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Philip Hall's problem on non-Abelian splitters

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Abstract

Philip Hall raised the following question which is stated in The Kourovka Notebook [**12**, p. 88]: *is there a non-trivial group which is isomorphic with every proper extension of itself by itself?* We will split the problem into two parts: we want to find non-commutative splitters, that are groups $G \neq 1$ with $\mathrm{Ext}\,(G, G) = 1$. The class of splitters fortunately is quite large so that extra properties can be added to G. We can consider groups G with the following properties: there is a complete group L with cartesian product $L^\omega \cong G$, $\mathrm{Hom}\,(L^\omega, S_\omega) = 0$ (S_ω the infinite symmetric group acting on ω) and $\mathrm{End}\,(L, L) = \mathrm{Inn}\, L \cup \{0\}$. We will show that these properties ensure that G is a splitter and hence obviously a Hall group in the above sense. Then we will apply a recent result from our joint paper [**9**] which also shows that such groups exist; in fact there is a class of Hall groups which is not a set.

1. *Introduction*

In one of his lecture courses at Cambridge in the 1960s Philip Hall investigated the following class of groups, which are characterized by our first:

Definition 1·1. We will say that a group G is a Hall group if any extension H of G is isomorphic to G provided G is normal in H and $H/G \cong G$.

John Lennox communicated in The Kourovka Notebook Hall's question concerning the existence of these Hall groups. We want to demonstrate the existence of Hall groups using some terminology which recently turned out to be of particular importance in module theory (and abelian groups) (see [**10**, **11**]).

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Definition 1·2. A group G is a weak splitter if any extension of G by G splits. In homological terms this is to say that $\operatorname{Ext}(G, G) = 1$, or equivalently any short exact sequence

$$1 \longrightarrow G \xrightarrow{\beta} H \xrightarrow{\alpha} G \to 1$$

gives raise to a splitting map $\gamma\colon G \to H$ such that $\gamma\alpha = \mathrm{id}_G$. Here maps are acting on the right. Hence $H = \ker\alpha \rtimes \operatorname{Im}\gamma \cong G \rtimes \operatorname{Im}\gamma$ and if $\operatorname{Im}\gamma$ is also normal in H, then we say that G is a splitter. Hence $G \lhd H$ with $H/G \cong G$ implies $H = G \times N$ for some normal subgroup $N \lhd H$ which is isomorphic to G.

Recall that $G = N \rtimes U$ is the semidirect product of N and U, where N is normal in G, and if also U is normal, then we write $N \times U$ for the direct product. In the case of abelian groups classical splitters are free abelian groups as well as torsion-free cotorsion groups, which has been well known for a considerable time (see [**4**]). Recall that an abelian group G^+ is cotorsion-free if O is the only cotorsion subgroup of G of equivalently G has no subgroups isomorphic to cyclic groups of order p, to the p-adic integers J_p for any prime p and/or to the rationals $\mathbb{Q}$ (see also [**1**]). An arbitrary group L is cotorsion-free if all its abelian subgroups are cotorsion-free. Other splitters have only recently been constructed (see [**10**, **11**]). They were also fundamental for solving the flat cover conjecture for modules. Here we will study non-commutative splitters, which are, however, obtained quite differently. Such groups will be based on the following special case:

Definition 1·3. We will say that a group $L \neq 1$ is rigid if the following holds:

(i) L has trivial centre $\mathfrak{z}L$ and is cotorsion-free.
(ii) $\operatorname{Hom}(L^\omega, S_\omega) = 0$.
(iii) $\operatorname{End} L = \operatorname{Inn} L \cup \{0\}$.

Here S_ω is the full symmetric group acting on a countable set $\omega = \{0, 1, 2, \dots\}$. $\operatorname{Aut} L$ and $\operatorname{Inn} L$ denote the automorphism group and the group of inner automorphisms of L, respectively. Moreover $\operatorname{Hom}(A, B)$ is the set of homomorphisms from group A to group B, where 0 is the zero-homomorphism mapping any element to 1; in particular $\operatorname{End} A = \operatorname{Hom}(A, A)$ is the near endomorphism ring of A. Any rigid group L has trivial centre and $\operatorname{Aut} L = \operatorname{Inn} L$, thus L is complete. By A^κ we denote the cartesian power over the cardinal κ of the group A. Moreover, using these natural definitions we have the possibility of finding Hall groups by means of our:

Main Theorem 1·4. *If L is a rigid group and $G \cong L^\omega$, then the following holds.*

(a) $\operatorname{Aut} G = \operatorname{Inn} G \rtimes S_\omega$.
(b) *G is a splitter.*
(c) $G \times G \cong G$.

