The Journal of Symbolic Logic

http://journals.cambridge.org/JSL

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The Journal of Symbolic Logic / Volume 72 / Issue 01 / March 2007, pp 226 - 242
DOI: 10.2178/jsl/1174668393, Published online: 12 March 2014

Link to this article: http://journals.cambridge.org/abstract_S0022481200005570

How to cite this article:
Saharon Shelah (2007). Power set modulo small, the singular of uncountable cofinality . The Journal of Symbolic Logic, 72, pp 226-242 doi:10.2178/jsl/1174668393

THE JOURNAL OF SYMBOLIC LOGIC
Volume 72, Number 1, March 2007

POWER SET MODULO SMALL, THE SINGULAR OF UNCOUNTABLE COFINALITY

SAHARON SHELAH

Abstract. Let μ be singular of uncountable cofinality. If $\mu > 2^{\mathrm{cf}(\mu)}$, we prove that in $\mathbb{P} = ([\mu]^\mu, \supseteq)$ as a forcing notion we have a natural complete embedding of Levy$(\aleph_0, \mu^+)$ (so $\mathbb{P}$ collapses μ^+ to $\aleph_0$) and even Levy$(\aleph_0, \boldsymbol{U}_{J^{\mathrm{bd}}_\kappa}(\mu))$. The "natural" means that the forcing $(\{p \in [\mu]^\mu : p \text{ closed}\}, \supseteq)$ is naturally embedded and is equivalent to the Levy algebra. Also if $\mathbb{P}$ fails the χ-c.c. then it collapses χ to $\aleph_0$ (and the parallel results for the case $\mu > \aleph_0$ is regular or of countable cofinality). Moreover we prove: for regular uncountable κ, there is a family $\boldsymbol{P}$ of $\mathfrak{b}_\kappa$ partitions $\bar{A} = \langle A_\alpha : \alpha < \kappa\rangle$ of κ such that for any $A \in [\kappa]^\kappa$ for some $\langle A_\alpha : \alpha < \kappa\rangle \in \boldsymbol{P}$ we have $\alpha < \kappa \Rightarrow |A_\alpha \cap A| = \kappa$.

§0. Introduction. This work on the one hand continue the celebrated work of the Czech school on the completion of the Boolean algebras $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$ solving some of their questions and on the other hand tries to confirm the "pcf is effective" thesis.

We may consider the completions of the Boolean Algebras $\mathscr{P}(\mu)/\{u \subseteq \mu : |u| < \mu\} = \mathscr{P}(\mu)/[\mu]^{<\mu}$. This is equivalent to considering the partial orders $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$, viewing them as forcing notions, so actually looking at their completion $\hat{\mathbb{P}}_\mu$, which are complete Boolean Algebras. Recall that forcing notions $\mathbb{P}^1, \mathbb{P}^2$ are equivalent iff their completions are isomorphic Boolean Algebras. The Czech school has investigated them, in particular, (letting $\ell(\mu)$ be 0 if $\mathrm{cf}(\mu) > \aleph_0$ and 1 if $\mu > \mathrm{cf}(\mu) = \aleph_0$, (and $\aleph_{\ell(\mu)} = \mathfrak{h}$ if $\mu = \aleph_0$) consider the questions:

$\otimes_1$ (a) is $\hat{\mathbb{P}}_\mu$ isomorphic to the completion of the Levy collapse Levy$(\aleph_{\ell(\mu)}, 2^\mu)$?
(b) which cardinals χ the forcing notion $\mathbb{P}_\mu$ collapse to $\aleph_{\ell(\mu)}$?
(c) is $\mathbb{P}_\mu$ (θ, χ)-nowhere distributive for $\theta = \aleph_{\ell(\mu)}$? This can be phrased as: for some $\mathbb{P}_\mu$-name $\underset{\sim}{f}$ of a function from $\aleph_{\ell(\mu)}$ to χ, for every $p \in \mathbb{P}_\mu$ for some $i < \theta$ the set $\{\alpha < \chi : p \nVdash \underset{\sim}{f}(i) \neq \alpha\}$ has cardinality χ.

The first, (a) is a full answer, (b) the second seems central for set theories, the last is sufficient if the density is right, to get the first. The case of collapsing seems central (it also implies clause (c)) so we repeat the summary from Balcar, Simon [BaSi95] of what was known of the collapse of cardinals by $\mathbb{P}_\mu$, i.e., $\otimes_1(b)$. Let $\chi \to_\mu \theta$ denote the fact that χ is collapsed to θ by $\mathbb{P}_\mu$

Received January 10, 2006.
This research was supported by the United States-Israel Binational Science Foundation. Publication 861.
I would like to thank Alice Leonhardt for the beautiful typing.

0022-4812/07/7201-0013/$2.70

$\boxtimes_1$ (i) for $\mu = \aleph_0, 2^{\aleph_0} \to_\mu \mathfrak{h}$, (but $\mathbb{P}_\mu$ adds no new sequence of length $< \mathfrak{h}$ so we are done), Balcar, Pelant, Simon [BPS],

(ii) for μ uncountable and regular, $\mathfrak{b}_\mu \to_\mu \aleph_0$, (hence $\mu^+ \to_\mu \aleph_0$), Balcar, Simon [BaSi88],

(iii) for μ singular with $\mathrm{cf}(\mu) = \aleph_0, 2^{\aleph_0} \to_\mu \aleph_1$, Balcar, Simon [BaSi95],

(iv) for μ singular with $\mathrm{cf}(\mu) \neq \aleph_0, \mathfrak{b}_{\mathrm{cf}(\mu)} \to_\mu \aleph_0$, Balcar, Simon [BaSi95];

under additional assumptions for singular cardinals more is known

(v) for μ singular with $\mathrm{cf}(\mu) = \aleph_0$ and $\mu^{\aleph_0} = 2^\mu, \mu^{\aleph_0} \to_\mu \aleph_1$, Balcar, Simon [BaSi88],

(vi) for μ singular with $\mathrm{cf}(\mu) \neq \aleph_0$ and $2^\mu = \mu^+, 2^\mu \to_\mu \aleph_0$, [BaSi88].

Now [BaSi95] finish with the following very reasonable conjecture.

0.1. *Conjecture* (Balcar and Simon). In ZFC: for a singular cardinal μ with countable cofinality, $\mu^{\aleph_0} \to_\mu \aleph_1$ and for a singular cardinal μ with an uncountable cofinality $\mu^+ \to_\mu \aleph_0$ (here we concentrate on the case $\mathrm{cf}(\mu) > \aleph_0$, see below).

Concerning the other questions they prove

$\boxtimes_2$ (i) Balcar, Franek [BaFr87]:
if $\mu > \mathrm{cf}(\mu) > \aleph_0, 2^{\mathrm{cf}(\mu)} = \mathrm{cf}(\mu)^+$ then $\mathbb{P}_\mu$ is (ω, μ^+)-nowhere distributive,

(ii) Balcar, Simon [BaSi89, 5.20, pg. 380]:
if $2^\mu = \mu^+$ and $2^{\mathrm{cf}(\mu)} = \mathrm{cf}(\mu)^+$ then $\mathbb{P}_\mu$ is equivalent:
to Levy$(\aleph_0, \mu^+)$ if $\mathrm{cf}(\mu) > \aleph_0$ and
to Levy$(\aleph_1, \mu^+)$ if $\mathrm{cf}(\mu) = \aleph_0$,

(iii) Balcar, Franek [BaFr87]:
if $2^\mu = \mu^+, \mu = \mathrm{cf}(\mu) > \aleph_0, J$ a μ-complete ideal on μ and J nowhere precipitous extending $[\mu]^{<\mu}$ then $\mathscr{P}(\mu)/J$ is equivalent to Levy$(\aleph_0, \mu^+)$; also the parallel of (ii).

So under G.C.H. the picture was complete; getting clause (ii) of $\boxtimes_2$, and, in ZFC for regular cardinals $\mu > \aleph_0$ the picture is reasonable, particularly if we recall that by Baumgartner [Ba]

$\boxtimes_3$ if $\kappa = \mathrm{cf}(\mu) < \theta = \theta^{<\theta} < \mu < \chi$ and $V \models$ G.C.H. for simplicity and $\mathbb{P}$ is forcing for adding χ Cohen subsets to θ then

(a) forcing with $\mathbb{P}$ collapses no cardinal, changes no cofinality, adds no new sets of $< \theta$ ordinals,

(b) in $V^{\mathbb{P}}$, $([\mu]^\mu, \supseteq)$ satisfies the μ_1^+-c.c. where $\mu_1 = (2^\mu)^V$.

Lately, Kojman, Shelah [KjSh:720] prove the conjecture 0.1 for the case when $\mu > \mathrm{cf}(\mu) = \aleph_0$; morever

$\boxtimes_4$ (i) if $\mu > \mathrm{cf}(\mu) = \aleph_0$ then Levy$(\aleph_1, \mu^{\aleph_0})$ can be completely embedded into the completion of $\mathbb{P}_\mu$. Moreover,

(ii) the embedding is "natural": Levy$(\aleph_1, \mu^{\aleph_0})$ is equivalent to $\mathbb{Q}_\mu = (\{A \subseteq \mu : A \text{ a closed subset of } \mu \text{ of cardinality } \mu\}, \supseteq) \lessdot \mathbb{P}_\mu$.

Here we continue [KjSh:720] in §1, [BaSi89] in §2 but make it self contained. Naturally we may add to the questions (answered positively for the case $\mathrm{cf}(\mu) = \aleph_0$ by [KjSh:720])

$\otimes_2$ (a) can we strengthen "$\mathbb{P}_\mu$ collapse χ to $\aleph_{\ell(\mu)}$" to "Levy$(\aleph_{\ell(\mu)}, \chi)$ is completely embeddable into $\mathbb{P}_\mu$ (really $\hat{\mathbb{P}}_\mu$)",

(b) can we find natural such embeddings.

We may add that by [BaSi95] the Baire number of $\mathscr{U}[\mu]$, the space of all uniform ultrafilters over uncountable μ is $\aleph_1$, except when $\mu > \mathrm{cf}(\mu) = \aleph_0$ and in that case it is $\aleph_2$ under some reasonable assumptions. By [KjSh:720] the Baire number of $\mathscr{U}[\mu]$ is always $= \aleph_2$ when $\mu > \mathrm{cf}(\mu) = \aleph_0$.

Our original aim in this work has been to deal with $\mu > \mathrm{cf}(\mu) > \aleph_0$, proving the conjecture of Balcar and Simon above (i.e., that μ^+ is collapsed to $\aleph_0$), first of all when $2^{\mathrm{cf}(\mu)} < \mu$ answering $\otimes_2(a) + (b)$ using pcf and (replacing μ^+ by $\mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}}(\mu)$). In fact this seems, at least to me, the best we can reasonably expect. But a posteriori we have more to say.

For $\mu = \kappa = \mathrm{cf}(\mu) > \aleph_0$, though by the above we know that some cardinal $> \mu$ is collapsed (that is $\mathfrak{b}_\kappa$), we do not know what occurs up to 2^μ or when the c.c. fails. This leads to the following conjecture, (stronger than the Balcar, Simon one mentioned above). Of course, it naturally breaks to cases according to μ.

