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LIFTING PROBLEM OF THE MEASURE ALGEBRA

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ABSTRACT
We prove the consistency of "$\mathcal{B}/I_{mz}$ does not split" (see Notation). We write the proof so that with the standard duality, also the consistency of "$\mathcal{B}/I_{fc}$ does not split" (i.e., replacing measure zero by first category, random by generic, etc.) is proved. The method is the oracle chain condition.

NOTATION. Let $\mathcal{B}$ be the family of Borel subsets of $(0,1)$. Every Borel se $\subseteq (0,1)$ has a definition ϕ (in the propositional calculus $L_{\omega_1,\omega}$), i.e., it acts on th proportional variables "$n \in r$". We let $A = \mathrm{Bo}[\phi]$ be the Borel set correspond ing to this definition. Notice that the answer to "$r \in \mathrm{Bo}[\phi]$" is absolute.

If in V, $B \in \mathcal{B}$, and $V[G]$ is a generic extension of V, then let $B^{V[G]}$ be th unique B_1 such that for some ϕ, $V \models$ "$B = \mathrm{Bo}(\phi)$".

Let I_{mz} be the family of $A \in \mathcal{B}$ of measure zero and I_{fc} be the family of $A \in \mathcal{B}$ which are of the first category. If a, b are reals, $0 \leqq a$, $b \leqq 1$, let

$$(a,b) = \{x : a < x < b \text{ or } b < x < a\}.$$

$[a,b]$ is defined similarly. Let I'_{mz} $[I'_{fc}]$ be the family of $A \subseteq (0,1)$ of measure zer [of the first category].

DEFINITION. If B is a Boolean algebra, I an ideal, we say that B/I splits i there is a homomorphism $h : B/I \to B$ such that $h(x/I)/I = x/I$. Equivalentl there is a homomorphism $h : B \to B$ with kernel I, $h(x) = x \bmod I$.

THEOREM. *It is consistent with* ZFC *that* $\mathcal{B}/\mathcal{B} \cap I_{mz}$ *does not split* (*if* ZFC *consistent*).

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REMARKS. (1) If CH holds then $\mathscr{B}/I_{\mathrm{mz}}$ splits (see Oxtoby [2]); in fact this holds for any $\aleph_1$-complete ideal.

(2) Note that $\mathscr{B}/I_{\mathrm{mz}}$ has a natural set of representatives:

$$h^m(X)=\left\{a \in R: 1=\lim_{\varepsilon\to 0}\,[m(X\cap(a-\varepsilon,a+\varepsilon))/2|\varepsilon|]\right\}$$

where $m(X)$ is the Lebesgue measure (of a set of reals).

Now for $X\in\mathscr{B}$, $h^m(X)\in\mathscr{B}$ and $h^m(X)=X \bmod I_{\mathrm{mz}}$ (see Oxtoby [2]).

(3) Note also that $\mathscr{B}+I'_{\mathrm{mz}}/I'_{\mathrm{mz}}$ splits where $\mathscr{B}+I'_{\mathrm{mz}}$ is the Boolean algebra generated by $\mathscr{B}$ and I'_{mz}. A function exemplifying it can be defined as follows: for each real r let E_r be an ultrafilter on $(0,1)$ such that if $A\subseteq(0,1)$ and $1=\lim_{\varepsilon\to 0}[m(A\cap(r-\varepsilon,r+\varepsilon))/2|\varepsilon|]$, then $A\in X$ and for every $X\in\mathscr{B}+I_{\mathrm{mz}}$

$$h(X)=\{r: X\in E_r,\ r\in(0,1)\}.$$

Clearly $h(X)=h^m(x) \bmod I_{\mathrm{mz}}$ hence $h(X)=X \bmod I_{\mathrm{mz}}$.

PROOF OF THE THEOREM. By [3] §2 (better represented in [4] IV, §1, §2, §3) the following lemma suffices (we use the notation from there):

MAIN LEMMA. *Let $\bar{M}$ be an $\aleph_1$-oracle (so* CH *and even $\diamondsuit_{\aleph_1}$ hold) and h a homomorphism from $\mathscr{B}$ to $\mathscr{B}$ with kernel I_{mz}, such that $h(X)=X$ mod I_{mz} for every $X\in\mathscr{B}$.*

Then there is a forcing P satisfying the $\bar{M}$-chain condition, and a P-name $\underset{\sim}{X}$ (of a Borel set) such that for every $G\subseteq P\times Q$ generic over V (where Q is Cohen forcing) there is no Borel set A in $V[G]$ satisfying:

(α) $A=\underset{\sim}{X}[G] \bmod I_{\mathrm{mz}}$ *in* $V[G]$,

(β) *for every* $B\in\mathscr{B}^V$, *if* $B^{V[G]}\subseteq\underset{\sim}{X}[G] \bmod I_{\mathrm{mz}}$ *then* $h(B)^{V[G]}\subseteq A$,

(γ) *for every* $B\in\mathscr{B}^V$, *if* $B^{V[G]}\cap\underset{\sim}{X}[G]=\emptyset \bmod I_{\mathrm{mz}}$ *then* $h(B)^{V[G]}\cap A=\emptyset$.

PROOF OF THE MAIN LEMMA. Let, in this proof, Se denote the set of sequences $\bar{a}=\langle a_i : i\leqq\omega\rangle$ such that the sequence is monotonic, $a_i\neq a_{i+1}$, for $i<\omega$, a_i is rational, but a_ω is irrational, and $\langle a_i : i<\omega\rangle$ converge to a_ω.

STAGE A — DEFINITION. Let $P=P(\langle\bar{a}^\alpha:\alpha<\beta\rangle)$ where $\beta\leqq\omega_1$, $\bar{a}^\alpha\in\mathrm{Se}$, a^α_ω pairwise distinct, denote the following forcing notion: $p\in P$ *iff* the following three conditions hold:

(a) $p=(U_p,f_p)$, where U_p is an open subset of $(0,1)$, $\mathrm{cl}(U_p)$ of measure $<1/2$, and f_p is a function from U_p to $\{0,1\}$;

(b) there are n, b_l, I_l such that $0=b_0<b_1<\cdots<b_{n-1}<b_n=1$ and $U_p=\bigcup_{l=0}^{n-1} I_l$, I_l an open subset of (b_l,b_{l+1}), and even $\mathrm{cl}(I_l)\subseteq(b_l,b_{l+1})$.

(c) I_l is either a rational interval, $f_p \restriction I$ constant, *or* I_l is, for some $\alpha < \beta$ and $n(l) < \omega$, $\bigcup_{n(l) \leqq m < \omega} (a^{\alpha}_{2m}, a^{\alpha}_{2m+1})$, $f_p \restriction (a^{\alpha}_{4m+2k}, a^{\alpha}_{4m+2k+1})$ is constantly k when $n(l) \leqq 2m + k$, $m < \omega$, $k \in \{0, 1\}$.

The order on P is: $p \leqq q$ iff $U_p \subseteq U_q$, $f_p \subseteq f_q$, and $\mathrm{cl}(U_p) \cap U_q = U_p$.

Last we let $\underset{\sim}{X}_P = \bigcup\{(a,b) : (a,b)$ a rational interval $\subseteq (0,1)$ and for some $p \in G_P$, $(a,b) \subseteq U_p$ and $f_p \restriction (a,b)$ is constantly zero$\}$.

STAGE B. We will define here a statement, in the next stage prove that it suffices to prove the main lemma, and later we shall prove it.

(St). Let $P_\delta = P(\langle \bar{a}^\alpha : \alpha < \delta \rangle)$, $\delta < \omega_1$, be given, as well as a countable M_δ, $P_\delta \in M_\delta$, a condition $(p^*, q^*) \in P_\delta \times Q$ and a $(P_\delta \times Q)$-name $\underset{\sim}{\phi}$ of a definition of a Borel set (this is a candidate for $h(\underset{\sim}{X}_P)$).

Then we can find $\bar{a}^\delta \in \mathrm{Se}$ such that, letting $P_{\delta+1} = P(\langle \bar{a}^\alpha : \alpha \leqq \delta \rangle)$, the following conditions hold:

(A) Every predense subset of P_δ which belongs to M_δ is a predense subset of $P_{\delta+1}$ (note that as M_δ is quite closed this implies the same for $(P_\delta \times Q, P_{\delta+1} \times Q)$).

