Generating ultrafilters in a reasonable way

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We continue investigations of *reasonable ultrafilters* on uncountable cardinals defined in Shelah [8]. We introduce a general scheme of generating a filter on λ from filters on smaller sets and we investigate the combinatorics of objects obtained this way.

0 Introduction

Reasonable ultrafilters were introduced in Shelah [8] in order to suggest a line of research that would in some sense repeat the beautiful theory created around the notion of *P-points on ω*. The definition of reasonable ultrafilters involves two conditions. The first one, called *the weak reasonability of an ultrafilter*, is a way to guarantee that we are not entering the realm of large cardinals: the considered ultrafilter is required to be very non-normal. Since this property will be also of some interest in the present paper, let us recall the following definition and observation.

Definition 0.1 (Shelah [8, Definition 1.4])

1. We say that a uniform ultrafilter D on λ is *weakly reasonable* if for every non-decreasing unbounded function $f \in {}^\lambda\lambda$ there is a club C of λ such that

$$\bigcup\{[\delta, \delta + f(\delta)) \,:\, \delta \in C\} \notin D.$$

2. Let D be an ultrafilter on λ, $C \subseteq \lambda$ be a club, and let $\langle \delta_\xi \,:\, \xi < \lambda \rangle$ be the increasing enumeration of $C \cup \{0\}$. We define

$$D/C = \{A \subseteq \lambda \,:\, \bigcup_{\xi \in A}[\delta_\xi, \delta_{\xi+1}) \in D\}.$$

(It is an ultrafilter on λ.) D/C will be called *the quotient of D by C*.

Observation 0.2 (Shelah [8, Observation 1.5]) *Let D be a uniform ultrafilter on a regular uncountable cardinal λ. Then the following conditions are equivalent:*

(A) *D is weakly reasonable.*

(B) *For every increasing continuous sequence $\langle \delta_\xi \,:\, \xi < \lambda \rangle \subseteq \lambda$ there is a club C^* of λ such that*

$$\bigcup\{[\delta_\xi, \delta_{\xi+1}) \,:\, \xi \in C^*\} \notin D.$$

(C) *For every club C of λ the quotient D/C does not extend the filter generated by clubs of λ.*

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The second part of the definition of reasonable ultrafilters is directly related to generalizing P-points to the context of weakly reasonable ultrafilters on an uncountable cardinal λ. To carry out this process we have to be somewhat creative in re-interpreting the property that any countable family of members of the ultrafilter has a pseudo-intersection in the ultrafilter. An interesting way of doing this is to impose some demands on how the ultrafilter on λ can be obtained from λ-sequences of objects on smaller cardinals (this approach was motivated by Rosłanowski and Shelah, see [4, §§5, 6].) For instance we may consider sequences $r = \langle (\alpha_\xi, d_\xi) : \xi < \lambda \rangle$ such that $\langle \alpha_\xi : \xi < \lambda \rangle$ is an increasing continuous sequence of ordinals below λ and d_ξ is an ultrafilter on the interval $[\alpha_\xi, \alpha_{\xi+1})$. For each such sequence r we look at the family of subsets of λ which are *eventually large in every interval* $[\alpha_\xi, \alpha_{\xi+1})$, that is, we consider the set

$$\mathrm{fil}(r) = \{A \subseteq \lambda : (\exists \zeta < \lambda)(\forall \xi > \zeta)(A \cap [\alpha_\xi, \alpha_{\xi+1}) \in d_\xi)\}.$$

(The set $\mathrm{fil}(r)$ is a filter on λ.) There is a natural quasi-order on sequences r as above: we say that $r \leq^* s$ if and only if $\mathrm{fil}(r) \subseteq \mathrm{fil}(s)$. Now the demand generalizing P-pointness may be phrased for an ultrafilter D on λ as follows: there is a $(< \lambda^+)$-directed (with respect to $\leq^*$) family H such that $D = \bigcup \{\mathrm{fil}(r) : r \in H\}$.

The ideas described above can be further generalized by allowing the use of arbitrary filters or just filters with some special properties in place of ultrafilters d_ξ. This immediately leads to the general scheme of generating filters on λ presented in our Definition 1.2.

This paper continues Shelah [8] and Rosłanowski and Shelah [3], but it is essentially self contained.

0.1 The content of the paper

In the first section we present our key definitions introducing *systems of local filters* and corresponding quasi-orders $\mathbb{Q}_\lambda^*(\mathcal{F}), \mathbb{Q}_\lambda^0(\mathcal{F})$. In Propositions 1.6, 1.8 we explain how those partial orders can be made $(< \lambda^+)$-complete (as we will be interested in directed subfamilies of $\mathbb{Q}_\lambda^*(\mathcal{F})$ and/or $\mathbb{Q}_\lambda^0(\mathcal{F})$). A directed subfamily H of $\mathbb{Q}_\lambda^*(\mathcal{F})$ and/or $\mathbb{Q}_\lambda^0(\mathcal{F})$ determines a filter $\mathrm{fil}(H)$ on λ. One of the basic questions is: how does the choice of H and/or of the system of local filters $\mathcal{F}$ influence the properties of the filter $\mathrm{fil}(H)$? Does the choice of $\mathcal{F}$ matter?

In Theorem 1.9 we show that ultrafilters generated by sufficiently directed generating systems are weakly reasonable, unless they are produced from a measurable ultrafilter. This result can be used as an argument that our re-interpretation of the P-pointness is very natural for weakly reasonable ultrafilters.

The second section is concerned with *the full system $\mathcal{F}^{\mathrm{ult}}$ of local ultrafilters* and the ultrafilters on λ generated by $H \subseteq \mathbb{Q}_\lambda^*(\mathcal{F}^{\mathrm{ult}})$. We show (see Theorem 2.3) that there may be weakly reasonable ultrafilters on λ generated by some $H' \subseteq \mathbb{Q}_\lambda^0(\mathcal{F})$ which cannot be obtained by the use of $\mathcal{F}^{\mathrm{ult}}$. Thus, in particular, the choice of the system of local filters may be important. Then we introduce more properties of families $H \subseteq \mathbb{Q}_\lambda^*(\mathcal{F}^{\mathrm{ult}})$ which are useful in generating ultrafilters on λ.

In the third section we introduce *pararegular filters* (Definition 3.1) and *the full system of local pararegular filters* $\mathcal{F}^{\mathrm{pr}}$. It occurs that filters $\mathrm{fil}(r)$ for $r \in \mathbb{Q}_\lambda^0(\mathcal{F}^{\mathrm{pr}})$ are related to generating numbers (in standard sense) of filters on λ (see Corollary 3.6, Proposition 3.8).

The referee of Shelah's paper [8] requested that the importance of the inaccessibility of λ in the assumptions of [8, Proposition 1.6(1)] is clarified. We pay this debt in the last section of the present paper and we show that one does need the assumption that λ is inaccessible for that result: Theorem 4.8 shows that consistently there is a very reasonable ultrafilter D on ω_1 such that Odd has a winning strategy in $\eth_D$.

