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Almost free groups and Ehrenfeucht–Fraïssé games for successors of singular cardinals

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Abstract

We strengthen nonstructure theorems for almost free Abelian groups by studying long Ehrenfeucht–Fraïssé games between a fixed group of cardinality λ and a free Abelian group. A group is called ε-game-free if the isomorphism player has a winning strategy in the game (of the described form) of length $\varepsilon \in \lambda$. We prove for a large set of successor cardinals $\lambda = \mu^+$ the existence of nonfree $(\mu \cdot \omega_1)$-game-free groups of cardinality λ. We concentrate on successors of singular cardinals. © 2002 Published by Elsevier Science B.V.

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1. Introduction

The problem of possible cardinals carrying a nonfree almost free Abelian group has already a long history, see e.g. [3]. Recall that if G is a group of cardinality λ, G is $L_{\infty\lambda}$-equivalent to a free Abelian group if and only if G is strongly λ-free [2].[3] Recall also that the $L_{\infty\lambda}$-equivalence is characterized by an Ehrenfeucht–Fraïssé game of length game of length ω ($<\lambda$ elements at the time). For an ordinal ε, a group

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G is called ε-game-free, if the "isomorphism player" has a winning strategy in the Ehrenfeucht–Fraïssé game of length ε between G and a free Abelian group (countable many elements at the time). Let μ be a cardinal. We study existence of nonfree groups of cardinality μ^+ which are ε-game-free with $\mu \leqslant \varepsilon < \mu^+$. We concentrate our attention on the case that μ is singular. This work continues [19], where the case that λ is a successors of a regular cardinal is studied. The following result is the main theorem of the paper (presented in a more general form in Section 5).

Theorem 1.1. *Let $\mathscr{C}_{\aleph_0}$ denote the smallest set of cardinals such that $\aleph_0$ is in $\mathscr{C}_{\aleph_0}$ and $\mathscr{C}_{\aleph_0}$ is closed under the operations $\lambda \mapsto \lambda^+$ and $\langle \lambda, \kappa \rangle \mapsto \lambda^{+\kappa+1}$. For every cardinal λ of the form μ^+ in $\mathscr{C}_{\aleph_0}$ with $\mu \geqslant \aleph_2$, there exists a nonfree $(\mu \cdot \omega_1)$-game-free group of cardinality λ.*

For an ordinal ε and λ of the form μ^+, $(\mu \cdot \varepsilon)$-game-freeness (the ordinal multiplication) of a group G implies that G is equivalent to a free Abelian group with respect to a "deep" infinitary language $L^{\theta}_{\infty\lambda}$, even a stronger language than $L_{\infty\lambda}$ [7,8,11]. So our results can be interpreted as strengthened nonstructure theorems for almost free Abelian groups.

Shelah proved in [14] that the question of existence of nonfree almost free Abelian groups is equivalent to a purely set theoretical question concerning existence of nonfree almost free families of countable sets. A family is almost free if all the subfamilies of cardinality strictly less than the whole family has a transversal. A transversal for a subfamily is an injective choice function whose domain is the subfamily. It is nowadays a standard custom to write that $\mathrm{NPT}(\lambda, \kappa)$ holds if there exists a family $\mathscr{S}$ of sets such that the elements of $\mathscr{S}$ have cardinality $\leqslant \kappa$, $\mathscr{S}$ is almost free, $\mathscr{S}$ is not free, and $\mathscr{S}$ has cardinality λ (in some papers $\mathrm{NPT}(\lambda, \kappa^+)$ is used instead of $\mathrm{NPT}(\lambda, \kappa)$). For some history of $\mathrm{NPT}(\lambda, \kappa)$ see, e.g., [16, II Section 0] and [9, Section 0]. Recall that $\mathrm{NPT}(\lambda, \kappa)$ fail for all $\lambda > \kappa$ if λ is a singular cardinal [12]. Recall also that by Shelah [17, Section 1], for every cardinal λ, $\mathrm{NPT}(\lambda, \aleph_0)$ holds iff there exists a nonfree, λ-separable group of cardinality λ.

A transversal game on a family $\mathscr{S}$ of sets is a two players game, where on each round i the first player chooses a set s_i from $\mathscr{S}$ and the second player, also called transversal player, must answer with an element x_i from s_i so that x_i is distinct from the elements x_j, $j < i$, the second player has chosen on the earlier rounds (Definition 2.2). That means that the transversal player must be able to choose an "extendable" transversal whose domain contains at least the sets chosen by the first player. A family $\mathscr{S}$ is called ε-game-free if the transversal player has a winning strategy in the transversal game of length ε on $\mathscr{S}$. As can be expected, the question "does there exist a nonfree ε-game-free group of cardinality λ", for various ε, is very closely related to the question "does there exist a nonfree ε-game-free family of countable sets having cardinality λ". For a detailed exposition of a transformation of an ε-game-free family into an ε-game-free group see [19, Section 4.1].

Let χ denote the first cardinal fixed point (the first cardinal κ with $\aleph_\kappa = \kappa$). Our main target is to prove, without any assumptions beyond ZFC, that for a given $\theta < \chi$, there are nonfree $\mu \cdot \theta$-game-free groups of cardinality μ^+ for unbounded many singular cardinals μ below χ. In [9, Section 1] it is proved that NPT$(\mu^+, \aleph_0)$ holds for unbounded many singular cardinals μ below χ. The main theorem follows from this result together with a proposition on "canonical" families of countable sets (Section 3).

We present definitions of λ-sets and λ-systems in Section 2 even though those could be found from [14, Section 3], [17, Appendix], or [3, VII.3A]. The reason for this is that we want the new terminology "canonical form of a λ-set" and "NPT$(\lambda, \aleph_0)$-skeletons of type $\bar{\lambda}$" (or briefly "incompactness skeletons") to be very clear for the reader. Moreover, we introduce a variant of these, called "NRT$(\lambda, \aleph_0)$-skeletons of type $\bar{\lambda}$", in Section 4.

In Section 3, we prove the main proposition. Namely, we show that any family $\mathscr{S}$ of countable sets of cardinality μ^+ with μ a singular cardinal, whose canonical form fulfills a certain "cofinality" condition, can be transformed into a family $\mathscr{S}'$ of countable sets such that $\mathscr{S}'$ is a nonfree μ-game-free family having cardinality μ^+. The reader may wonder why the conclusion of Proposition 3.1 is ε-game-free for every $\varepsilon < \mu$ instead of μ-game-free. The answer is that by [19, Lemma 4.23], ε-game-freeness for every $\varepsilon < \mu$ implies μ-game-freeness, when μ is a singular cardinal.

By [17, Section 3] it is consistent, relative to existence of two weakly compact cardinals, that for some uncountable regular cardinals $\kappa < \lambda$, both NPT(λ, κ) and NPT$(\kappa, \aleph_0)$ hold, even though, NPT$(\lambda, \aleph_0)$ does not hold. Hence, the ordinary notion of "almost free" is not strong enough for the transitivity conclusion: for all $\lambda_1 > \lambda_2 > \lambda_3$, if NPT$(\lambda_1, \lambda_2)$ and NPT(λ_2, λ_3) hold, then NPT(λ_1, λ_3) hold. In Section 4, we present special kind of families, called $R(\kappa)$-families, for which an analogical transitivity property hold (R refers to the notion NRT(λ, κ), which will be an analog of NPT(λ, κ)): for all $\lambda_1 > \lambda_2 > \lambda_3$, NRT$(\lambda_1, \lambda_2)$ and NRT(λ_2, λ_3) implies NRT(λ_1, λ_3). Note that if $\mathscr{S}$ exemplifies NPT(λ, κ), and $\mathscr{S}$ satisfies a stronger freeness notion, say the following property:

> whenever $\mathscr{R}$ is a subset of $\mathscr{S}$ of cardinality $< \lambda$, there are pairwise disjoint sets $\langle r_s \mid s \in \mathscr{R}$ such that $s \setminus r_s$ has cardinality $< \kappa$ for every $s \in \mathscr{R}$;

then $\mathscr{S}$ and any family exemplifying NPT$(\kappa, \aleph_0)$ can be amalgamated to get a family exemplifying NPT$(\lambda, \aleph_0)$. The point of introducing $R(\kappa)$-families and "$R_{\mathscr{I}}(\kappa)$-freeness" in Section 4 is that the existence of such a family is also necessary to build a canonical example of a family of countable sets. In other words, the largest nicely incompact subset of χ (which denotes the first cardinal fixed point), defined in [17, Section 2], coincides with the smallest set of cardinals below χ which contains $\aleph_0$ and is closed under "amalgamation of $R(\kappa)$-families" (Definition 5.1).

The main theorem of the paper is presented in Section 5. Namely we link together the pieces proved in [19, Section 4], Sections 3 and 4. For all nonzero $n < \omega$, it is possible to build nonfree $(\omega_n \cdot \omega_{n-1})$-game-free families of cardinality $\aleph_{n+1}$ level by level using the "old methods", see [19, Section 2]. To get an example of nonfree μ-game-free group of cardinality μ^+ with $\mu = \aleph_\omega$, one needs Proposition 3.1 and [9, Theorem 4 of Section 1]. To get examples for $\aleph_\omega < \mu < \aleph_{\omega+\omega}$, one may use an example for $\aleph_\omega$ and apply [19, Lemma 4.29], which just says that a certain type of an NPT$(\mu_2^+, \aleph_0)$-skeleton is $(\mu_2 \cdot \mu_1)$-game-free, if $\mu_2 = \mu_1^+$ and the "previous level" is μ_1-game-free (a group whose Γ-invariant consists of ordinals of cofinality $\geqslant \mu_2$ is always μ_2-game-free, but not necessarily $\mu_2 + \mu_2$-game-free [19, Section 2]). By [9, Theorem 4 of Section 1] and Proposition 3.1, one get examples for even larger μ's, e.g., $\mu = \aleph_{\omega_n}$ for any $n < \omega$. In fact it follows that such examples exist for unbounded many μ's below the first cardinal fixed point.

Suppose that G is an Abelian group of cardinality μ^+ whose Γ-invariant $\Gamma(G)$ contains stationary many θ cofinal ordinals, i.e., for some filtration $\langle G_\alpha \mid \alpha < \mu^+ \rangle$ of G, the set $\mathrm{cof}(\theta) \cap \{\alpha < \mu^+ \mid G_{\alpha+1}/G_\alpha$ is not free$\}$ is stationary in μ^+. Assume, for a while, that $\Gamma(G) \cap \mathrm{cof}(\theta)$ is in the ideal $I[\mu]^+$ of "all good subsets of μ^+" (see, e.g., [16, Analytical Guide Section 0]). Then by [19, Lemma 2.11], $(\mu \cdot (\theta + 1))$-game-freeness implies freeness for groups G of cardinality μ^+. Hence, Theorem 5.6 is very close to the optimal result provable from ZFC alone.

What happens if μ is singular and $\Gamma(G)$ is not in $I[\mu]^+$? (For the regular case see [19, Propositioin 3.4].) In Section 6, we collect together some facts about "good points with respect to a scale for μ" and we prove that, relative to the existence of a supercompact cardinal, it is possible to have a nonfree group G of cardinality μ^+ (e.g. $\mu = \aleph_{\omega_n}$ for any $n < \omega$), so that G is ε-game-free for every $\varepsilon < \mu^+$. It is also possible to obtain the "maximal game-freeness" without G being free by measuring the length of the game by trees.

2. Preliminaries

We denote the cofinality of an ordinal α by $\mathrm{cf}(\alpha)$ and the cardinality of a set X by $\mathrm{card}(X)$. The class of all ordinals of cofinality θ is denoted by $\mathrm{cof}(\theta)$. The set of all subsets of X of cardinality $< \lambda$ is denoted by $[X]^{<\lambda}$.

The definitions concerning λ-sets and λ-systems are from [14, Section 3]. There are slightly revised versions in [17, Appendix]. For the reader's convenience, we have chosen the notation so that it is compatible to [3, Section 3 of Chapter VII], and of course, compatible to [19]. The reason to represent the definitions is that in the later sections the concepts of "type of a λ-set" and "incompactness skeletons" have a central role.

Definition 2.1. Suppose $\mathscr{S}$ is a family of countable sets. A function T from $\mathscr{S}$ into $\bigcup \mathscr{S}$ is called a transversal for $\mathscr{S}$ when T is injective and for every $s \in \mathscr{S}$, $T(s) \in s$. If $\mathscr{S}$ is enumerated by $\{s_i \mid i \in I\}$ without repetition, then a transversal for $J \subseteq I$ means a transversal T for $\{s_i \mid i \in J\}$. When more convenient, we abbreviate $T(s_i)$ by $T(i)$.

A family $\mathscr{S}$ is called free if there exists a transversal for $\mathscr{S}$. For a cardinal κ, $\mathscr{S}$ is called κ-free, when every subfamily of $\mathscr{S}$ of cardinality $< \kappa$ is free. $\mathscr{S}$ is almost free when it is λ-free with $\lambda = \mathrm{card}(\mathscr{S})$.

For a subfamily $\mathscr{R}$ of $\mathscr{S}$, $\mathscr{S}/\mathscr{R}$ denotes the family $\{s \setminus \bigcup \mathscr{R} \mid s \in \mathscr{S} \setminus \mathscr{R}\}$.

Definition 2.2. Suppose $\mathscr{S}$ is a family of countable sets and ε is an ordinal. We call $\mathscr{S}$ ε-game-free if the second player, called player II, has winning strategy in the following two players "transversal game" denoted by $\mathrm{GT}_\varepsilon(\mathscr{S})$. A play of the game last at most ε rounds. On each round i the first player, called player I, chooses a countable subfamily $\mathscr{R}_i$ of $\mathscr{S}$. The second player, player II, must answer with a transversal T_i whose domain contains all the elements of $\mathscr{R}_i$ and which extends the transversals T_j, $j<i$, he has chosen on the earlier rounds. Player II wins a play if he succeeds to follow the given rules ε rounds.

Definition 2.3 (Shelah [14, Definition 3.1]). Suppose λ is an uncountable regular cardinal, and S is a nonempty set of finite sequences of ordinals which is closed under initial segments. Let S_f denote the "final nodes" of S, i.e., S_f is the smallest subset of S with $S=\{\eta\restriction m \mid \eta\in S_f \text{ and } m\leqslant \mathrm{lh}(\eta)\}$. A set S as above is called a λ-set if there exist uncountable regular cardinals $\langle\lambda_\rho \mid \rho\in S\backslash S_f\rangle$ such that
- $\lambda_\emptyset=\lambda$;
- for every $\rho\in S\backslash S_f$, $E^S_\rho=\{\alpha \mid \rho\frown\langle\alpha\rangle\in S\}$ is a stationary subset of λ_ρ (so $\rho\frown\langle\alpha\rangle\in S$ implies that $\alpha<\lambda_\rho$);
- for every $\rho\in S\backslash S_f$ and $\alpha\in E^S_\rho$, $\lambda_{\rho\frown\langle\alpha\rangle}\leqslant\mathrm{card}(\alpha)$ (so $\lambda_\rho>\lambda_{\rho\frown\langle\alpha\rangle}$).

The sequence $\langle\lambda_\rho \mid \rho\in S\backslash S_f\rangle$ is called the type of S. A λ-set S is said to have height n if n is a finite ordinal such that $\mathrm{lh}(\eta)=n$ for all $\eta\in S_f$.

Definition 2.4. Suppose S is a λ-set of type $\langle\lambda_\rho \mid \rho\in S\backslash S_f\rangle$. A subset S' of S is called a sub-λ-set of S if S' is a λ-set such that $S'_f\subset S_f$ and the type of S' is the restriction $\langle\lambda_\rho \mid \rho\in S'\backslash S'_f\rangle$ of the type of S. A subset I of S_f is small in S_f if the set $\{\eta\restriction m \mid \eta\in I \text{ and } m\leqslant\mathrm{lh}(\eta)\}$ is not a sub-λ-set of S.