0.2. *Conjecture.* If $\mu > \aleph_0$ and $\mathbb{P}_\mu$ does not satisfy the χ-c.c., then forcing with $\mathbb{P}_\mu$ collapse χ to $\aleph_{\ell(\mu)}$, see Definition 0.6 below.

Note that

0.3. *Observation.* If conjecture 0.2 holds for $\mu > \aleph_0$ then $\mathbb{P}_\mu$ is equivalent to a Levy collapse iff it fails the $d(\mathbb{P}_\mu)$-c.c. where $d(\mathbb{P}_\mu)$ is the density of $\mathbb{P}_\mu$.

Lastly, we turn to the results; by 1.16(1):

0.4. Theorem. *If $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$ and $\mu > 2^\kappa$ then $\mathbb{Q}_\mu$ (a natural complete subforcing of $\mathbb{P}_\mu$, forcing with closed sets) is equivalent to* Levy$(\aleph_0, \boldsymbol{U}_{J^{bd}_\kappa}(\mu))$.

By 1.17, 1.18 and 2.6 we have

0.5. Theorem. *Conjecture* 0.2 *holds except possibly when* $\aleph_0 < \mathrm{cf}(\mu) < \mu < 2^{\mathrm{cf}(\mu)}$.

We shall in a subsequent paper prove the Balcar, Simon conjecture fully, i.e., in all cases.

0.6. Definition. For $\mu > \aleph_0$ we define $\ell(\mu) \in \{0, 1\}$ by

$\ell(\mu) = 0$ if $\mathrm{cf}(\mu) > \aleph_0$,

$\ell(\mu) = 1$ if $\mu > \mathrm{cf}(\mu) = \aleph_0$

and may add

$\ell(\mu) = \alpha$ when $\mu = \aleph_0, \mathfrak{h} = \aleph_\alpha$.

We thank Menachem Kojman for discussions on earlier attempts, Shimoni Garti for corrections and Bohuslav Balcar and Petr Simon for improving the presentation.

§1. Forcing with closed set is equivalent to the Levy algebra.

1.1. Definition. (1) For $f \in {}^\kappa(\mathrm{Ord}\backslash\{0\})$ and ideal I on κ let

$$\boldsymbol{U}_I(f) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\sup \mathrm{Rang}(f)]^{\leq\kappa} \text{such that for every } g \leq f \text{ for some } u \in \mathscr{P} \text{ we have } \{i < \kappa : g(i) \in u\} \in I^+\}.$$

(2) Let $U_I(\lambda)$ means $U_I(f)$ where f is the function with domain $\mathrm{Dom}(I)$ which is constantly λ.

1.2. *Hypothesis.*
(a) μ is a singular cardinal,
(b) $\kappa = \mathrm{cf}(\mu) > \aleph_0$.

1.3. DEFINITION. (1) $\mathbb{P}_\mu$ is the following forcing notion

$$p \in \mathbb{P}_\mu \text{ iff } p \in [\mu]^\mu$$
$$\mathbb{P}_\mu \models p \leq q \text{ iff } p \supseteq q.$$

(2) $\mathbb{P}'_\mu$ is the forcing notion with the same set of elements and with the partial order

$$\mathbb{P}'_\mu \models p \leq q \text{ iff } |q \backslash p| < \mu.$$

(3) $\mathbb{Q}_\mu = \mathbb{Q}^0_\mu$ is $\mathbb{P}_\mu \restriction \{p \in \mathbb{P}_\mu : p \text{ is closed in the order topology of } \mu\}$.

1.4. CHOICE/DEFINITION. (1) Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of regular cardinals $> \kappa$ with limit μ.
(2) Let $\lambda_i^- = \cup\{\lambda_j : j < i\}$.
(3) For $p \in \mathbb{P}_\mu$ let $a(p) = \{i < \kappa : p \cap [\lambda_i^-, \lambda_i) \neq \emptyset\}$.
(4) $\mathbb{Q}^1_\mu = \{p \in \mathbb{P}_\mu : i < \kappa \Rightarrow |p \cap \lambda_i| < \lambda_i$ and for each $i \in a(p)$ the set $p \cap \lambda_i \backslash \lambda_i^-$ has no last element, is closed in its supremum and has cardinality $> |p \cap \lambda_i^-|\}$.
(5) For $p \in \mathbb{Q}^1_\mu$ let $\mathrm{ch}_p \in \prod_{i \in a(p)} \lambda_i$ be $\mathrm{ch}_p(i) = \cup\{\alpha + 1 : \alpha \in p \cap [\lambda_i^-, \lambda_i)\}$ and $\mathrm{cf}_p \in \prod_{i \in a(p)} \lambda_i$ be $\mathrm{cf}_p(i) = \mathrm{cf}(\mathrm{ch}_p(i))$.
(6) $\mathbb{Q}^2_\mu = \{p \in \mathbb{Q}^1_\mu : \mathrm{cf}_p(i) > |p \cap \lambda_i^-|$ for $i \in a(p)\}$.

1.5. CLAIM. (1) *$\mathbb{Q}^0_\mu, \mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$ are complete sub-forcings of $\mathbb{P}_\mu$.*
(2) *For $\ell = 0, 1, 2$ and $p, q \in \mathbb{Q}^\ell_\mu$ we have $p \Vdash_{\mathbb{Q}^\ell_\mu}$ "$q \in \underset{\sim}{G}$ iff $|p \backslash q| < \mu$ and similarly for $\mathbb{P}_\mu$.*
(3) *$\mathbb{Q}_\mu = \mathbb{Q}^0_\mu, \mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$ are equivalent, in fact $\mathbb{Q}^2_\mu$ a dense subset of $\mathbb{Q}^1_\mu$ and for $\ell = 0, 1, \{p/\approx : p \in \mathbb{Q}^\ell_\mu\}$ does not depend on ℓ where $\approx$ is the equivalence relation of $\mathbb{P}_\mu$, defined by $p_1 \approx p_2$ iff $(\forall q \in \mathbb{P}_\mu)(q \Vdash_{\mathbb{P}_\mu} p_1 \in \underset{\sim}{G} \Leftrightarrow q \Vdash_{\mathbb{P}_\mu} p_2 \in \underset{\sim}{G})$.*

PROOF. Easy.

Recall

1.6. CLAIM. (1) *$\mathbb{P}_\kappa$ can be completely embedded into $\mathbb{P}_\mu$ (naturally).*
(2) *$\mathbb{Q}_\mu$ can be completely embedded into $\mathbb{P}_\mu$ (naturally).*
(3) *$\mathbb{P}_\kappa$ is completely embeddable into $\mathbb{Q}_\mu$ (naturally).*

PROOF. (1) Known: just $a \in [\kappa]^\kappa$ can be mapped to $\cup\{[\lambda_i^-, \lambda_i) : i \in a\}$.
(2) By [KjSh:720, 2.2].
(3) Should be clear (map $A \in [\kappa]^\kappa$ to $\cup\{[\lambda_i^-, \lambda_i] : i \in A\}$). $\square_{1.6}$

1.7. CHOICE/DEFINITION. $\lambda_* = U_{J^{\mathrm{bd}}_\kappa}(\mu)$.

Recall

1.8. CLAIM. *Assume $\mu > 2^\kappa$.*
(1) $\lambda_* = \sup\{\mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu') : \kappa < \mu' \leq \mu, \mathrm{cf}(\mu') = \kappa\} = \sup\{\mathrm{tcf}(\prod_{i<\kappa} \lambda'_i/J^{bd}_\kappa) : \lambda'_i \in \mathrm{Reg} \cap (\kappa, \mu)$ *and* $\prod_{i<\kappa} \lambda'_i/J^{bd}_\kappa$ *has true cofinality*$\}$.

(2) *For every regular cardinal* $\theta \in [\mu, \lambda_*]$, *for some increasing sequence* $\langle \lambda_i^* : i < \kappa \rangle$ *of regulars* $\in (\kappa, \mu)$ *we have* $\theta = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i^*, <_{J_\kappa^{bd}})$.

(3) $\lambda_* = |\mathcal{S}|$ *where* $\mathcal{S} \subseteq [\mu]^\kappa$ *is a maximal almost disjoint family.*

PROOF. (1) Note that $J_\kappa^{\mathrm{bd}} \restriction A \approx J_\kappa^{\mathrm{bd}}$ if $A \in (J_\kappa^{\mathrm{bd}})^+$, we use this freely. By their definitions the second and third terms are equal. Also by the definition the second is smaller or equal to the first.

By [Sh:589, 1.1], the first $\lambda_* = \boldsymbol{U}_{J_\kappa^{\mathrm{bd}}}(\mu)$ is $\leq$ than the second number (well it speaks on $T^2_{J_\kappa^{\mathrm{bd}}}(\mu)$, instead $\boldsymbol{U}_{J_\kappa^{\mathrm{bd}}}(\mu)$ but as $2^\kappa < \mu$ they are the same).

(2) By [Sh:589, 1.1] we actually get the stronger conclusion.

(3) It follows easily from the definitions 1.1 and 1.7, and from the inequalities $2^\kappa < \mu < \lambda_*$. $\square_{1.8}$

1.9. CLAIM/DEFINITION. Fix a set $\mathscr{P} \subseteq [\mu]^\kappa$ exemplifying $\lambda_* = \boldsymbol{U}_{J_\kappa^{\mathrm{bd}}}(\mu)$.

(1) There is $\bar{C}^* = \langle C_\alpha^* : \alpha < \mu \rangle$ such that:
   (a) C_α^* is a subset of $[\lambda_i^-, \lambda_i)$ closed in its supremum when $\alpha \in [\lambda_i^-, \lambda_i)$,
   (b) if $i < \kappa, \gamma < \lambda_i$, γ is a regular cardinal and C is a closed subset of $[\lambda_i^-, \lambda_i)$ of order type γ^{++}, then for some $\alpha \in (\lambda_i^-, \lambda_i)$, $C_\alpha^* \subseteq C$ and $\mathrm{otp}(C_\alpha^*) = \gamma$.

(2) $\mathbb{Q}_\mu^3 = \{p \in \mathbb{Q}_\mu^2 :$ if $i \in a(p)$ then $p \cap [\lambda_i^-, \lambda_i) \in \{C_\alpha^* : \alpha \in (\lambda_i^-, \lambda_i)\}$ and for some $P \in \mathscr{P}$, $\{\alpha < \mu :$ for some $i < \kappa, p \cap [\lambda_i^-, \lambda_i) = C_\alpha^*\} \subseteq P\}$ is a dense subset of $\mathbb{Q}_\mu^1, \mathbb{Q}_\mu^2$, hence of $\mathbb{Q}_\mu$.

(3) For $p \in \mathbb{Q}_\mu^3$ let $\mathrm{cd}_p \in \prod_{i \in a(p)} \lambda_i$ be such that $\mathrm{cd}_p(i) \in (\lambda_i^-, \lambda_i)$ is the minimal $\alpha \in (\lambda_i^-, \lambda_i)$ such that $p \cap (\lambda_i^-, \lambda_i) = C_\alpha^*$. Notice that for every $p \in \mathbb{Q}_\mu^3$, there is some $P \in \mathscr{P}$ with $\mathrm{Rang}(\mathrm{cd}_p) \subseteq P$.

