*]}$ for $a \in M_t$ is well defined in 1.16(a).

The case $P_2(x) \in \bigcap_t p_t$ is easy and the case $P_1(x) \in \bigcap_t p_t$ is easy, too, by an assumption (and clause (c) of 1.16), so we can assume $P_0(x) \in \bigcap_t p_t(x)$.

Let $\mathscr{U} = \{i < \alpha : a_{s,i} \in P_0^{\mathfrak{C}}$ for every ($\equiv$ some) $s \in I\}$.

Now for every $i \in \mathscr{U}$:

$(*)^1_i$ The function $(s,t) \mapsto \mathrm{val}^{\mathfrak{C}}(a_{t,i} - a_{s,i})$ for $s <_I t$ satisfies one of the following:

Case (a)^1_i: it is constant.

Case (b)^1_i: it depends just on s and is a strictly monotonic (increasing, by $<_\Gamma$) function of s.

Case (c)^1_i: it depends just on t and is a strictly monotonic (decreasing, by $<_\Gamma$) function of t.

[Why? This follows by inspection or see the proof of $(*)^2_{i,j}$ below.]

For $\ell = -1, 0, 1$ let $\mathscr{U}_\ell := \{i \in \mathscr{U}$: if $\ell = 0, 1, -1$ then case (a)^1_i, (b)^1_i, (c)^1_i respectively of $(*)^1_i$ holds$\}$, so $\langle \mathscr{U}_{-1}, \mathscr{U}_0, \mathscr{U}_1\rangle$ is a partition of $\mathscr{U}$.

For $i, j \in \mathscr{U}_1$ we shall prove more:

$(*)^2_{i,j}$ We have $i, j \in \mathscr{U}_1$, and the function $(s,t) \mapsto \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i})$ for $s <_I t$ satisfies one of the following:

Case (a)$^2_{i,j}$: $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i})$ is constant.

Case (b)$^2_{i,j}$: $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i})$ depends only on s and is a monotonic (increasing) function of s and is equal to $\mathrm{val}^{\mathfrak{C}}(a_{s_1,i} - a_{s,i})$ when $s <_I s_1$.

Case (c)$^2_{i,j}$: $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i})$ depends only on t and is a monotonic (increasing) function of t and is equal to $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{t_1,j})$ when $t <_I t_1$.

[Why does $(*)^2_{i,j}$ hold? In this case we give a full check.

First, assume: for some (equivalently every) $t \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i})$: s satisfies $s <_I t\rangle$ is $<_\Gamma$-decreasing with s recalling that we have assumed I is a linear order with neither first nor last element. Choose $s_1 <_I s_2 <_I t$, so by the present assumption we have $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s_2,i}) <_\Gamma \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s_1,i})$, hence $\mathrm{val}^{\mathfrak{C}}((a_{t,j} - a_{s_2,i}) - (a_{t,j} - a_{s_1,i})) = \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s_2,i})$, which means $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s_2,i}) = \mathrm{val}^{\mathfrak{C}}(-(a_{s_2,i} - a_{s_1,i})) = \mathrm{val}^{\mathfrak{C}}(a_{s_2,i} - a_{s_1,i})$. So in the right side t does not appear, in the left side s_1 does not appear, hence by the equality the left side, $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s_2,i})$, does not depend on t and the right side, $\mathrm{val}^{\mathfrak{C}}(a_{s_2,i} - a_{s_1,i})$, does not depend on s_1, but as $i \in \mathscr{U}_1$ it does not depend on s_2. Together, by the indiscernibility for $s <_I t$ we have $\mathrm{val}^{\mathfrak{C}}(a_{t,i} - a_{s,i})$ is constant, i.e., case $(a)^2_{i,j}$ holds. So we can from now on assume: for each $t \in I$

the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : s$ satisfies $s <_I t\rangle$ is constant or for each $t \in I$ it is $<_\Gamma$-increasing with s.

Second, assume: for some (equivalently every) $s \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : t$ satisfies $s <_I t\rangle$ is $<_\Gamma$-decreasing with t. As above in the "first" situation, we can show that case $(a)^2_{i,j}$ holds. So from now on we can assume that for every $s \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : t$ satisfies $s <_I t\rangle$ is constant or for every $s \in I$ the sequence is $<_\Gamma$-increasing with s.

Third, assume: for some (equivalently every) $t \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : s$ satisfies $s <_I t\rangle$ is constant. This implies that $s <_I t \Rightarrow \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) = e_t$ for some $\bar{e} = \langle e_t : t \in I\rangle$. If for some (equivalently every) $s \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : t$ satisfies $s <_I t\rangle$ is constant, then clearly case $(a)^2_{i,j}$ holds, so we can assume this fails; so by the end of the "second" situation this sequence is $<_\Gamma$-increasing, hence $\langle e_t : t \in I\rangle$ is $<_\Gamma$-increasing. So most of the requirements in case $(c)^2_{i,j}$ hold; still we have to show that $t <_I t_1 \Rightarrow \mathrm{val}(a_{t,j} - a_{t_1,j}) = e_t$.

Let $s <_I t <_I t_1$. We know that $e_t <_\Gamma e_{t_1}$, which means that $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) <_\Gamma \mathrm{val}^{\mathfrak{C}}(a_{t_1,j} - a_{s,i})$. This implies that $\mathrm{val}^{\mathfrak{C}}((a_{t,j} - a_{s,i}) - (a_{t_1,j} - a_{s,i})) = \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i})$, which means that $\mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{t_1,j}) = \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) = e_t$ as required; so case $(c)^2_{i,j}$ holds and we are done (if the "third" situation holds).

Fourth, assume that for some (equivalently every) $s \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : t$ satisfies $s <_I t\rangle$ is constant. Then we proceed as in the "third" situation, getting case $(b)^2_{i,j}$ instead of case $(c)^2_{i,j}$.

So assume that none of the above occurs. Hence for every (equivalently some) $t \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : s$ satisfies $s <_I t\rangle$ is $<_\Gamma$-increasing (with s, by the "first" and "third" situations above), and for every (equivalently some) $s \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) : t$ satisfies $s <_I t\rangle$ is $<_\Gamma$-increasing (with t, by the "second" and "fourth" situations above).

Hence we have $s <_I t_1 <_I t_2 \Rightarrow \mathrm{val}^{\mathfrak{C}}(a_{t_1,j} - a_{s,i}) <_\Gamma \mathrm{val}^{\mathfrak{C}}(a_{t_2,j} - a_{s,i}) \Rightarrow \mathrm{val}^{\mathfrak{C}}(a_{t_1,j} - a_{s,i}) = \mathrm{val}^{\mathfrak{C}}((a_{t_2,j} - a_{s,i}) - (a_{t_1,j} - a_{s,i})) = \mathrm{val}(a_{t_2,j} - a_{t_1,j})$, hence $\mathrm{val}^{\mathfrak{C}}(a_{t_1,j} - a_{s,i})$ does not depend on s as s does not appear on the left side; but (see above) it is $<_\Gamma$-increasing with s, contradiction. So we have finished proving $(*)^2_{i,j}$.]

$(*)^3_i$ For each $i \in \mathscr{U}_1$, for some $t^*_i \in \{-\infty\} \cup I \cup \{+\infty\}$ we have:
(a)^3_i $\mathrm{val}^{\mathfrak{C}}(c - a_{s,i}) = \mathrm{val}^{\mathfrak{C}}(a_{t,i} - a_{s,i})$ when $s <_I t$ and $s \in I_{<t^*_i}$,
(b)^3_i $\langle \mathrm{val}^{\mathfrak{C}}(c - a_{s,i}) : s \in I_{>t^*_i}\rangle$ is constant, and if $r \in I_{>t^*_i}$ and $s <_I t$

are from $I_{>t^*_i}$ then $\mathrm{val}^{\mathfrak{C}}(c - a_{r,i}) <_\Gamma \mathrm{val}^{\mathfrak{C}}(a_{t,i} - a_{s,i})$,

(c)^3_i $\mathrm{ac}^{\mathfrak{C}}(c - a_{s,i}) = \mathrm{ac}^{\mathfrak{C}}(a_{t,i} - a_{s,i})$ when $s <_I t$ and $s \in I_{<t^*_i}$,

(d)^3_i $\langle \mathrm{ac}^{\mathfrak{C}}(c - a_{s,i}) : s \in I_{>t^*_i}\rangle$ is constant.

[Why? Recall the definition of $\mathscr{U}_1$ which appeared just after $(*)^1_i$, recalling that we are assuming I is a complete linear order; see $\boxdot_1(a)$.]

$(*)_4$ The set $J_1 = \{t^*_i : i \in \mathscr{U}_1\}$ has at most one member in I.

[Why? Otherwise we can find i, j from $\mathscr{U}_1$ such that $t^*_i \neq t^*_j$ are from I. Now apply $(*)^2_{i,j} + (*)^3_i + (*)^3_j$.]

So without loss of generality

$(*)_5$ J_1 is empty.

[Why? If not, let $J_0 = \{t_*\}$ and we can get enough to prove the claim for $I_{<t_*}$ and for $I_{>t_*}$.]

Now:

$\boxplus_1$ If $i \in \mathscr{U}_1$ and $t^*_i = \infty$ *then* for every $s_0 <_I s_1 <_I s_2 <_I s_3$ we have
(a) $\{\mathrm{val}^{\mathfrak{C}}(x - a_{s_3,i}) > \mathrm{val}^{\mathfrak{C}}(a_{s_2,i} - a_{s_1,i})\} \vdash p^{[*]}_{a_{s_0,i}}$ and
(b) c satisfies the formula in the left side; on $p^{[*]}_{a_{s_0,j}}$, see $\boxdot_4$.

[Why? By clause (b) of 1.16 and $(*)^3_i$ and reflect.]

Hence:

$\boxplus_2$ If $\mathscr{W}_1 = \{i \in \mathscr{U}_1 : t^*_i = \infty\}$ *then* $\boxtimes_{\mathscr{W}_1}$, where for $\mathscr{W} \subseteq \mathscr{U}$ we let:
$\boxtimes_{\mathscr{W}}$ if $s <_I t$ then $\boxtimes^{s,t}_{\mathscr{W}}$, where for $\mathscr{U}' \subseteq \mathscr{U}$:
$\boxtimes^{s,t}_{\mathscr{U}'}$ $\mathscr{U}' \subseteq \alpha, s, t \in I$ and $\mathbf{f}_{t,s}$ maps $\bigcup\{p^{[*]}_{a_{s,i}} : i \in \mathscr{U}'\}$ onto $\bigcup\{p^{[*]}_{a_{t,i}} : i \in \mathscr{U}'\}$.

[Why? Should be clear as $J_1 = \emptyset$ and the indiscernibility of $\langle \bar{a}_t : t \in I\rangle$ and $\boxplus_1$.]

$\boxplus_3$ Assume that: we have $i \in \mathscr{U}_1$ satisfying $t^*_i = -\infty$, and $j \in \mathscr{U}_1$ is such that $t^*_j = -\infty$ and $s, t \in I \Rightarrow \mathrm{val}^{\mathfrak{C}}(c - a_{t,j}) > \mathrm{val}^{\mathfrak{C}}(c - a_{s,i})$. *Then*:
$\odot_3$ if $s_0 <_I s_1 <_I s_2$, then $\{\mathrm{val}^{\mathfrak{C}}(x - a_{s_2,j}) > \mathrm{val}^{\mathfrak{C}}\{(c - a_{s_1,i})\} \vdash p^{[*]}_{a_{s_0,i}}$ and the formula on the left is satisfied by c.

[Why? Should be clear.]

Hence:

$\boxplus_4$ If for every $i \in \mathcal{U}_1$ satisfying $t^*_i = -\infty$ there is j as in the assumption of $\boxplus_3$ then $\boxtimes_{\mathscr{W}_2}$ holds for $\mathscr{W}_2 = \{i \in \mathscr{U}_1 : t^*_i = -\infty\}$.

[Why? As in $\boxplus_2$.]

Consider the assumption:

$\boxplus_5$ The hypothesis of $\boxplus_4$ fails and let $j(*) \in \mathscr{U}_1$ exemplify this (so, in particular, $t^*_{j(*)} = -\infty$). Let $\mathscr{W}_3 = \{i \in \mathscr{U}_1 : t^*_i = -\infty$ and $\mathrm{val}^{\mathfrak{C}}(c - a_{s,j(*)}) > \mathrm{val}^{\mathfrak{C}}(c - a_{t,i})$ for any $s,t \in I\}$ and $\mathscr{W}_4 = \{i \in \mathscr{U}_1 : t^*_i = -\infty$ and $i \notin \mathscr{W}_3\}$, so $j(*) \in \mathscr{W}_4$

$\boxplus_6$ If $\boxplus_5$ then $\boxtimes_{\mathscr{W}_3}$.

[Why? Similarly to the proof of $\boxplus_2$.]

$\boxplus_7$ If $\boxplus_5$ then:

(a) $\langle \mathrm{val}^{\mathfrak{C}}(c - a_{s,j}) : s \in I$ and $j \in \mathscr{W}_4\rangle$ is constant,

(b) $\mathrm{val}^{\mathfrak{C}}(c - a_{r,j(*)}) <_\Gamma \mathrm{val}^{\mathfrak{C}}(a_{t,i} - a_{s,i})$, hence $(p_s)^{[*]}_{a_{s,j(*)}} \vdash (p_s)^{[*]}_{a_{s,i}}$ when $i \in \mathscr{W}_4$ and $s <_I t \wedge r \in I$,

(c) for some finite $J_1 \subseteq I$ we have: if $s,t \in J \backslash J_1$ and $(\forall r \in J_1)(s <_I s \equiv r <_I t))$ then $\mathrm{tp}(\mathrm{val}^{\mathfrak{C}}(c - a_{s,j(*)}), M_s) = \mathbf{f}_{s,t}(\mathrm{tp}(\mathrm{val}^{\mathfrak{C}}(c - a_{t,j(*)}), M_t))$,

(d) for some finite $J_2 \subseteq I$ we have: if $s,t \in I \backslash J_2$ and $(\forall r \in J_r)(r <_I s \equiv r <_I t)$ then $\mathrm{tp}(\mathrm{ac}^{\mathfrak{C}}(c - a_{s,j(*)}), M_s) = \mathbf{f}_{s,t}(\mathrm{tp}(\mathrm{ac}^{\mathfrak{C}}(c - a_{t,j(*)}), M_t)$,

(e) for some finite $J_3 \subseteq I$ we have: if $s,t \in I \backslash J_3$ and $(\forall r \in J)(r <_I s \equiv r <_I t$, then $\boxtimes^{s,t}_{\mathscr{W}_4}$.

[Why? Let $i \in \mathscr{W}_4$; so $i \in \mathscr{W}_2$, hence $i \in \mathscr{U}_1$, which means that case $(\mathrm{b})^1_i$ of $(*)^1_i$ holds, so for each $t \in I$ the sequence $\langle \mathrm{val}^{\mathfrak{C}}(a_{t,i} - a_{s,i}) : s$ satisfies $s <_I t\rangle$ is $<_\Gamma$-increasing. Also, as $i \in \mathscr{W}_2$ clearly $t^*_i = -\infty$, hence by $(*)^3_i$ $(\mathrm{b})^3_i$ we have $\langle \mathrm{val}^{\mathfrak{C}}(c - a_{s,i}) : s \in I\rangle$ is constant; call it e_i. All this applies to $j(*)$, too. Now as $i \in \mathscr{W}_4$, we know that for some $s_1, t_1 \in I$ we have $\mathrm{val}^{\mathfrak{C}}(c - a_{s_1,j(*)}) \leq_\Gamma \mathrm{val}^{\mathfrak{C}}(c - a_{t_1,i})$, i.e., $e_{j(*)} \leq_\Gamma e_i$. By the choice of $j(*)$, for every $j \in \mathscr{U}_1$ such that $t^*_j = -\infty$, i.e., for every $j \in \mathscr{W}_2$ for some (equivalently every) $s,t \in I$, we have $\mathrm{val}^{\mathfrak{C}}(c - a_{s,j}) \leq \mathrm{val}^{\mathfrak{C}}(c - a_{t,j(*)})$. In particular, this holds for $j = i$, hence for some $s_2, t_2 \in I$ we have $\mathrm{val}^{\mathfrak{C}}(c - a_{s_2,i}) \leq \mathrm{val}^{\mathfrak{C}}(c - a_{t_2,j(*)})$, i.e., $e_i \leq_\Gamma e_{j(*)}$, so together with the previous sentence, $e_i = e_{j(*)}$, so clause (a) of $\boxplus_7$ holds. Also, the first phrase in clause (b) is easy (using $(*)^3_i$ $(\mathrm{b})^3_i$, second phrase); the second phrase of (b) follows because $e_i = e_{j(*)}$. For clause (c) note that it means $\mathrm{tp}(e_{j(*)}, M_s) = \mathbf{f}_{s,t}(\mathrm{tp}(e_{j(*)}, M_t))$ is strongly stable; for clause (d) note that $(*)^3_i(d)^3_i$ and $\mathrm{Th}(k^M)$ is strongly dependent.

Lastly, for clause (e) combine the earlier clauses.]

$\boxplus_8$ For some finite $J \subseteq I$, if $s,t \in I \backslash J$ and $(\forall r \in J)(r <_I s \equiv r <_I t)$ *then* $\boxtimes^{s,t}_{\mathscr{U}_1}$.

[Why? If the hypothesis of $\boxplus_3$ holds let $J = \emptyset$, and if it fails (so $\boxplus_5, \boxplus_6, \boxplus_7$ apply) let J be as in $\boxplus_7(e)$, so it partitions I to finitely many intervals. It is enough to prove $\boxtimes^{s,t}_{\mathscr{W}}$ for several $\mathscr{W} \subseteq \mathscr{U}_1$ which covers $\mathscr{U}_1$. Now by $\boxplus_2$ this holds for $\mathscr{W}_1 = \{i \in \mathscr{U}_1 : t^*_i = \infty\}$. If the assumption of $\boxplus_3$ holds we get the same for $\mathscr{W}_2$ by $\boxplus_4$, and if it fails we get it for $\mathscr{W}_3$ by $\boxplus_6$ and for $\mathscr{W}_4$ by $\boxplus_7(e)$ and the choice of J. Using $\mathscr{U}_1 = \mathscr{W}_1 \cup \mathscr{W}_2, \mathscr{W}_2 = \mathscr{W}_3 \cup \mathscr{W}_4$ we are done.]

As we can replace I by its inverse:

$\boxplus_9$ For some finite $J \subseteq I$, if $s, t \in I \backslash J$ and $(\forall r)(r <_I s \equiv r <_I t)$ then $\boxtimes^{s,t}_{\mathscr{U}_{-1}}$.

So we are left with $\mathscr{U}_0$. For $i \in \mathscr{U}_0$ let $e_{0,i} = \operatorname{val}(a_{t,i} - a_{s,i})$ for $s <_I t$, well defined by the definition of $\mathscr{U}_0$. Let $\mathscr{W}_5 := \{i \in \mathscr{U}_0$: for every (equivalently some) $s \neq t \in I$, $\operatorname{val}^{\mathfrak{C}}(c - a_{s,i}) < \operatorname{val}(a_{t,i} - a_{s,i})\}$ and let $\mathscr{W}_6 := \mathscr{U}_0 \backslash \mathscr{W}_5$.

Obviously:

$\boxplus_{10}$ We have $\boxtimes_{\mathscr{W}_5}$.

Easily:

$\boxplus_{11}$ If $i, j \in \mathscr{W}_6$ then case $(\mathrm{a})^2_{i,j}$ of $(*)^2_{i,j}$ holds.

[Why? By $(*)^2_{i,j}$ and as $i, j \in \mathscr{W}_6 \Rightarrow (*)^1_i(a)^1_i + (*)^1_j(a)^i_j$.]

$\boxplus_{12}$ If $i, j \in \mathscr{W}_6$ and $s \neq t \in I$, then $\operatorname{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) = e_{0,i}$.

[Why? As $\mathscr{W}_6 = \mathscr{U}_0 \backslash \mathscr{W}_5$.]

Hence:

$\boxplus_{13}$ $\langle e_{0,i} : i \in \mathscr{W}_6 \rangle$ is constant. Call the constant value e_*, so $s \neq t \in I \wedge i, j \in \mathscr{W}_6 \Rightarrow \operatorname{val}^{\mathfrak{C}}(a_{t,j} - a_{s,i}) = e_*$.

Easily:

$\boxplus_{14}$ For every $i \in \mathscr{W}_6$ the set $I_{i,c} := \{s \in I : \operatorname{val}^{\mathfrak{C}}(c - a_{s,i}) > e_*\}$ has at most one member.

$\boxplus_{15}$ Let $\mathscr{W}_7 := \{i \in \mathscr{W}_6 : I_{i,c} \neq \emptyset\}$ and let $\{t^{**}_i\} = I_{i,c}$ for $i \in \mathscr{W}_7$.

$\boxplus_{16}$ If $i, j \in \mathscr{W}_7$ then $t^{**}_i = t^{**}_j$.

[Why? Otherwise without loss of generality $t^{**}_i < t^{**}_j$ and let $t \in I$ be such that $t^{**}_i < t \wedge t^{**}_j < t$. Now $\operatorname{val}^{\mathfrak{C}}(c - a_{t^{**}_i,j}) > \operatorname{val}^{\mathfrak{C}}(a_{t,i} - a_{t^{**}_i,i}) = e_*$ and $\operatorname{val}^{\mathfrak{C}}(c - a_{t^{**}_j,j}) > \operatorname{val}^{\mathfrak{C}}(a_{t,j} - a_{t^{**}_j,j}) = e_*$, hence $e_* < \operatorname{val}^{\mathfrak{C}}((c - a_{t^{**}_i,i}) - (c - a_{t^{**}_j,j})) = \operatorname{val}^{\mathfrak{C}}(a_{t^{**}_j,j} - a_{t^{**}_i,i})$; but the last one is e_* by $\boxplus_{12}$, contradiction.]

$\boxplus_{17}$ Without loss of generality $\mathscr{W}_7 = \emptyset$.

[Why? E.g., as otherwise we can prove separately for $I_{<t^{**}_i}$ and for $I_{>t^{**}_i}$ for any $i \in \mathscr{W}_7$.]

$\boxplus_{18}$ If $i, j \in \mathscr{W}_6$ and $s \neq t \in I$ *then* $\mathrm{ac}^{\mathfrak{C}}(c - a_{t,j}) - \mathrm{ac}^{\mathfrak{C}}(c - a_{s,i}) = \mathrm{ac}^{\mathfrak{C}}(a_{s,i} - a_{t,j})$.

[Why? As $\mathrm{val}^{\mathfrak{C}}(c - a_{t,j}), \mathrm{val}^{\mathfrak{C}}(c - a_{s,i})$ and $\mathrm{val}^{\mathfrak{C}}(c_{s,i} - (c_{t,j})$ are all equal to e_*.]

The rest should be clear.

(3) For the ω-language the proof is similar. $\blacksquare_{1.17}$

2. Cutting indiscernible sequence and strongly$^+$ dependent

2.1. *Observation:* (1) The following conditions on T are equivalent, for $\alpha \geq \omega$.

(a) T is strongly dependent, i.e., $\aleph_0 = \kappa_{\mathrm{ict}}(T)$.

(b)$_\alpha$ If I is an infinite linear order, $\bar{a}_t \in {}^\alpha\mathfrak{C}$ for $t \in I$, $\mathbf{I} = \langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence and $C \subseteq \mathfrak{C}$ is finite, *then* there is a convex equivalence relation E on I with finitely many equivalence classes such that $sEt \Rightarrow \mathrm{tp}(\bar{a}_s, C) = \mathrm{tp}(\bar{a}_t, C)$.

(c)$_\alpha$ If $\mathbf{I} = \langle \bar{a}_t : t \in I \rangle$ is as above and $C \subseteq \mathfrak{C}$ is finite, *then* there is a convex equivalence relation E on I with finitely many equivalence classes such that: if $s \in I$ then $\langle \bar{a}_t : t \in (s/E) \rangle$ is an indiscernible sequence over C.

(2) We can add to the list in (1)

(b)$'_\alpha$ like (b)$_\alpha$, but C a singleton;

(c)$'_\alpha$ like (c)$_\alpha$, but the set C is a singleton.

(3) We can, in parts (1) and (2), clauses (c)$_\alpha$, (b)$_\alpha$, (b)$'_\alpha$, (c)$'_\alpha$, restrict ourselves to well order I.

(4) In parts (1), (2) and (3), given $\kappa = \kappa^{<\theta}, \theta > |T|$, in clauses (b)$_\kappa$, (c)$_\kappa$ and their parallels, we can add that "$\bar{a}_\alpha$ is the universe of a θ-saturated model"; moreover, we allow $\mathbf{I}$ to be:

(i) $\mathbf{I} = \langle \bar{a}_u : u \in [I]^{<\aleph_0} \rangle$ is indiscernible over A (see Definition 5.45(2)),

(ii) $\bar{a}_{\{t\}} = \bar{a}_t$,

(iii) each $\bar{a}_t$ is the universe of a θ-saturated model,

(iv) for some infinite linear orders I_{-1}, I_1 and some $\mathbf{I}' = \langle \bar{a}'_u : u \in [I_{-1} + I + I_1]^{<\aleph_0} \rangle$ indiscernible over $A = \mathrm{Rang}(\bar{a}_\emptyset)$, we have:

(α) $u \in [I]^{<\aleph_0} \Rightarrow \bar{a}'_u = \bar{a}_u$,

(β) for every $B \subseteq A$ of cardinality $< \theta$, every subtype of the type of $\langle \bar{a}_u : u \in [I_{-1} + I_1]^{<\aleph_0} \rangle$ over $\langle \bar{a}_u : u \in [I]^{<\aleph_0} \rangle$ of cardinality $< \theta$ is realized in A (we can use only A and $\langle \bar{a}_t : t \in I \rangle$, of course).

Remark: (1) Note that 2.8 below says more for the cases $\kappa_{\rm ict}(T) > \aleph_0$, so there is no point in dealing with it here.

(2) We can, in 2.1, add in $(\mathrm{b})_\alpha$, $(\mathrm{c})_\alpha$, $(\mathrm{b}_\alpha)'$, $(\mathrm{c}_\alpha)'$ "over a fixed A" by 1.4(3).

(3) By 1.10 we can translate this to the case of a family of indiscernible sequences.

Proof. (1) Let $\kappa = \omega$ (to serve in the proof of a subsequence observation).

$\neg(\mathrm{a}) \Rightarrow \neg(\mathrm{b})_\alpha$

Let $\lambda > \aleph_0$; as in the proof of 1.5, because we are assuming $\neg$(a), there are $\bar\varphi = \langle \varphi_i(\bar x, \bar y_i) : i < \kappa\rangle$ and $\langle \bar a^i_\alpha : i < \omega, \alpha < \lambda\rangle$ witnessing $\circledast^2_{\bar\varphi}$ from there.

For $\alpha < \lambda$ let $\bar a^*_\alpha \in \mathfrak{C}$ be the concatenation of $\langle \bar a^i_\alpha : i < \kappa\rangle$, possibly with repetitions, so it has length κ.

Let $\eta = \langle \omega n : n < \omega\rangle$ and $\bar b^*$ realizes $\{\varphi_n(x, \bar a^n_{\omega n}) \wedge \neg\varphi_n(x, \bar a^n_{\omega n+1}) : n < \omega\}$.

So for each n, $\mathrm{tp}(\bar a^n_{\omega n}, \bar b^*) \neq \mathrm{tp}(a^n_{\omega n+1}, \bar b^*)$, hence $\mathrm{tp}(\bar a^*_{\omega n}, \bar b^*) \neq \mathrm{tp}(\bar a^*_{\omega n+1}, \bar b^*)$. So any convex equivalence relation on λ as required (i.e., such that $\alpha E \beta \Rightarrow \mathrm{tp}(\bar a^*_\alpha, \bar b^*) = \mathrm{tp}(\bar a^*_\beta, \bar b^*)$) satisfies $n < \omega \Rightarrow \neg(\omega n) E(\omega n + 1)$; it certainly shows $\neg(\mathrm{b})_\alpha$.

$\neg(\mathrm{b})_\alpha \Rightarrow \neg(\mathrm{c})_\alpha$

Trivial.

$\neg(\mathrm{c})_\alpha \Rightarrow \neg(\mathrm{a})$

Let $\langle \bar a_t : t \in I\rangle$ and C exemplify $\neg(\mathrm{c})_\alpha$, and assume toward a contradiction that (a) holds. Without loss of generality I is a dense linear order (hence with neither first nor last element) and is complete and let $\bar c$ list C.

So

$(*)$ for no convex equivalence relation E on I with finitely many equivalence classes do we have $s \in I \Rightarrow \langle \bar a_t : t \in (s/E)\rangle$ is an indiscernible sequence over C.

We now choose $(E_n, I_n, \Delta_n, J_n)$ by induction on n such that

$\circledast$ (a) E_n is a convex equivalence relation on I such that each equivalence class is dense (so with no extreme member!) or is a singleton;
- (b) Δ_n is a finite set of formulas (each of the form $\varphi(\bar x_0, \ldots, \bar x_{m-1}, \bar y)$, $\ell g(\bar x_\ell) = \alpha$, for some $m, \ell g(\bar y) = \ell g(\bar c)$);
- (c) $I_0 = I, E_0$ is the equality, $\Delta_0 = \emptyset$;
- (d) I_{n+1} is one of the equivalence classes of E_n and is infinite;

(e) Δ_{n+1} is a finite set of formulas such that $\langle \bar{a}_t : t \in I_{n+1}\rangle$ is not Δ_{n+1}-indiscernible over C;
(f) $E_{n+1} \restriction I_{n+1}$ is a convex equivalence relation with finitely many classes, each dense (no extreme member) or singleton; if J is an infinite equivalence class of $E_{n+1} \restriction I_{n+1}$ then $\langle \bar{a}_t : t \in J\rangle$ is Δ_{n+1}-indiscernible over C and $|I_{n+1}/E_{n+1}|$ is minimal under those conditions;
(g) $E_{n+1} \restriction (I \backslash I_{n+1}) = E_n \restriction (I \backslash I_{n+1})$, so E_{n+1} refines E_n;
(h) we choose (Δ_{n+1}, E_{n+1}) such that, if possible, I_{n+1}/E_{n+1} has ≥ 4 members.

There is no problem in carrying the induction as T is dependent (see 2.2(1) below, which says more, or see [Sh:715, 3.4+Def. 3.3]).

For $n > 0, E_n \restriction I_n$ is an equivalence relation on I_n with finitely many equivalence classes, each convex; so as I is a complete linear order clearly

$(*)_1$ for each $n > 0$ there are $t^n_1 <_I \cdots < t^n_{k(n)-1}$ from I_n such that $s_1 \in I_n \wedge s_2 \in I_n \Rightarrow [s_1 E_n s_2 \equiv (\forall k)(s_1 < t^n_k \equiv s_2 < t^n_k \wedge s_1 > t^n_k \equiv s_2 > t^n_k)]$.

As $n > 0 \Rightarrow E_n \neq E_{n-1}$, clearly

$(*)_2$ $k(n) \geq 2$ and $|I_n/E_n| = 2k(n) - 1$,
$(*)_3$ $\{I_{n,\ell} : \ell < k(n)\} \cup \{\{t^n_\ell\} : 0 < \ell < k(n)\}$ are the equivalence classes of $E_n \restriction I_n$, where
$(*)_4$ for non-zero $n < \omega, \ell < k(i)$ we define $I_{n,\ell}$:
if $0 < \ell < k(n) - 1$ then $I_{n,\ell} = (t^n_\ell, t^n_{\ell+1})_{I_n}$,
if $0 = \ell$ then $I_{n,\ell} = (-\infty, t^n_\ell)_{I_n}$,
if $\ell = k(n) - 1$ then $I_{n,\ell} = (t^n_\ell, \infty)_{I_n}$.