Notation 0.3 Our notation is rather standard and compatible with that of classical textbooks (like Jech [1]).

1. Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta, \ldots$) and also by i, j (with possible sub- and superscripts). Cardinal numbers will be called κ, λ, μ (with possible sub- and superscripts). λ is always assumed to be an uncountable regular cardinal.

2. For two sequences η, ν we write $\nu \lhd \eta$ whenever ν is a proper initial segment of η, and $\nu \unlhd \eta$ when either $\nu \lhd \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.

3. We will use letters D, E, F, and d (with possible indexes) to denote filters on various sets. Typically, D will be a filter on λ (possibly an ultrafilter), while E, F will stand for filters on smaller sets. Also, in most cases d will be an ultrafilter on a set of size less than λ.

For a filter F of subsets of a set A, the family of all F-positive subsets of A is called F^+. (So $B \in F^+$ if and only if $B \subseteq A$ and $B \cap C \neq \emptyset$ for all $C \in F$.)

4. In forcing we keep the older convention that a stronger condition is the larger one. For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$-name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$-names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e. g. $\underset{\sim}{\tau}$, $\underset{\sim}{X}$).

1   Generating a filter from systems of local filters

Here we present the general scheme of generating a filter on a regular uncountable cardinal λ by using smaller filters. Our approach is slightly different from the one in [8, §2] and/or [3, §1], but the difference is notational only (see Remark 1.3 below).

Definition 1.1

1. A *system of local filters on* λ is a family $\mathcal{F}$ such that

   (a) all members of $\mathcal{F}$ are triples (α, Z, F) such that $Z \subseteq \lambda$, $|Z| < \lambda$, $\alpha = \min(Z)$, and F is a proper filter on Z;

   (b) the set $\{\alpha < \lambda : (\exists Z, F)((\alpha, Z, F) \in \mathcal{F})\}$ is unbounded in λ.

If above for each $(\alpha, Z, F) \in \mathcal{F}$, the set Z is infinite and F is a non-principal ultrafilter on Z, then we say that $\mathcal{F}$ *is a system of local non-principal ultrafilters*.

2. More generally, if Ψ is a property of filters, then a *system of local Ψ-filters on* λ is a system of local filters $\mathcal{F}$ such that for every $(\alpha, Z, F) \in \mathcal{F}$, the filter F has the property Ψ. *The full system of local Ψ-filters* is the family of *all* triples (α, Z, F) such that $\alpha < \lambda$, $\alpha \in Z \subseteq \lambda \setminus \alpha$, $|Z| < \lambda$, and F is a proper filter on Z with the property Ψ (assuming that it forms a system of local filters). The full system of local non-principal ultrafilters on λ is denoted by $\mathcal{F}_{\lambda}^{\mathrm{ult}}$ or just $\mathcal{F}^{\mathrm{ult}}$ (if λ is understood).

The next definition introduces the filters generated by some families of local filters. As we said in the introduction, our motivations have roots in forcings with norms and this suggested us to use sometimes a forcing-like notation (e. g. $\mathbb{Q}_{\lambda}^{*}$) similar to that of [4]. It is also worth noticing that some families of generators may be used as forcing notions – for instance $(\mathbb{Q}_{\lambda}^{0}, \leq^{*})$ is the forcing used in the end of [3, Section 1].

Definition 1.2 Let $\mathcal{F}$ be a system of local filters on λ.

1. We let $\mathbb{Q}_{\lambda}^{*}(\mathcal{F})$ be the family of all sets $r \subseteq \mathcal{F}$ such that

$$(\forall \xi < \lambda)(|\{(\alpha, Z, F) \in r : \alpha = \xi\}| < \lambda) \quad \text{and} \quad |r| = \lambda.$$

For $r \in \mathbb{Q}_{\lambda}^{*}(\mathcal{F})$ we define

$$\mathrm{fil}(r) = \{A \subseteq \lambda : (\exists \varepsilon < \lambda)(\forall (\alpha, Z, F) \in r)(\varepsilon \leq \alpha \Rightarrow A \cap Z \in F)\},$$

and we define a binary relation $\leq^{*} = \leq_{\mathcal{F}}^{*}$ on $\mathbb{Q}_{\lambda}^{*}(\mathcal{F})$ by

$$r_1 \leq_{\mathcal{F}}^{*} r_2 \text{ if and only if } (r_1, r_2 \in \mathbb{Q}_{\lambda}^{*}(\mathcal{F}) \text{ and}) \; \mathrm{fil}(r_1) \subseteq \mathrm{fil}(r_2).$$

2. We say that $r \in \mathbb{Q}_{\lambda}^{*}(\mathcal{F})$ *is strongly disjoint* if and only if

   (a) $(\forall \xi < \lambda)(|\{(\alpha, Z, F) \in r : \alpha = \xi\}| < 2)$, and

   (b) $(\forall (\alpha_1, Z_1, F_1), (\alpha_2, Z_2, F_2) \in r)(\alpha_1 < \alpha_2 \Rightarrow Z_1 \subseteq \alpha_2)$.

We let $\mathbb{Q}_{\lambda}^{0}(\mathcal{F})$ be the collection of all strongly disjoint elements of $\mathbb{Q}_{\lambda}^{*}(\mathcal{F})$.

3. We write $\mathbb{Q}_{\lambda}^{*}$, $\mathbb{Q}_{\lambda}^{0}$ for $\mathbb{Q}_{\lambda}^{*}(\mathcal{F}^{\mathrm{ult}})$, $\mathbb{Q}_{\lambda}^{0}(\mathcal{F}^{\mathrm{ult}})$, respectively (where, remember, $\mathcal{F}^{\mathrm{ult}}$ is the full system of local non-principal ultrafilters).

4. For a set $H \subseteq \mathbb{Q}_{\lambda}^{*}(\mathcal{F})$ we let $\mathrm{fil}(H) = \bigcup\{\mathrm{fil}(r) : r \in H\}$.

Remark 1.3

1. Note that if $r \in \mathbb{Q}_{\lambda}^{0}$, then there is $r' \in \mathbb{Q}_{\lambda}^{0}$ such that $\mathrm{fil}(r') = \mathrm{fil}(r)$ and for some club C of λ we have

$$\{(\alpha, Z) : (\exists d)((\alpha, Z, d) \in r')\} = \{(\alpha, [\alpha, \beta)) : \alpha \in C \, \& \, \beta = \min(C \setminus (\alpha + 1))\}.$$

Thus $\mathbb{Q}_{\lambda}^{0}$ is essentially the same as the one defined in [8, Definition 2.5].

2. If $H \subseteq \mathbb{Q}_{\lambda}^{*}(\mathcal{F})$ is $\leq^{*}$-directed, then $D = \mathrm{fil}(H)$ is a filter on λ which extends the filter of co-bounded sets. We may say that *the filter D is generated by H* or that *H is the generating system for D*.

