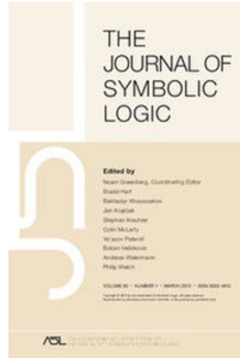

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MARTIN'S AXIOMS, MEASURABILITY AND EQUICONSISTENCY RESULTS

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Abstract. We deal with the consistency strength of ZFC + variants of MA + suitable sets of reals are measurable (and/or Baire, and/or Ramsey). We improve the theorem of Harrington and Shelah [2] repairing the asymmetry between measure and category, obtaining also the same result for Ramsey. We then prove parallel theorems with weaker versions of Martin's axiom (MA(σ-centered), (MA(σ-linked)), MA($\Gamma^+_{\aleph_0}$), MA(K)), getting Mahlo, inaccessible and weakly compact cardinals respectively. We prove that if there exists $r \in \mathbf{R}$ such that $\omega_1^{L[r]} = \omega_1$ and MA holds, then there exists a Δ^1_3-selective filter on ω, and from the consistency of ZFC we build a model for ZFC + MA(I) + every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey.

Table of Contents. §0. *Introduction.* We define $\Gamma^+_{\aleph_0}$, I, and other classes of partially ordered sets. We define MA($\Gamma^+_{\aleph_0}$), MA(I). etc. We define the basic notions used in this article.

§1. *A useful lemma.* We recall the proof of the following well-known lemma: If P is a forcing notion and $\Vdash_P$ "$\kappa = \aleph_1$" and for every P-name $\mathbf{r}$ for a real number there exist $Q_{\mathbf{r}} \lessdot P$ such that $\mathbf{r} \in V^{Q_{\mathbf{r}}}$ and $|Q_{\mathbf{r}}| < \kappa$ then $\Vdash_P$ "in $L(\mathbf{R})$ every set of reals is Lebesgue measurable, has the property of Baire and is Ramsey".

§2. MA($\Gamma^+_{\aleph_0}$) *and inaccessible cardinals.* We prove that the following theories are equiconsistent: (a) ZFC + there exists an inaccessible cardinal; (b) ZFC + MA($\Gamma^+_{\aleph_0}$) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$; (c) ZFC + MA($\Gamma^+_{\aleph_0}$) + every projective set of reals is Lebesgue measurable (Σ^1_3); (d) ZFC + MA($\Gamma^+_{\aleph_0}$) + every projective set of reals has the property of Baire (Σ^1_3); and (e) ZFC + MA($\Gamma^+_{\aleph_0}$) + every projective set of reals is Ramsey (Σ^1_3).

§3. MA(σ*-centered*). MA(σ*-linked*) *and Mahlo cardinals.* We prove that the following theories are equiconsistent: (a) ZFC + there exists a Mahlo cardinal; (b) ZFC + MA(σ-centered) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$; (c) ZFC + MA($\sigma$-centered) + every projective set of reals is Lebesgue measurable (Σ^1_3); (d) ZFC + MA(σ-centered) + every projective set of reals has the property of Baire (Σ^1_3); (e) ZFC + MA(σ-centered) + every projective set of reals is Ramsey (Σ^1_3); (f) ZFC + MA(σ-linked) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$; (g) ZFC + MA($\sigma$-linked) + every projective set of reals is Lebesgue measurable (Σ^1_3); (h) ZFC + MA(σ-linked) + every projective set of reals has the property of Baire (Σ^1_3); and (i) ZFC + MA(σ-linked) + every projective set of reals is Ramsey (Σ^1_3).

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§4. MA(K) *and weakly compact cardinals.* We prove that the following theories are equiconsistent: (a) ZFC + there exists a weakly compact cardinal; (b) ZFC + MA(K) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$; (c) ZFC + MA($K$) + every projective set of reals is Lebesgue measurable (Σ_3^1); (d) ZFC + MA(K) + every projective set of reals has the property of Baire (Σ_3^1); and (e) ZFC + MA(K) + every projective set of reals is Ramsey (Σ_3^1).

§5. MA *and a Δ_3^1-selective filter on ω.* We prove that if there exists a real r such that $\omega_1^{L[r]} = \omega_1$ then MA implies that there exists a Δ_3^1-selective filter on ω_1. This means that MA does not imply every Δ_3^1 set of reals is Lebesgue measurable (etc.). Then we prove that the following theories are equiconsistent: (a) ZFC + MA + there exists a weakly compact cardinal; (b) ZFC + MA + "every Δ_3^1-set of reals is Lebesgue measurable"; (c) ZFC + MA + "every Δ_3^1-set of reals has the property of Baire"; and (d) ZFC + MA + "every Δ_3^1-set of reals is Ramsey".

§6. MA(I) *and Δ_3^1-sets of reals.* We prove that the following theories are equiconsistent: (a) ZFC; and (b) ZFC + MA(I) + every Δ_3^1-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey.

§0. Introduction. In this article we will give exact equiconsistency results about problems involving variants of Martin's axiom and their relation with the measurability of some projective set of reals. We found that in all cases there is an exact symmetry between measurability, categoricity, and being Ramsey, and that these three properties are connected with the accessibility of $\aleph_1$ in L. The history of this problem begins with the famous article of Solovay [12], where, from an inaccessible cardinal, a model in which every projective set of real numbers is Lebesgue measurable, has the property of Baire and is Ramsey, was built. For a long time people worked in order to obtain this result without large cardinal assumptions, but Shelah [11] proved that if every Σ_3^1-set of reals is Lebesgue measurable then $\aleph_1$ must be an inaccessible cardinal in L, and also he proved that in order to obtain a model in which every projective set of reals has the property of Baire, a large cardinal assumption is not necessary. Therefore the problems of the measurability and of the categoricity of the projective set of reals are not equivalent from the point of view of ZFC.

In the same article of Solovay [12], it was remarked that from a weakly compact cardinal, Kunen and Solovay give a model for Martin's axiom where every projective set of reals is Lebesgue measurable, has the property of Baire and is Ramsey. In this direction, between Solovay's theorem and Shelah's theorem, in 1978 Harrington and Shelah [2] proved that if MA holds and either every Σ_3^1-set of reals is Lebesgue measurable or every Δ_3^1-set of reals has the property of Baire, then $\aleph_1$ must be a weakly compact cardinal in L, and again an asymmetry appears between measurability and categoricity. In §5 we correct this asymmetry by showing that if MA holds and every Δ_3^1-set of reals is Lebesgue measurable then $\aleph_1$ must be a weakly compact cardinal in L. Our proof also shows that if MA holds and every Δ_3^1-set of reals is Ramsey then $\aleph_1$ is a weakly compact cardinal in L. In fact, we prove that if there exists $r \in \mathbf{R}$ such that $\omega_1^{L[r]} = \omega_1$ then there exists a Δ_3^1-selective filter on ω. The idea of using filters on ω in this context was given by Raisonnier [10], who gave an elegant proof of Shelah's theorem using rapid filters on ω.

In Shelah [11] a model for "ZFC + every Δ^1_3-set of reals is Lebesgue measurable" was built from a model for ZFC, and in Ihoda [3] a model for "ZFC + every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey" was built from a model for ZFC; in §6, we present a new form of MA, namely MA(I), which seems to be maximal in order to obtain, from a model for ZFC, a model for MA(I) + "every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey".

Thinking about this, naturally the following problem appears: is the use of a weakly compact cardinal necessary in order to build a model for MA(I) + every projective set of reals is Lebesgue measurable (or Baire, or Ramsey)? The answer to this question is given in §4, where we prove that if MA for partially ordered sets satisfying Knaster's condition (MA(K)) holds, and every Σ^1_3-set of reals is Lebesgue measurable (or Baire, or Ramsey), then $\aleph_1$ is a weakly compact cardinal in L.

However, some weaker versions than MA(K) are known, for example MA(σ-linked) or MA(σ-centered), and the same question as above replacing MA(K) by MA(σ-linked) or MA(σ-centered) is answered in §3, where we prove that it is equivalent to the existence of a Mahlo cardinal. Here, we define the unbounded filters on ω, and we prove that these filters do not have the property of Baire and are not Ramsey. We conclude by proving that if MA(σ-centered) holds and $\aleph_1$ is not a Mahlo cardinal in L, then there exists a Σ^1_3-unbounded filter on ω.

From the work of Raisonnier [10], we know that in the presence of additivity of measure, the measurability of the Σ^1_3-set of reals, the categoricity of the Σ^1_3-set of reals, and the Ramsey property of the Σ^1_3-set of reals are equivalent, and at least the existence of an inaccessible cardinal is necessary in order to obtain this consistency. In Ihoda and Shelah [6] we introduce a weaker form of MA, which implies the additivity of measure, namely MA(Γ^+_0), and in §2 we will prove that the existence of an inaccessible cardinal is sufficient in order to give a model for MA(Γ^+_0) and every projective set of reals is Lebesgue measurable, has the property of Baire and is Ramsey.