Definition 2.5 (Shelah [14, Claim 3.2]). We say that a λ-set S of type $\bar\lambda$ has a canonical form when the following demands are fulfilled. S has height n^* and there exist sequences of cardinals $\langle\lambda'_n \mid n<n^*\rangle$ and $\langle\theta_n \mid n<n^*\rangle$ such that $\lambda'_0=\lambda_\emptyset$ and for every $\rho\in S\backslash S_f$ of length n,

(i) either both of the following two properties are satisfied:
  - the set E^S_ρ consist of regular limit cardinals and $\theta_n=0$ (in this case θ_n is called undefined);
  - if $n+1<n^*$ then for every $\alpha\in E^S_\rho$, $\lambda_{\rho\frown\langle\alpha\rangle}=\alpha$ and $\lambda_{n+1}=0$ (in this case λ'_{n+1} is called undefined);

(ii) or otherwise both of the following two properties are satisfied:
  - E^S_ρ is a subset of $\mathrm{cof}(\theta_n)$;
  - if $n+1<n^*$ then for every $\alpha\in E^S_\rho$, $\lambda_{\rho\frown\langle\alpha\rangle}=\lambda'_{n+1}$.

Fact 2.6 (Shelah [14, Claim 3.2]). *Every λ-set contains a sub-λ-set which is in a canonical form.*

Definition 2.7 (Shelah [14, Definition 3.4]). Suppose λ is an uncountable regular cardinal, S is a λ-set of type $\bar\lambda=\langle\lambda_\rho \mid \rho\in S\backslash S_f\rangle$. An indexed family

$$\bar A=\langle A_{\rho\frown\langle\alpha\rangle} \mid \rho\in S\backslash S_f \text{ and } \alpha<\lambda_\rho\rangle$$

is called a λ-system of type $\bar{\lambda}$, if $\Lambda_\emptyset$ is the empty set (only for technical reasons) and for every $\rho \in S \setminus S_f$ the sequence $\overline{\Lambda_\rho} = \langle \Lambda_{\rho \frown \langle \alpha \rangle} \mid \alpha < \lambda_\rho \rangle$ satisfies that

- $\overline{\Lambda_\rho}$ is a strictly increasing continuous chain of sets;
- the union of $\overline{\Lambda_\rho}$ has cardinality λ_ρ;
- each $\Lambda_{\rho \frown \langle \alpha \rangle}$ has cardinality $< \lambda_\rho$.

A λ-system is called disjoint when the sets $\bigcup_{\alpha < \lambda_\rho} \Lambda_{\rho \frown \langle \alpha \rangle}$, for all $\rho \in S \setminus S_f$, are disjoint.

Definition 2.8. Suppose $\lambda > \kappa$ are infinite regular cardinals. A tuple $\langle S, \bar{\Lambda}, \mathscr{S} \rangle$ is called an NPT(λ, κ)-skeleton (of type $\bar{\lambda}$ and of height n^*) when

- S is a λ-set of type $\bar{\lambda}$;
- S is in a canonical form and its height is n^*;
- the cardinals in $\bar{\lambda}$ are greater than κ;
- $\bar{\Lambda} = \langle \Lambda_{\rho \frown \langle \alpha \rangle} \mid \rho \in S \setminus S_f \text{ and } \alpha < \lambda_\rho \rangle$ is a disjoint λ-system of type $\bar{\lambda}$;
- $\mathscr{S}$ is a family of sets of cardinality $\leqslant \kappa$ enumerated by $\{s_\eta \mid \eta \in S_f\}$;
- $\mathscr{S}$ is based on $\bar{\Lambda}$, which means that for every $\eta \in S_f$,

$$s_\eta \subseteq \bigcup_{m \leqslant n^*} \Lambda_{\eta \restriction n}.$$

Additionally, the demands in [14, Claims 3.6 and 3.7] are fulfilled (in case $\kappa = \aleph_0$ they are essentially equivalent to the "beautifulness properties" presented in [3, Definition VII.3A.2(1–6)]). Define for every $\eta \in S_f$ and $m < n^*$, that

$$s_\eta^{m+1} = s_\eta \cap \Lambda_{n \restriction m+1}.$$

Remark. s_η^{m+1} is denoted by s_η^m in [14].

For the reader's convenience we repeat three of the additional demands (used in Section 3):

(A) For all $\eta \neq v \in S_f$, if $s_\eta \cap s_v \neq \emptyset$ then there is a unique $m < n^*$ such that $s_\eta \cap s_v = s_\eta^{m+1} \cap s_v^{m+1}$ and for every $l < n^*$ with $l \neq m$, $\eta(l) = v(l)$.

(B) For every $\eta \in S_f$ and $m < n^*$, if $\eta(m)$ has cofinality $\theta > \kappa$, then there is $n < n^*$ with $\lambda_{\eta \restriction n} = \theta$ ($\lambda_{\eta \restriction n}$ is from the type $\bar{\lambda}$ of S).

(C) In case $\kappa = \aleph_0$: For all $\eta \in S_f$ and $m < n^*$, the sets s_η^{m+1} have enumerations $\langle x_l^{\eta,m} \mid l < \omega \rangle$ with the following property: for every $v \in S_f$ and $n < \omega$, if $x_n^{v,m} \in s_\eta^{m+1}$ then the initial segments $\langle x_l^{v,m} \mid l \leqslant n \rangle$ and $\langle x_l^{\eta,m} \mid l \leqslant n \rangle$ are equal.

Definition 2.9. Suppose $\bar{\mu} = \langle \mu_\xi \mid \xi < \kappa \rangle$ is an increasing sequence of regular cardinals approaching μ such that $\mu_0 > \kappa^+$. For functions $f, g \in \prod_{\xi < \kappa} \mu_\xi$, we write that $f <^* g$ when f is eventually strictly less than g, i.e., if $\{\xi < \kappa \mid f(\xi) \geqslant f(\xi)\}$ is bounded in κ.

A pair $\langle \mu, \bar{f} \rangle$ is called a scale for μ when $\bar{\mu}$ is as above and $\bar{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ is a $<^*$-increasing cofinal sequence of functions in $\prod_{\xi < \kappa} \mu_\xi$ (cofinal means that for every $g \in \prod_{\xi < \kappa} \mu_i$ there exists α with $g <^* f_\alpha$).

Fact 2.10 (Shelah [16, Theorem II.1.5]). *For any singular cardinal μ, there exists a scale for μ. In other words, there is a sequence $\langle \mu_\xi \mid \xi < \kappa \rangle$ of regular cardinals ap-*

proaching μ *so that* $\prod_{\xi<\kappa}\mu_i$ *has true cofinality* μ^+ *under the partial order* $<^*$ (*denoted by* $<_{J_k^{\rm bd}}$ *in the reference*).

3. Transversals and game-free families

The paper [19] left open a question: if μ is a singular cardinal, does there exist a family $\mathscr{S}$ of countable sets of cardinality μ^+ which is nonfree and ε-game-free for all $\varepsilon<\mu$? Shelah has a very satisfactory answer to this question.

Proposition 3.1. *Suppose* $\langle S', \bar{\Lambda}', \mathscr{S}'\rangle$ *is an* $\mathrm{NPT}(\lambda,\aleph_0)$*-skeleton of type* $\bar{\lambda}=\langle\lambda_\rho\mid\rho\in S'\setminus S'_f\rangle$ *such that the following conditions hold*:
- μ *is a singular cardinal and* λ *equals* μ^+;
- *either* $\mathrm{cf}(\mu)=\aleph_0$ *or there is m below the height of S such that* $\lambda_\rho=\mathrm{cf}(\mu)$ *for every* $\rho\in S$ *of length m*;
- *the cardinal* θ *for which* $E_\emptyset^{S'}\subseteq\mathrm{cof}(\theta)$ *holds* (*and which exists by Definition* 2.8) *is such that* $\mathrm{cf}(\mu)\neq\theta$.

Then there exists $\mathrm{NPT}(\lambda,\aleph_0)$*-skeleton* $\langle S,\bar{\Lambda},\mathscr{S}\rangle$ *such that* $\mathscr{S}$ *is* ε*-game-free for every* $\varepsilon<\mu$. *Moreover, S is a sub-*λ*-set of* S' *and hence* $E_\emptyset^S\subseteq\mathrm{cof}(\theta)$ *holds* ($S=S'$ *if* $\bar{\lambda}$ *does not contain limit cardinals*).

The rest of the section is devoted to the proof of this proposition.

Suppose that S' has height n^* (by the canonical form height is well-defined). Now note that n^* must be at least 2: If μ has uncountable cofinality, this follows directly from our second demand. On the other hand, if μ has countable cofinality, the claim follows from the demand that $E_\emptyset^{S'}\nsubseteq\mathrm{cof}(\aleph_0)$ together with Definition 2.8(B).

Remark. If $n^*=1$ and $E_\emptyset^S\subseteq\mathrm{cof}(\aleph_0)$, then $\mathscr{S}$ would not be $\aleph_1$-game-free as explained in [19, Example 4.4].

We let λ_1 denote the regular cardinal given by Fact 2.6, i.e., for every $\alpha\in E_\emptyset^S$, $\lambda_{\langle\alpha\rangle}=\lambda_1$. By taking a suitable sub-λ-set S of S' if necessary, we may assume that

> if $\mathrm{cf}(\mu)$ is uncountable, then there is a fixed $m<n^*$ such that for every $\eta\in S_f$, $\lambda_{\eta\restriction m}=\mathrm{cf}(\mu)$.

If there is no limit cardinals in $\bar{\lambda}$, then Fact 2.6 guarantees that we can choose $S=S'$.

By Fact 2.10 choose a scale $\langle\mu,\bar{f}\rangle$ for μ (Definition 2.9), where $\bar{\mu}=\langle\mu_\xi\mid\xi<\mathrm{cf}(\mu)\rangle$ and $\bar{f}=\langle f_\alpha\mid\alpha<\lambda\rangle$.

The only modification of the family $\mathscr{S}'=\{s'_\eta\mid\eta\in S'_f\}$ needed is that the "first coordinate" $(s^1_\eta)'=s'_\eta\cap\Lambda'_{\eta\restriction 1}$ of each s'_η is slightly changed. The modification depends on the cofinality of μ as follows: Fix η from S_f. If μ has countable cofinality, then define

$$s^1_\eta=\{F_\eta\restriction l\mid l<\omega\},$$

where F_η is a fixed function from ω onto $\mathrm{ran}(f_{\eta(0)}) \times (s^1_\eta)'$. This definition ensures that the property Definition 2.8(C) becomes fulfilled. In order to choose the corresponding new λ-system, let $\Lambda_{\langle\alpha\rangle}$, for every $\alpha<\lambda$, be $[\mu \times \Lambda'\langle\alpha\rangle]^{<\aleph_0}$, and additionally, for all nonempty $\rho \in S \setminus S_f$ and $\alpha < \lambda_\rho$, define $\Lambda_{\rho \frown \langle\alpha\rangle}$ to be the old $\Lambda'_{\rho\frown\langle\alpha\rangle}$. Otherwise, for the fixed η, $\mathrm{cf}(\mu) = \lambda_{\eta\restriction m} > \aleph_0$ holds and we define

$$s^1_\eta = \{f_{\eta(0)}(\eta(m))\} \times (s^1_\eta)'$$

and for every $\alpha<\lambda$, $\Lambda_{\langle\alpha\rangle} = \mu \times \Lambda'_{\langle\alpha\rangle}$ (Definition 2.8(C) is satisfied since $(s^1_\eta)'$ satisfies it). Note that now $\langle S, \bar{\Lambda}, \mathscr{S}\rangle$ is an $\mathrm{NPT}(\lambda,\aleph_0)$-skeleton, i.e., it fulfills all the demands mentioned in Definition 2.8, because $\langle S', \bar{\Lambda}', \mathscr{S}'\rangle$ is an $\mathrm{NPT}(\lambda,\aleph_0)$-skeleton.

In order to show that player II has a winning strategy in the game $\mathrm{GT}_\varepsilon(\mathscr{S})$ for every $\varepsilon<\mu$, it suffices to describe a winning strategy for player II in a modified game of length σ for every regular cardinal σ with $\lambda>\sigma>\lambda_1$, where the rules of the modified game are exactly as in $\mathrm{GT}_\sigma(\mathscr{S})$, except that player II is demanded to choose a transversal only if the index of the round is a limit ordinal.

On every round $i<\sigma$ player II chooses two elementary submodels $\mathscr{M}_{i+1}$ and $\mathscr{N}_{i+1}$ of $\langle H(\chi), \in, \bar{f}, \bar{\mu}, S, \bar{\Lambda}, \mathscr{S}\rangle$ such that

- χ is some large enough regular cardinal;
- $\mathrm{card}(\mathscr{M}_{i+1}) = \mu$;
- $\mathrm{card}(\mathscr{N}_{i+1}) = \sigma$;
- $\mathscr{N}_{i+1} \subseteq \mathscr{M}_{i+1}$;
- $\mathscr{N}_{i+1}$ contains all the elements chosen by player I
- $\sigma+1 \subseteq \mathscr{N}_{i+1}$;
- $\mathscr{M}_{i+1}\cap\lambda$ is an ordinal denoted by $\delta_{i+1}\in\lambda$;
- $\mathscr{M}_i, \mathscr{N}_i \in \mathscr{N}_{i+1}$, where $\mathscr{M}_0=\mathscr{N}_0=\emptyset$ and for limit i, $\mathscr{M}_i = \bigcup_{j<i}\mathscr{M}_j$ and $\mathscr{N}_i = \bigcup_{j<i}\mathscr{N}_j$.

For the rest of this proof assume $i<\sigma$ to be a limit ordinal or zero, and suppose $\mathscr{M}_j$, $\mathscr{N}_j$ for each $j\leqslant i$ are chosen. Denote the set $\mathscr{S}\cap\mathscr{N}_i$ by $\mathscr{R}$. We show that

$$(3.1)\quad \mathscr{S}/\mathscr{R} \quad \text{is } \lambda\text{-free},$$

because then after $i+\omega<\sigma$ rounds, player II is able to continue (or start) with some transversal $T_{i+\omega}$ whose domain consist of the elements in $\mathscr{S}\cap\mathscr{N}_{i+\omega}$ (contains the elements chosen by player I so far) and satisfies that $T_i \subseteq T_{i+\omega}$ (where $T_0=\emptyset$ and for i which is a limit of limit ordinals, $T_i = \bigcup_{j<i} T_{j+\omega}$).

To check the details, we have to define some auxiliary notations (familiar from [19] or [3, VII.3A]): for every $\alpha<\lambda$,

$$\mathscr{S}_\alpha = \{s_\eta \mid \eta \in S_f \text{ and } \eta(0) < \alpha\},$$

$$S_f^{\langle\alpha\rangle} = \{\eta \in S_f \mid \eta(0) = \alpha\},$$

$$\mathscr{S}^{\langle\alpha\rangle} = \left\{ \bigcup_{0<l<n^*} s^{l+1}_\eta \mid \eta \in S_f^{\langle\alpha\rangle} \right\}.$$

Hence, e.g., $\mathscr{S} \cap \mathscr{M}_i$ equals to $\mathscr{S}_{\delta_i}$ for every $i < \sigma$. We also need the fact that when $\langle S, \bar{\Lambda}, \mathscr{S} \rangle$ is an NPT$(\lambda, \aleph_0)$-skeleton, for every α in $E^S_\emptyset$ and for all $I \subset S^{\langle \alpha \rangle}_f$ (recall smallness from Definition 2.4):

(3.2) I is small in $S^{\langle \alpha \rangle}_f$ iff there is a transversal T such that its domain is $\{s_\eta \mid \eta \in I\}$ and for every $\eta \in I$, $T(s_\eta) \notin s^1_\eta$.

Much more is proved in [14, Claim 3.8]. This simple fact is explained, e.g., in [19, Fact 4.21].

The proof of (3.1) is divided into several parts. Because we may assume that $E^S_\emptyset$ contains only limit ordinals, $\mathscr{S}/\mathscr{S}_{(\delta_i)+1}$ is λ-free. Since $\mathscr{R} \subseteq \mathscr{S}_{\delta_i}$ it suffices to show that $\mathscr{S}_{(\delta_i)+1}/\mathscr{R}$ is free. First we show that

(3.3) there is a transversal T for $\mathscr{S}^{\langle \delta_i \rangle}$ so that $\operatorname{ran}(T) \cap \bigcup \mathscr{R}$ is empty.