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Definition 1.4 Suppose that

(a) X is a non-empty set and F is a filter on X,

(b) F_x is a filter on a set Z_x (for $x \in X$).

We let

$$\bigoplus_{x \in X}^F F_x = \{A \subseteq \bigcup_{x \in X} Z_x \ : \ \{x \in X \ : \ Z_x \cap A \in F_x\} \in F\}.$$

(Clearly, $\bigoplus_{x \in X}^F F_x$ is a filter on $\bigcup_{x \in X} Z_x$.) If X is a linearly ordered set (e. g. it is a set of ordinals) with no maximal element and F is the filter of all co-bounded subsets of X, then we will write $\bigoplus_{x \in X} F_x$ instead of $\bigoplus_{x \in X}^F F_x$.

Proposition 1.5 (Cf. [8, Proposition 2.9])

1. *Let $\mathcal{F}$ be a system of local filters on λ and $p, q \in \mathbb{Q}_\lambda^*(\mathcal{F})$. Then $p \leq^* q$ if and only if there is $\varepsilon < \lambda$ such that*

$$(\forall(\alpha, Z, F) \in q)(\forall A \in F^+)(\alpha > \varepsilon \Rightarrow (\exists(\alpha', Z', F') \in p)(A \cap Z' \in (F')^+)).$$

2. *Let $p, q \in \mathbb{Q}_\lambda^*$. Then the following are equivalent:*

   (a) $p \leq^* q$.

   (b) *There is $\varepsilon < \lambda$ such that*

   $$(\forall(\alpha, Z, d) \in q)(\forall A \in d)(\alpha > \varepsilon \Rightarrow (\exists(\alpha', Z', d') \in p)(A \cap Z' \in d')).$$

   (c) *There is $\varepsilon < \lambda$ such that if $(\alpha, Z, d) \in q$, $\varepsilon \leq \alpha$, and $X = \{(\xi, Z', d') \in p \ : \ Z' \cap Z \neq \emptyset\}$, then $X \neq \emptyset$ and there is an ultrafilter e on X such that*

   $$d = \{A \cap Z \ : \ A \in \bigoplus^e\{d' \ : \ (\exists \xi, Z')((\xi, Z', d') \in X)\}\}.$$

The quasi-orders $(\mathbb{Q}_\lambda^*, \leq^*)$ and $(\mathbb{Q}_\lambda^0, \leq^*)$ are $(< \lambda^+)$-complete (cf. [8, Proposition 2.3(3)]). Moreover, by essentially the same argument we may show the following observation.

Proposition 1.6 *Assume that $\mathcal{F}$ is a system of local filters on λ such that*

$(\oplus)_\mathcal{F}^{\mathrm{sum}}$ *if $\kappa < \lambda$ is an infinite cardinal and a sequence $\langle(\alpha_\xi, Z_\xi, F_\xi) \ : \ \xi < \kappa\rangle \subseteq \mathcal{F}$ satisfies $(\forall \xi < \zeta < \kappa)(Z_\xi \subseteq \alpha_\zeta)$, then for some uniform filter F on κ we have $(\alpha_0, \bigcup_{\xi < \kappa} Z_\xi, \bigoplus_{\xi < \kappa}^F F_\xi) \in \mathcal{F}$.*

Then both $\mathbb{Q}_\lambda^*(\mathcal{F})$ and $\mathbb{Q}_\lambda^0(\mathcal{F})$ are $(< \lambda^+)$-complete (with respect to $\leq^*$).

It is worth noticing that in general $\mathbb{Q}_\lambda^*(\mathcal{F})$ and/or $\mathbb{Q}_\lambda^0(\mathcal{F})$ do not have to be even σ-complete. For instance, consider the full system of co-bounded filters $\mathcal{F}_0$; it consists of all triples (α, Z, F) such that $\alpha \in Z \subseteq \lambda \setminus \alpha$, $|Z| < \lambda$, $\sup(Z) \notin Z$, and F is the filter of all co-bounded subsets of Z. Let C consist of all ordinals $\alpha < \lambda$ divisible by $\omega \cdot \omega$, and for $\alpha \in C$ and $m < \omega$ let $Z_m^\alpha = [\alpha + m \cdot \omega, \alpha + m \cdot \omega + \omega)$ and F_m^α be the filter of co-bounded subsets of Z_m^α. For $n < \omega$ put

$$p_n = \{(\alpha + m \cdot \omega, Z_m^\alpha, F_m^\alpha) \ : \ \alpha \in C \ \& \ 0 < m < \omega \ \& \ 2^n \mid m\}.$$

Clearly $p_n \in \mathbb{Q}_\lambda^0(\mathcal{F}_0)$ and $p_n \leq^* p_{n+1}$ for all $n < \omega$. One may easily verify that the sequence $\langle p_n \ : \ n < \omega\rangle$ has no $\leq^*$-upper bound in $\mathbb{Q}_\lambda^*(\mathcal{F}_0)$.

There is a natural procedure which for a given system $\mathcal{F}$ of local filters on λ generates a system $\mathcal{F}^* \supseteq \mathcal{F}$ satisfying the condition $(\oplus)_\mathcal{F}^{\mathrm{sum}}$ of Proposition 1.6 (so then $\mathbb{Q}_\lambda^*(\mathcal{F}^*)$ and $\mathbb{Q}_\lambda^0(\mathcal{F}^*)$ are suitably complete).

Definition 1.7 Assume that

(a) $\mathcal{F}$ is a system of local filters on λ,

(b) $\bar{E} = \langle E_\kappa \ : \ \kappa$ is a cardinal and $\aleph_0 \leq \kappa < \lambda\rangle$, where each E_κ is a uniform filter on κ.

We define:

1. An $(\bar{E}, \mathcal{F})$-*block* is a pair $(T, \bar{D})$ such that

   (i) $T \subseteq {}^{<\omega}\lambda$ is a well-founded tree;

   (ii) if $\eta \in T \setminus \max(T)$, then $\{\xi < \lambda \ : \ \eta^\frown\langle\xi\rangle \in T\} = \kappa$ for some infinite cardinal $\kappa < \lambda$;

   (iii) $\bar{D} = \langle(\alpha_\eta, Z_\eta, F_\eta) \ : \ \eta \in \max(T)\rangle \subseteq \mathcal{F}$;

   (iv) if $\eta, \nu \in \max(T)$ and $\eta <_{\mathrm{lex}} \nu$, then $Z_\eta \subseteq \alpha_\nu$ (where $<_{\mathrm{lex}}$ is the lexicographic order of T).

2. For an $(\bar{E}, \mathcal{F})$-block $(T, \bar{D})$ we define a filter $\bar{D}(T)$ on $\bigcup\{Z_\eta : \eta \in \max(T)\}$ by induction on the rank of the tree T (where $\bar{D} = \langle(\alpha_\eta, Z_\eta, F_\eta) : \eta \in \max(T)\rangle$).

