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EXISTENTIALLY CLOSED STRUCTURES IN THE POWER OF THE CONTINUUM*

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Introduction

We introduce in this paper the notion of tree system. Under suitable conditions tree systems can be used to construct structures of the power of the continuum which are $\infty\omega$-equivalent to a prescribed countable structure. In particular tree systems allow a uniform approach to certain problems concerning existentially closed groups and skew fields, and universal locally finite groups. By means of tree system constructions we prove the following theorems:

Theorem A. *For every uncountable existentially closed group or skew field* $\mathfrak{M}$ *there is a structure* $\mathfrak{N}$ *with the following properties*:

(1) $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ *(so especially* $\mathfrak{N}$ *is an existentially closed group or skew field) and the cardinality of* $\mathfrak{N}$ *is* $2^{\aleph_0}$.

(2) *Every uncountable set of elements of* $\mathfrak{N}$ *contains an uncountable subset of pairwise noncommuting elements (in particular no uncountable substructure of* $\mathfrak{N}$ *is commutative).*

Theorem B. (1) *There is a universal locally finite group* $\mathfrak{N}$ *of power* $2^{\aleph_0}$ *such that every uncountable set of elements of* $\mathfrak{N}$ *contains an uncountable subset of pairwise noncommuting elements.*

(2) *There is a family* $(\mathfrak{N}_i)_{i<2^{\aleph_0}}$ *of universal locally finite groups each of cardinality* $2^{\aleph_0}$ *such that no uncountable group is embeddable in* $\mathfrak{N}_i$ *and* $\mathfrak{N}_j$ *whenever* $i<j<2^{\aleph_0}$.

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Theorem C. *There is a universal locally finite group $\mathfrak{N}$ of cardinality $2^{\aleph_0}$ such that no uncountable subgroup of $\mathfrak{N}$ belongs to any proper variety of groups.*

All statements in these theorems remain true when the cardinality of $\mathfrak{N}$ and of the $\mathfrak{N}_i$ is taken to be κ ($\aleph_0 < \kappa \leq 2^{\aleph_0}$) instead of $2^{\aleph_0}$.

The first section of the paper contains the definition of tree systems. A tree system consists of a collection of 'bottom' directed systems and a 'top' directed system whose structures are the direct limits of the bottom systems and whose morphisms are the inclusion mappings. We shall be interested in the direct limit of the top system which we call limit structure. By the special choice of the top system, the bottom systems are subjected to strong compatibility conditions. This makes it easier to get the limit structure to be $\infty\omega$-equivalent to a prescribed structure (whose finitely generated substructures are used to build up the bottom systems). On the other hand these conditions, being amalgamation properties, make it difficult to construct large limit structures. In Section 2 we prove some technical results which shall ensure the existence of tree systems with large limit structures under certain algebraic conditions. These results are used in Section 3 to construct tree systems whose limit structures satisfy the conditions on $\mathfrak{N}$ in Theorem A(1) and the corresponding part of B(1). In Section 4 the construction of Section 3 is refined in order to get limit structures satisfying the remaining conditions of the Theorems A and B. Section 5 is dedicated to the proof of Theorem C.

The first uniform treatment of uncountable existentially closed groups, skew fields and locally finite groups (note that universal locally finite groups are just the existentially closed structures in the class of locally finite groups) is due to Macintyre [6]. Macintyre proves Theorem A and B(1) assuming V = L. His results were considerably strengthened by Hickin [3] in the case of locally finite groups and by Shelah–Ziegler [8] in the case of groups. Hickin proves in ZFC the existence of a family $(\mathfrak{N}_i)_{i<2^{\aleph_1}}$ of universal locally finite groups of cardinality $\aleph_1$ with the property of Theorem B(2) and constructs a universal locally finite group which, among other properties, satisfies Theorem C for $\aleph_1$ instead of $2^{\aleph_0}$. Shelah–Ziegler proves Theorem A(1) for arbitrary uncountable cardinals instead of $2^{\aleph_0}$, and shows for instance that in every uncountable cardinal there is an existentially closed group $\mathfrak{N}$ which is $\infty\omega$-equivalent to a prescribed countable existentially closed group and enjoys the property that every uncountable subgroup of $\mathfrak{N}$ contains an uncountable free group. (If $\mathfrak{N}$ has at most $2^{\aleph_0}$ elements this could also be obtained using tree systems together with a result from [9] which allows to pull down certain infinite sets of formulas from groups to existentially closed subgroups.)

An earlier version of the present paper was written by the second author and circulated as a preprint since 1977. This preprint already contained proofs for the Theorems A and B. The present exposition is due to the first author. Although it describes a different approach, most of the essential ideas of the first four sections are explicit or implicit in Shelah's preprint. The central notion of tree system can be traced back to [7]. Section 5 is due to the first author.

We shall use the following notation: Structures are denoted by german capitals, and their universes by the corresponding latin capitals. If $\mathfrak{M}$ is a structure and A a subset of M we let $\langle A\rangle_{\mathfrak{M}}$ stand for the substructure of $\mathfrak{M}$ generated by A.

1. Tree systems

We fix a tree T with the following properties:

(T1) T is a subtree of $({}^{\omega+1 \geq}2, \subseteq)$, the tree consisting of all 0–1 sequences of length at most ω.

(T2) Every branch of T has length $\omega+1$.

(T3) For every integer n the following hold: If $\theta \in T_n$ (the nth level of T) then $\theta 0 \in T_{n+1}$. There is exactly one node $\eta \in T_n$ such that $\eta 1 \in T_{n+1}$. This node is called the *critical node* of T_n.

(T4) For every $\theta \in T \setminus T_\omega$ there is a critical node $\eta \in T$ with $\theta < \eta$.

By condition (T1) for each pair of distinct elements ρ, σ of T_ω there is an integer n such that $\rho \restriction n \neq \sigma \restriction n$.

It follows from (T1) and (T4) that T has continuum many branches, and so by (T2) we get $|T_\omega| = 2^{\aleph_0}$.

Notation. For $\beta \leqslant \alpha \leqslant \omega$ and $u \subseteq T_\alpha$ the set $\{\theta \restriction \beta \mid \theta \in u\}$ is denoted by $u \restriction \beta$. The expression $u \subseteq^* T_\alpha$ stands for $u \subseteq T_\alpha$ and $0 < |u| < \aleph_0$.

Now we extend the tree T to a partial ordering $(I, \leqslant)$ as follows:

(I1) $I = \{u \mid \exists \alpha \leqslant \omega\ (u \subseteq^* T_\alpha)\}$.

(I2) For $u, v \in I$ let $u \leqslant v$ iff $|u| = |v|$ and there are $\alpha \leqslant \beta \leqslant \omega$ such that $v \subseteq T_\beta$ and $u = v \restriction \alpha$.

T is embedded into $(I, \leqslant)$ by the mapping $\theta \mapsto \{\theta\}$. The minimal elements of I are $T_0 = \{\emptyset\}$ and those subsets of finite levels which contain both immediate successors of a critical node.

Let $(I, \leqslant)$ be the partial ordering satisfying (I1) and (I2). A pair $\mathcal{T} = ((\mathfrak{N}_u)_{u \in I}, (f_{uv})_{u \leqslant v \in I})$ is called a *tree system* if $\mathcal{T}$ satisfies the following conditions:

(TS1) For each $u \leqslant v \in I$, $\mathfrak{N}_u$ is an L-structure for a fixed language L, and f_{uv} is an embedding of $\mathfrak{N}_u$ into $\mathfrak{N}_v$.

(TS2) $((\mathfrak{N}_u)_{u<r}, (f_{uv})_{u \leqslant v < r})$ is a directed system for every $r \subseteq^* T_\omega$.

(TS3) If $u \subseteq^* v \subseteq^* T_n$ and $n < \omega$, then $\mathfrak{N}_u \subseteq \mathfrak{N}_v$.

(TS4) If $n < \omega$, $s \subseteq^* T_n$ and

$$\begin{array}{c} r \subseteq^* s \\ \vee\!\!/ \\ u \subseteq^* v \end{array}$$

then $f_{ur} = f_{vs} \restriction N_u$.

(TS5) $\mathfrak{N}_s = \xrightarrow{\lim} (\mathfrak{N}_u)_{u<s}$, and $((\mathfrak{N}_u)_{u\leqslant s}, (f_{uv})_{u\leqslant v\leqslant s})$ is a directed system for each $s \subseteq^* T_\omega$.

Remarks. (1) Call two sequences $t, t' \in {}^\omega(\bigcup_{n<\omega} N_{T_n})$ equivalent if t and t' disagree for finitely many arguments only. Denote the equivalence class of t by $[t]$. Let a be an element of N_s for $s \subseteq^* T_\omega$, and choose an integer m and an index $u \subseteq^* T_m$ such that $a = f_{us}a'$ for some $a' \in N_u$. Then clearly a can be identified with the equivalence class $[(f_{u,s\restriction n}a')_{m\leqslant n<\omega}]$. Using this convention, the following properties of $\mathcal{T}$ are easily deduced from (TS1)–(TS5):

(TS4′) For every $s \in I$: If

$$\begin{matrix} r \subseteq^* s \\ \vee\!\!/ \quad \vee\!\!/ \\ u \subseteq^* v \end{matrix}$$

then $f_{ur} = f_{vs} \restriction N_u$.

(TS3′) If $u \subseteq^* v \in I$, then $\mathfrak{N}_u \subseteq \mathfrak{N}_v$.

(2) Define for every $r \subseteq^* T_\omega$ a directed set $(J_r, \leqslant)$ as follows: $J_r = \{v \in I \mid \exists u < r\, (v \subseteq u)\}$, and $u \leqslant v$ iff there are $\alpha \leqslant \beta \leqslant \omega$ such that $v \subseteq T_\beta$ and $u \subseteq v \restriction \alpha$.

For $u \subseteq v$ let f_{uv} be the inclusion mapping from $\mathfrak{N}_u$ into $\mathfrak{N}_v$. Using remark (1) one sees that $\mathcal{T}$ is a tree system iff $\mathcal{T}$ satisfies (TS1) and the following conditions:

(TS6) For every $r \subseteq^* T_\omega$ the pair $\mathcal{T}_r = ((\mathfrak{N}_u)_{u\in J_r}, (f_{uv})_{u\leqslant v\in J_r})$ is a directed system.

(TS5′) $\mathfrak{N}_s = \xrightarrow{\lim} \mathcal{T}_s$ for every $s \subseteq^* T_\omega$.

Notation and terminology. For $\eta, \theta \in T$ we write $f_{\eta\theta}$ instead of $f_{\{\eta\}\{\theta\}}$. If η is the critical node of T_n the structure $\mathfrak{N}_{T_n\setminus\{\eta\}}$ is denoted by $\mathfrak{N}^*_{T_n}$.

If X is a nonempty subset of T_ω then by (TS3′) the pair $((\mathfrak{N}_r)_{r\subseteq^* X}, (\iota_{rs})_{r\subseteq^* s\subseteq^* X})$, where ι_{rs} is the inclusion mapping from $\mathfrak{N}_r$ to $\mathfrak{N}_s$, is a directed system with direct limit $\bigcup\{\mathfrak{N}_r \mid r \subseteq^* X\}$. We denote this limit by $\mathfrak{N}_X$ and refer to $\mathfrak{N}_{T_\omega}$ as to the *limit structure* of the tree system $\mathcal{T}$. Note that the structures $\mathfrak{N}_X$ for $\emptyset \neq X \subseteq T_\omega$ are uniquely determined by the $\mathfrak{N}_u$ and f_{uv} for subsets u, v of finite levels of T. For $a, b \in \bigcup\{N_u \mid u \in I\}$ and $n \leqslant \alpha \leqslant \omega$ we say that b is an *(n, α)-successor* of a (and a an *(n, α)-predecessor* of b) if there exist $u \subseteq^* T_n$ and $v \subseteq^* T_\alpha$ such that $u \leqslant v$ and $b = f_{uv}a$ (so especially $a \in N_u$ and $b \in N_v$).

For $\alpha \leqslant \omega$ and $A \subseteq \bigcup\{N_u \mid u \in I\}$ we let $S_\alpha(A)$ stand for the set of all elements of $\bigcup\{N_u \mid u \in I\}$ which have an (n, α)-predecessor in A for some $n \leqslant \alpha$. Clearly $S_\alpha(A)$ is finite if α and A are finite.

2. A framework for the basic construction

By condition (TS4′) we have in every tree system a certain amount of amalgamation. If we want to produce large limit structures we have to control this amalgamation in order to ensure that not too many elements are identified. In this section we develop the technical machinery for this purpose.

From now on, we denote by L a language, and by $\mathcal{K}$ a class of L-structures which is supposed to be closed under taking isomorphic images.

2.1. Definition. The class $\mathcal{K}$ is said to have *free amalgamation* if for all $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{B} \in \mathcal{K}$ with $\mathfrak{A} \cap \mathfrak{A}' = \mathfrak{B}$ there is a structure $\mathcal{P} \in \mathcal{K}$ such that $\mathfrak{A} \subseteq \mathcal{P}$, $\mathfrak{A}' \subseteq \mathcal{P}$ and whenever $f : \mathfrak{A} \to \mathfrak{A}$ and $f' : \mathfrak{A}' \to \mathfrak{A}'$ are embeddings which agree on $\mathfrak{B}$, there is an embedding $h : \mathcal{P} \to \mathcal{P}$ extending both f and f'.

2.2. Remarks. (1) Let $\mathcal{K}$—with embeddings as morphisms—be closed under taking free products with amalgamated submodels. It follows from the universal property of such products that $\mathcal{K}$ has free amalgamation, the free product of $\mathfrak{A}$ and $\mathfrak{A}'$ with $\mathfrak{B}$ amalgamated being a minimal choice for $\mathcal{P}$ in Definition 2.1. Examples for such classes $\mathcal{K}$ are the class of groups and the class of free ideal rings (fir). For firs see [2].

(2) Let $\mathcal{K}$ be the class of skew fields. Since every skew field is a fir, in the situation of Definition 2.1 we can form the free product $\mathcal{P}'$ of $\mathfrak{A}$ and $\mathfrak{A}'$ with $\mathfrak{B}$ amalgamated. $\mathcal{P}'$ being a fir has a field of fractions $\mathcal{P}$ which is universal w.r.t. specialisations [2, Theorem 4.C], and every embedding of $\mathcal{P}'$ into $\mathcal{P}'$ extends to an embedding of $\mathcal{P}$ into $\mathcal{P}$ [2, Theorem 4.3.3]. As a universal field of fractions, $\mathcal{P}$ is determined by $\mathcal{P}'$ up to isomorphism. $\mathcal{P}$ is called the field product of $\mathfrak{A}$ and $\mathfrak{A}'$ with $\mathfrak{B}$ amalgamated. *Notation*: $\mathcal{P} = \mathfrak{A} *_{\mathfrak{B}} \mathfrak{A}'$.