Remark. We do not claim that T witnesses $\mathscr{S}_{(\delta_i)+1}/\mathscr{S}_{\delta_i}$ to be free. It can happen that $\operatorname{ran}(T)$ contains elements from $\bigcup \mathscr{S}_{\delta_i}$.

Secondly we prove that

(3.4) T can be extended to a transversal T' for $\mathscr{S}_{(\delta_i)+1}/\mathscr{R}$.

Note that if $\delta_i \notin E^S_\emptyset$, then $\mathscr{S}_{(\delta_i)+1}/\mathscr{S}_{\delta_i}$ is empty, and (3.3) holds trivially. So suppose that $\delta_i \in E^S_\emptyset$. We want that

(3.5) there exists ξ^* such that $f_{\delta_i}(\xi) \notin \mathscr{N}_i$ for any $\xi \geqslant \xi^*$.

From this claim (3.3) follows in the following way: By the definition of the "new first coordinate" there are two different cases according to the cofinality of μ. If μ has countable cofinality then choose a transversal T' for $\{s'_\eta \mid \eta \in S^{\langle \delta_i \rangle}_f\}$ (s'_η is the "old first coordinate"), and define the desired transversal T for $\{s_\eta \mid \eta \in S^{\langle \delta_i \rangle}_f\}$ by setting for every $\eta \in S^{\langle \delta_i \rangle}_f$, that $T(s_\eta)$ is the finite restriction of F_η whose greatest element is the pair $\langle f_{\eta(0)}(\xi^*), T'(s'_\eta) \rangle$. Suppose then that $\operatorname{cf}(\mu)$ is uncountable. By the definition of s^1_η, when $\eta \in S^{\langle \delta_i \rangle}_f$, and the assumption (3.5), the following set is small in $S^{\langle \delta_i \rangle}_f$:

$$I_0 = \left\{ \eta \in S^{\langle \delta_i \rangle}_f \mid s^1_\eta \cap \left(\bigcup \mathscr{R} \right) \neq \emptyset \right\}.$$

Choose any transversal U_1 for $I_1 = S^{\langle \delta_i \rangle}_f \setminus I_0$ ($I_1 \subseteq S_f$ has cardinality $< \lambda$). By (3.2), $I'_1 = \{\eta \in I_1 \mid U_1(\eta) \notin s^1_\eta\}$ must be small in $S^{\langle \delta_i \rangle}_f$. Now the set $I_0 \cup I'_1$ is small in $S^{\langle \delta_i \rangle}_f$, and by (3.2) again, there is a transversal U_0 for $I_0 \cup I'_1$ with the property that $U_0(\eta) \notin s^1_\eta$ for all $\eta \in I_0 \cup I'_1$. Because of Definition 2.8(A), the union $T = U_0 \cup U_1 \restriction I_1 \setminus I'_1$ forms the desired transversal witnessing that (3.3) holds. So (3.5) implies (3.3).

Note that $\{\delta_j \mid j < i\}$ is a subset of $\mathscr{N}_i$. Define ζ^* to be the first index in $\operatorname{cf}(\mu)$ with $\mu_{\zeta^*} > \sigma$. For every $j \leqslant i$ define a function $h_j \in \prod_{\xi < \operatorname{cf}(\mu)} \mu_\xi$ by setting $h_j(\xi) = 0$ if $\xi < \zeta^*$, and $h_j(\xi) = \sup(\mu_\xi \cap \mathscr{N}_j)$ otherwise. Clearly, $h_k(\xi) < h_j(\xi)$ when $k < j \leqslant i$ and

$\xi \geqslant \zeta^*$. For every $j<i$ there is $\beta<\delta_i$ with $h_j <^* f_\beta$, since $h_j \in \mathcal{N}_i$ and $\mathcal{N}_i \cap \lambda \subseteq \delta_i$. Thus $h_j <^* f_{\delta_i}$ for every $j<i$. To prove (3.5) it suffices to show that $h_i \leqslant^* f_{\delta_i}$. For every $j<i$, there is the smallest $\xi_j < \mathrm{cf}(\mu) \setminus \zeta^*$ satisfying that $\{\xi < \mathrm{cf}(\mu) \mid h_j(\xi) \geqslant f_{\delta_i}(\xi)\} \subset \xi_j$. By the definition of h_j's, $\xi_k < \xi_j$ for every $k<j<i$. By the assumption $E^{S'}_\emptyset \cap \mathrm{cof}(\mathrm{cf}(\mu)) = \emptyset$, $\mathrm{cf}(i) = \mathrm{cf}(\delta_i) \neq \mathrm{cf}(\mu)$. Hence, there exists $\xi^* \geqslant \zeta^*$ such that $\xi_j \leqslant \xi^*$ for every $j<i$. By the definition of ξ_j's, for every $\xi \geqslant \xi^*$, $f_{\delta_i}(\xi) \geqslant \sup\{h_j(\xi) \mid j<i\} = h_i(\xi)$. Therefore (3.5) holds, and we have proved (3.3).

Next we prove that there is some transversal for $\mathcal{S}_{\delta_i}/\mathcal{R}$, and finally we explain how to find a transversal witnessing (3.4). In order to show that $\mathcal{S}_{\delta_i}/\mathcal{R}$ is free, it suffices to conclude that for every $j<i$,

(3.6) $\mathcal{S}_{\delta_{(j+1)}}/(\mathcal{S}_{\delta_j} \cup \mathcal{R})$

is free. Note that in this case δ_i might be in $E^S_\emptyset$ or in its complement, but in both cases $S^{\langle \delta_j \rangle}_f$ is a subset of $\mathcal{N}_i$ for every $j<i$ (because of the inequality $\sigma > \lambda_1 > \cdots > \lambda_\rho$, $\mathrm{lh}(\rho)>1$). Since $(\delta_j)+1$ is not in $E^S_\emptyset$, $\mathcal{S}_{\delta_{(j+1)}}/\mathcal{S}_{(\delta_j)+1}$ is free. Since $\mathcal{S}_{\delta_{(j+1)}}/\mathcal{S}_{(\delta_j)+1}$ is in $\mathcal{N}_i$, also

$$(\mathcal{S}_{\delta_{(j+1)}}/\mathcal{S}_{(\delta_j)+1})/((\mathcal{S}_{\delta_{(j+1)}}/\mathcal{S}_{(\delta_j)+1}) \cap \mathcal{N}_i) = \mathcal{S}_{\delta_{(j+1)}}/(\mathcal{S}_{(\delta_j)+1} \cup (\mathcal{R} \cap \mathcal{S}_{(\delta_j)+1}))$$

is free. Because both $\bigcup(\mathcal{S}_{(\delta_j)+1} \cap \mathcal{R}) = (\bigcup \mathcal{S}_{(\delta_j)+1}) \cap (\bigcup \mathcal{R})$ and $\mathcal{S}^{\langle \delta_j \rangle} \subseteq \mathcal{R}$ hold, we have that $\mathcal{S}_{\delta_{(j+1)}}/(\mathcal{S}_{\delta_j} \cup \mathcal{R}) = \mathcal{S}_{\delta_{(j+1)}}/(\mathcal{S}_{(\delta_j)+1} \cup (\mathcal{R} \cap \mathcal{S}_{(\delta_j)+1}))$. Hence (3.6) holds, and we have proved that $\mathcal{S}_{\delta_i}/\mathcal{R}$ is free.

Let X denote the set $E^S_\emptyset \cap \delta_i \setminus \mathcal{N}_i$. Suppose, for the moment, that for every $\alpha \in X$, the set

(3.7) $J_\alpha = \{\eta \in S^{\langle \alpha \rangle}_f \mid s_\nu \cap s_\eta \neq \emptyset$ for some $\nu \in S^{\langle \delta_i \rangle}_f\}$ is small in $S^{\langle \alpha \rangle}_f$.

By (3.2) choose a transversal U_α for J_α, $\alpha \in X$, so that $U_\alpha(\eta) \notin s^1_\eta$ hold for all $\eta \in J_\alpha$. By Definition 2.8(A) together with the fact that $X \cap \mathcal{N}_i$ is empty, $\mathrm{ran}(U_\alpha) \cap \bigcup \mathcal{R}$ is empty. Consequently, the desired extension T' in (3.4) can be $T \cup W \cup \bigcup_{\alpha \in X} U_\alpha$, where W is a restriction of some transversal witnessing that $\mathcal{S}_{\delta_i}/\mathcal{R}$ is free into the set $\mathcal{S}_{\delta_i} \setminus \{s_\eta \mid \eta \in J_\alpha$ for some $\alpha \in X\}$.

So it remains to prove (3.7). Fix some α from X. Since $f_\alpha <^* f_{\delta_i}$, there is ξ^* with $f_\alpha(\xi) < f_{\delta_i}(\xi)$ for each $\xi > \xi^*$. Assume that $\eta \in \bigcup_{\alpha \in X} S^{\langle \alpha \rangle}_f$ and $\nu \in S^{\langle \delta_i \rangle}_f$ are such that $s_\eta \cap s_\nu$ is nonempty. From Definition 2.8(A) it follows that $s^1_\eta \cap s^1_\nu \neq \emptyset$ and there is ξ with $\eta(m) = \nu(m) = \xi$. From the new definition of the "first coordinate" it follows that $f_\alpha(\xi) = f_{\delta_i}(\xi)$. Therefore $\xi \leqslant \xi^*$. This means that for every $\tau \in S$ of length m, the set $\{\eta \in S^\tau_f \mid s_\eta \cap s_\nu \neq \emptyset$ for some $\nu \in S^{\langle \delta_i \rangle}_f\}$ has cardinality $< \lambda_{\eta \restriction m} = \lambda_{\nu \restriction m} = \mathrm{cf}(\mu)$. Thus (3.7) holds and we have proved Proposition 3.1.

4. Building blocks of incompactness skeletons

In this section, $\lambda > \kappa$ are regular cardinals and $\bar{\kappa}$ is a decreasing sequence of regular cardinals. In the definitions below we present a variant "NRT(λ, κ)" of the notion

NPT(λ,κ). A simplest form of this variant is mentioned in [16, Fact 6.2(9)]. So we define analogical notions of "a transversal for a subfamily" and "a free subfamily over another subfamily". There are two main motivations. Firstly, even if both NPT(κ_1,κ_2) and NPT(κ_2,κ_3) hold, for regular cardinals $\kappa_1>\kappa_2>\kappa_3$, NPT$(\kappa_1,\kappa_3)$ does not need to hold:

Fact 4.1 (Shelah [18, Lemma 3.1]). *It is consistent with* ZFC *relative to existence of two weakly compact cardinals, that for some regular cardinals $\lambda>\kappa>\aleph_0$, both* NPT(λ,κ) *and* NPT$(\kappa,\aleph_0)$ *hold, but* NPT$(\lambda,\aleph_0)$ *does not hold.*

However, we show that an analogical "transitivity property" for NRT(λ,κ) holds, Corollary 4.19. Secondly, the introduced families and their canonical forms, called NRT$(\lambda,\aleph_0)$-skeletons, provide a unified picture of those construction methods of NPT$(\lambda,\aleph_0)$-skeletons presented in [9, Section 1] and [18, Section 2] (see Fact 4.6). We shall deal with the ideal $\mathscr{J}^{\mathrm{nst}}_{\bar\kappa}$ of "the product" of nonstationary ideals of fixed cardinals $\bar\kappa$ (the next definition) instead of considering just the ideal of the bounded subsets of κ. However, the reader may think in the beginning, if she or he wants, that $\bar\kappa=\langle\kappa\rangle$ and $\bar{\mathscr{I}}$ consist of the ideal of all bounded subsets of κ.

Remark. From Fact 4.5 below it follows that NRT$(\lambda,\bar\kappa)$-skeletons are necessary to "pump up" more complicated NPT$(\lambda,\aleph_0)$-skeletons. In fact we shall see in Definition 5.1 and Conclusion 5.3, that the definition of a nicely incompact set of regular cardinals from [17, Definition 2.1] can be expressed using NRT$(\lambda,\bar\kappa)$-skeletons as building blocks. All together (which means Fact 4.16, Definition 4.17, Lemma 4.18, and Lemma 5.2), NRT$(\lambda,\bar\kappa)$-skeletons are handy tools to divide and amalgamate NPT$(\lambda,\aleph_0)$-skeletons. The reader is advised to look at Section 5 to understand our goal, if she or he feel lost with the technicalities in this section.

Definition 4.2. Suppose $\bar\kappa=\langle\kappa_k\mid k<k^*\rangle$ is a nonempty decreasing sequence of infinite cardinals. A family $\mathscr{A}$ is called an $R(\bar\kappa)$-family ($R(\bar\kappa)$ comes from the notion NRT$(\lambda,\bar\kappa)$), when $\mathscr{A}$ fulfills the following demands:

- the elements of $\mathscr{A}$ are finite sequences of functions;
- $\mathscr{A}$ is enumerated by $\{\bar a_i\mid i<\mathrm{card}(\mathscr{A})\}$;
- each $\bar a_i$ is of the form $\langle a_{i,l}\mid l<\mathrm{lh}(\bar a_i)\rangle$;
- $a_{i,l}$'s have domain $\prod\bar\kappa$.

Definition 4.3. Suppose $\bar\kappa$ is as in Definition 4.2. A sequence $\bar{\mathscr{I}}=\langle\mathscr{I}_k\mid k<k^*\rangle$ is called a $\bar\kappa$-sequence of ideals, when each $\mathscr{I}_k$ is a proper ideal on κ_k containing all the bounded subsets of κ_k. For two ideals $\mathscr{I}_0$ and $\mathscr{I}_1$ on κ_0 and κ_1, respectively, the product $\mathscr{I}_0\times\mathscr{I}_1$ is the ideal of all subsets X of $\kappa_0\times\kappa_1$ such that

$$\{\alpha_0<\kappa_0\mid\{\alpha_1<\kappa_1\mid\langle\alpha_0,\alpha_1\rangle\in X\}\notin\mathscr{I}_1\}\in\mathscr{I}_0.$$

For a regular cardinal κ, let $\mathscr{J}^{\mathrm{bd}}_{\kappa}$ denote the ideal of all the bounded subsets of κ, and if κ is uncountable, let $\mathscr{J}^{\mathrm{nst}}_{\kappa}$ denote the ideal of all nonstationary subsets of κ. For a

$\bar{\kappa}$-sequence of ideals, let $\mathscr{J}_{\bar{\kappa}}^{\text{nst}}$ denote the product of the ideals in $\langle \mathscr{I}'_k \mid k < k^* \rangle$, where $\mathscr{I}'_k = \mathscr{J}_{\kappa_k}^{\text{nst}}$ if κ_k is uncountable, and otherwise, $\mathscr{I}_k = \mathscr{J}_{\aleph_0}^{\text{bd}}$.

Definition 4.4. Assume that $\bar{\mathscr{I}}$ is a $\bar{\kappa}$-sequence of ideals, $\mathscr{A}$ is an $R(\bar{\kappa})$-family, and $\mathscr{A}' = \{\bar{a}_i \mid i \in I\}$ is a subfamily of $\mathscr{A}$. A sequence $\mathbf{b} = \langle b_i \mid i \in I\rangle$ of functions is called an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}'$ if the following properties are satisfied for every $i \in I$:
- for some $l < \text{lh}(\bar{a}_i)$, $b_i \subset a_{i,l}$ and $\prod \bar{\kappa} \setminus \text{dom}(b_i)$ is in $\prod \bar{\mathscr{I}}$;
- the ranges of b_i, $i \in I$, are pairwise disjoint.

$\mathscr{A}'$ is called $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-free if there exists an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}'$ (the empty sequence is transversal for the empty family). A family $\mathscr{A}$ is almost $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-free, when every subfamily of cardinality $<\text{card}(\mathscr{A})$ is $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-free.

For a cardinal $\lambda > \max \bar{\kappa}$, we say that $\text{NRT}_{\bar{\mathscr{I}}}(\lambda, \bar{\kappa})$ holds, if there exists an almost $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-free $R(\bar{\kappa})$-family of cardinality λ which is not $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-free. In the case $\bar{\kappa} = \langle \kappa \rangle$ and $\bar{\mathscr{I}} = \langle \mathscr{J}_{\kappa}^{\text{bd}} \rangle$, $\text{NRT}(\lambda, \kappa)$ is an abbreviation for $\text{NRT}_{\bar{\mathscr{I}}}(\lambda, \bar{\kappa})$.

