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FULL REFLECTION OF STATIONARY SETS AT REGULAR CARDINALS

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Introduction. A stationary subset S of a regular uncountable cardinal κ *reflects fully at regular cardinals* if for every stationary set $T \subseteq \kappa$ of higher order consisting of regular cardinals there exists an $\alpha \in T$ such that $S \cap \alpha$ is a stationary subset of α. We prove that the Axiom of Full Reflection which states that every stationary set reflects fully at regular cardinals, together with the existence of n-Mahlo cardinals is equiconsistent with the existence of Π^1_n-indescribable cardinals. We also state the appropriate generalization for greatly Mahlo cardinals.

1. Results. It has been proved [7], [3] that reflection of stationary sets is a large cardinal property. We address the question of what is the largest possible amount of reflection. Due to complications that arise at singular ordinals, we deal in this paper exclusively with reflection at regular cardinals. (And so we deal with stationary subsets of cardinals that are at least Mahlo cardinals. If $\kappa \geq \aleph_3$ then there exist stationary sets $S \subseteq \{\alpha < \kappa : \text{cf } \alpha = \aleph_0\}$ and $T \subseteq \{\beta < \kappa : \text{cf } \beta = \aleph_1\}$, such that S does not reflect at any $\beta \in T$.)

If S is a stationary subset of a regular uncountable cardinal κ, then the *trace of S* is the set

$$Tr(S) = \{\alpha < \kappa : \quad S \cap \alpha \text{ is stationary in } \alpha\}$$

(and we say that S *reflects at* α). If S and T are both stationary, we define

$$S < T \text{ if for almost all } \alpha \in T, \quad \alpha \in Tr(S)$$

and say that S *reflects fully* in T. (Throughout the paper, "for almost all" means "except for a nonstationary set of points"). As proved in [4], $<$ is a well founded relation; the *order* $o(S)$ of a stationary set is the rank of S in this relation.

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If the trace of S is stationary, then clearly $o(S) < o(Tr(S))$. We say that *S reflects fully at regular cardinals* if its trace meets every stationary set T of regular cardinals such that $o(S) < o(T)$. In other words, if for all stationary sets T of regular cardinals,

$$o(S) < o(T) \text{ implies } S < T.$$

AXIOM OF FULL REFLECTION FOR κ. *Every stationary subset of κ reflects fully at regular cardinals.*

In this paper we investigate full reflection together with the existence of cardinals in the Mahlo hierarchy. Let *Reg* be the set of all regular limit cardinals $\alpha < \kappa$, and for each $\eta < \kappa^+$ let

$$E_\eta = Tr^\eta(Reg) - Tr^{\eta+1}(Reg)$$

(cf. [2]), and call κ *η-Mahlo* where $\eta \leq \kappa^+$ is the least η such that E_η is nonstationary. In particular,

$$E_0 = \text{inaccessible non-Mahlo cardinals}$$
$$E_1 = \text{1-Mahlo cardinals, etc.}$$

We also denote

$$E_{-1} = Sing = \text{the set of all singular ordinals } \alpha < \kappa.$$

It is well known [4] that each E_η, the ηth *canonical* stationary set is equal (up to the equivalence almost everywhere) to the set

$$\{\alpha < \kappa : \ \alpha \text{ is } f_\eta(\alpha)\text{-Mahlo}\}$$

where f_η is the *canonical* ηth function. A *κ^+-Mahlo* cardinal κ is called *greatly Mahlo* [2].

If κ is less than greatly Mahlo (or if it is greatly Mahlo and the canonical stationary sets form a maximal antichain) then Full Reflection for κ is equivalent to the statement

For every $\eta \geq -1$, every stationary $S \subseteq E_\eta$ reflects almost everywhere in every E_ν, $\nu > \eta$.

The simplest case of full reflection is when κ is 1-Mahlo; then full reflection states that every stationary $S \subseteq Sing$ reflects at almost every $\alpha \in E_0$. We will show that this is equiconsistent with the existence of a weakly compact cardinal.

More generally, we shall prove that full reflection together with the existence of n-Mahlo cardinals is equiconsistent with the existence of Π^1_n-indescribable cardinals.

To state the general theorem for cardinals higher up in the Mahlo hierarchy, we first give some definitions. We assume that the reader is familiar with Π^1_n-indescribability. A "formula" means a formula of second order logic for $\langle V_\kappa, \in\rangle$.

Definition. (a) A formula is $\Pi^1_{\eta+1}$ if it is of the form $\forall X \neg\varphi$ where φ is a Π^1_η formula.

(b) If $\eta < \kappa^+$ is a limit ordinal, a formula is Π^1_η if it is of the form $\exists\nu < \eta\, \varphi(\nu, \cdot)$ where $\varphi(\nu, \cdot)$ is a Π^1_ν formula.

For $\alpha \leq \kappa$ and $\eta < \kappa^+$ we define the satisfaction relation $\langle V_\alpha, \in\rangle \models \varphi$ for Π^1_η formulas in the obvious way, the only difficulty arising for limit η, which is handled as follows:

$$\langle V_\alpha, \in\rangle \models \exists\nu < \eta\ \varphi(\nu, \cdot) \quad \text{if} \quad \exists\nu < f_\eta(\alpha) \langle V_\alpha, \in\rangle \models \varphi(\nu, \cdot)$$

where f_η is the ηth canonical function.

Definition. κ is Π^1_η-*indescribable* ($\eta < \kappa^+$) if for every Π^1_η formula φ and every $Y \subseteq V_\kappa$, if $\langle V_\kappa, \in\rangle \models \varphi(Y)$ then there exists some $\alpha < \kappa$ such that $\langle V_\alpha, E\rangle \models \varphi(Y \cap V_\alpha)$.

κ is $\Pi^1_{\kappa^+}$-*indescribable* if it is Π^1_η-indescribable for all $\eta < \kappa^+$.

THEOREM A. *Assuming the Axiom of Full Reflection for κ, we have for every $\eta \leq (\kappa^+)^L$: Every η-Mahlo cardinal is Π^1_η-indescribable in L.*

THEOREM B. *Assume that the ground model satisfies $V = L$. There is a generic extension $V[G]$ that preserves cardinals and cofinalities (and satisfies GCH) such that for every cardinal κ in V and every $\eta \leq \kappa^+$:*

(a) If κ is Π^1_η-indescribable in V then κ is η-Mahlo in $V[G]$.

(b) $V[G]$ satisfies the Axiom of Full Reflection.

2. Proof of Theorem A. Throughout this section we assume full reflection. The theorem is proved by induction on κ. We shall give the proof for the finite case of the Mahlo hierarchy; the general case requires only minor modifications.

Let F^κ_0 denote the club filter on κ in L, and for $n > 0$, let F^κ_n denote the Π^1_n filter on κ in L, i.e. the filter on $P(\kappa) \cap L$ generated by the sets $\{\alpha < \kappa : L_\alpha \models \varphi\}$ where φ is a Π^1_n formula true in L_κ. If κ is Π^1_n-indescribable then F^κ_n is a proper filter. The Π^1_n ideal on κ is the dual of F^κ_n.

By induction on n we prove the following lemma which implies the theorem.