PROOF. (1) It is enough, for any limit $\delta \in (\lambda_i^-, \lambda_i)$ and regular θ, $\theta^+ < \mathrm{cf}(\delta)$, to find a family $\mathscr{P}_{\delta,\theta}$ of closed subsets of (λ_i^-, δ) of order type θ such that any club of δ contains (at least) one of them. This holds by guessing clubs, see [Sh:g, III,§2].

(2), (3) By the definitions. $\square_{1.9}$

1.10. CLAIM. (1) *If* $\mu > 2^\kappa$ (*or just* $\lambda_* \geq 2^\kappa$) *then* $\mathbb{Q}_\mu^2$ (*hence* $\mathbb{Q}_\mu^1$) *has a dense subset of cardinality* λ_*.

(2) *If* $\mu > 2^\kappa$ (*or just* $\lambda_* \geq 2^\kappa$) *then* $\mathbb{Q}_\mu^3$ *is a dense subset of* $\mathbb{Q}_\mu^1$ *and has cardinality* λ_*.

PROOF. (1) By part (2).

(2) By 1.9(2) it suffices to deal with $\mathbb{Q}_\mu^3$. The cardinality of the set $\mathscr{P}$ from 1.9 is λ_*. Whenever $p \in \mathbb{Q}_\mu^3$, then the function cd_p is uniquely determined by its range, because $i \in \mathrm{Dom}(\mathrm{cd}_p)$ iff $\mathrm{Rang}(\mathrm{cd}_p) \cap [\lambda_i^-, \lambda_i) \neq \emptyset$ and the value $\mathrm{cd}_p(i) = \alpha$ iff $\alpha \in [\lambda_i^-, \lambda_i) \cap \mathrm{Rang}(\mathrm{cd}_p)$. Also, the function cd_p uniquely determines p by $p = \bigcup \{C^*_{\mathrm{cd}_p(i)} : i \in \mathrm{Dom}(p)\}$. Since $\mathrm{Rang}(\mathrm{cd}_p) \subseteq P$, $P \in \mathscr{P}$, we get $|\mathbb{Q}_\mu^3| \leq 2^\kappa \cdot \lambda_* = \lambda_*$. $\square_{1.10}$

From now on (till the end of this section)

1.11. *Hypothesis.* $2^\kappa < \mu$ (in addition to 1.2). Recall (Claim 1.13(1) is Balcar, Simon [BaSi89, 1.15] and 1.13(2) is a variant).

1.12. DEFINITION. A forcing notion $\mathbb{P}$ is (θ, λ)-nowhere distributive when there are maximal antichains $\bar{p}^\varepsilon = \langle p_\alpha^\varepsilon : \alpha < \alpha_\varepsilon \rangle$ of $\mathbb{P}$ for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta, \lambda \leq |\{\alpha < \alpha_\varepsilon : p, p_\alpha^\varepsilon$ are compatible$\}|$.

1.13. CLAIM. (1) *If*

(a) $\mathbb{P}$ *is a forcing notion,* (θ, λ)*-nowhere distributive,*

(b) $\mathbb{P}$ *has density* λ,

(c) $\theta > \aleph_0 \Rightarrow \mathbb{P}$ *has a* θ*-complete dense subset*

then $\mathbb{P}$ *is equivalent to* $\mathrm{Levy}(\theta, \lambda)$.

(2) *If* $\mathbb{P}$ *is a forcing notion of density* λ *collapsing* λ *to* $\aleph_0$ *then* $\mathbb{P}$ *is equivalent to* $\mathrm{Levy}(\aleph_0, \lambda)$.

(3) *If* $\mathbb{P}$ *is a forcing notion of density* λ *and is* (θ, λ)*-nowhere distributive then* $\mathbb{P}$ *collapses* λ *to* θ *(and may or may not collapse* θ*).* $\square_{1.13}$

1.14. CLAIM. *Assume* $\langle b_\varepsilon : \varepsilon < \kappa \rangle$ *is a sequence of pairwise disjoint members of* $[\kappa]^\kappa$ *with union* b. *Then we can find an antichain* $\mathscr{I}$ *of* $\mathbb{Q}^3_\mu$ *such that:*

$(*)$ *if* $q \in \mathbb{Q}^3_\mu$ *and* $(\forall \varepsilon < \kappa)(a(q) \cap b_\varepsilon \in [\kappa]^\kappa)$, *then* q *is compatible with* $\lambda_* =: \boldsymbol{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$ *of the members of* $\mathscr{I}$.

PROOF. Let

$$\mathscr{I}^* = \{p \in \mathbb{Q}^3_\mu : \text{ we can find an increasing sequence } \langle i_\varepsilon : \varepsilon < \kappa \rangle \text{ such that } i_\varepsilon \in b_\varepsilon \backslash \varepsilon, a(p) \subseteq \{i_\varepsilon : \varepsilon < \kappa\} \text{ and } i_\varepsilon \in a(p) \Rightarrow p \cap [\lambda^-_{i_\varepsilon}, \lambda_{i_\varepsilon}) \text{ has order type } \lambda_\varepsilon\}.$$

Let $\mathscr{I}^* = \{p \in \mathbb{Q}^3_\mu$: for every $\varepsilon < \kappa$ we have $a(p) \cap b_\varepsilon \in [\kappa]^\kappa\}$.
Clearly

(a) $|\mathscr{I}^*| \leq \lambda_* = \boldsymbol{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$,
[Why? As $\mathscr{I}^* \subseteq \mathbb{Q}^3_\mu$.]

(b) if $\mathscr{I} \subseteq \mathscr{I}^*, |\mathscr{I}| < \lambda_*$ and $q \in \mathscr{I}^*$ then there is r such that $q \leq r \in \mathscr{I}^*$ and r is incompatible with every $p \in \mathscr{I}$.

[Why? Let $\theta = |\mathscr{I}| + \mu$ it is $< \lambda_*$, hence we can find an increasing sequence $\langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ of regular cardinals with limit μ such that $\prod_{\varepsilon < \kappa} \theta_\varepsilon / J^{\mathrm{bd}}_\kappa$ has true cofinality θ^+, this by 1.8 + the no hole lemma [Sh:g, II,§3]. By renaming without loss of generality $\theta_\varepsilon > \lambda_\varepsilon$.

Let $u = \{\varepsilon < \kappa : a(q) \cap b_\varepsilon \in [\kappa]^\kappa\}$, so we know that u is κ. For each $\varepsilon \in u$ we know that $a(q) \cap b_\varepsilon \in [\kappa]^\kappa$, and so for some $\zeta_\varepsilon < \kappa$ we have $\theta_\varepsilon < \lambda_{\mathrm{otp}(a(q) \cap \zeta_\varepsilon)}$. Now choose $i(\varepsilon) \in b_\varepsilon$ such that $i(\varepsilon) > \varepsilon \wedge i(\varepsilon) > \zeta_\varepsilon \wedge (\forall \varepsilon_1 < \varepsilon)(i(\varepsilon_1) < i(\varepsilon))$. As $q \in \mathbb{Q}^3_\mu$ it follows that $(q \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}))$ has order type $\geq \lambda_{\mathrm{otp}(a(q) \cap \zeta_\varepsilon)} > \theta_\varepsilon$. Let $C_{q,\varepsilon} = \{\alpha : \alpha \in q, \alpha \in [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)})$ and $\mathrm{otp}(q \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}) \cap \alpha)$ is $< \theta_\varepsilon\}$. Now for every $p \in \mathscr{I}^*$ the set $p \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}) \subseteq \cup\{[\lambda^-_i, \lambda_i) : i \in b_\varepsilon\}$ if non-empty has cardinality $\leq \lambda_\varepsilon$ which is $< \theta_\varepsilon$ hence $p \cap C_{q,\varepsilon}$ is a bounded subset of $C_{q,\varepsilon}$, call the lub $\alpha_{p,\varepsilon}$. As $\theta = |\mathscr{I}| + \mu < \mathrm{tcf}(\prod_{\varepsilon < \kappa} \theta_\varepsilon / J^{\mathrm{bd}}_\kappa)$ clearly there is $h \in \prod_{\varepsilon \in u} C_{q,\varepsilon}$ such that $p \in \mathscr{I}^* \Rightarrow \langle \alpha_{p,\varepsilon} : \varepsilon < \kappa \rangle <_{J^{\mathrm{bd}}_\kappa} h$ and let

$$r = \{\alpha : \text{ for some } \varepsilon \in u \text{ we have } \alpha \in C_{q,\varepsilon} \backslash h(\varepsilon) \text{ and } |C_{q,\varepsilon} \cap \alpha \backslash h(\varepsilon)| < \lambda_\varepsilon\}.$$

So r is as required in clause (b). (We can assume that $r \in \mathbb{Q}^3_\mu$, since by the density propositions of 1.10 we can find $r \leq r' \in \mathbb{Q}^3_\mu$ as required.) So clause (b) holds.]

As by 1.10(2) in the conclusion of the claim it is enough to deal with $q \in \mathbb{Q}^3_\mu$, there are only λ_* such q's so we can finish easily by diagonalization. $\square_{1.14}$

1.15. CLAIM. *The forcing notion $\mathbb{Q}^3_\mu$ is $(\mathfrak{b}_\kappa, \lambda_*)$-nowhere distributive.*

PROOF. Let $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ be such that: $\bar{A}_\alpha = \langle A_{\alpha,i} : i < \kappa \rangle$, $A_{\alpha,i} \in [\kappa]^\kappa$, $i < j \Rightarrow A_{\alpha,i} \cap A_{\alpha,j} = \emptyset$ and $(\forall B \in [\kappa]^\kappa)(\exists \alpha < \mathfrak{b}_\kappa)(\forall i < \kappa)[\kappa = |B \cap A_{\alpha,i}|]$, exists by 2.6(2) below. Hence for each $\alpha < \mathfrak{b}_\kappa$, $\mathcal{I}^*_\alpha \subseteq \mathbb{Q}^3_\mu$ as in 1.14 for the sequence $\bar{A}_\alpha$ exists. So $\langle \mathcal{I}^*_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ is a sequence of $\mathfrak{b}_\kappa$ antichains of $\mathbb{Q}^3_\mu$ and we shall show that it witnesses the conclusion. Now

⊛ if $q \in \mathbb{Q}^3_\mu$ then for some $\alpha < \mathfrak{b}_\kappa$ the set $\{p \in \mathcal{I}^*_\alpha : p \text{ compatible with } q \in \mathbb{Q}^3_\mu\}$ has cardinality λ_*.

Why? By the choice of $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ there is $\alpha < \mathfrak{b}_\kappa$ such that

$(*)$ $a(q) \cap A_{\alpha,i} \in [\kappa]^\kappa$ for every $i < \kappa$.