Here we note that many groups with (a) in Main Theorem 1·4 are constructed in [**2**, **3**, **6**, **7**] – in fact for arbitrary groups in place of S_ω, but (b), (c) are also crucial. We have an immediate:

Corollary 1·5. *If L is a rigid group, then L^ω is a Hall group.*

In view of Theorem 1·4 it remains to demonstrate the existence of a rigid group. In Section 3 we will provide the following result from [**9**] and sketch the essentials of its (lengthy) proof in a few lines. We also note that condition (ii) of Definition 1·3 does not elucidate on group L itself, only on its infinite cartesian power. It would be desirable for convenience and for aesthetic reasons to have a group-theoretic condition immediately on L. This is established in Proposition 2·1 and leads to our Main Theorem from [**9**] mentioned as Theorem 3·1 in the last section. Here are the parts needed for application in the proof of our Main Theorem 1·4.

THEOREM 1·6. *For any infinite cardinals $\kappa < \mu$ with κ regular, $\mu = \mu^\kappa$ and $\lambda = \mu^+ > 2^{\aleph_0}$ there is a group H with the following properties.*

(i) *H is a simple group of cardinality λ.*

(ii) *There is an element $h \in H$ such that any element of H is a product of at most 4 conjugates of h.*

(iii) *H is rigid.*

2. *Proof of Main Theorem* 1·4

Proof. Condition (c) is obvious because $G \times G = L^\omega \times L^\omega \cong L^\omega = G$.

In order to demonstrate (a) let S_ω act naturally on ω and write $G = \prod_{n\in\omega} Le_n$, hence

$$x = \prod_{n\in\omega} x_n e_n \quad \text{with} \quad x_n \in L \tag{2·1}$$

denotes a general element of G. Also let $[x] = \{n \in \omega : \ x_n \neq 1\}$ denote the support of x. Moreover

$$G_A = \{x \in G : [x] \subseteq A\} \subseteq G \quad \text{for any } A \subseteq \omega.$$

If $\pi \in S_\omega$, then π induces an automorphism of G (also denoted by π) given by

$$\pi : G \longrightarrow G \ \left(x = \prod_{n\in\omega} x_n e_n \longrightarrow x\pi = \prod_{n\in\omega} x_{n\pi} e_n \right).$$

Hence $S_\omega \subseteq \operatorname{Aut} G$ and if $g \in G$, then let

$$g^* : G \longrightarrow G \ \ (x \longrightarrow xg^* = g^{-1}xg)$$

denote the inner automorphism, conjugation by g. Hence

$$G^* = \operatorname{Inn} G = \{g^* : g \in G\}$$

is normal in $\operatorname{Aut} G$ and visibly $G^* \cap S_\omega = 1$. The semidirect product

$$G^* \rtimes S_\omega \subseteq \operatorname{Aut} G$$

is a subgroup of $\operatorname{Aut} G$ and we claim that the two groups are equal. Let $\sigma \in \operatorname{Aut} G$ be a given automorphism and $n, m \in \omega$. Then we define a homomorphism

$$\sigma_{nm} : L \longrightarrow L \ \ (x \longrightarrow xe_n \longrightarrow xe_n\sigma \longrightarrow (xe_n\sigma)_m),$$

where clearly $(xe_n\sigma)_m$ is the mth coordinate of $xe_n\sigma = \prod_{i\in\omega}(xe_n\sigma)_i e_i$ as follows from our equation (2·1). Using the canonical embedding

$$\iota_n : L \longrightarrow Le_n \subseteq G \ (x \longrightarrow xe_n)$$

and the canonical projection

$$\pi_m\colon G \longrightarrow L \quad \left(x = \prod_{n\in\omega} x_n e_n \longrightarrow x_m\right),$$

we can also say that

$$\sigma_{nm} = \iota_n \sigma \pi_m \in \operatorname{End} L.$$

From Definition 1·3(iii) we have $\sigma_{nm} \in \operatorname{Inn} L \cup \{0\}$ and if $\sigma_{nm} \neq 0$ then there is $t_{\sigma nm} \in L$ such that $\sigma_{nm} = t^*_{\sigma nm}$. Let

$$Y_\sigma = \{(n,m)\colon \exists\ t^*_{\sigma nm} = \sigma_{nm}\},$$

hence

$$\sigma_{nm} = 0 \Longleftrightarrow (n,m) \in \omega \times \omega \setminus Y_\sigma.$$