0.1. Definition. (i) Let A be a class of partially ordered sets. We say that MA(A) *holds* if and only if for every P in A satisfying the countable chain condition, and for every family $\langle D_i : i < \kappa \rangle$, $\kappa < 2^{\aleph_0}$, each D_i dense, there exists a directed subset $G \subset P$ such that $G \cap D_i \neq 0$ for every $i < \kappa$.

(ii) Clearly MA is MA(A), where A is the class of all partially ordered sets.

(iii) If P is a partially ordered set, we say that P has the *indestructible countable chain condition* if for every partially ordered set Q satisfying the countable chain condition we have that

$$\Vdash_Q \text{“}\langle P, \leqq \rangle \models \text{c.c.c.”}$$

Set $I = \{P: P \text{ has the indestructible countable chain condition}\}$.

(iv) If P is a partially ordered set, we say that P has *Knaster's condition* if, whenever $R \subset P$ is uncountable, there is an uncountable $R' \subset R$ such that every pair of members of R' are compatible.

Set $K = \{P: P \text{ has Knaster's condition}\}$.

(v) If P is a partially ordered set, we say that P is *σ-linked* if there exists $h: P \to \omega$ such that if $h(p) = h(q)$ then there exists $r \in P$ with $p \leq r$ and $q \leq r$.

Set σ-linked $= \{P: P \text{ is } \sigma\text{-linked}\}$.

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(vi) If P is a partially ordered set, we say that P is *σ-centered* if there exists $h: P \to \omega$ such that for every $p_1, \dots, p_n$ in P, if $h(p_1) = h(p_2) = \cdots = h(p_n)$ then there exists $r \in P$ such that $p_i \leq r$ for $1 \leq i \leq n$.

Set σ-centered $= \{P: P \text{ is } \sigma\text{-centered}\}$.

(vii) If P is a partially ordered set, we say that P is $\Gamma^+_{\aleph_0}$ if and only if P is a Σ^1_1 subset of $\mathbf{R}$ and $\leq_P$ is a Σ^1_1 subset of $\mathbf{R}^2$ and $\{(p,q): p \text{ is incompatible with } q\}$ is a Σ^1_1 subset of $\mathbf{R}^2$.

Set $\Gamma^+_{\aleph_0} = \{P: P \text{ is } \Gamma^+_{\aleph_0}\}$. A complete discussion of $\Gamma^+_{\aleph_0}$, MA($\Gamma^+_{\aleph_0}$), and Γ_λ, Γ^+_λ can be found in Ihoda and Shelah [6].

0.2. *Fact.* MA $\Rightarrow$ MA(I) $\Rightarrow$ MA(K) $\Rightarrow$ MA(*σ-linked*) $\Rightarrow$ MA(*σ-centered*). ■

0.3. *Fact.* (i) MA(*σ-linked*) *implies every* Σ^1_2*-set of reals is Lebesgue measurable.*

(ii) MA($\Gamma^+_{\aleph_0}$) *implies every* Σ^1_2*-set of reals is Lebesgue measurable.* ■

0.4. Remark. From (ii) we know that σ-centered $\not\subseteq \Gamma^+_{\aleph_0} \not\subseteq$ σ-centered; and in Ihoda and Shelah [6] we proved that $\Gamma_{\aleph_0^+} \subseteq I$, but we do not know if $\Gamma_{\aleph_0^+} \subseteq K$, or more still if MA ($\Gamma_{\aleph_0^+}$) is equivalent to additivity in measure. We think that it is well known when a subset of reals is Lebesgue measurable and when it has the property of Baire, so we will only define explicitly when a subset of $[\omega]^\omega = \{a \subseteq \omega: |a| = \aleph_0\}$ is Ramsey.

0.5. Definition. A subset $X \subseteq [\omega]^\omega$ is *Ramsey* if for every $a \in [\omega]^\omega$ there exists $b \subseteq a$ such that

$$\{c \subseteq b: |c| = \aleph_0\} = [b]^\omega \subseteq X \vee [b]^\omega \subseteq [\omega]^\omega - X.$$

This notion has been studied by many people, and more information on it can be found in Ihoda [4].

Finally we will use the symbol $P \Vdash$ "φ" or $\Vdash_P$ "φ" to say $\Phi \Vdash_P$ "φ", where Φ is the minimal member of P. All our notation is standard, and we will not make any special remarks on it.

§1. A useful lemma.

1.1. Lemma (Folklore). *Suppose that* (i) *κ is an inaccessible cardinal in* $V \models$ ZFC; (ii) *P satisfies the κ-c.c.*; (iii) $P \Vdash$ "$\kappa = \aleph_1$"; *and* (iv) *for every* $Q \subseteq P$, $|Q| < \kappa$, *there exists* P', $|P'| < \kappa$ *with* $Q \subseteq P' \lessdot P$. *Then for every* $G \subseteq P$ *generic over V we have that* $L(\mathbf{R})^{V[G]} \Vdash$ "*every set of reals is Lebesgue measurable, has the property of Baire and is Ramsey*".

Proof. Let $G \subseteq P$ be generic over V; we show that there exists $H \subseteq \text{Levy}(\aleph_0, <\kappa)$ generic over V such that $L(\mathbf{R})^{V[G]} = L(\mathbf{R})^{V[H]}$; then the conclusion follows from Solovay [12].

For every P-name r for a real number there exists $Q_r \lessdot P$ such that $r \in V^{Q_r}$ and $|Q_r| < \kappa$. Let $G_1 \subseteq \text{Levy}(\aleph_0, 2^{|P|})$ be generic over $V[G]$. Now working in $V[G][G_1]$, let $\langle r_n: n < \omega\rangle$ be a list of the P-names r for a real number which belongs to V. By induction on n we choose $P_n \in V$ such that

(i) $Q_i \lessdot P_n$ for $i < n$, and

(ii) $\Vdash_{P_{n+1}}$ "$|P_n| = \aleph_0$".

Now it is not hard to find $H \subseteq \text{Levy}(\aleph_0, <\kappa)$ generic over V such that $H \cap Q_i = G \cap Q_i$. This is possible because $H \subseteq \text{Levy}(\aleph_0, <\kappa)$ is generic over V if and only if every initial segment of H is generic over V. This concludes the proof of the lemma. ■

§2. MA($\Gamma^+_{\aleph_0}$) and inaccessible cardinals.

2.1. THEOREM. *The following theories are equiconsistent*:

(i) *ZFC + there exists an inaccessible cardinal.*

(ii) ZFC + MA($\Gamma_{\aleph_0^+}$) + *every projective set of reals is Lebesgue measurable* (Σ^1_3).

(iii) ZFC + MA($\Gamma_{\aleph_0^+}$) + *every projective set of reals has the property of Baire* (Σ^1_3).

(iv) ZFC + MA($\Gamma_{\aleph_0^+}$) + *every projective set of reals is Ramsey* (Σ^1_3).

PROOF. From Ihoda and Shelah [6] we know that MA($\Gamma^+_{\aleph_0}$) implies the additivity of measure; therefore if $\aleph_1$ is not an inaccessible cardinal in L, then there exists a real number r such that $\omega_1^{L[r]} = \omega_1$. Using these two facts, Raisonnier [10] gives a Σ^1_3-rapid filter, and this implies that (ii), (iii) and (iv) fail in the model. This proves (ii) $\to$ (i), (iii) $\to$ (i), and (iv) $\to$ (i).

Next, from a model for $(V = L)$ + there exists an inaccessible cardinal, we will force a model for ZFC + MA($\Gamma^+_{\aleph_0}$) + every projective set of reals is Lebesgue measurable, has the property of Baire and is Ramsey. This will suffice to prove the theorem.

Let V be a model for $(V = L)$ + there exists an inaccessible cardinal. Let κ be an inaccessible cardinal in V. We define the following α-stage iterated forcing notion $\bar{Q}_\alpha = \langle P_\beta; \mathbf{Q}_\beta : \beta < \alpha \rangle$ such that P_1 is coll($\aleph_0$, $<\kappa$) and if $\beta < \alpha$ then

$$P_\beta \Vdash \text{“}\mathbf{Q}_\beta \text{ belongs to } \Gamma^+_{\aleph_0} \text{ and satisfies the countable chain condition”}.$$

For β limit, P_β is the directed limit of $\bar{Q}_\beta$. Without loss of generality, we can assume that if α is κ^+ (κ^{++}, etc.), then

$$V^{P_{\kappa^+}} \models \text{“MA}(\Gamma^+_{\aleph_0}) + 2^{\aleph_0} = \aleph_2\text{”}.$$

So our problem is to show that in $V^{P_{\kappa^+}}$ every projective set of reals is Lebesgue measurable, has the property of Baire and is Ramsey. In order to give this we use Lemma 1.1 and the following fact.