Hence q fits the demand in 1.14 with $\bar{A}_\alpha$ here standing for $\langle b_\varepsilon : \varepsilon < \kappa \rangle$. Hence it is compatible with λ_* members of $\mathcal{I}^*_\alpha$ which, of course, shows that we are done. $\square_{1.15}$

1.16. *Conclusion.* (1) If $2^\kappa < \mu$ (and $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$, of course) then $\mathbb{Q}_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \lambda_*)$, i.e., they have isomorphic completions (recalling $\mathbb{Q}_\mu$ is naturally completely embeddable into the completion of $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$).

(2) If $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$ then $\mathbb{Q}_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \mu^\kappa)$.

(3) If μ is strong limit (singular of uncountable cofinality κ), then $\mathbb{P}_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \mu^\kappa) = \mathrm{Levy}(\aleph_0, 2^\mu)$.

PROOF. (1) By 1.10(1), $\mathbb{Q}^3_\mu$ has density (even cardinality) λ_* and by 1.15 it is $(\mathfrak{b}_\kappa, \lambda_*)$-nowhere distributive hence by 1.13(3), we know that $\mathbb{Q}^3_\mu$ collapses λ_* to $\mathfrak{b}_\kappa$. But $\mathbb{P}_\kappa$ is completely embeddable into $\mathbb{Q}^2_\mu$ (see 1.6(3)) and $\mathbb{P}_\kappa$ collapses $\mathfrak{b}_\kappa$ to $\aleph_0$ (see §2) and $\mathbb{Q}^3_\mu$ is dense in $\mathbb{Q}^2_\mu$. Together forcing with $\mathbb{Q}^3_\mu$ collapses λ_* to $\aleph_0$. As $\mathbb{Q}^3_\mu$ has density λ_*, by 1.13(2) we get that $\mathbb{Q}^2_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \lambda_*)$.

Lastly $\mathbb{Q}_\mu, \mathbb{Q}^3_\mu$ are equivalent by 1.5(3) + 1.9(2) so we are done.

(2) Recalling 1.8, by [Sh:g, VIII] we have $\lambda_* = \mu^\kappa$ (alternatively directly as in [Sh:506, §3]). Now apply part (1).

(3) By easy cardinal arithmetic $\mu^\kappa = 2^\mu$. Enough to check the demands in 1.13(2). Now as $\mathbb{Q}_\mu$ collapses λ_* to $\aleph_0$ by part (1) and $\mathbb{Q}_\mu$ can be completely embeddable into $\mathbb{P}_\mu$ (see 1.6(2)) clearly $\mathbb{P}_\mu$ collapses λ_* to $\aleph_0$. But $|\mathbb{P}_\mu| \leq |[\mu]^\mu| = 2^\mu$, so $\mathbb{P}_\mu$ has density $\leq 2^\mu$.

Lastly $\lambda_* = 2^\mu$ by [Sh:g, VIII]. So we are done. $\square_{1.16}$

1.17. CLAIM. *Assume that $\mathbb{P}_\mu$ does not satisfy the χ-c.c. Then forcing with $\mathbb{P}_\mu$ collapses χ to $\aleph_0$.*

PROOF. By the nature of the conclusion without loss of generality χ is regular. Now we can find $\bar{X}$ such that

$(*)_1$ (a) $\bar{X} = \langle X_\xi : \xi < \chi \rangle$,
(b) $X_\xi \in \mathbb{P}_\mu$,
(c) $X_\zeta \cap X_\xi \in [\mu]^{<\mu}$ for $\zeta \neq \xi < \chi$.

As $\mathbb{Q}_\mu \lessdot \mathbb{P}_\mu$, by the earlier proof (e.g., 1.16(1)) it suffices to prove that $\mathbb{P}_\mu$ collapses χ to λ_*. There exists $\boldsymbol{P} \subseteq \boldsymbol{P}^* = \{\bar{A} : \bar{A} = \langle A_\alpha : \alpha < \mu \rangle$, the A_α's are pairwise disjoint and each A_α belongs to $[\mu]^\mu\}$ such that $|\boldsymbol{P}| = \lambda_*$ and

$(*)_2$ for every $p \in \mathbb{P}_\mu$ there is a $\bar{A} \in \boldsymbol{P}$ such that $(\forall \alpha < \mu)[|A_\alpha \cap p| = \mu]$.

[Why? For each $i < \kappa$ fix some partition $\langle W_{i,\alpha} : \alpha < \lambda_i \rangle$ of λ_i into λ_i (pairwise disjoint) sets each of cardinality λ_i. Now for each $p \in \mathbb{P}_\mu$ we shall choose $\bar{A} = \bar{A}^p \in \boldsymbol{P}_\lambda$ as required in $(*)_2$ such that $\boldsymbol{P} := \{\bar{A}^p : p \in \mathbb{P}_\mu\}$ has cardinality $\leq \lambda_*$ this suffice; so fix $p \in \mathbb{P}_\mu$. By induction on $\varepsilon < \kappa$ we can find $\delta_\varepsilon < \mu$ of cofinality λ_ε^{++} such that $p \cap \delta_\varepsilon$ is unbounded in δ_ε and $\delta_\varepsilon > \cup\{\delta_\zeta : \zeta < \varepsilon\}$. There is a club C_ε^1 of δ_ε of order type λ_ε^{++} with $\min(C_\varepsilon^1) > \cup\{\delta_\zeta : \zeta < \varepsilon\}$. Let $C_\varepsilon^2 = \{\delta \in C_\varepsilon^1 : \delta$ is a limit ordinal such that C_ε^1 is unbounded in δ and has order type divisible by $\lambda_\varepsilon^+\}$, it is a club of δ_ε. But by the club guessing (see 1.9) there is C_ε^3 such that: $C_\varepsilon^3 \subseteq C_\varepsilon^2 (\subseteq C_\varepsilon^1)$ and $\mathrm{otp}(C_\varepsilon^3) = \lambda_\varepsilon$.

By the definition of $\mathbb{Q}_\mu^3$, there is some $a \in [\kappa]^\kappa$ such that $\bigcup\{C_\varepsilon^3 : \varepsilon \in a\} \in \mathbb{Q}_\mu^3$. Lastly, let us define $\bar{A}^p = \langle A_\alpha : \alpha < \mu \rangle$ by

$$A_\alpha = \cup\{[\beta, \min(C_\varepsilon^3 \backslash (\beta+1)) : \varepsilon \in a \text{ satisfies } \alpha < \lambda_\varepsilon \text{ and } \beta \in C_\varepsilon^3 \text{ and } \mathrm{otp}(C_\varepsilon^3 \cap \beta) \in W_{\varepsilon,\alpha}\}.$$

Easily $\langle A_\alpha : \alpha < \mu \rangle$ is as required in $(*)_2$, and since $\bar{A}$ is determined by an element of $\mathbb{Q}_\mu^3$, the cardinality $|\boldsymbol{P}| \leq |\mathbb{Q}_\mu^3| \leq \lambda_*$.]

Now for $\bar{A} \in \boldsymbol{P}$ we define a $\mathbb{P}_\mu$-name $\tau_{\bar{A}}$ as follows: for $\boldsymbol{G} \subseteq \mathbb{P}_\mu$ generic over $\boldsymbol{V}$,

$(*)_3$ $\tau_{\bar{A}}[\boldsymbol{G}] = \xi$ iff ξ is minimal such that $\cup\{A_\alpha : \alpha \in X_\xi\} \in \boldsymbol{G}$

clearly

$(*)_4$ $\tau_{\bar{A}}[\boldsymbol{G}]$ is defined in at most one way;

$(*)_5$ for every $p \in \mathbb{P}_\mu$ for some $\bar{A} \in \boldsymbol{P}$ for every $\xi < \chi$ we have $p \nVdash$ "$\tau_{\bar{A}} \neq \xi$".

[Why? Let $\bar{A} \in \boldsymbol{P}$ be such that $(\forall \alpha < \mu)(\mu = |p \cap A_\alpha|)$, it exists by $(*)_2$. Now we can find q satisfying $p \leq q \in \mathbb{P}_\mu$ such that $(\forall \alpha < \mu)(q \cap A_\alpha$ is a singleton) and for each $\xi < \chi$ let $q_\xi = \cup\{A_\alpha \cap q : \alpha \in X_\xi\}$. Clearly $\zeta < \xi \Rightarrow |X_\zeta \cap X_\xi| < \mu \Rightarrow \cup\{A_\alpha : \alpha \in X_\zeta\} \cap q_\xi \subseteq \cup\{A_\alpha \cap q_\xi : \alpha \in X_\zeta\} = \cup\{A_\alpha \cap q : \alpha \in X_\zeta \cap X_\xi\} \in [\mu]^{<\mu}$, hence $q_\xi \Vdash$ "$\xi = \tau_{\bar{A}}$".]

So

$(*)_6$ $\Vdash_{\mathbb{P}_\mu}$ "$\chi = \{\tau_{\bar{A}}[\boldsymbol{G}] : \bar{A} \in \boldsymbol{P}\}$".

Together clearly $\mathbb{P}_\mu$ collapses χ to $\lambda_* + |\boldsymbol{P}|$ which is $\leq \lambda_*$, so as said above we are done. $\square_{1.17}$

Lastly, concerning the singular μ_* of cofinality $\aleph_0$ so we forget the hypothesis 1.2, 1.11.

1.18. Claim. *If $\mu_* > \mathrm{cf}(\mu_*) = \aleph_0$ and $\mathbb{P}_{\mu_*}$ fails the χ-c.c., then $\mathbb{P}_{\mu_*}$ collapses χ to $\aleph_1$; note that in this case $\mathbb{Q}_{\mu_*}$ is equivalent to* Levy$(\aleph_1, \mu_*^{\aleph_0})$ *by* [*KjSh*:720].

Proof. Let $\lambda_* = \mu_*^{\aleph_0}$.

By Kojman, Shelah [KjSh:720], $\mathbb{P}_{\mu_*}$ collapses λ_* to $\aleph_1$ hence it suffices to prove that $\mathbb{P}_\mu$ collapse χ to λ_* assuming $\chi > \lambda_*$ (otherwise the conclusion is known). Let $\langle \lambda_n : n < \omega \rangle$ be a sequence of regular uncountable cardinals with limit μ_*. Now repeat the proof of 1.17 $\square_{1.18}$

§2. The regular uncountable case. We prove that (for κ regular uncountable), $\mathbb{P}_\kappa$ collapse λ to $\aleph_0$ iff $\mathbb{P}_\kappa$ fail the λ-c.c. This continues Balcar, Simon [BaSi88, 2.8] so we first re-represent what they do; the proof of 2.5 is made to help later. In the present notation they let $\lambda = \mathfrak{b}_\kappa$, let $\langle f_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ be a sequence exemplifying it; let $C_\alpha = \{\delta < \kappa : (\forall \beta < \delta)(f_\alpha(\beta) < \delta), \delta \text{ a limit ordinal}\}$ and let $B_\alpha = \kappa \backslash C_\alpha$, so $\langle B_\alpha : \alpha < \lambda \rangle$ is a (κ, λ)-sequence (see 2.4(1)), derive a good $(\kappa, {}^{\omega>}\lambda)$-sequence from it (see 2.4(2)), define $\alpha_n(A), \beta_n(A)$ and used the $A_{\eta,\delta,i}$'s to define the $\mathbb{P}_\kappa$-names $\underset{\sim}{\beta}_n$ and prove $\Vdash_{\mathbb{P}_\kappa}$ "$\{g^*(\underset{\sim}{\beta} n) : n < \omega\} = \mathfrak{b}_\kappa$" (see 2.5). We then prove the new result: if $\mathbb{P}_\kappa$ fail the χ-c.c. then it collapses χ to $\aleph_0$.