LEMMA 2.1. *Let $A \in L$ be a subset of κ that is in the Π^1_n ideal. Then $A \cap E_{n-1}$ is nonstationary.*

To see that the Lemma implies Theorem A, let $n \geq 1$, and letting $A = \kappa$, we have the implication

$$\kappa \text{ is in the } \Pi^1_n \text{ ideal in } L \Rightarrow E_{n-1} \text{ is nonstationary},$$

and so

$$\kappa \text{ is not } \Pi^1_n\text{-indescribable in } L \Rightarrow \kappa \text{ is not } n\text{-Mahlo}.$$

Proof. The case $n = 0$ is trivial (if A is nonstationary in L then $A \cap Sing$ is nonstationary). Thus assume that the statement is true for n, for all $\lambda \leq \kappa$, and let us prove it for $n+1$ for κ. Let A be a subset of κ, $A \in L$, and let φ be a Π^1_n formula such that for all $\alpha \in A$ there is some $X_\alpha \in L$, $X_\alpha \subseteq \alpha$, such that $L_\alpha \models \varphi(X_\alpha)$. Assuming that $A \cap E_n$ is stationary, we shall find an $X \in L$, $X \subseteq \kappa$, such that $L_\kappa \models \varphi(X)$. Let $B \supseteq A$ be the set

$$B = \{\alpha < \kappa : \exists X \in L\ L_\alpha \models \varphi(X)\},$$

and for each $\alpha \in B$ let X_α be the least such X (in L). For each $\alpha \in B$, $X_\alpha \in L_\beta$ where $\beta < \alpha^+$, and so let β be the least such β. Let $Z_\alpha \in \{0,1\}^\alpha \cap L$ be such that Z_α codes $\langle L_\beta, \in, X_\alpha \rangle$ (we include in Z_α the elementary diagram of the structure $\langle L_\beta, \in, X_\alpha \rangle$).

For every $\lambda \in E_n \cap B$, let

$$B_\lambda = \{\alpha < \lambda : \alpha \in B \text{ and } Z_\alpha = Z_\lambda \mid \alpha\}.$$

We have

$$\begin{aligned} B_\lambda \supseteq\ & \{\alpha < \lambda : Z_\lambda \mid \alpha \text{ codes } \langle L_\beta, \in, X \rangle \text{ where } \beta \text{ is the least } \beta \text{ and} \\ & \quad X \text{ is the least } X \text{ such that } L_\alpha \models \varphi(X) \text{ and } X = X_\lambda \cap \alpha\} \\ =\ & \{\alpha < \lambda : L_\alpha \models \psi(Z_\lambda \mid \alpha, X_\lambda \cap \alpha)\} \end{aligned}$$

where ψ is a $\Pi^1_n \wedge \Sigma^1_n$ statement, and hence B_λ belongs to the filter F^λ_n. By the induction hypothesis there is a club $C_\lambda \subseteq \lambda$ such that $B \cap E_{n-1} \supseteq B_\lambda \cap E_{n-1} \supseteq C_\lambda \cap E_{n-1}$.

LEMMA 2.2. *There is a club $C \subseteq \kappa$ such that $B \cap E_{n-1} \supseteq C \cap E_{n-1}$.*

Proof. If not then $E_{n-1} - B$ is stationary. This set reflects at almost all $\lambda \in E_n$, and since $B \cap E_n$ is stationary, there is $\lambda \in B \cap E_n$ such that $(E_{n-1} - B) \cap \lambda$ is stationary in λ. But $B \cap E_{n-1} \supseteq C_\lambda \cap E_{n-1}$, a contradiction. □

Definition 2.3. For each $t \in L \cap \{0,1\}^{<\kappa}$, let

$$S_t = \{\alpha \in E_{n-1} : t \subset Z_\alpha\}.$$

Since $B \cap E_{n-1}$ is almost all of E_{n-1}, there is for each $\gamma < \kappa$ some $t \in \{0,1\}^\gamma$ such that S_t is stationary.

LEMMA 2.4. *If $t, u \in \{0,1\}^{<\kappa}$ are such that both S_t and S_u are stationary then $t \subseteq u$ or $u \subseteq t$.*

Proof. Let $\lambda \in B \cap E_n$ be such that both $S_t \cap \lambda$ and $S_u \cap \lambda$ are stationary in λ. Let $\alpha, \beta \in C_\lambda$ be such that $\alpha \in S_t$ and $\beta \in S_u$. Since we have $t \subset Z_\alpha \subset Z_\lambda$ and $u \subset Z_\beta \subset Z_\lambda$, it follows that $t \subseteq u$ or $u \subseteq t$. □

COROLLARY 2.5. *For each $\gamma < \kappa$ there is $t_\gamma \in \{0,1\}^\gamma$ such that S_{t_γ} is almost all of E_{n-1}.*

COROLLARY 2.6. *There is a club $D \subseteq \kappa$ such that for all $\alpha \in D$, if $\alpha \in E_{n-1}$ then $\alpha \in B$ and $t_\alpha \subset Z_\alpha$.*

Proof. Let D be the intersection of C with the diagonal intersection of the witnesses for the S_{t_γ}. □

Definition. $Z = \bigcup\{t_\gamma : \gamma < \kappa\}$.

LEMMA 2.7. *For almost all $\alpha \in E_{n-1}$, $Z \cap \alpha = Z_\alpha$.*

Proof. By Corollary 2.6, if $\alpha \in D \cap E_{n-1}$ then $Z_\alpha = t_\alpha$. □

Now we can finish the proof of Lemma 2.1: The set Z codes a set $X \subseteq \kappa$ and witnesses that $X \in L$. We claim that $L_\kappa \models \varphi(X)$. If not, then the set $\{\alpha < \kappa : L_\alpha \models \neg\varphi(X \cap \alpha)\}$ is in the filter F_n^κ (because $\neg\varphi$ is Σ_n^1). By the induction hypothesis, $L_\alpha \models \neg\varphi(X \cap \alpha)$ for almost all $\alpha \in E_{n-1}$. On the other hand, for almost all $\alpha \in E_{n-1}$ we have $L_\alpha \models \varphi(X_\alpha)$ and by Lemma 2.7, for almost all $\alpha \in E_{n-1}, X \cap \alpha = X_\alpha$; a contradiction. □

3. Proof of Theorem B: Cases 0 and 1. The model is constructed by iterated forcing. (We refer to [5] for unexplained notation and terminology). Iterating with Easton support, we do a nontrivial construction only at stage κ where κ is a Mahlo cardinal.

Assume that we have constructed the forcing below κ, and denote it Q, and denote the model $V(Q)$; if $\lambda < \kappa$ then $Q \mid \lambda$ is the forcing below λ and $Q_\lambda \in V(Q \mid \lambda)$ is the forcing at λ. The rest of the proof will be to describe Q_κ. The forcing below κ has size κ and satisfies the κ-chain condition; the forcing at κ will be essentially $< \kappa$-closed (for every $\lambda < \kappa$ has a λ-closed dense set) and will satisfy the κ^+-chain condition. Thus cardinals and cofinalities are preserved, and stationary subsets of κ can only be made nonstationary by forcing at κ, not below κ and not after stage κ; after stage κ no subsets of κ are added.

By induction, we assume that Full Reflection holds in $V(Q)$ for subsets of all $\lambda < \kappa$. We also assume this for every $\lambda < \kappa$:

(a) If λ is inaccessible but not weakly compact in V then λ is non Mahlo in $V[Q]$.

(b) If λ is Π^1_1-indescribable but not Π^1_2-indescribable in V, then λ is 1-Mahlo in $V[Q]$.

(c) And so on accordingly.

Let E_0, E_1, E_2, etc. denote the subsets of κ consisting of all inaccessible non Mahlo, 1-Mahlo, 2-Mahlo etc. cardinals in $V[Q]$.

The forcing Q_κ will guarantee Full Reflection for subsets of κ and make κ into a cardinal of the appropriate Mahlo class, depending on its indescribability in V. (For instance, if κ is Π^1_2-indescribable but not Π^1_3-indescribable, it will be 2-Mahlo in $V(Q * Q_\kappa)$.)