2.2. *Claim.* (i) P_κ *satisfies* κ*-c.c.*

(ii) $V^{P_{\kappa^+}} \models \kappa = \aleph_1$.

(iii) *For every* $Q \subseteq P_{\kappa^+}$, $|Q| < \kappa$, *there exists* $P' \lessdot P_{\kappa^+}$ *such that* $Q \subseteq P'$ *and* $|P'| < \kappa$.

PROOF. (i) and (ii) are well known. Let $Q \subseteq P_{\kappa^+}$ be such that $|Q| < \kappa$. Using the κ-chain condition (and induction on α) we can find $\lambda < \kappa$, λ regular, and $S \subseteq \kappa^+ - 1$ such that

(a) $|S| < \kappa$ and S is closed.

(b) Inductively, on the ordinals in S, we prove and define, as in Ihoda and Shelah [6, §1],

(b1) coll ($\aleph_0$, $<\lambda$) $* \mathbf{P}_{S\restriction\beta} \lessdot P_{\kappa^+}$,

(b2) $\mathbf{Q}_\beta \in V^{\text{coll}(\aleph_0, <\lambda) * \mathbf{P}_{S\restriction\beta}}$,

(b3) coll($\aleph_0$, $<\lambda$) $* \mathbf{P}_{S\restriction\beta+1} =$ coll($\aleph_0$, $<\lambda$) $* \mathbf{P}_{S\restriction\beta} * \mathbf{Q}_\beta$, where $\beta \in S$, and

(b4) if β is a limit ordinal, then coll($\aleph_0$, $<\lambda$) $* \mathbf{P}_{S\restriction\beta}$ is the directed limit of the system.

Some special work is necessary in order to show why

$$\text{coll}(\aleph_0, <\lambda) * \mathbf{P}_{S\restriction\beta} \Vdash \text{“}\mathbf{Q}_\beta \text{ satisfies } \lambda\text{-c.c.”}$$

But this is exactly a particular case of Ihoda and Shelah [6, §3.14] (remember that coll($\aleph_0$, $<\lambda$) $* \mathbf{P}_{S\restriction\beta} \Vdash$ “$|\lambda| = \aleph_1$”), where we proved that for $P \in \Gamma^+_{\aleph_0}$, $P \Vdash$ “c.c.c.” is a strongly absolute property, for models containing the parameters of the definition of P.

(c) $Q \subseteq \mathrm{coll}(\aleph_0, <\lambda) * \mathbf{P}_S \lessdot P_{\kappa^+}$.

Because κ is an inaccessible cardinal in V we have that

$$|\mathrm{coll}(\aleph_0, <\lambda) * \mathbf{P}_S| < \kappa,$$

and this finishes the proof of 2.2 and 2.1. ■

REMARK. We have proved that cons(ZFC + there exists an inaccessible cardinal) implies cons(ZFC + MA($\Gamma^+_{\aleph_0}$) + every ordinal definable set of reals is Lebesgue measurable, has the property of Baire and is Ramsey).

§3. MA(σ-centered), MA(σ-linked) and Mahlo cardinals.

3.1. THEOREM. *The following theories are equiconsistent.*

(i) ZFC + *there exists a Mahlo cardinal.*

(ii) ZFC + MA(*σ-centered*) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$.

(iii) ZFC + MA(*σ-centered*) + *every projective set of reals is Lebesgue measurable* (Σ^1_3).

(iv) ZFC + MA(*σ-centered*) + *every projective set of reals has the property of Baire* (Σ^1_3).

(v) ZFC + MA(*σ-centered*) + *every projective set of reals is Ramsey* (Σ^1_3).

(vi) ZFC + MA(*σ-linked*) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$.

(vii) ZFC + MA(*σ-linked*) + *every projective set of reals is Lebesgue measurable* (Σ^1_3).

(viii) ZFC + MA(*σ-linked*) + *every projective set of reals has the property of Baire* (Σ^1_3).

(ix) ZFC + MA(*σ-linked*) + *every projective set of reals is Ramsey* (Σ^1_3).

PROOF. From (i) we will give a model for (ii), (iii), (iv), (v), (vi) (vii), (viii), and (ix). After that we prove that (ii) implies (i). Clearly (vi) implies (ii). From this, using rapid filters, we prove that (iii) implies (ii), (vii) implies (ii), (viii) implies (ii) and (ix) implies (ii). Lastly we introduce the notion of an unbounded filter on ω and prove that (iv) implies (ii) and (v) implies (ii).

Suppose (i); let $V = L$ and κ a Mahlo cardinal in L. We define the following α-stage iterated forcing notion:

$$\bar{Q}_\alpha = \langle P_\beta : \mathbf{Q}_\beta : \beta < \alpha \rangle$$

such that P_1 is $\mathrm{coll}(\aleph_0, <\kappa)$ and if $\beta < \alpha$ then $P_\beta \Vdash$ "$\mathbf{Q}_\beta$ is σ-linked, $\mathbf{h}_\beta$ witnesses this".

For β limit, P_β is the directed limit of $\bar{Q}_\beta$. Without loss of generality, we can assume that if α is κ^+ (κ^{++}, etc.) then

$$V^{P_{\kappa^+}} \models \text{"MA}(\sigma\text{-linked}) + 2^{\aleph_0} = \aleph_2 = \kappa^+\text{"}.$$

So our problem is to show that in $V^{P_{\kappa^+}}$ every projective set of reals is Lebesgue measurable, has the property of Baire and is Ramsey. Proving this we have that in $V^{P_{\kappa^+}}$ for every real number r, $\omega_1^{L[r]} < \omega_1$. In order to give this we will use Lemma 1.1 and the following fact.

3.2. *Claim.* (i) P_{κ^+} *satisfies* κ-*c.c.*

(ii) $V^{P_{\kappa^+}} \models \kappa = \aleph_1$.

(iii) *For every* $Q \subseteq \Gamma_{\kappa^+}$, $|Q| < \kappa$, *there exists* $P' \lessdot \Gamma_{\kappa^+}$ *such that* $Q \subseteq P'$ *and* $|P'| < \kappa$.

Proof. (i) and (ii) are clear. Let $Q \subseteq P_{\kappa^+}$ be such that $|Q| < \kappa$. As κ is Mahlo in V, there exists a stationary set $S \subseteq \kappa$ of inaccessible cardinals. Let $\langle M_i : i < \kappa\rangle$ be such that

$$M_i \prec \langle H(\chi), \in, <_\chi\rangle \quad \text{for } i < \kappa,$$

where χ is large enough and $<\chi$ is a well order of $H(\chi)$, and such that

(a) $Q \subseteq M_0$,
(b) $M_i \prec M_{i+1}$,
(c) if $\lambda = \bigcup \lambda \neq 0$ then $M_\lambda = \bigcup_{\beta<\lambda} M_\beta$,
(d) $\mathscr{P}(M_i) \in M_{i+1}$,
(e) $|M_i| < \kappa$,
(f) $P_{\kappa^+} \in M_0$, and
(g) $C = \{\delta : M_\delta \cap \kappa = \delta\}$ is a club subset of κ.

Therefore there exists $\lambda \in C \cap S$.

We will prove, by induction on $i < \kappa$, that

$$P_i \cap M_\lambda \stackrel{\text{def}}{=} P_i^\lambda \lessdot P_i.$$

This is sufficient for the claim.

First we prove by induction the following

3.3. *Claim. P_i^λ satisfies λ-c.c.*

Proof. The case $i = 0$ is clear.

$i = 1$. P_1^λ is $\mathrm{coll}(\aleph_0, <\lambda)$, which satisfies λ-c.c.

$i = \bigcup i \neq 0$ is well known.

$i = j + 1$. Let $G \subseteq P_j^\lambda$ be generic over V. Then, by the inductive hypothesis, $V[G] \models$ "λ is uncountable". It is sufficient to show that

$$V[G] \models \text{"}\mathbf{Q}_j[G] \text{ satisfies } \lambda\text{-c.c."}$$

But $M_\lambda[G] \models$ "$\mathbf{h}_j[G] : \mathbf{Q}_j[G] \to \omega$ witnesses $\mathbf{Q}_j[G]$ is σ-linked". So $V[G] \models$ "$\mathbf{h}_j[G]$ witnesses $\mathbf{Q}_j[G] \cap M_\lambda[G]$ is σ-linked".