2.1. *Context.* κ is a fixed regular uncountable cardinal.

2.2. DEFINITION. (1) Let $\mathfrak{b}^{\rm spc}_\kappa$ be the set of regular $\lambda > \kappa$ such that there is a $<_{J^{\rm bd}_\kappa}$-increasing sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of members of ${}^\kappa\kappa$ with no $\leq_{J^{\rm bd}_\kappa}$-upper bound in ${}^\kappa\kappa$.

(2) Let $\mathfrak{b}_\kappa = \operatorname{Min}(\mathfrak{b}^{\rm spc}_\kappa)$.

2.3. DEFINITION. (1) We say $\bar{B}$ is a (κ, λ)-sequence when
- (a) $\bar{B} = \langle B_\alpha : \alpha < \lambda \rangle$,
- (b) $B_\alpha \in [\kappa]^\kappa$ and $\kappa \backslash B_\alpha \in [\kappa]^\kappa$ and $B_{\alpha+1} \backslash B_\alpha \in [\kappa]^\kappa$,
- (c) for every $B \in [\kappa]^\kappa$ for some α, $B \cap B_\alpha \in [\kappa]^\kappa$,
- (d) $B_\alpha \subseteq^* B_\beta$ when $\alpha < \beta < \lambda$, i.e., $B_\alpha \backslash B_\beta \in [\kappa]^{<\kappa}$.

(2) We say that $\bar{B}$ is a $(\kappa, {}^{\omega>}\lambda)$-sequence when:
- (a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda \rangle$,
- (b) $B_\eta \in [\kappa]^\kappa$,
- (c) if $\eta_1 \lhd \eta_2 \in {}^{\omega>}\lambda$ then $B_{\eta_2} \subseteq^* B_{\eta_1}$ which means $B_{\eta_2} \backslash B_{\eta_1} \in [\kappa]^{<\kappa}$,
- (d) $B_{<>} = \kappa$,
- (e) if $\eta \in {}^{\omega>}\lambda$ and $A \in [B_\eta]^\kappa$ then for some $\alpha < \lambda$ we have $A \cap B_{\eta^\frown<\alpha>} \in [\kappa]^\kappa$,
- (f) if $\eta \in {}^{\omega>}\lambda$ and $\alpha < \beta < \lambda$ then $B_{\eta^\frown<\alpha>} \subseteq^* B_{\eta^\frown<\beta>}$ and $B_\eta \backslash B_{\eta^\frown<\alpha>} \in [\kappa]^\kappa$ and $B_{\eta^\frown<\beta>} \backslash B_{\eta^\frown<\alpha>} \in [\kappa]^\kappa$.

(3) For a $(\kappa, {}^{\omega>}\lambda)$-sequence $\bar{B}$ and $A \in [\kappa]^\kappa$ we try to define an ordinal $\alpha_k(A, \bar{B})$ by induction on $k < \omega$. If $\eta = \langle \alpha_\ell(A, \bar{B}) : \ell < k \rangle$ is well defined (holds for $k = 0$) and there is an $\alpha < \lambda$ such that $A \subseteq^* B_{\eta^\frown<\alpha>} \wedge (\forall \beta < \alpha)(A \cap B_{\eta^\frown<\beta>} \in [\kappa]^{<\kappa})$ then we let $\alpha_k(A, \bar{B}) = \alpha$; note that α, if exists, is unique. Let $n(A, \bar{B})$ be the $n \leq \omega$ such that $\alpha_\ell(A, \bar{B})$ is well defined iff $\ell < n$.

(4) We say that $(\bar{B}, \bar{\nu})$ is a $(\kappa, {}^{\omega>}\lambda)$-parameter when:
- (a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda \rangle$ is a $(\kappa, {}^{\omega>}\lambda)$-sequence,
- (b) $\bar{\nu}$ is an S^λ_κ-ladder which means that $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa \rangle$, ν_δ is an increasing sequence of ordinals of length κ with limit δ, where $S^\lambda_\kappa = \{\delta < \lambda : \operatorname{cf}(\delta) = \kappa\}$.

(5) We say $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter when (a)+(b) of part (4) holds and
- (c) if $A \in [\kappa]^\kappa$ then for some $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ and $A' \in [A]^\kappa$ we have
  - (α) $\alpha_\ell(A', \bar{B}) = \eta(\ell)$ for $\ell < n$,
  - (β) for κ many ordinals $\zeta < \kappa$ we have $(\forall \varepsilon < \zeta)(A' \cap B_{\eta^\frown<\nu_\delta(\zeta)>} \backslash B_{\eta^\frown<\nu_\delta(\varepsilon)>}$ belongs to $[\kappa]^\kappa$).

(6) $\bar{B}$ is a good $(\kappa, {}^{\omega>}\lambda)$-sequence if clause (a) of (4) and clause (c) of (5) holds for some S^λ_κ-ladder (see above). We say $\bar{B}$ is a weakly good sequence if clause (a) of (4) and clause (c)$^-$ of (5) which means that we ignore subclause (α) there. Similarly $(\bar{B}, \bar{\nu})$ is a weakly good $(\kappa, {}^{\omega>}\lambda)$-parameter.

Remark. Notice, if $\nu_1, \nu_2 \in {}^\kappa\delta$ are increasing and $\sup(\nu_1) = \sup(\nu_2) = \delta$, then $\{i < \kappa : \bigcup_{j<i} \nu_1(j) = \bigcup_{j<i} \nu_2(j)\}$ is a club of κ, so it doesn't matter which S^λ_κ-ladder you choose.

Note that for §1 we need no more than Claim 2.4 (actually the weakly good version is enough for §1 except presenting the proof that $\mathfrak{b}_\kappa$ is collapsed).

2.4. CLAIM. (1) *Assume $\lambda = \mathfrak{b}_\kappa$ or just $\lambda \in \mathfrak{b}^{spc}_\kappa$. Then λ is regular $> \kappa$ and there is a $\subseteq^*$-decreasing sequence $\langle C_\alpha : \alpha < \lambda\rangle$ of clubs of κ such that for no $A \in [\kappa]^\kappa$ do we have $\alpha < \lambda \Rightarrow A \subseteq^* C_\alpha \wedge |C_\alpha \setminus C_{\alpha+1}| = \kappa$. Hence $\langle \kappa \setminus C_\alpha : \alpha < \lambda\rangle$ is a (κ, λ)-sequence.*

(2) *Assume $\bar{C} = \langle C_\alpha : \alpha < \lambda\rangle$ is as above and $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa\rangle$ is an S^λ_κ-ladder, see Definition 2.3(4), clause (b) (such $\bar{\nu}$ always exists). Then $\bar{B} = \bar{B}_{\bar{C}}, \bar{f} = \bar{f}_{\bar{C}}$ are well defined and $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter where we define $\bar{B}$ and $\bar{f}$ as follows:*

⊛ (a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda\rangle$,

(b) $\bar{f} = \langle f_\eta : \eta \in {}^{\omega>}\lambda\rangle$,

(c) $B_{<>} = \kappa, f_{<>} = \mathrm{id}_\kappa$,

(d) $B_\eta \in [\kappa]^\kappa, f_\eta$ *is a function from B_η onto κ, non-decreasing, and not eventually constant,*

(e) *if the pair (B_ρ, f_ρ) is defined and $\alpha < \lambda$ we let*

$$B_{\rho^\frown<\alpha>} = \{\gamma \in B_\rho : f_\rho(\gamma) \in \kappa \setminus C_\alpha\},$$

(f) *if $\eta = \rho^\frown\langle\alpha\rangle$ and B_ρ, f_ρ and B_η are defined then we let $f_\eta : B_\eta \to \kappa$ be defined by $f_\eta(i) = \mathrm{otp}(C_\alpha \cap f_\rho(i))$ for each $i < \kappa$, hence*

(g) *if $\eta^\frown\langle\alpha\rangle \in {}^{\omega>}\lambda$ then $B_{\eta^\frown<\alpha>} \subseteq B_\eta$ and $i \in B_{\eta^\frown<\alpha>} \wedge f_\eta(i) > 0 \Rightarrow f_\eta(i) > f_{\eta^\frown<\alpha>}(i)$.*

PROOF. (1) Recall $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

By the definition of $\mathfrak{b}^{\mathrm{spc}}_\kappa$ there is an $<_{J^{\mathrm{bd}}_\kappa}$-increasing sequence $\langle f^*_\alpha : \alpha < \lambda\rangle$ of members of ${}^\kappa\kappa$ with no $\leq_{J^{\mathrm{bd}}_\kappa}$-upper bound from ${}^\kappa\kappa$. Let $C_\alpha := \{\delta < \kappa : \delta$ is a limit ordinal such that $(\forall\gamma < \delta)(f^*_\alpha(\gamma) < \delta)\}$.

Clearly

$(*)_1$ C_α is a club of κ,
[why? as κ is regular uncountable]

$(*)_2$ if $\alpha < \beta < \lambda$ then $C_\beta \subseteq^* C_\alpha$; i.e., $C_\beta \setminus C_\alpha \in [\kappa]^{<\kappa}$,
[why? as if $\alpha < \beta$ then $f^*_\alpha <_{J^{\mathrm{bd}}_\kappa} f^*_\beta$, i.e., for some $\varepsilon < \kappa, (\forall\zeta)(\varepsilon \leq \zeta < \kappa \Rightarrow f^*_\alpha(\zeta) < f^*_\beta(\zeta)) < \varepsilon$ hence $C_\beta \setminus (\varepsilon + 1) \subseteq C_\alpha$ as required]

$(*)_3$ for every club C of κ for some $\zeta < \lambda$ we have $C \setminus C_\zeta \in [\kappa]^\kappa$,
[why? as $\bar{f}$ has no $\leq_{J^{\mathrm{bd}}_\kappa}$-bound in ${}^\kappa\kappa$]

hence

$(*)_4$ for every unbounded subset A of κ for some $\zeta < \lambda$ we have $A \setminus C_\zeta \in [\kappa]^\kappa$.
[Why? Otherwise the closure of A contradicts $(*)_3$.]

Clearly wlog $\langle C_\alpha : \alpha < \lambda\rangle$ is as required.

Lastly, let $B_\alpha = \kappa \backslash C_\alpha$, it is easy to check that $\langle B_\alpha : \alpha < \lambda\rangle$ is a (κ, λ)-sequence.