As (see end of clause (f))) we cannot omit any t^n_ℓ $(\ell < k(n))$ and transitivity of equality of types, clearly

$(*)_5$ for each $\ell < k(n) - 1$ for some m and $\varphi = \varphi(x_0, \ldots, \bar{x}_{m-1}, \bar{y}) \in \Delta_n$ there are $s_0 <_I \cdots <_I s_{m-1}$ from $I_{n,\ell}$ and $s'_0 <_I \cdots <_I s'_{m-1}$ from $I_{n,\ell} \cup \{t^n_{\ell+1}\} \cup I_{n,\ell+1}$ such that $\mathfrak{C} \models \varphi[\bar{a}_{s_0}, \ldots, \bar{c}] \equiv \neg\varphi[a_{s'_0}, \ldots, \bar{c}]$.

Hence easily

$(*)_6$ $J \in \{I_{n,\ell} : \ell < k(n)\}$ iff J is a maximal open interval of I_n such that $\langle \bar{a}_t : t \in J\rangle$ is Δ_n-indiscernible over C.

By clause (h) and $(*)_6$,

$(*)_7$ if $k(n) < 4$ and $\ell < k(n)$, then $\langle a_t : t \in I_{n,\ell}\rangle$ is an indiscernible sequence over C, hence

$(*)_8$ if $k(n) < 4$, then for at most one $m > n$ do we have $I_m \subseteq I_n$.

Note that

$(*)_9$ $m < n \Rightarrow I_n \subset I_m \vee I_n \cap I_m = 0$.

Case 1: There is an infinite $u \subseteq \omega$ such that $\langle I_n : n \in u\rangle$ are pairwise disjoint.

For each $n \in u$ we can find $\bar{c}_n \in {}^{\omega >}C$ and $k_n < \omega$ (no connection to $k(n)$ from above!) and $\varphi(\bar{x}_0, \dots, \bar{x}_{k_n-1}, \bar{y}) \in \Delta_n$ such that $\langle \bar{a}_t : t \in I_n\rangle$ is not $\varphi_n(\bar{x}_0, \dots, \bar{x}_{k_n-1}, \bar{c})$-indiscernible (so $\ell g(\bar{x}_\ell) = \alpha$). So we can find $t^\ell_{n,0} < \cdots < t^\ell_{n,k_n-1}$ in I_n for $\ell = 1, 2$ such that $\models \varphi_n[\bar{a}_{n,t^\ell_0}, \dots, \bar{a}_{n,t^\ell_{k_n-1}}, \bar{c}_n]^{\mathrm{if}(\ell=2)}$. By minor changes in Δ_n, φ_n, without loss of generality $\bar{c}_n$ is without repetitions, hence without loss of generality $n < \omega \Rightarrow \bar{c}_n = \bar{c}_*$.

Without loss of generality Δ_n is closed under negation and, without loss of generality, $t^1_{k_n-1} <_I t^2_0$. We can choose $t^m_k \in I_n\, (m < \omega, m \notin \{1, 2\}, k < k_n)$ such that, for every $m < \omega, k < k_n$, we have $t^m_k <_I t^m_{k+1}, t^m_{k_n-1} <_I t^{m+1}_0$; let $\bar{a}^*_{n,m} = \bar{a}_{t^m_0}{}^\frown \cdots {}^\frown \bar{a}_{t^m_{k_n-1}}$ and let $\bar{x} = \langle x_i : i < \ell g(\bar{c}_*)\rangle$. So for every $\eta \in {}^\omega\omega$ the type $\{\neg\varphi_n(\bar{a}^*_{n,\eta(n)}, \bar{x}) \wedge \varphi_n(\bar{a}^*_{n,\eta(n)+1}, \bar{x}) : n < \omega\}$ is consistent. This is enough for showing $\kappa_{\mathrm{ict}}(T) > \aleph_0$.

Case 2: There is an infinite $u \subseteq \omega$ such that $\langle I_n : n \in u\rangle$ is decreasing.

For each $n \in u, E_n \restriction I_n$ has an infinite equivalence class J_n (so $J_n \subseteq I_n$) such that $n < m \wedge \{n, m\} \subseteq u \Rightarrow I_m \subseteq J_n$. By $(*)_8$, clearly for each $n \in u, k(n) \geq 4$, hence we can find $\ell(n) < k(n)$ such that $I'_n = (I_{n,\ell(n)} \cup \{t^n_{\ell,n}\} \cup I_{n,\ell(n)+1})$ is disjoint to J_m. Now $\langle I'_n : n \in u\rangle$ are pairwise disjoint and we continue as in Case 1.

By the Ramsey theorem at least one of the two cases occurs, so we are done.

(2) By induction on $|C|$.

(3), (4) Easy by now. $\blacksquare_{2.1}$

Recall

2.2. *Observation:* (1) Assume that T is dependent, $\langle \bar{a}_t : t \in I\rangle$ is an indiscernible sequence, Δ a finite set of formulas, $C \subseteq \mathfrak{C}$ finite. *Then* for some convex equivalence relation E on I with finitely many equivalence classes, each equivalence class in an infinite open convex set or is a singleton such that, for every $s \in I, \langle \bar{a}_t : t \in s/E\rangle$ is an Δ-indiscernible sequence over $\bigcup\{\bar{a}_t : t \in I \backslash (s/E)\} \cup C$.

(2) If I is dense and complete, there is the least fine such E. In fact, for J an open convex subset of I we have: J is an E-equivalence class *iff* J is a maximal open convex subset of I such that $\langle \bar{a}_t : t \in J \rangle$ is Δ-indiscernible over $C \cup \bigcup \{\bar{a}_t : t \in I \backslash J\}$.

(3) Assume that I is dense (with no extreme elements) and complete. Then there are $t_1 <_I \cdots < t_{k-1}$ such that, stipulating $t_0 = -\infty, t_k = \infty, I_\ell = (t_\ell, t_{\ell+1})_I$, we have

(a) $\langle \bar{a}_t : t \in I_\ell \rangle$ is indiscernible over C,
(b) if $\ell \in \{1, \dots, k-1\})$ and $t_\ell^- <_I t_\ell <_I t_\ell^+$, then $\langle a_t : t \in (t_\ell^-, t_\ell^+)_I \rangle$ is not Δ-indiscernible over C.

Proof. (1) See clause (b) of [Sh:715, Claim 3.2].

(2), (3) Done within the proof of 2.1 and see the proof of 2.10. $\blacksquare_{2.2}$

2.3. *Definition:* (1) We say that $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i) : i < \kappa \rangle$ witnesses $\kappa < \kappa_{\mathrm{ict},2}(T)$ *when* there are a sequence $\langle \bar{a}_{i,\alpha} : \alpha < \lambda, i < \kappa \rangle$ and $\langle \bar{b}_i : i < \kappa \rangle$ such that

(a) $\langle \bar{a}_{i,\alpha} : \alpha < \lambda \rangle$ is an indiscernible sequence over $\cup \{\bar{a}_{j,\beta} : j \in \kappa \backslash \{i\}$ and $\beta < \lambda\}$ for each $i < \kappa$,
(b) $\bar{b}_i \subseteq \cup \{\bar{a}_{j,\alpha} : j < i, \alpha < \lambda\}$,
(c) $p = \{\varphi_i(\bar{x}, \bar{a}_{i,0} {}^\frown \bar{b}_i), \neg \varphi_i(\bar{x}, \bar{a}_{i,1} {}^\frown \bar{b}_i) : i < \kappa\}$ is consistent (= finitely satisfiable in $\mathfrak{C}$).

(2) $\kappa_{\mathrm{ict},2}(T)$ is the first κ such that there is no witness for $\kappa < \kappa_{\mathrm{ict},2}(T)$.

(3) T is strongly^2 dependent (or strongly$^+$ dependent) if $\kappa_{\mathrm{ict},2}(T) = \aleph_0$.

(4) T is strongly^2 stable if it is strongly^2 dependent and stable.

2.4. *Observation:* If M is a valued field in the sense of Definition 2.3 and $|\Gamma^M| > 1$, *then* $T := \mathrm{Th}(M)$ is not strongly^2 dependent.

Proof. Let $a \in \Gamma^M$ be positive, $\varphi_0(x, a) := (\mathrm{val}(x) \geq a)$, $E(x, y, a) := (\mathrm{val}(x, y) \geq 2a)$ and $F(x, y) = x^2 + y$ (squaring in K^M). Now for $b \in \varphi_0(M, \bar{a})$, the function $F(-, b)$ is a (≤ 2)-to-1 function from $\varphi_0(M, a)$ to b/E. So we can apply [Sh:783, §4].

Alternatively, let $a_n \in \Gamma^M, a_n <_{\Gamma^M} a_{n+1}$ for $n < \omega$ be such that there are $b_{n,\alpha} \in K^M$ for $\alpha < \omega$ such that $\alpha < \beta < \omega \Rightarrow a_{n+1} > \mathrm{val}^M(b_{n,\alpha} - b_{n,\beta}) > a_n$ and $\mathrm{val}(b_{n,\alpha}) > a_n$. Without loss of generality, for each $n < \omega$ the sequence $\langle b_{n,\alpha} : \alpha < \omega \rangle$ is indiscernible over $\{b_{n_1,\alpha_1} : n_1 \in \omega \backslash \{n\}, \alpha < \omega\} \cup \{a_{n_1} : n_1 <$

$\omega\}$. Now for $\eta \in {}^\omega\omega$ clearly $p_\eta = \{\mathrm{val}(x - \Sigma\{a_{m,\eta(m)} : m < n\}) > a_n : n < \omega\}$; it is consistent, and we have an example. $\blacksquare_{2.4}$

Note that the definition of strongly[2] dependent here (in 2.3) is equivalent to the one in [Sh:783, 3.7(1)] by (a)⇔(e) of Claim 2.9 below.

The following example shows that there is a difference even among the stable T.

2.5. *Example:* There is a strongly[1] stable not strongly[2] stable T (see Definition 2.3).

Proof. Fix λ large enough. Let $\mathbb{F}$ be a field, let V be a vector space over $\mathbb{F}$ of infinite dimension, let $\langle V_n : n < \omega\rangle$ be a decreasing sequence of subspaces of V with V_n/V_{n+1} having infinite dimension λ and $V_0 = V$ and $V_\omega = \bigcap\{V_n : n < \omega\}$ have dimension λ. Let $\langle x^n_\alpha + V_{n+1} : \alpha < \lambda\rangle$ be a basis of V_n/V_{n+1} and let $\langle x^{\omega,i}_\alpha : i \in \mathbb{Z}$ and $\alpha < \lambda\rangle$ be a basis of V_ω. Let $M = M_\lambda$ be the following model:

(a) universe: V,
(b) individual constants: 0^V,
(c) the vector space operations: $x + y, x - y$ and cx for $c \in \mathbb{F}$,
(d) functions: F^M_1, a linear unary function: $F^M_1(x^n_\alpha) = x^{n+1}_\alpha, F^M_1(x^{\omega,i}_\alpha) = x^{\omega,i+1}_\alpha$,
(e) F^M_2, a linear unary function:
$$F^M_2(x^0_\alpha) = x^0_\alpha,\ F^M_2(x^{n+1}_\alpha) = x^n_\alpha \text{ and } F^M_2(x^{\omega,i}_\alpha) = x^{\omega,i-1}_\alpha,$$
(f) predicates: $P^M_n = V_n$, so P_n unary.

Now

$(*)_0$ for any models M_1, M_2 of $\mathrm{Th}(M_\lambda)$ with uncountable $\bigcap\{P^{M_\ell}_n : n < \omega\}$ for $\ell = 1, 2$, the set $\mathscr{F}$ exemplifies M_1, M_2 are $\mathbb{L}_{\infty,\aleph_0}$-equivalent where: $\mathscr{F}$ is the family of partial isomorphisms f from M_1 into M_2 such that, for some n, $\langle N_i : i < n \vee i = \omega\rangle$ we have:
  (a) $\mathrm{Dom}(f) = \bigoplus_{i<n} N_i \oplus N_\omega$,
  (b) $N_i \subseteq P^{M_1}_i$ is a subspace when $i < n \vee i = \omega$,
  (c) N_i is of finite dimension,
  (d) (α) $N_i \cap P^{M_1}_{i+1}$ if $i < n$ and $F^{M_1}_1(N_i) = N_{i+1}$ if $i + 1 < n$.
  (β) $N_i \cap \sum_{m>0} N_{i,m} = \{0\}$ when $i = w$ and $N_{i,0} := N_i, N_{i,m+1} := F^{M_1}_2(N_{i,m})$,
  (e) similar conditions on $N'_i = f(N_i)$ for $i < n \vee i = \omega$.

$(*)_1$ $T =$ $\mathrm{Th}(M_\lambda)$ has elimination of quantifiers

[Why? Easy.]

Hence

$(*)_2$ T does not depend on λ,

$(*)_3$ T is stable.

[Why? Because if N_1 is $\aleph_1$-saturated, $N_1 \prec N_2$, then $\{\mathrm{tp}(a, N_1, N_2) : a \in \mathfrak{C}\}$ has cardinality $\leq \|N_1\|^{\aleph_0}$ by $(*)_0$.]

Now

$(*)_4$ T is not strongly^2 dependent.

[Why? By 0.1. Alternatively, define a term $\sigma_n(y)$ by induction on $n : \sigma_0(y) = y, \sigma_{n+1}(y) = F_1(\sigma_n(y))$, and for $\eta \in {}^\omega\lambda$ increasing let

$$p_\eta(y) = \{P_1(y - \sigma_0(x^0_{\eta(0))})), P_2(y - \sigma_0(x^0_{\eta(0)}) - \sigma_1(x^1_{\eta(1)})), \ldots,$$
$$P_n(y - \Sigma\{\sigma_\ell(x^\ell_{\eta(\ell)}) : \ell < n\}), \ldots\}.$$

Clearly each p_η is finitely satisfiable in M_λ. Easily this proves that T is not strongly^2 stable I.]

So it remains to prove

$(*)_5$ T is strongly stable.

Why does this hold? We work in $\mathfrak{C} = \mathfrak{C}_T$. Let $\lambda \geq (2^\kappa)^+$ be large enough and $\kappa = \kappa^{\aleph_0}$. We shall prove $\kappa_{\mathrm{ict}}(T) = \aleph_0$ by the variant of $(\mathrm{b})'_\omega$ from 2.1(3); this suffices. Let $\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ be an indiscernible sequence over a set A such that $\ell g(\bar{a}_\alpha) \leq \kappa$. By 1.10, without loss of generality each $\bar{a}_\alpha$ enumerates the set of elements of an elementary submodel N_α of $\mathfrak{C}$ which includes A and is $\aleph_1$-saturated.

Without loss of generality ($I \cap \mathbb{Z} = \emptyset$ and):

$\boxdot_1$ for some $\bar{a}'_n (n \in \mathbb{Z}), A \supseteq c\ell(A' \cup \bigcup\{\bar{a}'_i : i \in \mathbb{Z}\})$, and $\langle \bar{a}'_n : n < 0\rangle \hat{}\langle \bar{a}_\alpha : \alpha < \lambda\rangle \hat{}\langle \bar{a}'_n : n \geq 0\rangle$ is an indiscernible sequence over A' and $\langle \bar{a}_\alpha : \alpha < \lambda\rangle \hat{}\langle A\rangle$ is linearly independent over A', A is the universe of N, N is $\aleph_1$-saturated and $N \cap N_\alpha$ is $\aleph_1$-saturated (and does not depend on α).

Hence by $(*)_0$

$\boxdot_2$ (a) $\alpha \neq \beta \wedge a_{\alpha,i} = a_{\beta,j} \Rightarrow a_{\alpha,i} = a_{\beta,i} \in A$,

(b) if $u \subseteq \lambda$ then $c\ell(\bigcup\{\bar{a}_\alpha : \alpha \in u\} \cup A\})$ is $\prec \mathfrak{C}$,

(c) if $u \subseteq \lambda$ is finite we get an $\aleph_1$-saturated model (not really used).

(We can use the stronger 2.1(4).) Easily

$\boxdot_3$ if $a \in N_\alpha, b \in c\ell(\bigcup\{N_\beta : \beta < \alpha\} \cup A)$ *then*:
(a) $a = b \Rightarrow a \in A$,
(b) $a - b \in P_n^{\mathfrak{C}} \Rightarrow (\exists c \in A)(a - c \in P_n^{\mathfrak{C}} \wedge b - c \in P_n^{\mathfrak{C}})$.

[Why? Let $b = \sigma^{\mathfrak{C}}(\bar{a}_{\beta_0}, \dots, \bar{a}_{\beta_{m-1}}, \bar{a}), \bar{a} \in {}^{\omega>}A, \sigma$ a term, $\beta_0 < \beta_1 < \cdots < \beta_{m-1} < \alpha$; then for every $k < \omega$ large enough $b' := \sigma^{\mathfrak{C}}(a'_k, \bar{a}'_{k+1}, \dots, \bar{a}_{k+m-1}, \bar{a})$ belongs to A (recalling $(*)_3 + \boxdot_1$) and, in Case (a), $a = b \Rightarrow a = b'$, and in Case (b), $a - b \in P_n^{\mathfrak{C}} \Rightarrow a - b' \in P_n^{\mathfrak{C}}$.]

$\boxdot_4$ If $a_\ell \in c\ell(\bigcup\{N_\alpha : \alpha \in u_\ell\} \cup A)$ and $u_\ell \subseteq \lambda$ for $\ell = 1, 2$ *then*:
(a) if $a_1 = a_2$, then for some $b \in c\ell(\bigcup\{N_\alpha : \alpha \in u_1 \cap u_2\} \cup A)$ we have $a_1 - b = a_2 - b \in A$;
(b) if $a_1 - a_2 \in P_n^{\mathfrak{C}}$, then for some $b \in c\ell(\{N_\alpha : \alpha \in u_1 \cap u_2\} \cup A)$ and $c \in A$ we have $a_2 - b - c \in P_n^{\mathfrak{C}}$ and $a_2 - b - c \in P_n^{\mathfrak{C}}$.

[Why? Similarly to $\boxdot_3$.]

Now let $c \in \mathfrak{C}$; the proof splits into cases.

Case 1: $c \in c\ell(\bigcup\{\bar{a}_\beta : \beta < \lambda\} \cup A)$.

So for some finite $u \subseteq \lambda, c \in c\ell(\bigcup\{\bar{a}_\beta : \beta \in u\})$; easily $\langle \bar{a}_\beta : \beta \in \lambda \backslash u \rangle$ is an indiscernible set over $A \cup \{c\}$, and we are done.

Case 2: For some finite $u \subseteq \lambda$, for every n for some $c_n \in c\ell(\bigcup\{\bar{a}_\beta : \beta \in u\} \cup A)$ we have $c - c_n \in P_n^M$ (but not case 1).

Clearly u is as required. (In fact, easily $c\ell(\{\bar{a}_\beta : \beta \in u\} \cup A)$ is $\aleph_1$-saturated (as u is finite, by $\boxdot_2$(c)), hence there is $c^* \in c\ell(\bigcup\{a_\beta : \beta \in u\} \cup A)$ such that $n < \omega \Rightarrow c^* - c_n \in P_n^M$.)

Case 3: Neither case 1 nor case 2 (less is needed).

Let $n(1) < \omega$ be maximal such that, for some $c_{n(1)} \in A$, we have $c - c_{n(1)} \in P_{n(1)}^M$ (for $n = 0$ every $c' \in A$ is O.K.; by not Case 2 such $n(1)$ exists).

Subcase 3A: There is $n(2) \in (n(1), \omega)$ and $c_{n(2)} \in c\ell(\{\bar{a}_\beta : \beta < \lambda\} \cup A)$ such that $c - c_{n(2)} \in P_{n(2)}^M$.

Let u be a finite subset of λ such that $c_{n(2)} \in c\ell(\{\bar{a}_\beta : \beta \in u\} \cup A)$; now u is as required (by $\boxdot_3 + \boxdot_4$ above).

Subcase 3B: Not subcase 3A.

Choosing $u = \emptyset$ works, because neither Case 1 nor Case 2 holds with $u = \emptyset$ and subcase 3A fails. $\blacksquare_{2.5}$

2.6. *Remark:* We can prove a claim parallel to 1.11, i.e., replacing strong dependent by strongly[2] dependent.

2.7. Claim: (1) *$\kappa_{ict,2}(T^{eq}) = \kappa_{ict,2}(T)$.*

(2) *If $T_\ell = \mathrm{Th}(M_\ell)$ for $\ell = 1, 2$, then $\kappa_{ict,2}(T_1) \geq \kappa_{ict,2}(T_2)$ when:*

(∗) *M_1 is (first order) interpretable in M_2.*

(3) *If $T' = \mathrm{Th}(\mathfrak{C}, c)_{c \in A}$, then $\kappa_{ict,2}(T') = \kappa_{ict,2}(T)$.*

(4) *If M is the disjoint sum of M_1, M_2 (or the product) and $\mathrm{Th}(M_1)$, $\mathrm{Th}(M_2)$ are strongly^2 dependent, then so is $\mathrm{Th}(M)$.*

Proof. Similar to 1.11. $\blacksquare_{2.7}$

Now $\kappa_{\mathrm{ict}}(T)$ is very close to being equal to $\kappa_{\mathrm{ict},2}(T)$.

2.8. Claim: (1) *If $\kappa = \kappa_{ict,2}(T) \neq \kappa_{ict}(T)$ then:*

(a) *$\kappa_{ict,2}(T) = \aleph_1 \wedge \kappa_{ict}(T) = \aleph_0$,*

(b) *there is an indiscernible sequence $\langle \bar{a}_t : t \in I \rangle$ with $\bar{a}_t \in {}^\omega\mathfrak{C}$ and $c \in \mathfrak{C}, I$ is dense complete for clarity, such that*

(∗) *for no finite $u \subseteq I$ do we have: if J is a convex subset of I disjoint to u then $\langle \bar{a}_t : t \in J \rangle$ is indiscernible over $\bigcup\{\bar{a}_t : t \in I \backslash J\} \cup \{c\}$.*

(2) *If T is strongly$^+$ dependent then T is strongly dependent.*

(3) *In the definition of $\kappa_{ict,2}(T)$, without loss of generality $m = 1$.*

Proof. (1) We use Observation 1.5. Obviously $\kappa_{\mathrm{ict}}(T) \leq \kappa_{\mathrm{ict},2}(T)$; the rest is proved together with 2.10 below.

(2) Easy.

(3) Similar to the proof of 1.7, or better use 2.10(1), (2). $\blacksquare_{2.8}$

2.9. Claim: *The following conditions on T are equivalent:*

(a) *$\kappa_{ict,2}(T) > \aleph_0$,*

(b) *we can find A and an indiscernible sequence $\langle \bar{a}_t : t \in I \rangle$ over A satisfying $\bar{a}_t \in {}^\omega\mathfrak{C}$ and $t_n \in I$ increasing with n and $\bar{c} \in {}^{\omega>}\mathfrak{C}$ such that, for every n, $t_n <_I t \Rightarrow \mathrm{tp}(\bar{a}_{t_n}, A \cup \bar{c} \cup \{\bar{a}_{t_m} : m < n\}) \neq \mathrm{tp}(\bar{a}_t, A \cup \bar{c} \cup \{\bar{a}_{t_m} : m < n\})$,*

(c) *similarly to (b), but $t_n <_I t \Rightarrow \mathrm{tp}(\bar{a}_{t_m}, A \cup \bar{c} \cup \{\bar{a}_s : s <_I t_n\}) \neq \mathrm{tp}(a_t, A \cup \bar{c} \cup \{\bar{a}_s : s <_I t_n\})$,*

(d) *we can find A and a sequence $\langle \bar{a}^n_t : t \in I_n \rangle, I_n$ an infinite order, such that $\langle \bar{a}^n_t : t \in I_n \rangle$ is indiscernible over $A \cup \bigcup\{\bar{a}^m_t : m \neq n, m < \omega, t \in I_n\}$ and, for some $\bar{c} \in {}^{\omega>}\mathfrak{C}$ for each $n, \langle \bar{a}^n_t : t \in I_n \rangle$ is not indiscernible over $A \cup \bar{c} \cup \bigcup\{\bar{a}^m_t : t \in I_m, m < n\}$,*

(e) *we can find a sequence* $\langle\varphi_n(x,\bar y_n,\dots,\bar y_0) : n<\omega\rangle$ *and* $\langle \bar a^n_\alpha : \alpha<\lambda, n<\omega\rangle$ *such that: for every* $\eta\in{}^\omega\lambda$ *the set*

$$p_\eta = \{\varphi_n(\bar x,\bar a^n_\alpha,\bar a^{n-1}_{\eta(n-1)},\dots,\bar a^0_{\eta(0)})^{\mathrm{if}(\alpha=\eta(n))} : n<\omega,\alpha<\lambda\}$$

is consistent.

Proof. Should be clear from the proof of 2.1 (more in 2.3). $\blacksquare_{2.9}$

2.10. *Observation:* (1) For any κ and $\zeta\ge\kappa$ we have (d)$\Leftrightarrow$(c)$_\zeta$ $\Rightarrow$(b)$_\zeta$ $\Leftrightarrow$(a); if, in addition, we assume $\neg(\aleph_0=\kappa_{\mathrm{ict}}(T)<\kappa=\aleph_1=\kappa_{\mathrm{ict},2}(T))$ *then* we have also (c)$_\zeta$ $\Leftrightarrow$(b)$_\zeta$, so all the following conditions on T are equivalent;

(a) $\kappa\ge\kappa_{\mathrm{ict}}(T)$,

(b)$_\zeta$ if $\langle\bar a_t : t\in I\rangle$ is an indiscernible sequence, I a linear order, $\bar a_t\in{}^\zeta\mathfrak C$ and $C\subseteq\mathfrak C$ is finite, *then* for some set $\mathscr P$ of $<\kappa$ initial segments of I we have:

(∗) if $s,t\in I$ and $(\forall J\in\mathscr P)(s\in J\equiv t\in J)$, then $\bar a_s,\bar a_t$ realizes the same type over C (if I is complete this means: for some $J\subseteq I$ of cardinality $<\kappa$, if $s,t\in I$ realizes the same quantifier free type over J in I, then $\bar a_s,\bar a_t$ realizes the same type over C),

(c)$_\zeta$ like (b), but strengthening the conclusion to: if $n<\omega, s_0<_I\dots<_I s_{n-1}, t_0<_I\dots<_I t_n$ and $(\forall\ell<n)(\forall k<n)(\forall J\in\mathscr P)[s_\ell\in J=t_k\in J]$, then $\bar a_{s_0}\hat{}\cdots\hat{}\bar a_{t_{n-1}}$ and $\bar a_{t_0}\hat{}\cdots\hat{}\bar a_{t_{n-1}}$ realize the same type over C,

(d) $\kappa\ge\kappa_{\mathrm{ict},2}(T)$.

(2) We can, in clauses (b)$_\zeta$ and (c)$_\zeta$, add $|C|=1$ and/or demand I is well ordered (for the last, use 1.10).

Proof. We shall prove various implications, which together obviously suffice (for 2.10 and 2.8(1) and 2.8(3)).

$\neg$(a)$\Rightarrow\neg$(b)$_\zeta$

Let $\lambda\ge\kappa$. As in the proof of 1.5 there are $\bar\varphi=\langle\varphi_i(\bar x,\bar y_i) : i<\kappa\rangle, m=\ell g(\bar x)$ and $\langle\bar a^i_\alpha : i<\kappa,\alpha<\lambda\rangle$ exemplifying $\circledast^2_{\bar\varphi}$ from 1.5, so necessarily $\bar a^\ell_\alpha$ is non-empty. Recall that $\ell g(\bar a^i_\alpha)$ is finite for $i<\kappa,\alpha<\lambda$. Let $\bar a^*_\alpha\in{}^\zeta\mathfrak C$ be $\bar a^0_\alpha\hat{}\bar a^1_\alpha\hat{}\cdots\hat{}\bar a'_\alpha$ where $\bar a'_\alpha$ has length $\zeta-\Sigma_{\ell<\kappa}\ell g(\bar a^i_\alpha)$ and is constantly the first member of $\bar a^0_\alpha$. Let $\bar c$ realize $p=\{\varphi_i(\bar x,\bar a_{2i})\wedge\neg\varphi_i(\bar x,\bar a_{2i+1}) : i<\kappa\}$.

Easily $\bar c$ (or pedantically $\mathrm{Rang}(\bar c)$) and $\langle\bar a^*_\alpha : \alpha<\lambda\rangle$ exemplify $\neg$(b)$_\zeta$.

(a)$\Rightarrow$(b)$_\zeta$.

If $\kappa=\aleph_0$, this holds by 2.1(1); in general, this holds by the proof of 2.1(1) and this is why there we use κ.

$\neg(\mathrm{b})_\zeta \Rightarrow \neg(\mathrm{c})_\zeta$

Obvious.

$\neg(\mathrm{a}) \Rightarrow \neg(\mathrm{d})$

The witness for $\neg$(a) is a witness for $\neg$(d).

$\neg(\mathrm{d}) \Rightarrow \neg(\mathrm{c})_\zeta$

Let $\langle \varphi_i(\bar{x}, \bar{y}_i) : i < \kappa \rangle$ witness $\neg(d)$, i.e., witness $\kappa < \kappa_{\mathrm{ict},2}(T)$, so there are $\langle \bar{a}_{i,\alpha} : \alpha < \lambda, i < \kappa \rangle$ and $\langle \bar{b}_i : i < \kappa \rangle$ satisfying clauses (a), (b), (c) of Definition 2.3. By Observation 1.10 we can find an indiscernible sequence $\langle \bar{a}^*_\alpha : \alpha < \lambda \times \kappa \rangle, \ell g(\bar{a}^*_\alpha) = \zeta_\kappa$, where $\zeta_j := \Sigma\{\ell g(\bar{y}_i) : i < j\}$ such that $i < \kappa \wedge \alpha < \lambda \Rightarrow \bar{a}^*_i \restriction [\zeta_i, \zeta_{i+1}) = \bar{a}^i_\alpha$. Now $\langle \bar{a}^*_\alpha : \alpha < \lambda \times \kappa \rangle, \bar{c}$ witness $\neg(c)_{\zeta_\kappa}$, because if $\mathscr{P}$ is as required in $(\mathrm{c})_{\zeta_\kappa}$ then easily $(\forall i < \kappa)(\exists J \in \mathscr{P})(J \cap [\lambda i, \lambda i + \lambda) \notin \{\emptyset, [\lambda i, \lambda i + \lambda)\}$, hence $|\mathscr{P}| \geq \kappa$. Now clearly $\zeta_\kappa \leq \zeta$, hence repeating the first element $(\zeta - \kappa)$ times we get $\langle \bar{b}^i_\alpha : \alpha < \lambda\kappa \rangle$, which together with $\bar{c}$ exemplify $\neg(\mathrm{c})_\zeta$.

It is enough to prove:

- $(*)$ assume $\neg(\mathrm{c})_\zeta$; then
  - (i) $\neg$(d),
  - (ii) $\neg$(a) except possibly when (a) + (b) of 2.8(1) holds, in particular $\aleph_0 = \kappa_{\mathrm{ict}}(T) < \kappa = \aleph_1 = \kappa_{\mathrm{ict},2}(T)$.

Toward this we can assume that

$\boxtimes$ T is dependent and $C, \langle \bar{a}_t : t \in I \rangle$ form a witness to $\neg(\mathrm{c})_\zeta$.