First, we see that existence of an $\text{NPT}(\lambda, \aleph_0)$-skeleton necessarily gives many almost free nonfree $R(\bar{\kappa})$-families (depending on the type of the skeleton). Later in Lemma 4.18 we shall see that existence of certain type of $R(\bar{\kappa})$-families is also sufficient condition for building $\text{NPT}(\lambda, \aleph_0)$-skeletons. More generally, "neat" $R(\bar{\kappa})$-families can be transformed into a form of an "$\text{NRT}(\lambda, \bar{\kappa})$-skeleton", which is an analog of $\text{NPT}(\lambda, \aleph_0)$-skeletons for $R(\bar{\kappa})$-families.

Fact 4.5. *Let $\langle S, \bar{\Lambda}, \mathscr{S} \rangle$ be an $\text{NPT}(\lambda, \aleph_0)$-skeleton of type $\bar{\lambda}$ and of height n^*. Suppose that there is a sequence $\langle \lambda_n \mid n < n^* \rangle$ such that for every $n < n^*$ and $\rho \in S$ of length n, $\lambda_n = \lambda_\rho$ holds ($\bar{\lambda}$ does not contain regular limit cardinals). Then for every $n < n^*$, $\text{NRT}_{\mathscr{J}_{\bar{\kappa}^n}^{\text{nst}}}(\lambda_n, \bar{\kappa}^n)$ holds, where $\bar{\kappa}^n = \langle \lambda_m \mid n < m < n^* \rangle \frown \langle \aleph_0 \rangle$ and $\mathscr{J}_{\bar{\kappa}^n}^{\text{nst}}$ is the ideal given in Definition 4.3 above. In fact, there exists an $\text{NRT}(\lambda_n, \bar{\kappa})$-skeleton of type $\langle \lambda_n \rangle$, see Definition 4.10.*

Proof. Fix $n < n^*$ and $\rho \in S \setminus S_f$ of length n. Let k^* denote the length of $\bar{\kappa}$, i.e., $k^* = n^* - n + 1$. Define a family $\mathscr{A}_\rho$ to be $\{\langle a_\alpha \rangle \mid \alpha \in E_\rho^S\}$, where each a_α is a function having domain $\prod \bar{\kappa}$ and for every $\tau \in \prod \bar{\kappa}$, the value $a_\alpha(\tau)$ is chosen in the following way. For fixed α and τ let $\eta_{\alpha,\tau} \in S_f$ be such that $\eta_{\alpha,\tau} \restriction n = \rho$, $\eta_{\alpha,\tau}(n) = \alpha$, and for every $k < k^* - 1$, $\eta_{\alpha,\tau}(n+k)$ is the $\tau(k)$'s member of $E^S_{\eta_{\alpha,\tau} \restriction n+k}$ in $\in$-order. Define $a_\alpha(\tau)$ to be the $\tau(k^*-1)$'s element of the countable set $s^{n+1}_{\eta_{\alpha,\tau}} = s_{\eta_{\alpha,\tau}} \cap \Lambda_{\rho \frown \langle \alpha \rangle}$ in the fixed enumeration of it, recall Definition 2.8(C).

By Fodor's lemma $\mathscr{A}_\rho$ cannot be $R_{\mathscr{J}_{\bar{\kappa}}^{\text{nst}}}(\bar{\kappa})$-free. On the other hand, $\mathscr{A}_\rho$ is almost $R_{\mathscr{J}_{\bar{\kappa}}^{\text{nst}}}(\bar{\kappa})$-free: Fix $\delta < \lambda_n$ and let I denote the set $\{\eta \in S_f^\rho \mid \eta(n) < \delta\}$. By [14, Claim 3.8(D)] we may choose a sequence $\langle u_\eta \mid \eta \in I \rangle$ of pairwise disjoint sets such that for every $\eta \in I$ there is an index l_η such that $n \leqslant l_\eta < n^*$ and u_η is an end segment of $s_\eta^{l_\eta+1}$. For every $\alpha < \delta$ the set

$$I^\alpha = \{\nu \in S_f^\rho \mid \nu(n) = \alpha \text{ and } l_\nu > n\}$$

must be small in $S_f^{\rho\frown\langle\alpha\rangle}$. It follows that the corresponding set $Y^\alpha = \{\tau \in \prod \bar\kappa \mid \eta_{\alpha,\tau} \in I^\alpha\}$ must be in $\mathscr{J}_{\bar\kappa}^{\mathrm{nst}}$. So we may define a transversal $\langle b_\alpha \mid \alpha < \delta\rangle$ for $\{\langle a_\alpha\rangle \mid \alpha < \delta\}$ by setting $b_\alpha = a_\alpha \restriction \prod \bar\kappa \setminus Y^\alpha$. □

There are many examples of nonfree almost free $R_{\bar{\mathscr{I}}}(\bar\kappa)$-families. Note that the examples below are NRT$(\lambda, \bar\kappa)$-skeletons of height 1 for some $\bar\kappa$ of length at most 2.

Fact 4.6. *Suppose λ is an uncountable regular cardinal.*

(a) [13, *Lemma* 23]. *If there exist a regular cardinal $\kappa < \lambda$ and a nonreflecting stationary subset E of $\lambda \cap \mathrm{cof}(\kappa)$ (which means for all $\alpha < \lambda$, $E \cap \alpha$ is nonstationary in α), then* NRT(λ, κ) *holds.*

(b) *If $\lambda = \kappa^+$ and κ is a regular cardinal, then* NRT(λ, κ) *holds (since $\lambda \cap \mathrm{cof}(\kappa)$ is a nonreflecting stationary set, or for the other cofinalities use $*(\kappa, \kappa)$ given in the references of the next item).*

(c) [10] *or* [3, *Theorem* VI.3.9 *and* VI.3.10]. *If μ is a singular cardinal of cofinality κ and $V = L$, then* NRT(μ^+, κ) *holds (because of $*(\mu, \kappa)$).*

(d) [13, *Lemma* 24]. *Suppose μ is a singular strong limit cardinal of cofinality κ such that $I[\mu]^+ = \mathscr{P}(\mu^+)$ (for a definition of $I[\mu]^+$, see e.g.* [15, *Definition* 2.1]). *Then for every regular $\theta < \mu$ with $\kappa \neq \theta$,* $\mathrm{NRT}_{\bar{\mathscr{I}}}(\lambda, \langle\theta, \kappa\rangle)$ *holds, where $\bar{\mathscr{I}} = \langle \mathscr{J}_\theta^{\mathrm{bd}}, \mathscr{J}_\kappa^{\mathrm{bd}}\rangle$.*

(e) [9, *Theorem* 4]. *Suppose $\kappa < \theta$ are regular cardinals and $\lambda = \theta^{+\kappa+1}$. Then* $\mathrm{NRT}_{\bar{\mathscr{I}}}(\lambda, \langle\theta, \kappa\rangle)$ *holds, where $\bar{\mathscr{I}} = \langle \mathscr{J}_\theta^{\mathrm{nst}}, \mathscr{J}_\kappa^{\mathrm{bd}}\rangle$.*

(f) [18, *Lemma* 1.16]. *Suppose κ is a regular cardinal such that for $\mu = \kappa^{+\kappa}$ there is a scale $\langle \bar\mu, \bar f\rangle$ whose good points $S^{\mathrm{gd}}[\bar f]$ contains a closed and unbounded subset of μ^+ (Definitions* 2.9 *and* 6.2). *Then for every regular θ with $\kappa < \theta < \mu$,* $\mathrm{NRT}_{\bar{\mathscr{I}}}(\mu^+, \langle\theta, \kappa\rangle)$ *holds, where $\bar{\mathscr{I}} = \langle \mathscr{J}_\theta^{\mathrm{nst}}, \mathscr{J}_\kappa^{\mathrm{bd}}\rangle$.*

(g) [16, *Claim* II.1.5A]. *If μ is a singular cardinal of cofinality κ and $\mathrm{pp}^*_{\mathscr{J}}(\mu) > \mu^+$, then* $\mathrm{NRT}_{\mathscr{J}}(\mu^+, \kappa)$ *holds. Moreover, when κ is uncountable, already $\mathrm{pp}(\mu) > \mu^+$ together with some weak assumptions imply* $\mathrm{NRT}_{\mathscr{J}_\kappa^{\mathrm{bd}}}(\mu^+, \kappa)$ [16, *Analytical Guide* 5.7(B) *and Sh371*; *Sections* 0 *and* 1].

(h) [16, *Theorem* II.6.3]. *If μ is a singular cardinal of countable cofinality and $\mathrm{cov}(\mu, \mu, \aleph_1, 2) > \mu^+$ (e.g., μ strong limit and $\mu^{\aleph_0} > \mu^+$), then* NRT$(\mu^+, \aleph_0)$ *holds.*

The rest of this section is essentially based on a similar analysis and a construction of a λ-system as in [14, Section 3]. Our presentation resembles [14, Appendix] (or [3, VII.3A]). The next definition yields an analog of the notion "a subfamily is free over an other subfamily".

Definition 4.7. Suppose $\mathscr{A}$ is an $R(\bar\kappa)$-family. A sequence $\langle \mathscr{A}_\alpha \mid \alpha < \lambda\rangle$ is called a filtration of $\mathscr{A}$ if it is a continuous increasing chain of subfamilies, such that the members have cardinality $< \lambda$ and the union of the members equals $\mathscr{A}$.

For every $\mathscr{A}' \subseteq \mathscr{A}$, $\bigsqcup \mathscr{A}'$ denotes the set $\bigcup_{\bar a \in \mathscr{A}'} \bigcup_{l < \mathrm{lh}(\bar a)} \mathrm{ran}(a_l)$.

Suppose $\bar{\mathscr{I}}$ is a fixed $\bar\kappa$-sequence of ideals and $\mathscr{A}_1, \mathscr{A}_2$ are subfamilies of $\mathscr{A}$. We denote by $\mathscr{A}_2/\mathscr{A}_1$ the family $\{\bar a \restriction L_{\bar a} \mid \bar a \in \mathscr{A}_2 \setminus \mathscr{A}_1$ and $L_{\bar a} \neq \emptyset\}$ ($\bar{\mathscr{I}}$ should be clear from the context), where $L_{\bar a}$ denotes those indices $l < \mathrm{lh}(\bar a)$ for which $\mathrm{ran}(a_l) \cap (\bigsqcup \mathscr{A}_1)$ is "small", i.e.,

$$L_{\bar{a}} = \left\{ l < \mathrm{lh}(\bar{a}) \mid \left\{ \tau \in \prod \bar{\kappa} \mid a_l(\tau) \in \bigsqcup \mathscr{A}_1 \right\} \in \prod \bar{\mathscr{I}} \right\}.$$

We say that $\mathscr{A}_2$ is free over $\mathscr{A}_1$, if the family $\mathscr{A}_2/\mathscr{A}_1$ is free (we omit the prefix $R_{\bar{\mathscr{I}}}(\bar{\kappa})$).

The next fact offers some basic facts needed to understand Lemma 4.9.

Fact 4.8. *Suppose $\mathscr{A}$ is an $R(\bar{\kappa})$-family and $\bar{\mathscr{I}}$ is a $\bar{\kappa}$-sequence of ideals.*
(a) *If $\langle \mathscr{A}_\alpha \mid \alpha < \delta \rangle$ is a continuous chain of subfamilies of $\mathscr{A}$ such that $\mathscr{A}_0$ is free and $\mathscr{A}_{\alpha+1}/\mathscr{A}_\alpha$ is free for every $\alpha < \delta$, then the family $\bigcup_{\alpha<\delta} \mathscr{A}_\alpha$ is free.*
(b) *Suppose $\mathscr{A}$ is of regular cardinality λ and $\mathscr{A}$ is free. Then for all filtrations $\langle \mathscr{A}_\alpha \mid \alpha < \lambda \rangle$ of $\mathscr{A}$, the set $\{\alpha < \lambda \mid \mathscr{A}/\mathscr{A}_\alpha \textit{ is free}\}$ contains a cub of λ.*

Proof. (a) Build an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for the union by induction on $\alpha < \delta$ using the following: Suppose $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \mathscr{A}_3$ are subfamilies of $\mathscr{A}$, $\mathbf{b}$ is an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}_2/\mathscr{A}_1$, and $\mathbf{c}$ is an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}_3/\mathscr{A}_2$. Then the concatenation $\mathbf{b} \frown \mathbf{c}$ is an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}_3/\mathscr{A}_1$, since w.l.o.g. for every $i \in I$, the intersection $\mathrm{ran}(b_i) \cap (\bigsqcup \mathscr{A}_1)$ is empty.

(b) Suppose $\langle b_i \mid i < \lambda \rangle$ is an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}$. By the demands on $\bar{\kappa}$ and $\bar{\kappa}$-sequence of ideals, each of the sets $\mathrm{ran}(b_i)$ has cardinality $\kappa_0 = \max \bar{\kappa}$. Let h_ξ, $\xi < \kappa_0$, be injective functions with domain λ such that for every $i < \lambda$, $h_\xi(i)$ is the ξ's element of $\mathrm{ran}(b_i)$ in some fixed enumeration. To prove the claim, it suffices to choose a cub C of λ so that for all $\alpha \in C$ and $i < \lambda$:

$$\text{if } h_\xi(i) \in \bigsqcup \mathscr{A}_\alpha \text{ for some } \xi < \kappa_0, \text{ then } \bar{a}_i \in \mathscr{A}_\alpha.$$

Then $\langle b_i \mid i < \lambda \text{ and } \bar{a}_i \notin \mathscr{A}_\alpha \rangle$ is an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal for $\mathscr{A}/\mathscr{A}_\alpha$. □

Lemma 4.9. *Suppose $\bar{\kappa}$ and $\bar{\mathscr{I}}$ are as in Definition* 4.4. *For all singular cardinals $\mu > \max \bar{\kappa}$, every almost $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-free family of cardinality μ is free, i.e.,* $\mathrm{NRT}_{\bar{\mathscr{I}}}(\mu, \bar{\kappa})$ *does not hold.*

Proof. The claim follows from [12, Theorem 2.1] or [1, Theorem in Section 0], because for a fixed family $\mathscr{A}$ of cardinality λ, the relation "$\mathscr{A}_2$ is free over $\mathscr{A}_1$" for subfamilies of $\mathscr{A}$ satisfies the demanded axioms. However, we briefly sketch why axioms I–V of [5, Theorem 5 in Section 4] hold. Define the set $S(\mathscr{A})$ of "the subalgebras of $\mathscr{A}$" to be the set of all subfamilies of $\mathscr{A}$. A subalgebra $\mathscr{A}_1 \in S(\mathscr{A})$ is free when $\mathscr{A}_1$ has an $R_{\bar{\mathscr{I}}}(\bar{\kappa})$-transversal, say $\mathbf{b}$, and a basis $\mathscr{F}$ of $\mathscr{A}_1$ is the set of all subalgebras $\mathscr{A}_2$ of $\mathscr{A}_1$ such that the corresponding restriction of $\mathbf{b}$ is a transversal for $\mathscr{A}_1/\mathscr{A}_2$ (the proof of Fact 4.8(b)). Note that a basis $\mathscr{F}$ is "fully closed unbounded above κ_0" as demanded in axiom II (use h_ξ's as in the proof of Fact 4.8(b)). Axioms I, III, and IV hold by the definition. Axiom V can be proved by a similar construction as in the proof of Fact 4.8(a). □

Our next task is to prove that $R(\bar\kappa)$-families can be transformed into a canonical form in the same way as families of countable sets are transformed into incompactness skeletons (Lemma 4.13). In order to succeed in the proof we have to assume the $R(\bar\kappa)$-family to be "neat" (Definition 4.12).