(B) There is $(p', r') \in P_{\delta+1} \times Q$, such that $(p^*, r^*) \leqq (p', r')$ and one of the following hold, for some n:

(B1) $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \notin \mathrm{Bo}[\underset{\sim}{\phi}]$, and $\bigcup_{n < m < \omega} (a^\delta_{4m+2}, a^\delta_{4m+3}) \cap \underset{\sim}{X} = \emptyset$" and $a^\delta_\omega \in h(\bigcup_{n<m<\omega} (a^\delta_{4m+2}, a^\delta_{4m+3}))$;

or

(B2) $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \notin \mathrm{Bo}(\underset{\sim}{\phi})$ and for $\bigcup_{n<m<\omega} (a^\delta_{4m}, a^\delta_{4m+1}) \subseteq \underset{\sim}{X}$" and $a^\delta_\omega \in h(\bigcup_{n<m<\omega} (a^\delta_{4m}, a^\delta_{4m+1}))$.

STAGE C. It is enough to prove the statement (St).

Remember that if $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}(\underset{\sim}{\phi})$" then this continues to hold if we replace $P_{\delta+1}$ by any forcing notion P, $P_{\delta+1} \subseteq P$, provided that some countably many maximal antichains of $P_{\delta+1}$ remain maximal antichains of P. So it is no problem to prove the main lemma.

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So from now on we concentrate on the proof of (St).

STAGE D — CHOOSING $\bar{a}^\delta$. So let P_δ, $\langle \bar{a}^\alpha : \alpha < \delta \rangle$, $\underset{\sim}{\phi}$, M_δ be given, choose λ big enough (i.e. $\lambda = \beth_8^+$), N an elementary submodel of $\langle H(\lambda), \in \rangle$ to which P_δ, $\langle \bar{a}^\alpha : \alpha < \delta \rangle$, $\underset{\sim}{\phi}$, M_δ, h belong, which is countable.

Choose a real a^δ_ω, which belongs to $(0,1) - \mathrm{cl}(U_{p^*})$ but does not belong to any Borel set of measure zero which belongs to N. This is possible as by demand (a) in the definition of $P(\langle \bar{a}^\alpha : \alpha < \delta \rangle)$, $\mathrm{cl}(U_{p^*})$ has measure $< 1/2$. So $(0,1) - \mathrm{cl}(U_{p^*})$

has positive measure, whereas the union of all measure zero Borel sets in N is a countable union hence has measure zero. So a^{δ}_{ω} is a random real over N and $N[a^{\delta}_{\omega}]$ is a model of enough set theory: ZF^{-} + "$\mathcal{P}^{\aleph}(\omega)$ exists" (where $\mathcal{P}(A)$ is the power set, $\mathcal{P}^{n+1}(A) = \mathcal{P}(\mathcal{P}^{n}(A))$) (those facts are now well known; see Jech [1]).

Clearly $a^{\delta}_{\omega} \in h((0, a^{\delta}_{\omega}))$ or $a^{\delta}_{\omega} \in h((a^{\delta}_{\omega}, 1))$, so w.l.o.g. the former occurs. It is also clear that for every $\varepsilon > 0$, $a^{\delta}_{\omega} \in h((a^{\sigma}_{\omega} - \varepsilon, a^{\delta}_{\omega}))$. [Otherwise, choose a rational b, $a^{\delta}_{\omega} - \varepsilon < b < a^{\delta}_{\omega}$, then $a^{\delta}_{\omega} \notin h((a^{\delta}_{\omega} - \varepsilon, a^{\delta}_{\omega})) \Rightarrow a^{\delta}_{\omega} \in h(0, b)$. Hence $a^{\delta}_{\omega} \in h(0, b) - (0, b)$, but this set has measure zero (by the properties of h) and obviously belong to N.]

Now let $\langle b_n : n < \omega \rangle \in N[a^{\delta}_{\omega}]$ be a strictly increasing sequence of rationals converging to a^{δ}_{ω}. Now in $N[a^{\delta}_{\omega}]$ we define a forcing notion R (the well-known dominating function forcing):

$$R = \{(f, g) : f \text{ a function from some } n < \omega \text{ to } \omega, \text{ satisfying } (\forall i < n) f(i) > i; \text{ and } g \text{ is a function from } \omega \text{ to } \omega\}.$$

The order is

$$(f, g) \leqq (f', g') \text{ iff } f \subseteq f', (\forall l) g(l) \leqq g'(l) \text{ and } (\forall i)[i \in \operatorname{Dom} f' \wedge i \neq \operatorname{Dom} f \Rightarrow f'(i) \geqq g(i)].$$

Let f^* be R-generic over N, so it is known that a finite change does not alter this property. We shall work for a while in the model $N[a^{\delta}_{\omega}][f^*]$. We define (in this model) a sequence of natural numbers $\langle n(l) : l < \omega \rangle$, defining $n(l)$ by induction on l. Let $n(0) = 0$, and $n(l+1) = f^*(n(l))$. Now we define for $m < 4$ and $k < \omega$ a set $A^k_m : A^k_m = \bigcup_{k \leq l < \omega} (b_{n(4l+m)}, b_{n(4l+m+1)})$.