Let $\bar{c}$ list C without repetitions and, without loss of generality, I is a dense complete linear order (so with no extreme elements). Let $\ell g(\bar{x}_\ell) = \zeta$ for $\ell < \omega$ be pairwise disjoint with no repetitions, of course, $\ell g(\bar{y}) = \ell g(\bar{c}) < \omega$ (pairwise disjoint), and let $\bar{\varphi} = \langle \varphi_i = \varphi_i(\bar{x}_0, \ldots, \bar{x}_{n(i)-1}, \bar{y}) : i < |T| \rangle$ list all such formulas in $\mathbb{L}(\tau_T)$. For each $i < |T|$, by 2.2(1), (2) there are $m(i) < \omega$ and $t_{i,1} <_I \cdots <_I t_{i,m(i)-1}$ as there and $m(i)$ is minimal, so stipulating $t_{i,0} = -\infty, t_{i,m(i)} = \infty$ we have:

- $(*)_1$ if $s'_0 <_I \cdots <_I s'_{m(i)-1}$ and $s''_0 <_I \cdots <_I s''_{m(i)-1}$ and s'_ℓ, s''_ℓ realize the same quantifier free type over $\{t_{i,1}, \ldots, t_{i,m(i)-1}\}$ in the linear order I for each $\ell < m(i)$, *then* $\mathfrak{C} \models$ "$\varphi_i[\bar{a}_{s'_0}, \ldots, \bar{a}_{s'_{m(i)-1}}, \bar{c}] \equiv \varphi_i[\bar{a}_{s''_0}, \ldots, \bar{a}_{s''_{m(i)-1}} \bar{c}]$".

For each $i < |T|$, for each $\ell \in \{1, \ldots, m(i)\}$ we can find $w_{i,\ell}$ such that

- $(*)_2$ (a) $w_{i,\ell} \subseteq I \backslash \{t_{i,\ell}\}$,
  - (b) $w_{i,\ell}$ is finite,

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(c) if $s_1 < t_{i,\ell(i)} < s_2$ then $\langle \bar{a}_t : t \in (s_1, s_2)_I \rangle$ is not $\{\varphi_i\}$-indiscernible over $C \cup \{\bar{a}_t : t \in w_i\}$. Moreover for some $n_i^* < m(i)$, letting $\Pi_i : \{0, \dots, m(i)-1\} \to \{0, \dots, m(i)-1\}$ be $\Pi_i(0) = n_i^*, \Pi_i(n_i^*) = 0$ and $\Pi_i(n) = n$ otherwise and letting $\varphi_i'(\bar{x}_0, \dots \bar{x}_{m(i)-1}, \bar{y}) = \varphi(\bar{x}_{\Pi_i(0)}, \dots, \bar{x}_{\Pi_i(m(i)-1)})$ for some $t_{i,n}^* \in w_{i,\ell(i)}$ for $n = 1, \dots, n(i)-1$ if $s_1 <_I t_{i,\ell(i)} < s_2$ then for some $t' \in (s_1, s_2)_I \backslash \{t\}$ we have $\models \varphi_i'[\bar{a}_{t'}, \bar{a}_{t*_{i,1}}, \dots, \bar{a}_{t*_{i,n(i)-1}}, \bar{c}] \equiv \neg\varphi_i'[\bar{a}_{t_{i,\ell(i)}}, \bar{a}_{t_{i,1}^*}, \dots, \bar{a}_{t_{i,n(i)-1}^*}, \bar{c}]$.

If the set $\{t_{i,k} : i < |T|, k = 1, \dots, m(i)-1\}$ has cardinality $< \kappa$ we are done, so assume that

$(*)_3$ $\{t_{i,\ell} : i < |T| \text{ and } \ell \in [1, m(i)]\}$ has cardinality $\geq \kappa$.

Case 1: $\kappa > \aleph_0$ (so we have to prove $\neg$(a)).

By the Hajnal free subset theorem and by $(*)_3$ there is $u_0 \subseteq |T|$ of order type κ such that $i \in u_0 \Rightarrow \{t_{i,\ell} : \ell = 1, \dots, m(i)-1\} \nsubseteq \{t_{j,\ell} : j \in u_0 \backslash \{i\}$ and $\ell = 1, \dots, m(j)-1\} \cup \bigcup\{w_{j,\ell} : j \in u \backslash \{i\}$ and $\ell \in (1, m(i))\}$.

There are $u \subseteq u_0$ of cardinality κ and a sequence $\langle \ell(i) : i \in u \rangle, 0 < \ell(i) < m(i)$ such that $\langle t_{i,\ell(i)} : i \in u \rangle$ is with no repetitions and disjoint to $\{t_{i,\ell} : i \in u$ and $\ell \neq \ell(i)\} \cup \bigcup\{w_{i,\ell(i)} : i \in u\}$. We shall now prove $\kappa < \kappa_{\text{ict}}(T)$; this gives $\neg$(a), $\neg$(d) so it suffices.

Clearly by 1.5 it suffices to show (λ any cardinality $\geq \aleph_0$; we can easily change the $\bar{a}_\ell^i$'s to have finite length preserving $(a) + (b)$ below):

$\boxdot_u$ there are $\bar{a}_\alpha^i \in {}^\zeta\mathfrak{C}$ for $i \in u, \alpha < \lambda$ and set A such that
- (a) $\langle \bar{a}_\alpha^i : \alpha < \lambda \rangle$ is an indiscernible sequence over $\bigcup\{\bar{a}_\beta^j : j \in u, j \neq i, \alpha < \lambda\} \cup A$,
- (b) $\langle \bar{a}_\alpha^i : \alpha < \lambda \rangle$ is not $\{\varphi_i\}$-indiscernible over $A \cup \bar{c}$.

By compactness it suffices to prove $\boxdot_v$ for any finite $v \subseteq u$ and $\lambda = \aleph_0$; also, we can replace λ by any infinite linear order.

We can find $\langle (s_{1,i}, s_{2,i}) : i \in v \rangle$ such that

$(*)_4$ $s_{1,i} <_I t_{i,\ell(i)} <_I s_{2,i}$ (for $i \in v$),

$(*)_5$ $(s_{1,i}, s_{2,i})_I$ is disjoint to $\bigcup\{(s_{1,j}, s_{2,j}) : j \in v \backslash \{i\}\} \cup \bigcup\{w_{j,\ell(j)} \in v\}$.

So $\langle \langle a_t^j : t \in (s_{1,j}, s_{2,j})_I \rangle : j \in v \rangle$ and choosing $A = \bigcup\{\bar{a}_t : t \in w_{i,\ell(i)}, i \in v\}$ are as required above. Thus we are done.

Case 2: $\kappa = \aleph_0$ so we have to prove $\neg$(d) and clause (ii) of $(*)$ and (for proving part (2) of the present 2.10) that, without loss of generality, $|C| = 1$.

We can find A and u:

$\boxdot^1$ (a) $A \subseteq C$,
(b) $u \subseteq I$ is finite,
(c) if $n < \omega$ and $t_0^\ell <_I \cdots <_I t_{n-1}^\ell$ for $\ell = 1, 2$ and $(\forall k < n)(\forall s \in u)$ $(t_k^1 = s \equiv t_k^2 = s \wedge t_k^1 <_I s \equiv t_k^2 <_I s)$, then $\bar{a}_{t_0^1} \char`^ \cdots \char`^ \bar{a}_{t_{n-1}^1}, \bar{a}_{t_0^2} \char`^ \cdots \char`^ \bar{a}_{t_{n-1}^2}$ realize the same type over A,
(d) if A', u' satisfies (a)+(b)+(c), then $|A'| \leq |A|$.

This is possible because C is finite and the empty set satisfies clauses (a), (b), (c) for A. By our present assumption $A \neq C$, so let $c \in C \backslash A$. Now we try to choose (i_k, ℓ_k, w_k) by induction on $k < \omega$:

$\circledast$ (a) $i_k < \kappa$,
(b) $1 \leq \ell_k \leq m(i_k) - 1$,
(c) $t_{i_k, \ell_k} \in I \backslash w_k$,
(d) $w_k \supseteq u \cup w_0 \cup \cdots \cup w_{k-1} \cup \{t_{i_0, k_0}, \ldots, t_{i_{k-1}, \ell_{k-1}}\}$,
(e) $w_k \subseteq I \backslash \{t_{i_k, \ell_k}\}$ is finite,
(f) if $s' <_I t_{i_k, \ell_k} <_I s''$, then $\langle \bar{a}_t : t \in (s', s'')_I \rangle$ is not indiscernible over $\{\bar{a}_s : s \in w_k\} \cup \{c\}$ moreover the parallel of $(*)_2(c)$ holds.

If we are stuck in k, then $w_{k-1} \in [I]^{<\aleph_0}$ when $k > 0$ and u when $k = 0$ show that $\langle \bar{a}_t : t \in I \rangle, A \cup \{c\}$ contradict the choice of A recalling we are assuming $\neg(\mathrm{c})_\zeta$. If we succeed, then we prove as in Case 1 that $\kappa_{\mathrm{ict},2}(\mathrm{Th}(\mathfrak{C}, a)_{a \in A}) > \aleph_0$, so by 1.4 we get $\kappa_{\mathrm{ict},2}(T) > \aleph_0$. So we have proved clause (d) completing the proof of 2.10; also clearly $(*)(b)$ holds hence we complete also the proof of 2.8 $\blacksquare_{2.10}$.

2.11. *Conclusion:* T is strongly^2 dependent by Definition 2.3 iff T is strongly^2 dependent by [Sh:783, §3,3.7], which means we say T is strongly^2 (or strongly$^+$) dependent when: if $\langle \bar{\mathbf{a}}_t : t \in I \rangle$ is an indiscernible sequence over A, $t \in I \Rightarrow \ell g(\bar{\mathbf{a}}_t) = \alpha$ and $\bar{b} \in {}^{\omega >}(\mathfrak{C})$ *then* we can divide I into finitely many convex sets $\langle I_\ell : \ell < k \rangle$ such that, for each ℓ, the sequence $\langle \bar{\mathbf{a}}_t : t \in I_\ell \rangle$ is an indiscernible sequence over $\{\bar{a}_s : s \in I \backslash I_\ell\} \cup A \cup \bar{b}$.

* * *

Discussion: Now we define "T is strongly^{2,*} dependent", parallel to 1.8, 1.9 at the end of §1.

2.12. *Definition:* (1) $\kappa_{\rm icu,2}(T)$ is the minimal κ such that, for no $m < \omega$ and $\bar\varphi = \langle \varphi_i(\bar x_i, \bar y_i) : i < \kappa\rangle$ with $\ell g(\bar x^i) = m \times n_i$, can we find $\bar a^i_\alpha \in {}^{\ell g(\bar y_i)}\mathfrak{C}$ for $\alpha < \lambda, i < \kappa$ and $\bar c_{\eta,n} \in {}^m\mathfrak{C}$ for $\eta \in {}^\kappa\lambda$ such that:

(a) $\langle \bar c_{\eta,n} : n < \omega\rangle$ is an indiscernible sequence over $\bigcup\{\bar a^i_\alpha : \alpha < \lambda, i < \kappa\}$,

(b) for each $\eta \in {}^\kappa\lambda$ and $i < \kappa$ we have $\mathfrak{C} \models \varphi_i(\bar c_{\eta,0}{}^\frown \cdots {}^\frown \bar c_{\eta,n_i-1}, \bar a^i_\alpha)^{\text{if } (\alpha=\eta(i))}$.

(2) If $\bar\varphi$ is as in (1), then we say that it witnesses $\kappa < \kappa_{\rm icu,2}(T)$.

(3) T is strongly^{1,*} dependent if $\kappa_{\rm icu}(T) = \aleph_0$.

2.13. Claim: (1) $\kappa_{icu,2}(T) \le \kappa_{ict,2}(T)$.

(2) *If* $\mathrm{cf}(\kappa) > \aleph_0$ *then* $\kappa_{icu,2}(T) > \kappa \Leftrightarrow \kappa_{ict,2}(T) > \kappa$.

(3) *The parallel of 1.4, 1.5, 1.7(2) holds.*

3. Ranks

3A. Rank for strongly dependent T.

3.1. *Explanation/Thesis:* (a) For stable theories we normally consider not just a model M (and, say, a type in it), but all its elementary extensions; we analyze them together.

(b) For dependent theories we should be more liberal, allowing one to replace M by $N^{[\bar a]}$ when $M \prec N \prec N_1, \bar a \in {}^{\ell g(\bar a)}(N_1)$ ($N^{[\bar a]}$ is the expansion of N by restrictions of the relation in N_1 definable with parameters from $\bar a$);

(c) this motivates some of the ranks below.

Such ranks relate to strongly^1 dependent, they have relatives for strongly^2 dependent.

Note that we can represent the $\mathfrak{x} \in K'_{\ell,m}$ (and ranks) close to [Sh:783, §1], particularly $\ell = 9$.

3.2. *Definition:* (1) Let $M_0 \le_A M_1$ for $M_0, M_1 \prec \mathfrak{C}$ and $A \subseteq \mathfrak{C}$ mean that:

(a) $M_0 \subseteq M_1$ (equivalently $M_0 \prec M_1$),

(b) for every $\bar b \in M_1$, the type $\mathrm{tp}(\bar b, M_0 \cup A)$ is f.s. (= finitely satisfiable) in M_0.

(2) Let $M_0 \le_{A,p} M_1$ for $M_0, M_1 \prec \mathfrak{C}, A \subseteq \mathfrak{C}$ and $p \in \mathbf{S}^{<\omega}(M_1 \cup A)$, or p is just a $(<\omega)$-type over $M_1 \cup A$, means that

(a) $M_0 \subseteq M_1$;

(b) if $\bar b \in M_1, \bar c \in M_0, \bar a_1 \in A, \bar a_2 \in A, \mathfrak{C} \models \varphi_1[\bar b, \bar a_1, \bar c]$ and $\varphi_2(\bar x, \bar b, \bar a_2, \bar c) \in p$ or is just a (finite) conjunction of members of p (e.g., empty), *then* for some

$\bar b' \in M_0$ we have $\mathfrak{C} \models \varphi_1[\bar b'_1, \bar a_1, \bar c]$ and $\varphi_2(\bar x, \bar b', \bar a_2, \bar c) \in p$, or is just a finite conjunction of members of p.

3.3. *Observation:* (1) $M_0 \leq_{A,p} M_1$ implies $M_0 \leq_A M_1$.

(2) If $p = \operatorname{tp}(\bar b, M_1 \cup A) \in \mathbf{S}^m(M_1 \cup A)$, *then* $M_0 \leq_{A,p} M_1$ iff $M_1 \leq_{A \cup \bar b} M_2$.

(3) If $M_0 \leq_A M_1 \leq_A M_2$, then $M_0 \leq_A M_2$.

(4) If $M_0 \leq_{A,p\restriction(M_1\cup A)} M_1 \leq_{A,p} M_2$, then $M_0 \leq_{A,p} M_2$.

(5) If the sequences $\langle M_{1,\alpha} : \alpha \leq \delta\rangle, \langle A_\alpha : \alpha \leq \delta\rangle$ are increasing continuous, δ a limit ordinal and $M_0 \leq_{A_\alpha} M_{1,\alpha}$ for $\alpha < \delta$, then $M_0 \leq_{A_\delta} M_{1,\delta}$. Similarly using $<_{A_\alpha,p_\alpha}$.

(6) If $M_1 \subseteq M_2$ and p is an m-type over $M_1 \cup A$, then $M_1 \leq_A M_2 \Leftrightarrow M_1 \leq_{A,p} M_2$.

Proof. Easy. ∎

3.4. *Discussion:* (1) Note that the ranks defined below are related to [Sh:783, §1]. An alternative presentation (for $\ell \in \{3,6,9,12\}$) is that we define M_A as $(M,a)_{a\in A}$ and $T_A = \operatorname{Th}(\mathfrak{C},a)_{a\in A}$, and we consider $p \in \mathbf{S}(M_A)$, and in the definition of ranks to extend A and p we use appropriate $q \in \mathbf{S}(N_B), M_A \prec N_A, A \subseteq B$. Originally, we prsented here many variants, but now we present only two ($\ell = 8,9$), retaining the others in §5A.

(2) We may change the definition, each time retaining from p only one formula with little change in the claims.

(3) We can define $\mathfrak{x} \in K_{\ell,m}$ such that it has also $N^{\mathfrak{x}}$, where $M^{\mathfrak{x}} \subseteq N^{\mathfrak{x}}(\prec \mathfrak{C}_T)$ and:

(A) change the definition of $\mathfrak{x} \leq^\ell_{\rm at} \mathfrak{y}$ to:
  (a) $N^{\mathfrak{y}} \subseteq N^{\mathfrak{x}}$,
  (b) $A^{\mathfrak{x}} \subseteq A^{\mathfrak{y}} \subseteq A^{\mathfrak{x}} \cup N^{\mathfrak{x}}$,
  (c) $M^{\mathfrak{x}} \subseteq M^{\mathfrak{y}} \subseteq N^{\mathfrak{x}}$,
  (d) $p^{\mathfrak{y}} \subseteq p^{\mathfrak{x}}$;

(B) change "$\mathfrak{y}$ explicitly $\bar\Delta$-split ℓ-strongly over $\mathfrak{x}$" according to, and replacing in Def 3.5(4) or Def. 5.1(4) clauses (e), (e)′ the type $p^{\mathfrak{x}'}$ by $p^{\mathfrak{x}}$,

(C) dp-rk$^m_{\bar\Delta,\ell}$ is changed accordingly.

So now dp-rk$^m_{\bar\Delta}$ may be any ordinal, hence 3.7 may fail, but the result in §4 becomes stronger, covering also some models of non-strongly dependent T.

3.5. *Definition:* (1) For $\ell = 8, 9$ let

$$K_{m,\ell} = \big\{\mathfrak{x} : \mathfrak{x} = (p, M, A), M \text{ a model } \prec \mathfrak{C}_T, A \subseteq \mathfrak{C}_T,$$
$$p \in \mathbf{S}^m(M \cup A), \text{ and if } \ell = 9 \text{ then } p \text{ is finitely satisfiable in } M\big\}.$$

If $m = 1$ we may omit it.

For $\mathfrak{x} \in K_{m,\ell}$ let $\mathfrak{x} = (p^{\mathfrak{x}}, M^{\mathfrak{x}}, A^{\mathfrak{x}}) = (p[\mathfrak{x}], M[\mathfrak{x}], A[\mathfrak{x}])$ and $m = m(\mathfrak{x})$, recalling $p^{\mathfrak{x}}$ is an m-type.

(2) For $\mathfrak{x} \in K_{m,\ell}$ let $N_{\mathfrak{x}}$ be $M^{\mathfrak{x}}$ expanded by $R_{\varphi(\bar{x},\bar{y},\bar{a})} = \{\bar{b} \in {}^{\ell g(\bar{y})}M : \varphi(\bar{x}, \bar{b}, \bar{a}) \in p\}$ for $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{a} \in {}^{\ell g(\bar{z})}A$ and $R_{\varphi(\bar{y},\bar{a})} = \{\bar{b} \in {}^{\ell g(\bar{y})}M : \mathfrak{C} \models \varphi[\bar{b}, \bar{a}]\}$ for $\varphi(\bar{y}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{a} \in {}^{\ell g(\bar{y})}\mathfrak{C}$; let $\tau_{\mathfrak{x}} = \tau_{N_{\mathfrak{x}}}$.

(2A) In parts (1) and (2): if we omit p we mean $p = \operatorname{tp}(\langle\rangle, M \cup A)$, therefore we can write N_A, a τ_A-model, so in this case $p = \{\varphi(\bar{b}, \bar{a}) : \bar{b} \in M, \bar{a} \in M$ and $\mathfrak{C} \models \varphi[\bar{b}, \bar{a}]\}$.

(3) For $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ let

- (α) $\mathfrak{x} \leq^{\ell}_{\mathrm{pr}} \mathfrak{y}$ means that $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ and
  - (a) $A^{\mathfrak{x}} = A^{\mathfrak{y}}$,
  - (b) $M^{\mathfrak{x}} \leq_{A[\mathfrak{x}]} M^{\mathfrak{y}}$,
  - (c) $p^{\mathfrak{x}} \subseteq p^{\mathfrak{y}}$,
  - (d) $M^{\mathfrak{x}} \leq_{A[\mathfrak{x}],p[\mathfrak{y}]} M^{\mathfrak{y}}$;
- (β) $\mathfrak{x} \leq^{\ell} \mathfrak{y}$ means that for some n and $\langle \mathfrak{x}_k : k \leq n\rangle, \mathfrak{x}_k \leq^{\ell}_{\mathrm{at}} \mathfrak{x}_{k+1}$ for $k < n$ and $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{x}_0, \mathfrak{x}_n)$, where
- (γ) $\mathfrak{x} \leq^{\ell}_{\mathrm{at}} \mathfrak{y}$ iff ($\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ and) for some $\mathfrak{x}' \in K_{m,\ell}$ we have
  - (a) $\mathfrak{x} \leq^{\ell}_{\mathrm{pr}} \mathfrak{x}'$,
  - (b) $A^{\mathfrak{x}} \subseteq A^{\mathfrak{y}} \subseteq A^{\mathfrak{x}} \cup M^{\mathfrak{x}'}$,
  - (c) $M^{\mathfrak{y}} \subseteq M^{\mathfrak{x}'}$,
  - (d) $p^{\mathfrak{y}} = p^{\mathfrak{x}'} \restriction (M^{\mathfrak{y}} \cup A^{\mathfrak{y}})$.

(4) For $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ we say that $\mathfrak{y}$ explicitly $\bar{\Delta}$-splits ℓ-strongly over $\mathfrak{x}$ *when*: $\bar{\Delta} = (\Delta_1, \Delta_2), \Delta_1, \Delta_2 \subseteq \mathbb{L}(\tau_T)$, and for some $\mathfrak{x}'$ and $\varphi(\bar{x}, \bar{y}) \in \Delta_2$ we have clauses (a),(b),(c),(d) of part (3)(γ) and

- (e) there are $\bar{\mathbf{b}}, \bar{\mathbf{a}}$ such that
  - (α) $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \omega + 1\rangle$ is Δ_1-indiscernible over $A^{\mathfrak{x}} \cup M^{\mathfrak{y}}$,
  - (β) $A^{\mathfrak{y}} \backslash A^{\mathfrak{x}} = \bigcup\{\bar{a}_i : i < \omega\}$; yes ω not $\omega + 1$! (note that "$A^{\mathfrak{y}} \backslash A^{\mathfrak{x}} =$ " and not "$A^{\mathfrak{y}} \backslash A^{\mathfrak{x}} \supseteq$ " as we use it in $(e)(\gamma)$ in the proof of 3.7),
  - (γ) $\bar{a}_i \in M^{\mathfrak{x}'}$ for $i < \omega + 1$ and $\bar{b} \in {}^{\omega >}(A^{\mathfrak{x}})$,
  - (δ) $\varphi(\bar{x}, \bar{a}_k{}^\frown\bar{b}) \wedge \neg\varphi(\bar{x}, \bar{a}_\omega{}^\frown\bar{b})$ belongs to $p^{\mathfrak{x}'}$ for $k < \omega$.

(5) We define $\text{dp-rk}^m_{\bar\Delta,\ell} : K_{m,\ell} \to \text{ Ord } \cup \{\infty\}$ by

(a) $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \geq 0$ always,
(b) $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \geq \alpha+1$ *iff* there is $\mathfrak{y} \in K_{m,\ell}$ which explicitly $\bar\Delta$-splits ℓ-strongly over $\mathfrak{x}$ and $\text{dp-rk}_{\bar\Delta,\ell}(\mathfrak{y}) \geq \alpha$,
(c) $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \geq \delta$ iff $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \geq \alpha$ for every $\alpha < \delta$ when δ is a limit ordinal.

This is clearly well defined. We may omit m from dp-rk as $\mathfrak{x}$ determines it.

(6) Let $\text{dp-rk}^m_{\bar\Delta,\ell}(T) = \bigcup\{\text{dp-rk}_{\bar\Delta,\ell}(\mathfrak{x}) : \mathfrak{x} \in K_{m,\ell}\}$; if $m = 1$ we may omit it.

(7) If $\Delta_1 = \Delta_2 = \Delta$ we may write Δ instead of (Δ_1, Δ_2). If $\Delta = \mathbb{L}(\tau_T)$ then we may omit it.

Remark: There are obvious monotonicity and inequalities.

3.6. *Observation:* (1) $\leq^\ell_{\text{pr}}$ is a partial order on $K_{m,\ell}$.

(2) $K_{m,9} \subseteq K_{m,8}$.
(3) if $\mathfrak{x}, \mathfrak{y} \in K_{m,9}$ then $\mathfrak{x} \leq^8_{\text{pr}} \mathfrak{y} \Leftrightarrow \mathfrak{x} \leq^9_{\text{pr}} \mathfrak{y}$.
(4) if $\mathfrak{x}, \mathfrak{y} \in K_{m,9}$ then $\mathfrak{x} \leq^8_{\text{at}} \mathfrak{y} \Leftrightarrow \mathfrak{x} \leq^9_{\text{at}} \mathfrak{y}$.
(5) if $\mathfrak{x}, \mathfrak{y} \in K_{m,9}$ then $\mathfrak{y}$ explicitly $\bar\Delta$-splits 8-strongly over $\mathfrak{x}$ iff $\mathfrak{y}$ explicitly $\bar\Delta$-splits 9-strongly over $\mathfrak{x}$.
(6) If $\mathfrak{x} \in K_{m,9}$, then $\text{dp-rk}^m_{\bar\Delta,9}(\mathfrak{x}) \leq \text{dp-rk}^m_{\bar\Delta,8}(\mathfrak{x})$.
(7) If $\bar a \in {}^m\mathfrak{C}$ and $\mathfrak{x} = (\text{tp}(\bar a, M \cup A), M, A)$, *then* $\mathfrak{x} \in K_{m,8}$.
(8) In part (7), if $\text{tp}(\bar a, M \cup A)$ is finitely satisfiable in M then also $\mathfrak{y} \in K_{m,9}$.
(9) If $\mathfrak{x} \in K_{m,\ell}$ and $\kappa > \aleph_0$, then there is $\mathfrak{y} \in K_{m,\ell}$ such that $\mathfrak{x} \leq^\ell_{\text{pr}} \mathfrak{y}$ and $M^{\mathfrak{y}}$ is κ-saturated; moreover, $M^{\mathfrak{y}}_{A[\mathfrak{y}],p[\mathfrak{y}]}$ is κ-saturated (hence in Definition 3.2(4), without loss of generality, $M^{\mathfrak{x}'}$ is $(|M^{\mathfrak{x}} \cup A^{\mathfrak{x}}|^+)$-saturated).

Proof. Easy. ∎

3.7. Claim: (1) *For each* $\ell = 8, 9$ *we have* $\text{dp-rk}_\ell(T) = \infty$ *iff* $\text{dp-rk}_\ell(T) \geq |T|^+$ *iff* $\kappa_{\text{ict}}(T) > \aleph_0$.

(2) *For each* $m \in [1, \omega)$, *the latter holds similarly using* $\text{dp-rk}^m_\ell(T)$, *hence the properties do not depend on such* m.

3.8. *Remark:* In the implications in the proof we allow more cases of ℓ.

Proof. Part (2) has the same proof as part (1) when we recall 1.7(1).

$\kappa_{\text{ict}}(T) > \aleph_0$ implies $\text{dp-rk}_\ell(T) = \infty$:

By the assumption there is a sequence $\bar\varphi = \langle \varphi_n(x, \bar y_n) : n < \omega \rangle$ exemplifying $\aleph_0 < \kappa_{\text{ict}}(T)$. Let $\lambda > \aleph_0$ and I be $\lambda \times \mathbb{Z}$ ordered lexicographically, and let

$I_\alpha = \{\alpha\} \times \mathbb{Z}$ and $I_{\geq\alpha} = [\alpha,\lambda) \times \mathbb{Z}$. As in 1.5, by the Ramsey theorem and compactness we can find $\langle \bar{a}^n_t : t \in I, n < \omega\rangle$ (in $\mathfrak{C}_T$) such that

$\circledast$ (a) $\ell g(\bar{a}^n_t) = \ell g(\bar{y}_n)$,
 (b) $\langle \bar{a}^n_t : t \in I\rangle$ is an indiscernible sequence over $\bigcup\{\bar{a}^m_t : m < \omega, m \neq n$ and $t \in I\}$,
 (c) for every $\eta \in {}^\omega I, p_\eta = \{\varphi_n(x, \bar{a}^n_t)^{\mathrm{if}(\eta(n)=t)} : n < \omega, t \in I\}$ is consistent (i.e., finitely satisfiable in $\mathfrak{C}$).

Choose a complete $T_1 \supseteq T$ with Skolem functions, and $M^* \models T_1$ expanding $\mathfrak{C}$ be such that in it $\langle \bar{a}^n_\alpha : t \in I, n < \omega\rangle$ satisfies $\circledast$ also in M^*; this exists by the Ramsey theorem. Let M^*_n be the Skolem hull in M* of $\bigcup\{\bar{a}^m_t : m < n, t \in I_1\} \cup \{\bar{a}^m_t : m \in [n,\omega)$ and $t \in I\}$ and let $M_n = M^*_n \restriction \tau(T)$. So we have $M_n \prec \mathfrak{C}$, which includes $\{\bar{a}^m_t : t \in I, m \in [n,\omega)\}$ such that $M_{n+1} \prec M_n$ and $\langle \bar{a}^n_t : t \in I_{\geq 2}\rangle$ is an indiscernible sequence over $M_{n+1} \cup \{\bar{a}^m_t : m < n, t \in I_1\}$, hence $\langle a^n_t : t \in I_2\rangle$ is an indiscernible sequence over $M_{n+1} \cup A_n$; the indiscernibility holds even in M^*, where $A_n = \{\bar{a}^m_t : m < n$ and $t \in I_1\}$. We delay the case $\ell = 9$. Let $\eta \in {}^\omega I$ be chosen as $\langle (2,i) : i < \omega\rangle$. Let $p \in \mathbf{S}(M_0)$ be such that it includes p_η.

Lastly, let $\mathfrak{x}_n = \mathfrak{x}'_n = (p_n, M_n, A_n)$, where $p_n = p \restriction (A_n \cup M_n)$. By 3.6(7) clearly $\mathfrak{x}_n \in K_\ell$.

It is enough to show that $\text{dp-rk}_\ell(\mathfrak{x}_n) < \infty \Rightarrow \text{dp-rk}_\ell(\mathfrak{x}_n) > \text{dp-rk}_\ell(\mathfrak{x}_{n+1})$, as by the ordinals being well ordered this implies that $\text{dp-rk}_\ell(\mathfrak{x}_n) = \infty$ for every n. By Definition 3.5(5) clause (b), it is enough to show (fixing $n < \omega$) that $\mathfrak{x}_{n+1}$ explicitly splits ℓ-strongly over $\mathfrak{x}_n$ using $\langle \bar{a}^n_{(1,i)} : i < \omega\rangle{}^\frown\langle \bar{a}^n_{(2,n)}\rangle$. To show this, see Definition 3.5(4); we use $\mathfrak{x}'_n := \mathfrak{x}_n$, clearly $\mathfrak{x}_n \leq^\ell_{\mathrm{pr}} \mathfrak{x}'_n$ as $\mathfrak{x}_n = \mathfrak{x}'_n \in K_\ell$, so clause (a), of Definition 3.5(3)(γ) holds. Also, $A^{\mathfrak{x}_n} \subseteq A^{\mathfrak{x}_{n+1}} \subseteq A^{\mathfrak{x}_n} \cup M^{\mathfrak{x}'_n}$ as $A^{\mathfrak{x}_{n+1}} = A^{\mathfrak{x}_n} \cup \{\bar{a}^n_t : t \in I_1\}$ and $\bigcup\{\bar{a}^n_t : t \in I_1\} \subseteq M^{\mathfrak{x}_n}$, so clause (b) of Definition 3.5(3)(γ) holds. Also, $M^{\mathfrak{x}_{n+1}} \subseteq M^{\mathfrak{x}'_n}$ and $p^{\mathfrak{x}_{n+1}} \supseteq p^{\mathfrak{x}'_n} \restriction (A^{\mathfrak{x}_n} \cup M^{\mathfrak{x}_{n+1}})$ and even $p^{\mathfrak{x}_{n+1}} = p^{\mathfrak{x}'_n} \restriction (A^{\mathfrak{x}_{n+1}} \cup M^{\mathfrak{x}_{n+1}})$ hold trivially, so also clause (c),(d) of Definition 3.5(3)(γ) holds.