   (i) If $\mathrm{rk}(T) = 0$, i.e., $T = \{\langle\rangle\}$, then $\bar{D}(T) = F_{\langle\rangle}$.

   (ii) Suppose $\mathrm{rk}(T) > 0$. Let $\kappa = \{\xi < \lambda : \langle\xi\rangle \in T\}$ (so $\aleph_0 \leq \kappa < \lambda$ is a cardinal). For $\xi < \kappa$ we put

$$T^\xi = \{\nu \in {}^{<\omega}\lambda : \langle\xi\rangle^\frown\nu \in T\} \quad \text{and} \quad \bar{D}^\xi = \langle(\alpha_\eta, Z_\eta, F_\eta) : \eta \in \max(T) \,\&\, \eta(0) = \xi\rangle.$$

Plainly, each $(T^\xi, \bar{D}^\xi)$ is an $(\bar{E}, \mathcal{F})$-block (and $\mathrm{rk}(T^\xi) < \mathrm{rk}(T)$). We define

$$\bar{D}(T) = \bigoplus\nolimits_{\xi<\kappa}^{E_\kappa} \bar{D}^\xi(T^\xi).$$

3. *The $\bar{E}$-closure of $\mathcal{F}$ is the family of all triples* (α, Z, D) such that $\alpha < \lambda$ and for some $(\bar{E}, \mathcal{F})$-block $(T, \bar{D})$ we have

$$Z = \bigcup\{Z_\eta : \eta \in \max(T)\} \quad \text{and} \quad D = \bar{D}(T) \quad \text{and} \quad \alpha = \min(Z)$$

(where $\bar{D} = \langle(\alpha_\eta, Z_\eta, F_\eta) : \eta \in \max(T)\rangle$).

Proposition 1.8 *Assume that*

(a) *$\mathcal{F}$ is a system of local filters on λ,*

(b) *$\bar{E} = \langle E_\kappa : \aleph_0 \leq \kappa < \lambda$ and κ is a cardinal$\rangle$, where each E_κ is a uniform filter on κ.*
Then the $\bar{E}$-closure of $\mathcal{F}$ is a system of local filters extending $\mathcal{F}$ and satisfying the condition $(\oplus)_{\mathcal{F}}^{\mathrm{sum}}$ of Proposition 1.6.

Suppose that a system $\mathcal{F}'$ of local filters on λ includes all triples $(\alpha, \{\alpha\}, d)$, where $\alpha < \lambda$ and d is the principal ultrafilter on $\{\alpha\}$. For a set $A \subseteq \lambda$ let $p_A = \{(\alpha, \{\alpha\}, d) \in \mathcal{F}' : \alpha \in A\} \in \mathbb{Q}_\lambda^0(\mathcal{F}')$. Note that $p_A \cap p_B = p_{A\cap B}$, so easily if D is a filter on λ extending the co-bounded filter, then $H^D \overset{\mathrm{def}}{=} \{p_A : A \in D\}$ is a $\leq^*$-directed family and $\mathrm{fil}(H^D) = D$. If D is a normal filter on λ, then H^D will be also $(< \lambda^+)$-directed (with respect to $\leq^*$). Consequently, if λ is a measurable cardinal, then we may find a system $\mathcal{F}$ of local filters on λ and a $(< \lambda^+)$-directed family $H \subseteq \mathbb{Q}_\lambda^0(\mathcal{F})$ such that $\mathrm{fil}(H)$ is an ultrafilter including all club subsets of λ (so $\mathrm{fil}(H)$ is not weakly reasonable). However, to have a quite directed family H such that $\mathrm{fil}(H)$ is a non-reasonable ultrafilter we do need a measurable cardinal.

Theorem 1.9 *Suppose that $\mathcal{F}$ is a system of local filters on λ, $\kappa \leq \lambda$, and $H \subseteq \mathbb{Q}_\lambda^*(\mathcal{F})$ is a $(< \kappa)$-directed family such that $\mathrm{fil}(H)$ is an ultrafilter. If $\mathrm{fil}(H)$ is not weakly reasonable, then for some club C^* of λ the quotient ultrafilter $\mathrm{fil}(H)/C^*$ is $(< \kappa)$-complete and it contains all clubs of λ.*

P r o o f. Assume that the family $H \subseteq \mathbb{Q}_\lambda^*(\mathcal{F})$ is $(< \kappa)$-directed and $\mathrm{fil}(H)$ is an ultrafilter which is not weakly reasonable. Let $\bar{\delta} = \langle\delta_\xi : \xi < \lambda\rangle$ be an increasing continuous sequence of ordinals below λ such that $\delta_0 = 0$ and for every club $C \subseteq \lambda$ we have that $\bigcup\{[\delta_\xi, \delta_{\xi+1}) : \xi \in C\} \in \mathrm{fil}(H)$. Now for a club C of λ and $p \in H$ put

$$S(p, C) = \{\xi \in C : (\exists(\alpha, Z, F) \in p)([\delta_\xi, \delta_{\xi+1}) \cap Z \in F^+)\}.$$

Claim 1.9.1 *For every club $C \subseteq \lambda$ and $p \in H$, the set $S(p, C)$ is stationary.*

P r o o f. Assume towards contradiction that $S(p, C)$ is non-stationary. So we may choose a club $C' \subseteq C$ of λ such that

$(*)_1 \qquad (\forall\xi \in C')(\forall(\alpha, Z, F) \in p)(Z \setminus [\delta_\xi, \delta_{\xi+1}) \in F)$.

Pick a club $C'' \subseteq C'$ such that

$(*)_2 \qquad (\forall(\alpha, Z, F) \in p)(\forall\xi \in C'')(\alpha < \delta_\xi \Rightarrow Z \subseteq \delta_\xi)$.

By the choice of $\bar{\delta}$ we know that $\bigcup\{[\delta_\xi, \delta_{\xi+1}) : \xi \in C''\} \in \mathrm{fil}(H)$, so necessarily

$$\bigcup\{[\delta_\xi, \delta_{\xi+1}) : \xi \in C''\} \in (\mathrm{fil}(p))^+.$$

Therefore we may pick $(\alpha, Z, F) \in p$ and $\xi \in C''$ such that $Z \cap [\delta_\xi, \delta_{\xi+1}) \in F^+$ (remember $(*)_2$), contradicting $(*)_1$. $\square$ (Claim 1.9.1)

Claim 1.9.2

1. If $p \leq^* q$, $p, q \in H$, and $C' \subseteq C$ are clubs of λ, then $|S(q, C') \setminus S(p, C)| < \lambda$.
2. If $A \subseteq \lambda$, then there are $p \in H$ and a club $C \subseteq \lambda$ such that either $S(p, C) \subseteq A$ or $S(p, C) \subseteq \lambda \setminus A$.