(3) Although there is in general no canonical choice for $\mathcal{P}$ in Definition 2.1, we shall write $\mathfrak{A} *_{\mathfrak{B}} \mathfrak{A}'$ for $\mathcal{P}$. This notation is justified by remarks (1) and (2), as we are mainly interested in groups and skew fields.

(4) In the class of groups the free product with amalgamation enjoys the so called *subamalgam property:* If, in the situation of Definition 2.1, we have $\mathcal{C} \subseteq \mathfrak{A}$, $\mathcal{C}' \subseteq \mathfrak{A}'$, and $\mathcal{C} \cap \mathfrak{B} = \mathcal{C}' \cap \mathfrak{B}$, then the subgroup of $\mathfrak{A} *_{\mathfrak{B}} \mathfrak{A}'$ generated by $C \cup C'$ is $\mathcal{C} *_{\mathcal{C} \cap \mathfrak{B}} \mathcal{C}'$.

2.3. Definition. Assume $\mathfrak{A} \in \mathcal{K}$. A triple $(a, \mathfrak{A}_0, \mathbb{B})$ is called a *separation in* $\mathfrak{A}$ if $a \in A$, $\mathfrak{A}_0 \subseteq \mathfrak{A}$; $\mathbb{B} \subseteq \mathcal{K}$ is a collection of substructures of $\mathfrak{A}$, and if there is an embedding $h : \mathfrak{A} \to \mathfrak{A}$ such that $ha \neq a$, $h \restriction A_0 = \mathrm{id}$ and h maps $\mathfrak{B}$ into $\mathfrak{B}$ for every $\mathfrak{B} \in \mathbb{B}$. We say that h is a *separating isomorphism for* $(a, \mathfrak{A}_0, \mathbb{B})$ *in* $\mathfrak{A}$.

In the class of groups the subamalgam property leads to the following pleasant situation: If f is an isomorphism with $\mathrm{dom}(f) = \mathfrak{A}$ and $\mathfrak{B}_0$ is a subgroup of $\mathfrak{A}$ such that $f \restriction B_0 = \mathrm{id}$ and $\mathfrak{A} \cap f\mathfrak{A} = \mathfrak{B}_0$, then for every subgroup $\mathfrak{A}_0 \subseteq \mathfrak{A}$ and every

$a \in A \setminus A_0$:

$$(*) \quad a \notin \langle A_0 \cup fA_0 \rangle_{\mathfrak{A} *_{\mathfrak{B}_0} f\mathfrak{A}} \quad \text{and} \quad fa \notin \langle A_0 \cup fA_0 \rangle_{\mathfrak{A} *_{\mathfrak{B}_0} f\mathfrak{A}}.$$

In Section 3 we shall define the morphisms f_{uv} of the tree system by piecewise composition from such isomorphisms f, and property $(*)$ would ensure that not too many elements are identified by amalgamation. Unfortunately $(*)$ is not true in all classes with free amalgamation. Now if $(a, A_0, \mathbb{B})$ is a separation, then clearly $a \in A \setminus A_0$, and the next proposition tells us that free amalgamation implies $(*)$ at least when $(a, A_0, \mathbb{B})$ is a separation for some $\mathbb{B}$ containing $\mathfrak{B}_0$.

2.4. Proposition. *Let $\mathfrak{A}, \mathfrak{B}_0 \in \mathcal{K}$, $\mathfrak{B}_0 \subseteq \mathfrak{A}$, and assume that $\mathcal{K}$ has free amalgamation. Then*

(i) *If f is an isomorphism defined on $\mathfrak{A}$ such that $\mathfrak{A} \cap f\mathfrak{A} = \mathfrak{B}_0$ and $f \restriction B_0 = \mathrm{id}$, and if $(a, \mathfrak{A}_0, \mathbb{B})$ is a separation in $\mathfrak{A}$ with $\mathfrak{B}_0 \in \mathbb{B}$, then $(a, \langle A_0 \cup fA_0 \rangle_{\mathcal{P}}, \tilde{\mathbb{B}})$ and $(fa, \langle A_0 \cup fA_0 \rangle_{\mathcal{P}}, \tilde{\mathbb{B}})$ are separations in $\mathcal{P} = \mathfrak{A} *_{\mathfrak{B}_0} f\mathfrak{A}$, where*

$$\tilde{\mathbb{B}} = \mathbb{B} \cup \{f\mathfrak{B} \mid \mathfrak{B} \in \mathbb{B}\} \cup \{\langle B \cup fB \rangle_{\mathcal{P}} \mid \mathfrak{B} \in \mathbb{B}\} \cup \{\mathfrak{A}, f\mathfrak{A}, \mathcal{P}\}.$$

Moreover there is a common separating isomorphism for both separations in $\mathcal{P}$.

(ii) If $\mathfrak{A}_0 = \mathfrak{B}_0$ *in* (i), *then also $(a, f\mathfrak{A}, \tilde{\mathbb{B}})$ and $(fa, \mathfrak{A}, \tilde{\mathbb{B}})$ are separations in $\mathcal{P}$.*

Proof. (i) Let h be a separating isomorphism for $(a, \mathfrak{A}_0, \mathbb{B})$ in $\mathfrak{A}$. Then fhf^{-1} is an embedding of $f\mathfrak{A}$ into $f\mathfrak{A}$. Since $\mathfrak{B}_0 \in \mathbb{B}$ the mappings h and fhf^{-1} agree on $\mathfrak{B}_0$. Since $\mathcal{K}$ has free amalgamation, h and fhf^{-1} have a common extension to an embedding $\tilde{h}$ of $\mathcal{P}$ into $\mathcal{P}$. One checks that $\tilde{h}a \neq a$, $\tilde{h}fa \neq fa$, $\tilde{h} \restriction \langle A_0 \cup fA_0 \rangle_{\mathcal{P}} = \mathrm{id}$, and $\tilde{h}\mathfrak{B} \subseteq \mathfrak{B}$ for every $\mathfrak{B} \in \tilde{\mathbb{B}}$.

(ii) Define $h_0 : \mathcal{P} \to \mathcal{P}$ and $h_1 : \mathcal{P} \to \mathcal{P}$ by $h_0 \restriction A = h$, $h_0 \restriction fA = \mathrm{id}$, $h_1 \restriction A = \mathrm{id}$, $h_1 \restriction fA = fhf^{-1}$. It is easily seen that h_0 resp. h_1 is a separating isomorphism for $(a, f\mathfrak{A}, \tilde{\mathbb{B}})$ resp. $(fa, \mathfrak{A}, \tilde{\mathbb{B}})$ in $\mathcal{P}$. $\square$

In Section 3 we want to construct tree systems using structures from sk($\mathfrak{M}$) (the class of all finitely generated structures embeddable in $\mathfrak{M}$) for $\mathfrak{M}$ existentially closed in $\mathcal{K}$. In order to control amalgamation we would like to apply Proposition 2.4 but we cannot expect the class sk($\mathfrak{M}$) to be closed under free products with amalgamated substructures, even if $\mathcal{K}$ contains all structures from sk($\mathfrak{M}$) (for instance being closed under taking substructures) and has free amalgamation. So we want to transfer the statements of Proposition 2.4 from $\mathcal{K}$ to sk($\mathfrak{M}$). Since $\mathfrak{M}$ is existentially closed this would be possible without loss if the validity of Proposition 2.4 were equivalent to the holding in an extension of $\mathfrak{M}$ of a finite set Γ of existential sentences with parameters from $\mathfrak{M}$. To get such a set Γ at all we must ensure that the isomorphisms occurring in Proposition 2.4 are in suitable way definable in the language of $\mathcal{K}$ with parameters from $\mathfrak{M}$. So we will limit ourselves to classes $\mathcal{K}$ which satisfy a HNN-type theorem. To keep Γ finite, we have to weaken Proposition 2.4 to a 'local' version.

2.5. Definition. Let $t(\boldsymbol{x}, y)$ be a term in L. We call $t(\boldsymbol{x}, y)$ an *automorphic term* if for every structure $\mathfrak{A} \in \mathcal{K}$ and every n-tuple $\boldsymbol{a}$ of elements of $\mathfrak{A}$ the term $t(\boldsymbol{a}, y)$ of $L(\boldsymbol{a})$ is interpreted in $(\mathfrak{A}, \boldsymbol{a})$ as an automorphism.

We say that $\mathcal{K}$ has *HNN-extensions* if there is an automorphic term $t(\boldsymbol{x}, y)$ with the property that for every $\mathfrak{A} \in \mathcal{K}$ and every partial isomorphism f from $\mathfrak{A}$ to $\mathfrak{A}$ there exists $\mathfrak{A}' \in \mathcal{K}$ and $\boldsymbol{a} \in A'^n$ such that $\mathfrak{A} \subseteq \mathfrak{A}'$ and the equation $t[\boldsymbol{a}, a] = fa$ holds true in $\mathfrak{A}'$ for every $a \in \operatorname{dom}(f)$.

Example. If $\mathcal{K}$ is the class of groups or skew fields, then xyx^{-1} is an automorphic term. The classical Higman–Neumann–Neumann theorem states that the class of groups has HNN-extensions. The class of skew-fields has HNN-extensions, too (for a proof, see Theorem 5.5.1 in [2]).

Before proving a modified version of Proposition 2.4 we state an auxiliary result:

2.6. Lemma. *Assume that $\mathcal{K}$ has free amalgamation and HNN-extensions. Let $\mathfrak{M} \in \mathcal{K}$ be existentially closed in $\mathcal{K}$. Then there is an automorphic term $t(\boldsymbol{x}, y)$ such that for every finitely generated substructure $\mathfrak{A} \subseteq \mathfrak{M}$ which is in $\mathcal{K}$, for every finite generating subset $\{a_1, \ldots, a_n\}$ of A and for every $a \in M$ the following holds*:

$$a \in A \quad \textit{iff} \quad \mathfrak{M} \models \forall \boldsymbol{x}(t[\boldsymbol{x}, a_1] = a_1 \wedge \cdots \wedge t[\boldsymbol{x}, a_n] = a_n) \rightarrow (t[\boldsymbol{x}, a] = a)).$$

Proof. (Following [5]). Take $t(\boldsymbol{x}, y)$ as in the definition of HNN-extension in 2.5. The 'only' direction is clear since $t(\boldsymbol{x}, y)$ is an automorphic term. For the 'if' direction assume $a \notin A$. Take an isomorphism f with $\operatorname{dom}(f) = M$, $f \restriction A = \mathrm{id}$, and $M \cap fM = A$. Since $\mathcal{K}$ has free amalgamation we get $\mathcal{P} = \mathfrak{M} *_{\mathfrak{A}} f\mathfrak{M} \in \mathcal{K}$, and f is a partial isomorphism from $\mathcal{P}$ to $\mathcal{P}$. Since $\mathcal{K}$ has HNN-extensions there is $\mathcal{P}' \in \mathcal{K}$ with $\mathcal{P} \subseteq \mathcal{P}'$ and $c_1, \ldots, c_m \in \mathcal{P}'$ such that $t[c_1, \ldots, c_m, a] = fa$ holds in $\mathcal{P}'$. So by the choice of f we get

$$\mathcal{P}' \models \exists \boldsymbol{x}(t[\boldsymbol{x}, a_1] = a_1 \wedge \cdots \wedge t[\boldsymbol{x}, a_n] = a_n \wedge t[\boldsymbol{x}, a] \neq a).$$

In this statement, $\mathcal{P}'$ can be replaced by $\mathfrak{M}$ since $\mathfrak{M}$ is existentially closed and $\mathfrak{M} \subseteq \mathcal{P}'$. □

Now we are ready for the crucial proposition.

2.7. Proposition. *Let $\mathcal{K}$ be closed under taking substructures and unions of chains. Assume that $\mathcal{K}$ has free amalgamation and HNN-extensions. Let $\mathfrak{M}$ be existentially closed in $\mathcal{K}$. Let $\mathfrak{A}, \mathfrak{B}_0$ be finitely generated substructures of $\mathfrak{M}$ with $\mathfrak{B}_0 \subseteq \mathfrak{A}$ and D an arbitrary finite subset of $A \setminus B_0$. Let Σ be a finite collection of separations in $\mathfrak{A}$ such that for every $(c, \mathscr{C}, \mathbb{B}) \in \Sigma$ the structure $\mathscr{C}$ is finitely generated and $\mathbb{B}$ is a finite collection of finitely generated substructures of $\mathfrak{A}$ with $\mathfrak{B}_0 \in \mathbb{B}$. Put $\Sigma = \{(c_S, \mathfrak{A}_S, \mathbb{B}_S) \mid S \in \Sigma\}$.*

Sh:83

Then there is an embedding $g:\mathfrak{A}\to\mathfrak{M}$ *and a finitely generated substructure* $\mathfrak{A}'\subseteq\mathfrak{M}$ *satisfying for every* $S\in\Sigma$ *the following conditions*:

(0) $\mathfrak{A}\cup g\mathfrak{A}\subseteq\mathfrak{A}'$.

(1) $g\restriction B_0=\mathrm{id}$, *and* $D\cap gA=A\cap gD=\emptyset$.

(2) *The triples* $(c_S,\langle A_S\cup gA_S\rangle_{\mathfrak{A}'},\tilde{\mathbb{B}}_S)$ *and* $(gc_S,\langle A_S\cup gA_S\rangle_{\mathfrak{A}'},\tilde{\mathbb{B}}_S)$ *are separations in* $\mathfrak{A}'$, *where* $\tilde{\mathbb{B}}_S$ *denotes the collection*

$$\mathbb{B}_S\cup\{g\mathfrak{B}\mid\mathfrak{B}\in\mathbb{B}_S\}\cup\{\langle B\cup gB\rangle_{\mathfrak{A}'}\mid\mathfrak{B}\in\mathbb{B}_S\}\cup\{\mathfrak{A}_S,\,g\mathfrak{A}_S,\,\mathfrak{A}'\}.$$

(3) *If* $\mathfrak{A}_S=\mathfrak{B}_0$, *then the triples*

$(c_S, gA,\tilde{\mathbb{B}}_S)$ *and* $(gc_S, A,\tilde{\mathbb{B}}_S)$ *with* $\tilde{\mathbb{B}}_S$ *as in* (2) *are separations in* $\mathfrak{A}'$.