So A^0_m $(m = 0, 1, 2, 3)$ is a partition of $(b_0, a^{\delta}_{\omega})$, hence for some unique $m(*)$, $a^{\delta}_{\omega} \in h(A^0_{m(*)})$. (Note that $A^k_m \in N[a^{\delta}_{\omega}][f^*]$, but $h \restriction N[a^{\delta}_{\omega}][f^*]$ does not necessarily belong to this model, so we determine $m(*)$ in V.) As we could have made a finite change in f^* (replacing $f^*(0)$ by $f^*(n(m(*)))$) we can assume $a^{\delta}_{\omega} \in A^0_0$.

As $a^{\delta}_{\omega} \in h((a^{\delta}_{\omega} - \varepsilon, a^{\delta}_{\omega}))$ for every ε, and as h is a homomorphism, $a^{\delta}_{\omega} \in h(A^k_0)$ for every k.

As a first try let us choose $\bar{a}^{\delta} = \langle b_{n(l)} : l < \omega \rangle ^\wedge \langle a^{\delta}_{\omega} \rangle$.

Stage E — Condition (A) of (St).

Subclaim. *Every predense subset $\mathcal{J}$ of P_δ which belongs to M_δ is a maximal antichain of $P_{\delta+1}$.*

Proof of the Subclaim. Let $p \in P_{\delta+1}$, $p \notin P_\delta$, so by $P_{\delta+1}$'s definition there are $q \in P_\delta$ and rational numbers c_0, c_1 and a natural number $l(0)$ such that

$$0 < c_0 < a^\delta_\omega < c_1 < 1, \quad c_0 < b_{n(4l(0))}, \quad c_0 > b_{n(4l(0))-1},$$

$$\mathrm{cl}(U_q) \cap [c_0, c_1] = \emptyset, \quad U_p = U_q \cup A_0^{l(0)} \cup A_2^{l(0)}, \quad f_p = f_q \cup 0_{A_0^{l(0)}} \cup 1_{A_2^{l(0)}}.$$

(0_A is the function with domain A which has constant value 0; similarly 1_A.)

FACT. Let $r \in P_\delta$, $\mathcal{J} \subseteq P_\delta$ be dense, $(c_0, c_1) \subseteq (0,1)$ be an open interval disjoint to U_r. Then

$$C = \{x \in (c_0, c_1) \text{: there is } r_1 \in \mathcal{J}, r_1 \geqq r, x \notin \mathrm{cl}(U_{r_1})\}$$

has measure $|c_1 - c_0|$.

PROOF OF THE FACT. The conclusion is equivalent to "$(c_0, c_1) - C$ has measure zero", so we can partition (c_0, c_1) into finitely many intervals and prove the conclusion for each of them. So w.l.o.g. the measure of (c_0, c_1) is $< 1/2$. Now for every $\varepsilon > 0$ we can find r_0, $r \leqq r_0 \in P_\delta$, $U_{r_0} \cap (c_0, c_1) = \emptyset$ and U_{r_0} has measure $\geqq 1/2 - \varepsilon$ (but of course $< 1/2$). As $\mathcal{J} \subseteq P_\delta$ is dense, there is $r_2 \in \mathcal{J}$, $r_0 \leqq r_2 \in P_\delta$. So (ignoring the sets $\mathrm{cl}(U_{r_l}) - U_{r_l}$, $l = 0, 2$ which have measure zero)

(i) $(c_0, c_1) - U_{r_2} \subseteq C$ (by C's definition);

(ii) $(c_0, c_1) \cap U_{r_2} \subseteq U_{r_2} - U_{r_0}$.

Hence

(iii) $m((c_0, c_1) - C) \leqq m((c_0, c_1) \cap U_{r_2}) \leqq m(U_{r_2} - U_{r_0}) \leqq m(U_{r_2}) - m(U_{r_0})$
$\leqq 1/2 - (1/2 - \varepsilon) = \varepsilon.$

As this holds for every ε we finish the proof of the fact.