P r o o f.
1. Pick $\gamma < \lambda$ so that

$$(\forall (\alpha, Z, F) \in q)(\forall A \in F^+)(\alpha > \gamma \Rightarrow (\exists (\alpha', Z', F') \in p)(A \cap Z' \in (F')^+))$$

(remember Proposition 1.5) and let $\gamma^* < \lambda$ be such that

$$\gamma < \gamma^* \quad \text{and} \quad (\forall (\alpha, Z, F) \in q)(\alpha \leq \gamma \Rightarrow Z \subseteq \gamma^*).$$

Suppose that $\xi \in S(q, C') \setminus \gamma^*$. Then $\xi \in C' \subseteq C$ and there exists $(\alpha, Z, F) \in q$ such that $[\delta_\xi, \delta_{\xi+1}) \cap Z \in F^+$. Since $\delta_\xi \geq \xi \geq \gamma^*$, we also have $\alpha > \gamma$ and hence there is $(\alpha', Z', F') \in p$ such that

$$[\delta_\xi, \delta_{\xi+1}) \cap Z \cap Z' \in (F')^+.$$

Hence we may conclude that $\xi \in S(p, C)$.

2. Assume $A \subseteq \lambda$. Let $A^* = \bigcup \{[\delta_\xi, \delta_{\xi+1}) : \xi \in A\}$. Since $\mathrm{fil}(H)$ is an ultrafilter, then either A^* or $\lambda \setminus A^*$ belongs to it. Suppose $A^* \in \mathrm{fil}(p)$ for some $p \in H$. Pick a club $C \subseteq \lambda$ such that

$(\odot)$ if $(\alpha, Z, F) \in p$ and $(\sup(Z) + 1) \cap C \neq \emptyset$, then $A^* \cap Z \in F$.

Suppose $\xi \in S(p, C)$, so $\xi \in C$ and for some $(\alpha, Z, F) \in p$ we have $[\delta_\xi, \delta_{\xi+1}) \cap Z \in F^+$. It follows from $(\odot)$ that $A^* \cap Z \in F$ and therefore $\xi \in A$. Thus $S(p, C) \subseteq A$.

If $\lambda \setminus A^* \in \mathrm{fil}(H)$, then we proceed in an analogous manner.   □ (Claim 1.9.2)

Let

$$D = \{A \subseteq \lambda : |S(p, C) \setminus A| < \lambda \text{ for some } p \in H \text{ and a club } C \subseteq \lambda\}.$$

By Claim 1.9.1, all members of D are stationary, and since H is directed we may use Claim 1.9.2, 1. to argue that D is a filter on λ. By Claim 1.9.2, 2. we see that D is an ultrafilter on λ (so it also contains all clubs as its members are stationary). Since H is $(< \kappa)$-directed and the intersection of $< \kappa$ many clubs is a club, we may conclude from Claim 1.9.2, 1. that D is a $(< \kappa)$-complete ultrafilter.

Let $C^* = \{\delta_\xi : \xi < \lambda\}$ (so it is a club of λ). To complete the proof of the theorem we are now going to show that $D = \mathrm{fil}(H)/C^*$. Since we already know that D is an ultrafilter, it suffices to show that $S(p, C) \in \mathrm{fil}(H)/C^*$ for every $p \in H$ and a club $C \subseteq \lambda$. So let $C \subseteq \lambda$ be a club, $p \in H$, and let $S^* = \bigcup \{[\delta_\xi, \delta_{\xi+1}) : \xi \in S(p, C)\}$. If $S^* \in \mathrm{fil}(H)$, then we are done, so assume $S^* \notin \mathrm{fil}(H)$. Since $\mathrm{fil}(H)$ is an ultrafilter and H is directed, we may find $q \in H$ such that $p \leq^* q$ and $\lambda \setminus S^* \in \mathrm{fil}(q)$. Let $\gamma < \lambda$ be such that

$$(\forall (\alpha, Z, F) \in q)(\gamma \leq \sup(Z) \Rightarrow Z \setminus S^* \in F).$$

Since $|S(q, C) \setminus S(p, C)| < \lambda$, we may pick $\xi \in S(q, C) \cap S(p, C)$ such that $\xi > \gamma$. Then $[\delta_\xi, \delta_{\xi+1}) \subseteq S^*$ but also there is $(\alpha, Z, F) \in q$ such that $[\delta_\xi, \delta_{\xi+1}) \cap Z \in F^+$, and thus also $S^* \cap Z \in F^+$. However,

$$\sup(Z) \geq \delta_\xi > \gamma,$$

so $Z \setminus S^* \in F$ by the choice of γ, a contradiction showing that $S^* \in \mathrm{fil}(H)$ as required.   □

2   Systems of local ultrafilters

In this section we are interested in the full system $\mathcal{F}^{\mathrm{ult}}$ of local ultrafilters on λ and $\mathbb{Q}^*_\lambda, \mathbb{Q}^0_\lambda$. The first question that one may ask is whether weakly reasonable ultrafilters on λ generated by some $H \subseteq \mathbb{Q}^0_\lambda(\mathcal{F})$ can be obtained by the use of $\mathbb{Q}^0_\lambda$. It occurs that it does matter which system of local filters we are using.

Definition 2.1 A filter F on a set Z is called an *unultra filter* if for every $A \in F^+$ there is $B \subseteq A$ such that both $B \in F^+$ and $A \setminus B \in F^+$. The full system of local unultra filters on λ will be denoted by $\mathcal{F}^{\mathrm{unu}}$. (Thus $\mathcal{F}^{\mathrm{unu}}$ consists of all triples (α, Z, F) such that $\emptyset \neq Z \subseteq \lambda$, $|Z| < \lambda$, $\alpha = \min(Z)$, and F is an unultra filter on Z.)

Observation 2.2

1. *If F is an unultra filter on Z, $A \in F^+$, then $F + A \stackrel{\mathrm{def}}{=} \{B \subseteq Z \,:\, B \cup (Z \setminus A) \in F\}$ is an unultra filter.*

2. *Suppose that ξ is a limit ordinal, $\{Z_\zeta \,:\, \zeta < \xi\}$ is a family of pairwise disjoint non-empty sets, F_ζ is a filter on Z_ζ (for $\zeta < \xi$). Then $\bigoplus_{\zeta < \xi} F_\zeta$ is an unultra filter on $\bigcup_{\zeta < \xi} Z_\xi$.* (*Remember the convention declared in the last sentence of Definition* 1.4.)