(4) *If* $\varphi(x,y)$ *is a quantifier-free formula in L, and if* $\mathfrak{G}*_{\mathcal{H}}\mathfrak{G}'\models\neg\varphi[a,a']$ *for all products* $\mathfrak{G}*_{\mathcal{H}}\mathfrak{G}'\in\mathcal{K}$ *and all* $a\in G\setminus H$ *and* $a'\in G'\setminus H$, *then* g *and* $\mathfrak{A}'$ *can be taken to satisfy the additional condition*

$$\mathfrak{A}'\models\neg\varphi[d,gd]\quad\textit{for all } d\in D.$$

Proof. We carry out the proof in four steps. First we show that the proposition holds when we replace $\mathfrak{M}$ in the conclusion by a suitable superstructure $\mathcal{P}$ of $\mathfrak{M}$. Second we find an existentially closed superstructure $\mathfrak{M}^*$ of $\mathcal{P}$ such that the validity of the proposition for $\mathcal{P}$ is equivalent to the fact that a certain set Γ of formulas with parameters from $\mathfrak{M}$ can be realized in $\mathfrak{M}^*$. Third we prove that Γ is already realized in $\mathfrak{M}$, and fourth we deduce from this that Proposition 2.7 is true.

First step. Take an isomorphism f defined on $\mathfrak{A}$ as in Proposition 2.4(i) and assume that $f\mathfrak{A}\cap\mathfrak{M}=\mathfrak{B}_0$. If we replace $\mathfrak{A}'$ by $\mathcal{P}=\mathfrak{A}*_{\mathfrak{B}_0}f\mathfrak{A}$ and g by f the statements (0), (1) and (4) of Proposition 2.7 obviously hold while (2) and (3) are true by Proposition 2.4.

Second step. Choose for every $S\in\Sigma$ a separating isomorphism h_S for $(c_S,\langle A_S\cup fA_S\rangle_{\mathcal{P}},\tilde{\mathbb{B}}_S)$ and $(fc_S,\langle A_S\cup fA_S\rangle_{\mathcal{P}},\tilde{\mathbb{B}}_S)$ in $\mathcal{P}$. If $(c_S,\mathfrak{B}_0,\mathbb{B}_S)$ is a separation take also isomorphisms h_S^0 resp. h_S^1 separating for $(c_S, f\mathfrak{A},\tilde{\mathbb{B}}_S)$ resp. $(fc_S,\mathfrak{A},\tilde{\mathbb{B}}_S)$. Consider the model $\mathfrak{M}'=\mathfrak{M}*_{\mathfrak{M}\cap\mathcal{P}}\mathcal{P}$. Then $\mathfrak{M}\subseteq\mathfrak{M}'$ and $\mathcal{P}\subseteq\mathfrak{M}'$. As the mappings f, h_S, h_S^0 and h_S^1 are partial isomorphisms in $\mathfrak{M}'$, we find a model $\mathfrak{M}^*\supseteq\mathfrak{M}'$, an automorphic term $t(\boldsymbol{x},y)$ and finite sequences $\boldsymbol{a}_f$, $\boldsymbol{a}_S$, $\boldsymbol{a}_S^0$, $\boldsymbol{a}_S^1$ of elements of $\mathfrak{M}^*$ such that for all $a\in A$ the following equations hold in $\mathfrak{M}^*$:

$$fa=t[\boldsymbol{a}_f,a],\qquad h_Sa=t[\boldsymbol{a}_S,a],\qquad h_S^0a=t[\boldsymbol{a}_S^0,a],\qquad h_S^1a=t[\boldsymbol{a}_S^1,a].$$

$\mathcal{K}$ is closed under taking unions of chains, so we can assume w.l.o.g. that $\mathfrak{M}^*$ is existentially closed. Thus Lemma 2.6 can be applied, and we get for every finitely generated substructure $\mathfrak{B}$ of $\mathfrak{M}^*$ a quantifier-free formula $\varphi_{\mathfrak{B}}(\boldsymbol{x},y)$ with parameters from $\mathfrak{B}$, such that for every $b\in M^*$ we have

$$\mathfrak{M}^*\models\forall\boldsymbol{x}\,\varphi_{\mathfrak{B}}[\boldsymbol{x},b]\quad\text{iff}\quad b\in B.$$

Now consider the following statements:

(S1) $f\restriction B_0=\mathrm{id}$.

(S2) $A\cap fD=\emptyset$.

(S3) $fA\cap D=\emptyset$.

$(\mathrm{S4})_S$ h_S is a separating isomorphism for $(c_S, \langle A_S\cup fA_S\rangle, \tilde{\mathbb{B}}_S)$.

$(\mathrm{S5})_S$ h_S is a separating isomorphism for $(fc_S, \langle A_S\cup fA_S\rangle, \tilde{\mathbb{B}}_S)$.

$(\mathrm{S6})_S$ h_S^0 is a separating isomorphism for $(c_S, f\mathfrak{A}, \tilde{\mathbb{B}}_S)$.

$(\mathrm{S7})_S$ h_S^1 is a separating isomorphism for $(fc_S, \mathfrak{A}, \tilde{\mathbb{B}}_S)$.

In the subsequent formulas ψ_k, different strings of variables are supposed to be disjoint and for a finitely generated structure B we denote by B' an arbitrary finite generating set. It is easily seen that (Sk) holds iff it is possible to satisfy ψ_k in $\mathfrak{M}^*$ after having assigned $\boldsymbol{a}_f$, $\boldsymbol{a}_S$, $\boldsymbol{a}_S^0$, $\boldsymbol{a}_S^1$ to $\boldsymbol{x}_f$, $\boldsymbol{x}_S$, $\boldsymbol{x}_S^0$, $\boldsymbol{x}_S^1$.

$$(\psi_1(\boldsymbol{x}_f))\quad \bigwedge_{t\in B_0'} t(\boldsymbol{x}_f, b)=b.$$

$$(\psi_2(\boldsymbol{x}_f, \boldsymbol{y}))\quad \bigwedge_{d\in D} \neg\varphi_{\mathfrak{A}}(\boldsymbol{y}, t(\boldsymbol{x}_f, d)).$$

$$(\psi_3(\boldsymbol{x}_f, \boldsymbol{z}))\quad \bigwedge_{d\in D} \neg\varphi'_{f\mathfrak{A}}(\boldsymbol{z}, d, \boldsymbol{x}_f).$$

Here $\varphi'_{f\mathfrak{A}}(\boldsymbol{z}, d, \boldsymbol{x}_f)$ is obtained from $\varphi_{f\mathfrak{A}}(\boldsymbol{z}, d)$ by substituting $t(\boldsymbol{x}_f, a)$ for every parameter fa $(a\in A)$ occurring in $\varphi_{f\mathfrak{A}}(\boldsymbol{z}, d)$.

In the next formula the following notation is used: For $\mathfrak{B}\in\mathbb{B}_S$ and $b\in B$ the elements $h_S b$ and $f^{-1}h_S fb$ are denoted by b' and b'' respectively.

$$(\psi_4(\boldsymbol{x}_f, \boldsymbol{x}_S))\quad t(\boldsymbol{x}_S, c_S)\neq c_s\wedge \bigwedge_{a\in A_S'} (t(\boldsymbol{x}_S, a)=a\wedge t(\boldsymbol{x}_S, t(\boldsymbol{x}_f, a))=t(\boldsymbol{x}_f, a))$$

$$\wedge \bigwedge_{\mathfrak{B}\in\mathbb{B}_S}\ \bigwedge_{b\in B'} (t(\boldsymbol{x}_S, b)=b'\wedge t(\boldsymbol{x}_S, t(\boldsymbol{x}_f, b))=t(\boldsymbol{x}_f, b'')).$$

$(\psi_5(\boldsymbol{x}_f, \boldsymbol{x}_S))$ is obtained from $\psi_4(\boldsymbol{x}_f, \boldsymbol{x}_S)$ by replacing the subformula $t(\boldsymbol{x}_S, c_S)\neq c_S$ by $t(\boldsymbol{x}_S, t(\boldsymbol{x}_f, c_s))\neq t(\boldsymbol{x}_f, c_S)$.

Similarly we get quantifier-free formulas $\psi_6(\boldsymbol{x}_f, \boldsymbol{x}_S^0)$ and $\psi_7(\boldsymbol{x}_f, \boldsymbol{x}_S^1)$ corresponding to $(\mathrm{S6})_S$ and $(\mathrm{S7})_S$.

Third step. Since all statements (Sk) $(1\leqslant k\leqslant 3)$ and $(\mathrm{S}j)_S$ $(4\leqslant j\leqslant 7,\ S\in\Sigma)$ hold, the formula

$$\psi:\quad \psi_1(\boldsymbol{x}_f)\wedge\psi_2(\boldsymbol{x}_f, \boldsymbol{y})\wedge\psi_3(\boldsymbol{x}_f, \boldsymbol{z})\wedge\bigwedge_{S\in\Sigma}(\psi_4(\boldsymbol{x}_f, \boldsymbol{x}_S)\wedge\psi_5(\boldsymbol{x}_f, \boldsymbol{x}_S)\wedge\psi_6(\boldsymbol{x}_f, \boldsymbol{x}_S^0)\wedge\psi_7(\boldsymbol{x}_f, \boldsymbol{x}_S^1))$$

is satisfied in $\mathfrak{M}^*$. Now ψ is quantifier-free, and $\mathfrak{M}$ is an existentially closed substructure of $\mathfrak{M}^*$. So ψ is already satisfied in $\mathfrak{M}$.

Fourth step. ψ can be satisfied in $\mathfrak{M}$ after having assigned certain sequences

$\boldsymbol{b}_f, \boldsymbol{b}_S, \boldsymbol{b}_S^0, \boldsymbol{b}_S^1$ of elements of $\mathfrak{M}$ to $\boldsymbol{x}_f, \boldsymbol{x}_S, \boldsymbol{x}_S^0, \boldsymbol{x}_S^1$. Define $g: A \to M$ by $gx = t(\boldsymbol{b}_f, x)$. g is an isomorphism since $t(\mathbf{y}, x)$ is an automorphic term. Define $\bar{h}_S, \bar{h}_S^0, \bar{h}_S^1: M \to M$ by $\bar{h}_S x = t(\boldsymbol{b}_S, x)$, $\bar{h}_S^0 x = t(\boldsymbol{b}_S^0, x)$ and $\bar{h}_S^1 x = t(\boldsymbol{b}_S^1, x)$. Put

$$\mathfrak{A}' = \langle A \cup gA \cup \boldsymbol{b}_S \cup \boldsymbol{b}_S^0 \cup \boldsymbol{b}_S^1 \rangle_{\mathfrak{M}}.$$

The sentence $\psi_1(\boldsymbol{b}_f) \wedge \exists \mathbf{y} \psi_2(\boldsymbol{b}_f, \mathbf{y}) \wedge \exists \mathbf{z} \psi_3(\boldsymbol{b}_f, \mathbf{z})$ is true in $\mathfrak{M}$. Using Lemma 2.6 we conclude that claim (1) of Proposition 2.7 holds. Considering ψ_3 and ψ_4 one sees that the restriction of $\bar{h}_S$ to $\mathfrak{A}'$ is a separating isomorphism as needed for claim (2). In the same fashion one proves (3), and (0) holds trivially.

In order to prove (4) for a fixed formula $\varphi(x, y)$, add to (S1)–(S7) the statement

$$\text{(S8)} \quad \mathcal{P} \models \bigwedge_{d \in D} \neg \varphi[d, fd],$$

and to (ψ1)–(ψ7) the corresponding formula

$$(\psi 8) \quad \bigwedge_{d \in D} \neg \varphi(d, t(\boldsymbol{x}_f, d)).$$

Then (4) is proven by substituting ψ by $\psi \wedge \psi 8$ in the third and fourth step of the above proof.

2.8. Corollary. *The conclusions of Proposition* 2.7 *hold for* $\mathfrak{M}$ *an existentially closed group or skew field.*

Proof. We have to show that the conditions of Proposition 2.7 are satisfied if $\mathcal{K}$ is taken to be the class of all groups or the class of all skew fields of given characteristic. Now either class is closed under taking unions of chains. By an appropriate choice of the language L we can ensure that both classes are also closed under taking substructures. It was already mentioned in 2.2(1), 2.2(2) and in the example following 2.5 that both classes have free amalgamation and HNN-extensions. □

Proposition 2.7 cannot be applied to the case of a universal locally finite group $\mathfrak{M}$, since the class of locally finite groups does not have free amalgamation.

But in this case we do not need separations since a result by Baumslag [1] enables us to transfer the subamalgam property from the class of all groups to the class of finite subgroups of $\mathfrak{M}$. Baumslag's result says that a free product $\mathcal{P}$ of groups with amalgamated subgroups is residually finite whenever the factors are finite. By definition, $\mathcal{P}$ is residually finite if for an arbitrary finite subset $E \subseteq P$ there is a homomorphism h from $\mathcal{P}$ onto a finite group such that $h \restriction E$ is injective.

The following proposition is intended to replace Proposition 2.7 in the case of locally finite groups. The additional condition 2.9(4) has no analogue in the situation of Proposition 2.7 and will play an important role in generating nonisomorphic models.

2.9. Proposition. *Let $\mathfrak{M}$ be a universal locally finite group, and let $\mathfrak{A}$ and $\mathfrak{B}_0$ be finite subgroups of $\mathfrak{M}$ such that $\mathfrak{B}_0 \subseteq \mathfrak{A}$. Then there is an embedding $g:\mathfrak{A}\to\mathfrak{M}$ with the following properties:*

(1) $A \cap gA = B_0$, *and* $g \restriction B_0 = \mathrm{id}$.

(2) *For every* $a \in A$ *and every subgroup* $\mathfrak{A}_0 \subseteq \mathfrak{A}$: $a \in \langle A_0 \cup gA_0 \rangle_{\mathfrak{M}}$ *iff* $a \in A_0$ *iff* $ga \in \langle A_0 \cup gA_0 \rangle_{\mathfrak{M}}$.