The forcing Q_κ is an iteration of length κ^+ with $< \kappa$-support of forcing notions that shoot a club through a given set. We recall ([1], [7], [6]) how one shoots a club through a single set, and how such forcing iterates: Given a set $B \subseteq \kappa$, the conditions for shooting a club through B are closed bounded sets p of ordinals such that $p \subseteq B$, ordered by end-extension. In our iteration, the B will always include the set Sing of all singular ordinals below κ, which guarantees that the forcing is essentially $< \kappa$-closed. One consequence of this is that at stage α of the iteration, when shooting a club through (a name for) a set $B \in V(Q * Q_\kappa \mid \alpha)$, the conditions can be taken to be sets in $V(Q)$ rather than (names for) sets in $V(Q * Q_\kappa \mid \alpha)$.

We use the standard device of iterated forcing: as Q_κ satisfies the κ^+-chain condition, it is possible to enumerate all names for subsets of κ such that the βth name belongs to $V(Q * Q_\kappa \mid \beta)$, and such that each name appears cofinally often in the enumeration. We call this a *canonical enumeration.*

We use the following two facts about the forcing:

LEMMA 3.1. *If we shoot a club through B, then every stationary subset of B remains stationary.*

Proof. See [5], Lemma 7.38. □

LEMMA 3.2. *If B contains a club, then shooting a club through B has a dense set that is a $< \kappa$-closed forcing (and so preserves all stationary sets).*

Proof. Let $C \subseteq B$ be a club, and let $D = \{p : \max(p) \in C\}$. □

Remark. There is a unique forcing of size κ that is $< \kappa$-closed (and nontrivial), namely the one adding a Cohen subset of κ. We shall henceforth call every forcing that has such forcing as a dense subset *the Cohen forcing* for κ.

We shall describe the construction of Q_κ for the cases when κ is respectively inaccessible, weakly compact and Π^1_2-indescribable, and then outline the general case. Some details in the three low cases have to be handled separately from the general case.

Case 0. Q_γ for γ which is Mahlo but not weakly compact.

We assume that we have constructed $Q \mid \gamma$, and construct Q_γ in $V(Q \mid \gamma)$. To construct Q_γ, we first shoot a club through the set *Sing* and then do an iteration of length γ^+ (with $< \gamma$-support), where at the stage α we shoot a club through B_α where $\{B_\alpha : \alpha < \gamma^+\}$ is a canonical enumeration of all potential subsets of γ such that $B_\alpha \supseteq$ *Sing*. As *Sing* contains a club, γ is in $V(Q * P)$ non-Mahlo. As Q_γ is essentially $< \kappa$-closed, κ remains inaccessible.

In this case, Full Reflection for subsets of γ is (vacuously) true.

This completes the proof of Case 0. We shall now introduce some machinery that (as well as its generalization) we need later.

Definition 3.3. Let γ be an inaccessible cardinal. An *iteration of order* 0 (for γ) is an iteration of length $< \gamma^+$ such that at each stage α we shoot a club through some B_α with the property that $B_\alpha \supseteq$ *Sing*.

LEMMA 3.4.

(a) If P and R are iterations, and P is of order 0 *then $P \Vdash$ (R is of order* 0) *if and only of R is of order* 0.

(b) If $\dot{R}$ is a P-name then $P * \dot{R}$ is an iteration of order 0 *if and only if P is an iteration of order* 0 *and $P \Vdash$ ($\dot{R}$ is an iteration of order* 0).

(c) If $A \subseteq$ Sing is stationary and P is an iteration of order 0 *then $P \Vdash A$ is stationary.*

Proof. (a) and (b) are obvious, and (c) is proved as follows: Consider the forcing R that shoots a club through *Sing*. R is an iteration (of length 1) of order 0, and $R * P \Vdash A$ is stationary, because R preserves A by Lemma 3.1, and forces that P is the iterated Cohen forcing (by Lemma 3.2). Since R commutes with P, we note that A is stationary in some extension of the forcing extension by P, and so $P \Vdash A$ is stationary. □

We stated Lemma 3.4 in order to prepare ground for the (less trivial) generalization. We remark that "P is an iteration of order 0" is a first order property over V_γ (using a subset of V_γ to code the length of the iteration). The following lemma, that does not have an analog at higher cases, simplifies somewhat the handling of Case 1.

LEMMA 3.5. *If γ has a Π^1_1 property φ and P is a $<\gamma$-closed forcing, then $P \Vdash \varphi(\gamma)$.*

Proof. Let $\varphi(\gamma) = \forall X \sigma(X)$, where σ is a 1st order property. Toward a contradiction, let $p_0 \in P$ and $\dot{X}$ be such that $p_0 \Vdash \neg\sigma(\dot{X})$. Construct a descending γ-sequence of conditions $p_0 \geq p_1 \geq \cdots \geq p_\alpha \geq \cdots$ and a continuous sequence $\gamma_0 < \gamma_1 < \cdots < \gamma_\alpha < \cdots$ such that for each α, $p_\alpha \Vdash \neg\sigma(\dot{X} \cap \gamma_\alpha)$, and that p_α decides $\dot{X} \cap \gamma_\alpha$; say $p_\alpha \Vdash \dot{X} \cap \gamma_\alpha = X_\alpha$. Let $X = \bigcup_{\alpha<\gamma} X_\alpha$. There is a club C such that for all $\alpha \in C$, $\sigma(X \cap \alpha)$. This is a contradiction since for some $\alpha \in C$, $\gamma_\alpha = \alpha$. □

Case 1. λ is Π^1_1-indescribable but not Π^1_2-indescribable.

We assume that $Q \mid \lambda$ has been defined, and we shall define an iteration Q_λ of length λ^+. The idea is to shoot clubs through the sets $Sing \cup (Tr(S) \cap E_0)$, for all stationary sets $S \subseteq Sing$ (including those that appear at some stage of the iteration). Even though this approach would work in this case, we need to do more in order to assure that the construction will work at higher cases. For that reason we use a different approach.

At each stage of the iteration, we define a filter F_1 on E_0, such that the filters all extend the Π^1_1 filter on λ in V, that the filters get bigger as the iteration progresses, and that sets that are positive modulo F_1 remain positive (and therefore stationary) at all later stages. The iteration consists of shooting clubs through sets B such that $B \supseteq Sing$ and $B \cap E_0 \in F_1$, so that eventually every such B is taken care of. The crucial property of F_1 is that whenever S is a stationary subset of $Sing$, then $Tr(S) \cap E_0 \in F_1$. Thus at the end of the iteration, every stationary subset of $Sing$ reflects fully. Of course, we have to show that the filter F_1 is nontrivial, that is that in $V(Q \mid \lambda)$ the set E_0 is positive mod F_1.

We now give the definition of the filter F_1 on E_0. The definition is nonabsolute enough so that F_1 will be different in each model $V(Q \mid \lambda * Q_\lambda \mid \alpha)$ for different α's.

Definition 3.6. Let C_λ denote the forcing that shoots a club through *Sing*.

If φ is a Π^1_1 formula and $X \subseteq \lambda$, let

$$B(\varphi,X) = \{\gamma \in E_0 : \varphi(\gamma, X \cap \gamma)\}$$

The filter F_1 is generated by the sets $B(\varphi,X)$ for those φ and X such that $C_\lambda \Vdash \varphi(\lambda,X)$. A set $A \subset E_0$ is *positive* (or *1-positive*), if for every Π^1_1 formula φ and every $X \subseteq \lambda$, if $C_\lambda \Vdash \varphi(\lambda,X)$ then there exists a $\gamma \in A$ such that $\varphi(\gamma, X \cap \gamma)$.

Remarks.

(1) The filter F_1 extends the club filter (which is generated by the sets $B(\varphi,X)$ where φ is first-order). Hence every positive set is stationary.