Theorem 2.3 *Assume $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. There exists a $\leq^*$-increasing sequence*

$$\langle p_\xi \,:\, \xi < \lambda^+ \rangle \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{unu}})$$

such that

(a) $\mathrm{fil}(\{p_\xi \,:\, \xi < \lambda^+\})$ *is a weakly reasonable ultrafilter on λ, but*

(b) *there is no $p \in \mathbb{Q}^0_\lambda$ with $\mathrm{fil}(p) \subseteq \mathrm{fil}(\{p_\xi \,:\, \xi < \lambda^+\})$.*

P r o o f. Fix enumerations

1. $\langle Y_\zeta \,:\, \zeta < \lambda^+ \text{ and } \zeta \text{ is limit} \rangle$ of all subsets of λ, and

2. $\langle r_\zeta \,:\, \zeta < \lambda^+ \text{ and } \zeta \text{ is limit} \rangle$ of $\mathbb{Q}^0_\lambda$, and

3. $\langle \bar{\delta}^\zeta \,:\, \zeta < \lambda^+ \text{ and } \zeta \text{ is limit} \rangle$ of all the increasing continuous sequences $\bar{\delta}^\zeta = \langle \delta^\zeta_\alpha \,:\, \alpha < \lambda \rangle$ of ordinals below λ.

By induction on $\xi < \lambda^+$ we choose $p_\xi \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{unu}})$ so that the following conditions are satisfied for every limit ordinal $\zeta < \lambda^+$.

(o) For $n < \omega$, the element $p_n \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{unu}})$ is

$$\{(\alpha, Z_\alpha, F_\alpha) \,:\, \alpha < \lambda \text{ is limit}, Z_\alpha = [\alpha, \alpha + \omega), \text{ and } F_\alpha \text{ is the filter of co-finite subsets of } Z_\alpha\}.$$

(i) If $\mathrm{cf}(\zeta) < \lambda$, then for some increasing and cofinal in ζ sequence $\langle \zeta_i \,:\, i < \mathrm{cf}(\zeta) \rangle$, for every $(\alpha, Z, F) \in p_\zeta$, there is a sequence $\langle (\alpha_i, Z_i, F_i) \,:\, i < \mathrm{cf}(\zeta) \rangle$ such that

   1) $(\alpha_i, Z_i, F_i) \in p_{\zeta_i}$;
   2) $Z_i \subseteq \alpha_j$ for $i < j < \mathrm{cf}(\zeta)$;
   3) $Z = \bigcup_{i < \mathrm{cf}(\zeta)} Z_i$, and $F = \bigoplus \{F_i \,:\, i < \mathrm{cf}(\zeta)\}$.

(ii) If $\mathrm{cf}(\zeta) = \lambda$, then for some increasing and cofinal in ζ sequence $\langle \zeta_i \,:\, i < \lambda \rangle$, we have:

   if $(\alpha, Z, F) \in p_\zeta$ and $\mathrm{otp}(\{\alpha' < \alpha \,:\, (\exists Z', F')((\alpha', Z', F') \in p_\zeta)\}) = j$,
   then $(\alpha, Z, F) \in p_{\zeta_j}$ and $(\forall i < j)(\forall A \in F^+)(\exists (\beta, W, D) \in p_{\zeta_i})(A \cap W \in D^+)$.

(iii) If $|\{(\alpha, Z, F) \in p_\zeta \,:\, Y_\zeta \cap Z \in F^+\}| = \lambda$, then

$$p_{\zeta+1} = \{(\alpha, Z, F + [Y_\zeta \cap Z]) \,:\, (\alpha, Z, F) \in p_\zeta \,\&\, Y_\zeta \cap Z \in F^+\},$$

and otherwise $p_{\zeta+1} = \{(\alpha, Z, F) \in p_\zeta \,:\, Z \setminus Y_\zeta \in F\}$.

(iv) $p_{\zeta+2} \subseteq p_{\zeta+1}$ and for some club C of λ, for every $\beta \in C$ we have:

   1) $Z \subseteq \delta^\zeta_\beta$ whenever $(\alpha, Z, F) \in p_{\zeta+2}$, $\alpha < \delta^\zeta_\beta$;
   2) $\delta^\zeta_{\beta+1} < \min(\alpha \geq \delta^\zeta_\beta \,:\, (\exists Z, F)((\alpha, Z, F) \in p_{\zeta+2}))$.

(v) $p_{\zeta+3} = \{(\alpha, Z, F + A_\alpha) \,:\, (\alpha, Z, F) \in p_{\zeta+2}\}$, where for every $(\alpha, Z, F) \in p_{\zeta+2}$ the set $A_\alpha \in F^+$ is such that $(\forall (\beta, Y, d) \in r_\zeta)(A_\alpha \cap Y \notin d)$.

(vi) $p_{\zeta+3+n} = p_{\zeta+3}$ for all $n < \omega$.

(vii) For every $r \in \mathbb{Q}^0_\lambda$ and $\xi < \lambda^+$, if $(\alpha, Z, F) \in p_\xi$ and $A \in F^+$, then there is $A' \subseteq A$ such that

$$A' \in F^+ \quad \text{and} \quad (\forall (\beta, Y, d) \in r)(A' \cap Y \notin d).$$

Conditions (o) – (vi) fully describe how the construction is carried out and Proposition 1.5, Observation 2.2 imply that $\langle p_\xi : \xi < \lambda^+\rangle \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{unu}})$ is $\leq^*$-increasing. However, we have to argue that the demand in (vii) is satisfied, as it is crucial for the possibility of satisfying the demand in (v). Let $r \in \mathbb{Q}^0_\lambda$. By induction on $\xi < \lambda$ we show that for every $(\alpha, Z, F) \in p_\xi$ we have:

$(\boxdot)_{(\alpha,Z,F)}$ if $A \in F^+$, then there is $A' \subseteq A$ such that $A' \in F^+$ and $(\forall(\beta, Y, d) \in r)(A' \cap Y \notin d)$.

(For a set A' as above we will say that *it works for F and r*.)

S t e p $\xi < \omega$.

Note that for each limit ordinal $\alpha < \lambda$ there is at most one $(\beta, Y, d) \in r$ such that $Y \cap [\alpha, \alpha + \omega)$ is infinite. Assume $A \subseteq [\alpha, \alpha + \omega)$ is infinite. Considering any two disjoint infinite sets $A', A'' \subseteq A$ we easily see that one of them must work for the filter of co-finite subsets of $[\alpha, \alpha + \omega)$ and r.

S t e p $\xi = \zeta + n + 1$, $\zeta < \lambda^+$ is limit, $n < \omega$.

If $(\alpha, Z, F) \in p_{\zeta+n}$, $A \in F^+$, $A \subseteq A^*$, and $A' \subseteq A$ works for F and r, then also A' works for $F + A^*$ and r.

S t e p $\xi = \zeta < \lambda^+$ is limit.

Suppose that $(\alpha, Z, F) \in p_\zeta$. If $\mathrm{cf}(\zeta) = \lambda$, then $(\alpha, Z, F) \in p_{\xi'}$ for some $\xi' < \zeta$ (see (ii)) so the inductive hypothesis applies directly. So assume $\mathrm{cf}(\zeta) < \lambda$. Then $Z = \bigcup_{i < \mathrm{cf}(\zeta)} Z_i$ and $F = \bigoplus\{F_i : i < \mathrm{cf}(\zeta)\}$ for some sequence $\langle(\alpha_i, Z_i, F_i) : i < \mathrm{cf}(\zeta)\rangle$ such that

1. $(\boxdot)_{(\alpha_i, Z_i, F_i)}$ holds for each $i < \mathrm{cf}(\zeta)$;
2. $Z_i \subseteq \alpha_j$ for $i < j < \mathrm{cf}(\zeta)$.