(3) *If* $\varphi(x, y)$ *is a quantifier-free formula of L, and if* $\mathcal{F} *_{\mathcal{H}} \mathcal{F}' \models \neg\varphi[a, b]$ *for all products of groups* $\mathcal{F} *_{\mathcal{H}} \mathcal{F}'$ *and all* $a \in F \setminus H$, $b \in F' \setminus H$, *then g can be taken to satisfy the additional condition:*

$$\langle A \cup gA \rangle_{\mathfrak{M}} \models \neg\varphi[a, gb] \quad \text{for all } a, b \in A \setminus B_0.$$

(4) *There is a quantifier-free formula* $\psi(x, y)$ *such that* $\langle A \cup gA \rangle_{\mathfrak{M}} \models \psi[a, b]$ if $\{a, b\} \subseteq A$ *or* $\{a, b\} \subseteq gA$, *and* $\langle A \cup gA \rangle_{\mathfrak{M}} \models \neg\psi[a, b]$ *if* $a \in A \setminus B_0$ *and* $b \in gA \setminus B_0$ or $b \in A \setminus B_0$ *and* $a \in gA \setminus B_0$.

Proof. (1) Let f be an isomorphism defined on $\mathfrak{A}$ such that $A \cap fA = B_0$ and $f \restriction B_0 = \mathrm{id}$. Put $\mathcal{P} = \mathfrak{A} *_{\mathfrak{B}_0} f\mathfrak{A}$. Define a finite subset E of P as follows:

$$E = \{w[\boldsymbol{a}] \mid \boldsymbol{a} \text{ a sequence of elements of } A \cup fA,$$
$$w(\boldsymbol{x}) \text{ a word of length } \leqslant l\}.$$

By a word we mean a term in the language whose nonlogical constants are $\cdot$ and $^{-1}$, and the integer l will be fixed later, but in any case $l \geqslant 1$ (and so $A \cup fA \subseteq E$). According to Baumslag's result we can choose a homomorphism h defined on P with finite range such that $h \restriction E$ is injective. Since every finite group can be embedded in $\mathfrak{M}$ we can assume that $h\mathcal{P} \subseteq \mathfrak{M}$, and by the ω-homogeneity of $\mathfrak{M}$ we can take h such that $h \restriction A = \mathrm{id}$. So, putting $h\mathcal{P} = \mathfrak{G}'$ and $hf = g'$, we have $\mathfrak{G}' = \langle A \cup g'A \rangle_{\mathfrak{M}}$, and g' satisfies (1) as a $A \cup fA \subseteq E$. Now let $\mathfrak{G}$ stand for the direct product $\mathfrak{G}' \times \mathfrak{G}'$, and let $g_0 : \mathfrak{A} \to \mathfrak{G}$ and $g_1 : g'\mathfrak{A} \to \mathfrak{G}$ be embeddings defined by $g_0 : a \to (a, a)$ and $g_1 : g'a \to (a, g'a)$. Using again the fact that $\mathfrak{M}$ contains isomorphic copies of all finite groups and is ω-homogeneous we find an embedding $k : \mathfrak{G} \to \mathfrak{M}$ such that kg_0 is the identity on $\mathfrak{A}$. Put $g = kg_1g'$. Then g satisfies (1).

(2) Assume that $a \in \mathfrak{A} \cap \langle A_0 \cup gA_0 \rangle_{\mathfrak{M}}$. We have to show that $a \in A_0$. We can write a as a product $a_1 g b_1 \cdots a_n g b_n$ for appropriate $a_1, b_1, \ldots, a_n, b_n \in A_0$. Applying k^{-1} we get

$$(a, a) = (a_1, a_1)(b_1, g'b_1) \cdots (a_n, a_n)(b_n, g'b_n),$$

and so $a = a_1 b_1 \cdots a_n b_n \in A_0$. If $a \in A$ and $ga = a_1 g b_1 \cdots a_n g b_n$ the same argument yields $g'a = a_1 g' b_1 \cdots a_n g' b_n$ and therefore $fa = a_1 f a_1 \cdots a_n f a_n$, and we deduce $a \in A_0$ from the subamalgam property of $\mathfrak{A} *_{\mathfrak{B}_0} f\mathfrak{A}$.

(3) Take $\varphi(x, y)$ as in (3). Let $\varphi'(x, y)$ be a formula in disjunctive normal form such that $\varphi'(x, y)$ is equivalent to $\neg\varphi(x, y)$, and every atomic component of $\varphi'(x, y)$ is of the form $w(x, y) = 1$. Let l_0 be the maximal length of a word

occurring in $\varphi'(x, y)$. Choose $l \geq l_0$. Due to the special form of $\varphi'(x, y)$ the claim of (3) follows immediately once we have shown that $\langle A \cup gA\rangle_{\mathfrak{M}} \models \chi[a, gb]$ if $\mathcal{P} \models \chi[a, fb]$ for formulas $\chi(x, y)$ of the form $w(x, y) = 1$ or $w(x, y) \neq 1$.

If $\mathcal{P} \models w[a, fb] \neq 1$, then $\mathfrak{G}' \models w[a, g'b] \neq 1$ by the choice of l. Clearly $\mathfrak{G} \models w[(a, a), (b, g'b)] \neq 1$ iff $\mathfrak{G}' \models w[a, b] \neq 1$ or $\mathfrak{G}' \models w[a, g'b] \neq 1$. So we get $\mathfrak{G} \models w[(a, a), (b, g'b)] \neq 1$, and applying k we conclude that $\langle A \cup gA\rangle_{\mathfrak{M}} \models w[a, gb] \neq 1$. If $\mathcal{P} \models w[a, fb] = 1$, then $\mathfrak{G} \models w[a, g'b] = 1$ since h is a homomorphism. Anticipating Lemma 5.2 we get $\mathcal{P} \models w[a, fb] = w[a, b]$, and therefore $\mathfrak{G} \models w[a, g'b] = w[a, b] = 1$. As above we conclude that $\langle A \cup gA\rangle_{\mathfrak{M}} \models w[a, gb] = 1$.

(4) Take for $\psi(x, y)$ the formula $(xy)^n = 1$ with $n = |A|$. Choose $l \geq 2n$. Obviously $\langle A \cup gA\rangle_{\mathfrak{M}} \models (ab)^n = 1$ if $\{a, b\} \subseteq A$ or $\{a, b\} \subseteq gA$. In the other case we can assume $a \in A$, $b = ga'$ for some $a' \in A \setminus B_0$. Then $fa' \in fA \setminus B_0$, and $a \cdot fa'$ has infinite order in $\mathcal{P}$. So by the choice of l we have $\mathfrak{G}' \models (a \cdot g'a)^n \neq 1$, and as in the proof of (3) we conclude that $\langle A \cup gA\rangle_{\mathfrak{M}} \models (a \cdot ga')^n \neq 1$. □

Remark. In the proof of Theorem B(1) we shall apply Lemma 2.9(3) for $xy = yx$ at the place of $\varphi(x, y)$.

So $\varphi'(x, y)$ is of the form $xyx^{-1}y^{-1} \neq 1$, and in this case we do not need Lemma 5.2 to get 2.9(3).

3. The basic construction

In this section L is a language and $\mathcal{K}$ a class of L-structures. $\mathcal{K}$ is assumed either to have free amalgamation and HNN-extensions and to be closed under taking substructures and unions of chains, or to be the class of locally finite groups. We consider a countable existentially closed structure $\mathfrak{M} \in \mathcal{K}$ (so in the case of locally finite groups $\mathfrak{M}$ is universal locally finite).

We are going to add some conditions to the axioms (TS1)–(TS5) for tree systems. This conditions describe a sequence f_n of embeddings of $\mathfrak{N}_{T_n}$ into $\mathfrak{M}$. Every morphism f_{uv} of the tree system will be composed from restrictions of the f_n and from identities. The main result of the section says that the embeddings f_n and the structures $\mathfrak{N}_{T_n}$ can be chosen to produce a limit structure $\mathfrak{N}_{T_\omega} \equiv_{\infty\omega} \mathfrak{M}$ with $|N_{T_\omega}| = 2^{\aleph_0}$. First we add two conditions which are sufficient to get $|N_{T_\omega}| = 2^{\aleph_0}$.

(TS6)

(i) $\mathfrak{N}_{\{\emptyset\}} = \mathfrak{N}_{T_0}$ is a finitely generated substructure of $\mathfrak{M}$ containing a finitely generated substructure $\mathfrak{B}$. f_0 is an embedding of $\mathfrak{N}_{T_0}$ into $\mathfrak{M}$ such that $f_0 \restriction B = \mathrm{id}$.

(ii) Let $0 < n < \omega$. Then f_n is an embedding of $\mathfrak{N}_{T_n}$ into $\mathfrak{M}$ such that $f_n \restriction N^*_{T_n} = \mathrm{id}$. For $n \geq 0$, $\mathfrak{N}_{T_{n+1}}$ is a finitely generated substructure of $\mathfrak{M}$ containing $N_{T_n} \cup f_n N_{T_n}$.

(iii) For $0 \leq n \leq \omega$, $w \subseteqq^* T_{n+1}$, $w' = w \restriction n$ and $\eta \in T_n$ critical the following holds:

(a) $\mathfrak{N}_w = \mathfrak{N}_{w'}$ and $f_{w'w} = \mathrm{id}$ if $\eta 1 \notin w$,

(b) $\mathfrak{N}_w = f_n \mathfrak{N}_{w'}$ and $f_{w'w} = f_n \restriction N_{w'}$ if $\eta 0 \notin w$,

Sh:83

(c) $\mathfrak{N}_w = \langle N_{w\setminus\{\eta 1\}} \cup N_{w\setminus\{\eta 0\}}\rangle_{\mathfrak{M}}$
$= \langle N_{w'} \cup f_n N_{w'}\rangle_{\mathfrak{M}}$ if $\{\eta 0, \eta 1\} \subseteq w$.

(TS7) There is an element $c \in N_{T_0}$ such that for all $n < \omega$, all $u \subseteq^* T_n$ and all $\theta \in T_n$ we have $f_{\emptyset\theta} c \in N_u$ iff $\theta \in u$.

3.1. Proposition. *For every existentially closed structure $\mathfrak{M} \in \mathcal{K}$ there is a tree system $((\mathfrak{N}_u)_{u\in I}, (f_{uv})_{u\leqslant v\in I})$ satisfying* (TS6) *and* (TS7). *In this system, for every $n < \omega$ and every finitely generated $\mathfrak{N} \subseteq \mathfrak{M}$, $\mathfrak{N}_{T_{n+1}}$ can be chosen to contain $\mathfrak{N}$.*

Proof. First we consider the case when $\mathcal{K}$ satisfies the conditions of Proposition 2.7. We are going to define a tree system $((\mathfrak{N}_u)_{u\in I}, (f_{uv})_{u\leqslant v\in I})$ satisfying (TS6), a nonempty subset C_0 of $\mathfrak{N}_{\{\emptyset\}} = \mathfrak{N}_{T_0}$ and a finitely generated substructure $\mathfrak{B}$ of $\mathfrak{N}_{T_0}$ such that

(1) $(c, \mathfrak{B}, \{\mathfrak{B}\})$ is a separation in $\mathfrak{N}_{T_0}$ for every $c \in C_0$.
(2) For every $n < \omega$, $u \subseteq^* T_n$ and $\theta \in T_n$ we have $f_{\emptyset\theta} c \in N_u$ iff $\theta \in u$ for all $c \in C_0$.

We use induction on the level n of T. For $n = 0$ choose a finitely generated $\mathfrak{A} \subseteq \mathfrak{M}$ containing a finitely generated proper substructure $\mathfrak{B}$. Apply Proposition 2.7(1) for $\mathfrak{B}_0 = \mathfrak{B}$ and $\Sigma = \emptyset$ to get an embedding $g: \mathfrak{A} \to \mathfrak{M}$ such that $\mathfrak{A} \cap g\mathfrak{A} = \mathfrak{B}$ and $g \restriction B = \mathrm{id}$. Since $\mathfrak{M}$ is existentially closed in $\mathcal{K}$ and $\mathcal{K}$ has HNN-extensions there is a finitely generated substructure $\mathfrak{A}'$ of $\mathfrak{M}$ containing $A \cup gA$ such that g extends to an automorphism $\bar{g}$ of $\mathfrak{A}'$. Put $\mathfrak{A}' = \mathfrak{N}_{T_0}$, and take for C_0 any nonempty finite subset of $A \setminus B$. Then clearly (1) is satisfied with $\bar{g}$ as separating isomorphism, and (2) holds trivially for $n = 0$.

Now assume that the tree system has been constructed up to the level n. We use Proposition 2.7 as follows:

For $n = 0$ put $\mathfrak{A} = \mathfrak{N}_{T_0}$, $\mathfrak{B}_0 = \mathfrak{B}$ and

$$\Sigma = \{(c, \mathfrak{B}, \{\mathfrak{B}\}) \mid c \in C_0\}.$$

For $n > 0$ put $\mathfrak{A} = \mathfrak{N}_{T_n}$, $\mathfrak{B}_0 = \mathfrak{N}^*_{T_n}$ and

$$\Sigma = \{(c, \mathfrak{N}_u, \{\mathfrak{N}_v \mid v \subseteq^* T_n\}) \mid c \in C_n, u \subseteq^* T_n, (c, \mathfrak{N}_u, \{\mathfrak{N}_v \mid v \subseteq^* T_n\}) \text{ is a separation in } \mathfrak{N}_{T_n}\}.$$

Now set $f_n = g$ and $C_{n+1} = C_n \cup f_n C_n$. For a given finitely generated substructure $\mathfrak{N} \subseteq \mathfrak{M}$ we choose $\mathfrak{N}_{T_{n+1}}$ to be a finitely generated substructure of $\mathfrak{M}$ containing $N_{T_n} \cup f_n N_{T_n} \cup N$. Then it follows from Proposition 2.7(1) that (TS6) (i) and (ii) are satisfied. By (TS6) (ii) the conditions (a) and (b) of (TS6) (iii) are compatible and so for every $w \subseteq^* T_{n+1}$ we can define $\mathfrak{N}_w$ and $f_{w'w}$ according to (TS6) (iii). A routine checking shows that the system defined by this induction has a unique extension to a tree system satisfying (TS6) and condition (1) stated at the beginning of this proof. In order to fulfill condition (2) we have to make the choice of $\mathfrak{N}_{T_{n+1}}$ more precise to ensure the existence of sufficiently many separations.