Lastly, $\neg\varphi_n(x, \bar{a}^n_{(1,i)})$ for $i < \omega, \varphi_n(x, \bar{a}_{(2,n)})$ belongs to p_η, hence to $p^{\mathfrak{x}_{n+1}}$, hence by renaming also clause (e) from Definition 3.5(4) holds. So we are done.

We are left with the case $\ell = 9$. For the proof above to work we need just that $p(\in \mathbf{S}(M_0))$ satisfies $n < \omega \Rightarrow p \restriction (M_n \cup A_n)$ is finitely satisfiable in M_n. Toward this, without loss of generality, for each n there is a function symbol $F_n \in \tau(M^*)$ such that: if $\eta \in {}^n I$ then $c_\eta := F^{M^*}_n(\bar{a}^0_{\eta(0)}, \ldots, \bar{a}^{n-1}_{\eta(n-1)})$ realizes $\{\varphi_m(x, \bar{a}^m_t)^{\mathrm{if}(t=\eta(m))} : m < n$ and $\alpha < \lambda\}$, so F_n has arity $\Sigma\{\ell g(\bar{y}_m) : m < n\}$.

Let D be a uniform ultrafilter on ω and let $c_\omega \in \mathfrak{C}$ realize $p^* = \{\psi(x,\bar b) : \bar b \subseteq M_0, \psi(x,\bar y) \in \mathbb{L}(\tau_{M^*})$ and $\{n : \mathfrak{C} \models \psi(c_{\eta\restriction n},\bar b)\} \in D\}$, so clearly $p = \operatorname{tp}(c_\omega, M_0, \mathfrak{C}) \in \mathbf{S}(M_0)$ extends $\{\varphi_n(x,\bar a^m_t)^{\mathrm{if}(t=\eta(n))} : n<\omega$ and $t \in I\}$. Therefore we have just to check that $p_n = p \restriction (A_n \cup M_n)$ is finitely satisfiable in M_n, so let $\vartheta(\bar x,\bar b) \in p_n$; thus we can find $k(*) < \omega (\subseteq \mathbb{Z})$ such that $\bar b$ is included in the Skolem hull $M^*_{n,k(*)}$ of $\cup\{\bar a^m_{(1,a)} : m<n$ and $a \in \mathbb{Z} \wedge a < k(*)\} \cup \{\bar a^m_t : m \in [n,\omega), t \in I\}$ inside M^*.

Let $\nu \in {}^\omega\lambda$ be defined by

$$\nu(m) = \eta(m) \text{ for } m \in [n,\omega)$$

$$\nu(m) = (1, k(*)+m) \text{ for } m<n.$$

By the indiscernibility:

$(*)_1$ for every $n, \mathfrak{C} \models \psi(c_{\eta\restriction n},\bar b) \equiv \psi(c_{\nu\restriction n},\bar b)$,

and by the choice of p

$(*)_2$ $\{n : \mathfrak{C} \models \psi(c_{\eta\restriction n},\bar b)\}$ is infinite, but clearly

$(*)_3$ $c_{\eta\restriction m} \in M_n$ for $m<\omega$.

Together we are done.

dp-rk$_\ell(T) = \infty$ implies dp-rk$_\ell(T) \ge |T|^+$:

Trivial.

dp-rk$_\ell(T) \ge |T|^+ \Rightarrow \kappa_{\mathrm{ict}}(T) > \aleph_0$:

We choose by induction on n sequences $\bar\varphi^n$ and $\langle \mathfrak{x}^n_\alpha : \alpha < |T|^+\rangle, \langle \mathbf{\bar a}^n_\alpha, A^n_\alpha : \alpha < |T|^+\rangle$ such that:

$\circledast_n$ (a) $\bar\varphi^n = \langle \varphi_m(x,\bar y_m) : m<n\rangle$; that is $\bar\varphi^n = \langle \varphi^n_m(x,\bar y^n_m) : m<n\rangle$ and $\varphi^n_m(x,\bar y^n_m) = \varphi^{n+1}_m(x,\bar y^{m+1}_m)$ for $m<n$, so we call it $\varphi_m(x,\bar y_m)$.

(b) $\mathfrak{x}^n_\alpha \in K_\ell$ and dp-rk$_\ell(\mathfrak{x}^n_\alpha) \ge \alpha$.

(c) $\mathbf{\bar a}^n_\alpha = \langle \bar a^{n,m}_{\alpha,k} : k<\omega, m<n\rangle$, where the sequence $\bar a^{n,m}_{\alpha,k}$ is from $A^{\mathfrak{x}^n_\alpha}$.

(d) For each $\alpha < |T|^+$ and $m<n$ the sequence $\langle \bar a^{n,m}_{\alpha,k} : k<\omega\rangle$ is indiscernible over $\bigcup\{\bar a^{n,i}_{\alpha,k} : i<n, i \ne m, k<\omega\} \cup M^{\mathfrak{x}^n_\alpha} \cup A^n_\alpha$.

(e) We have $\bar b^{n,m}_\alpha \subseteq A^{\mathfrak{x}^n_\alpha} = \bigcup\{\bar a^{n,i}_{\alpha,k} : i<m, k<\omega\} \cup A^n_\alpha$ for $m<n$ such that: if $\eta \in {}^n\omega$ and $m<n \Rightarrow \bar b^{n,m}_\alpha \subseteq \bigcup\{\bar a^{n,i}_{\alpha,k} : i<m, k<\eta(i)\} \cup A^n_\alpha$, *then* $(p^{\mathfrak{x}^n_\alpha} \restriction M^{\mathfrak{x}^n_\alpha}) \cup \{\neg\varphi_m(\bar a^{n,m}_{\alpha,\eta(m)}, \bar b^{n,m}_\alpha) \wedge \varphi_m(\bar x, \bar a^{n,m}_{\alpha,\eta(m)+1}, \bar b^{n,m}_\alpha) : m<n\}$ is finitely satisfiable in $\mathfrak{C}$.

For $n=0$ this is trivial by the assumption rk-dp$_\ell(T) \ge |T|^+$; see Definition 3.5(6) (and 3.5(7)).

For $n+1$, for every $\alpha < |T|^+$ (as $\text{rk-dp}_\ell(\mathfrak{x}^n_{\alpha+1}) > \alpha$ by Definition 3.5(5)) we can find $\mathfrak{z}^n_\alpha, \mathfrak{y}^n_\alpha, \varphi^n_\alpha(x, \bar{y}^n_\alpha), \langle \bar{a}^{n,*}_{\alpha,k} : k < \omega \rangle$ such that Definition 3.5(4) is satisfied, with $(\mathfrak{x}^n_{\alpha+1}, \mathfrak{z}^n_\alpha, \mathfrak{y}^n_\alpha, \varphi^n_\alpha(x, \bar{y}_\alpha), \langle \bar{a}^{n,*}_{\alpha,k} : k < \omega \rangle)$ here standing for $(\mathfrak{x}, \mathfrak{x}', \mathfrak{y}, \varphi(x, \bar{y}), \langle \bar{a}_k : k < \omega \rangle)$ there such that $\text{rk-dp}_\ell(\mathfrak{y}^n_\alpha) \geq \alpha$, and we also have $\bar{a}^{n,*}_{\alpha,\omega}, \bar{b}^{n,*}_\alpha$ here standing for $\bar{a}_\omega, \bar{b}$ there. So for some formula $\varphi_n(x, \bar{y}_n)$ the set $S_n = \{\alpha < |T|^+ : \varphi^n_\alpha(x, \bar{y}^n_\alpha) = \varphi_n(x, \bar{y}_n)\}$ is unbounded in $|T|^+$, so $\bar{\varphi}^{n+1}$ is well defined, hence clause (a) of $\circledast_{n+1}$ holds.

For $\alpha < |T|^+$, let $\beta_n(\alpha) = \operatorname{Min}(S_n \backslash \alpha)$ and let $\mathfrak{x}^{n+1}_\alpha = \mathfrak{y}^n_{\beta(\alpha)}$ so clause (b) of $\circledast_{n+1}$ holds. Let $\langle \bar{a}^{n+1,m}_{\alpha,k} : k < \omega \rangle$ be $\langle \bar{a}^{n,m}_{\beta(\alpha)+1,k} : k < \omega \rangle$ if $m < n$ and $\langle \bar{a}^{n,*}_{\beta(\alpha),k} : k < \omega \rangle$ if $m = n$, and let $A^{n+1}_\alpha = A^n_{\beta(\alpha)+1}$, so clauses (c) + (d) from $\circledast_{n+1}$ hold. Also, we let $\bar{b}^{n+1,m}_\alpha$ be $\bar{b}^{n,m}_{\beta(\alpha)+1}$ if $m < n$ and $\bar{b}^{n,*}_{\beta(\alpha)}$ if $m = n$. Next, we check clause (e) of $\circledast_{n+1}$.

Let $\eta \in {}^{n+1}\omega$ be as required in sub-clause (γ) of clause (e) of $\circledast_{n+1}$ and let α be any member of S. By the induction hypothesis

$$(p^{\mathfrak{x}^n_{\alpha+1}} \restriction M^{\mathfrak{x}^n_{\alpha+1}}) \cup \{\neg\varphi(x, \bar{a}^{n,m}_{\alpha,\eta(m)}), \bar{b}^{n,m}_\alpha) \wedge \varphi(x, \bar{a}^{n,m}_{\alpha,\eta(m)+1}, \bar{b}^{n,m}_\alpha) : m < n\}$$

is finitely satisfiable in $\mathfrak{C}$.

By clause (d) of 3.5(3)(α) it follows that

$$(p^{\mathfrak{z}^n_\alpha} \restriction M^{\mathfrak{z}^n_\alpha}) \cup \{\neg\varphi(x, \bar{a}^{n,m}_{\alpha+1,\eta(m)}), b^{n,m}_\alpha) \wedge \varphi(x, \bar{a}^{n,m}_{\alpha+1,\eta(m)+1}) : m < n\}$$

is finitely satisfiable in $\mathfrak{C}$ (i.e., we use $M^{\mathfrak{x}^n_{\alpha+1}} \leq_{A[\mathfrak{z}^n_\alpha], p[\mathfrak{z}^n_\alpha] \restriction M[\mathfrak{z}^n_\alpha]} M^{\mathfrak{z}^n_\alpha}$, which suffices; we use freely the indiscernibility).

Hence, by monotonicity, the set

$$\begin{aligned}(p^{\mathfrak{z}^n_\alpha} \restriction (M^{\mathfrak{y}^n_\alpha} \cup \{\bar{a}^{n+1,m}_{\alpha,k} : k \leq \eta(n) \text{ or } k = \omega\} \cup A^n_{\alpha+1})\\ \cup \{\neg\varphi(\bar{x}, \bar{a}^{n+1,m}_{\alpha,\eta(m)}, \bar{b}^{m,n}_\alpha) \wedge \varphi(x, \bar{a}^{n+1,m}_{\alpha,\eta(m)+1}; \bar{b}^{n,m}_\alpha) : m < n\}\end{aligned}$$

is finitely satisfiable in $\mathfrak{C}$.

Similarly,

$$\begin{aligned}(p^{\mathfrak{z}^n_\alpha} \restriction (M^{\mathfrak{y}^n_\alpha}) \cup \{\neg\varphi(x, \bar{a}^{n+1,m}_{\alpha,\eta(n)}, \bar{b}^{n+1,n}_\alpha) \wedge \varphi(x, \bar{a}^{n+1,n}_{\alpha,\omega})\}\\ \cup \{\neg\varphi(x, \bar{a}^{n+1,n}_{\alpha,\eta(m)}, \bar{b}^{n+1,m}_\alpha) \wedge \varphi(x, \bar{a}^{n+1,m}_{\alpha,\eta(m)+1}, \bar{b}^{n+1,m}_\alpha) : m < n\}\end{aligned}$$

is finitely satisfiable in $\mathfrak{C}$.

But $\bar{a}^{n+1,m}_{\alpha,\omega}, \bar{a}^{n+1,n}_{\alpha,\eta(n)+1}$ realize the same type over a set including all the relevant elements, so we can replace above the first $(\bar{a}^{n+1,n}_{\alpha,\omega})$ by the second $(\bar{a}^{n+1,n}_{\alpha,\eta(m)+1})$, so we are done proving clause (e) of $\circledast_{n+1}$.

Having carried out the induction it suffices to show that $\bar{\varphi} = \langle \varphi_n(x, \bar{y}_n) : n < \omega \rangle$ exemplifies that $\kappa_{\rm ict}(T) > \aleph_0$; for this it suffices to prove the assertion $\circledast^2_{\bar{\varphi}}$ from 1.5(1). By compactness, it suffices for each n to find $\langle \bar{a}^{n,m}_k : k < \omega \rangle$ for $m < n$ in $\mathfrak{C}$ such that $\ell g(\bar{a}^{n,m}_k) = \ell g(\bar{y}_n), \langle \bar{a}^{n,m}_k : k < \omega \rangle$ is indiscernible over $\bigcup \{\bar{a}^{n,i}_k : k < \omega, i < n, i \neq m\}$ for each $m < n$ and $\mathfrak{C} \models (\exists x)[\bigwedge_{m<n} (\neg \varphi(x, \bar{a}^{n,m}_0) \wedge \varphi(x, \bar{a}^{n,m}_1)]$.

We choose $\bar{a}^{n,m}_k = \bar{a}^{n,m}_{\alpha,k(*)+k} {}^\frown \bar{b}^{n,m}_\alpha$, where $k(*)$ is large enough such that $\bigcup \{\bar{b}^{n,m}_\alpha : m < n\} \subseteq \bigcup \{\bar{a}^{n,m}_{\alpha,k} : m < n$ and $k < k(*)\}$ and let $\alpha = 0$; clearly we are done. $\blacksquare_{3.7}$

3.9. *Observation:* (1) If $\mathfrak{x} \in K_\ell$ and $|T| + |A^{\mathfrak{x}}| \leq \mu < \|M^{\mathfrak{x}}\|$, *then* for some $M_0 \prec M^{\mathfrak{x}}$ we have $\|M_0\| = \mu$ and for every $\mathfrak{y} \leq^\ell_{\rm pr} \mathfrak{x}$ satisfying $M_0 \subseteq M^{\mathfrak{y}}$ we have $\text{dp-rk}_\ell(\mathfrak{y}) = \text{dp-rk}_\ell(\mathfrak{x})$.

(1A) If $\text{dp-rk}_\ell(\mathfrak{x}) < \infty$ then it is $< |T|^+$. Similarly, $\text{dp-rk}_\ell(T)$ (with $(2^{|T|})^+$ this is easier).

(1B) If $\text{dp-rk}_{\bar{\Delta},\ell}(\mathfrak{x}) < \infty$ then it is $< |\Delta_1 \cup \Delta_2|^+ + \aleph_1$.

(2) If $\mathfrak{x} \leq^\ell_{\rm pr} \mathfrak{y}$ then $\text{dp-rk}_\ell(\mathfrak{x}) \geq \text{dp-rk}_\ell(\mathfrak{y})$.

(3) If $\mathfrak{x} \leq^\ell_{\rm pr} \mathfrak{y}$ and $\mathfrak{z}$ explicitly splits ℓ-strongly over $\mathfrak{y}$, *then* $\mathfrak{z}$ explicitly splits ℓ-strongly over $\mathfrak{x}$.

(4) The previous parts hold for $m > 1$, too.

Proof. (1) We do not need a really close look at the rank for this. First, fix an ordinal ζ.

We can choose a vocabulary $\tau_{\zeta,\alpha,m}$ of cardinality $|A| + |\zeta| + |T|$ such that:

$\circledast_1$ for any set A fixing a sequence $\bar{\mathbf{a}} = \langle a_\beta : \beta < \alpha \rangle$ listing the elements of $A, M \prec \mathfrak{C}$ and $p \in \mathbf{S}^m(M \cup \{a_\beta : \beta < \alpha\}), M_{A,p}$, or more exactly $M_{\bar{\mathbf{a}},p}$, is a $\tau_{\zeta,\alpha,m}$-model;

we let

$\circledast_2$ (a) $\text{ds}(\zeta) = \{\eta : \eta$ a decreasing sequence of ordinals $< \zeta\}$,

(b) $\Gamma_\zeta = \{u : u$ is a subset of $\text{ds}(\zeta)$ closed under initial segments$\}$ and $\Gamma_\infty = \bigcup \{\Gamma_\zeta : \zeta$ an ordinal$\}$,

(c) for $u \in \Gamma_\zeta$ let $\Xi^m_u = \{\bar{\varphi} : \bar{\varphi}$ has the form $\langle \varphi_n(\bar{x}, \bar{y}_\eta) : \eta \in u \rangle$ where $\bar{x} = \langle x_\ell : \ell < m \rangle, \varphi_\eta(\bar{x}, \bar{y}_n) \in \mathbb{L}(\tau_T)\}$, and

$\circledast_3$ there are functions $\Phi_{\alpha,m}$ for $m < \omega, \alpha$ an ordinal, satisfying:

(a) if $u \in \Gamma_\infty, \alpha \in$ Ord and $\bar{\varphi} \in \Xi^m_u$, *then* $\Phi_{\alpha,m}(u)$ is a set of first order sentences,

(b) $\Phi_{\alpha,m}(u)$ is a set of first order sentences,

(c) if $\mathfrak{x} \in K_{m,\ell}$ and $\bar{\mathbf{a}} = \langle a_\beta : \beta < \alpha \rangle$ list $A^{\mathfrak{x}}$, then $\text{dp-rk}_\ell(\mathfrak{x}) \geq \zeta$ iff $\text{Th}(M_{\bar{\mathbf{a}},p[\mathfrak{x}]}) \cup \Phi_{\alpha,m}(\bar{\varphi})$ is consistent for some $\bar{\varphi} \in \Xi^m_{\text{ds}(\zeta)}$,

(d) if $\bar{\varphi}, \bar{\psi}$ are isomorphic (see below), then $\Phi_{\alpha,m}(\bar{\varphi})$ is consistent iff $\Phi_{\alpha,m}(\bar{\psi})$ is,

where

$\circledast_4$ $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{y}_\eta) : \eta \in u \rangle, \bar{\psi} = \langle \psi_\eta(\bar{x}, \bar{z}_\eta) : \eta \in v \rangle$ are isomorphic when there is a one-to-one mapping function h from u onto v preserving lengths, being initial segments, and its negation such that $\varphi_\eta(\bar{x}, \bar{y}_\eta) = \psi_{h(\eta)}(\bar{x}, \bar{z}_{h(\eta)})$ for $\eta \in u$.

[Why $\circledast_3$? Just reflect on the definition.]

Now if $\zeta = \text{dp-rk}_\ell(\mathfrak{x})$ has cardinality $\leq \mu$ (e.g., $\zeta < |T|^+$), part (1) should be clear. In the remaining case, if $\mu \geq |T|^+$, by (1A) we are done and otherwise use the implicit characterization of "$\infty = \text{dp-rk}_\ell(\mathfrak{x})$".

(1A) Now the proof is similar to the third part of the proof of 3.7(1) and is standard. We choose by induction on n a formula $\varphi_n(\bar{x}, \bar{y}_n) < |T|^+$ for some decreasing sequence $\eta^*_{m,\alpha}$ of ordinals $> \alpha$ of length n; we have

$\odot$ $\Phi_{n,\alpha}(\bar{\varphi}^n)$ is consistent with $\text{Th}(M^{\mathfrak{x}^n_\alpha}_{\bar{\mathbf{a}}[\mathfrak{x}^n_\alpha],p[\mathfrak{x}^n_\alpha]})$ where $\text{Dom}(\bar{\varphi}^{n,\alpha}) = \{\eta^*_{n,\alpha} \restriction \ell : \ell \leq n\}$ and $\varphi^{n,\alpha}_{\eta_{n,\alpha}\restriction\ell}(\bar{x}, \bar{y}^{n,\alpha}_{\eta_{n,\alpha}\restriction\ell}) = \varphi_\ell(\bar{x}, \bar{y}_\ell)$ for $\ell < n$.

The induction should be clear and clearly is enough.

(1B) Similarly.

(2) We prove by induction on the ordinal ζ that $\text{dp-rk}_\ell(\mathfrak{y}) \geq \zeta \Rightarrow \text{dp-rk}_\ell(\mathfrak{x}) \geq \zeta$. For $\zeta = 0$ this is trivial, and for ζ a limit ordinal this is obvious. For ζ successor order, let $\zeta = \xi + 1$ so there is $\mathfrak{z} \in K_\ell$ which explicitly splits ℓ-strongly over $\mathfrak{y}$ by part (3) and the definition of dp-rk_ℓ; we are done.

(3) Easy, as $\leq^{\text{pr}}_\ell$ is transitive.

(4) Similarly. $\blacksquare_{3.9}$

* * *

3B. Ranks for strongly$^+$ dependent T. We now deal with a relative of Definition 3.5 relevant for "strongly$^+$ dependent".

3.10. *Definition:* (1) For $\ell \in \{14, 15\}$ we define $K_{m,\ell} = K_{m,\ell-6}$ (and if $m = 1$ we may omit it and $\leq^\ell_{\text{pr}} = \leq^{\ell-6}_{\text{pr}}, \leq^\ell_{\text{at}} = \leq^{\ell-6}_{\text{at}}, \leq^\ell = \leq^{\ell-6}$).

(2) For $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ we say that $\mathfrak{y}$ explicitly $\bar{\Delta}$-splits ℓ-strongly over $\mathfrak{x}$ when $\bar{\Delta} = (\Delta_1, \Delta_2), \Delta_1, \Delta_2 \subseteq \mathbb{L}(\tau_T)$ and for some $\mathfrak{x}'$ and $\varphi(\bar{x}, \bar{y}) \in \Delta_2$ with $\ell g(\bar{x}) = m$ we have clauses (a),(b),(c),(d) of clause (γ) of Definition 3.5(3), and

(e)$''$ there are $\bar{b}, \bar{\mathbf{a}}$ such that

- (α) $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \omega \rangle$ is Δ_1-indiscernible over $A^{\mathfrak{x}} \cup M^{\mathfrak{y}}$,
- (β) $A^{\mathfrak{y}} \supseteq A^{\mathfrak{x}} \cup \{\bar{a}_i : i < \omega\}$,
- (γ) $\bar{b} \subseteq A^{\mathfrak{x}}$ and $\bar{a}_i \in M^{\mathfrak{x}}$ for $i < \omega$,
- (δ) $\varphi(\bar{x}, \bar{a}_0 {}^\frown \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{a}_1 {}^\frown \bar{b}) \in p^{\mathfrak{x}'}$.

(3) $\text{dp-rk}^m_\ell(T) = \bigcup\{\text{dp-rk}_\ell(\mathfrak{x}) + 1 : \mathfrak{x} \in K_\ell\}$.

(4) If $\Delta_1 = \Delta = \Delta_2$ we may write Δ instead of $\bar{\Delta}$, and if $\Delta = \mathbb{L}(\tau_T)$ we may omit Δ. Lastly, if $m = 1$ we may omit it.

Similarly to 3.6.

3.11. *Observation:* (1) If $\mathfrak{x}, \mathfrak{y} \in K_{15}$, then "$\mathfrak{y}$ explicitly $\bar{\Delta}$-splits 15-strongly over $\mathfrak{x}$" iff "$\mathfrak{y}$ explicitly $\bar{\Delta}$-splits 14-strongly over $\mathfrak{x}$".

(2) If $\mathfrak{x} \in K_{m,15}$ then $\text{dp-rk}^m_{\bar{\Delta},15}(\mathfrak{x}) \leq \text{dp-rk}^m_{\bar{\Delta},14}(\mathfrak{x})$.

Proof. Easy by the definition. $\blacksquare$

3.12. Claim: (1) *For $\ell = 14$ we have* $\text{dp-rk}_\ell(T) = \infty$ *iff* $\text{dp-rk}_\ell(T) \geq |T|^+$ *iff* $\kappa_{ict,2}(T) > \aleph_0$.

(2) *For each $m \in [1, \omega)$, similarly using* $\text{dp-rk}^m_\ell(T)$.

(3) *The parallel of 3.9 holds (for $\ell = 14, 15$).*

Proof. (1) $\kappa_{\text{ict},2}(T) > \aleph_0$ implies $\text{dp-rk}_\ell(T) = \infty$.

As in the proof of 3.7.

$\text{dp-rk}_\ell(T) = \infty \Rightarrow \text{dp-rk}_\ell(T) \geq |T|^+$ for any ℓ.

Trivial.

$\text{dp-rk}_\ell(T) \geq |T|^+ \Rightarrow \kappa_{\text{ict},2}(T)$.

We repeat the proof of the parallel statement in 3.7, and we choose $\bar{b}$ but not $\bar{a}^{n+1,n}_{\alpha,\omega}$.

(2) By part (1) and 2.8(3).

(3) A similar proof. $\blacksquare_{3.12}$

4. Existence of indiscernibles

Now we arrive at our main result.

4.1. Theorem: (1) *Assume*

(a) $\ell \in \{8, 9\}$,
(b) $\infty > \zeta(*) = \text{dp-rk}^m_\ell(T)$ *so* $\zeta(*) < |T|^+$,
(c) $\lambda_* = \beth_{2\times(\zeta(*)+1)}(\mu)$,
(d) $\bar{a}_\alpha \in \mathfrak{C}_T$ *for* $\alpha < \lambda^+_*, \ell g(\bar{a}_\alpha) = m$,
(e) $A \subseteq \mathfrak{C}_T, |A| + |T| \le \mu$.

Then *for some* $u \in [\lambda^+_*]^{\mu^+}$, *the sequence* $\langle \bar{a}_\alpha : \alpha \in u\rangle$ *is an indiscernible sequence over* A.

(2) *If* T *is strongly dependent, then for some* $\zeta(*) < |T|^+$ *part (1) holds, i.e., if clauses (c),(d),(e) from there hold, then the conclusion there holds.*

4.2. *Remark:* (0) This works for $\ell = 14, 15, 17, 18$, too; see §5A.

(1) A theorem in this direction is natural as small dp-rk points to definability and if the relevent types increase with the index and are definable, say over the first model, then it follows that the sequence is indiscernible.

(2) The $\beth_{2\times(\zeta+1)}(\mu)$ is more than needed; we can use $\lambda^+_{\zeta(*)}$ where we define $\lambda_\zeta = \mu + \Sigma\{(2^{\lambda_\xi})^+ : \xi < \zeta\}$ by induction on ζ.

(3) We may like to have a one-model version of this theorem. This will be dealt with elsewhere.

Proof. (1) Clearly $\mathfrak{x} \in K_{m,\ell} \Rightarrow p^{\mathfrak{x}} \in \mathbf{S}^m(A^{\mathfrak{x}} \cup M^{\mathfrak{x}})$ and we shall use clause (e) of Definition 3.5(4).

By 3.6(6), it is enough to prove this for $\ell = 9$, but the proof is somewhat simpler for $\ell = 8$, so we carry the proof for $\ell = 8$ but say what more is needed for $\ell = 9$. We prove by induction on the ordinal ζ that (note that the M_α's are increasing but not necessarily the p_α's; this is not an essential point as by decreasing somewhat the cardinals we can regain it):

- $(*)_\zeta$ if the sequence $\mathbf{I} = \langle \bar{a}_\alpha : \alpha < \lambda^+\rangle$ satisfies $\boxtimes_\zeta$ below, then for some $u \in [\lambda^+]^{\mu^+}$ the sequence $\langle \bar{a}_\alpha : \alpha \in u\rangle$ is an indiscernible sequence over A where (below, the 2 is an overkill, in particular for successor of successor, but for limit ζ we "catch our tail"):
- $\boxtimes_\zeta$ there are $\lambda, B, \bar{M}, \bar{p}$ such that
  - (a) $\lambda = \lambda^{\beth_{2(\xi+1)}(\mu)}$ for every $\xi < \zeta$,
  - (b) $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ and $M_\alpha \prec \mathfrak{C}_T$ is increasing continuous (with α),
  - (c) M_α has cardinality $\le \lambda$,

(d) $\bar{a}_\alpha \in {}^m(M_{\alpha+1})$ for $\alpha < \lambda^+$,
(e) $p_\alpha = \operatorname{tp}(\bar{a}_\alpha, M_\alpha \cup A \cup B)$,
(f) $B \subseteq \mathfrak{C}, |B| \leq \aleph_0$,
(g) $\mathfrak{x}_\alpha = (p_\alpha, M_\alpha, A \cup B)$ belongs to $K_{m,\ell}$ and satisfies $\text{dp-rk}^m_\ell(\mathfrak{x}_\alpha) < \zeta$.

[Why is this enough? We apply $(*)$ for the case $\zeta = \zeta(*)$ so $\lambda = \lambda_*$, and we choose $M_\alpha \prec \mathfrak{C}$ of cardinality λ by induction on $\alpha < \lambda^+$ such that M_α is increasing continuous, $\{\bar{a}_\beta : \beta < \alpha\} \subseteq M_\alpha$.]

If $\ell = 8$, fine; if $\ell = 9$, it seems that we have a problem with clause (g). That is, in checking $\mathfrak{x}_\alpha \in K_{n,\ell}$ we have to show that "p_α is finitely satisfiable in M_α". But this is not a serious one: in this case note that for some club E of λ^+, for every $\alpha \in E$, the type we have $\operatorname{tp}(a_\alpha, M_\alpha \cup A \cup B)$ is finitely satisfiable in M_α. So letting $M'_\alpha = M_{\alpha'}, a'_\alpha = \bar{a}_{\alpha'}$ when $\alpha < \lambda^+, \alpha' \in E$ and $\operatorname{otp}(C \cap \alpha') = \alpha$ and similarly $p'_\alpha = \operatorname{tp}(\bar{a}_{\alpha'}, M_\alpha, \mathfrak{C})$ we can use $\langle (a'_\alpha, M'_\alpha, p'_\alpha) : \alpha < \lambda^+ \rangle$, so we are done.

So let us carry the induction; arriving at ζ we let $\theta_\ell = \beth_{2 \times \zeta + \ell}(\mu)$, for $\ell < 3$; note that $\theta_{\ell+1}^{\theta_\ell} = \theta_\ell$ and $\lambda^{\theta_2} = \lambda$. Let χ be large enough and let $\mathfrak{B} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality λ such that $\mathfrak{C}, \bar{M}, \bar{p}, \bar{\mathbf{a}}, B, A$ belong to $\mathfrak{B}$ and $\lambda + 1 \subseteq \mathfrak{B}$ and $Y \subseteq \mathfrak{B} \wedge |Y| \leq \theta_2 \wedge \lambda^{|Y|} = X \Rightarrow Y \in \mathfrak{B}$. Let $\delta(*) = \mathfrak{B} \cap \lambda^+$ so, without loss of generality, $\operatorname{cf}(\delta(*))$ satisfies $\lambda^{\operatorname{cf}(\delta(*))} > \lambda$. Let $\zeta^* = \text{dp-rk}(p_{\delta(*)}, M_{\delta(*)}, A \cup B)$ and $\theta = \theta_1$, hence $\lambda = \lambda^{\theta^+}$. We try by induction on $\varepsilon \leq \theta^+ + \theta^+$ to choose $(N_{\alpha_\varepsilon}, \alpha_\varepsilon)$ such that:

$\circledast_\varepsilon$ (a) $\alpha_\varepsilon < \delta(*)$ is increasing with ε,
(b) $N_\varepsilon <_{A \cup B, p_{\alpha(*)}} M_{\delta(*)}$ is increasing continuous with ε,
(c) N_ε has cardinality θ,
(d) $\xi < \varepsilon \Rightarrow a_{\alpha_\xi} \in N_{\alpha_\varepsilon}$,
(e) $\bar{a}_{\alpha_\varepsilon}$ realizes $p_{\delta(*)} \restriction (N_{\alpha_\varepsilon} \cup A \cup B)$
(f) if $p_{\delta(*)}$ splits over $N_\varepsilon \cup A \cup B$, then $p_{\delta(*)} \restriction (N_{\alpha_{\varepsilon+1}} \cup A \cup B)$ splits over $N_\varepsilon \cup A \cup B$,
(g) $(p_{\alpha_\varepsilon} \restriction (N_{\alpha_\varepsilon} \cup A \cup B), N_{\alpha_\varepsilon}, A \cup B) <_{\text{pr}} (p_{\delta(*)}, M_{\delta(*)}, A \cup B)$ and they (have to) have the same dp-rk,
(h) $N_\varepsilon \subseteq M_{\alpha_\varepsilon}$ (but not used).