(2) The property "A is 1-positive" is Π^1_2.

LEMMA 3.7. *In $V(Q \mid \lambda)$, E_0 is positive.*

Proof. We recall that in V, λ is Π^1_1-indescribable, and E_0 is the set of inaccessible, non-weakly-compact cardinals. Let $Q = Q \mid \lambda$. So let φ be a Π^1_1 formula, let $\dot{X}$ be a Q - name for a subset of λ, and assume that $V(Q * C_\lambda) \models \varphi(\lambda, \dot{X})$. The statement that $Q * C_\lambda \Vdash \varphi(\lambda, \dot{X})$ is a Π^1_1 statement (about Q, C and $\dot{X}$). By Π^1_1-indescribability, this reflects to some $\gamma \in E_0$ (as E_0 is positive in the Π^1_1 filter). Since $Q \cap V_\gamma = Q \mid \gamma$ and since $Q \mid \gamma$ satisfies the γ-chain condition, the name $\dot{X}$ reflects to the $Q \mid \gamma$-name for $\dot{X} \cap \gamma$. Also $C_\lambda \cap V_\gamma = C_\gamma$. Hence

$$Q \mid \gamma * C_\gamma \Vdash \varphi(\gamma, \dot{X} \cap \gamma).$$

What we want to show is that $V(Q) \models \varphi(\gamma, \dot{X} \cap \gamma)$. Since forcing above γ does not add subsets of γ it is enough to show that $V(Q \mid \gamma * Q_\gamma) \models \varphi$. However, C_γ was the first stage of Q_γ (see Case 0), and the rest of Q_γ is the iterated Cohen forcing for γ. By Lemma 3.5, if φ is true in $V(Q \mid \gamma * C_\gamma)$, then it is true in $V(Q \mid \gamma * Q_\gamma)$. □

LEMMA 3.8. *If $S \subseteq Sing$ is stationary, then the set $\{\gamma \in E_0 : S \cap \gamma$ is stationary$\}$ is in F_1.*

Proof. The property $\varphi(\lambda, S)$ which states that S is stationary is Π^1_1. If we show that $C_\lambda \Vdash \varphi(\lambda, S)$, then $\{\gamma \in E_0 : \varphi(\gamma, S \cap \gamma)\}$ is in F_1. But forcing with C_λ preserves stationarity of S, by Lemma 3.1. □

Definition 3.9. An *iteration of order* 1 (for λ) is an iteration of length $< \lambda^+$ such that at each stage α we shoot a club through some B_α such that $B_\alpha \supseteq Sing$ and $B_\alpha \cap E_0 \in F_1$.

Remark. If we include the witnesses for $B_\alpha \cap E_0 \in F_1$ as parameters in the definition, i.e. φ_α, X_α such that $C_\lambda \Vdash \varphi_\alpha(\lambda, X_\alpha)$ and $B_\alpha \cap E_0 \supseteq \{\gamma \in E_0 : \varphi(\gamma, X_\alpha \cap \gamma)\}$, then the property "$P$ is an iteration of order 1" is Π^1_1.

We shall now give the definition of Q_λ:

Definition 3.10. Q_λ is (in $V(Q(\lambda))$ an iteration of length λ^+, such that for each $\alpha < \lambda^+$, $Q_\lambda \mid \alpha$ is an iteration of order 1, and such that each potential B is used as B_β at cofinally many stages β.

We will now show that both "$B \in F_1$" and "A is positive" are preserved under iterations of order 1:

LEMMA 3.11. *If $B \in F_1$ and P is an iteration of order 1 then $P \Vdash B \in F_1$. Moreover, if (φ, X) is a witness for $B \in F_1$, then it remains a witness after forcing with P.*

Proof. Let $B \supseteq B(\varphi, X)$ where φ is Π^1_1 and $C_\lambda \Vdash \varphi(\lambda, X)$, and let P be an iteration of order 1. As P does not add bounded subsets, $B(\varphi, X)$ remains the same, and so we have to verify that $P \Vdash (C_\lambda \Vdash \varphi)$. However, C_λ commutes with P, and moreover, C_λ forces that P is the Cohen forcing (because after C_λ, P shoots clubs through sets that contain a club, see Lemma 3.2). By Lemma 3.5, $C_\lambda \Vdash \varphi$ implies that $C_\lambda \Vdash (P \Vdash \varphi)$. □

LEMMA 3.12. *If $A \subseteq E_0$ is positive and P is an iteration of order 1 then $P \Vdash A$ is positive.*

We postpone the proof of this crucial lemma for a while. We remark that the assumption under which Lemma 3.12 will be proved is that the model in which we are working contains $V(Q \mid \lambda)$; this assumption will be satisfied in the future when the Lemma is applied.

LEMMA 3.13.

(a) If P and R are iterations, and P is of order 1 then $P \Vdash$ (R is of order 1) if and only if R is of order 1.

(b) If $\dot{R}$ is a P-name then $P * \dot{R}$ is an iteration of order 1 if and only if P is an iteration of order 1 and $P \Vdash$ ($\dot{R}$ is an iteration of order 1).

(c) Every iteration of order 1 is an iteration of order 0.

Proof. Both (a) and (b) are consequences of Lemma 3.12. The decision whether a particular stage of the iteration R satisfies the definition of being of order 1 depends only on whether $B_\alpha \in F_1$, which does not depend on P.

(c) is trivial. □

COROLLARY 3.14. *In* $V(Q \mid \lambda * Q_\lambda), E_0$ *is stationary (so* λ *is 1-Mahlo), and every stationary* $S \subseteq Sing$ *reflects fully in* E_0.

Proof. Suppose that E_0 is not stationary. Then it is disjoint from some club C, which appears at some stage $\alpha < \lambda^+$ of the iteration Q_λ. So E_0 is nonstationary in $V(Q \mid \lambda * Q_\lambda \mid (\alpha + 1))$. This is a contradiction, since E_0 is positive in that model, by Lemmas 3.7 and 3.12.

If S is a stationary subset of *Sing*, then $S \in V(Q \mid \lambda * Q_\lambda \mid \alpha)$ for some α and so by Lemma 3.8, $B = Tr(S) \cap E_0 \in F_1$ (in that model). Hence B remains in F_1 at all later stages, and eventually, $B = B_\alpha$ is used at stage α, that is we produce a club C so that $B \supseteq C \cap E_0$. Since Q_λ adds no bounded subsets of λ, the trace of S remains the same, and so S reflects fully in $V(Q \mid \lambda * Q_\lambda)$. □

Proof of Lemma 3.12. Let φ be a Π^1_1 property, and let $\dot{X}$ be a P-name for a subset of λ. Let $p \in P$ be a condition that forces that $C_\lambda \Vdash \varphi(\lambda, \dot{X})$. We are going to find a stronger $q \in P$ and a $\gamma \in A$ such that q forces $\varphi(\gamma, \dot{X} \cap \gamma)$.

P is an iteration of order 1, of length α. At stage β of the iteration, we have $P \mid \beta$-names $\dot{B}_\beta$, φ_β and $\dot{X}_\beta$ for a set $\supseteq Sing$, a Π^1_1 formula, and a subset of λ such that $P \mid \beta$ forces that $C_\lambda \Vdash \varphi_\beta(\lambda, \dot{X}_\beta)$ and that $\dot{B}_\beta \supseteq \{\gamma \in E_0 : \varphi_\beta(\gamma, \dot{X}_\beta \cap \gamma)\}$, and we shoot a club through $\dot{B}_\beta$.