Put $\mathfrak{N}_\emptyset = \mathfrak{B}$ and $\mathbb{B}_n = \{\mathfrak{N}_v \mid v \subseteq T_n\}$. Define

$$\Sigma_0 = \{(c, \mathfrak{B}, \mathbb{B}_0) \mid c \in C_0\}$$

and

$$\begin{aligned}\Sigma_{n+1} = \{&(c, \mathfrak{N}_u, \mathbb{B}_{n+1}),\ (f_n c, \mathfrak{N}_u, \mathbb{B}_{n+1}) \mid u \subseteq^* T_{n+1},\\ &(c, \mathfrak{N}_{u\restriction n}, \mathbb{B}_n) \in \Sigma_n\} \cup \{(c, f_n \mathfrak{N}_{T_n}, \mathbb{B}_{n+1}),\\ &(f_n c, \mathfrak{N}_{T_n}, \mathbb{B}_{n+1}) \mid (c, \mathfrak{N}^*_{T_n}, \mathbb{B}_n) \in \Sigma_n\} \quad \text{for } n \geq 0.\end{aligned}$$

We want to choose the structures $\mathfrak{N}_{T_{n+1}}$ so that all triples from Σ_{n+1} are separations in $\mathfrak{N}_{T_{n+1}}$. Assume that this has been done for all $k<n$. If in Σ_{n+1} we replace $\mathbb{B}_{n+1}$ by $\mathbb{B}_{n+1} \setminus \{N_{T_{n+1}}\}$, we get a set Σ'_{n+1} which by Proposition 2.7(2) and 2.7(3) is a set of separations in a finitely generated substructure $\mathfrak{N}'$ of $\mathfrak{M}$ with $N_{T_n} \cup f_n N_{T_n} \subseteq N'$. Choose for every $S \in \Sigma'_{n+1}$ a separating isomorphism h_S. Since $\mathfrak{M}$ is existentially closed in $\mathcal{K}$ and $\mathcal{K}$ has HNN-extensions there is a finitely generated substructure $\bar{\mathfrak{N}}$ of $\mathfrak{M}$ containing $\mathfrak{N}'$ and a prescribed finitely generated $\mathfrak{N} \subseteq \mathfrak{M}$ such that every h_S extends to an automorphism of $\bar{\mathfrak{N}}$. Put $\bar{\mathfrak{N}} = \mathfrak{N}_{T_{n+1}}$. Then every triple from Σ_{n+1} is a separation in $\mathfrak{N}_{T_{n+1}}$.

Now in order to establish Proposition 3.1 it is sufficient to show that the tree system defined above satisfies condition (2). In view of (TS3) it is sufficient to prove $\theta \notin u \Rightarrow f_{\emptyset\theta} c \notin N_u$ for $\theta \in T_n$, $u \subseteq^* T_n$ and $c \in C_0$, and, again by (TS3), this is equivalent to $f_{\emptyset\theta} c \notin N_{T_n \setminus \{\theta\}}$ for every $c \in C_0$ and $\theta \in T_n$. So it suffices to show that $(f_{\emptyset\theta} c, \mathfrak{N}_{T_n \setminus \{\theta\}}, \{\mathfrak{N}_v \mid v \subseteq^* T_n\})$ is a separation in $\mathfrak{N}_{T_n}$. For $n=0$ this is trivially true by the choice of C_0 and the convention $\mathfrak{B} = \mathfrak{N}_\emptyset$. Now pick $c \in C_0$ and put $f_{\emptyset\theta} c = c_\theta$ for $\theta \in T$. Let $\theta \in T_{n+1}$, η critical in T_n, and denote the immediate predecessor of θ by θ'.

Case 1: $\theta' \neq \eta$. Then $f_{\emptyset\theta} c = f_{\theta'\theta} c_{\theta'} = c_{\theta'}$ by (TS6) (iii a). Since $\{\eta 0, \eta 1\} \subseteq T_{n+1} \setminus \{\theta\}$ we get by (TS6) (iii c):

$$\mathfrak{N}_{T_{n+1} \setminus \{\theta\}} = \langle N_{T_n \setminus \{\theta'\}} \cup f_n N_{T_n \setminus \{\theta'\}} \rangle_{\mathfrak{M}}.$$

By our choice of $\mathfrak{N}_{T_{n+1}}$ we conclude that the triple $(f_{\emptyset\theta} c, \mathfrak{N}_{T_{n+1} \setminus \{\theta\}}, \{\mathfrak{N}_v \mid v \subseteq^* T_{n+1}\})$ is a separation in $\mathfrak{N}_{T_{n+1}}$, using the induction assumption for n.

Case 2: $\theta = \eta 1$. Then $c_\theta = f_n c_\eta$, and $\mathfrak{N}_{T_{n+1} \setminus \{\theta\}} = \mathfrak{N}_{T_n}$ by (TS6) (iii). By induction assumption the triple $(c_\eta, \mathfrak{N}^*_{T_n}, \{\mathfrak{N}_v \mid v \subseteq^* T_n\})$ is a separation in $\mathfrak{N}_{T_n}$. Due to the special choice of $\mathfrak{N}_{T_{n+1}}$ we conclude that $(f_\eta c_\eta, \mathfrak{N}_{T_n}, \{\mathfrak{N}_v \mid v \subseteq^* {}_{T_{n+1}}\})$ is a separation, and this triple is equal to $(c_\theta, \mathfrak{N}_{T_{n+1} \setminus \{\theta\}}, \{\mathfrak{N}_v \mid v \subseteq^* T_{n+1}\})$.

Case 3: $\theta = \eta 0$. This is handled in the same way as the second case. This completes the proof of Proposition 3.1 for $\mathcal{K}$ as in Proposition 2.7.

If $\mathcal{K}$ is the class of locally finite groups, then let $\mathfrak{N}_{T_0}$ be a nontrivial finite subgroup of $\mathfrak{M}$, and $\mathfrak{B}$ a proper subgroup of $\mathfrak{N}_{T_0}$. Put $\mathfrak{B} = \mathfrak{N}_\emptyset$ and assume that f_k and $\mathfrak{N}_{T_{k+1}}$ are defined for $k<n$. Apply Proposition 2.9 to $\mathfrak{A} = \mathfrak{N}_{T_n}$ and $\mathfrak{B}_0 = \mathfrak{N}^*_{T_n}$. Let f_n be a mapping with the properties of g in Proposition 2.9. Let $\mathfrak{N}_{T_{n+1}}$ be a finite subgroup of $\mathfrak{M}$ containing $N_{T_n} \cup f_n N_{T_n}$ and a prescribed finite subgroup $\mathfrak{N}$ of

$\mathfrak{M}$. So we get sequences $(f_n)_{n<\omega}$ and $(\mathfrak{N}_{T_n})_{n<\omega}$ which determine a tree system satisfying (TS6).

For the proof that (TS7) holds in this system it is sufficient to show that $\theta \notin u \Rightarrow f_{\emptyset\theta}c \notin N_u$ for all $n<\omega$, $u \subseteq^* T_n$, $\theta \in T_n$ and $c \in N_{T_0} \backslash B$.

For $n=0$ this is trivial, so assume $n \geqslant 1$. Let θ' be the immediate predecessor of θ, and put $u'=u \upharpoonright n$. Denote the element $f_{\emptyset\theta}c$ by c_θ.

Case 1: $\theta=\eta 0$. Then $c_\theta = f_{n,\eta 0}c_\eta = c_\eta$. If $\eta 1 \notin u$, then $N_{u'}=N_u$, and $\eta \notin u'$. So by assumption $c_\eta \notin N_{u'}$ or equivalently $c_\theta \notin N_u$. If $\eta 1 \in u$, then $N_u = f_n N_{u'}$. By assumption $c_\eta \notin N^*_{T_n}$, so by Proposition 2.9(1) we get $c_\theta = c_\eta \notin N_u$.

Case 2: $\theta = \eta 1$. Then $c_\theta = f_n a_\eta$. Since $\eta 1 \notin u$ we have $N_u = N_{u'}$, and Proposition 2.9(1) yields $c_\theta \notin N_u$.

Case 3: $\theta' \neq \eta$. Then we have $c_{\theta'} \in N^*_{T_n}$, and $c_\theta = c_{\theta'}$. The subcase $\{\eta 0, \eta 1\} \nsubseteq u$ is trivial. If $\{\eta 0, \eta 1\} \subseteq u$, then $\mathfrak{N}_u = \langle N_u, \bigcup f_n N_{u'} \rangle_{\mathfrak{M}}$. By Proposition 2.9(2) we get $c_\theta \in N_u$ iff $c_\theta \in N_{u'}$. But $\theta' \notin u'$ follows from $\theta \notin u$ and $\theta' \neq \eta$. So $c_\theta = c_{\theta'} \notin N_{u'}$ by assumption, and we conclude that $c_\theta \notin N_u$. □

In order to exclude tree systems whose limit structures are not $\infty\omega$-equivalent to $\mathfrak{M}$ we add the following condition.

(TS8) $(A_n)_{n<\omega}$ is an ascending chain of finite subsets of M such that $\bigcup\{A_n \mid n<\omega\} = M$ and $A_0 \subseteq N_{T_0} \cdot (D_n)_{n<\omega}$ is an ascending chain of finite subsets of M with the following properties:

(a) $D_n \subseteq N_{T_n}$.

(b) $D_0 = A_0$, and $D_{n+1} = A_{k_n}$, where k_n is the first integer k such that

(i) $A_{n+1} \cup S_n(D_n) \cup f_n S_n(D_n) \subseteq A_k$.

(ii) Every partial isomorphism whose field is contained in $A_{n+1} \cup S_n(D_n) \cup f_n S_n(D_n)$ extends to a partial isomorphism whose domain is $A_{n+1} \cup S_n(D_n) \cup f_n S_n(D_n)$ and whose range is contained in A_k.

3.2. Proposition. *For every countable existentially closed structure $\mathfrak{M} \in \mathcal{K}$ there is a tree system satisfying* (T6)–(T8).

Proof. If $\mathcal{K}$ is the class of locally finite groups $\mathfrak{M}$ is the countable universal locally finite group and therefore ω-homogeneous. If $\mathcal{K}$ satisfies the conditions of Proposition 2.7, then $\mathcal{K}$ has HNN-extensions, and so $\mathfrak{M}$ is ω-homogeneous, too. Thus condition (b)(ii) of (TS8) can be satisfied, and the claim follows by induction from Proposition 3.1 with $\mathfrak{N} = \langle D_{n+1} \rangle_{\mathfrak{M}}$ in the induction step. □

3.3. Definition. A nonempty subset X of T_ω is *full* if for all $r \subseteq^* X$ and all $u<r$ there is $s \subseteq^* X$ such that $r \subseteq s$ and $T_n < s$, where T_n is the level of T containing u.

3.4. Proposition. *For every tree system satisfying* (TS6)–(TS8) *the following hold*:

(a) *If $X \subseteq T_\omega$ is full, then $\mathfrak{N}_X \equiv_{\infty\omega} \mathfrak{M}$. Especially $\mathfrak{N}_{T_\omega} \equiv_{\infty\omega} \mathfrak{M}$.*

Sh:83

(b) *Let* X_0 *be the set* $\{\rho \in T_\omega \mid \exists m < \omega\, \forall n < \omega\, (n > m \Rightarrow \rho(n) = 0)\}$. *Then* $\mathfrak{N}_X \equiv_{\infty\omega} \mathfrak{M}$ *whenever* $X_0 \subseteq X \subseteq T_\omega$.

Proof. (b) Assume $u \subseteq^* X$, $v \subseteq^* T_n$, and $v < u$. If $v = T_n$, then T_n has a successor which contains u and is a subset of X, namely u itself. If $v \neq T_n$, take $w \subseteq X_0$ such that $T_n \setminus v < w$. Then $T_n < u \cup w$ and $u \cup w \subseteq X$ since $X_0 \subseteq X$. So X is full and the claim follows from (a).

(a) We show that the set Π of all finite partial isomorphisms from $\mathfrak{M}$ into $\mathfrak{N}_X$ has the extension property. So let g be an element of Π with domain A and range B. We must find for every $a \in M$ and $b \in N_X$ mappings $g_a, g_b \in \Pi$ such that $g \subseteq g_a$, $g \subseteq g_b$, $\mathrm{dom}(g_a) = A \cup \{a\}$ and $\mathrm{rg}(g_b) = B \cup \{b\}$. There exist $r \subseteq^* X$, $j < \omega$, $B' \subseteq N_{T_j}$ and $b' \in N_{T_j}$ such that $T_j < r$, $B = f_{T_j,r} B'$ and $b = f_{T_j,r} b'$ (Here an inessential use is made of the fullness of X.) Choose $n \geqslant j$ such that $A \cup B' \cup \{a, b'\} \subseteq D_n$, and put $u = r \upharpoonright n$, $w = r \upharpoonright n+1$. The set $f_{T_j,u}(B' \cup \{b'\})$ is contained in $S_n(D_n)$. Since either $f_{uw} = \mathrm{id}$ or $f_{uw} = f_n$, the set $f_{T_j,w}(B' \cup \{b'\})$ is contained in $S_n(D_n) \cup f_n S_n(D_n)$. We also have $A \cup \{a\} \subseteq S_n(D_n)$ since $D_n \subseteq S_n(D_n)$. So g induces via $f_{T_j,r}^{-1}$ and $f_{T_j,w}$ a partial isomorphism h in $S_n(D_n) \cup f_n S_n(D_n)$ with $\mathrm{dom}(h) = A$ and $\mathrm{rg}(h) = f_{T_j,w} B'$. By (TS8) the mapping h^{-1} extends to a partial isomorphism h_b^{-1} in $\mathfrak{N}_{T_{n+1}}$ with $\mathrm{dom}(h_b^{-1}) = B' \cup \{b'\}$. Since X is full, the diagram

$$\begin{array}{l} T_j \leqslant w < r \\ \quad\;\; \cap\!\mid \\ \quad\; T_{n+1} \end{array}$$

can be completed to a diagram

$$\begin{array}{l} T_j \leqslant w < r \\ \quad\;\; \cap\!\mid \;\; \cap\!\mid \\ \quad\; T_{n+1} < s \end{array}$$

Now put $g_b c = f_{T_{n+1},s} h_b c$ for all $c \in \mathrm{rg}(h_b^{-1})$. Using (TS4') one sees that g_b is as required. In the same way, extend h to h_a with $\mathrm{dom}(h_a) = A \cup \{a\}$, and put $g_a c = f_{T_{n+1},s} h_a c$ for all $c \in A \cup \{a\}$. □

3.5. Remark. For $\rho \in T_\omega$ let $\rho[n]$ stand for $\rho(k)$, where k is the level of the nth critical node contained in the branch of ρ. Put

$$X_\rho = \{\sigma \in T_\omega \mid \exists m < \omega\, \forall n < \omega\, (m < n \Rightarrow \sigma[n] = \rho[n])\}.$$

Clearly Proposition 3.4(b) is also true for X_ρ instead of X_0, and all X_ρ are countable.