Now we define $A_n = \{m \in \omega\colon (n,m) \in Y_\sigma\}$ and claim that

$$\text{the } A_n\ (n \in \omega) \text{ are pair-wise disjoint.} \tag{2·2}$$

If $m \in A_{n_1} \cap A_{n_2}$ and $n_1 \neq n_2$ then $(n_1, m), (n_2, m) \in Y_\sigma$. Take any $x_i \in L$ and let $y_i = x_i e_{n_i}$ $(i = 1, 2)$. Hence $[y_1, y_2] = 1$ in G and $[L\sigma_{n_1 m}, L\sigma_{n_2 m}] = 1$ follows from $L\sigma_{n_i m} = Le_{n_i}\sigma\pi_m \subseteq L$. If $L\sigma_{n_i m} = L$, then $[L, L] = 1$ and L would be abelian, which contradicts $\mathfrak{z}L = 1 \neq L$. If $L\sigma_{n_i m} \neq L$ then $\sigma_{n_i m} = 0$ by Definition 1·3(iii), which contradicts $(n_i, m) \in Y_\sigma$ and (2·2) is shown.

Next we observe:

$$\text{If } x = \prod_{i\in\omega} x_i e_i \in G \quad\text{and}\quad x_n = 1 \quad\text{then } x\sigma\pi_m = 1 \quad\text{for all } m \in A_n. \tag{2·3}$$

To see this, note that $x \in G_{\omega\setminus\{n\}}$, $G = G_{\{n\}} \times G_{\omega\setminus\{n\}}$ and $x\sigma\pi_m \in G_{\omega\setminus\{n\}}\sigma\pi_m$. From

$$[G_{\{n\}}, G_{\omega\setminus\{n\}}] = 1$$

follows that $G_{\omega\setminus\{n\}}\sigma\pi_m$ and $G_{\{n\}}\sigma\pi_m$ commute, and $m \in A_n$ implies that

$$G_{\{n\}}\sigma\pi_m = L\sigma_{nm} = L.$$

Hence $G_{\omega\setminus\{n\}}\sigma\pi_m = 1$ from $\mathfrak{z}L = 1$ and in particular $x\sigma\pi_m = 1$.

Now we want to show that

$$|A_n| = 1 \quad \text{for all } n \in \omega. \tag{2·4}$$

If $A_n = \varnothing$ and $1 \neq x \in L$, then $xe_n\sigma\pi_m = 1$ for all $m \in \omega$, hence $xe_n\sigma = 1$ and $1 \neq xe_n \in \ker\sigma$ contradicts $\sigma \in \operatorname{Aut} G$. If $|A_n| > 1$, then choose $m_1 \neq m_2 \in A_n$ and define $C = G_{\omega\setminus\{m_1,m_2\}}$ and $D = G_{\{m_1,m_2\}}$. Hence $G = C \times D$, $D \cong L \times L$ and consider the canonical projection $\pi_D\colon G \to D$. From (2·3) follows $\sigma\pi_D \restriction C = 0$, and $\iota_n\sigma\pi_D$ maps L into D. This map is an isomorphism as follows from $m_1, m_2 \in A_n$, hence $Le_n \cong Le_{m_1} \oplus Lm_{n_2}$. The right-hand side has the outer automorphism switching coordinates, while $\operatorname{Aut} L = L^*$ by Definition 1·3(iii). This is impossible, and $|A_n| = 1$ follows.