If there exists in $V[G]$ an uncountable antichain of $\mathbf{Q}_j[G] \cap M_\lambda[G]$ of cardinality λ, this implies that λ is countable in $V[G]$, and this is impossible. This finishes the proof of the claim. ∎

Therefore $P_{\kappa^+}^\lambda$ satisfies λ-c.c. Thus every maximal antichain A of $P_{\kappa^+}^\lambda$ lies in M_λ (λ inaccessible and $M_\lambda = \bigcup_{\beta<\lambda} M_\beta$); as $M_\lambda \prec \langle H(\chi), \in, <_\chi\rangle$ this implies that A is a maximal antichain of P_{κ^+} in V. Thus $P_{\kappa^+}^\lambda \lessdot P_{\kappa^+}$, $|P_{\kappa^+}^\lambda| \leq |M_\lambda| < \kappa$ and $Q \subseteq P_{\kappa^+}^\lambda$. This concludes the proof of Claim 3.2. ∎

So we have proved that (i) $\to$ (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix).

3.4. *Claim.* (ii) $\to$ (i).

Proof. Let V be a model satisfying

$$V \models \text{"MA}(\sigma\text{-centered}) + (\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)\text{"}.$$

We will show that in this case $\aleph_1$ is Mahlo in L.

Suppose that $\aleph_1$ is not a Mahlo cardinal in L. Hence there exists $C \in L$, C a club subset of $\aleph_1^V$, such that every element of C is singular in L. Therefore there exists $A \subseteq \omega_1$ such that

(*) For every $\delta \in \omega_1$, $L[A \cap \delta] \models |\delta| = \aleph_0$.

(For example, if $\{0\} \cup C = \{\delta_i : i < \omega_1\}$ then $A \cap [\delta_i, \delta_i + \omega]$ encodes a well-order of ω of type δ_{i+1}.)

So in $L[A \cap \delta]$ there exists a sequence $\eta_\delta = \langle \eta_\delta(i) : i < \omega \rangle$ such that

(i) $\eta_\delta(n) < \eta_\delta(m)$ if $n < m$,

(ii) $\lim_{n \to \omega} \eta_\delta(n) = \delta$, and

(iii) for every n, $\eta_\delta(n) = n \bmod(\omega)$.

3.5. Definition. Let $B \subseteq \omega_1$ be such that every member of B is a limit ordinal. We define Q_B to be the partially ordered set $Q_B = \{f : \omega_1 \to 2 : f$ is a partial function and $f^{-1}\{1\}$ is finite and if $\delta \in \lim \omega_1 - B$ then $\operatorname{Dom} f \cap \{\eta_\delta(i) : i < \omega\}$ is finite, and $f^{-1}\{0\} \cap \bigcup\{\{\eta_\delta(i) : i < \omega\} : \delta \in B\}$ is a finite union of sets of the form $\{\eta_\delta(i) : n < i < \omega\}$. Therefore for a generic f for Q_B we have:

if $\delta \in \beta$ then there exists $n \in \omega$ such that $f(\eta_\delta(i)) = 0$ if and only if $i \geq n$,

if $\delta \notin B$ then $\{i : f(\eta_\delta(i)) = 1\}$ is infinite.

$(Q_B, \leq)$ is $(Q_B, \subseteq)$.

3.6. *Claim. Q_B is σ-centered.*

Proof. Let $\langle r_i : i < \omega_1 \rangle$ be a sequence of ω_1-many subsets of ω such that for $i \neq j < \omega_1$ we have $r_i \neq r_j$. For every $f \in Q_B$ we define

$$W_1(f) = f^{-1}(\{1\}),$$
$$n(f) = \operatorname{Max}\{\operatorname{rest}(i, \omega) : i \in W_1(f)\} + 1,$$
$$\operatorname{rest}(i, j) = \operatorname{Min}\{\alpha : (\exists \beta)(i = j\,\beta + \alpha)\},$$
$$W_0(f) = \{i \in f^{-1}(\{0\}) : \operatorname{rest}(i, \omega) \leq n(f)\},$$
$$W(f) = W_0(f) \cup W_1(f).$$

3.7. *Fact. Let f_1, f_2 be in Q_B, let $n(f_1) = n(f_2)$, and let $f_1 \restriction W(f_1)$ and $f_2 \restriction W(f_2)$ be compatible. Then f_1 and f_2 are compatible in Q_B.*

Proof. Clear. ■

Now we define (i) $\kappa(f) =$ the minimal κ such that $\{r_i \restriction \kappa : i \in W(f)\}$ are pairwise distinct, and (ii) $r_l(f) = \{r_i \restriction \kappa(f) : i \in W_l(f)\}$, $l = 0, 1$.

3.8. *Fact. For $n_0 \in \omega$, $\kappa \in \omega$, $R_0 \in [[\omega]^{<\omega}]^{<\omega}$ and $R_1 \in [[\omega]^{<\omega}]^{<\omega}$ the set*

$$Q_B(n_0, \kappa, R_0, R_1) = \{f \in Q_B : n(f) = n_0, \kappa(f) = \kappa, r_0(f) = R_0, r_1(f) = R_1\}$$

is directed.

Proof. Let f_1, f_2 be in $Q_B(n_0, \kappa, R_0, R_1)$. By Fact 3.7 it is sufficient to show that $f_1 \restriction W(f_1)$ and $f_2 \restriction W(f_2)$ are compatible. If this is false then there exist δ, n such that $n < n_0$, $\eta_\delta(n) \in W(f_1) \cap W(f_2)$ and

$$\text{(a)}\quad f_1(\eta_\delta(n)) = 0, \qquad \text{(b)}\quad f_2(\eta_\delta(n)) = 1.$$

So $\eta_\delta(n) \in W_0(f_1) \cap W_1(f_2)$.

From (a) we have

$$r_{\eta_\delta(n)} \restriction \kappa \in r_0(f_1) = R_0 = r_0(f_2).$$

By the choice of κ, we have $f_2(\eta_\delta(n)) = 0$. And this contradicts (b). ■

Therefore we have proved that Q_B is σ-centered.

Now by induction on $i < \omega$, for every $\nu \in {}^{i}\omega$, we define $f_\nu: \omega_1 \to \{0, 1\}$.

(0) $f_{\langle\ \rangle}$ is the characteristic function of A.

$(i + 1)$ For every $\nu \in {}^{i}\omega$ and $j < \omega$ we define

$$B(\nu^\frown\langle j\rangle) = \{\delta \in \lim \omega_1 : f_\nu(\delta + j) = 0\}.$$

Using MA (σ-centered) and $Q_{B(\nu^\frown\langle j\rangle)}$ pick $f_{\nu^\frown\langle j\rangle}$ satisfying $f_\nu(\delta + j) = 0$ if and only if there exists $n \in \omega$ such that $f_{\nu^\frown\langle j\rangle}(\eta_\delta(i)) = 0$ for every $i \geq n$. Now let r_0 be

$$r_0 = \{\langle \nu, f_\nu \restriction \omega\rangle : \nu \in {}^{\omega>}\omega\}$$

Clearly r_0 is encoded by a real number.

3.9. *Claim. In* $L[r_0]$ *we can compute, for all* $\delta \in \lim \omega_1^v$.

$$F_\delta = \{\langle \nu, f_\nu \restriction \delta\rangle : \nu \in {}^{\omega>}\omega\}.$$

Proof. By induction. (0) $\delta = \omega$; this is encoded by r_0.

(1) If δ is a limit of limit ordinals, then the conclusion is clear from the inductive hypothesis

$$f_\delta = \left\{\left\langle \nu, \bigcup_{\beta<\delta}\left\{f_\nu \restriction \beta\right\}\right\rangle : \nu \subset {}^{\omega>}\omega\right\}.$$

(2) $\delta = (\gamma + \omega)$. (i) $f_\nu(\gamma + j) = 0$ if and only if

$$(\exists m \forall i \geq m)(f_{\nu^\frown\langle j\rangle}(\eta_\gamma(i)) = 0)$$

as $\eta_\gamma(i) < \gamma$. By the induction hypothesis we know the true value of

$$(\exists m \forall i \geq m)(f_{\nu^\frown\langle j\rangle}(\eta_\gamma(i)) = 0). \quad \blacksquare$$

Therefore $f_{\langle\ \rangle} \in L[r_0]$, and this implies that $A \in L[r_0]$, and this says that for every $\delta \in \omega_1^V$

$$L[r_0] \models |\delta| = \aleph_0.$$

And this finishes the proof of Claim 3.4. $\blacksquare$

3.10. *Claim. From the hypothesis of* (iii) *or* (vii) *or* (viii) *or* (ix) *we have* (ii).

Proof. In all cases the following holds: every Σ_2^1 set of reals is Lebesgue measurable.

So by Raisonnier [10] if there exists $r \in \mathbf{R}$ such that $\omega_1^{L[r]} = \omega_1$, then there exists a Σ_3^1-subset of reals which is not Lebesgue measurable, and there exists a Σ_3^1-subset of reals which does not have the property of Baire and there exists a Σ_3^1-subset of reals which is not Ramsey. $\blacksquare$

3.11. Definition. A filter $\mathscr{F}$ on ω is *unbounded* if and only if for every function $f: \omega \to \omega$ there exists an $a \in \mathscr{F}$ such that

$$(\exists^\infty \kappa)(f(\kappa) < f_a(\kappa)),$$

where $f_a: \omega \to a$ is a one-to-one and onto increasing function.