3.6. Theorem. *Let* $\mathcal{K}$ *be either the class of locally finite groups or any class of L-structures which is closed under taking substructures and unions of chains and has free amalgamation and HNN-extensions.*

Then for every existentially closed countable structure $\mathfrak{M} \in \mathcal{K}$ there exists $\mathfrak{N} \in \mathcal{K}$ such that $\mathfrak{N} \equiv_{\infty\omega} \mathfrak{M}$ and $|N| = 2^{\aleph_0}$.

Proof. For $\mathfrak{M}$ countable existentially closed in $\mathcal{K}$ take a tree system satisfying (TS6)–(TS8). Then $\mathfrak{N}_{T_\omega} \equiv_{\infty\omega} \mathfrak{M}$ by Proposition 3.4. Take $c \in N_{T_0}$ satisfying (TS7), and assume $\rho \neq \sigma \in T_\omega$. Then $\rho \upharpoonright n \neq \sigma \upharpoonright n$ for some $n < \omega$, and so $f_{\emptyset,\rho\upharpoonright n} c \neq f_{\emptyset,\sigma\upharpoonright n} c$ by (TS7). Put $u = \{\rho \upharpoonright n, \sigma \upharpoonright n\}$ and $v = \{\rho, \sigma\}$. Then $u < v$, and $f_{\emptyset\sigma} c = f_{uv} f_{\emptyset,\sigma\upharpoonright n} c \neq f_{uv} f_{\emptyset,\rho\upharpoonright n} c = f_{\emptyset\rho} c$. Thus $|N_X| = |X|$ for every infinite $X \subseteq T_\omega$; especially $|N_{T_\omega}| = 2^{\aleph_0}$. □

Since the class of groups and the class of skew fields satisfy the conditions of Proposition 2.7 we get Theorem A(1) and the corresponding part of B(1) from Theorem 3.6.

4. Refinements of the basic construction

We go on extending the axiom system (TS1)–(TS5) to produce limit structures with particular properties. $\mathcal{K}$ and $\mathfrak{M}$ are as in the preceding section.

(TS9)

(i) $N_{T_n} \cap f_n(D_n \setminus N^*_{T_n}) = (D_n \setminus N^*_{T_n}) \cap f_n N_{T_n} = \emptyset$.

(ii) Let $(\varphi_n)_{n<\omega}$ be a fixed sequence of quantifier free formulas $\varphi(x, y)$ with the property that $\mathfrak{G} *_{\mathcal{H}} \mathfrak{G}' \models \neg\varphi[a, a']$ for all products $\mathfrak{G} *_{\mathcal{H}} \mathfrak{G}' \in \mathcal{K}$ and all $a \in G \setminus H$, $a' \in G' \setminus H$. Put $\psi_n = \bigvee_{k \leq n} \varphi_k$.

Then $\mathfrak{N}_{T_{n+1}} \models \neg\psi_n[d, f_n d']$ for all $d, d' \in D_n \setminus N^*_{T_n}$ and all $n < \omega$.

4.1. Proposition. *For every countable existentially closed $\mathfrak{M} \in \mathcal{K}$ there is a tree system satisfying* (TS6)–(TS9).

Proof. By the proofs of 3.1 and 3.2 one gets tree systems satisfying (TS6)–(TS8) for every sequence $(f_n)_{n<\omega}$ chosen according to Proposition 2.7 resp. 2.9. So we can take f_n with $D = D_n \setminus N^*_{T_n}$ in 2.7 and $\varphi(x, y) = \psi_n$ in 2.7(4) resp. 2.9(3). Then (TS9) is satisfied, too. □

4.2. Lemma. *In a tree system satisfying* (TS6)–(TS9) *the following statements hold:*

(a) *If $n \leq m < \omega$ and $a \in D_n$, then all (n, m)-successors of a are elements of D_m.*

(b) *For every element $a \in N_{T_\omega}$ there exists $n < \omega$ such that a has an (n, ω)-predecessor in D_n.*

(c) *If $a \in N_{T_\omega}$ and a has an (n, ω)-predecessor in D_n, then for all $m \geq n$ the structure $\mathfrak{N}_{T_m}$ contains exactly one (m, ω)-predecessor a' of a, and a' is an element of D_m.*

(d) *Let a, b be elements of N_{T_ω} having (n, ω)-predecessors in D_n. Let $\varphi(x, y)$ be*

Sh:83

among the first n members of the enumeration in (TS9) (*ii*), *and assume that* $\mathfrak{N}_{T_\omega} \models \varphi[a, b]$. *Then* $\mathfrak{N}_{T_m} \models \varphi[a', b']$ for all $m \geq n$ and (m, ω)-*predecessors* a', b' *of* a, b.

Notation. For a and m as in (c) the unique (m, ω)-predecessor of a is denoted by $a_{(m)}$.

Proof. (a) By induction on m. If c is an $(n, m+1)$-successor of a, then by (TS2) and (TS6) (iii) either $c = b$ or $c = f_m b$ for some (n, m)-successor b of a. By induction assumption $b \in D_m$, so $c \in D_m \cup f_m D_m$, and (TS8) (b)(i) yields $c \in D_{m+1}$.

(b) For some $j < \omega$ the element a has a (j, ω)-predecessor, say b. This follows from (TS2) and (TS5). As $b \in M = \bigcup \{D_n \mid n < \omega\}$, we have $b \in D_{n-1}$ for some $n > j$. Every (j, n)-successor of b is in D_n, being an element of $S_{n-1}(D_{n-1}) \cup f_{n-1} S_{n-1}(D_{n-1})$. But among the (j, n)-successors of b there is an (n, ω)-predecessor of a.

(c) Assume for contradiction that $m \geq n$, and that a has distinct (m, ω)-predecessors b, c given by $a = f_{T_{m,r}} b = f_{T_{m,s}} c$. For $j \geq m$ put $b_j = f_{T_{m,r \restriction j}} b$, and $c_j = f_{T_{m,s \restriction j}} c$. By statement (a) of this lemma we can assume $b_j \in D_j$ for all $j \geq m$. For sufficiently large integers j we get $r \restriction j \cup s \restriction j < r \cup s$, and so $b_j = c_j$. Thus there is a maximal integer $k \geq m$ with $b_k \neq c_k$. Using (TS7) (iii) and the maximality of k we conclude that $b_k, c_k \in N_{T_k} \setminus N^*_{T_k}$, especially $b_k \in D_k \setminus N^*_{T_k}$, and that either $b_{k+1} = b_k$ and $c_{k+1} = f_k c_k$, or $b_{k+1} = f_k c_k$ and $c_{k+1} = c_k$. In both cases (TS9) (i) yields $b_{k+1} \neq c_{k+1}$, a contradiction.

(d) Using the fact that $\varphi(x, y)$ is quantifier free we can apply the same argument as in (c). The final contradiction arises from (TS9) (ii). □

4.3. Proposition. *For all tree systems satisfying* (TS6)–(TS9) *the following holds*: *For every uncountable subset* A *of* N_{T_ω} *and every* $n < \omega$ *there is an uncountable subset* $B \subseteq A$ *such that* $\mathfrak{N}_{T_\omega} \vdash \neg \varphi_n[a, b]$ *for all distinct* $a, b \in B$.

Proof. Let $A \subseteq N_{T_\omega}$ be uncountable. Applying part (b) of Lemma 4.2 we get an uncountable subset B of A, an integer n and an element a of D_n such that every $b \in B$ is an (n, ω)-successor of a. By Lemma 4.2(a) we can assume $n > k$ for a given integer k. Assume for contradiction the existence of $b \neq b' \in B$ with $\mathfrak{N}_{T_\omega} \models \varphi_k[b, b']$. Since $b_{(n)} = b'_{(n)} = a$, and $b_{(j)} \neq b'_{(j)}$ for all sufficiently large j, there is a minimal $m > n$ such that $b_{(m)} \neq b'_{(m)}$. Like in the proof of Lemma 4.2(c) we conclude that $b_{(m-1)} = b'_{(m-1)} \in D_{m-1} \setminus N^*_{T_{m-1}}$, and so by (TS9) we get $\mathfrak{N}_{T_m} \models \neg \psi_{m-1}[b_{(m)}, b'_{(m)}]$. As we have chosen $k < n \leq m-1$ the formula ψ_{m-1} has the form $\varphi_k \vee \psi$. It follows that $\mathfrak{N}_{T_m} \models \neg \varphi_k[b_{(m)}, b'_{(m)}]$, contradicting Lemma 4.2(d). □

Now we add a last condition which will be used in connection with locally finite groups.

(TS10) For every $n<\omega$ there is a quantifier free formula $\psi(x, y)$ such that

$$\mathfrak{N}_{T_{n+1}} \models \psi[a, b] \qquad \text{if } \{a, b\} \subseteq D_n \text{ or } \{a, b\} \subseteq f_n D_n$$

$$\mathfrak{N}_{T_{n+1}} \models \neg\psi[a, f_n b] \quad \text{if } \{a, b\} \subseteq D_n \setminus N^*_{T_n}.$$

4.4. Proposition. *For the countable universal locally finite group* $\mathfrak{M}$ *there is a tree system satisfying* (TS6)–(TS10).

Proof. Since the embeddings f_n have been chosen according to Proposition 2.9 this follows from Proposition 4.1 and 2.9(4). □

4.5. Proposition. *For every tree system satisfying* (TS6)–(TS10) *the limit structure* $\mathfrak{N}_{T_\omega}$ *contains continuum many substructures, each of cardinality* $2^{\aleph_0}$ *and each* $\infty\omega$*-equivalent to* $\mathfrak{M}$*, such that no uncountable structure is embeddable in two of them.*

Proof. We call $n<\omega$ a splitting level for $X \subseteq T_\omega$ if there are $\rho, \sigma \in X$ such that $\rho \upharpoonright n = \sigma \upharpoonright n$ and $\rho \upharpoonright n+1 \neq \sigma \upharpoonright n+1$. Denote by Sp($X$) the set of all splitting levels for X.

Claim. There is a family $(Y_i)_{i<2^{\aleph_0}}$ of subsets of T_ω, each of cardinality $2^{\aleph_0}$, such that $\mathrm{Sp}(Y_i) \cap \mathrm{Sp}(Y_j)$ is finite for $i \neq j$.

Proof of the claim. Consider the subtree T' of T consisting of all critical nodes of T. Let γ be a function which assigns to each integer n the level of the critical node of T_n in T'. γ cannot be constant on an infinite set. Thus $\mathrm{Sp}(X) \cap \mathrm{Sp}(Y)$ is finite if $\gamma\,\mathrm{Sp}(X) \cap \gamma\,\mathrm{Sp}(Y)$ is finite.

Since T' is a full binary tree we find for every infinite subset $S \subseteq \omega$ a subset $X \subseteq T_\omega$ such that $|X| = 2^{\aleph_0}$ and $\gamma\,\mathrm{Sp}(X) = S$. So we get the desired family starting from an almost disjoint family of continuum many infinite subsets of ω.

Now take a family $(Y_i)_{i<2^{\aleph_0}}$ as described in the claim, and let X_0 be as in Proposition 3.4(b). Put $Z_i = X_0 \cup Y_i$. By Proposition 3.4 we know that $\mathfrak{N}_{Z_i} \equiv_{\infty\omega} \mathfrak{M}$ for all $i<2^{\aleph_0}$, and as a consequence of (TS7) the structures $\mathfrak{N}_{Z_i}$ have power $2^{\aleph_0}$. Assume for contradiction that $Z = Z_i \neq Z_j = Z'$, and that there is an isomorphism f from an uncountable substructure $\mathfrak{A} \subseteq \mathfrak{N}_Z$ into $\mathfrak{N}_{Z'}$.

Since $\mathrm{Sp}(Z \setminus X_0) \cap \mathrm{Sp}(Z' \setminus X_0)$ is finite we can fix n big enough to ensure that $\mathrm{SP}(Z \setminus X_0) \cap \mathrm{Sp}(Z' \setminus X_0) \subseteq n$. According to Lemma 4.2(a) and (b) we choose $k \geq n$ and $a' \in D_k$ such that the set A' of (k, ω)-successors of a' in A is uncountable. In the same way we find $m \geq k$, $b^* \in D_m$ and an uncountable subset B of A' such that all elements of fB are (m, ω)-successors of b^*. By Lemma 4.2(c) all (m, ω)-predecessors of elements of B are contained in the finite set D_m, and so there is an uncountable subset $C \subseteq B$ and an element $a^* \in D_m$ such that all elements of C are (m, ω)-successors of a^*. Clearly all elements of fC are (m, ω)-successors of b^*. Fix for every $a \in C$ subsets $u_a \subseteq^* T_\omega$ and $v_a \subseteq^* T_\omega$ such that $a = f_{T_{m,u_a}} a^*$ and $fa = f_{T_{m,v_a}} b^*$. Since X_0 is countable we find an uncountable subset $C' \subseteq C$ such that $u_a \cap X_0 = u_b \cap X_0$ and $v_a \cap X_0 = v_b \cap X_0$ for all $a, b \in C'$. Choose $a \neq a' \in C'$. Let a_j, b_j stand for $f_{T_{m,u_a \upharpoonright j}} a^*$, $f_{T_{m,v_a \upharpoonright j}} a^*$, and similarly for a'_j, b'_j.

Let $\mu(a, a')$ resp. $\mu(fa, fa')$ be the first integer j such that there is $w \subseteq^* T_\omega$ with $a = f_{T_{j+1,w}} a_{j+1}$ and $a' = f_{T_{j+1,w}} a'_{j+1}$ resp. $fa = f_{T_{j+1,w}} b_{j+1}$ and $fa' = f_{T_{j+1,w}} b'_{j+1}$.