In the next step we show that

$$\bigcup_{n\in\omega} A_n = \omega. \tag{2·5}$$

Otherwise there is $1 \neq y \in G$ with $[y] \cap \bigcup_{n\in\omega} A_n = \emptyset$. Let $x \in G$ be the pre-image of y under the automorphism σ, hence $x\sigma = y$. We want to show that $x = \prod_{n\in\omega} x_n e_n = 1$, which is a contradiction. By (2·4) we can write $A_n = \{m\}$, hence

$$1 = y_m = x\sigma\pi_m = x_n\sigma_{nm} = x_n t^*_{nm}$$

using $m \notin [y]$ and (2·3), for all $n \in \omega$. We get $x_n = 1$ and $x = 1$ follows. From (2·4) and (2·5) follows that

$$\pi = \pi_\sigma : \omega \longrightarrow \omega \quad (n \to m) \quad \text{if } A_n = \{m\}$$

is a permutation $\pi \in S_\omega$, and it is easy to see that

$$\pi_{\sigma^{-1}} = (\pi_\sigma)^{-1}. \tag{2·6}$$

We view $\pi_\sigma \in \operatorname{Aut} G$ as described at the beginning. If we replace σ by $\sigma' = \sigma\pi_\sigma^{-1}$ then $\pi_{\sigma'} = \mathrm{id}$ is obvious and σ' acts component-wise on Le_n , inducing conjugations t^*_n for each $n \in \omega$. If $t = \prod_{n\in\omega} t_n e_n \in G$ and $D = \times_{n\in\omega} Le_n$ is the restricted direct product in G then $\sigma' \restriction D = t^* \restriction D$. Since L is cotorsion-free, $G = L^\omega$ is also cotorsion-free and thus stout as in [**5**]. It follows from [**5**, p. 49, theorem 4.1(4)] and $\sigma' \restriction D = t^* \restriction D$ that $\sigma' = t^*$ on all of G thus $\sigma = t^*\pi_\sigma \in G^* \rtimes S_\omega$ and (*a*) is shown.

It remains to demonstrate (*b*). Consider

$$1 \longrightarrow G \xrightarrow{\ \mathrm{id}\ } H \xrightarrow{\ \beta\ } M \to 1 \quad \text{with} \quad M \cong G$$

and let $\gamma : M \to H$ be a map of representatives for β in H, that is $x\gamma\beta = x$ for all $x \in M$. If $x \in M$, then $(x\gamma)^*$ is an inner automorphism of H which induces an automorphism $\alpha = (x\gamma)^* \restriction G$ of $G \lhd H$. From (*a*) we have $\alpha = y^*_x\pi_x$ for some $y_x \in G$ and $\pi_x \in S_\omega$. If we replace γ by $\gamma' : M \to H$ with $x\gamma' = y_x^{-1}(x\gamma)$, then γ' is again a coset representation for β and if we call the new map γ again, then $y_x = 1$ for all $x \in M$. Recall that

$$(x\gamma)^* \restriction G = \pi_x \quad \text{for all } x \in M,$$

and consider the map

$$\pi : G \longrightarrow S_\omega \,(x \longrightarrow \pi_x).$$

It is easy to check that π is a homomorphism, hence $\pi \in \operatorname{Hom}(G, S_\omega) = 0$ by Definition 1·3(ii), which sends every x to the identity in S_ω. This is to say that $x\gamma \in \mathfrak{c}_H G$ for all $x \in H$. Let $C = \mathfrak{c}_H G \subseteq H$ denote the centralizer of G in H. From $\mathfrak{z}G = 1$ it follows that $G \cap C = 1$; moreover by the above, H is generated by the normal subgroups G and C, hence $H = G \times C$. By the exact sequence above $\beta \restriction C : C \to M \cong G$ is an isomorphism and we arrive at $H \cong G \times G$; hence G is a splitter.

Proposition 2·1. *Let κ be a cardinal and K and L be groups with the following properties.*

(*a*) *L is simple and $|K| < |L|$, moreover*

(*b*) *$\exists\ g \in L, m \in \omega$ such that any $x \in L$ is product of at most m conjugates of g.*

Then $\operatorname{Hom}(L^\kappa, K) = 0$.

We derive:

COROLLARY 2·2. *If L is a simple group of cardinality $> 2^{\aleph_0}$ such that* $\operatorname{End} L = \operatorname{Inn} L \cup \{0\}$ *and condition* (*b*) *of Proposition* 2·1 *holds, then L is rigid and L^ω is a Hall group.*

Proof of the corollary. We can apply Proposition 2·1 for $K = S_\omega$, hence

$$\operatorname{Hom}(L^\omega, S_\omega) = 0.$$

The group L is simple and large, so $\mathfrak{z}L = 1$ and L is rigid. By Corollary 1·5 it follows that L^ω is a Hall group.