3.12. *Fact. The following are equivalent for a filter* $\mathscr{F}$ *on* ω:

(i) $\mathscr{F}$ *is unbounded.*

(ii) *For every* $f: \omega \to \omega$, *there exists an* $a \in \omega$ *such that*

$$(\exists^\infty \kappa \in \omega)([\kappa, f(\kappa)) \cap a = \varnothing).$$

Proof. (i) → (ii). Let $f: \omega \to \omega$ be an increasing function, and, for every $n \in \omega$, $n < f(n)$. Let g be defined by $g(n) = f^{2n}(n)$. As $\mathscr{F}$ is unbounded, there exists $a \in \mathscr{F}$ satisfying

$$(*) \qquad (\exists^\infty n)(g(n) < f_a(n)).$$

Suppose that there exists $\kappa_0 \in \omega$ such that for every $\kappa_0 \leq \kappa \in a$, $[\kappa, f(\kappa)) \cap a \neq 0$. Then $f_a(\kappa) \leq f^{\kappa_0 + \kappa}(k)$, and this contradicts $(*)$.

(ii) → (i). Trivial. ∎

3.13. *Fact. Every nonprincipal ultrafilter on* ω *is unbounded.* ∎

Therefore, from ZFC we can obtain unbounded filters on ω.

3.14. *Fact. Let* $\mathscr{F}$ *be an unbounded filter. Then* $\operatorname{char}\mathscr{F} = \{\operatorname{char}_a \in 2^\omega : a \in \mathscr{J}\}$ *does not have the property of Baire, where* $\operatorname{char}_a(n) = 1$ *if and only if* $n \in a$.

Proof. It is well known that if $\operatorname{char}\mathscr{F}$ has the property of Baire then it is a meager set. In order to get a contradiction let $\langle T_n : n < \omega\rangle$ be a succession of nowhere dense sets of 2^ω. We define $f: \omega \to \omega$ by setting $f(n) =$ *minimal* $\kappa \in \omega$ such that for every $\eta \in {}^n2$, there exists $\mu \in {}^2\kappa$, extending η, such that for every $l \in n$, $[\mu] \cap T_l = \varnothing$, where $[\mu] = \{h \in 2^\omega : \mu \subseteq h\}$.

Now by hypothesis there exists $a \in \mathscr{F}$ such that $\{n \in \omega : [n, f(n) + 1 \cap a = \varnothing\}$ is an infinite subset of ω; let b be infinite such that for $n \in b$, $[n, f(n) + 1) \cap a = \varnothing$. Using this, we can define, by induction on κ. $\langle \eta_\kappa : \kappa < \omega\rangle$ satisfying

(i) $\eta_\kappa \in {}^{f_b\kappa}2$,

(ii) $\kappa_1 < \kappa_2$ implies $\eta_{\kappa_1} \subset \eta_{\kappa_2}$,

(iii) $n \in a \cap f_s(\kappa)$ implies $\eta_\kappa(n) = 1$,

(iv) $[\eta_\kappa] \cap \bigcup_{i<\kappa} T_i = \varnothing$, and

(v) $\eta_0 = \operatorname{char}_{a \cap f_b(0)}$.

Now if $\operatorname{char}_c = \bigcup \eta_\kappa$, then $a \subseteq c \in \mathscr{F}$ and $\operatorname{char}_c \notin \bigcup_i T_i$. This implies that $\mathscr{F}$ is not meager. ∎

3.15. DEFINITON (MATHIAS [9]). For an infinite subset a of ω, set

$$\bar{a} = \{n : n < f_a(0)\} \cup \{n : (\exists m \in \omega)(f_a(2m + 1) < n < f_a(2m + 2))\},$$
$$\underline{a} = \{n : (\exists m)(f_a(2m) < n \leq f_a(2m + 1))\}.$$

Clearly $\bar{a} = \omega - \underline{a}$. For a filter $\mathscr{F}$ on ω, set $\bar{\mathscr{F}} = \{a : \bar{a} \in \mathscr{F}\}$ and $\underline{\mathscr{F}} = \sim\bar{\mathscr{F}}$. Clearly $\bar{\mathscr{F}} \cap \underline{\mathscr{F}} = \varnothing$, and if $a \in \bar{\mathscr{F}}$ and $n \in a$, then $a - \{n\} \in \underline{\mathscr{F}}$.

3.16. *Claim. If* $\mathscr{F}$ *is an unbounded filter on* ω, *then* $\bar{\mathscr{F}}$ *is not a Ramsey subset of* $[\omega]^\omega$.

Proof. We need to prove that every infinite subset a of ω has an infinite subset in $\bar{\mathscr{F}}$. But this is not hard, using $b \in \mathscr{F}$ satisfying $(\exists^\infty n \in \omega)((n, f_a(2n)) \cap b = \varnothing)$. ∎

3.17. THEOREM. *Let* V *be a model for* ZFC + MA(σ-*centered*), *and suppose that there exists a real number* r *such that* $\omega_1^{L[r]} = \omega_1$. *Then there exists a* Σ_3^1-*unbounded filter on* ω.

PROOF. Set $X = L[r] \cap 2^\omega$. Let $h: 2^\omega \times 2^\omega \to \omega$ be the following function:

$$h(x, y) = \inf\{n : x \restriction n \neq y \restriction n\}.$$

For a relation $R \subseteq 2^\omega \times 2^\omega$ we define

$$RX = \{n \in \omega : (\exists xy)(\langle x, y\rangle \in X^2 \cap R \wedge h(x, y) = n)\}.$$

Now we define the following filter on ω: $\mathscr{F} = \{RX: R \cap X^2$ is an equivalence relation on X with countable many equivalence classes, and R is Borel$\}$.

3.18. *Claim.* $\mathscr{F}$ *is a* Σ^1_3*-set of reals.*

Proof. $a \in \mathscr{F}$ if and only if $(\exists R \exists x)((1) \wedge (2) \wedge (3) \wedge (4) \wedge (5))$, where

(1) R is a Borel relation (Π^1_1),

(2) $(\forall x_1 x_2 x_3)(x_1 \notin L[r] \vee x_2 \notin L[r] \vee x_3 \notin L[r]$
$\vee [(\langle x_1, x_2\rangle \in R \wedge \langle x_2, x_3\rangle \in R \rightarrow \langle x_2, x_3\rangle \in R) \wedge \langle x_1, x_1\rangle \in R$
$\wedge (\langle x_1, x_2\rangle \in R \rightarrow \langle x_2, x_1\rangle \in R)](\Pi^1_2)$,

(3) $(\forall y)(y \notin L[r] \vee (\exists n)(x(n) R y)(\Pi^1_2)$

(here x encodes an ω-sequence of reals),

(4) $(\forall n)(n \notin a \vee (\exists x_1 x_2)(x_1 \in L[r] \wedge x_2 \in L[r]$
$\wedge \langle x_1, x_2\rangle \in R \wedge h(x_1, x_2) = n)(\Sigma^1_2)$,

(5) $(\forall x_1 x_2)(x_1 \notin L[r] \vee x_2 \notin L[r] \vee \langle x_1, x_2\rangle \notin R \vee h(x_1, x_2) \in a)(\Pi^1_2)$. ■

This is the best possible—see §6. ($\mathscr{F}$ is not Δ^1_3.)

3.19. *Claim* (MA(*σ-centered*)). $\mathscr{F}$ *is unbounded.*

Proof. We define the partially ordered set $Q = \{(E, Y, s)$: s is a finite subset of ω, and Y is a finite subset of X, and $E: Y \rightarrow \omega$ satisfies $(h(x, y) \leq \mathrm{Max}(s) \wedge E(x) = E(y) \rightarrow h(x, y) \in s)\}$. The order is given by $(E_1, Y_1, s_1) \leq (E_2, Y_2, s_2)$ if and only if $E_1 \subseteq E_2$ and $Y_1 \subseteq Y_2$ and $s_1 = s_2 \cap (\mathrm{Max} + 1)$.