CONTINUATION OF THE PROOF OF THE SUBCLAIM. Let $\mathcal{J}_1 = \{r \in P_\delta : (\exists q_1 \in \mathcal{J})(q_1 \leqq r)\}$. Now for every $k > n(4l(0))$ let

$T_k = \{t \colon t \in P_\delta$, U_t is the union of finitely many intervals whose endpoints are from $\{b_l : n(4l(0)) \leqq l < k\}$ and $m(U_q \cup U_t) < 1/2\}$.

So T_k is finite, and for every $t \in T_k$, $q \leqq q \cup t \in P_\delta$, and $a^\delta_\omega \notin \mathrm{cl}(U_t)$. Now in the model N we can define, for each k, $t \in T_k$,

$$D_t = \{x \in (0,1) \text{: there is } r \in \mathcal{J}_1, r \geqq q \cup t, x \notin \mathrm{cl}(U_r)\}.$$

By the fact we have proved, we know that $(0,1) - D_t - U_t$ has measure zero (note, $\mathrm{cl}(U_t) - U_t$ has measure zero). As $a^\delta_\omega \in (0,1) - U_t$ and a^δ_ω does not belong to any Borel set of measure zero which belongs to N, clearly,

$$(*) \qquad \text{for every } k \geqq n(4l(0)) \text{ and } t \in T_k,\ a^\delta_\omega \in D_t.$$

So for each $k \geqq n(4l(0))$ and $t \in T_k$, there is $r_t \subseteq P_\delta$, $r_t \in \mathcal{J}_1$ such that $a^\delta_\omega \notin \mathrm{cl}(U_{r_t})$. Hence for some $g(t) < \omega$, $[b_{g(t)}, a^\delta_\omega] \cap \mathrm{cl}(U_{r_t}) = \emptyset$ and $(a^\delta_\omega - b_{g(t)}) < 1/2 - m(U_{r_t})$. As T_k is finite, we can define $g : \omega \to \omega$, $g(k) = \mathrm{Max}\{g(t) : t \in T_k\}$. Clearly $g \in N[a^\delta_\omega]$. Hence for every large enough l, $g(l) < f^*(l)$ (see Stage D for f^*'s definition). So for every large enough l, $g(n(l)) < n(l+1)$. Choose large enough l, let $k = n(4l)+1$, $U_t = U_p \upharpoonright [b_{n(4l(0))}, b_k]$, $f_t = f_p \upharpoonright U_t$, $t = (U_t, f_t)$ and it belongs to T_k. Now r_t, p are compatible, so we finish the proof of the subclaim.

This really proves part (A) of (St).

Stage F — Condition (B) of (St). Remember $\underset{\sim}{\phi}$ is a $P_\delta \times Q$-name of a Borel set, so it is also a $P_{\delta+1} \times Q$-name. Now let

$$p^*_1 = (U_{p^*} \cup A^k_0 \cup A^k_2, f_{p^*} \cup 0_{A^k_0} \cup 1_{A^k_1}).$$

Clearly for k large enough $p^*_1 \in P_{\delta+1}$, so $p^*_1 \in N[a^\delta_\omega][f^*]$ and $(p^*_1, q^*) \geqq (p^*, q^*)$, so there is $(p', r') \geqq (p^*_1, q^*)$ forcing an answer to "$a^\omega_\delta \in \mathrm{Bo}[\underset{\sim}{\phi}]$", i.e. $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\phi}]$" or $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \notin \mathrm{Bo}[\underset{\sim}{\phi}]$". If the second possibility holds, then (B2) holds, so condition (B) of (St) holds (remember we have made $a^\delta_\omega \in h(A^k_0)$ for every k in stage D).

So suppose $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\phi}]$". Observe that the truth value of such a statement can be computed in $N[a^\delta_\omega][f^*]$ (i.e., we get the same result in the universe and in this countable model). But f^* is R-generic over $N[a^\delta_\omega]$. So if something holds, then there is $(f_0, g_0) \in R$ such that:

(α) $(f_0, g_0) \Vdash_R$ "$(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\phi}]$"",

(β) $f_0 \subseteq f^*$, $(\forall i < \omega)[i \notin \mathrm{Dom} f_0 \Rightarrow g_0(i) \leqq f^*(i)]$.

(Note the definition of $P_{\delta+1}$ depends on f^*.)

So if we can change f^* in finitely many places, maintaining (β), it will still be true that $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\phi}]$". But we can do it in such a way that for some k,

$$A^k_0 = A^l_2, \qquad A^k_2 = A^l_0,$$

so now (B1) will hold.

In any case (B) of (St) holds, hence we finish the proof of (St), hence we finish the proof of the main lemma and of the theorem.

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