Clearly we can carry the definition. Now the proof splits into two cases.

Case 1: For $\xi = \theta^+, p_{\delta(*)}$ does not split over $N_{\alpha_\xi} \cup A \cup B$.

By clause (e) of $\circledast_\varepsilon$ clearly $\varepsilon \in [\xi, \xi + \theta^+) \Rightarrow \operatorname{tp}(\bar{a}_{\alpha_\varepsilon}, N_\varepsilon \cup A \cup B)$ does not split over $N_{\alpha_\xi} \cup A \cup B$ and increases with ε. As $\langle N_{\xi+\varepsilon} : \varepsilon < \theta \rangle$ is increasing and

$\bar{a}_{\alpha_\varepsilon} \in N_{\varepsilon+1}$, it follows that $\mathrm{tp}(\bar{a}_{\alpha_\varepsilon}, N_{\theta^+} \cup \{\bar{a}_\beta : \beta \in [\theta^+, \varepsilon)\} \cup A \cup B\}$ does not split over $N_{\theta_1^+} \cup A \cup B$. Hence by [Sh:c, I,§2] the sequence $\langle \bar{a}_{\alpha_j} : j \in [\xi, \xi + \theta^+)\rangle$ is an indiscernible sequence over $N_{\alpha_\xi} \cup A \cup B$ so, as $M^+ \leq \theta^+$, we are done.

Case 2: For $\xi = \theta^+$, $p_{\delta(*)}$ splits over $N_{\alpha_\xi} \cup A \cup B$.

So we can find $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ and $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}(M_{\delta(*)} \cup A \cup B)$ realizing the same type over $N_{\alpha_\xi} \cup A \cup B$ and $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p_{\delta(*)}$. So, without loss of generality, $\bar{b} = \bar{b}' {}^\frown \bar{d}, \bar{c} = \bar{c}' {}^\frown \bar{d}$ where $\bar{d} \in {}^{\omega>}(A \cup B)$ and $\bar{b}', \bar{c}' \in {}^{m(*)}(M_{\delta(*)})$ for some $m(*)$. As $N_{\alpha_\xi} <_{A \cup B} M_{\delta(*)}$ (see clause (b) of $\circledast_\xi$) clearly there is D, an ultrafilter on ${}^{m(*)}(N_\xi)$ such that $\mathrm{Av}(N_\xi \cup A \cup B, D) = \mathrm{tp}(\bar{b}', N_\xi \cup A \cup B) = \mathrm{tp}(\bar{c}', N_\xi \cup A \cup B)$.

Without loss of generality $\{\bar{b}'' \in {}^{m(*)}(N_{\alpha_\xi}) : \neg\varphi(\bar{x}, \bar{b}'', \bar{d}) \in p_{\delta(*)}\}$ belongs to D, as otherwise we can replace $\varphi, \bar{b}', \bar{c}'$ by $\neg\varphi, \bar{c}', \bar{b}'$.

Let $M_* = (M_{\delta(*)})_{A \cup B \cup \{\bar{a}_{\delta(*)}\}}$ and let $M^+ \prec \mathfrak{C}$ be such that $M_{\delta(*)} \subseteq M^+$ and, moreover, $(M_*)_{A \cup B \cup \{\bar{a}_{\delta(*)}\}} \prec M^+_{A \cup B \cup \{\bar{a}_{\delta(*)}\}}$ and the latter is λ^+-saturated. Clearly, letting $p^+_\delta = (\mathrm{tp}(\bar{a}_{\delta(*)}, M^+ \cup A \cup B)$ and $\mathfrak{x}^+_{\delta(*)} = (p^+_{\delta(*)}, M^+_{\delta(*)}, A \cup B)$ we have $\mathfrak{x}_{\delta(*)} \leq_{\mathrm{pr}} \mathfrak{x}^+_{\delta(*)}$. Note that $\varepsilon < \xi \Rightarrow (p_{\alpha_\varepsilon} \restriction (N_{\alpha_\varepsilon} \cup A \cup B), N_{\alpha_\varepsilon}, A \cup B) \leq_{\mathrm{pr}} \mathfrak{x}_{\delta(*)}$.

We can find $\langle \bar{b}_\alpha : \alpha < \omega + \omega\rangle$ such that $\bar{b}_\alpha \in {}^{m(*)}(M^+)$ realizes $\mathrm{Av}(N_{\alpha_\xi} \cup A \cup B \cup \{\bar{b}_\beta : \beta < \alpha\}, D)$ and, without loss of generality, $\bar{b}_\omega = \bar{b}'$.

We would like to apply the induction hypothesis to $\zeta' = \text{dp-rk}(\mathfrak{x}_{\delta(*)})$, so let:

$\boxdot$ (a) $\lambda' = \theta$,
(b) $a'_\varepsilon = a_{\alpha_\varepsilon}$ for $\varepsilon < \theta^+$,
(c) $M'_\varepsilon = N_\varepsilon$,
(d) $p'_\varepsilon = \mathrm{tp}(\bar{a}_{\alpha_\varepsilon}, N_\varepsilon)$,
(e) $B' = B \cup \{\bar{b}_\alpha : \alpha < \omega + \omega\}$,
(f) $A' = A$.

We can apply the induction hypothesis to ζ', i.e., use $(*)_{\zeta'}$: for some $u' \subseteq \theta^+$ of cardinality μ^+ the sequence $\langle a'_\varepsilon : \varepsilon \in u'\rangle$ is indisernible over A, hence the set $u := \{\alpha_\varepsilon : \varepsilon \in u'\}$ has cardinality μ^+ and the sequence $\langle a_\alpha : \alpha \in u\rangle$ is indiscernible over A, so we are done.

But we have to check that the demands from $\boxtimes_{\zeta'}$ hold (for θ^+), $\bar{M}' = \langle M'_\varepsilon : \varepsilon < \theta^+\rangle$, $\bar{p}' = \langle p'_\varepsilon : \varepsilon < \theta^+\rangle$.

Clause (a): As $\theta = \beth_{2 \times \zeta^* + 1}(\mu)$, clearly for every $\xi < \zeta^*$ we have $\theta = \theta^{\beth_{2 \times (\xi+1)}}$, hence $\theta = \theta^{\beth_{2 \times (\xi+1)}}$.

Clause (b): By $\circledast_\varepsilon$(b), $\bar{M}$ is increasing continuous.

Clause (c): By $\circledast_\varepsilon$(c).

Clause (d): By $\circledast_\varepsilon$(d).

Clause (e): By the choice of p'_ε.

Clause (f): By the choice of B'.

Clause (g): Clearly $\mathfrak{x}'_\varepsilon \in K_{m,\ell}$, but why do we have dp-rk$(\mathfrak{x}'_\varepsilon) < \zeta^*$? This is equivalent to dp-rk$(\mathfrak{x}'_\varepsilon) <$ dp-rk$(\mathfrak{x}_{\delta(*)})$.

Recall $\mathfrak{x}_{\delta(*)} \le_{\mathrm{pr}} \mathfrak{x}^+_{\delta(*)}$ and $\mathfrak{x}'_\varepsilon$ explicitly split ℓ-strongly over $\mathfrak{x}_{\delta(*)}$, hence by the definition of dp-rk we get dp-rk$(\mathfrak{x}'_\varepsilon) <$ dp-rk$(\mathfrak{x}_{\delta(*)})$.

What about the finitely satisfiable property of p' when $\ell = 9$? For some club E of θ^+, $\varepsilon \in E \Rightarrow$ tp$(\bar a_{\alpha_\varepsilon}, N_{\alpha_\varepsilon} \cup A \cup B')$ is finitely satisfiable in N_{α_ε}.

(2) By 3.7, dp-rk$^m_\ell(T) < |T|^+$ for $\ell = 8$, so we can apply part (1). $\blacksquare_{4.1}$

5. Concluding remarks

We comment on some things here which we intend to continue elsewhere, so the various parts ((A),(B),...) are not so connected.

(A). Ranks for dependent theories. We note some generalizations of §3, so Definition 3.5 is replaced by

5.1. *Definition:* (1) For $\ell = 1, 2, 3, 4, 5, 6, 8, 9, 11, 12$ (but not 7, 10), let

$$\begin{aligned} K_{m,\ell} = \big\{\mathfrak{x} : &\mathfrak{x} = (p, M, A), M \text{ a model } \prec \mathfrak{C}_T, A \subseteq \mathfrak{C}_T,\\ &\text{if } \ell \in \{1,4\} \text{ then } p \in \mathbf{S}^m(M), \text{ if } \ell \notin \{1,4\} \text{ then}\\ &p \in \mathbf{S}^m(M \cup A), \text{ and if } \ell = 3,6,9,12 \text{ then}\\ &p \text{ is finitely satisfiable in } M\big\}. \end{aligned}$$

If $m = 1$ we may omit it.

For $\mathfrak{x} \in K_{m,\ell}$ let $\mathfrak{x} = (p^{\mathfrak{x}}, M^{\mathfrak{x}}, A^{\mathfrak{x}}) = (p[\mathfrak{x}], M[\mathfrak{x}], A[\mathfrak{x}])$ and $m = m(\mathfrak{x})$, recalling $p^{\mathfrak{x}}$ is an m-type.

(2) For $\mathfrak{x} \in K_{m,\ell}$ let $N_{\mathfrak{x}}$ be M expanded by $R_{\varphi(\bar x,\bar y,\bar a)} = \{\bar b \in {}^{\ell g(\bar y)}M : \varphi(\bar x, \bar b, \bar a) \in p\}$ for $\varphi(\bar x, \bar y, \bar z) \in \mathbb{L}(\tau_T), \bar a \in {}^{\ell g(\bar z)}A$ and $\ell = 1,4 \Rightarrow \bar a = \langle\rangle$ and $R_{\varphi(\bar y,\bar a)} = \{\bar b \in {}^{\ell g(\bar y)}M : \mathfrak{C} \models \varphi[\bar b, \bar a]\}$ for $\varphi(\bar y, \bar z) \in \mathbb{L}(\tau_T), \bar a \in {}^{\ell g(\bar y)}\mathfrak{C}\}$; let $\tau_{\mathfrak{x}} = \tau_{N_{\mathfrak{x}}}$.

(2A) If we omit p we mean $p =$ tp$(\langle\rangle, M \cup A)$ so we can write N_A, a τ_A-model, so in this case $p = \{\varphi(\bar b, \bar a) : \bar b \in M, \bar a \in M$ and $\mathfrak{C} \models \varphi[\bar b, \bar a]\}$.

(3) For $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ let

(α) $\mathfrak{x} \le^\ell_{\rm pr} \mathfrak{y}$ means that $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ and
- (a) $A^{\mathfrak{x}} = A^{\mathfrak{y}}$,
- (b) $M^{\mathfrak{x}} \le_{A[\mathfrak{x}]} M^{\mathfrak{y}}$,
- (c) $p^{\mathfrak{x}} \subseteq p^{\mathfrak{y}}$,
- (d) if $\ell = 1,2,3,8,9$ then $M^{\mathfrak{x}} \le_{A[\mathfrak{x}],p[\mathfrak{y}]} M^{\mathfrak{y}}$ (for $\ell = 1$ this follows from clause (b)).

(β) $\mathfrak{x} \le^\ell \mathfrak{y}$ means that for some n and $\langle \mathfrak{x}_k : k \le n\rangle, \mathfrak{x}_k \le^\ell_{\rm at} \mathfrak{x}_{k+1}$ for $k < n$ and $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{x}_0, \mathfrak{x}_n)$ where

(γ) $\mathfrak{x} \le^\ell_{\rm at} \mathfrak{y}$ iff ($\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ and) for some $\mathfrak{x}' \in K_{m,\ell}$ we have
- (a) $\mathfrak{x} \le^\ell_{\rm pr} \mathfrak{x}'$,
- (b) $A^{\mathfrak{x}} \subseteq A^{\mathfrak{y}} \subseteq A^{\mathfrak{x}} \cup M^{\mathfrak{x}'}$,
- (c) $M^{\mathfrak{y}} \subseteq M^{\mathfrak{x}'}$,
- (d) $\ell \in \{1,4\} \Rightarrow p^{\mathfrak{y}} = p^{\mathfrak{x}'} \restriction M^{\mathfrak{y}}$ and $\ell \notin \{1,4\} \Rightarrow p^{\mathfrak{y}} = p^{\mathfrak{x}'} \restriction (M^{\mathfrak{y}} \cup A^{\mathfrak{y}})$.

(4) For $\mathfrak{x}, \mathfrak{y} \in K_{m,\ell}$ we say that $\mathfrak{y}$ explicitly $\bar\Delta$-splits ℓ-strongly over $\mathfrak{x}$ *when*: $\bar\Delta = (\Delta_1, \Delta_2), \Delta_1, \Delta_2 \subseteq \mathbb{L}(\tau_T)$ and, for some $\mathfrak{x}'$ and $\varphi(\bar x, \bar y) \in \Delta_2$, we have clauses (a),(b),(c),(d) of part (3)(γ) and

- (e) when $\ell \in \{1,2,3,4,5,6\}$, in $A^{\mathfrak{y}}$ there is a Δ_1-indiscernible sequence $\langle \bar a_k : k < \omega\rangle$ over $A^{\mathfrak{x}} \cup M^{\mathfrak{y}}$ such that $\bar a_k \in {}^{\omega>}(M^{\mathfrak{x}'})$ and $\varphi(\bar x, \bar a_0), \neg\varphi(\bar x, \bar a_1) \in p^{\mathfrak{x}'}$ and $\bar a_k \subseteq A^{\mathfrak{y}}$ for $k < \omega$,
- (e)$'$ when $\ell = 8,9,11,12$ there are $\bar b, \bar{\mathbf{a}}$ such that
  - (α) $\bar{\mathbf{a}} = \langle \bar a_i : i < \omega + 1\rangle$ is Δ_1-indiscernible over $A^{\mathfrak{x}} \cup M^{\mathfrak{y}}$,
  - (β) $A^{\mathfrak{y}} \backslash A^{\mathfrak{x}} = \{\bar a_i : i < \omega\}$; yes ω not $\omega + 1$! (note that "$A^{\mathfrak{x}} =$ " and not "$A^{\mathfrak{y}} \backslash A^{\mathfrak{x}} \supseteq$ " as we use it in (e)(γ) in the proof of 3.7),
  - (γ) $\bar b \subseteq A^{\mathfrak{x}}$ and $\bar a_i \in M^{\mathfrak{x}'}$ for $i < \omega + 1$,
  - (δ) $\varphi(\bar x, \bar a_k{}^\frown \bar b) \wedge \neg\varphi(\bar x, \bar a_\omega{}^\frown \bar b))$ belongs [4] to $p^{\mathfrak{x}'}$ for $k < \omega$.

(5) We define $\text{dp-rk}^m_{\bar\Delta,\ell} : K_{m,\ell} \to \text{ Ord } \cup \{\infty\}$ by

- (a) $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \ge 0$ always,
- (b) $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \ge \alpha + 1$ *iff* there is $\mathfrak{y} \in K_{m,\ell}$ which explicitly $\bar\Delta$-splits ℓ-strongly over $\mathfrak{x}$ and $\text{dp-rk}_{\bar\Delta,\ell}(\mathfrak{y}) \ge \alpha$,
- (c) $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \ge \delta$ iff $\text{dp-rk}^m_{\bar\Delta,\ell}(\mathfrak{x}) \ge \alpha$ for every $\alpha < \delta$ when δ is a limit ordinal.

These are clearly well defined. We may omit m from dp-rk as $\mathfrak{x}$ determines it.

(6) Let $\text{dp-rk}^m_{\bar\Delta,\ell}(T) = \bigcup\{\text{dp-rk}_{\bar\Delta,\ell}(\mathfrak{x}) : \mathfrak{x} \in K_{m,\ell}\}$; if $m = 1$ we may omit it.

[4] This explains why $\ell = 7, 10$ are missing.

(7) If $\Delta_1 = \Delta_2 = \Delta$ we may write Δ instead of (Δ_1, Δ_2). If $\Delta = \mathbb{L}(\tau_T)$ then we may omit it.

(8) For $\mathfrak{x} \in K_{m,\ell}$ let $\mathfrak{x}^{[*]} = (p^{\mathfrak{x}} \restriction M^{\mathfrak{x}}, M^{\mathfrak{x}}, A^{\mathfrak{x}})$.

So Observation 3.6 is replaced by

5.2. *Observation:* (1) $\le^{\ell}_{\rm pr}$ is a partial order on K_ℓ.

(2) $K_{m,\ell(1)} \subseteq K_{m,\ell(2)}$ when $\ell(1), \ell(2) \in \{1,2,3,4,5,6,8,9,11,12\}$ and $\ell(1) \in \{1,4\} \Leftrightarrow \ell(2) \in \{1,4\}$ and $\ell(2) \in \{3,6,9,12\} \Rightarrow \ell(1) \in \{3,6,9,12\}$.

(2A) $K_{m,\ell(1)} \subseteq \{\mathfrak{x}^{[*]} : \mathfrak{x} \in K_{m,\ell(2)}\}$ when $\ell(1) \in \{1,4\}, \ell(2) \in \{1,\dots,6,8,9,11,12\}$.

(2B) In (2A) equality holds if $x(\ell(1), \ell(2)) \in \{(1,2),(1,3),(4,5),(4,6)\}$.

(3) $\mathfrak{x} \le^{\ell(1)}_{\rm pr} \mathfrak{y} \Rightarrow \mathfrak{x} \le^{\ell(2)}_{\rm pr} \mathfrak{y}$ when $(\ell(1), \ell(2))$ is as in (2) and $\ell(2) \in \{2,3,8,9\} \Rightarrow \ell(1) \in \{2,3,8,9\}$.

(3A) $\mathfrak{x} \le^{\ell(1)}_{\rm pr} \mathfrak{y} \Rightarrow \mathfrak{x}^{[*]} \le^{\ell(1)}_{\rm pr} \mathfrak{y}^{[*]}$ when the pair $(\ell(1), \ell(2))$ is as in (2B).

(4) $\mathfrak{x} \le^{\ell(1)}_{\rm at} \mathfrak{y} \Rightarrow \mathfrak{x} \le^{\ell(2)}_{\rm at} \mathfrak{y}$ when $(\ell(1), \ell(2))$ are as in part (3) (hence (2)).

(4A) $\mathfrak{x} \le^{\ell(1)}_{\rm at} \mathfrak{y} \Rightarrow \mathfrak{x}^{[*]} \le^{\ell(2)}_{\rm at} \mathfrak{y}$ if $(\ell(1), \ell(2))$ are as in part (2A).

(5) $\mathfrak{y}$ explicitly $\bar\Delta$-splits $\ell(1)$-strongly over $\mathfrak{x}$ implies $\mathfrak{y}$ explicitly $\bar\Delta$-splits $\ell(2)$-strongly over $\mathfrak{x}$ when the pair $(\ell(1), \ell(2))$ is as in parts (2),(3) and $\ell(1) \in \{1,2,3,4,5,6\} \Leftrightarrow \ell(2) \in \{1,2,3,4,5,6\}$.

(6) Assume $(\ell(1), \ell(2))$ is as in parts (2),(3),(5). If $\mathfrak{x} \in K_{m,\ell(1)}$ then dp-rk$^m_{\bar\Delta,\ell(1)}(\mathfrak{x}) \le$ dp-rk$^m_{\bar\Delta,\ell(2)}(\mathfrak{x})$, i.e.,

$$\{\ell(1), \ell(2)) \in \{(3,2),(2,5),(3,5),(6,5),(3,6)\} \cup \{(9,8),(8,11),(9,11),(12,11),(9,12)\}.$$

(7) Assume $\bar a \in {}^m\mathfrak{C}$ and $\mathfrak{y} = (\mathrm{tp}(\bar a, M \cup A), M, A)$ and $\mathfrak{x} = (\mathrm{tp}(\bar a, M \cup A), M, A)$. Then

(a) $\mathfrak{x}^{[*]} = \mathfrak{y}^{[*]}$,

(b) $\mathfrak{x} \in K_{m,1} \cap K_{m,4}$,

(c) $\mathfrak{y} \in K_{m,2} \cap K_{m,5} \cap K_{m,8} \cap K_{m,11}$,

(d) if $\mathrm{tp}(\bar a, M \cup A)$ is finitely satisfiable in M then also $\mathfrak{y} \in K_{m,3} \cap K_{m,6} \cap K_{m,9} \cap K_{m,12}$.

(8) If $\mathfrak{x} \in K_{m,\ell(2)}$, then dp-rk$_{\ell^m(2)}(\mathfrak{x}^{[*]}) \le$ dp-rk$_{\ell^m(2)}(\mathfrak{x})$ when the pair $(\ell(1), \ell(2))$ is as in part (2A).

(9) If $\mathfrak{x} \in K_{m,\ell}$ and $\kappa > \aleph_0$, then there is $\mathfrak{y} \in K_{m,\ell}$ such that $\mathfrak{x} \le^{\ell}_{\rm pr} \mathfrak{y}$ and $M^{\mathfrak{y}}$ is κ-saturated; moreover, $M^{\mathfrak{y}}_{A[\mathfrak{y}],p[\mathfrak{y}]}$ is κ-saturated (hence in Definition 3.2(4), without loss of generality, $M^{\mathfrak{x}'}$ is $(|M^{\mathfrak{x}} \cup A^{\mathfrak{x}}|^+)$-saturated).

5.3. Claim: *In 3.7 we can allow $\ell = 1, 2, 5$ (in addition to $\ell = 8, 9$).*

Proof. Similar but:

$\kappa_{\rm ict}(T) > \aleph_0$ implies dp-rk$_\ell(T) = D$ when $\ell \in \{1, 2, 4, 5, 8, 9, 11, 12\}$:

(A) Let $A_n = \cup\{\bar a^m_t : m < n, t \in I_2\}$ if $\ell < 7$ and, if $\ell > 7, A_n = \{\bar a^m_t : m < n$ and $t \in I_1\}$.

(B) "$\mathfrak{x}_{n+1}$ explicitly split ℓ-strongly over $\mathfrak{x}_n$" using $\langle \bar a^n_{(2,n+i)} : i < \omega\rangle$ if $\ell < 7$ and $\langle a^n_{(1,i)} : i < \omega\rangle\hat{\ }\langle \bar a^n_{2,n}\rangle$ if $\ell > 7$.

(C) Similarly in "Lastly...": Lastly, if $\ell < 7$, $\varphi_n(x, \bar a^n_{(1,n)}), \neg\varphi_n(x, \bar a^n_{(1,n+1)})$ belongs to $p^{\mathfrak{x}'_n}$ and even $p^{\mathfrak{x}_{n+1}}$, and if $\ell > 7, \varphi_n(x, \bar a^n_{(1,n)})$ for $n < \omega$, $\neg\varphi_n(x, \bar a_{(2,n)})$ belongs to p_η, hence to $p^{\mathfrak{x}_{n+1}}$, so by renaming also clause (e) or (e)$^-$ from Definition 3.5(4) holds. Thus we are done.

dp-rk$_\ell(T) \geq |T|^+ \Rightarrow \kappa_{\rm ict}(T) > \aleph_0$ when $\ell = 1, 2, 3, 5, 6, 8, 9$:

(D) In $\circledast_n(e)$ we use

(E) (α) if $\ell \in \{2, 3, 5, 6\}$ and $m < n, k < \omega$, then $\varphi_m(x, \bar a^{n,m}_{\alpha,k}) \in p^{\mathfrak{x}^n_\alpha} \Leftrightarrow k = 0$ hence $\neg\varphi_m(x, \bar a^{n,m}_{\alpha,k}) \in p^{\mathfrak{x}^n_\alpha} \Leftrightarrow k \neq 0$ for $k < 2$;

(β) if $\ell = 1$ then $p^{\mathfrak{x}^n_\alpha} \cup \{\varphi_m(x, \bar a^{n,m}_{\alpha,k})^{{\rm if}(k=0)} : m < n, k < 2\}$ is consistent,

(γ) if $\ell = 8, 9$ we also have $\bar b^{n,m}_\alpha \subseteq A^{\mathfrak{x}^n_\alpha} = \bigcup\{\bar a^{n,i}_{\alpha,k} : i < m, k < \omega\} \cup A^n_\alpha$ for $m < n$ such that: if $\eta \in {}^n\omega$ and $m < n \Rightarrow \bar b^{n,m}_\alpha \subseteq \bigcup\{\bar a^{n,i}_{\alpha,k} : i < m, k < \eta(i)\} \cup A^n_\alpha$, then $(p^{\mathfrak{x}^n_\alpha} \restriction M^{\mathfrak{x}^n_\alpha}) \cup \{\varphi_m(\bar a^{n,m}_{\alpha,\eta(m)}, \bar b^{n,m}_\alpha) \wedge \neg\varphi_m(\bar x, \bar a^{n,m}_{\alpha,\eta(m)+1}, \bar b^{n,m}_\alpha) : m < n\}$ is finitely satisfiable in $\mathfrak{C}$.

(F) In checking clause (e) of $\circledast_{n+1}$

Case $\ell = 1$: We know that

$$p^{\mathfrak{x}^n_{\alpha+1}} \cup \{\varphi_m(x, \bar a^{n,m}_{\alpha,k})^{{\rm if}(k=0)} : m < n \text{ and } k < 2\}$$

is consistent. As $\mathfrak{x}^n_{\alpha+1} \leq^\ell_{\rm pr} \mathfrak{z}^n_\alpha$ by clause (α)(d) of Definition 3.5(3), we know that $q^{n+1}_\alpha := p^{\mathfrak{z}^n_\alpha} \cup \{\varphi_m(x, \bar a^{n,m}_{\alpha+1,k})^{{\rm if}(k=0)} : m < n$ and $k < 2\}$ is consistent. But $\varphi_n(x, \bar a^{n+1,m}_{\alpha,k}) = \varphi_n(x, \bar a^{n,m}_{\alpha+1,k}) \in q^{n+1}_\alpha$ for $k < 2, m < n$ and $\varphi_n(x, \bar a^{n+1}, m_{\alpha,k})^{{\rm if}(k=0)} = \varphi_n(x, \bar a^{n,*}_{\alpha,k})^{{\rm if}(k=0)} \in q^{n+1}_\alpha$ and $p^{\mathfrak{x}^{n+1}_\alpha} \subseteq p^{\mathfrak{z}^n_\alpha} \subseteq q^{n+1}_\alpha$, hence $p^{\mathfrak{x}^{n+1}_\alpha} \cup \{\varphi(x, \bar a^{n,m}_{\alpha,k})^{{\rm if}(k=0)} : m \leq n$ and $k < 2\}$, being a subset of q^{n+1}_α, is consistent, as required (this argument does not work for $\ell = 4$).

Case 2: $\ell \in \{2, 3, 5, 6\}$. Straightforward.

Case 3: $\ell \in \{8, 9\}$. ∎

As before

5.4. *Observation:* Like 3.9 for $\ell = 1,2,3,4,5,6,8,9,11,12$.

5.5. *Definition:* In Definition 3.10 we allow $\ell = 17, 18$.

5.6. *Observation:* (1) If "$\mathfrak{y}$ explicitly $\bar{\Delta}$-splits $\ell(1)$-strongly over $\mathfrak{x}$", then "$\mathfrak{y}$ explicitly $\bar{\Delta}$-splits $\ell(2)$-strongly over $\mathfrak{x}$" when

$$(\ell(1),\ell(2)) \in \{(15,14),(14,17),(18,17),(15,18)\} \cup \{(\ell,\ell+12) : \ell = 2,3,5,6\}.$$

(2) If $\mathfrak{x} \in K_{m,\ell(1)}$, then $\text{dp-rk}^m_{\Delta,\ell(1)}(\mathfrak{x}) \le \text{dp-rk}^m_{\Delta,\ell(2)}(\mathfrak{x})$ when $(\ell(1),\ell(2))$ is as above.

Proof. Easy by the definition. $\blacksquare$

5.7. Claim: (1) *In 3.12(3) we allow* $\ell = 17, 18$.

(2) "$\text{dp-rk}_\ell(T) \ge |T|^+ \Rightarrow \kappa_{ict}(T) \ge \aleph_1$"*; we allow* $\ell = 14, 15, 17, 18$.

5.8. Theorem: *In 4.1 we can allow*

(a) $\ell \in \{8,9,11,12\}$ *and even* $\ell \in \{14,15,17,18\}$.

Proof. Similar to 4.1. $\blacksquare_{5.8}$

We can try to use ranks as in §3 for T which are just dependent. In this case it is natural to revise the definition of the rank to make it more "finitary", say in Definition 3.5(4), clauses (e),(e)′ replace $\langle \bar{a}_k : k < \omega\rangle$ by a finite long enough sequence.

Meanwhile just note

5.9. Claim: *Let* $\ell = 1,2,3,5,6$ *[and even* $\ell = 14,15,17,18$*]. For any finite* $\Delta \subseteq \mathbb{L}(\tau_T)$ *we have: for every finite* Δ_1, $\text{rk}_{\Delta_1,\Delta,\ell}(T) = \infty$ iff *for every finite* Δ_1, $\text{rk}_{\Delta_1,\Delta,\ell}(T) \ge \omega$ iff *some* $\varphi(x,\bar{y}) \in \Delta$ *has the independence property.*

Proof. Similar proof to 3.7, 5.3.

Let $\langle \bar{a}_\alpha : \alpha < \omega\rangle \subseteq M$ be indiscernible.

Let $\varphi(\bar{x},\bar{a}_0), \neg\varphi(\bar{x},\bar{a}_1) \in p$ exemplify "p splits strongly over $A_\varepsilon = \bigcup\{M_{\alpha_\varepsilon} : \zeta < \varepsilon\} \cup A \cup B$ so $\text{tp}(\bar{a}_0, A_\varepsilon) = \text{tp}(\bar{a}_1, A_\varepsilon)$. Let $A^+ = A \cup \bar{a}_0 \cup a_1$ and we find $u \subseteq \{\alpha_\varepsilon : \varepsilon < \theta_1^+\}$ as required:

(∗) there is $N^+ \prec M, \|N^*\| \le \theta$ such that $N^* \prec N \prec M \Rightarrow \text{dp-rk}(A, p \restriction (N^* \cup A), N^*) = \text{dp-rk}(A,p,M)$. $\blacksquare_{5.9}$

5.10. *Question:* (1) Can such local ranks help us prove some weak versions of "every $p \in \mathbf{S}_\varphi(M)$ is definable"? (Of course, the first problem is to define such "weak definability"; see [Sh:783, §1]).

(2) Does this help for indiscernible sequences?