Definition 4.10. Suppose $\bar\kappa$ and $\bar{\mathscr{I}}$ are as in Definition 4.4 and λ is a regular cardinal greater than $\max\bar\kappa$. A tuple $\langle S, \bar\Lambda, \mathscr{A}\rangle$ is called an NRT$(\lambda,\bar\kappa)$-skeleton of type $\bar\lambda$, when the following conditions hold (recall the definitions of a canonical λ-set and a disjoint λ-system from Section 2):

- S is a λ-set of type $\bar\lambda = \langle \lambda_\rho \mid \rho \in S\setminus S_f\rangle$;
- S has a canonical form and its height is $n^* < \omega$;
- $\bar\Lambda = \langle \Lambda_{\rho\frown\langle\alpha\rangle} \mid \rho\in S\setminus S_f$ and $\alpha<\lambda_\rho\rangle$ is a disjoint λ-system;
- $\mathscr{A}$ is an $R(\bar\kappa)$-family of cardinality λ;
- $\mathscr{A}$ is enumerated by $\{\bar a_\eta \mid \eta\in S_f\}$;
- every sequence in $\mathscr{A}$ has a fixed length $l^*<\omega$, where $l^*\geqslant n^*$;
- $\mathscr{A}$ is based on $\bar\Lambda$, i.e., for all $\eta\in S_f$ and $l<l^*$, $\mathrm{ran}(a_{\eta,l})\subseteq \bigcup_{m\leqslant \mathrm{lh}(\eta)} \Lambda_{\eta\restriction m}$;
- $\mathscr{A}$ is almost $R_{\mathscr{I}_{\bar\kappa}^{\mathrm{nst}}}(\bar\kappa)$-free ($\mathscr{I}_{\bar\kappa}^{\mathrm{nst}}$ given in Definition 4.3).

Moreover, analogously to the definition of an NPT(λ,κ)-skeleton, we demand that for every $n<n^*$ and $\rho\in S$ of length n the following conditions hold:

When the cardinal θ_n, given by the canonical form of S (Definition 2.5), is well-defined:

- $\theta_n > \max\bar\kappa$ implies that there is $m<n^*$ such that the cardinal λ_m, given by the canonical form of S, is well-defined and $\lambda_m=\theta_n$;
- if $\aleph_0<\theta_n\leqslant\max\bar\kappa$ then $\theta_n\in\mathrm{ran}(\bar\kappa)$.

The index set l^* can be partitioned into n^* blocks $\langle L^{n+1}\mid n<n^*\rangle$ (analogously to the partition $\langle s_\eta^{n+1}\mid n<n^*\rangle$ of a set in an NPT$(\lambda,\aleph_0)$-skeleton) such that for every $n<n^*$, $l\in L^{n+1}$, and $\eta\in S_f$, $\mathrm{ran}(a_{\eta,l})\subseteq\Lambda_{\eta\restriction n+1}$ (by the disjointness of $\bar\Lambda$, for all $l\in L^{n+1}$ and $l'\in L^{n'+1}$, $\mathrm{ran}(a_{\eta,l})\cap\mathrm{ran}(a_{\eta',l'})=\emptyset$ whenever $\eta\restriction n\neq\eta'\restriction n$).

For all ρ in $S\setminus S_f$, $\langle S^\rho,\bar\Lambda^\rho,\mathscr{A}^\rho\rangle$ denotes the NRT$(\lambda_\rho,\bar\kappa)$-skeleton of type $\langle\lambda_\tau\mid\tau\in S^\rho\setminus S_f\rangle$, where

- S^ρ denotes the set $\{\eta\in S\mid\rho\subseteq\eta\}$;
- $\bar\Lambda^\rho$ is the restriction of $\bar\Lambda$ according to the new index set;
- $\mathscr{A}^\rho$ denotes the family $\{\bar a_\eta\restriction L^{>\mathrm{lh}(\rho)}\mid\eta\in S_f^\rho\}$, where

$$L^{>\mathrm{lh}(\rho)} = \bigcup_{\mathrm{lh}(\rho)\leqslant n<n^*} L^{n+1}.$$

Lastly, for every $\rho\in S$ of length $n>0$ and $\gamma<\rho(n-1)$, there are strictly less than λ_ρ indices $\eta\in S_f$ such that $\rho\subseteq\eta$ and for some $l\in L^{n+1}$, the set $\{\tau\in\prod\bar\kappa\mid a_{\eta,l}(\tau)\in\Lambda_{\rho\frown\langle\gamma\rangle}\}$ is not in $\mathscr{I}_{\bar\kappa}^{\mathrm{nst}}$.

Fact 4.11. *If $\langle S,\bar\Lambda,\mathscr{A}\rangle$ is an* NRT$(\lambda,\bar\kappa)$-*skeleton, then there is no injective choice function for the family $\{s_\eta\mid\eta\in S_f\}$, where each s_η is the set $\bigcup_{l<\mathrm{lh}(\bar a_\eta)}\mathrm{ran}(a_{\eta,l})$. In particular, there is no $R_{\mathscr{I}_{\bar\kappa}^{\mathrm{nst}}}(\bar\kappa)$-transversal for $\mathscr{A}$.*

Proof. The given family is based on the λ-system $\bar{\Lambda}$. Hence, the claim follows from [14, Claim 3.5] (or [3, VII.3.6]). $\square$

Definition 4.12. For technical reasons (needed in the proof of Lemma 4.13), we say that an $R(\bar{\kappa})$-family $\mathscr{A}$ is neat, when for all $\bar{a}, \bar{a}' \in \mathscr{A}$, $l < \mathrm{lh}(\bar{a})$, and $l' < \mathrm{lh}(\bar{a}')$: $a_l \neq a'_{l'}$ implies that the set

$$\left\{\tau(0) \mid \tau \in \prod \bar{\kappa} \text{ and } a_l(\tau) = a'_{l'}(\tau)\right\}$$

is a nonstationary subset of κ_0, if κ_0 is uncountable, and otherwise, the set is finite.

Remark. We do not demand neatness in the definition of an $\mathrm{NRT}(\lambda, \bar{\kappa})$-skeleton. Therefore, Facts 4.6(d)–(f) are examples of $\mathrm{NRT}(\lambda, \bar{\kappa})$-skeletons.

In the definition of the neatness the most natural demand would be that "intersection of two different coordinates" is small in the $\mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}$ sense, but such a definition would cause problems in the next lemma, since $\mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}$ is not κ_0-complete when $\bar{\kappa}$ has length > 1.

Lemma 4.13. *If $\mathscr{A}$ is a neat family witnessing that $\mathrm{NRT}_{\mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}}(\lambda, \bar{\kappa})$ holds, then there is an $\mathrm{NRT}(\lambda, \bar{\kappa})$-skeleton.*

Proof. We define an $\mathrm{NRT}(\lambda, \bar{\kappa})$-skeleton $\langle S', \bar{\Lambda}', \mathscr{A}' \rangle$ following the same ideas as in [14, Claim 3.3] or [14, Appendix: Proposition 4] (or [3, Proposition VII.3.7]), except that at the end we need a different trick.

Let $\lambda_\emptyset$ be λ itself. By Fact 4.8(a) there is a filtration $\langle \mathscr{A}_{\langle\alpha\rangle} \mid \alpha < \lambda_\emptyset \rangle$ of $\mathscr{A}$ and a stationary subset $E_\emptyset$ of λ such that for every $\alpha \in E_\emptyset$, $\mathscr{A}_{\langle\alpha+1\rangle} / \mathscr{A}_{\langle\alpha\rangle}$ is not free. For every $\alpha \in E_\emptyset$, choose $\mathscr{B}_{\langle\alpha\rangle} \subseteq \mathscr{A}_{\alpha+1} \setminus \mathscr{A}_\alpha$ so that $\lambda_{\langle\alpha\rangle} = \mathrm{card}(\mathscr{B}_{\langle\alpha\rangle}) < \lambda_\emptyset$ is the smallest cardinal such that $\mathscr{B}_{\langle\alpha\rangle}$ is not free over $\mathscr{A}_\alpha$.

Suppose $\lambda_{\langle\alpha\rangle} > \kappa_0$ holds. By the choice of $\mathscr{B}_{\langle\alpha\rangle}$, the family $\mathscr{B}_{\langle\alpha\rangle} / \mathscr{A}_{\langle\alpha\rangle}$ witnesses that $\mathrm{NRT}_{\mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}}(\lambda_{\langle\alpha\rangle}, \bar{\kappa})$ holds. By Lemma 4.9 $\lambda_{\langle\alpha\rangle}$ is a regular cardinal. By Fact 4.8(a) there is a filtration $\langle \mathscr{A}_{\langle\alpha,\beta\rangle} \mid \beta < \lambda_\alpha \rangle$ of $\mathscr{B}_{\langle\alpha\rangle}$ and a stationary subset $E_{\langle\alpha\rangle}$ so that $\mathscr{A}_{\langle\alpha,\beta+1\rangle}$ is not free over $\mathscr{A}_{\langle\alpha,\beta\rangle} \cup \mathscr{A}_{\langle\alpha\rangle}$. Hence, we may continue choosing a subset $\mathscr{B}_{\langle\alpha,\beta\rangle}$ of $\mathscr{A}_{\langle\alpha,\beta+1\rangle} \setminus (\mathscr{A}_\alpha \cup \mathscr{A}_{\langle\alpha,\beta\rangle})$ having the smallest possible cardinality $\lambda_{\langle\alpha,\beta\rangle} < \lambda_{\langle\alpha\rangle}$ for which $\mathscr{B}_{\langle\alpha,\beta\rangle}$ is not free over $\mathscr{A}_{\langle\alpha,\beta\rangle} \cup \mathscr{A}_{\langle\alpha\rangle}$.

Assume η is a sequence of ordinals such that its length is $n > 0$ and $\lambda_{\eta\restriction n-1} > \kappa_0 \geqslant \lambda_\eta$. Such sequences η form the final nodes S'_f of the desired λ-set S'. Suppose also that all the sets $\mathscr{A}_{\eta\restriction m}$, for nonzero $m \leqslant n$, together with $\mathscr{B}_\eta$ are already chosen so that $\mathscr{B}_\eta$ is not free over $\mathscr{A}^*_\eta$, where $\mathscr{A}^*_\eta$ is an abbreviation for $\bigcup_{0<m\leqslant n} \mathscr{A}_{\eta\restriction m}$. To simplify our explanations, let $\bar{a}/\mathscr{A}^*_\eta$ denote the sequence (possibly empty) in the singleton $\{\bar{a}\}/\mathscr{A}^*_\eta$, and on the other hand, let $\bar{a} \sqcap \mathscr{A}^*_\eta$ denote the "complementary" sequence $\langle a_l \mid l < \mathrm{lh}(\bar{a}) \text{ and } l \notin \mathrm{dom}(\bar{a}/\mathscr{A}^*_\eta)\rangle$. Now we would like to define that the new $R(\bar{\kappa})$-family is $\mathscr{A}' = \{\bar{a}'_\eta \mid \eta \in S'_f\}$, where each $\bar{a}'_\eta$ is $\bar{a} \sqcap \mathscr{A}^*_\eta$ for some $\bar{a} \in \mathscr{B}_\eta$. However, we should choose $\bar{a}'_\eta$ so that $\mathscr{A}'$ becomes based on a λ-system.

The desired λ-system $\bar{\Lambda}'$ is defined by setting for every $\rho \in S' \setminus S'_f$ and $\alpha < \lambda_\rho$, $\Lambda'_{\rho \frown \langle \alpha \rangle} = \bigsqcup \mathscr{A}_{\rho \frown \langle \alpha \rangle}$

Assume that $\mathscr{A}$ is enumerated by $\{\bar{a}_i \mid i < \lambda\}$. Let for every $i < \lambda$, $X_{\eta,i,l}$ be those indices for which $a_{i,l}$ takes value in the defined λ-system, i.e.,

$$X_{\eta,i,l} = \left\{ \tau \in \prod \bar{\kappa} \mid a_{i,l}(\tau) \in \bigsqcup \mathscr{A}^*_\eta \right\}.$$

We get a better candidate for $\bar{a}'_\eta$ by considering the sequence $\langle a_{i,l} \restriction X_{\eta,i,l} \mid l < \mathrm{lh}(\bar{a})$ and $l \notin \mathrm{dom}(\bar{a}/\mathscr{A}^*_\eta)\rangle$. Even this sequence is problematic. The domains of the coordinates are not equal to $\prod \bar{\kappa}$ as demanded in the definition of an $R(\bar{\kappa})$-family. (Why the domain should be of that form? Otherwise we run into difficulties when we combine two skeletons together in Lemma 4.18.) We can fix this in the same way as we did in the proof of Fact 4.5. Let $\pi_{\eta,i,l}$ be the following "continuous" map from $\prod \bar{\kappa}$ onto $X_{\eta,i,l}$ defined for every $\sigma \in \prod \bar{\kappa}$ by

> $\pi_{\eta,i,l}(\sigma) = \tau$ iff for each $k < \mathrm{lh}(\bar{\kappa})$, $\tau(k)$ is the $\sigma(k)'$s ordinal (in $\in$-order) of the set $\{\tau'(k) \mid \tau' \in X_{\eta,i,l}$ and $\tau' \restriction k = \tau \restriction k\}$.

The property of these maps used below is that if $X_{\eta,i,l}$ is not in $\mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}$, then for every $B \subseteq \prod \bar{\kappa}$:

$$\text{(A)} \quad \prod \bar{\kappa} \setminus B \in \mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}} \text{ iff } \pi^{-1}_{\eta,i,l}[X_{\eta,i,l} \setminus B] \in \mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}.$$

A new candidate for $\bar{a}'_\eta$ could be $\langle a_{i,l} \restriction X_{\eta,i,l} \circ \pi_{\eta,i,l} \mid a_{i,l} \in \bar{a}_i \sqcap \mathscr{A}_\eta^{*} \rangle$ for some i such that $\bar{a}_i \in \mathscr{B}_\eta$. But then we face "the real problem", why should such an $\mathscr{A}'$ be almost free? A final definition of $\bar{a}'_\eta$ will be a concatenation of finite number of sequences of the lastly given form. How to choose a suitable finite set?

We claim that the neatness of the original $\mathscr{A}$ guarantees that there is no "choice function" picking different functions from the sequences in $\mathscr{B}_\eta/\mathscr{A}_\eta^*$, i.e., there is no injective function f with domain $\mathscr{B}_\eta/\mathscr{A}_\eta^*$ such that $f(\bar{a}) \in \bar{a}$ for each $\bar{a}$ in the domain. Namely, assume that such a f exists and $\langle \bar{a}_\xi \mid \xi < \lambda_\eta \rangle$ enumerates $\mathscr{B}_\eta$. By induction on $\xi < \lambda_\eta$ define b_ξ to be the restriction $f(\bar{a}_\xi) \restriction (\prod \bar{\kappa} \setminus A_\xi)$, where

$$A_\xi = \left\{ \tau \in \prod \bar{\kappa} \mid f(\bar{a}_\xi)(\tau) \in \bigcup_{\zeta < \xi} \mathrm{ran}(f(\bar{a}_\zeta)) \right\}.$$

Since $\lambda_\eta \leqslant \kappa_0$ and for every $\xi \neq \zeta$, the set $\{\tau(0) \mid \tau \in \prod \bar{\kappa}$ and $f(\bar{a}_\xi)(\tau) = f(\bar{a}_\zeta)(\tau)\}$ is nonstationary in κ_0, A_ξ is in $\mathscr{J}^{\mathrm{nst}}_{\bar{\kappa}}$. Hence, $\langle b_\xi \mid \xi < \lambda_\eta \rangle$ is a transversal for $\mathscr{B}_\eta/\mathscr{A}_\eta^*$, contrary to the choice of $\mathscr{B}_\eta$.

Because the sequences in $\mathscr{B}_\eta$ are finite, the standard compactness argument (for the first-order logic) yields a finite subset F_η of $\mathscr{B}_\eta$ such that $F_\eta/\mathscr{A}_\eta^*$ does not have such a choice function (the neatness is needed because a similar argument cannot be applied to transversals).