Let ψ be the following statement (about V_λ and a relation on V_λ that codes a model of size λ including the relevant parameters and satisfying enough axioms of ZFC; the relation will also insure that the model M below has the properties that we list):

> *P is an iteration of length α, at each stage shooting a club through $\dot{B}_\beta \supseteq Sing$, and $p \Vdash \varphi(\lambda, \dot{X})$ and for every $\beta < \alpha$, $P \mid \beta \Vdash \varphi_\beta(\lambda, \dot{X}_\beta)$.*

First we note that ψ is a Π^1_1 property. Secondly, we claim that $C_\lambda \Vdash \psi$: In the forcing extension by C_λ, P is still an iteration etc., and $p \Vdash \varphi$ and $P \mid \beta \Vdash \varphi_\beta$ because in the ground model, $p \Vdash (C_\lambda \Vdash \varphi)$ and $P \mid \beta \Vdash (C_\lambda \Vdash \varphi_\beta)$, and C_λ commutes with P.

Thus, since A is positive in the ground model, there exists some $\gamma \in A$ such that $\psi(\gamma,$ parameters $\cap V_\gamma)$. This gives us a model M of size γ, and its transitive collapse $N = \pi(M)$, with the following properties:

(a) $M \cap \lambda = \gamma$,

(b) $P, p, \dot{X} \in M$ and $M \models P$ is an iteration given by $\{\dot{B}_\beta : \beta < \alpha\}$,

(c) $p \Vdash \varphi(\gamma, \pi(\dot{X}))$ (the forcing $\Vdash$ is in $\pi(P)$),

(d) $\forall \beta < \alpha$, if $\beta \in M$, then $\pi(P \mid \beta) \Vdash \varphi_\beta(\gamma, \pi(\dot{X}_\beta))$.

It follows that $\pi(P)$ is an iteration on γ (or order 0), of length $\pi(\alpha)$, that at stage $\pi(\beta)$ shoots a club through $\pi(\dot{B}_\beta)$. Also, $p \Vdash \pi(\dot{X}) = \dot{X} \cap \gamma$ (forcing in P).

SUBLEMMA 3.12.1. *There exists an N-generic filter $G \ni p$ on $\pi(P)$ such that if $X \subseteq \gamma$ denotes the G-interpretation $\pi(\dot{X})/G$ of $\pi(\dot{X})$, and for each $\beta \in M, X_\beta = \pi(\dot{X}_\beta)/G$, then $\varphi(\gamma, X)$ and $\varphi_\beta(\gamma, X_\beta)$ hold.*

Proof. We assume that $V(Q \mid \lambda)$ is a part of our universe, and that no subsets of γ have been added after Q_γ. So it suffices to find G in $V(Q \mid \gamma * Q_\gamma)$. Note also that $E_0 \cap \gamma$ is nonstationary (as γ was made non Mahlo by Q_γ). Since $\pi(P)$ is an iteration of order 0, since *Sing* contains a club, and because $\pi(P)$ has size γ, it is the Cohen forcing for γ, and therefore isomorphic to the forcing at each stage of the iteration Q_γ except the first one (which is C_γ).

There is $\eta < \gamma^+$ such that $V(Q \mid \gamma * Q_\gamma \mid \eta)$ contains $\pi(P), \pi(\dot{X})$, all members of N, and all $\pi(\dot{X}_\beta), \beta \in M$. Also, the statements $p \Vdash \varphi(\gamma, \pi(\dot{X}))$ and $\pi(P \mid \beta) \Vdash \varphi_\beta(\gamma, \pi(\dot{X}_\beta))$, being Π^1_1 and true, are true in $V(Q \mid \gamma * Q_\gamma \mid \eta)$. As $\pi(P)$ (below p) as well as the $\pi(P \mid \beta)$ are isomorphic to the η^{th} stage $Q_\gamma(\eta)$ of Q_γ, and we do have a generic filter for $Q_\gamma(\eta)$ over $V(Q \mid \gamma * Q_\gamma \mid \eta)$, we have a G that is N-generic for $\pi(P)$ and $\pi(P \mid \beta)$. If we let $X = \pi(\dot{X})/G$ and $X_\beta = \pi(\dot{X})/G$, then in $V(Q \mid \gamma * Q_\gamma \mid (\eta + 1))$ we have $\varphi(\gamma, X)$ and $\varphi_\beta(\gamma, X_\beta)$. Since the rest of the iteration Q_γ is the iterated Cohen forcing, we use Lemma 3.5 again to conclude that $\varphi(\gamma, X)$ and $\varphi_\beta(\gamma, X_\beta)$ are true in $V(Q \mid \gamma * Q_\gamma)$, hence are true. $\square$

Now let $H = \pi^{-1}(G)$ and for every $\beta \in M$ let $B_\beta = \pi(\dot{B}_\beta)/G$. By induction on $\beta \in M$, we construct a condition $q \leq p$ (with support $\subseteq M$) as follows: For each $\xi \in M$, let $q(\xi) = H_\xi \cup \{\lambda\}$. This is a closed set of ordinals. At stage β, $q \mid \beta$ a condition by the induction hypothesis, and $q \mid \beta \supseteq H \mid \beta$ (consequently, $q \mid \beta$ forces $\dot{X}_\beta \cap \gamma = X_\beta$ and $\dot{B}_\beta \cap \gamma = B_\beta$). H_β is a closed set of ordinals, cofinal in γ, and $H_\beta \subseteq B_\beta$. We let $q(\beta) = H_\beta \cup \{\gamma\}$. In order that $q \mid (\beta + 1)$ is a condition it is necessary that $q \mid \beta \Vdash \gamma \in \dot{B}_\beta$. But by Sublemma 3.12.1 we have $\varphi_\beta(\gamma, X_\beta)$, so this is forced by P (which does not add subsets of γ), and since $q \mid \beta \Vdash X_\beta = \dot{X}_\beta \cap \gamma$, we have $q \mid \beta \Vdash \varphi_\beta(\gamma, \dot{X}_\beta \cap \gamma)$. But this implies that $q \mid \beta \Vdash \gamma \in \dot{B}_\beta$. Hence $q \mid (\beta + 1)$ is a condition, which extends $H \mid (\beta + 1)$.

Therefore q is a condition, and since $q \supseteq H$, we have $q \Vdash \dot{X} \cap \gamma = X$. But $\varphi(\gamma, X)$ holds by Sublemma 3.12.1., so it is forced by q, and so $q \Vdash \varphi(\gamma, \dot{X} \cap \gamma)$, as required. $\square$

4. Case 2 and up. Let κ be Π^1_2-indescribable but not Π^1_3-indescribable. Below κ, we have four different types of limit cardinals in V:

$$\begin{aligned} Sing &= \text{the singular cardinals} \\ E_0 &= \text{inaccessible not weakly compact} \\ E_1 &= \Pi^1_1\text{- but not } \Pi^1_2\text{-indescribable} \\ \text{the rest} &= \Pi^1_2\text{-indescribable} \end{aligned}$$

We shall prove a sequence of lemmas (and give a sequence of definitions), analogous to 3.6–3.14. Whenever possible, we use the same argument; however, there are some changes and additional complications.

Definition 4.1. A Π^1_2 formula φ is *absolute* for $\lambda \in E_1$ if for every $\alpha < \lambda^+$ and every $X \in V(Q \mid \lambda * Q_\lambda \mid \alpha)$,

(1) $V(Q \mid \lambda * Q_\lambda \mid \alpha) \models$ (for every iteration R of order 1, $\varphi(\lambda, X)$ iff $R \Vdash \varphi(\lambda, X)$),

(2) $V(Q \mid \lambda * Q_\lambda \mid \alpha) \models \varphi(\lambda, X)$ implies $V(Q \mid \lambda * Q_\lambda) \models \varphi(\lambda, X)$, and

(3) $V(Q \mid \lambda * Q_\lambda \mid \alpha) \models \neg\varphi(\lambda, X)$ implies $V(Q \mid \lambda * Q_\lambda) \models \neg\varphi(\lambda, X)$.