5.11. *Definition:* We define $K^x_{m,\ell}$ and dx-rk$^m_{\Delta,\ell}$ for $x = \{p, c, q\}$ as follows:

(A) for $x = p$: as in Definition 3.5(4),(5), 5.1(4),(5),

(B) for $x = c$: as in Definition 3.5(4),(5), 5.1(4),(5) but we demand that in clause (e),(e)′ of part (4) that $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ is contradictory;

(C) for $x = q$: as in Definition 3.5(4),(5), 5.1(4),(5) but in clauses (e),(e)′ of part (4) we have $\bar{a}_\alpha$ from $A^{\mathfrak{y}}$ for $\alpha < \omega + \omega$ such that $\{\varphi(x, a_\alpha)^{\text{if}(\alpha<\omega)} : \alpha < \omega + \omega\} \subseteq p^{\mathfrak{x}'}$ and in (e′) we have $\bar{a}_n$ from $A^{\mathfrak{y}}$ and $\mathbf{a}_{\omega+n}$ from $M^{\mathfrak{x}'}$. In detail:

(e) when $\ell \in \{1, 2, 3, 4, 5, 6\}$, in $A^{\mathfrak{y}}$ there is a Δ_1-indiscernible sequence $\langle \bar{a}_k : k < \omega \rangle$ over $A^{\mathfrak{x}} \cup M^{\mathfrak{y}}$ such that $\bar{a}_k \in {}^{\omega>}(M^{\mathfrak{x}'})$ for $\alpha < \omega$ and $\varphi(\bar{x}, \bar{a}_k), \neg\varphi(\bar{x}, \bar{a}_{\omega+k}) \in p^{\mathfrak{x}'}$ and $\bar{a}_k, \bar{a}_{\omega+k} \subseteq A^{\mathfrak{y}}$ for $k < \omega$,

(e)′ when $\ell = 8, 9, 11, 12$ there are $\bar{b}, \bar{\mathbf{a}}$ such that
- (α) $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \omega + \omega \rangle$ is Δ_1-indiscernible over $A^{\mathfrak{x}} \cup M^{\mathfrak{y}}$,
- (β) $A^{\mathfrak{y}} \supseteq A^{\mathfrak{x}} \cup \{\bar{a}_i : i < \omega + \omega\}$,
- (γ) $\bar{b} \subseteq A^{\mathfrak{x}}$ and $\bar{a}_i \in M^{\mathfrak{x}'}$ for $i < \omega + \omega$,
- (δ) $\varphi(\bar{x}, \bar{a}_k{}^\frown\bar{b}) \wedge \neg\varphi(\bar{x}, \bar{a}_\omega{}^\frown\bar{b})$ belongs [5] to $p^{\mathfrak{x}'}$ for $k < \omega$.

5.12. *Question:* Does Definition 5.11 help concerning Question 5.10?

5.13. *Discussion:* We can imitate §3 with dc-rk or dq-rk instead of dp-rk and use appropriate relatives of $\kappa_{\text{ict}}(T)$. But compare with §4.

* * *

(B). Minimal theories (or types). It is natural to look for the parallel of minimal theories (see end of the introduction).

A subsequent work of E. Firstenberg and the author [FiSh:E50], using [Sh:757] (see better [Sh:E63]), considered a generalization of "uni-dimensional stable T". The generalization says (see 5.22(1)):

5.14. *Definition:* (1) T is uni-dp-dimensional when: (T is a dependent theory and) if $\mathbf{I}, \mathbf{J}$ are infinite non-trivial indiscernible sequences of singletons, *then* $\mathbf{I}, \mathbf{J}$

[5] This explains why $\ell = 7, 10$ are missing.

have finite distance, see below, *or* $\mathbf{I}$ and $\mathbf{J}^*$ do, recalling $\mathbf{J}^*$ is the inverse of $\mathbf{J}$ (i.e., we invert the order).

(2) (From [Sh:93]) For indiscernible sequences $\mathbf{I}, \mathbf{J}$ over A we say that they are immediate A-neighbours *if* $\mathbf{I} + \mathbf{J}$ is an indiscernible sequence over A or $\mathbf{J} + \mathbf{I}$ is an indiscernible sequence over A. They have distance $\leq n$ if there are $\mathbf{I}_0, \ldots, \mathbf{I}_n$ such that $\mathbf{I} = \mathbf{I}_0, \mathbf{J} = \mathbf{I}_n$ and $\mathbf{I}_\ell, \mathbf{I}_{\ell+1}$ are immediate A-neighbors (so indiscernible over A) for $\ell < n$. They are neighbors [6] if they have distance $\leq n$ for some n.

(3) If $\mathbf{I}$ is an infinite indiscernible sequence over A, then $\mathbf{C}_A(\mathbf{I}) = \bigcup\{\mathbf{I}' : \mathbf{I}', \mathbf{I}$ have finite A-distance.

Discussion: Note that for $\mathrm{Th}(\mathbb{Q}, <)$, the first order theory of the rational order, any two increasing infinite sequences of elements are of distance 2. If we forget above to have the "or $\mathbf{I}, \mathbf{J}^*$ of finite distance", we shall get two classes up to the relevant equivalence.

5.15. *Problem:* (1) Does uni-dp-dimensional theories have a dimension theory?

(2) Can we characterize them?

(3) If $p \in \mathbf{S}^m(A)$, is there an indiscernible sequence $\mathbf{I} \subseteq p(\mathfrak{C})$ based on A?, i.e., such that $\{F(\mathbf{C}_A(\mathbf{I})) : F$ an automorphism of $\mathfrak{C}$ over $A\}$ has cardinality $< \mathfrak{C}$ (equivalently $\leq 2^{|T|+|A|}$) as is the case for simple theories.

We can try another generalization.

5.16. *Hypothesis* (till 5.23)*:* Let ℓ be as in Definitions 3.5 and 5.1.

5.17. *Definition:* T is dp$^\ell$-minimal when dp-rk$^\ell(\mathfrak{x}) \leq 1$ for every $\mathfrak{x} \in K_\ell$, i.e., $K_{m,\ell}$ for $m = 1$.

5.18. *Remark:* For this property, T and T^{eq} may differ. Probably, if we add only finitely many sorts, the "finite rank, i.e., dp-rk$^\ell(\mathfrak{x}) < n^* < \omega$ for every $\mathfrak{x} \in K_\ell$", is preserved.

5.19. *Observation:* T is dp$^\ell$-minimal *when*: for every infinite indiscernible sequence $\langle \bar{a}_t : t \in I\rangle, I$ complete, $\bar{a}_t \in {}^\alpha\mathfrak{C}$ and element $c \in \mathfrak{C}$ there is $\{t\} \subseteq I$ as

[6] We may prefer the local version: for every finite $\Delta \subseteq \mathbb{L}(\tau_T)$ and finite $A' \subseteq A$ (or $A' = A$) there are $\mathbf{I}', \mathbf{J}'$ realizing the Δ-type over A' of $\mathbf{I}, \mathbf{J}$, respectively, such that $\mathbf{I}', \mathbf{J}'$ are (infinite) indiscernible sequences over A' (or A) and have distance over A'.

in 2.1 (i.e., a singleton or the empty set if you like) when $\ell \le 12$, and as in 2.9 when $\ell \in \{14, \dots\}$.

Proof. Should be clear. $\blacksquare_{5.19}$

5.20. Claim: (1) *For* $\ell = 1, 2$ *we have* T *is* dp^ℓ*-minimal* when*: there are no* $\langle \bar{a}^i_n : n < \omega \rangle$ *and* $\varphi_i(x, \bar{y}_i)$ *such that*

(a) *for* $i = 1, 2$, $\langle \bar{a}^i_n : n < \omega \rangle$ *is an indiscernible sequence over* $\bigcup \{\bar{a}^{3-i}_n : n < \omega\}$,
(b) *for some* $b \in \mathfrak{C}$ *we have* $\models \varphi_1(b, \bar{a}^1_0) \wedge \neg\varphi_2(b, \bar{a}^1_1) \wedge \varphi_2(b, \bar{a}^2_0) \wedge \neg\varphi_2(b, \bar{a}^2_1)$.

(2) *Similarly for* $\text{rk-dp}^\ell(\mathfrak{x}) \le n (< \omega)$, *i.e., if we replace* 1 *by* n *in Definition 5.17.*

Proof. Straightforward. $\blacksquare$

5.21. *Problem:* (1) Are dp^ℓ-minimal theories T similar to o-minimal theories?
(2) Characterize the dp^ℓ-minimal theories of fields.
(3) What are the implications between "dp^ℓ-minimal" for the various ℓ?
(4) As above also for uni-dp-dimensionality.

5.22. Claim: (1) *For* $\ell = 1, 2$ *the theory* T, $\mathrm{Th}(\mathbb{R})$, *the theory of real closed fields is* dp^ℓ*-minimal; similarly for any o-minimal theory.*
(2) $\mathrm{Th}(\mathbb{R})$ *is* dp^ℓ*-minimal for* $\ell = 1, 2$, *similarly for any o-minimal theory.*
(3) *For prime* p, *the first order theory of the* p*-adic field is* dp^1*-minimal.*

Proof. (1) As in [FiSh:E50].
(2) Repeat the proof in [Sh:783, 3.3](6).
(3) By the proof of 1.17. $\blacksquare_{5.22}$

5.23. *Remark:* If T is a theory of valued fields with elimination of field quantifier (see Definition 1.14(1),(2)) and $k^{\mathfrak{C}_T}$ is infinite, this fails. However, if $\Gamma^{\mathfrak{C}_T}, k^{\mathfrak{C}_T}$ are dp^1-minimal *then* the dp-rk for T are ≤ 2.

Another direction is:

5.24. *Definition:* (1) We say that a type $p(\bar{x})$ is content minimal *when*:

(a) $p(\bar{x})$ is not algebraic,
(b) if $q(\bar{x})$ extends $p(\bar{x})$ and is not algebraic then $\Phi_{q(\bar{x})} = \Phi_{p(\bar{x})}$; see below.

(2) $\Phi_{p(\bar{x})} = \{\varphi(\bar{x}_0, \dots, \bar{x}_{n-1}) : \bigcup \{p(\bar{x}_\ell) : \ell < n\} \cup \{\varphi(\bar{x}_1, \dots, \bar{x}_n)\}$ is consistent (see [Sh:93]).

5.25. *Question:* Can we define a reasonable dimension for such types, at least for T dependent or even strongly dependent?

* * *

(C). LOCAL RANKS FOR SUPER DEPENDENT AND INDISCERNIBLES. Note that the original motivation of introducing "strongly dependent" in [Sh:783] was to solve the equation: X/dependent = superstable/stable. However, (the various variants of) strongly dependent, when restricted to the family of stable theories, gives classes which seem to be interesting but are not the class of superstable T. So the original question remains open. Now returning to the search for "super-dependent" we may consider another generalization of superstable.

5.26. *Definition:* (1) We define $\text{lc-rk}^m(p,\lambda) = \text{lc}_0\text{-rk}^m(p,\lambda)$ for types p which belong to $\mathbf{S}^m_\Delta(A)$ for some $A(\subseteq \mathfrak{C})$ and finite $\Delta(\subseteq \mathbb{L}(\tau_T))$.

It is an ordinal or infinity and

(a) $\text{lc-rk}^m(p,\lambda) \geq 0$ always,

(b) $\text{lc-rk}^m(p,\lambda) \geq \alpha = \beta+1$ *iff* for every $\mu < \lambda$ there are finite $\Delta_1 \supseteq \Delta$ and pairwise distinct $q_i \in \mathbf{S}^m_{\Delta_1}(A)$ extending p such that $i < 1+\mu \Rightarrow \text{lc-rk}^m(q_i,\lambda) \geq \beta$,

(c) $\text{lc-rk}^m(p,\lambda) \geq \delta, \delta$ a limit ordinal *iff* $\text{lc-rk}^m(p) \geq \alpha$ for every $\alpha < \delta$.

(2) For $p \in \mathbf{S}^m(A)$ let[7] $\text{lc-rk}^m(p,\lambda)$ be $\min\{\text{lc-rk}^m(p,\lambda) \restriction \Delta : \Delta \subseteq \mathbb{L}(\tau_T)$ finite$\}$.

(3) Let $\text{lc-rk}^m(T,\lambda) = \bigcup\{\text{lc-rk}^m(p,\lambda)+1 : p \in \mathbf{S}^m(A), A \subset \mathfrak{C}\}$.

(4) If we omit λ we mean $\lambda = |T|^{++}$.

5.27. *Discussion:* There are other variants and they are naturally connected to the existence of indiscernibles (for subsets of ${}^m\mathfrak{C}$, concerning subsets of ${}^{|T|}\mathfrak{C}$); probably representability is also relevant ([CoSh:919], [Sh:F705]).

5.28. CLAIM: (1) *The following conditions on T are equivalent (for all $\lambda > |T|^+$):*

(a)$_\lambda$ *for every A and $p \in \mathbf{S}^m_\Delta(A)$ we have* $\text{lc-rk}^m(p,\lambda) < \infty$,

(b)$_\lambda$ *for some $\alpha^* < |T|^+$, for every A and $p \in \mathbf{S}^m_\Delta(A)$ we have* $\text{lc-rk}^m(p,\lambda) < \alpha^*$,

[7] Easily, if $\Delta_1 \subseteq \Delta_2 \subseteq \mathbb{L}(\tau_T)$ are finite and $p_2 \in \mathbf{S}^m_{\Delta_2}(A)$ and $p_1 = p_2 \restriction \Delta_1$ *then* $\text{lc-rk}^m(p_1) \geq \text{lc-rk}^m(p_2)$. So $\text{lc-rk}^m(p,\lambda)$ is well defined.

(c)$_\lambda$ *there is no increasing chain* $\langle \Delta_n : n < \omega\rangle$ *of finite subsets of* $\mathbb{L}(\tau_T)$ *and* A *and* $\langle p_\eta : \eta \in {}^{\omega>}\lambda\rangle$ *such that* $p_\eta \in \mathbf{S}^m_{\Delta_{\ell g(\eta)}}(A)$ *and* $\nu \triangleleft \eta \Rightarrow p_\nu \subseteq p_\eta$, *and if* $\eta_1 \neq \eta_2$ *are from* ${}^n\lambda$ *then* $p_{\eta_1} \neq p_{\eta_2}$,

(c)$_{\aleph_0}$ *like (c)*$_\lambda$ *with* $\langle p_\eta : \eta \in {}^{\omega>}\omega\rangle$.

(2) *Similarly restricting ourselves to* $A = |M|$.

Proof. Easy. $\blacksquare_{5.28}$

Closely related is

5.29. *Definition:* (1) We define $\mathrm{lc}_1 - \mathrm{rk}^m(p, \lambda)$ for types $p \in \mathbf{S}^m(A)$ for $A \subseteq \mathfrak{C}$ as an ordinal or infinitely by:

(a) $\mathrm{lc}_1 - \mathrm{rk}^m(p, \lambda) \geq 0$ always,

(b) $\mathrm{lc}_1 - \mathrm{rk}^m(p, \lambda) \geq \alpha = \beta + 1$ *iff* for every $\mu < \lambda$ and finite $\Delta \subseteq \mathbb{L}(\tau_T)$ we can find pairwise distinct $q_i \in \mathbf{S}^m(A)$ for $i < 1 + \mu$ such that $p \restriction \Delta \subseteq q_i$ and $\mathrm{lc}_1 - \mathrm{rk}^m(q_i, \lambda) \geq \beta$,

(c) $\mathrm{lc}_1 - \mathrm{rk}^m(p, \lambda) \geq \delta$ for δ a limit ordinal iff $\mathrm{lc}_1 - \mathrm{rk}^m(p) \geq \alpha$ for every $\alpha < \delta$.

(2) If $\lambda = \beth_2(|T|)^{++}$ we may omit it.

5.30. Claim: (1) *The following conditions on* T *are equivalent when* $\mu > \lambda = \beth_2(|T|)^+$:

(a)$_\mu$ *for every* A *and* $p \in \mathbf{S}^m(A)$ *we have* $\mathrm{lc}_1 - \mathrm{rk}^m(p, \mu) < \infty$,

(b)$_\mu$ *for some* $\alpha^* < \beth_2(|T|)^+$, *for every* A *and* $p \in \mathbf{S}^m(A)$ *we have* $lc_1 - \mathrm{rk}^m(p, \mu) < \alpha^*$,

(c)$_\lambda$ *for no* A *do we have a non-empty set* $\mathbf{P} \subseteq \mathbf{S}^m(A)$ *such that, for every* $p \in \mathbf{P}$ *and finite* $\Delta \subseteq \mathbb{L}(\tau_T)$, *for some finite* Δ_1 *the set* $\{q \restriction \Delta_1 : q \in \mathbf{P}$ *and* $q \restriction \Delta = p \restriction \Delta\}$ *has cardinality* $\geq \lambda$,

(d)$_\lambda$ *letting* $\Xi = \bigcup\{\Xi_n : n < \omega\}, \Xi_n = \{\bar\Lambda : \bar\Lambda$ *is a sequence of length* n *of finite sets of formulas* $\varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T), \ell g(\bar x) = m\}$ *there is no* $\langle \Delta_{\bar\Lambda} : \bar\Lambda \in \Xi\rangle$, *where* $\Delta_{\bar\Lambda}$ *is a finite set of formulas such that: for every* λ *we can find* A *and* $\langle p_{\bar\Lambda,\eta} : \bar\Lambda \in \Xi$ *and* $\eta \in {}^{\ell g(\bar\Lambda)}\lambda\rangle$ *such that:*

(α) $p_{\bar\Lambda,\bar\eta} \in \mathbf{S}^m(A)$,

(β) *if* $\bar\Lambda \in \Xi_n, \eta \in {}^n\lambda$ *and* $\bar\Lambda' = \bar\Lambda{}^\frown\langle\Lambda_n\rangle \in \Xi_{n+1}$, *then* $p_{\bar\Lambda',\eta^\frown\langle\alpha\rangle} \restriction \Lambda_n = p_{\bar\Lambda,\eta} \restriction \Lambda_n$ *for* $\alpha < \lambda$ *and* $\langle p_{\bar\Lambda',\eta^\frown\langle\alpha\rangle} \restriction \Delta_{\bar\Lambda'} : \alpha < \lambda\rangle$ *are pairwise distinct,*

(e)$_\lambda$ *for some* $\langle \Delta_{\bar\Lambda} : \bar\Lambda \in \Xi\rangle$ *as above the set* $T \cup \Gamma_\lambda$ *is inconsistent, where* Γ *is non-empty and:*

(α) *if* $\bar\Lambda = \Xi_{n+1}, \eta \in {}^{n+1}\lambda$ *and* $\varphi(\bar x, \bar y) \in \Lambda_n$, *then*

$$(\forall \bar y)\Big[\bigwedge_{\ell < \ell g(\bar y)} P(y_\ell) \to (\varphi(\bar x_{\bar\Lambda,\eta}, \bar y) \equiv \varphi(\bar x_{\bar\Lambda\restriction n, \eta\restriction n}, \bar y))\Big],$$

(β) *if* $\bar\Lambda \in \Xi_{n+1}, \eta \in {}^n\lambda$ *and* $\alpha < \beta < \lambda$, *then*

$$\bigvee_{\varphi(x,\bar y)\in\Delta_{\bar\Lambda}} (\exists \bar y)\Big(\bigwedge_{\ell<\ell g(\bar y)} P(y_\ell) \wedge (\varphi(x_{\bar\Lambda,\eta\hat{}\langle\alpha\rangle} : \bar y)) \equiv \neg\varphi(\bar x_{\bar\Lambda,\eta\hat{}\langle\beta\rangle}, \bar y)\Big).$$

(2) *Similarly restricting ourselves to the cases* $A = |M|$, *i.e.,* A *is the universe of some* $M \prec \mathfrak{C}$.

Proof. We will elaborate elsewhere, using [893, Th 2.16, 335]. $\blacksquare_{5.30}$

5.31. *Definition:* (1) We define $\mathrm{lc}_2 - \mathrm{rk}^m(p,\lambda)$ and $\mathrm{lc}_3 - \mathrm{rk}^m(p,\lambda)$ like $\mathrm{lc}_0 - \mathrm{rk}^m(p,\lambda)$ and $\mathrm{lc}_1 - \mathrm{rk}^m(p,\lambda)$, respectively, replacing "$\Delta \subseteq \mathbb{L}(\tau_T)$ is finite" by "$\Delta \subseteq \mathbb{L}(\tau_T)$ and $\mathrm{arity}(\Delta) < \omega$" where:

(2) $\mathrm{arity}(\varphi)$ = the number of free variables of φ, $\mathrm{arity}(\Delta) = \sup\{\mathrm{arity}(\varphi) : \varphi \in \Delta\}$ (if we use the objects $\varphi(\bar x)$ we may use $\mathrm{arity}(\varphi(\bar x)) = \ell g(\bar x)$).

5.32. Claim: *The parallel of 5.28, 5.30 for Definition 5.31.*

Remark: In particular, the rank $\mathrm{lc}_3 - \mathrm{rk}^m$ seems related to the existence of indiscernibility, i.e.,

5.33. Conjecture: (1) *Assume* lc_ℓ-$rk^m(T) < \infty$ *for some* $\ell \le 3$. *We can prove (in ZFC!) that for every cardinal* μ, *for some* λ *we have* $\lambda \to (\mu)_T$.

(2) *Moreover,* λ *is not too large, say is less than* $\beth_{\omega+1}(\mu + |T|)^{++}$ *(or just* $< \beth_{(2^\mu)^+}$*).*

* * *

(D). Strongly[2] stable fields. A reasonable aim is to generalize the characterization of the superstable complete theories of fields. Macintyre [Ma71] proved that every infinite field whose first order theory is $\aleph_0$-stable, is algebraically closed. Cherlin [Ch78] proves that every infinite division ring whose first order theory is superstable, is commutative, i.e., is a field so algebraically closed. Cherlin and Shelah [ChSh:115] prove "any superstable theory $\mathrm{Th}(K)$, K an infinite field, is the theory of algebraically closed fields" (and this is true even for division rings). More generally, we would like to replace stable by dependent and/or superstable by strongly dependent or at least strongly^2 stable (or another variant).

Of course, for strongly dependent we should allow at least the following cases (in addition to the algebraically closed fields): the first order theory of the real field (not problematic, as it is the only one with finite non-trivial Galois groups), the p-adic field for any prime p and the first order theories covered by 1.17(2), i.e., $\mathrm{Th}(K^{\mathbb{F}})$ for such $\mathbb{F}$.

Hence

5.34. Conjecture: (a) *If K is an infinite field and $T = Th(K)$ is strongly^2 dependent (i.e., $\kappa_{ict,2}(T) = \aleph_0$),* then *$K$ is an algebraically closed field (not strongly!!).*

(b) *Similarly for division rings.*

(c) *If K is an infinite field and $T = Th(K)$ is strongly^1 dependent,* then *K is finite or algebraically closed or real closed or elementary equivalent to $K^{\mathbb{F}}$ for some $\mathbb{F}$ as in 1.17(2) (like the p-adics) or a finite algebraic extension of such a field.*

(d) *Similar to (c) for division rings.*

Of course it is even better to answer 5.35(1):

5.35. *Question:* (1) Characterize the fields with dependent first order theory.

(2) At least “strongly dependent” (or another variant; see (E), (F) below).

(3) Suppose M is an ordered field and $T = \mathrm{Th}(M)$ is dependent (or strongly dependent). Can we characterize?

Remark: But we do not know this even for stability. So adopting strongly dependent as our context we look to what we can do.

5.36. Claim: *For a dependent T and group G interpreted in the monster model $\mathfrak{C}$ of T, for every $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ there is $n_\varphi < \omega$ such that, if α is finite, $\langle \bar{a}_i : i < \alpha \rangle$ is such that $G \cap \varphi(\mathfrak{C}, \bar{a}_i)$ is a subgroup of G,* then *their intersection is the intersection of some $\leq n_\varphi$ of them.*

Remark: If T is stable this holds also for infinite α by the Baldwin–Saxl theorem [BaSx76].

Proof. See Kaplan-Shelah [KpSh:993]. ∎

5.37. Claim: *If the complete theory T is strongly^2 dependent,* then *"finite kernel implies almost surjectivity", which means that if in $\mathfrak{C}, G$ is a definable group, π a definable homomorphism from G into G with finite kernel, then* $(G : \mathrm{Rang}(\pi))$ *is finite.*

Proof. By a general result from [Sh:783, 3.8, 4.5] quoted here as 0.1. $\blacksquare_{5.37}$

5.38. Claim: *Being strongly$^\ell$ dependent is preserved under interpretation.*

Proof. By 1.4, 2.7. $\blacksquare_{5.38}$

Hence the proof in [ChSh:115] works "except" the part on "translating the connectivity", which relies on ranks not available here.

However, if T is stable this is fine, hence we deduce

5.39. *Conclusion:* If K is an infinite field and $\mathrm{Th}(K)$ is strongly^2 stable, *then* T is algebraically closed.

5.40. Claim: *Let p be a prime. Then T is not strongly dependent if T is the theory of differentially closed fields of characteristic p* or *T is the theory of some separably closed fields of characteristic p which is not algebraically closed.*

Proof. The second case implies the first because, if $\tau_1 \subseteq \tau_1, T_2$ a complete $\mathbb{L}(\tau_2)$-theory which is strongly dependent, then so is $T_1 = T_2 \cap \mathbb{L}(\tau_1)$. So let M be a $\aleph_1$-saturated separably closed field of characteristic p which is not algebraically closed. Let $\varphi_n(x) = (\exists y)(y^{p^n} = x)$ and $p_*(x) = \{\varphi_n(x) : n < \omega\}$ and let xE_ny mean $\varphi_n(x-y)$, so E^M_n is an equivalent relation.

Let $\langle a_\alpha : \alpha < \omega_1\rangle$ be an indiscernible set such that $\alpha < \beta < \omega_1 \Rightarrow a_\beta - a_\alpha \notin \varphi_1(M)$.

Let $\psi_n(x, y_0, y_1, \dots, y_{n-1}) = (\exists z)[\varphi_n(z) \wedge x = y_0 + y_1^p + \dots + y_{n-1}^{p^{n-1}} + z]$.

Now by our understanding of $\mathrm{Th}(M)$:

⊛ (a) if $b_\ell \in M$ for $\ell < n$ then $M \models (\exists x)\psi_n(x, b_0, \dots, b_{n-1})$,
(b) in M we have $\psi_{n+1}(x, y_0, \dots, y_n) \vdash \psi_n(x, y_0, \dots, y_{n-1})$,
(c) in M we have, if $\psi_n(b, a_{\alpha_0}, \dots, a_{\alpha_{n-1}}) \wedge \psi_n(b, a_{\beta_0}, \dots, a_{\beta_{n-1}})$, then $\bigwedge_{\ell<n} \alpha_\ell = \beta_\ell$.

[Why? Clause (a) holds because, if $b_\ell \in M$ for $\ell < n$, then $a = b_0 + b_1^p + \dots + b_{n-1}^{p^{n-1}}$ exemplifies "$\exists x$". Clause (b) holds because if $M \models \psi_{n+1}[a, b_0, \dots, b_{n-1}, b_n]$ as witnessed by $z \mapsto d$, then $M \models \psi_n[a, b_0, \dots, b_{n-1}]$ as witnessed by $z \mapsto d + b_n^{p^n}$, which $\in \varphi_n(M)$ as $\varphi_n(M)$ is closed under addition, and $d \in \varphi_n(M)$

by $d \in \varphi_{n+1}(M) \subseteq \varphi_n(M)$ and $b_n^{p^n} \in \varphi_n(M)$ as b_n witnesses it. Lastly, to prove clause (c) assume that for $\ell = 1,2$ we have $d^\ell = d_\ell^{p^n} \in \varphi_n(M), b = a_{\alpha_0} + a_{\alpha_1}^p + a_{\alpha_2}^{p^2} + \cdots + a_{\alpha_{n-1}}^{p^{n-1}} + d_1^{p^n}$ and $b = a_{\beta_0} + a_{\beta_1}^p + a_{\beta_2}^{p^2} + \cdots + a_{\beta_{n-1}}^{p^{n-1}} + d_2^{p^n}$. We prove this by induction on n. For $n = 0$ this is trivial, $n = m+1$ substituting, etc., we get $a_{\alpha_0} - a_{\beta_0} = (a_{\beta_1}^p - a_{\alpha_1}^p) + \cdots + (a_{\beta_{n-1}}^{p^{n-1}} - a_{\alpha_{n-1}}^{p^{n-1}}) + (d_2^{p^n} - d_1^{p^n}) \in \varphi_1(M)$, so by an assumption on $\langle a_\gamma : \gamma < \omega_1 \rangle$ it follows that $\alpha_0 = \beta_0$. As there are unique p-th roots the original equation implies $a_{\alpha_1} + a_{\alpha_2}^p + \cdots + a_{\alpha_{n-2}}^{p^{n-2}} + d_1^{p^n} = a_{\beta_1} + a_{\beta_2}^p + \cdots + a_{\beta_{n-2}}^{p^{n-2}} + d_2^{p^n}$, and we use the induction hypothesis.]

So together:

$\odot$ for every $\eta \in {}^\omega(\omega_1)$, there is $b_\eta \in M$ such that
- (α) $M \vdash \psi_n(b_\eta, a_{\eta(0)}, \ldots, a_{\eta(n-1)})$, hence
- (β) if $n < \omega, \nu \in {}^n(\omega_1), \nu \neq \eta \restriction n$ then $M \models \neg\psi_n(b_\eta, a_{\nu(0)}, \ldots, a_{\nu(m-1)})$.

This suffices. $\blacksquare_{5.40}$

* * *

(E). STRONGLY[3] DEPENDENT. It is still not clear which versions of strong dependent (or stable) will be most interesting. Another reasonable version is strongly[3] dependent, but see more below. It has parallel properties and is natural. Hopefully, at least some of those versions allows us to generalize weight (see [Sh:c, V,§3]); we intend to return to it elsewhere. Meanwhile, note:

5.41. *Definition:* (1) T is strongly[3] dependent if $\kappa_{\mathrm{ict},3}(T) = \aleph_0$ (see below).

(2) $\kappa_{\mathrm{ict},3}(T)$ is the first κ such that the following [8] holds:

if γ is an ordinal, $\bar{a}_\alpha \in {}^\gamma(M_{\alpha+1})$ for $\alpha < \delta, \langle \bar{a}_\alpha : \alpha \in [\beta, \delta) \rangle$ is an indiscernible sequence over M_β for $\beta < \delta$ and $\beta_1 < \beta_2 \Rightarrow M_{\beta_1} \prec M_{\beta_2} \prec \mathfrak{C}$ and $\bar{c} \in {}^{\omega>}\mathfrak{C}$ and $\mathrm{cf}(\delta) \geq \kappa$, such that if $n < \omega, \alpha_{\ell,0} < \cdots \alpha_{\ell,n-1} < k$ for $\ell = 1,2$, $\alpha_{1,i} \leq \alpha_{2,i}$ for $1 < n$ and $\bar{b}^1 \subseteq M_{\alpha^*_{1,n-1}}$ there is $\bar{b}^2 \subseteq M_{\alpha^*_{2,n-1}}$ such that $\bar{a}_{\alpha_{1,0}} \wedge \cdots \wedge \bar{a}_{\alpha_{1,n-1}} \wedge \bar{b}^1$ and $\bar{a}_{\alpha_{2,0}} \wedge \cdots \wedge \bar{a}_{\alpha_{2,n-1}} \wedge \bar{b}^2$ realize the same type, *then* for some $\beta < \kappa, \langle \bar{a}_\alpha : \alpha \in [\beta, \delta) \rangle$ is an indiscernible sequence over $M_\beta \cup \bar{c}$.

(3) We say T is strongly$^\ell$ stable if T is strongly$^\ell$ dependent and is stable.