Let $\alpha^* = \sup(Z)$ and let $A \in F^+$. Now we consider three cases.

C a s e A: For some $\alpha' < \alpha^*$ we have $(\forall(\beta, Y, d) \in r)(Y \cap [\alpha', \alpha^*) = \emptyset)$.

Plainly, the set $A' = A \setminus \alpha'$ works for F and r.

C a s e B: For some $(\beta, Y, d) \in r$ we have $\beta < \alpha^* \leq \sup(Y)$.

For each $i < \mathrm{cf}(\zeta)$ such that $A \cap Z_i \in (F_i)^+$ choose disjoint sets $A_i^0, A_i^1 \in (F_i)^+$ included in $A \cap Z_i$ (remember each F_i is an unultra filter) and let

$$A^\ell = \bigcup\{A_i^\ell : i < \mathrm{cf}(\zeta) \,\&\, A \cap Z_i \in (F_i)^+\} \setminus \beta \subseteq A$$

(for $\ell < 2$). Both $A^0 \in F^+$ and $A^1 \in F^+$, and one of these two sets works for F and r.

C a s e C: For each $\alpha' < \alpha^*$ there is $(\beta, Y, d) \in r$ such that $\alpha' < \beta < \sup(Y) < \alpha^*$.

Let $A \in F^+$. Then the set $I = \{i < \mathrm{cf}(\zeta) : A \cap Z_i \in (F_i)^+\}$ is unbounded in $\mathrm{cf}(\zeta)$. Using the assumptions of the current case we may choose an increasing sequence $\langle i_j : j < \mathrm{cf}(\zeta)\rangle \subseteq I$ such that for every $(\beta, Y, d) \in r$ there is at most one $j < \mathrm{cf}(\zeta)$ such that $Z_{i_j} \cap Y \neq \emptyset$. For each $j < \mathrm{cf}(\zeta)$ pick $A'_{i_j} \in (F_{i_j})^+$ included in $A \cap Z_{i_j}$ which works for F_{i_j} and r, and then put $A' = \bigcup_{j < \mathrm{cf}(\zeta)} A'_{i_j}$. $\square$

Problem 2.4 *Is it provable in* ZFC *that for some system $\mathcal{F}$ of local filters on λ there exists a $\leq^*$-directed family $H \subseteq \mathbb{Q}^*_\lambda(\mathcal{F})$ such that*

(a) $\mathrm{fil}(H)$ *is a weakly reasonable ultrafilter on λ, but*

(b) *there is no $\leq^*$-directed family $H' \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{ult}})$ such that $\mathrm{fil}(H) = \mathrm{fil}(H')$?*

The assumption that a generating system $H \subseteq \mathbb{Q}^*_\lambda(\mathcal{F})$ is directed is an easy way to ensure that $\mathrm{fil}(H)$ is a filter on λ. However, if we work with $H \subseteq \mathbb{Q}^*_\lambda$ we may consider alternative ways of guaranteeing this.

Definition 2.5 For $p \in \mathbb{Q}^*_\lambda$ let

$$\Sigma(p) = \{(\alpha, Z, d) \in \mathcal{F}^{\mathrm{ult}} : (\forall A \in d)(\exists(\alpha', Z', d') \in p)(A \cap Z' \in d')\}.$$

Observation 2.6

1. *If $p, q \in \mathbb{Q}^*_\lambda$, then $p \leq^* q$ if and only if $|q \setminus \Sigma(p)| < \lambda$.*

2. *If $p \in \mathbb{Q}^*_\lambda$ and $(\alpha, Z, d) \in \Sigma(p)$, then for some $\{(\alpha_x, Z_x, d_x) : x \in X\} \subseteq p$ and an ultrafilter e on X we have*

$$d = \{A \subseteq Z : A \cap \bigcup_{x \in X} Z_x \in \bigoplus^e\{d_x : x \in X\}\}.$$

Definition 2.7 We say that a non-empty family $H \subseteq \mathbb{Q}_\lambda^*$ is

(a) *big* if for each $\mathcal{D} \subseteq \mathcal{F}^{\mathrm{ult}}$ there is $q \in H$ such that either $q \subseteq \mathcal{D}$ or $q \cap \mathcal{D} = \emptyset$;

(b) *linked* if for each $p_0, \ldots, p_n \in H$, $n < \omega$, we have

$$|\{\alpha \,:\, (\exists Z, d)((\alpha, Z, d) \in \Sigma(p_0) \cap \cdots \cap \Sigma(p_n))\}| = \lambda.$$

The property introduced in Definition 2.7(a) resembles the bigness of creating pairs (cf. [4, Section 2.2]), so the use of the term "big" seemed natural. The name "linked" is motivated by Observation 2.8, 1. below.

Observation 2.8

1. *If $H \subseteq \mathbb{Q}_\lambda^*$ is linked, then*

   (a) *for each $p_0, \ldots, p_n \in H$, $n < \omega$, there is $q \in \mathbb{Q}_\lambda^*$ which is $\leq^*$-above all $p_0, \ldots, p_n$;*

   (b) $\mathrm{fil}(H)$ *has the finite intersection property.*

2. *If $H \subseteq \mathbb{Q}_\lambda^*$ is linked and big, then $\mathrm{fil}(H)$ is an ultrafilter on λ.*

For basic information on the ideal $\mathbf{M}_{\lambda,\lambda}^\lambda$ of meager subsets of $^\lambda\lambda$ and its covering number we refer the reader e. g. to Matet, Rosłanowski, and Shelah [2, §4]. Let us recall the following definition.

Definition 2.9

1. The space $^\lambda\lambda$ is endowed with the topology obtained by taking as basic open sets $\emptyset$ and O_s for $s \in {}^{<\lambda}\lambda$, where $O_s = \{f \in {}^\lambda\lambda \,:\, s \subseteq f\}$.

2. The $(<\lambda^+)$-complete ideal of subsets of $^\lambda\lambda$ generated by nowhere dense subsets of $^\lambda\lambda$ is denoted by $\mathbf{M}_{\lambda,\lambda}^\lambda$.

3. $\mathrm{cov}(\mathbf{M}_{\lambda,\lambda}^\lambda)$ is the minimal size of a family $\mathcal{A} \subseteq \mathbf{M}_{\lambda,\lambda}^\lambda$ such that $\bigcup \mathcal{A} = {}^\lambda\lambda$.