We say that φ is *absolute* if it is absolute for all $\lambda \in E_1$, $\lambda < \kappa$.

Definition 4.2. If φ is a Π^1_2 formula and $X \subseteq \kappa$, let

$$B(\varphi, X) = \{\lambda \in E_1 : \varphi(\lambda, X \cap \lambda)\}.$$

The filter F_2 is generated by the sets $B(\varphi, X)$ where φ is an absolute Π^1_2 formula and X is such that $R \Vdash \varphi(\kappa, X)$, for all iterations R of order 1.

A set $A \subseteq E_1$ is *positive* (2-positive) if for any absolute Π^1_2 formula φ and every $X \subseteq \kappa$, if every iteration R of order 1 forces $\varphi(\kappa, X)$, then there exists a $\lambda \in A$ such that $\varphi(\lambda, X \cap \lambda)$.

Remark. The property "A is 2-positive" is Π^1_3.

LEMMA 4.3. *In $V(Q \mid \kappa)$, E_1 is positive.*

Proof. Let $Q = Q \mid \kappa$. Let φ be an absolute Π^1_2 formula, and let $\dot{X}$ be a Q-name for a subset of κ, and assume that in $V(Q)$, $R \Vdash \varphi(\kappa, \dot{X})$ for all order-1 iterations R. In particular, (taking R the empty iteration), $V(Q) \models \varphi(\kappa, \dot{X})$.

Using the Π^1_2-indescribability of κ in V, there exists a $\lambda \in E_1$ such that $V(Q \mid \lambda) \models \varphi(\lambda, \dot{X} \cap \lambda)$. In order to prove that $V(Q) \models \varphi(\lambda, \dot{X} \cap \lambda)$, it is enough to show that $V(Q \mid \lambda * Q_\lambda) \models \varphi(\lambda, \dot{X} \cap \lambda)$. This however is true because φ is absolute for λ. □

LEMMA 4.4. *The property "S is 1-positive" of a set $S \subseteq E_0$ is an absolute Π^1_2 property, and is preserved under forcing with iterations of order 1.*

Proof. The preservation of "1-positive" under iterations of order 1 was proved in Lemma 3.12. To show that the property is absolute for all $\lambda \in E_1$, first assume that $S \in V(Q \mid \lambda * Q_\lambda \mid \alpha)$ is 1-positive. Since all longer initial segments of the iteration Q_λ are iterations of order 1, hence order 1 iterations over $Q_\lambda \mid \alpha$ (by Lemma 3.13), S is 1-positive in each $V(Q \mid \lambda * Q_\lambda \mid \beta)$, $\beta > \alpha$. However, the property "S is 1-positive" is Π^1_2, and so it also holds in $V(Q \mid \lambda * Q_\lambda)$, because every subset of λ in that model appears at some stage β. (We remark that this argument, using Π^1_2, does not work in higher cases).

Conversely, assume that S is not 1-positive in $V(Q \mid \lambda * Q_\lambda \mid \alpha)$. There exists a Π^1_1 formula φ and some $X \subseteq \lambda$ such that $\varphi(\gamma, X \cap \gamma)$ fails for all $\gamma \in S$, while $C_\lambda \Vdash \varphi(\lambda, X)$. The rest of the argument is the same as the one in Lemma 3.11: Let $P = Q_\lambda/(Q_\lambda \mid \alpha)$; C_λ commutes with P and forces that P is the iterated Cohen forcing. Hence by Lemma 3.5, $P \Vdash (C_\lambda \Vdash \varphi)$, i.e. $V(Q \mid \lambda * Q_\lambda) \models (C_\lambda \Vdash \varphi)$. Therefore S is not 1-positive in $V(Q \mid \lambda * Q_\lambda)$. (Again, this argument does not work in higher cases.). □

LEMMA 4.5. *The property "R is an iteration of order 1" is an absolute* Π^1_2 *property, and is preserved under forcing with iterations of order 1. Moreover, in* $V(Q \mid \lambda * Q_\lambda)$, *if R is an iteration of order 1, then R is the Cohen forcing.*

Proof. The preservation of the property under iterations of order 1 was proved in Lemma 3.13. If R is an iteration of order 1 in $V(Q \mid \lambda * Q_\lambda \mid \alpha)$, shooting clubs through $\dot{A}_0, \dot{A}_1, \dot{A}_2$, etc., then R embeds in Q_λ above α as a subiteration, i.e. there are β_0, β_1, etc. such that $\dot{A}_0 = B_{\beta_0}, \dot{A}_1 = B_{\beta_1}$, etc. Moreover, there is some $\gamma > \alpha$ such that the A_0, A_1, A_2, etc. all contain a club. Hence R is the Cohen forcing in $V(Q \mid \lambda * Q_\lambda \mid \gamma)$. Therefore R is the Cohen forcing in $V(Q \mid \lambda * Q_\lambda)$, and consequently an iteration of order 1. As for the absoluteness downward, we give the proof for iterations of length 2. Let $M_\infty = V(Q \mid \lambda * Q_\lambda)$, let $R = (R_0, R_1)$ be an iteration given by A_0 and $\dot{A}_1 \in M_\infty(R_0)$, such that in M_∞, $A_0 \in F_1$ and $R_0 \Vdash \dot{A}_1 \in F_1$. Let $R \in M_\alpha = V(Q \mid \lambda * Q_\lambda \mid \alpha)$. We will show that in M_α, R is an iteration of order 1, and that in M_∞, R is the Cohen forcing.

First, since $A_0 \in F_1$ is absolute, there is a $\beta > \alpha$ such that $M_\beta \models A_0$ contains a club and such that $A_0 = B_\beta$ (B_β is the set used at stage β of the iteration Q_λ). Since $M_\beta \models$ (R_0 is Cohen), we have $M_\infty \models R_0$ is Cohen.

Now, in M_∞ we have $R_0 \Vdash \dot{A}_1 \in F_1$. We claim that in M_β, $R_0 \Vdash \dot{A}_1 \in F_1$. Then it follows that R is an iteration of order 1 in M_β.

It remains to prove the claim. Let $\dot{X}$ denote $\dot{A}_1$, let $\varphi(\dot{X})$ denote the absolute Π^1_2 property $\dot{A}_1 \in F_1$ and let C denote the Cohen forcing. We recall that $M_{\beta+1} = M_\beta(C)$.

SUBLEMMA 4.5.1. *Let* $\dot{X}$ *be a C-name in* M_β, *and assume that* $M_{\beta+1} = M_\beta(C)$. *If* $C \Vdash \varphi(\dot{X})$ *in* M_∞, *then* $C \Vdash \varphi(\dot{X})$ *in* M_β.

Proof. Let P be the forcing such that $M_\infty = M_{\beta+1}(P)$, and assume, toward a contradiction, that $C \Vdash \varphi(\dot{X})$ in M_∞ but $C \Vdash \neg\varphi(\dot{X})$ in M_β. Let $G_C \times G_P \times H$ be a generic on $C * P * C$, and let $X = \dot{X}/H$. Let $C = C_1 \times C_2$ where both C_1 and C_2 are Cohen, and consider the generic $H \times G_C \times G_P$ on $C_1 \times C_2 \times P = C \times P$ (it is a generic because since H is generic over $G_C \times G_P$, $G_C \times G_P$ is generic over H).