We can assume $\mu(fa, fa') \leq \mu(a, a')$. (Otherwise replace f by f^{-1} in the whole argument.) Put $\mu(a, a') = p$. Assuming $a_{p+1} = a_p$ and $a'_{p+1} = a'_p$ or $a_{p+1} = f_p a_p$ and $a'_{p+1} = f_p a'_p$ we get a contradiction with the minimality of p. So we can suppose by symmetry that $a_p = a_{p+1} \neq f_p a_p$, and $a'_{p+1} = f_p a'_p \neq a'_p$. So $\{a_p, a'_p\} \cap N^*_{T_p} = \emptyset$, and by Lemma 4.2 we have $\{a_p, a'_p\} \subseteq D_P$. We also conclude that $f_{u_a \upharpoonright p, u_a \upharpoonright p+1} \neq f_p$, and $f_{u_{a'} \upharpoonright p, u_{a'} \upharpoonright p+1} \neq \mathrm{id}$; thus by (TS6) (iii) we know that $\eta 1 \in u_{a'} \upharpoonright p+1$ and $\eta 0 \in u_a \upharpoonright p+1$. Consequently $p \in \mathrm{Sp}(Z)$. Since $u_a \cap X_0 = u_{a'} \cap X_0$, and $u_{a \upharpoonright p} < u_a$, $u_{a' \upharpoonright p} < u_{a'}$, no branch through η has the maximal element in X_0, and therefore $p \in \mathrm{Sp}(Z \setminus X_0)$. As $\mathrm{Sp}(Z \setminus X_0) \cap \mathrm{Sp}(Z' \setminus X_0) \subseteq n \leq p$ it follows that $p \notin \mathrm{Sp}(Z' \setminus X_0)$. Since $v_a \cap X_0 = v_{a'} \cap X_0$ we have either $b_{P+1} = b_P$ and $b'_{P+1} = b'_P$ or $b_{P+1} = f_P b_P$ and $b'_{P+1} = f_P b'_P$.

So by Lemma 4.2 either $\{b_{P+1}, b'_{P+1}\} \subseteq D_P$ or $\{b_{P+1}, b'_{P+1}\} \subseteq f_P D_P$. Let $\psi(x, y)$ be a formula as in (TS10). Then $\mathfrak{N}_{T_{p+1}} \models \psi[b_{p+1}, b'_{p+1}]$ and $\mathfrak{N}_{T_{p+1}} \models \neg\psi[a_{p+1}, a'_{p+1}]$.

Because of $\mu(a, a') = p$, the pair (a_{p+1}, a'_{p+1}) is sent to (a, a') by an embedding, and the same holds for the pairs (b_{p+1}, b'_{p+1}) and (fa, fa') since $\mu(fa, fa') \leq \mu(a, a')$ by assumption. As ψ is quantifier-free we get $\mathfrak{A} \models \neg\psi[a, a']$, and $f\mathfrak{A} \models \psi[fa, fa']$, contradicting the assumption that f is an isomorphism. □

Proof of the theorems A and B. Theorem A(1) and a part of B(1) were proven in Section 3. A(2) and the remaining part of B(1) follow from Propositions 4.1 and 4.3, where $\varphi_n(x, y)$ is taken to be the formula $xy = yx$ for all $n < \omega$. Theorem B(2) is immediate from Proposition 4.4 and Proposition 4.5. □

4.6. Remarks. (1) If we take for $\varphi_n(x, y)$ the formula $(xy)^n \neq 1$, then we get by Proposition 4.3 existentially closed groups and universal locally finite groups of power $2^{\aleph_0}$ without uncountable subgroups of finite exponent. The $\varphi_n(x, y)$ can also be chosen to yield e.g. existentially closed groups and skew fields and universal locally finite groups of power $2^{\aleph_0}$ without uncountable nilpotent substructures.

(2) Lemma 4.2 can be used to prove the following theorem: *Let $\mathfrak{M}$ be a countable existentially closed group or skew field, or the countable universal locally finite group. Let $\mathfrak{A} \subseteq \mathfrak{M}$ be a maximal commutative submodel. Then there is $\mathfrak{N} \equiv_{\infty\omega} \mathfrak{M}$ such that $|N| = 2^{\aleph_0}$ and $\mathfrak{A}$ is a maximal commutative submodel of $\mathfrak{N}$.*

Clearly this remains true when $2^{\aleph_0}$ is replaced by κ ($\aleph_0 < \kappa \leq 2^{\aleph_0}$). Boffa [10] proved the special case $\kappa = \aleph_1$ for skew fields.

For a proof of this result assume that $\mathfrak{B} \subseteq \mathfrak{M}$ is not contained in any finitely generated substructure of $\mathfrak{M}$. We are going to prove that there is $\mathfrak{N} \equiv_{\infty\omega} \mathfrak{M}$ of power $2^{\aleph_0}$ such that $C_{\mathfrak{M}}(\mathfrak{B}) = C_{\mathfrak{N}}(\mathfrak{B})$, where $C_{\mathfrak{M}}(\mathfrak{B}) = \{a \in M \mid [a, b] = 1$ for all $b \in B\}$. Our result follows immediately, since $\mathfrak{A}$ being maximal commutative cannot be contained in a finitely generated substructure of $\mathfrak{M}$, and $\mathfrak{A}$ satisfies the equation $C_{\mathfrak{M}}(\mathfrak{A}) = \mathfrak{A}$. By the assumption on $\mathfrak{B}$ we can choose the sets D_{n+1} in

(TS8) such that there is $b_{n+1} \in B$ with $b_{n+1} \in D_{n+1} \setminus \langle N_{T_n} \cup f_n N_{T_n} \rangle_{\mathfrak{M}}$. Let the formula $xy = yx$ be the first member of the enumeration of (TS9). For $0 \leq n \leq m < \omega$ denote by f_{nm} the inclusion of $\mathfrak{N}_{T_n}$ in $\mathfrak{N}_{T_m}$.

By renaming $\mathfrak{N}_{T_\omega}$ we can assume that $\xrightarrow{\lim} (f_{nm} \mathfrak{N}_{T_n})_{m<\omega} = \mathfrak{N}_{T_n}$, and that the corresponding embedding $f_{n\omega}$ is the inclusion of $\mathfrak{N}_{T_n}$ in $\mathfrak{N}_{T_\omega}$. By this renaming we obtain a structure $\mathfrak{N} \equiv_{\infty\omega} \mathfrak{M}$ of power $2^{\aleph_0}$ with $\mathfrak{M} = \bigcup \{\mathfrak{N}_{T_n} \mid n < \omega\} \subseteq \mathfrak{N}$. We want to show that $\mathfrak{N}$ satisfies the claim. Let $c \in C_{\mathfrak{N}}(\mathfrak{B})$. It is sufficient to prove $c \in M$.

By Lemma 4.2(b) we can pick $n < \omega$ such that c has an (n, ω)-predecessor in D_n. So by Lemma 4.2(c) the elements $c_{(k)}$ exist for $n \leq k \leq \omega$.

Moreover each b_m is a (k, ω)-predecessor of itself and an element of D_k for all $k \geq m$, and the condition $c \in C_{\mathfrak{N}}(\mathfrak{B})$ implies $\mathfrak{N} \models [c, b_m] = 1$ for all $m < \omega$. So we can apply Lemma 4.2(d) and conclude $\mathfrak{N}_{T_k} \models [c_{(k)}, b_m] = 1$ whenever $n \leq m \leq k < \omega$.

Assume for contradiction that $c_{(k)} \neq c_{(k+1)}$ for some $k \geq n$. Then $c_{(k)} \in D_k \setminus N^*_{T_k}$, and $c_{(k+1)} = f_k c_{(k)}$. Since $b_k \in D_k \setminus N^*_{T_k}$ by definition, condition (TS9) yields $\mathfrak{N}_{T_{k+1}} \models [c_{(k+1)}, b_k] \neq 1$, contradicting the conclusion from Lemma 4.2(d). This completes the proof of the theorem.

The theorem remains true if $\mathfrak{A}$ is replaced by a countable collection $\{\mathfrak{A}_n \mid n < \omega\}$.

5. The proof of Theorem C

For an arbitrary integer l we can add to Proposition 2.9 the following condition:

$(5)_l$ *If $w(x_1, \ldots, x_n)$ is a word of length at most l, then $\langle A \cup gA \rangle_{\mathfrak{M}} \models w[a_1, \ldots, a_n] \neq 1$ whenever $a_1, \ldots, a_n \in A \cup gA$ and $\mathfrak{A} *_{\mathfrak{B}_0} g\mathfrak{A} \models w[a_1, \ldots, a_n] \neq 1$.*

Proof. We use the notation of the proof of Proposition 2.9. By the universal property of free products with amalgamated substructures the mapping $\mu : A \cup gA \to P$ defined by $\mu a = a$ and $\mu g a = f a$ for all $a \in A$ extends to an isomorphism between $\mathfrak{N} *_{\mathfrak{B}_0} g\mathfrak{A}$ and $\mathscr{P}$.

So in $(5)_l$ we can replace $\mathfrak{A} *_{\mathfrak{B}_0} g\mathfrak{A} \models w[a_1, \ldots, a_n] \neq 1$ by $\mathscr{P} \models w[\mu a_1, \ldots, \mu a_n] \neq 1$. The proof of this modified statement is essentially contained in the proof of 2.9(3). □

We are going to show that every tree system which satisfies the conditions (TS6)–(TS10) and a new condition (TS11) has a limit structure for which Theorem C holds. So in fact we prove a stronger statement, namely that every structure $\mathfrak{N}_i$ in Theorem B(2) can be chosen to satisfy the conclusion of Theorem C. For the statement of (TS11) we fix a function $\lambda \in {}^{\omega}\omega$.

(TS11) For every $n<\omega$ the following hold:

(i) $\langle C\cup f_nC\rangle_{\mathfrak{N}_{T_{n+1}}}\cap\mathfrak{N}_{T_n}=\mathscr{C}$ and $\langle C\cup f_nC\rangle_{\mathfrak{N}_{T_{n+1}}}\cap f_n\mathfrak{N}_{T_n}=f_n\mathscr{C}$ for every subgroup $\mathscr{C}\subseteq\mathfrak{N}_{T_n}$.

(ii) $\mathfrak{N}_{T_{n+1}}\models w[a,b]\neq 1$ whenever $a,b\in N_{T_n}\cup f_nN_{T_n}$, and $w(x,y)$ is a word of length at most $\lambda(n)$ such that

$$\mathfrak{N}_{T_n} *_{\mathfrak{N}^*_{T_n}} f_n\mathfrak{N}_{T_n}\models w[a,b]\neq 1.$$

5.1. Proposition. *For the countable universal locally finite group $\mathfrak{M}$ there is a tree system satisfying* (TS6)–(TS11).

Proof. For every $n<\omega$ choose f_n according to Proposition 2.9 with the additional condition $(5)_{\lambda(n)}$, and take D and $\varphi(x,y)$ as in the proof of Proposition 4.1. □

Now it is sufficient to prove for every limit structure $\mathfrak{N}$ of a tree system satisfying (TS6)–(TS11) that no uncountable subgroup of $\mathfrak{N}$ is contained in a proper variety. So we have to find for every uncountable group $\mathfrak{A}\subseteq\mathfrak{N}$ and every nontrivial word $w(x_1,\ldots,x_n)$ ($w(\boldsymbol{x})$ is called nontrivial if $\forall\boldsymbol{x}\ (w(\boldsymbol{x})=1)$ is not a theorem of group theory) elements $a_1,\ldots,a_n\in A$ such that $\mathfrak{A}\models w[a_1,\ldots,a_n]\neq 1$. Using the fact that outside of any proper variety there is a finite group, and that every finite group is embeddable in a two generator group it is easily seen that it suffices to consider words $w(x,y)$ in two variables.

For the proof of Theorem C we need a couple of technical lemmas.

5.2. Lemma. *Let $\mathfrak{A}$ be a finite group, f an isomorphism defined on $\mathfrak{A}$ such that $\mathfrak{A}\cap f\mathfrak{A}=\mathfrak{B}$ and $f\restriction B=\mathrm{id}$. Consider $\mathfrak{A}, f\mathfrak{A}$ and $\mathfrak{B}$ as subgroups of $\mathfrak{A} *_{\mathfrak{B}} f\mathfrak{A}$. If $a_1,a_1',\ldots,a_n,a_n'$ are elements of A, and the product $b=a_1(fa_1')\cdots a_n(fa_n')$ is an element of B, then $b=f(a_1a_1'\cdots a_na_n')=a_1a_1'\cdots a_na_n'$.*

Proof. The last equality is clear since $b\in B$. The claim $b=f(a_1a_1'\cdots a_na_n')$ is proved by induction on n.

For $n=1$ we have $b=a_1fa_1'$, and the claim follows from Schreier's normal form theorem for free products with amalgamated subgroups (see [4]). Now assume $n>1$. It follows from $b\in B$ by Schreier's theorem that there is $m\leqslant n$ such that $a_m\in B$ or $a_m'\in B$.

Case 1: $a_m\in B$ and $m\geqslant 1$. Then $(fa_{m-1}')a_m(fa_m')=f(a_{m-1}'a_ma_m')$. Put $\bar a_{m-1}=a_{m-1}'a_ma_m'$. Since $\bar a_{m-1}\in A$ we can apply the induction assumption to

$$b=a_1(fa_1')\cdots a_{m-1}(f\bar a_{m-1})a_{m+1}(fa_{m+1}')\cdots a_nfa_n'.$$

Case 2: $a_m'\in B$ and $m<n$. This is proved by the same argument.

Case 3: $a_1\in B$. Then $(fa_1')a_2(fa_2')\cdots a_n(fa_n')\in B$, and so there exists k $(1<k\leqslant n)$ such that $a_k\in B$ or j $(1\leqslant j\leqslant n)$ such that $a_j'\in B$. If $a_k\in B$ we are back in the first case. If $a_j'\in B$ and $j<n$ we are in the second case. If finally

$a'_n \in B$, then $(fa'_1)a_2 \cdots (fa'_{n-1})a_n \in B$, and thus $a_n^{-1}(fa'^{-1}_{n-1}) \cdots a_2^{-1}(fa'^{-1}_1) \in B$, and we can apply the induction assumption. The fourth case, namely $a'_n \in B$, is handled similarly. □

5.3. Lemma. *In every tree system satisfying* (TS6)–(TS11) *the following holds: Let* $n \leq \alpha \leq \omega$, $a, b \in N_{T_\alpha}$, $a', b' \in \mathfrak{N}_{T_n}$. *Assume that* a' *resp.* b' *is an* (n, ω)*-predecessor of* a *resp.* b. *If* $w(x, y)$ *is a word of length at most* $\lambda(n)$ *and* $\mathfrak{N}_{T_\alpha} \models w[a, b] = 1$, *then* $\mathfrak{N}_{T_n} \models w[a', b'] = 1$.