(4) We define $\kappa_{\mathrm{ict},3,*}(T)$ and strongly^{3,*} dependent and strongly^{3,*} stable as in the parallel cases (see Definitions 1.8 and 2.12), i.e., above we replace $\bar{c}$ by $\langle \bar{c}_n : n < \omega \rangle$ indiscernible over $\bigcup\{M_\beta : \beta < \delta\}$.

---

[8] We may consider replacing δ by a linear order and ask for $< \kappa$ cuts.

5.42. Claim: (1) *If T is strongly$^{\ell+1}$ dependent* then T *is strongly$^\ell$ dependent for $\ell = 1, 2$.*

(2) *T is strongly$^\ell$ dependent iff T^{eq} is; moreover, $\kappa_{ict,\ell}(T) = \kappa_{ict,\ell}(T^{eq})$.*

(3) *If T_1 is interpretable in T_2* then *$\kappa_{ict,\ell}(T_1) \le \kappa_{ict,\ell}(T_2)$.*

(4) *If $T_2 = \mathrm{Th}(\mathfrak{B}_{M,MA})$ (see* [Sh:783, §1]*) and $T_1 = \mathrm{Th}(M)$* then *$\kappa_{ict,\ell}(T_2) = \kappa_{ict,\ell}(T_1)$.*

(5) *T is not strongly^3 dependent* iff *we can find $\bar\varphi = \langle \varphi_n(\bar x_0, \bar x_1, \bar y_n) : n < \omega\rangle$, $m = \ell g(\bar x_0)$, and for any infinite linear order I we can find an indiscernible sequence $\langle \bar a_t, \bar b_\eta : t \in I, \eta \in {}^{\omega>}I$ increasing$\rangle$ (see Definition 5.45 below) such that for any increasing sequence $\eta \in {}^\omega I$, the set $\{\varphi_n(\bar x_0, \bar a_s, \bar b_{\eta\restriction n})^{\mathrm{if}(s=\eta(n))} : n < \omega$ and $\eta(n-1) <_I s \in I$ if $n > 0\}$ of formulas is consistent (or use just $s = \eta(n), \eta(n)+1$ or $\eta(n) \le_I s$, it does not matter).*

(6) *The parallel of parts (1)–(5) hold with strongly^{3,*} instead of strongly^3. In particular, (parallel to part (5)) we have T is not strongly^{3,*} dependent* iff *we can find $\bar\varphi = \langle \varphi_n(\bar x_0, \dots, \bar x_{k(n)}, \bar y_n) : n < \omega\rangle, m = \ell g(\bar x)$, and for any infinite linear order I we can find an indiscernible sequence $\langle \bar a_t, \bar b_{\eta,t} : t \in I, \eta \in {}^{\omega>}I$ increasing$\rangle$ (see 5.45) such that for any increasing $\eta \in {}^\omega I$,*

$$\begin{aligned}&\{\varphi(\bar x_0, \bar a_s, \bar b_{\eta\restriction n})^{\mathrm{if}(s=\eta(n))} : n < \omega \text{ and } \eta(n-1) <_I s \text{ if } n > 0\}\\ &\cup \{\psi(\bar x_{i_0}, \dots, \bar x_{i_{m-1}}, \bar c) = \psi(\bar x_{j_0}, \dots, \bar x_{j_{m-1}}, \bar c) : m < \omega, i_0 < \dots < i_{m-1} < \omega,\\ &\qquad j_0 < \dots < j_{m-1} < \omega \text{ and } \bar c \subseteq \bigcup\{\bar a_s, b_\rho : s \in I, \rho \in {}^{\omega>}I \text{ increasing}\}\}\end{aligned}$$

is consistent.

Proof. (1)–(4). Easy.

(5), (6) Easy, see [Sh:F918]. $\blacksquare_{5.42}$

Recall that this definition applies to stable T (i.e., Definition 5.41(3)).

5.43. *Observation:* The theory T is strongly^3 stable *iff*: T is stable and we cannot find $\langle M_n : n < \omega\rangle, \bar c \in {}^{\omega>}\mathfrak{C}$ and $\bar{\mathbf{a}}_n \in {}^\omega(M_{n+1})$ such that:

(a) M_n is $\mathbf{F}^a_\kappa$-saturated,

(b) M_{n+1} is $\mathbf{F}^a_\kappa$-prime over $M_n \cup \bar{\mathbf{a}}_n$,

(c) $\mathrm{tp}(\bar{\mathbf{a}}_n, M_n)$ does not fork over M_0,

(d) $\mathrm{tp}(\bar c, M_n \cup \bar{\mathbf{a}}_n)$ forks over M_n.

Proof. Easy. $\blacksquare_{5.43}$

5.44. Conjecture: *For strongly^3 stable T we have dimension theory (including weight) close to the one for superstable theories (as in* [Sh:c, V]*). We may try to deal with it in* [Sh:839]*; it is related to* §5 *G below.*

(F). Representability and strongly$_4$ dependent. In [Sh:897] we deal with T being fat or lean. We say a class K of models is fat when, for every ordinal α, there are a regular cardinal λ and non-isomorphic models $M, N \in K_\lambda$ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent, where $\mathrm{EF}^+_{\alpha,\lambda}$ is a strong version of "the isomorphism player has a winning strategy in a strong version of the Ehrenfuecht–Frässe game of length λ". We prove there that, consistently, if T is not strongly stable and $T_1 \supseteq T$, then $\mathrm{PC}(T_1, T)$ is fat (in a work in preparation [Sh:F918] we show that it suffices to assume "T is not strongly$_4$-stable"; see below).

Cohen-Shelah [CoSh:919] deals with the stable case [Sh:F705], a work in preparation, we hope to deal with representability. The weakest form (for $\mathfrak{k}$ a class of index models, e.g., linear orders) is, e.g., first order T is weakly $\mathfrak{k}$-represented when for every model M of T and, say, a finite set $\Delta \subseteq \mathbb{L}(\tau_T)$ we can find an index model $I \in \mathfrak{k}$ and sequence $\langle \bar{a}_t : t \in I\rangle$ of finite sequences from $M^{\mathfrak{C}}$ (or just singletons) which is Δ-indiscernible, i.e., (see below) such that $|M| \subseteq \{a_t : t \in I\}$.

This is a parallel to stable and superstable when we play with essentially the arity of the functions of $\mathfrak{k}$ and the size of Δ's considered. The thesis is that T is stable *iff* it, essentially, can be represented for essentially $\mathfrak{k}$ the class of sets and parallel representability for $\mathfrak{k}$ derived for order characterize versions of the class of dependent theories. We also define $\mathfrak{k}$-forking, i.e., replace linear orders by other index sets. Meanwhile, [CoSh:919], has fulfilled those hopes for stable T but [KpSh:975] shows that for general dependent T the hopes fail. We define

5.45. *Definition:* (1) For any structure I we say that $\langle \bar{a}_t : t \in I\rangle$ is indiscernible (in $\mathfrak{C}$ over A) when: $\ell g(\bar{a}_t)$ depends only on the quantifier type of t in I and: if $n < \omega$ and $\bar{s} = \langle s_0, s_1, \dots, s_{n-1}\rangle, \bar{t} = \langle t_0, \dots, t_{n-1}\rangle$ realize the same quantifier-free type in I *then* $\bar{a}_{\bar{t}} := \bar{a}_{t_0}\char`^ \cdots \char`^\bar{a}_{t_{n-1}}$ and $\bar{a}_{\bar{s}} = \bar{a}_{s_0}\char`^ \cdots \char`^\bar{a}_{s_{n-1}}$ realize the same type (over A) in $\mathfrak{C}$.

(2) We say that $\langle \bar{b}_u : u \in [I]^{<\aleph_0}\rangle$ is indiscernible (in $\mathfrak{C}$) (over A) similarly: if $n < \omega, w_0, \dots, w_{m-1} \subseteq \{0, \dots, n-1\}$ and $\bar{s} = \langle s_\ell : \ell < n\rangle$, $\bar{t} = \langle t_\ell : \ell < n\rangle$ realize the same quantifier-free types in I and $u_\ell = \{s_k : k \in w_\ell\}, v_\ell = \{t_k : k \in w_\ell\}$ *then* $\bar{a}_{u_0}\char`^ \cdots \char`^\bar{a}_{u_{n-1}}, \bar{a}_{v_0}\char`^ \cdots \char`^\bar{a}_{v_{n-1}}$ realize the same type in $\mathfrak{C}$ (over A).

(3) We may use $\operatorname{incr}(<\omega, I)$ instead of $[I]^{<\aleph_0}$, where $\operatorname{incr}({}^\alpha I) = \operatorname{incr}_\alpha(I) = \operatorname{incr}(\alpha, I) = \{\rho : \rho$ is an increasing sequence of length α of members of $I\}$. We can use $< \alpha$ or $\leq \alpha$; clearly the difference between $\operatorname{incr}(<\omega, I)$ and $[I]^{<\aleph_0}$ is notational only (when we have order).

5.46. *Definition:* (1) We say that the m-type $p(\bar{x})$ does (Δ, n)-ict divide over A (or (Δ, n)-ict[1] divide over A) *when*: there are an indiscernible sequence $\langle \bar{a}_t : t \in I\rangle, I$ an infinite linear order and $s_0 <_I t_0 \leq_I s_1 <_I t_1 <_I \cdots \leq_I s_{n-1} <_I t_{n-1}$ such that

$\circledast_1$ $p(\bar{x}) \vdash$ "$\operatorname{tp}_\Delta(\bar{x}\hat{\ }\bar{a}_{s_\ell}, A) \neq \operatorname{tp}_\Delta(\bar{x}\hat{\ }\bar{a}_{t_\ell}, A)$" for $\ell < n$.

(2) We say that the m-type $p(\bar{x})$ (Δ, n)-ict^2-divides over A *when*, above, we replace $\circledast_1$ by:

$\circledast_2$ $p(\bar{x}) \vdash$ "$\operatorname{tp}_\Delta(\bar{x}\hat{\ }\bar{a}_{s_\ell}, \bigcup\{\bar{a}_{s_k} : k < \ell\} \cup A) \neq \operatorname{tp}_\Delta(\bar{x}\hat{\ }\bar{a}_{t_\ell}, \bigcup\{\bar{a}_{s_k} : k < \ell\} \cup A)$" for $\ell < n$.

(3) We say that the m-type $p(\bar{x})$ (Δ, n)-ict^3-divides over A *when*, above, $(\langle \bar{a}_t : t \in I \cup \operatorname{incr}(<n, I)\rangle$ is indiscernible over A and we replace $\circledast_1$ by

$\circledast_3$ $p(\bar{x}) \vdash$ "$\operatorname{tp}_\Delta(\bar{x}\hat{\ }\bar{a}_{s_\ell}, \bar{a}_{\langle s_0,\ldots,s_{\ell-1}\rangle} \cup A) \neq \operatorname{tp}_\Delta(\bar{x}\hat{\ }\bar{a}_{t_\ell}, \bar{a}_{\langle s_0,\ldots,s_{\ell-1}\rangle} \cup A)$" for $\ell < n$.

(4) We say that the m-type $p(\bar{x})$ (Δ, n)-ict^4-divides over A *when* there are $n^* < \omega$ and sequence $\langle \bar{a}_\eta : \eta \in \operatorname{inc}(\leq n^*, I)\rangle$ indiscernible over A such that (where $\operatorname{comp}(I)$ is the completion of the linear order I):

if $\bar{c}$ realizes $p(\bar{x})$, then for no set $J \subseteq \operatorname{comp}(I)$ with $\leq n$ members is the sequence $\langle \bar{a}_\eta : \eta \in \operatorname{inc}(\leq n^*, I^+)\rangle$ Δ-indiscernible over A, where $I^+ = (I, P_t)_{t\in J}$ and $P_t := \{s \in I : s < t\}$. Note that if T is stable, we can equivalently require $J \subseteq I$ and use $P_t := \{t\}$.

(5) For $k \in \{1, 2, 3, 4\}$ we say that the m-type $p(\bar{x})$ (Δ, n)-ict^k-forks over A *when* for some sequence $\langle \psi_\ell(\bar{x}, \bar{a}_\ell) : \ell < \ell(*) < \omega\rangle$ we have

(a) $p(\bar{x}) \vdash \bigvee_{\ell<\ell(*)} \psi_\ell(\bar{x}, \bar{a}_i)$,

(b) $\psi_\ell(\bar{x}, \bar{a}_\ell)$ (Δ, n)-ict^k-divides over A.

If $k = 1$ we may omit it; if $\Delta = \mathbb{L}(\tau_T)$ we may omit it.

(6) We define $\operatorname{ict}^k - \operatorname{rk}^m(p)$, an ordinal or ∞, as follows (easily well defined): $\operatorname{ict}^k - \operatorname{rk}^m(p) \geq \alpha$ *iff* p is an m-type and, for every finite $q \subseteq p$, finite $A \subseteq \operatorname{Dom}(p)$ and $n < \omega$ and $\beta < \alpha$, there is an m-type r extending q which $(\mathbb{L}(\tau_T), n) - \operatorname{ict}^k$-forks over A with ict^k-$\operatorname{rk}^m(r) \geq \beta$. If ict^k-$\operatorname{rk}^m(r) \ngeq \beta + 1$, we say that n witnesses this if the demand above for this n fails. If $n + 1$ is the minimal witness, let $n = \operatorname{ict}^k - \operatorname{wg}^n(r)$.

(7) $\kappa^m_{k,\mathrm{ict}}(T)$ is the first $\kappa \geq \aleph_0$ such that, for every $p \in \mathbf{S}^m(B), B \subseteq \mathfrak{C}$, there is a set $A \subseteq B$ of cardinality $< \kappa$ such that p does not ict^k-fork over A. Omitting m means for some $m < \omega$; note that we write $\kappa_{k,\mathrm{ict}}(T)$ to distinguish it from Definition 2.3 of $\kappa_{\mathrm{ict},2}$.

(8) T is strongly$_k$ dependent [stable] if $\kappa_{k,\mathrm{ict}}(T) = \aleph_0$ [and T is stable].

(9) We define $\kappa_{k,\mathrm{ict},*}(T)$ in a parallel way, i.e., now $p(\bar{x})$ is the type of an indiscernible sequence of m-tuples and T is strongly$_{k,*}$ dependent [stable] if it is dependent [stable] and $\kappa_{k,\mathrm{ict},*}(T) = \aleph_0$.

5.47. Claim: (1) *For dependent T, the following conditions are equivalent:*

(a) $\kappa_{4,\mathrm{ict},*}(T) > \aleph_0$*; see Definition 5.46(4),(7),(9).*

(b) *There are $m, \langle(\Delta_\ell, n_\ell) : \ell < \omega\rangle, I, \mathbf{J}$ such that:*

 (α) *$\Delta_\ell \subseteq \mathbb{L}(\tau_T)$ finite and $n_\ell < \omega$ and $n_\ell > \ell$ for $\ell < \omega$,*

 (β) *I is an infinite linear order with increasing ω-sequence of members,*

 (γ) *$\mathbf{J} = \langle \bar{a}_\rho : \rho \in \mathrm{inc}_{<\omega}(I)\rangle$ is an indiscernible sequence with $\bar{a}_\rho \in {}^\omega\mathfrak{C}$,*

 (δ) *for $\eta \in {}^\omega I$ an increasing sequence, for some $\bar{c}_\ell \in {}^m\mathfrak{C}(\ell < \omega)$ we have:*

  (i) *$\langle \bar{c}_\ell : \ell < \omega\rangle$ is an indiscernible sequence over $\bigcup\{\bar{a}_\rho : \rho \in \mathrm{incr}(I, < \omega)\}$,*

  (ii) *if J is the completion of the linear order I,* then *for no finite $J_0 \subseteq J$ do we have: if $n < \omega$ and $\rho^\ell_0, \dots, \rho^\ell_{n-1} \in \mathrm{incr}(I, < \omega)$ for $\ell = 1, 2$ are such that $\ell g(\rho^1_m) = \ell g(\rho^2_m)$ for $m < n$ and $\rho^1_0 \hat{} \cdots \hat{} \rho^1_{n-1}$ and $\rho^2_0 \hat{} \cdots \hat{} \rho^2_{n-1}$ realize the same quantifier free type over J_0 in J, then $\bar{a}_{\rho^1_0} \hat{} \cdots \hat{} \bar{a}_{\rho^1_{n-1}}, \bar{a}_{\rho^2_0} \hat{} \cdots \hat{} \bar{a}_{\rho^2_{n-1}}$ realize the same Δ_ℓ-type over $\bigcup\{\bar{c}_\ell : \ell < \omega\}$ in $\mathfrak{C}$.*

(c) *The natural rank is always $< \infty$.*

(2) *For dependent T the following conditions are equivalent:*

(a) $\kappa^m_{4,ict}(T) > \aleph_0$,

(b) *like (b) is part (1), only $\langle \bar{c}_\ell : \ell < \omega\rangle$ is replaced by one m-tuple $\bar{c}$,*

(c) $\mathrm{ict}^4 - \mathrm{rk}^m(\bar{x} = \bar{x}) = \infty$,

(d) $\mathrm{ict}^4 - \mathrm{rk}^m(\bar{x} = \bar{x}) \geq |T|^+$.

(3) *Similarly (just simpler) for $k = 1, 2, 3$ instead* 4.

Proof. Straightforwad, but for part (2) see details in Cohen and Shelah [CoSh:E65, §2]. $\blacksquare_{5.47}$

5.48. *Question:* (1) Can we characterize the T such that the ict^k-rk^1 rank of the formula $x = x$ is 1?

(2) Do we have ict^ℓ-$\mathrm{rk}^m(\bar{x} = \bar{x}) = \infty$ iff ict^ℓ-$\mathrm{rk}^1(x = x) = \infty$, i.e., can we in part (2) say that the properties do not depend on m? The positive answer will appear in Cohen and Shelah [CoSh:E65].

Now

5.49. *Observation:* (1) For $k = 1, 2, 3$, if $p(\bar{x})$ (Δ, n)-ict^k forks over A, *then* $p(\bar{x})$ (Δ, n)-ict^{k+1} forks over A.

(2) If T is strongly_{k+1} dependent/stable, *then* T is strongly_k dependent/stable.

(3) For $k \in \{1, 2, 3, 4\}$, if T is strongly k dependent/stable, *then* T is strongly k dependent/stable; if T_1 is interpretable in T_2 and T_2 is strongly_k dependent/stable, *then* so is T_1.

(4) Assume T is stable. If $p \in \mathbf{S}^m(B)$ does not fork over $A \subseteq B$, *then* ict^k-$\mathrm{rk}^m(p) = \mathrm{ict}^k - \mathrm{rk}^m(p \restriction A)$.

Remark: Also, the natural inequalities concerning ict_k-$\mathrm{rk}^n(-)$ follow by 5.49(1). The parallel of 5.49 holds for types of indiscernible sequences over A.

Proof. Straightforward. Details on the proof of part (3) for $k = 1$, see [CoSh:E65, 12] $\blacksquare_{5.49}$

5.50. *Example:* (1) There is a stable NDOP, NOTOP, not multi-dimensional countable complete theory which is not strongly^2 dependent.

(2) $T = \mathrm{Th}({}^{\omega_1}(\mathbb{Z}_2), E_n)_{n<\omega}$ is as above, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ as an additive group; $E_n = \{(\eta, \nu) : \eta, \nu \in {}^{\omega_1}(\mathbb{Z}_2)\}$ are such that $\eta \restriction (\omega n) = \nu \restriction (\omega n)$.

(3) As in part (1) but T is not strongly dependent.

Remark: This is [Sh:897, 0.2]. It shows that the theorem there adds more cases.

Proof. (1) By part (2).

(2) So let M_0 be the additive group $({}^{\omega_1}(\mathbb{Z}_2), +)$ where $+$ is coordinatewise addition and, for $\alpha \leq \omega$, let $M_\alpha = ({}^{\omega_1}(\mathbb{Z}_2), P_n)_{n<\alpha}$, where $P_n = \{\eta \in {}^{\omega_1}(\mathbb{Z}_2) : \eta \restriction (\omega n)\}$ is constantly zero and $E_n = \{(\eta, \nu) : \eta, \nu \in {}^{\omega_1}(\mathbb{Z}_2)$ are such that $\eta \restriction (\omega n) = \nu \restriction (\omega n)\}$ and $M'_\alpha = ({}^{\omega_1}(\mathbb{Z}_2), E_n)_{n<\alpha}$. Hence M'_α, M_α are bi-interpretable, so we shall use M_α. Let $T = \mathrm{Th}(M_\omega)$ and let $T_\alpha = \mathrm{Th}(M_\alpha)$. So for a model N of T_α is just an abelian group in which every element has

order 2, with distinguished subgraph P_n^N for $n < \alpha$, hence a vector space over the field $\mathbb{Z}_2$ and P_n^N decrease with n.

T is stable:

For $n < \omega$, a model of T_n is determined by finitely many dimensions: $(P_k^N : P_{k+1}^N)$ for $k < n$ (where E_0^N is interpreted as the equality), so T_n is superstable not multi-dimensional.

Hence T necessarily is stable.

T is strongly dependent not strongly^2 dependent:

As in 2.5; in fact it is strongly dependent.

T is not multi-dimensional:

If N is an $\aleph_1$-saturated model of T, then it is determined by the following dimension as vector spaces over $\mathbb{Z}_2$, for $n < \omega$:

$(*)_1$ P_n^N / P_{n+1}^N,

$(*)_2$ $\bigcap_{n<\omega} P_n^N$.

Each corresponds to a regular type (in $\mathfrak{C}_T^{\rm eq}$).

T has NDOP:

Follows from non-multi-dimensionality.

T has NOTOP:

Assume $N_\ell \prec \mathfrak{C}_T$ is $\aleph_1$-saturated, $N_0 \prec N_\ell$ for $\ell = 0, 1, 2$ such that $\mathrm{tp}(N_1, N_2)$ does not fork over N_0. Let A be the subgroup of $\mathfrak{C}$ generated by $N_1 \cup N_2$ and let $N_3 = \mathfrak{C}_T \restriction A$. Easily $N_3 \prec \mathfrak{C}_T$; moreover, N_3 is $\aleph_1$-saturated.

By [Sh:c, XII] this suffices.

(3) Expand M_α by $Q_m = \{\eta \in {}^{\omega_1}(\mathbb{Z}_2) : \eta \restriction [\omega m, \omega m + \omega) \text{ is constantly zero}\}$ for $m < n$. $\blacksquare_{5.50}$

(G). Strongly$_3$ stable and primely minimal types.

5.51. *Hypothesis:* T is stable (throughout §5 G).

5.52. *Definition:* [T stable] We say $p \in \mathbf{S}^\alpha(A)$ is primely regular (usually $\alpha < \omega$) *when*: if $\kappa > |T| + |\alpha|$ is a regular cardinal, the model M is κ-saturated, the type $\mathrm{tp}(\bar{a}, M)$ is parallel to p (or just a stationarization of it) and N is κ-prime over $M + \bar{a}$ and $\bar{b} \subseteq {}^{\kappa>}N \backslash {}^{\kappa>}M$, *then* $\mathrm{tp}(\bar{a}, M + \bar{b})$ is κ-isolated; equivalently [9] N is κ-prime over $M + \bar{b}$.

5.53. Claim: (1) *Definition 5.52 is equivalent to: there are $\kappa, M, \bar{a}, N$ as there.*

[9] Because N is κ-prime over $M + \bar{a} + \bar{c}$ whenever $\bar{c} \in {}^{\kappa>}N$.

(2) *We can in part (1) replace "$\kappa > |T| + |\alpha|$ regular, κ-prime" by "$\mathrm{cf}(\kappa) \geq \kappa(T), \mathbf{F}^a_\kappa$-prime", respectively.*

Proof. Straightforward. $\blacksquare_{5.53}$

Now (recalling Definition 5.41 and Observation 5.43)

5.54. Claim (T is strongly$_3$ stable): *If* $\mathrm{cf}(\kappa) \geq \kappa_r(T)$ *and* $M \prec N$ *are* $\mathbf{F}^a_\kappa$-*saturated,* then *for some* $a \in N \backslash M$ *the type* $\mathrm{tp}(a, M)$ *is primely regular.*

Proof. The reader can note that by easy manipulations, without loss of generality $\kappa = \mathrm{cf}(\kappa) > |T|$; in fact, by this we can use tp instead of stp, etc.

Let $\alpha_* = \min\{\mathrm{ict}^3 - \mathrm{rk}(\mathrm{tp}(a, M)) : a \in N\backslash M\}$, and let $a \in N\backslash M$ and $\varphi_*(x, \bar{d}_*) \in \mathrm{tp}(a, M)$ be such that $\alpha_* = \mathrm{ict}^3 - \mathrm{rk}(\{\varphi_*(x, \bar{d}_*)\})$.

We try to choose N_ℓ, a_ℓ, B_ℓ by induction on $\ell < \omega$ such that

$\boxplus_\ell$ (a) $M \prec N_\ell \prec N$ and $a_\ell \in N_\ell \backslash M$;
(b) N_ℓ is $\mathbf{F}^a_\kappa$-primary over $M + a_\ell$ and $a_0 = a$;
(c) if $\ell = m + 1$ *then*
(α) $N_\ell \prec N_m$ and $\mathrm{tp}(a_m, M + a_\ell)$ is not $\mathbf{F}^a_\kappa$-isolated,
(β) N_m is $\mathbf{F}^a_\kappa$-primary over $N_\ell + a_m$,
(γ) N_ℓ is $\mathbf{F}^a_\kappa$-constructible over $N_{\ell+1} + a_0$;
(d) (α) $B_\ell \subseteq N_\ell$,
(β) $a_\ell \in B_\ell$,
(γ) $|B_\ell| < \kappa$,
(δ) every $\mathbf{F}^a_\kappa$-isolated type $q \in \mathbf{S}^{<\omega}(M \cup B_\ell)$ has no extension in $\mathbf{S}^{<\omega}(M \cup \bigcup\{B_m : m \leq \ell\})$ which forks over $M \cup B_\ell$,
(ε) B_ℓ is $\mathbf{F}^a_\kappa$-atomic over $M + a_\ell$.

Let (N_ℓ, a_ℓ) be defined iff $\ell < 1 + \ell(*) \leq \omega$; clearly $\ell(*) \geq 0$.

$\boxtimes_1$ If $\ell(*) < \omega$, then $\mathrm{tp}(a_{\ell(*)}, M)$ is primely regular.

[Why? If not, then for some $b \in N_{\ell(*)} \backslash M$ we have $\mathrm{tp}(a_{\ell(*)}, M + b)$ is not $\mathbf{F}^a_\kappa$-isolated.

We try to choose $\bar{b}'_\varepsilon$ by induction on $\varepsilon < \kappa$ such that

($\boxtimes_{1.1}$) (α) $\bar{b}'_0 = \langle b \rangle$,
(β) $\bar{b}'_\varepsilon \in {}^{\omega >}(N_{\ell(*)})$,
(γ) $\mathrm{tp}(\bar{b}'_\varepsilon, M \cup \bigcup\{\bar{b}'_\zeta : \zeta < \varepsilon\} \cup \{b\}\}$ is $\mathbf{F}^a_\kappa$-isolated,
(δ) $\mathrm{tp}(\bar{b}'_\varepsilon, M \cup \bigcup\{\bar{b}'_\zeta : \zeta < \varepsilon\} \cup \{b, a_k, \ldots, a_{\ell(*)}\}$ is $\mathbf{F}^a_\kappa$-isolated for $k = \ell(*), \ldots, 0$,

(ε) $\mathrm{tp}(\bar{a}, M \cup \bigcup\{\bar{b}'_\zeta : \zeta \le \varepsilon\})$ forks over $M \cup \bigcup\{\bar{b}_\zeta : \zeta < \varepsilon\}$ for some $\bar{a} \in {}^{\omega>}(B_{\ell(*)})$ when $\varepsilon > 0$.

We are stuck for some $\varepsilon(*) < \kappa$ because $|B_{\ell(*)}| < \kappa$, and let $B' = \bigcup\{\bar{b}'_\varepsilon : \varepsilon < \varepsilon(*)\}$. Now we can find an $\mathbf{F}^a_\kappa$-saturated N' which is $\mathbf{F}^a_\kappa$-constructible over $M + B'$ and $\mathbf{F}^a_\kappa$-saturated N'' which is $\mathbf{F}^a_\kappa$-constructible over $N' \cup B_{\ell(*)}$. By the choice of B', the model N' is $\mathbf{F}^a_\kappa$-constructible also over $M \cup B_{\ell(*)} \cup B'$ (by the same construction), hence N'' is $\mathbf{F}^a_\kappa$-constructible over $M + B_{\ell(*)} + B'$.

Clearly N'' is $\mathbf{F}^a_\kappa$-prime over $M + B_{\ell(*)} + B'$ and $N_{\ell(*)}$ is $\mathbf{F}^a_\kappa$-prime over $M + B_{\ell(*)} + B'$ (as $B' \subseteq N_{\ell(*)}$, see clause (β) above, and B' has cardinality $< \kappa$). So there is an isomorphism f from N'' onto $N_{\ell(*)}$ over $M \cup B_{\ell(*)} \cup B$. Renaming, without loss of generality $f = \mathrm{id}_{N''}$ so $N'' = N_{\ell(*)}$.

Lastly, we shall show that (N', b, B') is a legal choice for $(N_{\ell(*)+1}, a_{\ell(*)+1}, B_{\ell(*)+1})$.

Why? The non-obvious clauses are (c)(β), (γ) and (d) of $\boxplus_{\ell(*)+1}$.

First, for clause (d) obviously $B' \subseteq |N'|, b \in N'$ and $|B'| < \kappa$, so (d)(α), (β), (γ) hold and clause (d)(ε) holds by the clause $\boxplus_{1.1}(\gamma)$. As for (d)(δ), assume $q \in \mathbf{S}^{<\omega}(M \cup B')$ is $\mathbf{F}^a_\kappa$-isolated, let $\bar{c} \in {}^{\omega>}(N')$ realize q, and let $B_q \subseteq M \cup B'$ be of cardinality $< \kappa$ such that $\mathrm{stp}(\bar{c}, B_q) \vdash \mathrm{stp}(\bar{c}, M \cup B')$. Now we have $\mathrm{stp}(\bar{c}, M \cup B') \vdash \mathrm{stp}(\bar{c}, M \cup B_{\ell(*)} \cup B')$, as otherwise we can find $\bar{c}'_\ell$ in $\mathfrak{C}$ realizing $\mathrm{stp}(\bar{c}, B_q)$. hence $\mathrm{stp}(\bar{c}, M \cup B')$ for $\ell = 1, 2$ such that $\mathrm{stp}(\bar{c}_1, M \cup B_{\ell(*)} \cup B') \neq \mathrm{stp}(\bar{c}_2, M \cup B_{\ell(*)} \cup B')$; so for some finite $\bar{a} \subseteq B_{\ell(*)}, \bar{d} \subseteq M$ we have $\mathrm{stp}(\bar{c}, \bar{d} \cup \bar{a} \cup B') \neq \mathrm{stp}(\bar{c}_2, \bar{d} \cup \bar{a} \cup B')$. Now without loss of generality $\bar{c}_1, \bar{c}_2$ are from $N_{\ell(*)}$, contradicting the choice of $\varepsilon(*)$. Let $\bar{\mathbf{b}}$ list B' without repetitions, so by the induction hypothesis $\mathrm{stp}(\bar{\mathbf{b}}\hat{\ }\bar{c}, M \cup B_{\ell(*)}) \vdash \mathrm{stp}(\bar{\mathbf{b}}\hat{\ }\bar{c}, M \cup B_0 \cup \cdots \cup B_{\ell(*)})$, hence $\mathrm{stp}(\bar{c}, M \cup B_{\ell(*)} \cup \bar{\mathbf{b}}) \vdash \mathrm{stp}(\bar{c}, M \cup B_0 \cup \cdots \cup B_{\ell(*)} \cup \bar{\mathbf{b}})$, so by the choice of $\bar{\mathbf{b}}$ and the previous sentence really clause (d)(δ) holds for the choice of $(N_{\ell(*)+1}, a_{\ell(*)+1}, B_{\ell(*)+1})$ above.