Proof of Proposition 2·1. Let $g \in L$ and $m \in \omega$ be as in (*b*). If $G = L^\kappa$ and σ is any homomorphism from $\operatorname{Hom}(G, K)$, then consider any $x = \prod_{i<\kappa} e_i x_i \in G$. We want to show that $x\sigma = 1$. By (*b*) these coordinates $x_i \in L$ of x can be expressed as

$$x_i = \prod_{j<k_i} g^{y_{ij}} \quad (i < \kappa)$$

as products of $k_i \leqslant m$ conjugates of g. If $A_k = \{i < \kappa \colon k_i = k\}$, then $\kappa = \bigcup_{k \leqslant m} A_k$ is a decomposition of κ and we may assume $A_k = \emptyset$ for each $k \in \omega \setminus \{k_i;\ i \in \kappa\}$. Using the earlier notation we show that $G = G_{A_1} \times \cdots \times G_{A_m}$ is expressed as a direct product. If $\bar{g} = \prod_{i<\kappa} e_i g \in G$ is the canonical diagonal element of $g \in L$, then using the elements $y_{ij}, x, \bar{g}$ we get new elements

$$\bar{y}_{kj} = \prod_{t \in A_k} e_t y_{kj}, \quad \bar{g}_k = \prod_{t \in A_k} e_t g, \quad \bar{x}_k = \prod_{t \in A_k} e_t x_t \in G_{A_k},$$

which are restrictions of the old ones to A_k. In particular

$$z_{kj} = \bar{g}_k^{\bar{y}_{kj}} \in G_{A_k}, \quad \bar{x}_k = \prod_{j<k} z_{kj} \quad \text{and} \quad x = \prod_{k \leqslant m} \bar{x}_k. \tag{2·7}$$

Consider the canonical diagonal homomorphisms

$$\iota_k : L \longrightarrow G_{A_k} \quad \left(x \longrightarrow \prod_{t \in A_k} e_t x \right) \qquad (k \leqslant m)$$

and note that $g\iota_k = \bar{g}_k$, hence

$$\bar{g}_k \sigma = g \iota_k \sigma = 1 \quad \text{from } \iota_k \sigma \in \operatorname{Hom}(L, K),$$

which is trivial because $|K| < |L|$ and L is simple. From (2·7) follows $x\sigma = 1$ and $\operatorname{Hom}(G, L) = 0$.

3. *A result from* [**9**]

It remains to find simple groups L of cardinality $|L| > 2^{\aleph_0}$ such that $\operatorname{End} L = \operatorname{Inn} L \cup \{0\}$ with the extra property that there are $g \in L, m \in \omega$ and any element of L is the product of at most m conjugates of g.

Relatives of these groups are constructed in [**8**, and references therein] for instance in [**2**, **3**, **6**, **7**]. However, they are not good enough for our purpose in this paper. When working on [**9**] we noticed the connection to the Hall problem and extended the construction in order to incorporate its use above.

THEOREM 3·1. *Let $\mathscr{A}$ be a family of suitable groups and $\kappa < \mu$ be infinite cardinals such that κ is regular uncountable and $\mu^\kappa = \mu$. Then we can find a group H of cardinality $\lambda = \mu^+$ such that the following holds.*

(i) *H is simple. Moreover, if $1 \neq g \in H$, then any element of H is a product of at most four conjugates of g.*

(ii) *Any $A \in \mathscr{A}$ is a subgroup of H and if $\mathscr{A}$ is not empty, then $H[\mathscr{A}] = H$, where $H[\mathscr{A}]$ is the subgroup of H generated by all subgroups of H isomorphic to A. If $\mathscr{A}$ is empty, then we may assume that H is cotorsion-free.*

(iii) *Any monomorphism $\varphi : A \to H$ for some $A \in \mathscr{A}$ is induced by some $h \in H$, that is there is some $h \in H$ such that $\varphi = h^* \restriction A$.*

(iv) *If $A' \subseteq H$ is an isomorphic copy of some $A \in \mathscr{A}$, then the centralizer $\mathfrak{c}_H A'$ is trivial.*

(v) *Any monomorphism $H \to H$ is an inner automorphism.*

For our application we can assume $\mathscr{A} = \varnothing$. Otherwise the following definition is needed.