(2) Clearly $\bar{B}_{\bar{C}}, \bar{f}_{\bar{C}}$ are well defined and $(\bar{B}, \bar{\nu})$ is a $(\kappa, {}^{\omega>}\lambda)$-parameter and clauses (a)–(g) of $\circledast$ holds. Why is it good? Toward contradiction assume that it is not, so choose $A \in [\kappa]^\kappa$ which exemplify the failure of clause (c) of Definition 2.3(5) and define

$$\mathscr{T}_0 = \mathscr{T}_A^0 = \{\eta \in {}^{\omega>}\lambda : \text{there is } A' \in [A]^\kappa \text{ such that } \mathrm{nat}(\eta)\rangle \text{ is well defined and equal to } \eta\}.$$

and define

$$\mathscr{T}_1 = \mathscr{T}_A^1 := \{\eta \in \mathscr{T}_A^0 : \text{for every } k < \ell g(\eta) \text{ there are } < \kappa \text{ ordinals } \alpha < \eta(k) \text{ such that } (\eta \restriction k)^\frown\langle\alpha\rangle \in \mathscr{T}_0\}.$$

Clearly

$(*)_1$ $\mathscr{T}_0 \supseteq \mathscr{T}_1$ are non-empty subsets of ${}^{\omega>}\lambda$ (in fact $<> \in \mathscr{T}_1 \subseteq \mathscr{T}_0$),

$(*)_2$ $\mathscr{T}_0, \mathscr{T}_1$ are closed under initial segments.

For $\eta \in \mathscr{T}_\ell$ let $\mathrm{Suc}_{\mathscr{T}_\ell}(\eta) = \{\rho \in \mathscr{T}_\ell : \ell g(\rho) = \ell g(\eta) + 1 \text{ and } \eta \lhd \rho\}$.

We define $A_\eta \in [B_\eta]^\kappa$ for $\eta \in \mathscr{T}_1$ by induction on $\ell g(\eta)$:

$(*)_3$ (a) $A_{<>} = A$,

(b) if A_ν is defined and $\nu^\frown\langle\alpha\rangle \in \mathscr{T}_1$ then we let

$$A_{\nu^\frown<\alpha>} = A_\nu \cap B_{\nu^\frown<\alpha>} \backslash \bigcup\{B_{\nu^\frown<\beta>} : \beta < \alpha \text{ and } \nu^\frown\langle\beta\rangle \in \mathscr{T}_1\}.$$

Now

$(*)_4$ if $\nu \in \mathscr{T}_1$ then

(a) if $B \in [A]^\kappa$ and $\langle \alpha_\ell(B, \bar{B}) : \ell < \ell g(\nu)\rangle$ is well defined and equal to ν then $B \subseteq^* A_\nu$,

(b) $A_\nu \backslash \cup\{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_j}(\nu)\}$ has cardinality $< \kappa$ for $j = 1$ (actually $j = 0$ is O.K., too).

[Why? First we can prove clause (a) by induction on $\ell g(\nu)$ using the definition of $\mathscr{T}_1$ and clause (c) of 2.3(2). Second, we can prove clause (b) from it.]

$(*)_5$ $|\mathscr{T}_1| \geq \kappa$

[Why? Otherwise by $(*)_4$ the set $A' := \cup\{A_\nu \backslash \cup\{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_0}(\nu)\} : \nu \in \mathscr{T}_1\}$ is a subset of κ of cardinality $< \kappa$ and by clause (d) of $\circledast$ of the present claim also $A'' = \cup\{f_\nu^{-1}\{0\} : \nu \in \mathscr{T}_1\}$ is a subset of κ of cardinality $< \kappa$. So we can choose $j \in A \backslash (A' \cup A'')$. Now we try to choose $\nu_n \in \mathscr{T}_1$ by induction on n such that $\ell g(\nu_n) = n, \nu_{n+1} \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$ and $j \in A_{\nu_n}$. So $\nu_0 = <>$ belongs to $\mathscr{T}_1$ by $(*)_1 + (*)_2$. Now assume ν_n is well defined, then $\mathrm{Suc}_{\mathscr{T}_0}(\nu_n) = \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$.

(Why? Otherwise, as $\mathscr{T}_1 \subseteq \mathscr{T}_0$, there is an α with $\nu_n^\frown\langle\alpha\rangle \in \mathrm{Suc}_{\mathscr{T}_0}(\nu_n) \backslash \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$. Hence by the definition of $\mathscr{T}_1$ the set $u := \{\beta < \alpha : \nu_n^\frown\langle\beta\rangle \in \mathscr{T}_0\}$ has cardinality $\geq \kappa$ but then $\beta \in u \wedge |\beta \cap u| < \kappa \Rightarrow \nu_n^\frown\langle\beta\rangle \in \mathscr{T}_1$ which implies that $|\mathrm{Suc}_{\mathscr{T}_1}(\nu_n)| \geq \kappa$, contradiction to the "otherwise").

Now $j \notin A', A' \supseteq A_{\nu_n} \backslash \cup\{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)\}$, but $j \in A_{\nu_n}$ hence clearly $j \in \cup\{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)\}$, so we can choose ν_{n+1} as required. As $j \in A_{\nu_n} \subseteq B_{\nu_n}$ by $(*)_3(b)$ above, clearly $f_{\nu_n}(j)$ is well defined (for each $n < \omega$). As $j \notin A''$ and $f_{\nu_n}^{-1}\{0\} \subseteq A''$, so $j \notin f_{\nu_n}^{-1}\{0\}$, necessarily $f_{\nu_n}(j) \neq 0$ and so $f_{\nu_n}(j) > f_{\nu_{n+1}}(j)$

by the choice of $f_{\nu_{n+1}}$ in clauses (g) of $\circledast$. Hence $\langle f_{\nu_n}(j) : n < \omega\rangle$ is decreasing, contradiction. So $(*)_5$ holds.]

Let $n < \omega$ be maximal such that $|\mathscr{T}_1 \cap {}^{n\geq}\lambda| < \kappa$, it exists as $|\mathscr{T}_1| \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $n = 0 \Rightarrow |\mathscr{T}_1 \cap {}^{n\geq}\lambda| = 1 < \kappa$, and let $\eta \in \mathscr{T}_1 \cap {}^n\lambda$ be such that $\mathrm{Suc}_{\mathscr{T}_1}(\eta)$ has $\geq \kappa$ members; it exists as κ is regular. We can choose an increasing sequence $\langle \alpha_i : i < \kappa\rangle$ of ordinals such that α_i is the i-th member of the set $\{\alpha < \lambda : \eta^\frown\langle\alpha\rangle \in \mathscr{T}_1\}$ and let $A_i \in [A]^\kappa$ be such that $\langle \alpha_\ell(A_i, \bar{B}) : \ell \leq n\rangle = \eta^\frown\langle\alpha_i\rangle$ and let $\delta = \cup\{\alpha_i : i < \kappa\}$, so $\delta \in S^\lambda_\kappa$. Let

$$A_* = \cup\{A_i : i < \kappa\} \cap B_\eta \setminus \cup\{A_{(\eta\restriction\ell)^\frown\langle\gamma\rangle} : \ell < \ell g(\eta) \text{ and } \gamma < \eta(\ell) \text{ and } (\eta \restriction \ell)^\frown\langle\gamma\rangle \in \mathscr{T}_1\}$$

(note that that number of (ℓ, γ) as mentioned above is $< \kappa$).

Clearly $\alpha_\ell(A_*, \bar{B}) = \eta(\ell)$ for $\ell < \ell g(\eta)$ hence $\alpha_\ell(A_* \cap A_i, \bar{B}) = \eta(\ell)$ for $i < \kappa$, $\ell < n$ so clause (α) of (c) of Definition 2.3(5) holds, as well as clause (β) because $\alpha_n(A_* \cap A_i, \bar{B}) = \alpha_i$ for $i < \kappa$. $\square_{2.4}$

2.5. Claim. *If there is a good $(\kappa, {}^{\omega>}\lambda)$-parameter and $\lambda_1 \in \mathfrak{b}^{spc}_\kappa$ then the forcing notion $\mathbb{P}_\kappa$ collapses λ_1 to $\aleph_0$.*

Proof. Let $(\bar{B}, \bar{\nu})$ be a good $(\kappa, {}^{\omega>}\lambda)$-parameter.

Note

$\circledast_1$ if $A_1 \subseteq A_2$ are from $[\kappa]^\kappa$ and $\alpha_\ell(A_2, \bar{B})$ is well defined then $\alpha_\ell(A_1, \bar{B})$ is well defined and equal to $\alpha_\ell(A_2, \bar{B})$, recalling Definition 2.3(3).

Let $\bar{h} = \langle h_\gamma : \gamma < \lambda_1\rangle$ exemplify $\lambda_1 \in \mathfrak{b}^{\mathrm{spc}}_\kappa$, i.e., is as in Definition 2.2 and without loss of generality $[i < j < \kappa \Rightarrow i < h_\gamma(i) < h_\gamma(j)]$. For each $\delta \in S^\lambda_\kappa$ and $\eta \in {}^{\omega>}\lambda$ let $A_{\eta,\delta,i} = B_{\eta^\frown\langle\nu_\delta(i+1)\rangle} \setminus \cup\{B_{\eta^\frown\langle\nu_\delta(j+1)\rangle} : j < i\}$ for $i < \kappa$ so $\langle A_{\eta,\delta,i} : i < \kappa\rangle$ are pairwise disjoint subsets of κ (each of cardinality κ). For $A \in [\kappa]^\kappa$ we try to define an ordinal $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h})$ as follows:

$\circledast_2$ $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$ iff for some $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ we have $\langle \alpha_\ell(A, \bar{B}) : \ell \leq n\rangle = \eta^\frown\langle\delta\rangle$ so in particular is well defined and $A \subseteq^* \cup\{A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa\}$ but for every $\beta < \gamma$ we have $A \cap \cup\{A_{\eta,\delta,i} \cap h_\beta(i) : i < \kappa\} \in [\kappa]^{<\kappa}$.

Next we define a $\mathbb{P}_\kappa$-name $\underset{\sim}{\beta}_n = \underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h})$ by:

$\circledast_3$ for $\boldsymbol{G} \subseteq \mathbb{P}_\kappa$ generic over $\boldsymbol{V}$: $\underset{\sim}{\beta}_n[\boldsymbol{G}] = \gamma$ iff for some $A \in \boldsymbol{G}$ we have $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$ or there is no such A and $\gamma = 0$.

Now

$\circledast_4$ if $A \in [\kappa]^\kappa$ and (and $\mathscr{T}^0_A, \mathscr{T}^1_A$) n, η, δ are chosen as in the proof of 2.4(2), then $u := \{\beta < \lambda_1 : A \Vdash_{\mathbb{P}_\kappa} \text{“}\underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h}) \neq \beta\text{”}\}$ is a κ-closed unbounded subset of λ_1.