Second, concerning clause (c)(β) of $\boxplus_{\ell(*)+1}$, by the sentence after the choices of B', N' above we know that N' is $\mathbf{F}^a_\kappa$-constructively over $M \cup B_{\ell(*)} \cup B'$, so clearly $\mathrm{stp}(N', M \cup B') \vdash \mathrm{stp}(N', M \cup B' \cup B_{\ell(*)})$, hence $\mathrm{stp}(B_{\ell(*)}, M \cup B') \vdash \mathrm{stp}(B_{\ell(*)}, N')$, so easily $\mathrm{stp}(B_{\ell(*)}, M \cup B' \cup \{a_{\ell(*)}\}) \vdash \mathrm{stp}(B_{\ell(*)}, N')$.

Now $B_{\ell(*)} \cup B'$ is $\mathbf{F}^a_\kappa$-atomic over $M \cup \{a_{\ell(*)}\}$, being $\subseteq N_{\ell(*)}$, recalling $\boxplus_{\ell(*)}$(b) holds. Therefore hence $B_{\ell(*)}$ is $\mathbf{F}^a_\kappa$-atomic over $M \cup B' \cup \{a_{\ell(*)}\}$, hence by the previous sentence $B_{\ell(*)}$ is $\mathbf{F}^a_\kappa$-atomic over $N' + a_{\ell(*)}$; but $|B_{\ell(*)}| < \kappa$, hence it is $\mathbf{F}^a_\kappa$-constructible over $N' + a_{\ell(*)}$. As N'' is $\mathbf{F}^a_\kappa$-constructible over $B_{\ell(*)} \cup N'$ by its choice (and $a_{\ell(*)} \in B_{\ell(*)}$ by $\boxplus_{\ell(*)}(\mathrm{d})(\beta)$), clearly N'' is also $\mathbf{F}^a_\kappa$-constructible over $N' \cup \{a_{\ell(*)}\}$ as required in (c)(β).

Clause $\boxplus_\ell(\mathrm{c})(\gamma)$ means that $N_{\ell(*)} = N''$ is $\mathbf{F}^a_\kappa$-constructible over $N' + a_{\ell(*)}$. Now $N_{\ell(*)} = N''$ is $\mathbf{F}^a_\kappa$-constructible over $B_{\ell(*)} \cup N'$ and $\bar{a} \in {}^{\omega>}(N_{\ell(*)})$ implies $\mathrm{stp}(\bar{a}, B_{\ell(*)} \cup N') \vdash \mathrm{stp}(\bar{a}, B_0 \cup \cdots \cup B_{\ell(*)} \cup N')$, hence by monotonicity $\mathrm{stp}(\bar{a}, B_{\ell(*)} \cup N') \vdash \mathrm{stp}(\bar{a}, a_0 + B_{\ell(*)} + N')$, so by the same construction $N_{\ell(*)} = N''$ is $\mathbf{F}^a_\kappa$-constructible over $a_0 + B_{\ell(*)} + N'$. As $B_{\ell(*)} \subseteq N_{\ell(*)}, |B_{\ell(*)}| < \kappa$, it is enough to show that $B_{\ell(*)}$ is $\mathbf{F}^a_\kappa$-atomic over $a_0 + N'$, and this is proved as in the proof of clause (d)(δ) above. So indeed (N', b, B') is a legal choice for $(N_{\ell(*)+1}, a_{\ell(*)+1}, B_{\ell(*)+1})$. But this contradicts the choice of $\ell(*)$, so we have finished proving $\boxtimes_1$.]

$\boxtimes_2$ If $\ell = m+1 < 1 + \ell(*)$, *then* $\mathrm{tp}(a_m, N_\ell)$ is not orthogonal to M.

[Why? Toward a contradiction assume $\mathrm{tp}(a_m, N_\ell) \perp M$. So we can find $A_\ell \subseteq N_\ell$ of cardinality $<\kappa$ such that $\mathrm{tp}(\langle a_0, \ldots, a_m\rangle, A_\ell)$ is stationary, $\mathrm{tp}(\langle a_0, \ldots, a_m\rangle, N_\ell)$ does not fork over A_ℓ and $\mathrm{tp}(A_\ell, M)$ does not fork over $C_\ell := A_\ell \cap M$ and $\mathrm{tp}(A_\ell, C_\ell)$ is stationary, and $a_\ell \in A_\ell$ and (recalling N_ℓ is $\mathbf{F}^a_\kappa$-primary over $M + a_\ell$) we have $\mathrm{stp}(A_\ell, C_\ell + a_\ell) \vdash \mathrm{stp}(A_\ell, M + a_\ell)$; it follows that $\mathrm{tp}(M, A_\ell)$ does not fork over C_ℓ. As $\mathrm{tp}(a_m, M + A_\ell)$ is parallel to $\mathrm{tp}(a_m, N_\ell)$ and to $\mathrm{tp}(a_m, A_\ell)$ and $\mathrm{tp}(a_m, N_\ell) \perp M$ is assumed, we get that all three types are orthogonal to M. It follows that $\mathrm{stp}(a_m, A_\ell) \vdash \mathrm{stp}(a_m, M + A_\ell)$, but recall $a_\ell \in A_\ell$, so $\mathrm{stp}(a_m, A_\ell) \vdash \mathrm{stp}(a_m, M + a_\ell)$. As $|A_\ell| < \kappa$ this implies that $\mathrm{tp}(a_m, M + A_\ell)$ is $\mathbf{F}^a_\kappa$-isolated. But recall $\mathrm{stp}(A_\ell, C_\ell + a_\ell) = \mathrm{stp}(A_\ell, (A_\ell \cap M) + a_\ell) \vdash \mathrm{stp}(A_\ell, M + a_\ell)$. Together $\mathrm{stp}(a_m + A_\ell, C_\ell + a_\ell) \vdash \mathrm{stp}(a_m + A_\ell, M + a_\ell)$, hence $\mathrm{tp}(a_m, M + a_\ell)$ is $\mathbf{F}^a_\kappa$-isolated, contradicting $\boxtimes_\ell(c)(\alpha)$.]

To complete the proof by $\boxtimes_1$ it suffices to show $\ell(*) < \omega$, so toward a contradiction assume:

$\boxtimes_3$ $\ell(*) = \omega$.

As we are assuming $\boxtimes_3$, we can find $\langle N^+_\ell : \ell < \ell(*) = \omega\rangle$ such that:

$\odot_1$ (a) $N_\ell \prec N^+_\ell$,
(b) N_ℓ is saturated, e.g., of cardinality $\|N\|^{|T|}$,
(c) $N^+_{\ell+1} \prec N^+_\ell$,
(d) $\mathrm{tp}(N^+_\ell, N)$ does not fork over N_ℓ,
(e) $(N^+_\ell, c)_{c \in N_\ell \cup N^+_{\ell+1}}$ is saturated.

[Why? We can choose N^+_ℓ by induction on ℓ. For $\ell = 0$ it is obvious, and for $\ell = m+1$ we choose N'_ℓ while satisfying the relevant demands in $\odot_1$ on N^+_ℓ, and then choose N'_m satisfying the relevant demands on (N^+_ℓ, N^+_m). Lastly, by the uniqueness of the saturated model there is an isomorphism f_ℓ from N'_m onto N^+_m over N_m, and let $N_\ell = f_\ell(N'_\ell)$.]

Next, for $\ell < \ell(*)$ we can find $\mathbf{I}_\ell$ such that:

$\odot_2$ (a) $\mathbf{I}_\ell \subseteq N^+_\ell \backslash N^+_{\ell+1}$,

(b) $\mathbf{I}_\ell$ is independent over $(N^+_{\ell+1}, M)$, (i.e., $c \in \mathbf{I}_\ell \Rightarrow$ $\mathrm{tp}(c, N^+_{\ell+1})$ does not fork over M and $\mathbf{I}$ is independent over $N^+_{\ell+1}$),

(c) $\mathrm{tp}(N^+_\ell, N^+_{\ell+1} \cup \mathbf{I}_\ell)$ is almost orthogonal to M,

(d) if $c \in \mathbf{I}_\ell$, then either $c \in \varphi_*(\mathfrak{C}, \bar{d}_*)$ or $\mathrm{tp}(c, M)$ is orthogonal to $\varphi_*(x, \bar{d}_*)$, i.e., to every $q \in \mathbf{S}(M)$ to which $\varphi_*(x, \bar{d}_*)$ belongs,

(e) if $q \in \mathbf{S}(N^+_{\ell+1})$ does not fork over M and $\varphi_*(x, \bar{d}_*) \in q$ or q is orthogonal to $\varphi_*(x, \bar{d}_*)$, *then* the set $\{c \in \mathbf{I}_\ell : c \text{ realizes } q\}$ has cardinality $\|N_\ell\|$,

(f) we let $\mathbf{I}'_\ell = \mathbf{I}_\ell \cap \varphi_*(\mathfrak{C}, \bar{d}_*)$.

[Why possible? As $(N^+_\ell, c)_{c \in N^+_{\ell+1}})$ is saturated.]

Now for $\ell < \ell(*)$,

$\odot_3$ $\mathbf{I}_\ell$ is not independent over $(N^+_{\ell+1} + a, N^+_{\ell+1})$.

[Why? Recall $a = a_0$. Assume toward a contradiction that

$(*)_{3.1}$ $\mathbf{I}_\ell$ is independent over $(N^+_{\ell+1} + a, N^+_{\ell+1})$.

As by clause (b) of $\odot_2$ we have $\mathrm{tp}(\mathbf{I}_\ell, N^+_{\ell+1})$ does not fork over M, it follows that $\mathbf{I}_\ell$ is independent over $(N^+_{\ell+1}+a, M)$. Also, by $(*)_{3.1}$ we know that $\mathrm{tp}(a, N^+_{\ell+1} \cup \mathbf{I}_\ell)$ does not fork over $N^+_{\ell+1}$. Also, $\mathrm{tp}(a, N^+_{\ell+1})$ does not fork over $N_{\ell+1}$ (because $a \in N$ and $\mathrm{tp}(N^+_{\ell+1}, N)$ does not fork over $N_{\ell+1}$ by $\odot_1$(d)). Together it follows that

$(*)_{3.2}$ $\mathrm{tp}(a, N^+_{\ell+1} + \mathbf{I}_\ell)$ does not fork over $N_{\ell+1}$.

Recall that $\mathrm{tp}(N_\ell, N^+_{\ell+1})$ does not fork over $N_{\ell+1}$ (by $\odot_1$(d) because $N_\ell \prec N$ using symmetry) and $\mathrm{tp}(a, N_\ell \cup N^+_{\ell+1})$ does not fork over N_ℓ similarly, hence $\mathrm{tp}(N_\ell + a, N^+_{\ell+1})$ does not fork over $N_{\ell+1}$, hence

$(*)_{3.3}$ $\mathrm{tp}(N_\ell, N^+_{\ell+1} + a)$ does not fork over $N_{\ell+1} + a$.

Recall N_ℓ is $\mathbf{F}^a_\kappa$-constructible over $N_{\ell+1} + a$ (by $\boxplus_{\ell+1}(c)(\gamma)$), N_ℓ is $\mathbf{F}^a_\kappa$-saturated and $\mathrm{tp}(N^+_{\ell+1}, N_\ell + a)$ does not fork over $N_{\ell+1}$. Clearly

$(*)_{3.4}$ N_ℓ is also $\mathbf{F}^a_\kappa$-constructible over $N^+_{\ell+1} + a$ (even by the same construction).

As $\mathrm{tp}(a, N^+_{\ell+1} + \mathbf{I}_\ell)$ does not fork over $N_{\ell+1}$ and $N^+_{\ell+1}$ is $\mathbf{F}^a_\kappa$-saturated, it follows that

$(*)_{3.5}$ $\mathrm{tp}(N_\ell, N^+_{\ell+1} + \mathbf{I}_\ell)$ does not fork over $N^+_{\ell+1}$, hence over $N_{\ell+1}$.

But by $\odot_2$ clause (c), for every $\bar{d} \in {}^{\omega>}(N^+_\ell)$ the type $\mathrm{tp}(\bar{d}, N^+_{\ell+1} + \mathbf{I}_\ell)$ is almost orthogonal to M, hence recalling $N_\ell \subseteq N^+_\ell$,

$(*)_{3.6}$ $\mathrm{tp}(N_\ell, N^+_{\ell+1} + \mathbf{I}_\ell)$ is almost orthogonal to M (this does not depend on $\odot_{3.1} - \odot_{3.5}$ so can be used later).

Hence by $(*)_{3.5} + (*)_{3.6}$ we have

$(*)_{3.7}$ $\mathrm{tp}(N_\ell, N_{\ell+1})$ is almost orthogonal to M.

But $N_{\ell+1}$ is $\mathbf{F}^a_\kappa$-saturated, so this implies

$(*)_{3.8}$ $\mathrm{tp}(N_\ell, N_{\ell+1})$ is orthogonal to M.

But by $\boxplus_\ell$(b)

$(*)_{3.9}$ $a_\ell \in N_\ell$.

By $\boxtimes_2$ we have

$(*)_{3.10}$ $\mathrm{tp}(a_\ell, N_{\ell+1})$ is not orthogonal to M.

Together $(*)_{3.8} + (*)_{3.9} + (*)_{3.10}$ give a contradiction, so $(*)_{3.1}$ fails, hence $\odot_3$ holds.]

Now (recalling clause (f) of $\odot_2$)

$\odot_4$ $\mathbf{I}'_\ell$ is not independent over $(N^+_{\ell+1} + a, N^+_{\ell+1})$.

[Why? By $\odot_3$ + clauses (b)+(d) of $\odot_2$, recalling that $a \in \varphi_*(\mathfrak{C}, \bar{d}_*)$, by the choice of a in the beginning of the proof of 5.54.]

$\odot_5$ For each n, $\mathrm{tp}(a, N^+_n)$ $(\mathbb{L}(\tau_T), n)$-ict^3-forks over M.

[Why? By 5.55 below, with $\mathbf{I}_\ell, N^+_{n-\ell}$ here standing for $\mathbf{I}_{n-\ell-1}, N_\ell$ there, clause (d) there holds by $\odot_3$ here; M, A there stand for M, M here, clauses (a),(b),(c) there hold by $(*)_{3.6}$ here (recalling that $(*)_{3.6}$ does not depend on $\odot_{3.1} - \odot_{3.5}$.]

$\odot_6$ $\alpha_* >$ $\mathrm{ict}^3 - \mathrm{rk}(\mathrm{tp}(a, N^+_n))$ for every $n < \omega$.

[Why? By the choice of $\varphi_*(x, \bar{d}_*), a, \alpha_*$ in the beginning of the proof we have $\alpha^* =$ $\mathrm{ict}^3 - \mathrm{rk}(\mathrm{tp}(a, M))$, and by $\odot_5$ and the definition of $\mathrm{ict}^3 - \mathrm{rk}(-)$ this follows.]

$\odot_7$ For each n, $\mathrm{tp}(a, N^+_{n+1})$ is not orthogonal to M.

[Why? By $\odot_2(b) + \odot_4$.]

Hence we can find $q \in \mathbf{S}(M)$ such (for any n):

$\odot_8$ (a) some automorphism of $\mathfrak{C}$ over $\bar{d}_*$ maps $\mathrm{tp}(a, N_n)$ to a type parallel to q,
(b) $\mathrm{ict}^3 - \mathrm{rk}(q) < \alpha_*$,
(c) q and $\mathrm{tp}(a, N_{n+1})$ are not orthogonal,
(d) if $q' \subseteq q, |q'| < \kappa$ then $q'(N) \nsubseteq M$ [actually clause (d) follows by (c)].

This contradicts the choice of α_*; so $\ell(*) < \omega$ and we are done. $\blacksquare_{5.54}$

5.55. Claim: *Assume T is stable. A sufficient condition for*

$$\text{“}\mathrm{tp}(a, N_n)\ (\Delta, n) - \mathrm{ict}^3\text{-divides over } A\text{”}$$

is:

⊛ (a) $\langle N_\ell : \ell \leq n\rangle$ *is* $\prec$*-increasing,*
(b) $A \subseteq M \prec N_0$,
(c) $\mathbf{I}_\ell \subseteq N_{\ell+1} \backslash N_\ell$ *is independent over* (N_ℓ, M) *for* $\ell < n$,
(d) $\mathrm{tp}(a, N_\ell \cup \mathbf{I}_\ell)$ *forks over* $N_{\ell+1}$,
(e) $\mathrm{tp}(N_{\ell+1}, N_\ell + \mathbf{I}_\ell)$ *is almost orthogonal to* M.

Proof. Left to the reader, noting that $\langle \mathbf{I}_\ell : \ell < n\rangle$ are pairwise disjoint (by clauses (a) +(c)) and $\cup\{\mathbf{I}_\ell : \ell < n\}$ is independent). $\blacksquare_{5.55}$

5.56. *Remark:* (1) We may give more details on the last proof and intend to continue the investigation of the theory of regular types (in order to get good theory of weight) in this context somewhere else.

(2) We can use essentially 5.55 to define a variant of the rank for stable theory. So 5.55 can be written to use it and hence 5.57 connects the two ranks.

5.57. Claim: *Assume $k \in \{3, 4\}$ and ict^k-$\mathrm{rk}(T) < \infty$; see Definition 5.46(6).*

If $\mathrm{cf}(\kappa) \geq |T|^+$ *or less and $M \prec N$ are κ-saturated,* then *for some $a, \varphi(x, \bar{a}), n^*$ we have:*

⊛ (a) $a \in N \backslash M$,
(b) *if T is stable, the type $p = \mathrm{tp}(a, M)$ is primely regular,*
(c) $\bar{a} \in {}^{\omega>}M$ *and* $\varphi(x, \bar{a}) \in p$,
(d) $\omega \times (\mathrm{wict}^k\text{-}\mathrm{rk}(\varphi(x, \bar{a}))) + (\mathrm{ict}^k - \mathrm{wg}(\varphi(x, \bar{a})))$ *is minimal.*

Proof. We choose $a, \varphi_*(x, \bar{d}_*), \alpha, n_*$ such that:

⊛ (a) $a \in N \backslash M$,
(b) $\bar{d}_* \subseteq M$,
(c) $\mathfrak{C} \models \varphi[a, \bar{d}_*]$,
(d) $\alpha = \mathrm{ict}^k - \mathrm{rk}(\{\varphi_*(x, \bar{d}_*)\})$,
(e) under clauses (a)–(d), the ordinal α is minimal,
(f) n_* witness $\alpha + 1 \not\leq \ \mathrm{ict}^k - \mathrm{rk}(\{\varphi(x, \bar{d}_*)\})$,
(g) under clauses (a)–(f) the number $n_*(< \omega)$ is minimal.

Clearly there are such $a, \varphi_*(x, \bar{c}), \alpha$ and n_*. Then we try to choose (N_ℓ, a_ℓ) by induction on $\ell < \omega$ such that $\boxplus_\ell$ from the proof of 5.54 holds. But now we can prove similarly that $\ell(*) \leq n_*$. However, still $\mathrm{tp}(a, N_{\ell(*)})$ is not orthogonal to M.

[Why? We can choose $N_0^+, \dots, N_{\ell(*)}^+, \mathbf{I}_0, \dots, \mathbf{I}_{\ell(*)-1}$ as in $\odot_2 + \odot_3$ in the proof of 5.53 and prove $\odot_3$ there, which implies the statement above. As $\varphi_*(x, \bar{d}_*) \in \mathrm{tp}(a, N_{\ell(*)})$, it follows that $\varphi(N_{\ell(*)}, \bar{c}) \not\subseteq M$ and any $a' \in \varphi(N_{\ell(*)}, \bar{c}) \backslash M$ is as required.]

This is enough. $\blacksquare_{5.57}$

Similarly to Definition 5.46:

5.58. *Definition:* Let T be stable.

(1) For an m-type $p(\bar{x})$ we define $\mathrm{sict}^3\text{-}\mathrm{rk}^m(p(\bar{x}))$ as an ordinal or ∞ by defining when $\mathrm{ict}^3\text{-}\mathrm{rk}^m(p(\bar{x})) \geq \alpha$ for an ordinal α by induction on α:
$(*)^\alpha_{p(\bar{x})}$ $\mathrm{sict}^3\text{-}\mathrm{rk}^m(p(\bar{x})) \geq \alpha$ iff for every $\beta < \alpha$ and finite $q(\bar{x}) \subseteq p(x)$ and $n < \omega$ we have:
$(**)^{\beta,n}_{q(\bar{x})}$ we can find $\langle M_\ell : \ell \leq n \rangle, \langle \mathbf{I}_\ell : \ell < n \rangle$ and $\bar{a}$ such that:
(a) $M_\ell \prec \mathfrak{C}$ is $\mathbf{F}^a_{\kappa_1(T)}$-saturated,
(b) $M_\ell \prec M_{\ell+1}$,
(c) $q(\bar{x})$ is an m-type over M_0,
(d) $\bar{a}$ realizes $q(\bar{x})$ and $\beta \leq \ \mathrm{sict}^3 - \mathrm{rk}(\mathrm{tp}(\bar{a}, M_n)) \geq \beta$,
(e) $\mathbf{I}_\ell \subseteq {}^{\omega>}(M_{\ell+1})$ is independent over (M_ℓ, M_0),
(f) $\mathbf{I}_\ell$ is not independent over $(M_\ell + \bar{a}, M_0)$ (clearly, without loss of generality, $\mathbf{I}_\ell$ is a singleton).

(2) If $\mathrm{sict}^3\text{-}\mathrm{rk}^m(p(\bar{x})) = \alpha < \infty$, then we let $\mathrm{sict}^3\text{-}\mathrm{wg}^m(p(\bar{x}))$ be the maximal n such that, for every finite $q(\bar{x}) \subseteq p(\bar{x})$, we have $(**)^{\alpha,n}_{q(\bar{x})}$.

(3) Above instead of $\mathrm{sict}^3\text{-}\mathrm{rk}(\mathrm{tp}(\bar{a}, A))$ we may write $\mathrm{sict}^3\text{-}\mathrm{rk}^m(\bar{a}, A)$; similarly for $\mathrm{scit}^3\text{-}\mathrm{wg}^m(\bar{a}, A)$; if $m = 1$ we may omit it.

5.59. Claim: (1) *T is strongly$_3$ stable* iff *T is stable and* $\mathrm{sict}^3\text{-}\mathrm{rk}^m(p(\bar{x})) < \infty$ *for every m-type $p(\bar{x})$.*

(2) *For every type $p(\bar{x})$ there is a finite $q(\bar{x}) \subseteq p(\bar{x})$ such that* ($\mathrm{sict}^3\text{-}\mathrm{rk}(p(\bar{x}))$, $\mathrm{sict}^3\text{-}\mathrm{wg}(p(\bar{x})) = \mathrm{sict}^3 - \mathrm{rk}(q(\bar{x}))$, $\mathrm{sict}^3\text{-}\mathrm{wg}(q(\bar{x}))$).

(3) *If $p(\bar{x}) \vdash q(\bar{x})$, then* $\mathrm{sict}^3\text{-}rk(p(\bar{x})) \leq \mathrm{sict}^3\text{-}\mathrm{rk}(q(\bar{x}))$, *and if equality holds* then $\mathrm{sict}^3\text{-}\mathrm{wg}^m(p(\bar{x})) \leq \mathrm{sict}^3\text{-}\mathrm{wg}^m(q(\bar{x}))$.

(4) *(T stable) If $p(\bar{x})$, $q(\bar{x})$ are stationary parallel types,* then $\mathrm{sict}^3\text{-}\mathrm{rk}^m(p(\bar{x})) = \mathrm{sict}^3\text{-}\mathrm{rk}^m(q(\bar{x}))$, *etc. If $\bar{a}_1, \bar{a}_1$ realizes $p \in \mathbf{S}^m(A)$, then* $\mathrm{sict}^3\text{-}\mathrm{rk}^m(\mathrm{stp}(\bar{a}_1, A)) = \mathrm{sict}^3\text{-}\mathrm{rk}^m(\mathrm{stp}(\bar{a}_2, A))$. *Similarly for* $\mathrm{sict}^3\text{-}\mathrm{wg}^m$. *Also, automorphisms of $\mathfrak{C}$ preserve* $\mathrm{sict}^3\text{-}\mathrm{rk}^m$ *and* $\mathrm{sict}^3\text{-}\mathrm{wg}$.

5.60. Claim: *$p(\bar{x})$ (Δ, n)-*ict^3 *forks over A for every n* when*:*

$\odot$ (a) *G is a definable group over A (in $\mathfrak{C}$),*

(b) *$b \in G$ realizes a generic type of G from $\mathbf{S}(A)$, as was proved to exist in* [Sh:783, 4.11], *or T stable,*

(c) *$p(\bar{x}) \in \mathbf{S}^{<\omega}(A + b)$ forks over A.*

Remark: We may have said it in §5 F.

Proof of 5.60. Straightforward. ∎

5.61. *Conclusion:* Assume T is strongly$_3$ dependent.

If G is a type-definable group in $\mathfrak{C}_T$, *then* there is no decreasing sequence $\langle G_n : n < \omega \rangle$ of subgroups of G such that $(G_n : G_{n+1}) = \bar{\kappa}$ for every n.

5.62. *Remark:* (1) In 5.60 we can replace "ict^3" by "ict^4" and also by suitable variants for stable theories.

(2) Similarly in 5.61.

(H). T is n-dependent. On related problems and background see [Sh:702, 2.9–2.20], (but, concerning indiscernibility, it speaks about finite tuples, i.e., $\alpha < \omega$ in 5.71, which affect the definitions and the picture). On a consequence of "T is 2-dependent" for definable subgroups in $\mathfrak{C}$ (and more, e.g., concerning 5.64), see [Sh:886].

5.63. *Definition:* (1) A (complete first order) theory T is n-independent when clause $(a)^n$ in 5.64 below holds.

(2) The negation is n-dependent.

5.64. *Problem:* Sort out the relationships between the following candidates for "T is n-independent" (T is order order complete; also, we can fix φ; omitting m we mean 1):

(a)^n Some $\varphi(\bar{x}, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n-1})$ is n-independent, i.e., (a)^n_m for some m.

(a)^n_m Some $\varphi(\bar{x}, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n-1})$ is n-independent where $\ell g(\bar{x}) = m$, *where*:

$\odot$ $\varphi(\bar{x}, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n-1})$ is n-independent when there are $\bar{a}^\ell_\alpha \in {}^{\ell g(\bar{y}_\ell)}\mathfrak{C}$ for $\alpha < \lambda, \ell < n$ and $\langle \varphi(\bar{x}, \bar{a}^0_{\eta(0)}, \dots, \bar{a}^{n-1}_{\eta(n-1)}) : \eta \in {}^n\lambda$ is increasing$\rangle$ is an independent (sequence of formulas).

(b)^n_m There is an indiscernible sequence $\langle \bar{a}_\alpha : \alpha < \lambda \rangle, \varphi = \varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$, $m = \ell g(\bar{x}), \ell g(\bar{y}_\ell) = \ell g(\bar{a}_\alpha)$ for $\ell < n, \alpha < \lambda$ and $\bar{c} \in {}^{\ell g(\bar{x})}\mathfrak{C}$ such that: if $k < n$ and $\langle R_\ell : \ell < \ell(*) \rangle$ is a finite sequence of k-place relations on λ, *then* for some sequence $\bar{t}, \bar{s} \in {}^n\lambda$ realizing the same quantifier free type in $(\lambda, <, R_0, R_1, \dots, R_{\ell(\alpha)})$ we have $\mathfrak{C} \models \varphi[\bar{b}, \bar{a}_{s_0}, \dots, \bar{a}_{s_{n-1}}] \wedge \neg\varphi[\bar{b}, \bar{a}_{t_0}, \dots, \bar{a}_{t_{n-1}}]$.

(c)^n_m For some $\varphi = \varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}), \ell g(\bar{x}) = m$, for every $j \in [1, \omega)$, for infinitely many k there are $\bar{a}^\ell_i \in {}^{\ell g(\bar{y})}\mathfrak{C}$ for $i < k, \ell < n$ such that $|\{p \cap \{\varphi(\bar{x}, \bar{a}^0_{i_0}, \dots, \bar{a}^{n-1}_{i_{n-1}}) : i_\ell < k \text{ for } \ell < n\} : p \in \mathbf{S}^m(\bigcup\{\bar{a}^\ell_i : \ell < n, i < k\}|\}| \geq 2^{k^{n-1} \times m}$.

Remark: We can phrase (b)^n_m, (c)^n_m as alternative definitions of

$$\text{“}\varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}) \text{ is } n\text{-independent”}.$$

So in (b)^n_m it is better to have n indiscernible sequences.

5.65. *Observation:* If $\varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$ satisfies clause (a)^n, *then* it satisfies a strong form of clause (c)^n (for every k) and the number is $\geq 2^{k^n}$.

Remark: Clearly Observation 5.65 can be read as a sufficient condition for being n-dependent, e.g.:

5.66. *Conclusion:* T is n-dependent *when*: for every m, ℓ and finite $\Delta \subseteq \mathbb{L}(\tau_T)$ for infinitely many $k < \omega$ we have $|A| \leq k \Rightarrow |\mathbf{S}^m_\Delta(A)| < 2^{(k/\ell)^n}$.

5.67. *Question:* (1) Can we get clause (a) from clause (c)?
(2) Can we use it to prove (a)$^n_1$ $\equiv$(a)^n_m?

5.68. *Observation:* In 5.64, if clause (a) then clause (b).

5.69. *Question:* Does (b) imply (a)?

5.70. Claim: *If T satisfies $(a)^n$ for every n,* then*: if $\lambda \nrightarrow (\mu)_2^{<\omega}$ then $\lambda \nrightarrow_T (\mu)_{\aleph_0}$ where:*

5.71. *Definition:* We say that $\lambda \rightarrow_T (\mu)_\alpha$ when: if $\bar{a}_i \in {}^\alpha(\mathfrak{C}_T)$ for $i < \lambda$ *then* for some $\mathscr{U} \in [\lambda]^\mu$ the sequence $\langle \bar{a}_i : i \in \mathscr{U} \rangle$ is an indiscernible sequence in $\mathfrak{C}_T$.

Remark: (1) Note that for $\alpha < \omega$ this property behaves differently.

(2) Of course, if $\theta = 2^{|\alpha|+|T|}$ and $\lambda \rightarrow (\mu)_\theta^{<\omega}$ then $\lambda \rightarrow_T (\mu)_\alpha$.

(3) See on the non-2-independent T and definable groups in [Sh:886].

5.72. Conjecture: *Assume $\neg(a)^n$ (or another variant of n-dependent).* Then *ZFC* $\vdash \forall\alpha\forall\mu\exists\lambda(\lambda \rightarrow_T (\mu)_\alpha)$.

5.73. *Question:* Can we phrase and prove a generalization of the type-decomposition theorems for dependent theories ([Sh:900]) to n-dependent theories T, e.g., when $(\lambda_{\ell+1}^{\lambda_\ell}) = \lambda_{\ell+1}$ for $\ell < n, \mathfrak{B}_\ell \prec (\mathscr{H}(\bar{\kappa}^+), \in, <^*_{\bar{\kappa}^+})$ has cardinality λ_ℓ, $[\mathfrak{B}_{\ell+1}]^{\lambda_\ell} \subseteq \mathfrak{B}_\ell, \{\mathfrak{C}_T, \mathfrak{B}_{\ell+1}, \ldots, \mathfrak{B}_n\} \in \mathfrak{B}_\ell$.

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