In M_β, C_1 forces φ false, hence $\varphi(X)$ is false in $M_\beta[H]$. Since $\neg\varphi$ is preserved by Cohen forcing (in fact by all order-1 iterations), so $\varphi(X)$ is false in $M_\beta[H\times G_C]$. Now φ is absolute (between $M_{\beta+1}$ and M_∞) and so $\varphi(X)$ is false in $M_\beta[H\times G_C\times G_P]$. On the other hand, since $C \Vdash \varphi(\dot{X})$ in M, we have $M_\beta[G_C\times G_P\times H] \models \varphi(\dot{X}/H)$, so φ is true in $M_\beta[G_C\times G_P\times H]$, a contradiction. □

LEMMA 4.6. *If $S \subseteq E_0$ is 1-positive, then the set*

$$\{\lambda \in E_1 : S\cap\lambda \textit{ is 1-positive}\}$$

is in F_2. Therefore $Tr(S)\cap E_1 \in F_2$.

Proof. The first sentence follows from the definition of F_2 because "S is 1-positive" is absolute Π^1_2 and if S is positive then it is positive after every order 1 iteration. The second sentence follows, since 1-positive subsets of λ are stationary. □

Definition 4.7. An *iteration of order* 2 (for κ) is an iteration of length $<\kappa^+$ that at each stage α shoots a club through some B_α such that $B_\alpha \supseteq Sing$, $B_\alpha\cap E_0 \in F_1$, and $B_\alpha\cap E_1\in F_2$.

Remarks.

(1) An iteration of order 2 is an iteration of order 1.

(2) If we include the witnesses for B_α to be in the filters, then the property "P is an iteration of order 2" is Π^1_3.

Definition 4.8. Q_κ is (in $V(Q \mid \kappa)$) an iteration of length κ^+, such that for each $\alpha<\kappa^+$, $Q_\kappa \mid \alpha$ is an iteration of order 2, and such that each potential B is used as B_β at cofinally many stages β.

LEMMA 4.9. *If $B\in F_2$ and P is an iteration of order 2 then $P\Vdash B\in F_2$ (and a witness (φ,X) remains a witness).*

Proof. Let $B\supseteq B(\varphi,X)$ where φ is an absolute Π^1_2, and every iteration of order 1 forces φ; let P be an iteration of order 2. Since P does not add subset of κ, $B(\varphi,X)$ remains the same and φ remains absolute. Thus it suffices to verify that for each P-name $\dot{R}$ for an order 1 iteration, $P\Vdash(\dot{R}\Vdash\varphi)$. However, P is an iteration of order 1, so by Lemma 3.13, $P*\dot{R}$ is an iteration of order 1, and by the assumption on φ, $P*\dot{R}\Vdash\varphi$. □

LEMMA 4.10. *If $A\subseteq E_1$ is 2-positive and P is an iteration of order 2 then $P\Vdash A$ is 2-positive.*

Proof. Let φ be an absolute Π^1_2 property, let $\dot{X}$ be a P-name for a subset of κ and let $p \in P$ force that for all order-1-iterations R, $R \Vdash \varphi(\kappa, \dot{X})$. We want a $q \leq p$ and a $\lambda \in A$ such that $q \Vdash \varphi(\lambda, \dot{X} \cap \lambda)$.

P is an iteration of order 2 that at each stage β (less than the length of P) shoots a club through a set $\dot{B}_\beta$ such that $P \mid \beta$ forces that

(1) $\dot{B}_\beta \supseteq Sing$,

(2) $\dot{B}_\beta \cap E_0 \supseteq \{\gamma \in E_0 : \varphi^1_\beta(\gamma, \dot{X}_\beta \cap \gamma)\}$, and

(3) $\dot{B}_\beta \cap E_1 \supseteq \{\lambda \in E_1 : \varphi^2_\beta(\lambda, \dot{Y}_\beta \cap \lambda)\}$,

where $\dot{X}_\beta$ and $\dot{Y}_\beta$ are names for subsets of κ, the φ^1_β are Π^1_1 formulas (with some extra property that make P an order-1 interation) and the φ^2_β are absolute (in $V(Q \mid \kappa)$) Π^1_2 properties, and $P \mid \beta$ forces that $\forall R$ (if R is an iteration of order 1 then $R \Vdash \varphi^2_\beta(\kappa, \dot{Y}_\beta)$).

We shall reflect, to some $\lambda \in E_1$, the Π^1_2 statement ψ that states (in addition to a first order statement in some parameter that produces the model M below):

(a) P is an iteration of order 1 using the $\varphi^1_\beta, \dot{X}_\beta, \varphi^2_\beta, \dot{Y}_\beta$,

(b) $p \Vdash \varphi(\kappa, \dot{X})$,

(c) for every $\beta <$ length(P), $P \mid \beta \Vdash \varphi^2_\beta(\kappa, \dot{Y}_\beta)$.

First we note that ψ is a Π^1_2 property. Secondly, we claim that ψ is absolute for every $\lambda \in E_1$. Being an iteration of order 1 is absolute by Lemma 4.5. That (b) and (c) are absolute will follow once we show that if φ is an absolute Π^1_2 property and R an iteration of order 1, then "$R \Vdash \varphi$" is absolute:

SUBLEMMA 4.10.1. *Let φ be absolute for λ, let $\alpha < \lambda^+$, $X, R \in V(Q \mid \lambda * Q_\lambda \mid \alpha)$ be a subset of λ and an iteration of order 1. Then the property $R \Vdash \varphi(\lambda, X)$ is absolute between $M_\alpha = V(Q \mid \lambda * Q_\lambda \mid \alpha)$ and $M_\infty = V(Q \mid \lambda * Q_\lambda)$.*

Proof. Let $M_\alpha \models (R \Vdash \varphi(\lambda, X))$. Then $M_\alpha \models \varphi(\lambda, X)$ and by absoluteness, $M_\infty \models \varphi(\lambda, X)$. If in M_∞, $R \Vdash \neg\varphi(\lambda, X)$, then because R is in M_∞ the Cohen forcing, there is (by Sublemma 4.5.1) some $\beta > \alpha$ such that R is the Cohen forcing in M_β and $M_\beta \models (R \Vdash \neg\varphi)$. By absoluteness again, $M_\beta \models \neg\varphi$, a contradiction. □

Thus ψ is an absolute Π^1_2 property. Next we show that if R is an iteration of order 1 then R forces $\psi(\kappa$, parameters):

(a) $R \Vdash$ (P is an iteration of order 1), by Lemma 3.13.

(b) R commutes with P, and by the assumption of the proof, $p \Vdash (R \Vdash \varphi(\kappa, \dot{X}))$. Hence $R \Vdash (p \Vdash \varphi(\kappa, \dot{X}))$.

(c) For every β, R commutes with $P \mid \beta$, and by the assumption on φ^2_β, $P \mid \beta \Vdash (R \Vdash \varphi^2_\beta(\kappa, \dot{Y}_\beta))$. Hence $R \Vdash (P \mid \beta \Vdash \varphi^2_\beta(\kappa, \dot{Y}_\beta))$.

Now since A is 2-positive in the ground model, there exists a $\lambda \in A$ such that $\psi(\lambda,$ parameters $\cap V_\lambda)$. This gives us a model M of size λ, and its transitive collapse $N = \pi(M)$, with the following properties:

(a) $M \cap \kappa = \lambda$,

(b) $P, p, \dot{X} \in M$,

(c) $\pi(P)$ is an iteration of order 1 for λ,

(d) $p \Vdash \varphi(\lambda, \pi(\dot{X}))$,

(e) $\forall \beta \in M \quad \pi(P \mid \beta) \Vdash \varphi^2_\beta(\lambda, \pi(\dot{Y}_\beta))$.

The rest of the proof is analogous to the proof of Lemma 3.12, as long as we prove the analog of Sublemma 3.12.1: after that, the proof is Case 1 generalizes with the obvious changes.