3.20. *Fact.* $\langle Q, \leq\rangle$ *is σ-centered.*

Proof. For every finite $s \subseteq \omega$, $k = \max(s) + 1$, $n \in \omega$ and $\mu_l \subseteq {}^k 2$, for $l < n$, satisfying $(\forall \eta \neq \mu \in u_l)(h(\eta, \mu) \in s)$, set

$$Q_{s, \langle u_l : l < n\rangle} = \{(E, Y, s): (x \in Y \rightarrow E(x) < n) \wedge (E(x) = l \rightarrow x \restriction \max(s) + 1 \subset u_l\}.$$

Then $Q_{s, \langle u_l : l < n\rangle}$ is a directed subset of Q and $Q = \bigcup\{Q_{s, \langle u_l : l < n\rangle}\}$. ■

Now let f be an increasing function from ω to ω, and set

$$D^m_f = \{(E, Y, s): (\exists n)(m < n \wedge s \cap [n, f(n)) = \varnothing \wedge f(n) < \max(s))\}.$$

Then D^m_f is a dense subset of Q, and, for every $x \in X$, $D_x = \{(E, Y, s): x \in Y\}$ is a dense subset of Q. Applying MA(σ-centered), we see that there exist $E: X \rightarrow \omega$, and $a \subseteq \omega$ such that for every x, $y \in X$ if $E(x) = E(y)$ then $h(x, y) \in a$; and this implies that $a \in \mathscr{F}$, (MA(σ-centered) implies $E = R \cap x^2$ for an appropriate Borel relation). By using D^m_f we prove that

$$(\exists^\infty n \in a)([n, f(n)) \cap a = \varnothing). \quad ■$$

Now using 3.14, 3.16 and 3.17, we prove the following fact.

3.21. *Claim. If* (iv) or (v) *of* 3.1 *holds, then* (ii) *holds in the model.* ■

This finishes the proof of the theorem. We remark only that we have proved that from (i) we can obtain a model where (ii)–(ix) hold simultaneously for every ordinal definible set of reals.

§4. MA(K) and weakly compact cardinals.

4.1. THEOREM. *The following theories are equiconsistent.*

(i) ZFC + *there exists a weakly compact cardinal.*

(ii) ZFC + MA(K) + $(\forall r \in \mathbf{R})(\omega_1^{L[r]} < \omega_1)$.

(iii) ZFC + MA(K) + *every projective set of reals is Lebesgue measurable* (Σ_3^1).

(iv) ZFC + MA(K) + *every projective set of reals has the property of Baire* (Σ_3^1).

(v) ZFC + MA(K) + *every projective set of reals is Ramsey* (Σ_3^1).

PROOF. The proof of (i) → (ii), (iii), (iv), (v) was given by Harrington and Shelah [2]. The proof that (iii) or (iv) or (v) implies (ii) is similar to §3. We need only to show that (ii) implies (i). In order to give this, we will prove that the coding forcing given in Harrington and Shelah [2] satisfies Knaster's condition. Clearly this is sufficient. The coding forcing is essentially the forcing notion which forces that an Aronszajn tree is special. We recall this forcing notion.

4.2. DEFINITION. Let T be an Aronszajn tree on $\aleph_1$. Let $P(T)$ be the following partially ordered set:

$$p \in P(T) \text{ if and only if } p\colon x \to Q = \text{the rationals},$$

where $x \subseteq T$ is finite and p is order preserving; $P(T)$ is ordered by extension.

4.3. *Fact. If T is an Aronszajn tree on $\aleph_1$ and if for every uncountable $R \subseteq T$ there exists an uncountable $R' \subseteq R$ such that every pair of members of R' are incompatible, then $P(T)$ satisfies Knaster's condition.*

Proof. Given an uncountable subset of $P(T)$, by a delta-system argument we can find $\langle p_\alpha\colon \alpha < \omega_1\rangle$ such that $\operatorname{dom}(p_\alpha) = \hat{x} \cup x_\alpha$, where $\hat{x}$, $\langle x_\alpha\colon \alpha < \omega_1\rangle$ are pairwise disjoint. By thinning this sequence if necessary, we may assume that, for $\alpha < \beta$, $(a \in x_\alpha, b \in x_\beta \Rightarrow \text{height}(a) < \text{height}(b))$. We may also assume that $p_\alpha \restriction \hat{x} = p_\beta \restriction \hat{x}$, and that the x_α's have the same cardinality, say n. Let $a(1,\alpha), \ldots, a(n,\alpha)$ list the elements of x_α, and for $i \in [1,n]$ let

$$p_\alpha(a(i,\alpha)) = p_\beta(a(i,\beta)).$$

It is not hard to see that it is sufficient to show that for $0 \le i \ne j \le n$ the sets $\langle a(i,\alpha)\colon \alpha < \omega_1\rangle \cup \langle a(j,\alpha)\colon \alpha < \omega_1\rangle$ are pairwise incompatible. By our hypothesis we can obtain that, for $i \in [1,n]$, $\langle a(i,\alpha)\colon \alpha < \omega_1\rangle$ are pairwise incompatible. If $\langle a(i,\alpha)\colon \beta < \alpha < \omega_1\rangle \cup \langle a(j,\alpha)\colon \beta < \alpha < \omega_1\rangle$ are not pairwise incompatible, for every $\beta < \omega_1$, then we can find $\langle \beta_\alpha\colon \alpha < \omega_1\rangle$ and $\langle \beta'_\alpha\colon \alpha < \omega_1\rangle$ such that $\alpha_1 < \alpha_2$ implies $\beta_{\alpha_1} < \beta'_{\alpha_1} < \beta_{\alpha_2} < \beta'_{\alpha_2}$ and $a(i,\beta_{\alpha_1}) <_T a(j,\beta'_{\alpha_1})$. We claim that the sets

$$\langle a(i,\beta'_\alpha)\colon \alpha < \omega_1\rangle \cup \langle a(j,\beta'_\alpha)\colon \alpha < \omega_1\rangle$$

are pairwise incompatible. If this is false, then there exists $\alpha_1 < \alpha_2$ such that either (i) or (ii) holds:

$$\text{(i)}\quad a(i,\beta'_{\alpha_1}) <_T a(j,\beta'_{\alpha_2}); \qquad \text{(ii)}\quad a(j,\beta'_{\alpha_1}) <_T a(i,\beta'_{\alpha_2}).$$

However, (i) implies that $a(i,\beta'_{\alpha_1})$ and $a(i,\beta_{\alpha_2})$ are compatible, and (ii) implies that $a(i,\beta_{\alpha_1})$ and $a(i,\beta'_{\alpha_2})$ are compatible, and this is a contradiction. ■

4.4. *Fact. Let V be a model for ZFC; $T \in V$ an Aronszajn tree on $\aleph_1$. Suppose that for every $V' \supseteq V$, if, in V', T has a branch, then, in V', $\aleph_1^V$ is countable. Then, in V, $P(T)$ has the Knaster condition.*

Proof. By 4.3 it is sufficient to show that for every uncountable $R \subseteq T$, there exists an uncountable $R' \subseteq R$ of pairwise incompatible elements. If this does not hold, then we can show that there exists a Souslin tree $T' \subseteq T$. And forcing with $\langle T_j \leq_T \rangle$ we obtain a c.c.c. extension of V in which T has a branch. ■

In order to finish the proof of 4.1 we remark that the Aronszajn tree used in the coding process of Harrington and Shelah [2] satisfies the condition of 4.4. ■

§5. MA **and a Δ^1_3-selective filter on ω.**

5.1. Definition. (i) A filter $\mathscr{F}$ on ω is *selective* if and only if for every $a_n \in \mathscr{F}, n \in \omega$, there exists $\{i_n : n \in \omega\} \in \mathscr{F}$ such that $i_n \in a_n$ for every $n \in \omega$.

(ii) A filter $\mathscr{F}$ on ω is *rapid* if and only if for every $f: \omega \to \omega$ there exists an $a \in \mathscr{F}$ such that, for every $n \in \omega$, $\operatorname{card}(f(n) \cap a) < n$. Without loss of generality we suppose that the filter of cofinite subsets of ω is contained in all our filters.

5.2. *Fact. Every selective filter on ω is a rapid filter on ω.*

Proof. Easy. ■

Theorem. *If there exists a $\Sigma^1_n(\Pi^1_n, \Delta^1_n)$ rapid filter, then*

(a) *there exists a $(\Sigma^1_n(\Pi^1_n, \Delta^1_n)$-subset of reals which is not Lebesgue measurable,*

(b) *there exists a $\Sigma^1_n(\Pi^1_n, \Delta^1_n)$-subset of reals which does not have the property of Baire, and*

(c) *there exists a $\Sigma^1_n(\Pi^1_n, \Delta^1_n)$-subset of reals which is not Ramsey.*

Proof. Parts (a) and (b) were given by Talagrand [13]; part (c) by Mathias [9]. ■

Further information on rapid filters can be find in Ihoda [5], and their connection with κ_σ-regularity appears in Ihoda [3].