Now we can define for every $\eta \in S'_f$, that I_η is such that $\{\bar{a}_i \mid i \in I_\eta\} = F_\eta$ and

$$\bar{a}'_\eta = \langle a_{i,l} \circ \pi_{\eta,i,l} \mid i \in I_\eta \text{ and } a_{i,l} \in \bar{a}_i \sqcup \mathscr{A}^*_\eta \rangle.$$

We show first that $\mathscr{A}'=\{\bar{a}'_\eta \mid \eta\in S'_f\}$ is almost free. Fix J to be a subset of S'_f having cardinality $<\lambda$. Since the given $\mathscr{A}$ is almost free, there is a transversal $\langle b_i \mid i\in\bigcup_{\eta\in J} I_\eta\rangle$ for $\{\bar{a}_i \mid i\in\bigcup_{\eta\in J} I_\eta\}$. We define a transversal $\langle b'_\eta \mid \eta\in J\rangle$ for $\{\bar{a}'_\eta \mid \eta\in J\}$. Fix an arbitrary η from J. Assume, toward a contradiction, that for every $i\in I_\eta$, b_i is a restriction of the coordinate a_{i,l_i} in $\bar{a}_i/\mathscr{A}^*_\eta$. But then for all $i\neq j\in I_\eta$, $a_{i,l_i}\neq a_{j,l_j}$ contrary to the choice of F_η. Therefore, there is $i\in I_\eta$ such that for some coordinate $a_{i,l}$ from the nonempty sequence $\bar{a}_i\sqcap\mathscr{A}^*_\eta$, b_i is a restriction of $a_{i,l}$. Thus $X_{\eta,i,l}$ is not in $\mathscr{J}^{\text{nst}}_{\bar{\kappa}}$. By (A), $\prod\bar{\kappa}\backslash\text{dom}(b_i)\in\mathscr{J}^{\text{nst}}_{\bar{\kappa}}$ implies that the set $\prod\bar{\kappa}\backslash Y_{\eta,i}$ is in $\mathscr{J}^{\text{nst}}_{\bar{\kappa}}$, where $Y_{\eta,i}={\pi_{\eta,i,l}}^{-1}[\text{dom}(b_i)\cap X_{\eta,i,l}]$. Hence, we may define $b'_\eta=b_i\circ\pi_{\eta,i,l}\restriction Y_{\eta,i}$.

It remains to show that all the other demands listed in Definition 4.10 can be fulfilled. Exactly as for the NPT$(\lambda,\aleph_0)$-skeletons, S', $\bar{\Lambda}'$, and $\mathscr{A}'$ can be modified so that S' is in the canonical form of type $\bar{\lambda}$ and height n^*, and moreover, $\bar{\Lambda}'$ is disjoint.

By choosing a suitable sub-λ-set of S' if necessary [14, Claim 3.2(1)], we may assume that for all $n<n^*$ and $\eta,\nu\in S'_f$, the sets $\{l<\text{lh}(\bar{a}'_\eta)\mid \text{ran}(a'_{\eta,l})\subseteq\Lambda'_{\eta\restriction n+1}\}$ and $\{l<\text{lh}(\bar{a}'_\nu)\mid \text{ran}(a'_{\nu,l})\subseteq\Lambda'_{\eta\restriction n+1}\}$ are equal. Hence, there exists l^* such that $\text{lh}(\bar{a}'_\eta)=l^*$ for every $\eta\in S'_f$, and also, the required partition $\langle L^{n+1}\mid n<n^*\rangle$ of l^* exists.

When $\aleph_0<\theta_n<\min\bar{\lambda}$ is well-defined one can add θ_n into $\bar{\kappa}$ (see Fact 4.16 below), if not yet appeared there. So assume that θ_n has the index k in $\bar{\kappa}$ and define $a''_{\eta,l}(\tau)=\langle a'_{\eta,l}(\tau),\alpha_{\tau(k)}\rangle$ for every $l<l^*$ and $\tau\in\prod\bar{\kappa}$, where $\langle\alpha_\xi\mid\xi<\kappa_k\rangle$ is an increasing sequence of ordinals cofinal in $\eta(n)$ (change $\bar{\Lambda}'$ accordingly).

For the rest of the properties (covering the case $\theta_n>\max\bar{\kappa}$), a suitable modification procedure is described in a very detailed way in [3, Theorem VII.3A.5] (e.g., the case that $\theta_n>\min\bar{\lambda}$ and $\theta_n\notin\bar{\lambda}$). □

Corollary 4.14. (a) *Suppose $\langle S,\bar{\Lambda},\mathscr{A}\rangle$ is an* NRT$(\lambda,\aleph_0)$*-skeleton of type $\bar{\lambda}$. Then there is a family $\mathscr{S}$ of countable sets such that the triple*

$$\langle S,\langle[\Lambda_\rho]^{<\aleph_0}\mid\rho\in S\backslash S_f\rangle,\mathscr{S}\rangle$$

forms an NPT$(\lambda,\aleph_0)$*-skeleton.*

(b) *For every uncountable cardinal λ,* NPT$(\lambda,\aleph_0)$*-holds if and only if there is a neat family exemplifying that* NRT$(\lambda,\aleph_0)$ *holds.*

Proof. The reader may wonder why a family witnessing that NRT$(\lambda,\aleph_0)$-holds is not "straightforwardly" an example of a family witnessing that NPT$(\lambda,\aleph_0)$-holds. The answer is that, because $R(\aleph_0)$-transversal is a stronger notion than the standard transversal, the nonfreeness in the $R(\aleph_0)$-sense does not guarantee the nonfreeness in the standard sense.

(a) The existence of an NPT$(\lambda,\aleph_0)$-skeleton follows from Fact 4.11 together with [14, Claim 3.8] or [3, VII.3A.6]. To see that the structure of the λ-system can be almost preserved, consider a family $\{s_\eta\mid\eta\in S_f\}$ such that each coordinate (w.r.t. to the slightly modified λ-system) s^{n+1}_η corresponds to a "well-chosen" subset of all finite sequences of elements in $\bigcup_{l\in L^{n+1}}\text{ran}(a_{\eta,l})$ (the reason to use finite sequences is the same as in the proof of Proposition 3.1).

(b) From left to right the claim follows from [14, Claim 3.8] or [3, VII.3A.6]. The other direction follows from Lemma 4.13 and (a). □

The following two small facts are needed in the proof of Lemma 4.18.

Fact 4.15. *For all* NRT$(\lambda,\bar\kappa)$*-skeletons* $\langle S,\bar\Lambda,\mathscr{A}\rangle$ *and* $I\subseteq S_f$, *I is small in* S_f *if and only if there exists an* $R_{\mathscr{I}^{\rm nst}_{\bar\kappa}}(\bar\kappa)$*-transversal for* $\{\bar a_\eta \mid \eta\in I\}$.

Proof. A proof of this fact is similar to a proof of the property (3.2) in Section 3. □

Fact 4.16. *If there is an* NRT$(\lambda,\bar\kappa)$*-skeleton of type* $\bar\lambda$, *then for every regular cardinal* $\chi<\min\bar\lambda$, *there exists an* NRT$(\lambda,\bar\chi)$*-skeleton of type* $\bar\lambda$, *where* $\mathrm{ran}(\bar\chi)=\mathrm{ran}(\bar\kappa)\cup\{\chi\}$.

Proof. Suppose $\bar\chi=\langle\chi_k\mid k<k^*\rangle$ is a decreasing enumeration of $\mathrm{ran}(\bar\kappa)\cup\{\chi\}$ and K denotes the index set $\{k<k^*\mid\chi_k\neq\chi\}$. For every $\eta\in S_f$ and $l<\mathrm{lh}(\bar a_\eta)$, replace the old coordinate $a_{\eta,l}$ with a new one $a'_{\eta,l}$, where $a'_{\eta,l}$ has domain $\prod\bar\chi$ and for every $\tau\in\prod\bar\chi$, $a'_{\eta,l}(\tau)=a_{\eta,l}(\tau\restriction K)$. Define a new family $\mathscr{A}'$ to be $\{\bar a'_\eta\mid\eta\in S_f\}$, where $\bar a'_\eta$ is $\langle a'_{\eta,l}\mid l<\mathrm{lh}(\bar a_\eta)\rangle$. The only problem is that $\mathscr{A}'$ should be almost $R_{\mathscr{I}^{\rm nst}_{\bar\chi}}(\bar\chi)$-free. However, for any $X\subseteq\prod\bar\kappa$, $\prod\bar\kappa\setminus X\in\mathscr{I}^{\rm nst}_{\bar\kappa}$ implies that $\{\tau\in\prod\bar\chi\mid\tau\restriction K\notin X\}$ is in $\mathscr{I}^{\rm nst}_{\bar\chi}$ as well. Hence, every $R_{\mathscr{I}^{\rm nst}_{\bar\kappa}}(\bar\kappa)$-transversal $\langle b_\eta\mid\eta\in I\rangle$ can be straightforwardly transformed into the form of an $R_{\mathscr{I}^{\rm nst}_{\bar\chi}}(\bar\chi)$-transversal $\langle b'_\eta\mid\eta\in I\rangle$, where each $\mathrm{dom}(b'_\eta)$ equals $\{\tau\in\prod\bar\chi\mid\tau\restriction K\in\mathrm{dom}(b_\eta)\}$. □

In the light of the previous fact and for purposes of the forthcoming "transitivity" lemma, it makes sense to define "compatible skeletons".

Definition 4.17. Suppose $\langle S',\bar\Lambda',\mathscr{A}'\rangle$ is an NRT$(\lambda,\bar\kappa)$-skeleton of type $\bar\lambda$ and $\langle S'',\bar\Lambda'',\mathscr{A}''\rangle$ is an NRT$(\sigma,\bar\chi)$-skeleton of type $\bar\sigma$. We say that $\langle S',\bar\Lambda',\mathscr{A}'\rangle$ is compatible with $\langle S'',\bar\Lambda'',\mathscr{A}''\rangle$ if the following conditions are satisfied:
(A) $\min\bar\lambda>\max\bar\sigma$;
(B) for all cardinals $\kappa\in\mathrm{ran}(\bar\kappa)$, if $\kappa\geqslant\min\bar\sigma$ then there is n below the height of S'' such that for every $\rho\in S''$ of length n, $\sigma_\rho=\kappa$.

Lemma 4.18. *Suppose that an* NRT$(\lambda,\bar\kappa)$*-skeleton* $\langle S',\bar\Lambda',\mathscr{A}'\rangle$ *of type* $\bar\lambda$ *is compatible with an* NRT$(\sigma,\bar\chi)$*-skeleton* $\langle S'',\bar\Lambda'',\mathscr{A}''\rangle$ *of type* $\bar\sigma$. *Then there exists an* NRT$(\lambda,\bar\kappa\cup\bar\chi)$*-skeleton* $\langle S,\bar\Lambda,\mathscr{A}\rangle$ *of type* $\bar\lambda\frown\bar\sigma$.

Proof. By Fact 4.16, we may assume that $\bar\chi$ is an end segment of $\bar\kappa$ (does not have any effect on compatibility and if $\bar\chi$ is $\langle\aleph_0\rangle$, just make sure that $\aleph_0$ is the last element of $\bar\kappa$). By the assumption on $\bar\sigma$, let $N\subseteq n''$ be such that the initial segment of $\bar\kappa$ of cardinals not in $\bar\chi$ is equal to $\langle\sigma_n\mid n\in N\rangle$. Thus, for every $\eta''\in S''_f$ and $\tau\in\prod\bar\chi$, the concatenation $\eta''\restriction N\frown\tau$ is a sequence in $\prod\bar\kappa$.

We define $\langle S,\bar\Lambda,\mathscr{A}\rangle$ in a straightforward manner by "concatenating" $\langle S',\bar\Lambda',\mathscr{A}'\rangle$ and several copies of $\langle S'',\bar\Lambda'',\mathscr{A}''\rangle$. So S_f is defined to be $\{\eta'\frown\eta''\mid\eta'\in S'_f$ and $\eta''\in S''_f\}$, and S is $\{\eta\restriction n \mid n<n^*\}$, where $n^*=n'+n''$, n' is the height of S', and n'' is the height of S''.

To define the other components, fix an arbitrary $\eta=\eta'\frown\eta''$ from S_f (where this notation means $\eta'\in S'_f$ and $\eta''\in S''_f$). Let l^* denote $l'+l''$, where l' and l'' are the lengths of the sequences in $\mathscr{A}'$ and $\mathscr{A}''$, respectively.

Define $\bar{a}_\eta=\langle a_{\eta,l} \mid l<l^*\rangle$, by setting for each $l<l'$ and $\tau\in\prod\bar{\chi}$,

$$a_{\eta,l}(\tau)=\langle l,\eta'',a'_{\eta',l}(\eta''\restriction N\frown\tau)\rangle$$

and for every l with $l'\leqslant l<l^*$ and $\tau\in\prod\bar{\chi}$,

$$a_{\eta,l}(\tau)=\langle l,\eta',a''_{\eta'',l-l'}(\tau)\rangle.$$

Define a λ-system by setting for every $\rho'\in S'\setminus S'_f$ and $\alpha<\lambda_{\rho'}$,

$$\Lambda_{\rho'\frown\langle\alpha\rangle}=l'\times S''_f\times\Lambda_{\rho'\frown\langle\alpha\rangle}$$

and for every $\eta'\in S'_f$, $\rho''\in S''\setminus S''_f$, and $\alpha<\sigma_{\rho''}$, set

$$\Lambda_{\eta'\frown\rho''\frown\langle\alpha\rangle}=l^*\times\{\eta'\}\times\Lambda''_{\rho''\frown\langle\alpha\rangle}.$$

Clearly S and $\bar{\Lambda}$ have the desired form and $\mathscr{A}=\{\bar{a}_\eta \mid \eta\in S_f\}$ is based on $\bar{\Lambda}$. So to prove that $\langle S,\bar{\Lambda},\mathscr{A}\rangle$ is an NRT(λ,θ)-skeleton, it remains to show that $\mathscr{A}$ is almost $R_{\mathscr{I}^{\text{nst}}_{\bar{\chi}}}(\bar{\chi})$-free.

Fix $I\subseteq S_f$ of cardinality $<\lambda$. We define an $R_{\mathscr{I}^{\text{nst}}_{\bar{\chi}}}(\bar{\chi})$-transversal $\langle b_\eta \mid \eta\in I\rangle$ for $\{\bar{a}_\eta \mid \eta\in I\}$. The set $I'=\{\eta\restriction n' \mid \eta\in I\}$ has cardinality $<\lambda$. Therefore, there is an $R_{\mathscr{I}^{\text{nst}}_{\bar{\kappa}}}(\bar{\kappa})$-transversal $\langle b_{\eta'} \mid \eta'\in I'\rangle$ for $\{\bar{a}'_{\eta'} \mid \eta'\in I'\}$. Fix now $\eta'\in I'$. Define $J_{\eta'}$ to be the set of all $\eta''\in S''_f$ such that

$$\left\{\tau\in\prod\bar{\chi} \mid \eta''\restriction N\frown\tau\notin\operatorname{dom}(b_{\eta'})\right\}\in\mathscr{I}^{\text{nst}}_{\bar{\chi}}.$$

Since $\prod\bar{\kappa}\setminus\operatorname{dom}(b_{\eta'})$ is in $\mathscr{I}^{\text{nst}}_{\bar{\kappa}}$ and $\bar{\chi}$ is an end segment of $\bar{\kappa}$, the complement $K_{\eta'}=S''_f\setminus J_{\eta'}$ must be small in S''_f (remember, $X\subseteq S''_f$ is small in S''_f if $\{\eta''\restriction N \mid \eta''\in X\}$ is in $\mathscr{I}^{\text{nst}}_{\bar{\kappa}\restriction k}$, where k is the largest index below $\operatorname{lh}(\bar{\kappa})$ with $\kappa_k>\max\bar{\chi}$). By Fact 4.15 there exists an $R_{\mathscr{I}^{\text{nst}}_{\bar{\chi}}}(\bar{\chi})$-transversal $\langle b_{\eta',\eta''} \mid \eta''\in K_{\eta'}\rangle$ for $\{\bar{a}''_{\eta''} \mid \eta''\in K_{\eta'}\}$.