SUBLEMMA 4.10.2. *There exists an N-generic filter $G \ni p$ on $\pi(P)$ such that if $X = \pi(\dot{X})/G$ and $Y_\beta = \pi(\dot{Y}_\beta)/G$ for each $\beta \in M$, then $\varphi(\lambda, X)$ and $\varphi^2_\beta(\lambda, Y_\beta)$ hold.*

Proof. We find G in $V(Q \mid \lambda * Q_\lambda)$. Since $\pi(P)$ is an iteration of order 1 and φ is absolute, there is an $\alpha < \lambda^+$ such that $V(Q \mid \lambda * Q_\lambda \mid \alpha)$ contains $\pi(\dot{X}), \pi(\dot{Y}_\beta)$ $(\beta \in M)$ and the dense sets in N, thinks that $\pi(P)$ is the Cohen forcing, such that the forcing $Q_\lambda(\alpha)$ at stage α is the Cohen forcing, and (by absoluteness and by Sublemma 4.10.1) $Q_\lambda(\alpha)$ (or $\pi(P)$) forces $\varphi(\lambda, \pi(\dot{X}))$ and $\varphi^2_\beta(\lambda, \pi(\dot{Y}_\beta))$. The generic filter on $Q_\lambda(\alpha)$ yields a generic G such that $V(Q \mid \lambda * Q_\lambda \mid (\alpha + 1)) \models \varphi(\lambda, X)$ and $\varphi^2_\beta(\lambda, Y_\beta)$ where $X = \pi(\dot{X})/G$, $Y_\beta = \pi(\dot{Y}_\beta)/G$. By absoluteness again, $\varphi(\lambda, X)$ and $\varphi^2_\beta(\lambda, Y_\beta)$ hold in $V(Q \mid \lambda * Q_\lambda)$, and hence they hold. □

LEMMA 4.11.

(a) *If P and R are iterations, and P is of order 2, then $P \Vdash$ (R is of order 2) if and only if R is order 2.*

(b) *If $\dot{R}$ is a P-name then $P * \dot{R}$ is an iteration of order 2 if and only if P is an iteration of order 2 and $P \Vdash$ ($\dot{R}$ is an iteration of order 2).*

Proof. By Lemma 4.10 (just as Lemma 3.13 follows form Lemma 3.12). □

COROLLARY 4.12. *In $V(Q \mid \kappa * Q_\kappa)$, E_1 is stationary, every stationary $S \subseteq E_0$ reflects fully, and every stationary $T \subseteq E_1$ reflects fully.*

Proof. The first part follows from Lemma 4.3 and 4.10. The second part is a consequence of Lemmas 3.8 and 4.6 and the construction that destroys non-1-positive as well as all non-2-positive sets. □

This concludes Case 2. We can now go on to Case 3 (and in an analogous way, to higher cases), with only one difficulty remaining. In analogy with definition 4.2

we can define a filter F_3 and the associated with it 3-positive sets. All the proofs of Chapter 4 will generalize from Case 2 to Case 3, with the exception of Lemma 4.4 which proved that "1-positive" is an absolute Π^1_2 property. The proof does not generalize, as it uses, in an essential way, the fact that the property is Π^1_2, while "2-positive" is a Π^1_3 property.

However, we can replace the property "$A \subseteq E_1 \cap \kappa$ is 2-positive" by another Π^1_3 property that is absolute for κ, and that is equivalent to the definition 4.2 at all stages of the iteration Q_κ except possibly at the end of the iteration. The new property is as follows:

(4.13) *Either Full Reflection fails for some $S \subseteq Sing \cap \kappa$ and A is 2-positive, or Full Reflection holds for all subsets of Sing and A is stationary.*

"Full Reflection" for $S \subseteq Sing$ means that $E_0 - Tr(S)$ is nonstationary. It is a Σ^1_1 property of S, and so (4.13) is Π^1_3. We claim that Full Reflection fails at every intermediate stage of Q_κ. Hence (4.13) is equivalent to "2-positive" at the intermediate stages. At the end of Q_κ, every 2-positive set becomes stationary, and every non-2-positive set becomes nonstationary. Hence (4.13) is absolute.

Since for every $\alpha < \kappa^+$, the size of $Q \mid \kappa * Q_\kappa \mid \alpha$ is κ, the following lemma verifies our claim:

LEMMA 4.14. *Let κ be a Mahlo cardinal, and assume $V = L[X]$ where $X \subseteq \kappa$. Then there exists a stationary set $S \subseteq Sing \cap \kappa$ such that for every $\gamma \in E_0$, $S \cap \gamma$ is nonstationary.*

Proof. We define $S \subseteq Sing$ by induction on $\alpha < \kappa$. Let $\alpha \in Sing$ and assume $S \cap \alpha$ has been defined. Let $\eta(\alpha)$ be the least $\eta < \alpha^+$ such that $L_\eta[X \cap \alpha]$ is a model of ZFC$^-$ and $L_\eta[X \cap \alpha] \models \alpha$ is not Mahlo. Let

$$\alpha \in S \text{ iff } L_{\eta(\alpha)}[X \cap \alpha] \models S \cap \alpha \text{ is nonstationary.}$$

First we show that S is stationary.

Assume that S is nonstationary. Let $\nu < \kappa^+$ be such that $S \in L_\nu[X]$ and $L_\nu[X] \models S$ is nonstationary. Also, since κ is Mahlo, we have $L_\nu[X] \models \kappa$ is Mahlo. Using a continuous elementary chain of submodels of $L_\nu[X]$, we find a club $C \subseteq \kappa$ and a function $\nu(\xi)$ on C such that for every $\xi \in C$,

$$L_{\nu(\xi)}[X \cap \xi] \models \xi \text{ is Mahlo and } S \cap \xi \text{ is not stationary.}$$

If $\alpha \in Sing \cap C$, then because α is Mahlo in $L_{\nu(\alpha)}[X \cap \alpha]$ but non Mahlo in $L_{\eta(\alpha)}[X \cap \alpha]$, we have $\nu(\alpha) \leq \eta(\alpha)$. Since $S \cap \alpha$ is nonstationary in $L_{\nu(\alpha)}[X \cap \alpha]$, it is nonstationary in $L_{\eta(\alpha)}[X \cap \alpha]$. Therefore $\alpha \in S$, and so $S \supseteq Sing \cap C$ contrary to the assumption that S is nonstationary.

Now let $\gamma \in E_0$ be arbitrary and let us show that $S \cap \gamma$ is nonstationary. Assume that $S \cap \gamma$ is stationary. Let $\delta < \gamma^+$ be such that $S \cap \gamma \in L_\delta[X \cap \gamma]$, that $L_\delta[X \cap \gamma] \models S \cap \gamma$ is stationary and that $L_\delta[X \cap \gamma] \models \gamma$ is not Mahlo. There is a club $C \subseteq \gamma$ and a function $\delta(\xi)$ on C such that for every $\xi \in C$,

$$L_{\delta(\xi)}[X \cap \xi] \models \xi \text{ is not Mahlo and } S \cap \xi \text{ is stationary.}$$

Since $S \cap \gamma$ is stationary, there is an $\alpha \in S \cap C$. Because $\eta(\alpha)$ is the least η such that α is not Mahlo in $L_{\eta(\alpha)}[X \cap \alpha]$, we have $\eta(\alpha) \leq \delta(\alpha)$. But $S \cap \alpha$ is nonstationary in $L_{\eta(\alpha)}[X \cap \alpha]$ and stationary in $L_{\delta(\alpha)}[X \cap \alpha]$, a contradiction. □

Now with the modification given by (4.13), the proofs of Chapter 4 go through in the higher cases, and the proof of Theorem B is complete.

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