5.3. Theorem. *Let V be a model for* ZFC + MA, *and suppose that there exists a real number r such that $\omega_1^{L[r]} = \omega_1$. Then there exists a Δ^1_3-selective filter on ω.*

Proof. (This proof was inspired by a similar construction given by Raisonnier [10].) Set $X = L[r] \cap 2^\omega$. Let $h: 2^\omega \times 2^\omega \to \omega$ be the function

$$h(x, y) = \inf\{n : x \upharpoonright n \neq y \upharpoonright n\}.$$

For a relation $R \subseteq 2^\omega \times 2^\omega$ we define

$$RX = \{n \in \omega : (\exists xy)(\langle x, y\rangle \in X^2 \cap R \wedge h(x, y) = n)\}.$$

Now we define the following filter on ω: $\mathscr{F} = \{RX : R \cap X^2$ is an equivalence relation on X with countable many equivalence classes, and R is Borel$\}$.

5.4. *Claim* (MA). *$\mathscr{F}$ is a Δ^1_3-set of reals.*

Proof. (i) $a \in \mathscr{F}$ if and only if $(\exists R \exists x)((1) \wedge (2) \wedge (3) \wedge (4) \wedge (5))$, where

(1) $$R \text{ is a Borel relation } (\Pi^1_2),$$

(2) $$(\forall x_1 x_2 x_3)(x_1 \notin L[r] \vee x_2 \notin L[r] \vee x_3 \notin L[r] \vee [(\langle x_1, x_2\rangle \in R \vee \langle x_2, x_3\rangle \in R \to \langle x_1, x_3\rangle \in R) \wedge (\langle x_1, x_1\rangle \in R) \vee (\langle x_1, x_2\rangle \in R \to \langle x_2, x_1\rangle \in R)])(\Pi^1_2),$$

(3) $$(\forall y)(y \notin L[r] \vee (\exists n)(x(n) R y))(\Pi^1_2)$$

(here x encodes a ω-sequence of reals),

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(4) $(\forall n)(n \notin a \vee (\exists x_1 x_2)(x_1 \in L[r] \wedge x_2 \in L[r] \wedge \langle x_1, x_2\rangle \in R \wedge h(x_1, x_2) = n))(\Sigma^1_2)$,

(5) $(\forall x_1 x_2)(x_1 \notin L[r] \vee x_2 \notin L[r] \vee \langle x_1, x_2\rangle \notin R \vee h(x_1, x_2) \in a)(\Pi^1_2)$.

Therefore $\mathscr{F}$ is a Σ^1_3-set of reals.

Now we need to show that $\sim\mathscr{F}$ is also a Σ^1_3-set of reals.

5.5. *Claim* (MA). *For a subset $a \subseteq \omega$, the following assertions are equivalent:*

(1) *There is no equivalence relation R on X satisfying* $(*)$ $(\forall x, y)(\langle x, y\rangle \in R \rightarrow h(x, y) \in a)$ *and R has countable many equivalence classes.*

(2) *If $P_a = \{f: f$ is a finite function from X to ω such that $f(x) = f(y) \rightarrow h(x, y) \in a\}$, then $\langle P_a, \subseteq\rangle$ does not satisfy the countable chain condition.*

(3) *There exists $\langle\langle x^\alpha_l: l < n\rangle: \alpha < \omega_1\rangle$ such that*

$$(\forall \alpha < \beta < \omega_1)(\exists l)(h(x^\alpha_l, x^\beta_l) \notin a).$$

Proof. (1) $\rightarrow$ (2) is clear from MA.

(2) $\rightarrow$ (3). By hypothesis there exists $\langle f_\alpha: \alpha < \omega_1\rangle \subseteq P_a$ such that $\alpha < \beta < \omega_1$ implies $f_2 \cup f_\beta \notin P_a$. Without loss of generality, there exists $\kappa < \omega$ and

(a) $\operatorname{Dom} f_\alpha = \langle x^\alpha_1, \ldots, x^\alpha_m\rangle$,

(b) $\langle x^\alpha_1 \restriction \kappa, \ldots, x^\alpha_m \restriction \kappa\rangle$ are pairwise distinct,

(c) $\alpha \neq \beta$ implies $\langle x^\alpha_1 \restriction \kappa, \ldots, x^\alpha_m \restriction \kappa\rangle = \langle x^\beta_1 \restriction \kappa, \ldots, x^\beta_m \restriction \kappa\rangle$,

(d) $\operatorname{Dom} f_\alpha \cap \operatorname{Dom} f_\beta = \varnothing$, and

(e) for every $\alpha < \beta < \omega_1$ there exists $l < m$ such that $h(x^\alpha_l, x^\beta_l) \notin a$.

Proof of (e). For every α set $\langle n^\alpha_1, \ldots, n^\alpha_m\rangle$ such that, for every $l < m$, $f_\alpha(x^\alpha_l) = n^\alpha_l$. This defines a partition of ω_1 in ω-many equivalence classes. Therefore there exists $A \subseteq \omega_1$, $|A| = \aleph_1$, and for every $\alpha < \beta$ in A we have that

$$\langle n^\alpha_1, \ldots, n^\alpha_m\rangle = \langle n^\beta_1, \ldots, n^\beta_m\rangle.$$

By hypothesis $f_\alpha \cup f_\beta \notin P$, so there exists $l < m$ satisfying $h(x^\alpha_l, x^\beta_l) \notin a$.

(3) $\rightarrow$ (1). Let $\langle\langle x^\alpha_l: l < m\rangle: \alpha < \omega_1\rangle$ be given by (3). Suppose that there exists an equivalence relation R on X witnessing to $(*)$. Set

$$F(\langle x^\alpha_l: l < m\rangle) = \langle [x^\alpha_1]_R, \ldots, [x^\alpha_m]_R\rangle.$$

As R has countable many classes, there exists $\alpha < \beta < \omega_1$ satisfying

$$\langle [x^\alpha_1]_R, \ldots, [x^\alpha_m]_R\rangle = \langle [x^\beta_1]_R, \ldots, [x^\beta_m]_R\rangle.$$

By hypothesis there exists $l < m$ such that $h(x^\alpha_l, x^\beta_l) \notin a$, and this contradicts the choice of R. ■

Now, using 5.5(3) and MA, we have that

(ii) $a \notin \mathscr{F}$ if and only if

$$(\exists A \subseteq \aleph_1)(\langle L_{\aleph_1}[r], \in, a, A\rangle \models \phi),$$

where ϕ is some first-order sentence. By having A absorb some Skolem function for ϕ we may assume that ϕ is Π_1. By MA, any $A \subseteq \aleph_1$ can be coded by a real, say c, and the uncoding process is Δ_1 over $\langle L_{\aleph_1}[r], \in, c\rangle$. So $a \notin \mathscr{F}$ if and only if $(\exists c \subseteq \omega)(c$ codes $A \subseteq \aleph_1$ and $\langle L_{\aleph_1}[r], \in, a, A\rangle \models \phi)$; and this expression is seen to be Σ^1_3. This concludes the proof of Claim 5.4. ■

5.6. *Claim* (MA). *If $a_n \in \mathscr{F}$ for $n < \omega$, then there exists $a = \{\kappa_n : n < \omega\} \in \mathscr{F}$ such that $\kappa_n \in a_n$ for $n < \omega$. Therefore, under* MA, *$\mathscr{F}$ is a selective filter on ω.*

Proof. Without loss of generality, $a_n \supseteq a_{n+1}$ for every $n < \omega$. Let E_n be an equivalence relation witnessing $a_n \in \mathscr{F}$. Set $Q = \{\langle f, Y, \langle \kappa_l : l < \omega\rangle\rangle : Y \subseteq X$ and f is a finite function and $f: Y \to \omega$ and ($y_1 \neq y_2 \in Y$ and $f(y_2) = f(y_2)$ implies $h(y_1, y_2) \in \langle \kappa_l : l < n\rangle$) and, for $l < n$, $\kappa_l \in a_l\}$.

5.7. *Fact. Q satisfies the countable chain condition.*

PROOF. Let $\langle\langle f_\alpha, Y_\alpha, <\kappa_l^\alpha : l < n_\alpha\rangle\rangle : \alpha < \omega_1\rangle$ be a subset of Q. Then we can assume

$$Y_\alpha = \{y_1^\alpha, \ldots, y_m^\alpha\}, \quad \text{where } m \text{ does not depend on } \alpha, \tag{1}$$

$$n_\alpha = n, \tag{2}$$

$$\text{there exists } j \text{ such that } \langle y_1^\alpha \restriction j, \ldots, y_m^\alpha \restriction j\rangle \text{ are pairwise distinct,} \tag{3}$$

and if $\alpha, \beta \in \omega_1$ then

$$\langle y_1^\alpha \restriction j, \ldots, y_m^\alpha \restriction j\rangle = \langle y_1^\beta \restriction j, \ldots, y_m^\beta \restriction j\rangle, \tag{4}$$

$$\langle f_\alpha(y_1^\alpha), \ldots, f_\alpha(y_m^\alpha)\rangle = \langle f_\beta(y_1^\beta), \ldots, f_\beta(y_m^\beta)\rangle, \tag{5}$$

$$\langle \kappa_l^\alpha : l < n\rangle = \langle \kappa_l^\beta : l < n\rangle \stackrel{\text{def}}{=} \langle \kappa_l : l < n\rangle, \tag{6}$$

$$\langle y_l^\alpha \restriction E_{n+m+8} : 1 \le l \le m\rangle = \langle y_l^\beta \restriction E_{n+m+8} : 1 \le l \le m\rangle. \tag{7}$$

Now let $\alpha \neq \beta$ be in ω_1, and set

$$Y \stackrel{\text{def}}{=} Y_1 \cup Y_2, \qquad f \stackrel{\text{def}}{=} f_1 \cup f_2,$$

$$\langle \kappa_l : n \le l < n + m + 8\rangle \stackrel{\text{def}}{=} \{h(y_l^\alpha, y_l^\beta) : 1 \le l \le m\}.$$

(8) *Fact.* $\{\kappa(y_l^\alpha, y_l^\beta) : 1 \le l \le m\} \subseteq a_{n+m+8}$.