[Why? We know that $w := \{i < \kappa : A \cap A_{\eta,\delta,i} \in [\kappa]^\kappa\}$ has cardinality κ. Why is u “unbounded”? For any $\gamma_1 < \lambda_1$, we define a function $h \in {}^\kappa\kappa$ as follows, $h(i)$ is the minimal $i_1 < \kappa$ such that for some $i_0, i < i_0 < i_1$ the set $A \cap A_{\eta,\delta,i_0} \cap i_1 \setminus h_{\gamma_1}(i_0)$ is not empty, clearly h is well defined because $|w| = \kappa$. So for some $\gamma_2 \in (\gamma_1, \lambda_1)$ the set $v := \{i < \kappa : h(i) < h_{\gamma_2}(i)\}$ has cardinality κ. Let C be the club $\{\delta < \kappa : \delta$ is a limit ordinal and $i < \delta \Rightarrow h(i) < \delta \wedge h_{\gamma_2}(i) < \delta\}$ and let $\langle \alpha_\varepsilon : \varepsilon < \kappa\rangle$ list

$C \cup \{0\}$ increasing order κ and let $A' = \cup\{A \cap A_{\eta,\delta,i} \cap [\alpha_\varepsilon, \alpha_{\varepsilon+1}) : i < \kappa, \varepsilon < \kappa$ and $\alpha_\varepsilon \le i < \alpha_{\varepsilon+1}\}$, now $A' \in [\kappa]^\kappa$ (really $i < j < \kappa \Rightarrow i < h_{\gamma_1}(i) < h_{\gamma_2}(j)$). So $\mathbb{P}_\kappa \models$ "$A \le A'$" and $A' \Vdash$ "$\underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h}) \in (\gamma_1, \gamma_2]$", recalling that the h_γ's are increasing. Why "the set u is κ-closed" (that is the limit of any increasing sequence of length κ of members belong to it)? Easy, too.]

Let $\langle S_\varepsilon : \varepsilon < \lambda_1\rangle$ be pairwise disjoint stationary subsets of $S^{\lambda_1}_\kappa$ and define $g^* : \lambda_1 \to \lambda_1$ by $g^*(\gamma) = \varepsilon$ if $\gamma \in S_\varepsilon \vee (\gamma \in \lambda_1 \setminus \bigcup_{\zeta<\lambda_1} S_\zeta \wedge \varepsilon = 0)$. So

$\circledast_5$ for every $p \in \mathbb{P}_\kappa$ for some n, for every $\varepsilon < \lambda_1$, $p \nVdash$ "$g^*(\underset{\sim}{\beta}_n) \ne \varepsilon$"

so we are done. $\square_{2.5}$

Now we arrive to the main point.

2.6. MAIN CLAIM. (1) *If $\mathbb{P}_\kappa$ does not satisfy the χ-c.c. then forcing with $\mathbb{P}_\kappa$ collapses χ to $\aleph_0$.*

(2) *There is $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa\rangle$ such that $\bar{A}_\alpha = \langle A_{\alpha,i} : i < \kappa\rangle$ is a sequence of pairwise disjoint subsets of κ each of cardinality κ (without loss of generality each is a partition of κ) such that for every $B \in [\kappa]^\kappa$ for some $\alpha < \mathfrak{b}_\kappa$ we have $i < \kappa \Rightarrow \kappa = |A_{\alpha,i} \cap B|$; i.e., for every $i < \kappa$ not just for κ many $i < \kappa$.*

(3) *In part (2) we can replace $\mathfrak{b}_\kappa$ by $\lambda \in \mathfrak{b}^{spc}_\kappa$ (so $\lambda = \kappa^+ \Rightarrow \mathfrak{b}_\kappa = \kappa^+$).*

PROOF. The proof is divided to two cases.

1. CASE. $\lambda \in \mathfrak{b}^{spc}_\kappa, \lambda > \kappa^+$.

So λ is regular $> \kappa^+$ and a good $(\kappa, {}^{\omega>}\lambda)$ sequence $\bar{B}$ exists (by 2.4).

Let $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa\rangle$ be such that $\nu_\delta \in {}^\kappa\delta$ is increasing continuous with limit δ and $\bar{\nu}$ guesses clubs (i.e., for every club C of λ, for stationarily many $\delta \in S^\lambda_\kappa$ we have $\mathrm{Rang}(\nu_\delta) \subseteq C$); exists by [Sh:g, III,§2] because $\lambda = \mathrm{cf}(\lambda) > \kappa^+$. As $\bar{B}$ is a good $(\kappa, {}^{\omega>}\lambda)$-sequence, $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter.

Let $\langle h_\alpha : \alpha < \lambda\rangle$ exemplify $\lambda \in \mathfrak{b}^{spc}_\kappa$ and h_α is increasing.

For $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$ and $i < \kappa$, recall that $A_{\eta,\delta,i} = B_{\eta^\frown\langle \nu_\delta(i+1)\rangle} \setminus \cup\{B_{\eta^\frown\langle \nu_\delta(j+1)\rangle} : j < i\}$ and let $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}), \underset{\sim}{\beta}_n = \underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h})$ be defined as in the proof of 2.5. For $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \delta^* \in S^\lambda_\kappa$ and $i < \kappa$ and $\gamma < \lambda$ let $B^*_{\eta,\delta,\gamma} := \cup\{A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa\}$. So clearly (for each $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$) the sequence $\langle B^*_{\eta,\delta,\gamma} : \gamma < \lambda\rangle$ is $\subseteq^*$-increasing. Let $A^*_{\eta,\delta,\delta^*,i} := B^*_{\eta,\delta,\nu_{\delta^*}(i+1)} \setminus \cup\{B^*_{\eta,\delta,\nu_{\delta^*}(j+1)} : j < i\}$. So $\langle A^*_{\eta,\delta,\delta^*,i} : i < \kappa\rangle$ are pairwise disjoint subsets of κ. Note that (by the proof of 2.5 but not used) for each pair (η,δ) as above for some club $E_{\eta,\delta}$ of λ, for every $\delta^* \in S^\lambda_\kappa \cap E_{\eta,\delta}$ and $i < \kappa, A^*_{\eta,\delta,\delta^*,i}$ has cardinality κ. We shall show during the proof of (1) that $\{\langle A^*_{\eta,\delta,\delta^*,i} : i < \kappa\rangle : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \delta^* \in S^\lambda_\kappa\}$ is as required in part (2), so this will prove part (2) when $\mathfrak{b}_\kappa > \kappa^+$ and also part (3) when $\lambda > \kappa^+$.

Let $\langle X^*_\xi : \xi < \chi\rangle$ be an antichain of $\mathbb{P}_\kappa$, it exists by the assumption. We now for η, δ, δ^* as above define $\mathbb{P}_\kappa$-names $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$: for $\boldsymbol{G} \subseteq \mathbb{P}_\kappa$ generic over $\boldsymbol{V}$ we let:

$\circledast_0$ $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}[\boldsymbol{G}] = \xi$ iff for some $A \in \boldsymbol{G}, n < \omega$ and $\eta \in {}^n\lambda$ and $\delta, \delta^* \in S^\lambda_\kappa$ we have:

(a) $\langle \alpha_\ell(A, \bar{B}) : \ell < n\rangle = \eta$ so in particular is well defined,

(b) $\alpha_n(A, \bar{B}) = \delta \in S^\lambda_\kappa$,

(c) $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \delta^* \in S^\lambda_\kappa$,

(d) $A \cap A^*_{\eta,\delta,\delta^*,i}$ has at most one member for each $i < \kappa$,

(e) $A \subseteq \cup\{A^*_{\eta,\delta,\delta^*,i} : i \in X^*_\xi\}$.

Note that demands (a),(b),(c) are natural but actually not being used; with them we could have defined the $\mathbb{P}_\kappa$-names $\underset{\sim}{\gamma}_n$ which is $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$ when defined. Now clearly

$\circledast_1$ $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$ is a $\mathbb{P}_\kappa$-name of an ordinal $< \chi$ (may have no value),

$\circledast_2$ for every $p \in \mathbb{P}_\kappa$ for some $\eta \in {}^{\omega>}\lambda$ and $\delta, \delta^* \in S^\lambda_\kappa$, for every $\varepsilon < \chi$ there is q such that $p \leq q \in \mathbb{P}_\kappa$ and $q \Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} = \varepsilon$".

[Why? We start as in the proof of 2.5. First there are $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ such that $p \cap A_{\eta,\delta,i} \in [\kappa]^\kappa$ for κ many ordinals $i < \kappa$. Second, there is a club C_p of λ such that: if $\beta < \gamma < \lambda$ are from C_p then $p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa$. Indeed, $C_p = \{\gamma < \lambda : (\forall \beta < \gamma)\, p \cap B^*_{\eta,\delta,\gamma} \setminus B^*_{\eta,\delta,\beta}$ is from $[\kappa]^\kappa\}$ is as required.

Now by the choice of $\bar{\nu}$, i.e., club guessing, there is $\delta^* \in \operatorname{acc}(C_p) \cap S^\lambda_\kappa$ such that $(\forall i < \kappa)(\nu_{\delta^*}(i) \in C_p)$. So (as we have used $\nu_{\delta^*}(i+1), \nu_{\delta^*}(j+1)$ in the definition of $A^*_{\eta,\delta,\delta^*,i}$)

$$i < \kappa \Rightarrow p \cap A^*_{\eta,\delta,\delta^*,i} \in [\kappa]^\kappa.$$

This fulfills the promise needed for proving part (2) in the present case 1. Choose $\zeta_i \in p \cap A^*_{\eta,\delta,\delta^*,i}$ for $i < \kappa$. Now for every $\xi < \chi$ let $q_\xi = \{\zeta_i : i \in X^*_\xi\}$. Recall that $\langle X^*_\zeta : \zeta < \chi\rangle$ is an antichain in $\mathbb{P}_\kappa$. Clearly for $\xi < \chi$ we have $\mathbb{P}_\kappa \models$ "$p \leq q_\xi$" and $q_\xi \Vdash$ "$\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} = \xi$"; so we have finished proving $\circledast_2$.]

This is enough for proving

$\circledast_3$ forcing with $\mathbb{P}_\kappa$ collapse χ to $\aleph_0$.

[Why? By $\circledast_1 + \circledast_2$ we know that $\Vdash_{\mathbb{P}_\kappa}$ "$\chi = \{\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$ and $\delta^* \in S^\lambda_\kappa\}$", so it is forced that $|\chi| \leq |\lambda|$. As we already have by 2.5 that $\Vdash_{\mathbb{P}_\kappa}$ "$|\lambda| = \aleph_0$", we are done.]