Definition 3·2. Let A be any group with trivial center and view $A \subseteq \mathrm{Aut}\,(A)$ as inner automorphisms of A. Then A is called *suitable* if the following conditions hold:

(i) $A \neq 1$ is a finite group.

(ii) If $A' \subseteq \mathrm{Aut}\,(A)$ and $A' \cong A$ then $A' = A$.

(iii) $\mathrm{Aut}\,(A)$ is complete.

Note that $\mathrm{Aut}\,(A)$ has trivial centre because A has trivial centre. Hence the last condition only requires that $\mathrm{Aut}\,(A)$ has no outer automorphisms. It also follows from this that any automorphism of A extends to an inner automorphism of $\mathrm{Aut}\,(A)$. Recall the easy observation from [**8**] which is a consequence of the classification of finite simple groups:

All finite simple groups are suitable.

Also note that there are many well-known examples of suitable groups which are not simple. Just apply Wieland's theorem on automorphism towers of finite groups with trivial centre (see [**14**]).

The proof of Theorem 3·1 is a transfinite induction building the group H (which has cardinal $\lambda = \mu^+$ the successor cardinal of μ) as a union of a chain of subgroups H_α of cardinality μ. The inductive steps are separated by four disjoint stationary subsets S_i $(0 \leqslant i \leqslant 3)$ of λ, where ordinals in $S_0 \cup S_1 \cup S_2$ are limit ordinals of cofinality ω while ordinals in S_3 have cofinality κ. Passing from H_α to $H_{\alpha+1}$ now depends on the position of α. If α does not belong to one of these stationary subsets, then $H_{\alpha+1} = H_\alpha * \alpha\mathbb{Z}$ is just a free product of H_α with a (new) infinite cyclic group $\alpha\mathbb{Z}$. If α belongs to one of the first three stationary subsets, then HNN-extensions are used as in [**13**]. In case $\alpha \in S_0$ we must deal with the conjugacy problem for condition (i) of the theorem. Here it is enough to ensure that all elements of infinite order are conjugate and this is what HNN is designed for. Similarly we can deal with conditions (ii) and (iii) by free products with amalgamated subgroups using $\alpha \in S_1$ and $\alpha \in S_2$, respectively. An enumeration of elements with repetitions ensures that nothing is overlooked. Condition (iv), which is not needed here, is pure group theory, a book-keeping proof by transfinite induction. The more complicated demand is condition (v) of the theorem which is a strengthening for completion. As there are obviously more possible monomorphisms to deal with than elements in the group, a combinatorial principle is needed, a Black Box must be applied which still allows us to deal with the possible injections one after another while α runs with repetitions through S_3 (a set of 'only' λ elements) while enumerating partial injective maps on the set H. The basic tool is that the group H is built in such a way that there are many elements $h \in H_\alpha$ with large abelian centralizers $\mathfrak{c}_{H_\alpha}(h)$ of cardinality κ. These centralizers can be arranged to come from a rigid family of abelian groups, this is to say from a theorem shown two decades ago for abelian groups in [**1**, p. 465]:

THEOREM 3·3. *For each subset $X \subseteq \kappa$ of the set (the cardinal) κ there is an $\aleph_1$-free abelian group G_X of cardinal κ such that the following holds.*

$$\mathrm{Hom}\,(G_X, G_Y) = \begin{cases} \mathbb{Z}: & \text{if } X \subseteq Y \\ 0: & \text{if } X \nsubseteq Y. \end{cases}$$

The proof of the theorem on abelian groups uses an earlier Black Box from Shelah (see also [**1**] for more details).

An abelian group is $\aleph_1$-free if all its countable subgroups are free abelian. These abelian groups are also visibly cotorsion-free. By the indicated construction inductively it follows for $\mathscr{A} = \varnothing$ that each H_α hence H is cotorsion-free. Theorem 3·3 ensures that many centralizers are algebraically very different. And as centralizers must be mapped under monomorphisms into a centralizer, an idea often used for characterizing certain (automorphism) groups, it perhaps seems convincing that such a covering with a rigid system of abelian centralizers almost forces monomorphisms to be conjugated. The details, however, are time consuming and are done in [**9**].

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