Proof. By (7), $y_l^\alpha E_{n+m+8} y_l^\beta$ for $1 \le l \le m$. This implies that $h(y_l^\alpha, y_l^\beta) \in a_{n+m+8}$ for $1 \le l \le m$. ■

(9) *Fact.* $1 \le l_1 \le l_2 \le m$ *implies* $h(y_{l_1}^\alpha, y_{l_2}^\beta) \in \langle \kappa_l : l < m\rangle$.

Proof. $h(y_{l_1}^\alpha, y_{l_2}^\beta) = h(y_{l_1}^\alpha, y_{l_2}^\alpha) \in \langle \kappa_l : l < m\rangle$. ■

Therefore, using that $a_{n+m+8} \subseteq a_l$ for $n \le l < n + m + 8$, we have that $\langle f, Y, \langle \kappa_l : l < n + m + 8\rangle\rangle$ is a condition in Q extending both, $\langle f_\alpha, Y_\alpha, \langle \kappa_l : l < n\rangle\rangle$ and $\langle f_\beta, Y_\beta, \langle \kappa_l : l < n\rangle\rangle$. ■

Now, using MA, for this Q we can $a \in \mathscr{F}$ satisfying the requirement of Claim 5.6. ■

5.8. COROLLARY. (i) MA *does not imply every Δ_3^1-set of reals is Lebesgue measurable.*

(ii) MA *does not imply every Δ_3^1-set of reals is Ramsey.*

(iii) MA *does not imply every Δ_3^1-set of reals has the property of Baire* (*Harrington and Shelah*).

5.9. COROLLARY. *The following theories are equiconsistent:*

(i) ZFC + MA + *there exists a weakly compact cardinal.*

(ii) ZFC + MA + "*every Δ_3^1-set of reals is Lebesgue measurable*".

(iii) ZFC + MA + "*every Δ_3^1-set of reals is Ramsey*".

(iv) ZFC + MA + "*every Δ_3^1-set of reals has the property of Baire*" (*Harrington and Shelah* [2]).

§6. MA(I) and Δ^1_3-sets of reals.

6.1. THEOREM. *The following theories are equiconsistent:*

(i) ZFC.

(ii) ZFC + MA(I) + *every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey.*

PROOF. (ii) $\to$ (i) is clear.

(i) $\to$ (ii). Let $V = L$. Let $\bar{Q} = \langle P_\beta; \mathbf{Q}_\beta: \beta < \omega_2\rangle$ be an ω_2-stage iterated forcing notion satisfying:

$$P_\beta \Vdash \text{“}\mathbf{Q}_\beta \in I\text{”} \quad \text{for every } \beta < \omega_2, \tag{1}$$

$$\text{if } \beta = \bigcup\beta \neq 0 \text{ then } P_\beta \text{ is the directed limit of } \langle P_\alpha; \mathbf{Q}_\alpha: \alpha < \beta\rangle, \tag{2}$$

$$P_{\omega_2} \Vdash \text{“MA}(I)\text{”}. \tag{3}$$

6.2. *Claim. Suppose that $\beta < \alpha < \omega_2$, and $\mathbf{r}$ is a P_α-name of a generic object for $\mathbf{Q}_\beta$ over V^{P_β}. Denote by $P_{\alpha-\beta}[\mathbf{r}]$ the interpretation of $P_{\alpha-\beta+1}$ in $V^{P_\beta * \mathbf{r}}$,* i.e. *$P_{\alpha-\beta}[\mathbf{r}]$ is a P_α-name of a forcing notion. Then*

$$P_\alpha \Vdash \text{“}P_{\alpha-\beta}[\mathbf{r}] \in I\text{”}$$

($\mathbf{r}$ *also is viewed as the Boolean algebra generated by this name over P_β).*

Proof. $P_\alpha \cong P_\beta * \mathbf{r} * (P_\alpha \backslash P_\beta * \mathbf{r})$, where

$$P_\beta * \mathbf{r} \Vdash \text{“}P_\alpha \backslash P_\beta * \mathbf{r} \text{ satisfies c.c.c.”}$$

We know that $P_{\alpha-\beta}[\mathbf{r}]$ has a $P_\beta * \mathbf{r}$-name and

$$P_\beta * \mathbf{r} \Vdash \text{“}P_{\alpha-\beta}[\mathbf{r}] \in I\text{”}$$

(we will prove that $P_{\beta+1} \Vdash$ “$P_{\alpha-\beta+1} \in I$”). Therefore

$$P_\beta * \mathbf{r} * (P_\alpha \backslash P_\beta * \mathbf{r}) * \mathbf{R} \Vdash \text{“}P_{\alpha-\beta}[\mathbf{r}] \text{ satisfies c.c.c.”}$$

for every $\mathbf{R}$ a P_α-name of a c.c.c. forcing notion, and this implies that

$$P_\alpha \Vdash \text{“}P_{\alpha-\beta}[\mathbf{r}] \in I\text{”}. \quad \blacksquare$$

So we need to show the following fact.

6.3. *Claim.* $P_{\beta+1} \Vdash$ “$P_{\alpha-\beta+1} \in I$”.

Proof. By induction over $\alpha - \beta + 1 = \gamma$. The case $\gamma = 1$ is the construction of P_α.

$\gamma = \bigcup\gamma \neq 0$. If $\mathrm{cof}(\gamma) = \omega$, then by directed limit. If $\mathrm{cof}(\gamma) = \omega_1$ then use a Δ-system.

$\gamma = \delta + 1$. By hypothesis we know that

$$\text{(i)} \quad P_{\beta+1} \Vdash \text{“}\mathbf{P}_\delta \in I\text{”}, \qquad \text{(ii)} \quad P_{\beta+1} * P_\delta \Vdash \text{“}\mathbf{Q}_{\beta+1+\delta} \in I\text{”}.$$

Let $\mathbf{R}$ be a $P_{\beta+1}$-name of a c.c.c.-forcing notion. We need to prove that

$$P_{\beta+1} * \mathbf{R} \Vdash \text{“}\mathbf{P}_\delta * \mathbf{Q}_{\beta+1+\delta} \text{ satisfies c.c.c.”}$$

By (ii), $P_{\beta+1} * \mathbf{P}_\delta * \mathbf{R} \Vdash$ “$\mathbf{Q}_{\beta+1+\delta}$ satisfies c.c.c.”; but, as $\mathbf{R}$ and $\mathbf{P}_\delta$ belong to $V^{P_{\beta+1}}$,

$$P_{\beta+1} * \mathbf{P}_\delta * \mathbf{R} \cong P_{\beta+1} * \mathbf{R} * \mathbf{P}_\delta,$$

and using (ii) we obtain the conclusion. $\blacksquare$

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(4) For every β there exists $\alpha > \beta$ such that $P_\alpha \Vdash$ "$\mathbf{Q}_\alpha$ is random real forcing".

(5) For every β there exists $\alpha > \beta$ such that $P_\alpha \Vdash$ "$\mathbf{Q}_\alpha$ is Cohen real forcing".

(6) For every β there exists $\alpha > \beta$ such that $P_\alpha \Vdash$ "$\mathbf{Q}_\alpha$ is a Mathias real from a Ramsey ultrafilter".

(7) $\bar{Q}$ is sufficiently generic.

Following Ihoda [3], using (1)–(7) and 6.2 we can show that:

$P_{\omega_2} \Vdash$ "*every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey*". ■

From Harrington and Shelah [2] we can extract the following corollary: "MA(I) + *every Aronszajn tree is special* + $(\exists r \in \mathbf{R})(\omega_1^{L[r]} = \omega_1)$ *implies that there exists a Δ^1_3-set of reals which does not have the property of Baire.*"

We do not know if MA(I) + "every Aronszajn tree is special" implies MA.

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