2. CASE. $\mathfrak{b}_\kappa = \kappa^+$.

Let[1] $\lambda = \kappa^+$ and $\bar{B}$ be a good $(\kappa, {}^{\omega>}\lambda)$-sequence. Let $\langle S_\varepsilon : \varepsilon < \kappa\rangle$ be a partition of $S^{\kappa^+}_\kappa$ to (pairwise disjoint) stationary sets. For $\alpha < \kappa^+$ let $\langle u^\alpha_i : i < \kappa\rangle$ be an increasing continuous sequence of subsets of α each of cardinality $< \kappa$ with union α and without loss of generality $\alpha < \beta \Rightarrow (\forall^* i < \kappa)(u^\alpha_i = u^\beta_i \cap \alpha)$. Let $\bar{h} = \langle h_\beta : \beta < \kappa^+\rangle$ exemplifying $\kappa^+ \in \mathfrak{b}^{\rm spc}_\kappa$ be such that each h_β is strictly increasing, $(\forall i) h_\beta(i) > i$ and let $C_\beta = \{\delta < \kappa : \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta\}$ and let $(\bar{B}, \bar{\nu})$ be a good $(\kappa, {}^{\omega>}\lambda)$-parameter; exists by 2.4(2). Now for $\eta \in {}^{\omega>}\lambda$ and $\delta \in S^\lambda_\kappa$ and define $A_{\eta,\delta,i} (i < \kappa), B^*_{\eta,\delta,\gamma} (\gamma < \lambda)$ as in Case 1. Now for $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define the sequence $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha\rangle$ by

$$Y_{\eta,\delta,\alpha,\beta,\gamma} = \cup\{B^*_{\eta,\delta,\gamma} \cap [i, \operatorname{Min}(C_\beta \backslash (i+1))) \cup \{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma \cap u^\alpha_i\} : i \in C_\beta \text{ satisfy } \gamma \in u^\alpha_i\}.$$

[1] Actually, we can make this case to cover Case 1, too: for $\delta_* \in S^\lambda_{\kappa^+}$ choose C'_{δ_*} a club of δ_* of order type κ^+. Now for each δ we can repeat the construction of names from the proof of Case 2, for each $p \in \mathbb{P}_\kappa$ for some δ_* we succeed to show $\circledast$ below.

So $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha \rangle$ is a sequence of pairwise disjoint subsets of κ and for $\varepsilon < \kappa$ let

$$Z_{\eta,\delta,\alpha,\beta,\varepsilon} = \cup\{Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma \in S_\varepsilon \cap \alpha\}.$$

Clearly

$\boxdot_1$ $\langle Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon < \kappa \rangle$ is a sequence of pairwise disjoint subsets of κ

possibly for some quadruple $(\eta,\delta,\beta,\zeta)$ we do not have $(\forall \varepsilon < \kappa)[|Z_{\eta,\delta,\alpha,\beta,\varepsilon}| = \kappa]$. However the quadruples $(\eta,\delta,\alpha,\beta)$ for which this fail, cannot satisfy the desired property in part (2), so we can just omit them.

We shall show during the proof of (1) that

$$\left\langle \langle Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon < \kappa \rangle : \eta \in {}^{\omega>}\lambda,\ \delta \in S^\lambda_\kappa,\ \alpha < \lambda,\ \beta < \lambda \right\rangle$$

exemplify part (2).

Let $\langle X^*_\xi : \xi < \chi \rangle$ be a family of sets from $[\kappa]^\kappa$ such that the intersection of any two have cardinality $< \kappa$, it exists as $\mathbb{P}_\kappa$ fail the χ-c.c.. For each $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define a $\mathbb{P}_\kappa$-name $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}$ as follows:

$\boxdot_2$ for $\boldsymbol{G} \subseteq \mathbb{P}_\kappa$ generic over $\boldsymbol{V}$, $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}[\boldsymbol{G}] = \xi$ iff

- (α) for some $A \in \boldsymbol{G}$ we have
  - (a) $\varepsilon < \kappa \Rightarrow A \cap Z_{\eta,\delta,\alpha,\beta,\varepsilon}$ has at most one member,
  - (b) $A \subseteq \cup\{Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon \in X^*_\xi\}$
- (β) if for no $A \in \boldsymbol{G}$ does (a)+(b) hold and $\xi = 0$.

Clearly

$\boxdot_3$ $\underset{\sim}{\tau}_{\eta,\gamma,\alpha,\beta}$ is a well defined ($\mathbb{P}_\kappa$-name) (by $\boxdot_2$).

Now

$\boxdot_4$ for every $p \in \mathbb{P}_\kappa$, for some $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+, \beta < \kappa^+$ we have: for every $\xi < \chi$ for some $q \in \mathbb{P}_\kappa$ above p we have $q \Vdash$ "$\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta} = \xi$" and $\varepsilon < \kappa \Rightarrow |Z_{\eta,\delta,\alpha,\beta,\varepsilon} \cap p| = \kappa$.

As in Case 1, this is enough for proving that $\mathbb{P}_\kappa$ collapse χ to $\lambda = \kappa^+$. But by 2.5 we already know that forcing with $\mathbb{P}_\kappa$ collapses κ^+ to $\aleph_0$ and so we are done.

Note: we can eliminate η from the $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}$, but not worth it. So we are left with proving $\boxdot_4$.

Why does $\boxdot_4$ hold? First, as in the earlier cases, find $\eta \in {}^{\omega>}\lambda$ and $\delta \in S^\lambda_\kappa$ such that $p \cap A_{\eta,\delta,i} \in [\kappa]^\kappa$ for κ ordinals $i < \kappa$. Second, for some club C_p of λ we have $\beta < \gamma \wedge \beta, \gamma \in C_p \Rightarrow p \cap B^*_{\eta,\delta,\gamma} \setminus B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa$. As S_ε (for $\varepsilon < \kappa$) is a stationary subset of λ and C_p a club of λ for each $\varepsilon < \kappa$ we can choose $\gamma^*_\varepsilon \in S_\varepsilon \cap C_p$. Hence there is $\alpha^* < \kappa^+$ large enough such that $\varepsilon < \kappa \Rightarrow \gamma^*_\varepsilon < \alpha^* \in C_p$. Now define a function $h : \kappa \to \kappa$ by induction on i, as follows:

$$h(i) = \mathrm{Min}\{j : j \in (i,\kappa) \text{ and } i_1 < i \Rightarrow h(i_1) < j \text{ and if } \gamma \in u^{\alpha^*}_i \cap S_\varepsilon \text{ then } p \cap (i,j) \cap B^*_{\eta,\delta,\gamma} \setminus \cup\{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma \cap u^{\alpha^*}_i\} \text{ is not empty}\}.$$

it is well defined as for a given $i < \kappa$ the number of pairs (γ,ε) such that $\gamma \in u^{\alpha^*}_i \cap S_\varepsilon$ is $< \kappa$ and is increasing; next we define

$$C = \{j < \kappa : j \text{ is a limit ordinal such that } i < j \Rightarrow h(i) < j\}.$$

Clearly C is a club of κ and let $h' \in {}^{\kappa}\kappa$ be defined by $h'(i) = h(\mathrm{Min}(C \setminus (i+1))$. By the choice of $\langle h_\beta : \beta < \lambda \rangle$ there is $\beta < \lambda$ such that for κ many ordinals $i < \kappa, h'(i) < h_\beta(i)$. Recall that $C_\beta = \{\delta < \kappa : \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta\}$.

So $W_1 = \{i < \kappa : h'(i) < h_\beta(i)\}$, by the choice of β clearly $W_1 \in [\kappa]^\kappa$. Let $\langle i^0_j : j < \kappa \rangle$ be an enumeration of the club $C \cap \mathrm{acc}(C_\beta)$ of κ in an increasing order, so clearly $\mathscr{U} := \{j < \kappa : W_1 \cap [i^0_j, i^0_{j+1}) \neq \emptyset\}$ is unbounded in κ. For each $j \in \mathscr{U}$ let $i^2_j \in W_1 \cap [i^0_j, i^0_{j+1})$, then let $i^1_j = \sup(C_\beta \cap (i^2_j + 1))$, it is well defined as $i^0_j \in C \cap \mathrm{acc}(C_\beta)$, and so $i^0_j \leq i^1_j$ and let $i^3_j = \min(C_\beta \setminus (i^2_j + 1))$ so $i^2_j < i^3_j$ and

$$i^0_j \leq i^1_j \leq i^2_j < h(i^2_j) < h'(i^2_j) < h_\beta(i^2_j) < i^3_j. \tag{$*$}$$

[why? as said above, by the choice of i^1_j, by the choice of h, by the choice of the pair (C, h'), by $i^2_j \in W_1$, by the choice of i^3_j resp.]

$$i^1_j < i^3_j \text{ are successive members of } C_\beta \tag{$*'$}$$

[why? both are members of C_β by their choices, hence it is enough to prove that $C_\beta \cap (i^1_j, i^3_j) = \emptyset$. But $C_\beta \cap (i^1_j, i^2_j) = \emptyset$ by the choice of i^1_j and $C_\beta \cap (i^2_j, i^3_j) = \emptyset$ by the choice of i^3_j, so we are done.

Now for each $\varepsilon < \kappa$ we know that $\gamma^*_\varepsilon \in \alpha^* \cap S_\varepsilon \cap C_p \subseteq \alpha^* = \cup\{u^{\alpha^*}_i : i < \kappa\}$ and $\langle u^{\alpha^*}_i : i < \kappa \rangle$ is $\subseteq$-increasing hence for some $j(\varepsilon) < \kappa$ if $j \in \mathscr{U} \setminus j(\varepsilon)$ then $\gamma^*_\varepsilon \in u^{\alpha^*}_{i^0_j}$ hence by the choice of $h(i^1_j)$ and $(*)$ we have $p \cap (i^1_j, i^3_j) \cap B^*_{\eta,\delta,\gamma^*_\varepsilon} \setminus \cup \{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma^*_\varepsilon \cap u^{\alpha^*}_{i^0_j}\}$ is not empty; but $i^1_j < i^3_j$ are successive members of C_β by $(*)'$, so the definition of $Y_{\eta,\delta,\alpha^*,\beta,\gamma^*_\varepsilon}$ implies that $p \cap Y_{\eta,\delta,\alpha^*,\beta,\gamma^*_\varepsilon} \cap [i^1_j, i^3_j) \neq \emptyset$.

As this holds for every large enough $j \in \mathscr{U}$ i.e., for every $j \in \mathscr{U} \setminus j(\varepsilon)$ and $\mathscr{U} \in [\kappa]^\kappa$ it follows that $p \cap Y_{\eta,\delta,\alpha^*,\beta,\gamma^*_\varepsilon} \in [\kappa]^\kappa$. By the definition of $Z_{\eta,\delta,\alpha^*,\beta,\varepsilon}$ it follows that $p \cap Z_{\eta,\delta,\alpha^*,\beta,\varepsilon} \in [\kappa]^\kappa$.

We have proved this for every $\varepsilon < \kappa$.

Choose $\zeta_\varepsilon \in p \cap Z_{\eta,\delta,\alpha^*,\beta,\varepsilon}$. Now for each $\xi < \chi$ let

$$q_\xi = \{\zeta_\varepsilon : \varepsilon \in X^*_\xi\}.$$

So clearly:

$$\xi < \chi \Rightarrow \mathbb{P}_\kappa \models \text{“}p \leq q_\xi\text{” and } q_\xi \Vdash_{\mathbb{P}_\kappa} \text{“}\underset{\sim}{\tau}_{\eta,\delta,\alpha^*,\beta} = \xi\text{”}. \qquad \square_{2.6}$$

2.7. *Conclusion.* If κ is regular uncountable and $\mathbb{P}_\kappa$ fail the 2^κ-c.c. then $\mathrm{comp}(\mathbb{P}_\kappa)$ is isomorphic to the completion of $\mathrm{Levy}(\aleph_0, 2^\kappa)$.

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