Consider $\eta\in S_f$. If η'' is in $J_{\eta'}$, then define

$$b_\eta=a_{\eta,l}\restriction\left\{\tau\in\prod\bar{\chi} \mid \eta''\restriction N\frown\tau\in\operatorname{dom}(b_{\eta'})\right\},$$

where $l<l'$ is the index with $b_{\eta'}\subseteq a'_{\eta',l}$. Otherwise η'' is in $K_{\eta'}$, and we can define

$$b_\eta=a_{\eta,l}\restriction\operatorname{dom}(b_{\eta',\eta''}).$$

where $l=l'+m$ and $m<l''$ is the index with $b_{\eta',\eta''}\subseteq a''_{\eta'',l}$. $\square$

Corollary 4.19. *For all regular cardinals $\lambda>\kappa>\chi$, if* NRT(λ,κ) *and* NRT(κ,χ) *hold, then* NRT(λ,χ) *holds.*

5. Conclusion

Now, we may put the pieces together. By Definition 2.5, if λ is a regular cardinal below the possible first regular limit cardinal and S is a λ-set in the canonical form of height n^* and type $\langle \lambda_\rho \mid \rho \in S \setminus S_f \rangle$, then there exist sequences $\bar{\lambda} = \langle \lambda_n \mid n < n^* \rangle$ and $\bar{\theta} = \langle \theta_n \mid n < n^* \rangle$ such that for every $\rho \in S \setminus S_f$ of length n, both $\lambda_\rho = \lambda_n$ and $E^S_\rho \subseteq \operatorname{cof}(\theta_n)$ hold. For the rest of this section we assume that all the types of skeletons are given in this simplified form.

Remark. For simplicity we have chosen $\prod \bar{\kappa}$ to be the domain of the functions appearing in the elements of $R(\bar{\kappa})$-families. Hence, we must restrict ourselves below the possible first regular limit cardinal. It is possible to replace $\prod \bar{\kappa}$ with a "full" κ_0-set and $\mathscr{J}^{\text{nst}}_{\bar{\kappa}}$ with "small sets" w.r.t. the fixed κ_0-set, but that makes the notation unnecessarily complicated.

Definition 5.1. Let $\mathscr{C}_{\text{NRT}}$ denote the smallest set of cardinals such that
- $\aleph_0$ is in $\mathscr{C}_{\text{NRT}}$;
- if there exists an $\text{NRT}(\lambda, \bar{\kappa})$-skeleton of type $\bar{\lambda}$ such that there is no regular limit cardinal below λ^+ and $\operatorname{ran}(\bar{\kappa}) \subseteq \mathscr{C}_{\text{NRT}}$, then $\operatorname{ran}(\bar{\lambda}) \subseteq \mathscr{C}_{\text{NRT}}$.

Remark. One could expect that, analogously to Fact 4.5 and because of the transitivity property Lemma 4.18, it would suffice to consider $\text{NRT}(\lambda, \bar{\kappa})$-skeletons of height 1 only (i.e., that in the definition above we could assume $\bar{\lambda}$ has length 1). However, an analogous proof does not work for $\text{NRT}(\lambda, \bar{\kappa})$-skeletons because, even in a fixed "level n", a transversal may choose from several possible coordinates $a_{\eta,l}$, $l \in L^{n+1}$, contrary to the $\text{NPT}(\lambda, \aleph_0)$-case, where on each level n there is only one coordinate, namely s^{n+1}_η.

The benefit of Fact 4.5 is that we can separate levels of $\text{NPT}(\lambda, \aleph_0)$-skeletons to independent building blocks and combine those blocks in various ways. Of course in this countable case, we apply Corollary 4.14 (i.e., the coordinates $a_{\eta,l}$, $l \in L^{n+1}$, of level n in the definition of an $R(\aleph_0)$-family can be amalgamated to a single coordinate).

Lemma 5.2. *Suppose $\sigma < \lambda$ are cardinals in $\mathscr{C}_{\text{NRT}}$. There exists an* $\text{NPT}(\lambda, \aleph_0)$*-skeleton of type $\bar{\lambda}$ such that if σ is uncountable, $\sigma \in \operatorname{ran}(\bar{\lambda})$.*

Proof. If λ is the first uncountable cardinal in $\mathscr{C}_{\text{NRT}}$, i.e., $\lambda = \aleph_1$, then the claim holds by the definition. Suppose $\langle S^0, \bar{\Lambda}^0, \mathscr{A}^0 \rangle$ is an $\text{NRT}(\lambda, \bar{\kappa})$-skeleton of type $\bar{\chi}$, $\operatorname{ran}(\bar{\chi}) \cup \operatorname{ran}(\bar{\kappa}) \subseteq \mathscr{C}_{\text{NRT}}$, and the claim holds for all cardinals below λ. Note that the problematic case is $\max \bar{\chi} > \sigma > \min \bar{\chi}$.

By the induction hypothesis there is an $\text{NPT}(\max \bar{\kappa}, \aleph_0)$-skeleton $\langle S^1, \bar{\Lambda}^1, \mathscr{S}^1 \rangle$ of type $\bar{\theta}$ such that $\operatorname{ran}(\bar{\kappa}) \subseteq \operatorname{ran}(\bar{\theta})$ and if $\max \bar{\kappa} > \sigma > \aleph_0$ then $\sigma \in \operatorname{ran}(\bar{\theta})$ holds too. Since $\langle S^0, \bar{\Lambda}^0, \mathscr{A}^0 \rangle$ and $\langle S^1, \bar{\Lambda}^1, \mathscr{S}^1 \rangle$ are compatible, it follows from Lemma 4.18 and Corollary 4.14, that there is an $\text{NPT}(\lambda, \aleph_0)$-skeleton $\langle S^2, \bar{\Lambda}^2, \mathscr{S}^2 \rangle$ of type $\bar{\chi} \frown \bar{\theta}$.

Suppose $\sigma > \min \bar{\chi}$ (hence $\sigma \notin \mathrm{ran}(\bar{\theta})$) and n is the largest index with $\chi_n > \sigma > \chi_{n+1}$. By the induction hypothesis there is an $\mathrm{NPT}(\sigma, \aleph_0)$-skeleton $\langle S^3, \bar{\Lambda}^3, \mathscr{S}^3 \rangle$ of such a type that it contains the sequence $\bar{\sigma} = \langle \sigma \rangle \frown \langle \chi_m \mid n < m < n^* \rangle \frown \bar{\theta}$. By applying Facts 4.5 and 4.16 to $\langle S^2, \bar{\Lambda}^2, \mathscr{S}^2 \rangle$, there is an $\mathrm{NRT}(\lambda, \bar{\sigma})$-skeleton $\langle S^4, \bar{\Lambda}^4, \mathscr{A}^4 \rangle$ of type $\langle \chi_m \mid m \leqslant n \rangle$. Now $\langle S^3, \bar{\Lambda}^3, \mathscr{S}^3 \rangle$ and $\langle S^4, \bar{\Lambda}^4, \mathscr{A}^4 \rangle$ are compatible, and hence the claim follows from Lemma 4.18 together with Corollary 4.14. □

Conclusion 5.3.
(a) *For all cardinals* λ *in* $\mathscr{C}_{\mathrm{NRT}}$, $\mathrm{NPT}(\lambda, \aleph_0)$ *hold. In particular*, $\mathscr{C}_{\mathrm{NRT}}$ *is a nicely incompact set of regular cardinals in the sense of* [18, *Definition* 2.1].
(b) *If K is a nicely incompact set of regular cardinals below the possible first inaccessible cardinal, then* $K \subseteq \mathscr{C}_{\mathrm{NRT}}$.
(c) *All the cardinals from the set* $\mathscr{C}_{\aleph_0}$ *defined in* [9, *Theorem* 1 *of Section* 1] *belong to* $\mathscr{C}_{\mathrm{NRT}}$ (*or look at Theorem* 1.1). *Particularly, all the regular cardinals below* $\aleph_{\omega \cdot \omega+1}$ *are in* $\mathscr{C}_{\mathrm{NRT}}$, and $\mathscr{C}_{\mathrm{NRT}}$ *is cofinal below the first cardinal fixed point.*

Proof. (a) By Lemma 5.2.

(b) Suppose $\bar{\chi}$ is a type of some $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton. By induction on increasing order of the cardinals in $\bar{\chi}$, apply Fact 4.5 to show that $\mathrm{ran}(\bar{\chi}) \subseteq \mathscr{C}_{\mathrm{NRT}}$.

(c) By Facts 4.6(b) and (c). □

Our final conclusion concerns game-free groups. As we have seen in Proposition 3.1, game-freeness of an $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton $\langle S, \bar{\Lambda}, \mathscr{S} \rangle$ is closely connected to the possible value of $E^S_\emptyset$. Hence, we have to look at a little bit more restricted set of nicely incompact cardinals.

Definition 5.4. Let $\mathscr{C}_{\mathrm{GT}}$ denote the set of all cardinals λ in $\mathscr{C}_{\mathrm{NRT}}$ such that λ appears in a type of some $\mathrm{NRT}(\lambda, \bar{\kappa})$-skeleton $\langle S, \bar{\Lambda}, \mathscr{A} \rangle$ satisfying that $\bar{\kappa} \subseteq \mathscr{C}_{\mathrm{GT}}$, and moreover, if λ is a successor of a singular cardinal μ, then $E^S_\emptyset \cap \mathrm{cof}(\mathrm{cf}(\mu)) = \emptyset$.

Fact 5.5. *All the cardinals from the set* $\mathscr{C}_{\aleph_0}$ (*Conclusion* 5.3(c)) *belong to* $\mathscr{C}_{\mathrm{GT}}$. *In fact, all the examples in Fact* 4.6, *except the first two of them, yield skeletons fulfilling the cofinality demand in the definition of* $\mathscr{C}_{\mathrm{GT}}$.

Theorem 5.6. *Suppose* μ *is a cardinal such that both* $\mathrm{cf}(\mu)$ *and* $\lambda = \mu^+$ *are in* $\mathscr{C}_{\mathrm{GT}}$. *Then there exists a nonfree* $(\mu \cdot \theta)$*-game-free group of cardinality* λ, *where* $\theta < \mu$ *is a regular cardinal such that if* μ *is a successor of a regular cardinal then* $\mu = \theta^+$, *and otherwise* $\theta \neq \mathrm{cf}(\mu)$.

Proof. The proof proceeds by induction on increasing order of the cardinals in $\mathscr{C}_{\mathrm{GT}}$. By Väisänen [19, Lemmas 4.17, 4.23, and 4.29] it suffices to show existence of $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton such that if μ is regular then $\mathscr{S}$ is μ-game-free, and if μ is singular, then $\mathscr{S}$ is ε-game-free for every $\varepsilon < \mu$.

The case successor of a regular cardinal follows from [19, Fact 4.6 and Lemma 4.29], since as an induction hypothesis, we may assume that there exists a σ-game-free

NPT($\mu, \aleph_0$)-skeleton $\langle S, \bar{\Lambda}, \mathscr{S} \rangle$, where σ is a cardinal such that $\mu = \sigma^+$ and if σ is a regular cardinal, then $\sigma = \theta^+$ and $E^S_\emptyset \subseteq \mathrm{cof}(\theta)$.

The case successor of a singular cardinal follows from Conclusion 5.3 and Proposition 3.1, since $\mathscr{C}_{\mathrm{GT}} \subseteq \mathscr{C}_{\mathrm{NRT}}$ and a modification of Lemma 5.2 for $\mathscr{C}_{\mathrm{GT}}$ holds (thus, the demands of Proposition 3.1 can be fulfilled). □

6. On cub-game and game-free groups

In this section, μ is a singular cardinal, κ denotes the cofinality of μ, and λ is the successor cardinal of μ. We study existence of nonfree ε-game-free families of cardinality λ, when $\varepsilon > (\mu \cdot \mu) + \mu$. Our tools for that are some known and modified results about "cub-game" for successors of singular cardinals. An analog study for successors of regular cardinals, which is an easier case, is carried out in [19, Section 2].

Definition 6.1. Suppose λ is an uncountable regular cardinal $A \subseteq \lambda$, and $\varepsilon < \lambda$ is an ordinal. For notational purposes let x denote a set of regular cardinals below λ (x tells in which "cofinalities the limits are checked"). We denote by $\mathrm{GC}^x_\varepsilon(A, \lambda)$ the following two players cub-game. The players, player I (also called "outward" player) and Player II (also called "inward" player), choose in turns a sequence $\langle \alpha_i \mid i < \varepsilon \rangle$ of ordinals such that

- Player I chooses ordinals $\alpha_0 < \lambda$ and $\alpha_{i+2} < \lambda$, for $i < \varepsilon$, with $\alpha_{i+2} > \alpha_{i+1}$;
- if i is a limit ordinal, player I must choose the ordinal $\alpha_i = \sup_{j<i} \alpha_j$;
- when α_i is defined, player II must pick some ordinal $\alpha_{i+1} < \lambda$ with $\alpha_{i+1} > \alpha_i$.

Player II wins a play if for all limit ordinals i for which $\mathrm{cf}(i) \in x$, α_i belongs to A.

We say that player II wins $\mathrm{GC}^x_\varepsilon(A, \lambda)$ if there exists a winning strategy for player II in $\mathrm{GC}^x_\varepsilon(A, \lambda)$. For a regular cardinal θ below λ, $\mathrm{GC}^\theta_\varepsilon(A, \lambda)$ is a shorthand for $\mathrm{GC}^{\{\theta\}}_\varepsilon(A, \lambda)$.

Definition 6.2 (Shelah [18, Definition 1.9], Magidor and Shelah [9, Definition 2 of Section 1], Shelah [15, Section 1]). Suppose $\langle \bar{\mu}, \bar{f} \rangle$ is a scale for a singular cardinal μ of cofinality κ (Definition 2.9), where $\bar{\mu} = \langle \mu_\xi \mid \xi < \kappa \rangle$ and $\bar{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$. We denote by $S^{\mathrm{gd}}[\bar{f}]$ the set of all "good points α w.r.t. $\bar{f}$", i.e., α's below λ such that for some unbounded set $A \subseteq \alpha$ and $\zeta < \kappa$, $f_\alpha(\xi) < f_\beta(\xi)$ holds whenever $\alpha < \beta \in A$ and $\zeta \leqslant \xi < \kappa$. Let $S^{\mathrm{ngd}}[\bar{f}]$ denote the set of all limit ordinals $\alpha < \lambda$ such that $\alpha \notin S^{\mathrm{gd}}[\bar{f}]$ and $\mathrm{cf}(\alpha) > \kappa = \mathrm{cf}(\mu)$.

Proposition 6.3. *Assume that the following conditions are fulfilled*:

- *μ is a singular cardinal, $\lambda = \mu^+$, and $\mathrm{cf}(\mu) = \kappa$;*
- *there is a scale $\langle \bar{\mu}, \bar{f} \rangle$ for μ such that $S^{\mathrm{ngd}}[\bar{f}]$ is stationary in λ;*
- *there exists an $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton $\langle S, \bar{\Lambda}, \mathscr{S} \rangle$ of type $\bar{\lambda}$ such that $E^S_\emptyset \subseteq S^{\mathrm{ngd}}[\bar{f}]$;*
- *if κ is uncountable then there is n below the height of S such that for every $\rho \in S$ of length n, $\lambda_\rho = \kappa$ (where λ_ρ is from the type $\bar{\lambda}$).*

Then there exists a nonfree group of cardinality λ, which is δ-game-free for every $\delta < \lambda$ (and even in a "longer" game).

Proof. By Proposition 3.1 we may assume that $\langle S, \bar{\Lambda}, \mathscr{S}\rangle$ is chosen so that $\mathscr{S}$ is ε-game-free for every $\varepsilon < \mu$. By Väisänen [19, Lemma 4.23] we know that $\mathscr{S}$ is μ-game-free. Fix a δ below λ. As in [19, Lemma 4.29], we can fix a filtration $\langle \mathscr{S}_\alpha \mid \alpha < \lambda\rangle$ of $\mathscr{S}$, and ensure using a suitable bookkeeping, that after any "block" of μ moves by the players of the transversal game $\mathrm{GT}_\delta(\mathscr{S})$, there is some $\alpha < \lambda$ such that the elements chosen by the players are from $\mathscr{S}_\alpha$, and moreover, all the elements of $\mathscr{S}_\alpha$ has been chosen.