Proof. By (TS5) the number α can be supposed to be finite. So the claim easily reduces to the case $\alpha = n+1$. In this case either $a = a'$ or $a = f_n a'$, and similarly for b. If $a = a'$ and $b = b'$ or if $a = f_n a'$ and $b = f_n b'$, then the claim is obviously true. So we can assume by symmetry that $a = a'$ and $b = f_n b'$.

If $\mathfrak{N}_{T_{n+1}} \models w[a, b] = 1$, then $\mathfrak{N}_{T_{n+1}} \models w[a', f_n b'] = 1$. Thus by (TS11) we get $\mathfrak{N}_{T_n} * \mathfrak{N}^*_{T_n} f_n \mathfrak{N}_{T_n} \models w[a', f_n b'] = 1$. Now apply Lemma 5.2 for $b = 1$ and conclude that

$$\mathfrak{N}_{T_n} * \mathfrak{N}^*_{T_n} f_n \mathfrak{N}_{T_n} \models w[a', f_n b'] = w[a', b'].$$

Therefore $\mathfrak{N}_{T_n} \models w[a', b'] = 1$. □

From now on we fix $n < \omega$, $a^* \in N_{T_n}$, and a set $A \subseteq N_{T_\omega}$ of (n, ω)-successors of a^*. By $\tilde{A}_j$ $(n \leq j \leq \omega)$ we denote the set of those (n, j)-successors of a^* which have (j, ω)-successors in A.

5.4. Definition. Let $n \leq j < \omega$. A node $\eta \in T_j$ is called (a^*, A)*-critical* if η is critical and there are elements $a, b \in \tilde{A}_j \setminus N^*_{T_j}$ such that $\{a, f_j b\} \subseteq \tilde{A}_{j+1}$.

5.5. Lemma. *There is a subtree* T^A *of* T *with the following properties:*

(i) *For every* $a \in A$ *there is* $u \subseteq^* T^A_\omega$ with $a \in N_u$.

(ii) *Whenever* $j \geq n$, $\theta \in T^A \cap T_j$, *and no* $\subseteq$*-successor of* θ *is* (a^*, A)*-critical, then* T^A *contains just one branch through* θ.

Proof. Let $\mathfrak{B}$ be the set of all branches t of T satisfying one of the subsequent conditions:

(α) Every node of $t \setminus T_\omega$ has an (a^*, A)-critical successor in T.

(β) If $n \leq j$ and η is a critical node in $t \cap T_j$ which has no (a^*, A)-critical $\subseteq$-successor in T, then $\eta 1 \in t$ iff there exists $a \in \tilde{A}_j \setminus N^*_{T_j}$ such that $f_j a \in \tilde{A}_{j+1}$.

Put $T^A = \bigcup \mathfrak{B}$. The tree T^A clearly satisfies condition (ii) of the lemma. Take $u \subseteq^* T_\omega$ and $a \in A$ such that $a = f_{T_n u} a^*$. Denote by T^u the subtree of T having the nodes of u as endpoints, say $T^u = \bigcup \{t_0, \ldots, t_{k-1}, t'_k, \ldots, t'_p\}$ with $t_0, \ldots, t_{k-1} \in \mathfrak{B}$, $t'_k, \ldots, t'_p \notin \mathfrak{B}$. Let $k \leq i \leq p$. The branch t'_i contains a minimal critical node θ on a level above $n-1$ such that θ has no (a^*, A)-critical $\subseteq$-successor in T. Let t_j be the unique branch in $\mathfrak{B}$ such that $\theta \in t_j$. Put

$$v = (t_0 \cup \cdots \cup t_{k-1} \cup t_k \cdots \cup t_p) \cap T^A_\omega.$$

In order to complete the proof of the lemma it is sufficient to show $f_{T_n,u}a^* = f_{T_n,v}a^*$. If this equation were false there would be a minimal integer $r \geq n$ such that $f_{T_n,u\restriction r+1}a^* \neq f_{T_n,v\restriction r+1}a^*$. Then

$$f_{T_n,u\restriction r}a^* = f_{T_n,v\restriction r}a^* := b, \quad \text{and} \quad f_{u\restriction r,u\restriction r+1}b \neq f_{v\restriction r,v\restriction r+1}b.$$

So $b \in N_{T_r} \setminus N^*_{T_r}$. We can assume that $f_{u\restriction r,u\restriction r+1} = \text{id}$, and $f_{v\restriction r,v\restriction r+1} = f_r$ (else $f_{u\restriction r,u\restriction r+1} = f_r$ and $f_{v\restriction r,v\restriction r+1} = \text{id}$ and the argument is essentially the same). Then for some $i \geq k$ we get $\eta 0 \in t'_i$, $\eta 1 \in t_i$, where η denotes the critical node of T_r. It follows from the definition of t_i that no successor of η is (a^*, A)-critical. So by (8) there is $a \in N_{T_r} \setminus N^*_{T_r}$ with $f_r a \in A_{r+1}$. Since η is not (a^*, A)-critical we conclude that $b \notin A_{r+1}$. So $f_{T_n,u\restriction r+1}a^*$ has no $(r+1, \omega)$-successor in A; in particular $f_{T_n,u}a^* \notin A$, a contradiction. □

5.6. Lemma. *Suppose $m \geq n$; $\theta \in T_m$; $a, b \in \tilde{A}_m$; $c = w'[a, b] \in N_{T_m} \setminus N_{T_m\setminus\{\theta\}}$. Let $u \subseteq^* T_m$ be such that $a, b \in N_u$, and let $u < v \subseteq T_\omega$* such that *$\{f_{uv}a, f_{uv}b\} \subseteq A$. Put $c_j = f_{u,v\restriction j}c$ for $j \geq m$. Let $\eta \geq \theta$ be an (a^*, A)-critical node such that for all $\theta \leq \sigma < \eta$ the node σ is not (a^*, A)-critical.*

If $\theta \leq \rho \leq \eta$ and *$\rho \in T_j$, then $c_j \notin N_{T_j\setminus\{\rho\}}$.*

Proof. The proof uses induction on $j \geq m$. For $j = m$ there is nothing to show. Assume $\theta \leq \rho < \eta$, $\rho \in T_j$, $\eta \in T_l$ and $c_j \notin N_{T_j\setminus\{\rho\}}$.

Case 1: ρ is not critical. Then $\rho 0 \leq \eta$, and we must show that $c_{j+1} \notin N_{T_{j+1}\setminus\{\rho 0\}}$. This follows from (TS11) (i) since

$$N_{T_{j+1}\setminus\{\rho 0\}} = \langle N_{T_j\setminus\{\rho\}} \cup f_j N_{T_j\setminus\{\rho\}} \rangle_{\mathfrak{N}_{T_{j+1}}}, \quad \text{and} \quad c_{j+1} \in \{c_j, f_j c_j\}.$$

Case 2: ρ is critical. We suppose $\rho 0 \leq \eta$. The subcase $\rho 1 \leq \eta$ is parallel. Assume for contradiction that $c_{j+1} \in N_{T_{j+1}\setminus\{\rho 0\}}$. We have $N_{T_{j+1}\setminus\{\rho 0\}} = f_j N_{T_j}$. As $c_j \notin N_{T_j\setminus\{\rho\}} = N^*_{T_j}$ it follows that $c_{j+1} = f_j c_j \neq c_j$. Put $a_j = f_{u,v\restriction j}a$ and $b_j = f_{u,v\restriction j}b$. Then $c_j = w'[a_j, b_j]$, and so either $f_j a_j \neq a_j$ or $f_j b_j \neq b_j$. Since moreover $a_j, b_j \in \tilde{A}_j$, and ρ is not (a^*, A)-critical, it follows that $\tilde{A}_{j+1} \subseteq N_{T_{j+1}\setminus\{\rho 0\}}$. Consequently $\tilde{A}_l \subseteq N_{T_l\setminus\{\eta\}}$, contradicting the assumption that η is (a^*, A)-critical. □

Now we are ready to prove Theorem C. Let $\mathfrak{N}_{T_\omega}$ be the limit structure of a tree system satisfying (TS6)–(TS10), and (TS11) for suitable λ. Let $\mathscr{C}$ be an uncountable subgroup of $\mathfrak{N}_{T_\omega}$, and assume for contradiction that $\mathscr{C} \models \forall x\, \forall y\, (w(x, y) = 1)$ for some nontrivial word $w(x, y)$.

Let k be the length of $w(x, y)$. Since $\mathscr{C}$ is uncountable we can choose an integer n and an element a^* of N_{T_n} such that the set A of (n, ω)-successors of a^* in A is uncountable. Take T^A with the properties of Lemma 5.5. Let $\mathfrak{B}_1$ be the set of branches t of T^A such that every node of $t \setminus T_\omega$ has an (a^*, A)-critical successor in T (and thus in T^A), and $\mathfrak{B}_2$ the set of the remaining branches of T^A. It follows easily from property (ii) of Lemma 5.5 that $\mathfrak{B}_2$ is countable. Choose for each

$a \in A$ a subset $u_a \subseteq^* T^A_\omega$ such that $T_n < u_a$ (this is possible since we can assume that $\mathfrak{B}_1 \cup \mathfrak{B}_2 = \mathfrak{B}$ as in the proof of Lemma 5.5) and $a = f_{T_n, u_a} a^*$. Let $\mathfrak{B}_a$ be the set of branches of T^A with endpoints in u_a. Since $\mathfrak{B}_2$ has only countably many finite subsets there is an uncountable subset $B \subseteq A$ and a finite subset $\mathfrak{B}_0 \subseteq \mathfrak{B}_2$ such that $\mathfrak{B}_a \cap \mathfrak{B}_2 = \mathfrak{B}_0$ for all $a \in B$. There is an integer $m > k$ such that all (a^*, A)-critical nodes of $\bigcup \mathfrak{B}_0$ are contained in $\bigcup \{T_i \mid i < m-1\}$. Moreover we can assume that m is, for some $\tilde{a} \neq \tilde{b} \in B$, the minimal integer such that there exists $v \subseteq u_{\tilde{a}} \cup u_{\tilde{b}}$ and $a, b \in \tilde{A}_m$ with $\tilde{a} = f_{v \restriction m, v} a$. and $\tilde{b} = f_{v \restriction m, v} b$. (Here we use that B is uncountable.) Because of the minimality of m we find $a', b' \in \tilde{A}_{m-1}$ with $a = a' \neq f_{m-1} a'$ and $b = f_{m-1} b' \neq b'$ or conversely. So the critical node $\eta \in T_{m-1}$ is (a^*, A)-critical. Consequently η cannot be contained in a branch from $\mathfrak{B}_2$ by the choice of m. So η has an (a^*, A)-critical successor $\tilde{\eta} \neq \eta$ in T^A, and we may assume that ρ is not (a^*, A)-critical for $\eta < \rho < \tilde{\eta}$. Denote by c the product $a \cdot b$ in $\mathfrak{N}_{T_m}$. Let the function λ in (TS11) be defined by $\lambda(i) = 6\,|N_{T_i}|$. In $\mathfrak{N}_{Tm-1} *_{\mathfrak{N}^*_{T_{m-1}}} f_{m-1} \mathfrak{N}_{T_{m-1}}$ the product $a \cdot b$ has infinite order, and the length of $(x \cdot y)^{2|N_{T_{m-1}}|}$ is $4\,|N_{T_{m-1}}|$ and thus inferior to $\lambda(m-1)$. Hence we conclude using (TS11) (ii) that $\mathfrak{N}_{T_m} \models c^{2|N_{T_{m-1}}|} \neq 1$ and so $c^2 \notin N_{T_{m-1}}$ and $c^2 \notin f_m N_{T_{m-1}}$. Otherwise stated $c^2 \notin N_{T_m \setminus \{\eta 0\}}$ and $c^2 \notin N_{T_m \setminus \{\eta 1\}}$. Now $\tilde{\eta}$ is a successor of $\eta 0$ or of $\eta 1$. Lemma 5.6 with $w'[a, b] = a \cdot b$, $\theta = \eta 0$ resp. $\theta = \eta 1$ and $\rho = \eta = \tilde{\eta}$ yields $f_{v \restriction m, v \restriction r} c^2 \in N_{T_r} \setminus N^*_{T_r}$, where r is the level of $\tilde{\eta}$. Put $f_{v \restriction m, v \restriction r} c = \tilde{c}$, and assume $f_{v \restriction m, v \restriction r+1} c = \tilde{c}$. (The other case, namely $f_{v \restriction m, v \restriction r+1} c = f_r \tilde{c}$, is handled similarly.) Since $\tilde{\eta}$ is (a^*, A)-critical there exists $d \in \tilde{A}_r \setminus N^*_{T_r}$ such that $f_r d$ has a $(r+1, \omega)$-successor in $\mathscr{C}$, say $\bar{d}$. The element $\tilde{c}$ has a $(r+1, \omega)$-successor in $\mathscr{C}$, too, namely $\bar{c} = f_{v \restriction m, v} c$. Put $v(x, y) = w((xyx)^2, (xy)^2)$. By assumption $\mathscr{C} \models v[\bar{c}, \bar{d}] = 1$. The length of $v(x, y)$ is at most $6k$, and $k \leqslant m \leqslant r \leqslant |N_{T_r}|$. Hence the length of $v(x, y)$ is bounded by $\lambda(r)$, and by Lemma 5.3 we get $\mathfrak{N}_{T_{r+1}} \models v[\tilde{c}, f_r d] = 1$. On the other hand $\tilde{c} \in N_{T_r} \setminus N^*_{T_r}$, and $f_r d \in f_r N_{T_r} \setminus N^*_{T_r}$. Moreover $\tilde{c}^2 \in N_{T_r} \setminus N^*_{T_r}$. It is not hard to see that under these conditions $(\tilde{c}(f_r d)\tilde{c})^2$ and $(\tilde{c} f_r d)^2$ freely generate a free subgroup of $\mathscr{P} = \mathfrak{N}_{T_r} *_{\mathfrak{N}^*_{T_r}} f_r \mathfrak{N}_{T_r}$. Since $w(x, y)$ is a nontrivial word we have therefore

$$\mathscr{P} \models w[(\tilde{c}(f_r d)\tilde{c})^2, (\tilde{c} f_r d)^2] = v[\tilde{c}, f_r d] \neq 1.$$

Applying (TS11) (ii) we conclude that $\mathfrak{N}_{T_{r+1}} \models v[\tilde{c}, f_r d] \neq 1$, a contradiction. □

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