Suppose θ is a regular cardinal below μ such that $E^S_\emptyset \subseteq \mathrm{cof}(\theta)$. Note that $E^S_\emptyset \subseteq S^{\mathrm{ngd}}[\bar{f}]$ implies $\theta > \kappa$. By Lemmas 6.7 and 6.8, player II has a winning strategy in the cub-game $\mathrm{GC}^\theta_\delta(S^{\mathrm{gd}}[\bar{f}], \lambda)$.

During the transversal game $\mathrm{GT}_\delta(\mathscr{S})$, player II can additionally use his winning strategy in the cub-game $\mathrm{GC}^\theta_\delta(S^{\mathrm{gd}}[\bar{f}], \lambda)$ to ensure that after arbitrary many rounds of the blocks of μ moves in the transversal game, the elements chosen by the players are exactly the elements in $\mathscr{S}_\alpha$ for some α which is not in $E^S_\emptyset = \{\beta < \lambda \mid \mathscr{S}/\mathscr{S}_\beta$ is not λ-free$\}$ $\subseteq S^{\mathrm{ngd}}[\bar{f}] \cap \mathrm{cof}(\theta)$.

By [19, Lemma 4.25], the family $\mathscr{S}/\mathscr{S}_\alpha = \{s \setminus \bigcup \mathscr{S}_\alpha \mid s \in \mathscr{S} \setminus \mathscr{S}_\alpha\}$ is μ-game-free. Therefore, player II can continue the transversal game $\mathrm{GT}_\delta(\mathscr{S})$ one more round of the block of μ moves. During these new μ moves player II uses his bookkeeping and winning strategy in the cub-game again, and so on, up to all the required δ moves. Now, the claim follows from [19, Lemma 4.17]. □

Before changing the subject to the winning strategies in the cub-game, we ask: for which singular cardinals μ can the demands of the last proposition be fulfilled? We need few lemmas before the conclusion.

Lemma 6.4 (Shelah [13, Claim 27]). *Suppose χ is a supercompact cardinal, μ is a singular cardinal, $\kappa = \mathrm{cf}(\mu)$, $\kappa < \chi < \mu$, $\lambda = \mu^+$, and $\langle \bar{\mu}, \bar{f}\rangle$ is a scale for μ. There exists a singular cardinal $\sigma < \mu$ such that $\mathrm{cf}(\sigma) = \kappa$ and $S^{\mathrm{ngd}}[\bar{f}] \cap \mathrm{cof}(\sigma^+)$ is stationary in λ.*

Proof. Let us first show that $S^{\mathrm{ngd}}[\bar{f}]$ is stationary in λ. Suppose, contrary to this claim, that C is a cub of λ with $C \cap S^{\mathrm{ngd}}[\bar{f}] = \emptyset$ and j is an embedding from V onto an inner model M satisfying that $j(\xi) = \xi$ for every $\xi < \chi$, $j(\chi) \geqslant \lambda$, and $[M]^{\leqslant\lambda} \subseteq M$. Let δ be $\sup j[\lambda]$. The set $j[\lambda]$ is in M and δ is in $j(C)$, because C is a cub of λ, $\delta < \sup j(C) = j(\lambda)$, and $j[\lambda] \subseteq j(C)$. So δ is good w.r.t. $j(\bar{f})$ for $j(\mu)$ in M and $j[\lambda]$ is a cofinal subset of δ. By [9, Lemma 6] there exists a cofinal subset A of $j[\lambda]$ and ζ witnessing the goodness of δ. Define A' to be the set $j^{-1}[A]$. Then for every $\alpha < \beta \in A'$, the inequality $f_\alpha(\zeta) < f_\beta(\zeta)$ holds, since j is an elementary embedding, $j(\zeta) = \zeta$, $j(\alpha)$, $j(\beta) \in A$, and $(j(f))_{j(\alpha)}(\zeta) < (j(f))_{j(\beta)}(\zeta)$. This is a contradiction, since A' has cardinality λ, and the set $\{f_\alpha(\zeta) \mid \alpha < \lambda\}$ has cardinality $\mu_\zeta < \mu < \lambda$ (where μ_ζ is from $\bar{\mu}$).

Now the desired conclusion follows from the fact that λ is a successor of the singular cardinal μ of cofinality κ and δ has cofinality λ in M: If $\alpha \in C$ whenever $\sup C \cap \alpha = \alpha$ and α has cofinality ρ, where ρ is a successor of a singular cardinal having cofinality κ, then $j[C]$ has this property too. So the assumption $C \cap S^{\mathrm{ngd}}[\bar{f}] = \emptyset$ leads to a contradiction as above. □

Lemma 6.5. *Suppose κ is a regular cardinal, $\mu = \aleph_\kappa > \kappa$, $\lambda = \mu^+$, $\langle \bar{\mu}, \bar{f} \rangle$ is a scale for μ, and moreover, $2^\kappa = \kappa^+$. Then $S^{\text{ngd}}[\bar{f}] \subseteq \text{cof}(\kappa^+)$ and $S^{\text{ngd}}[\bar{f}] \cap \alpha$ is nonstationary in α, for every $\alpha < \lambda$.*

Proof. Suppose α below λ has cofinality $\sigma > \kappa^+$. We show that α is good (of course, all the ordinals of cofinality $<\kappa$ are good). By Shelah [16, II.1.2A(3)], $2^\kappa = \kappa^+$ implies that there exists an exact upper bound g of $\langle f_\beta \mid \beta < \alpha \rangle$ (i.e., g is an $<^*$-upper bound for $\langle f_\beta \mid \beta < \alpha \rangle$ such that for every $h \in \prod \bar{\mu}$, $h <^* g$ implies that $h <^* f_\beta$ for some $\beta < \alpha$).

Now, argue as in [9, Case 1 of the proof of Lemma 5 in Section 1]: By Magidor and Shelah [9, Lemma 7] the set $\{\xi < \kappa \mid \text{cf}(g(\xi)) > \sigma\}$ is in $\mathscr{J}_\kappa^{\text{bd}}$. Assume, toward a contradiction, that α is not good. By Magidor and Shelah [9, Lemma 6] $\langle \text{cf}(g(\xi)) \mid \xi < \kappa \rangle$ is not eventually constant, even though, its range is a subset of $\aleph_\kappa \cap (\sigma + 1)$ modulo $\mathscr{J}_\kappa^{\text{bd}}$. Since there are $<\kappa$ cardinals in the range of this sequence, there must exists different cardinals σ_1 and σ_2 such that both $\{\xi < \kappa \mid \text{cf}(g(\xi)) = \sigma_1\} \notin \mathscr{J}_\kappa^{\text{bd}}$ and $\{\xi < \kappa \mid \text{cf}(g(\xi)) = \sigma_2\} \notin \mathscr{J}_\kappa^{\text{bd}}$ hold. However, by [9, Lemma 8], $\sigma_1 = \sigma = \sigma_2$ must hold, a contradiction.

The claim on nonreflection follows from the definition of goodness: if α is in $S^{\text{gd}}[\bar{f}]$, then there is a closed unbounded subset of α consisting of ordinals in $S^{\text{gd}}[\bar{f}]$ only. □

Using [15, Fact 4.2] (similarly to [13, Conclusion 29]) we get the following conclusion.

Lemma 6.6. *Suppose χ is a supercompact cardinal,* GCH *holds, $\kappa < \aleph_\kappa$ is a regular cardinal below χ, μ is $\chi^{+\kappa}$, and λ is μ^+. Then there is a forcing extension, where ZFC +* GCH *holds, no bounded subset of κ is added, μ is the singular cardinal $\aleph_\kappa$, $\lambda = \mu^+$, and all the assumptions of Proposition* 6.3 *hold too.*

Proof. If $\text{NPT}(\kappa, \aleph_0)$ does not hold, shoot a nonreflecting stationary subset F of $\kappa \cap \text{cof}(\aleph_0)$ by a forcing notion described, e.g., in [17, Proof of Lemma 3.1]. This is for building the desired $\text{NPT}(\lambda, \aleph_0)$-skeleton at the end of this proof. Then no bounded subset of κ is added, χ remains supercompact, and GCH still holds.

By Lemma 6.4, there is a cardinal θ which is a successor of a singular cardinal $\sigma < \lambda$ so that $\text{cf}(\sigma) = \kappa$ and $S^{\text{ngd}}[\bar{f}] \cap \text{cof}(\theta)$ is stationary in λ. Let E_1 denote this stationary set. Note that by the form of σ and χ, $\sigma < \sigma^+ = \theta < \chi$ holds.

Using Levy collapse $\text{Col}(\kappa, <\sigma)$, collapse all the cardinals below σ to κ. Because of GCH and $\text{cf}(\sigma) = \kappa$, $\text{Col}(\kappa, <\sigma)$ has cardinality σ in V. Recall that $\text{Col}(\kappa, <\sigma)$ is κ-complete. So in $V^{\text{Col}(\kappa, <\sigma)}$, no bounded subset of κ is added, κ is a regular cardinal, F is a nonreflecting stationary subset of κ, E_1 is still a stationary subset of λ, σ has cardinality κ, θ is the cardinal κ^+, and GCH holds. Moreover, the κ-completeness and $\text{card}(\text{Col}(\kappa, <\sigma)) = \sigma < \theta < \lambda$ implies that $\langle \bar{\mu}, \bar{f} \rangle$ is still a scale for μ, and as mentioned in [15, 4.2(2)], if we let E_2 denote the set $S^{\text{ngd}}[\bar{f}]$ in $V^{\text{Col}(\kappa, <\sigma)}$, $E_1 = E_2$ modulo a cub of λ in $V^{\text{Col}(\kappa, <\sigma)}$.

Now use the Levy collapse $\text{Col}(\theta, <\chi)$, to collapse all the cardinals between $\theta = \kappa^+$ and χ. Since in $V^{\text{Col}(\kappa, <\sigma)}$, χ is still a strongly inaccessible cardinal, $\text{Col}(\theta, <\chi)$ has χ-c.c. Consequently, in the final extension, E_2 is still a stationary subset of λ, $\chi = \theta^+ =$

κ^{++}, $\mu = \aleph_\kappa$ (by the assumption $\kappa < \aleph_\kappa$), and $\lambda = \mu^+$. Since $\mathrm{Col}(\theta, <\chi)$ is also θ-complete, it follows that in the final extension, no bounded subset of κ is added, $\langle \bar{\mu}, \bar{f} \rangle$ remains as a scale for μ, GCH holds, $\theta = \kappa^+$, F is still a nonreflecting stationary subset of κ, and by [15, 4.2(2)] again, $S^{\mathrm{ngd}}[\bar{f}] = E_2$ modulo a cub of λ. Consequently $S^{\mathrm{ngd}}[\bar{f}]$ is stationary in $\lambda = \aleph_{\kappa+1}$.

It remains to show that the required type of $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton exists. By the properties of F, Facts 4.6(a) and (b), we get an $\mathrm{NPT}(\theta, \aleph_0)$-skeleton in whose type κ appears, if κ is uncountable. By Lemma 6.15, Fact 4.6(a), and Lemma 4.18 we get an $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton $\langle S, \bar{A}, \mathscr{S} \rangle$ in whose type θ appears, and also κ is in the type if κ is uncountable. Furthermore, the set $E^S_\emptyset$ equals $\lambda \cap \mathrm{cof}(\theta)$. Therefore, by shrinking $E^S_\emptyset$ to $S^{\mathrm{ngd}}[\bar{f}']$, one gets the required $\mathrm{NPT}(\lambda, \aleph_0)$-skeleton. □

In "extreme cases" $S^{\mathrm{gd}}[\bar{f}]$ does not contain a closed and unbounded subset of λ (for more cases, see [15, Fact 1.7] and [4, Claim 4.3]). However, $S^{\mathrm{gd}}[\bar{f}]$ is "almost" a cub of λ, because there is a winning strategy for the "inward" player in a very long cub-game.

Lemma 6.7. *Suppose $\lambda = \mu^+$ and $\mu > \mathrm{cf}(\mu) = \kappa$. We denote by $\mathrm{GC}^{\neq\kappa}_\varepsilon(A, \lambda)$ the game $\mathrm{GC}^x_\varepsilon(A, \lambda)$, where x is the set of all other regular cardinal below λ except κ. For every $\varepsilon < \mu$, player II has a winning strategy in the game $\mathrm{GC}^{\neq\kappa}_\varepsilon(S^{\mathrm{gd}}[\bar{f}], \lambda)$.*

Proof. First of all let ζ^* be the least index below κ with $\varepsilon < \mu_{\zeta^*}$. Suppose that the players has already chosen the ordinals $\langle \alpha_j \mid j \leqslant i \rangle$ and Player II should choose α_{i+1}. Assume that Player II has picked during the earlier rounds also functions $\langle h_j \mid j < i \text{ is odd} \rangle$ satisfying for each odd j that

- $h_j \in \prod_{\xi<\kappa} \mu_\xi$;
- for every odd $k < j$ and $\xi \in \kappa \setminus \zeta^*$, both $f_{\alpha_k}(\xi) < h_j(\xi)$ and $h_k(\xi) < h_j(\xi)$ hold;
- $h_j <^* f_{\alpha_j}$.

Firstly, Player II defines h_{i+1} by setting $h_{i+1}(\xi) = 0$ for all $\xi < \zeta^*$, and otherwise,

$$h_{i+1}(\xi) = \left(\sup_{j<i \text{ odd}} (\max\{f_{\alpha_j}(\xi), h_j(\xi)\}) \right) + 1.$$

Then h_{i+1} is in $\prod_{\xi<\kappa} \mu_\xi$, since μ_ξ is a regular cardinal greater than ε when $\xi \geqslant \zeta^*$. Secondly, player II picks α_{i+1} to be the least ordinal β above α_i satisfying that $h_{i+1} <^* f_\beta$. Such an ordinal exists because $\bar{f}$ is cofinal in $\prod_{\xi<\kappa} \mu_\xi$.

So it remains to show that for a limit i of cofinality not equal to κ, $\alpha_i = \sup\{\alpha_j \mid j < i\}$ belongs to $S^{\mathrm{gd}}[\bar{f}]$. If $\mathrm{cf}(i) < \kappa$, α_i is good. So assume that $\mathrm{cf}(i) > \kappa$. Let I be a cofinal subset of i having order type $\mathrm{cf}(i)$ and consisting of odd ordinals only (the moves of player II). For every $j \in I$, define $\xi_j < \kappa$ to be the smallest index with $h_j <_{\xi_j} f_{\alpha_j}$. Since $\mathrm{cf}(i) > \kappa$, there is a cofinal subset J of I and ζ such that for all $j \in J$, $\xi_j = \zeta$. But then $A = \{\alpha_j \mid j \in J\}$ is a cofinal subset of α_i and A together with ζ witness that α_i is good, since for all $k < j$ from J and for all $\xi \geqslant \zeta$,

$$f_{\alpha_k}(\xi) < h_j(\xi) < f_{\alpha_j}(\xi). \qquad \square$$

Remark. If $2^\mu = \lambda$ then the conclusion of the previous lemma follows from the definition of $I[\lambda]$ too, as explained in detail, e.g., in [6, Lemma 2.1].

Lemma 6.8. *Assume $\kappa \leqslant \mu$ are cardinals, λ is μ^+, $A \subseteq \lambda$, and x a set of regular cardinals below λ. Suppose that player* II *wins* $\mathrm{GC}^x_\varepsilon(A,\lambda)$ *for every $\varepsilon < \mu$, and moreover, if μ is a regular cardinal, player* II *wins* $\mathrm{GC}^x_{\mu+1}(A,\lambda)$. *Then player* II *wins* $\mathrm{GC}^x_\sigma(A,\lambda)$ *for every $\sigma < \lambda$.*

Proof. This is a known fact, presented e.g. in [19, Section 2]. □

It is again a known fact that the inward player has a winning strategy even in "much longer" games of the form $\mathrm{GC}^x_T(A,\lambda)$, where the length of the game is measured by the tree T of closed subsets of A of order type $\alpha+1$, $\alpha<\lambda$ (ordered by the end extension). Details are presented in [19, Section 2].

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