Theorem 2.10 *Assume that $\lambda = \lambda^{<\lambda} \geq \aleph_1$ and $\mathrm{cov}(\mathbf{M}_{\lambda,\lambda}^\lambda) = 2^\lambda$. Then there exists a linked and big family $H \subseteq \mathbb{Q}_\lambda^0$ such that $\mathrm{fil}(H)$ is a weakly reasonable ultrafilter.*

P r o o f. The proof is very similar to that of [8, Theorem 2.14]. Let χ be a sufficiently large regular cardinal and let $N \prec \mathcal{H}(\chi)$ be such that $|N| = \lambda$ and $^{<\lambda}N \subseteq N$. Put $\mathcal{F}_N^{\mathrm{ult}} = \mathcal{F}^{\mathrm{ult}} \cap N$. We will inductively construct a linked and big family H included in $\mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}}) \subseteq \mathbb{Q}_\lambda^0(\mathcal{F}^{\mathrm{ult}})$. The following two claims are the key points of the inductive process. Below, "linked" means "linked as a subfamily of $\mathbb{Q}_\lambda^*$" (i. e., it is the notion introduced in Definition 2.7(b)).

Claim 2.10.1 *Assume that $H_0 \subseteq \mathbb{Q}_\lambda^*(\mathcal{F}_N^{\mathrm{ult}})$ is linked, $|H_0| < \mathrm{cov}(\mathbf{M}_{\lambda,\lambda}^\lambda)$, and $\mathcal{D} \subseteq \mathcal{F}^{\mathrm{ult}}$. Then there is*

$$q \in \mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}}) \subseteq \mathbb{Q}_\lambda^0$$

such that $H_0 \cup \{q\}$ is linked and either $q \subseteq \mathcal{D}$ or $q \cap \mathcal{D} = \emptyset$.

P r o o f. We consider two cases.

C a s e A: For every $n < \omega$, $p_0, \ldots, p_n \in H_0$, and $\beta < \lambda$, there is

$$(\alpha, Z, d) \in \Sigma(p_0) \cap \cdots \cap \Sigma(p_n) \cap \mathcal{D} \cap \mathcal{F}_N^{\mathrm{ult}}$$

such that $\beta < \alpha$.

Let $\mathcal{T}_0$ be the family of all sequences η such that

(i) $\mathrm{lh}(\eta) < \lambda$;

(ii) if $\xi < \mathrm{lh}(\eta)$, then $\eta(\xi) \in \mathcal{D} \cap \mathcal{F}_N^{\mathrm{ult}}$;

(iii) if $\xi < \xi' < \mathrm{lh}(\eta)$, $\eta(\xi) = (\alpha, Z, d)$, $\eta(\xi') = (\alpha', Z', d')$, then $Z \subseteq \alpha'$.

It follows from the assumptions of the current case that $\mathcal{T}_0$ is a λ-branching tree (remember $|\mathcal{F}_N^{\mathrm{ult}}| = \lambda$). Moreover, for each $p_0, \ldots, p_n \in H_0$ we have

$$\{\varrho \in \lim(\mathcal{T}_0) \,:\, (\exists \zeta < \lambda)(\forall \xi > \zeta)(\varrho(\xi) \notin \Sigma(p_0) \cap \cdots \cap \Sigma(p_n))\} \in \mathbf{M}_{\lambda,\lambda}^\lambda.$$

Hence (as $|H_0| < \mathrm{cov}(\mathbf{M}_{\lambda,\lambda}^\lambda)$) we may pick $\varrho \in \lim(\mathcal{T}_0)$ such that for every $p_0, \ldots, p_n \in H_0$, $n < \omega$, we have

$$|\{\xi < \lambda \,:\, \varrho(\xi) \in \Sigma(p_0) \cap \cdots \cap \Sigma(p_n)\}| = \lambda.$$

Let $q = \{\varrho(\xi) \,:\, \xi < \lambda\}$. Then $q \in \mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}}) \subseteq \mathbb{Q}_\lambda^0$, $H_0 \cup \{q\}$ is linked, and $q \subseteq \mathcal{D}$.

C a s e B: Not Case A.

Then for some $p_0^*, \ldots, p_m^* \in H_0$ and $\beta < \lambda$ we have

$$(\forall(\alpha, Z, d) \in \Sigma(p_0^*) \cap \cdots \cap \Sigma(p_m^*) \cap \mathcal{F}_N^{\mathrm{ult}})(\alpha > \beta \Rightarrow (\alpha, Z, d) \notin \mathcal{D}).$$

It follows from the choice of N that if $p_0, \ldots, p_n \in \mathbb{Q}_\lambda^*(\mathcal{F}_N^{\mathrm{ult}})$ and $(\alpha, Z, d) \in \Sigma(p_0) \cap \cdots \cap \Sigma(p_n)$, then there are Z', d' such that $(\alpha, Z', d') \in \Sigma(p_0) \cap \cdots \cap \Sigma(p_n) \cap \mathcal{F}_N^{\mathrm{ult}}$.

Consequently, we may repeat arguments of the previous case replacing in clause (ii) $\mathcal{D} \cap \mathcal{F}_N^{\mathrm{ult}}$ by $\mathcal{F}_N^{\mathrm{ult}} \setminus \mathcal{D}$. Then we obtain $q \in \mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}}) \subseteq \mathbb{Q}_\lambda^0$ such that $H_0 \cup \{q\}$ is linked and $q \cap \mathcal{D} = \emptyset$. □ (Claim 2.10.1)

Claim 2.10.2 *Assume that $H_0 \subseteq \mathbb{Q}_\lambda^*(\mathcal{F}_N^{\mathrm{ult}}) \subseteq \mathbb{Q}_\lambda^*$ is linked, $|H_0| < \mathrm{cov}(\mathbf{M}_{\lambda,\lambda}^\lambda)$, and $\langle \delta_\xi : \xi < \lambda \rangle \subseteq \lambda$ is an increasing continuous sequence. Then there are $p \in \mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}})$ and a club C^* of λ such that*

(a) *$H_0 \cup \{p\}$ is linked;*

(b) *$\bigcup\{[\delta_{\xi+1}, \delta_\zeta) : \xi < \zeta$ are successive members of $C^*\} \in \mathrm{fil}(p)$.*

P r o o f. This is essentially [8, Claim 2.14.4]. □ (Claim 2.10.2)

Now we employ a bookkeeping device to construct inductively a sequence $\langle q_\xi : \xi < 2^\lambda \rangle \subseteq \mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}})$ such that

1. for each $\zeta < 2^\lambda$ the family $\{q_\xi : \xi < \zeta\}$ is linked;

2. if $\mathcal{D} \subseteq \mathcal{F}_N^{\mathrm{ult}}$, then for some $\xi < 2^\lambda$ we have $q_\xi \subseteq \mathcal{D}$ or $q_\xi \cap \mathcal{D} = \emptyset$;

3. if $\langle \delta_\xi : \xi < \lambda \rangle \subseteq \lambda$ is increasing continuous, then for some $\varepsilon < 2^\lambda$ and a club C^* of λ we have that

$$\bigcup\{[\delta_{\xi+1}, \delta_\zeta) : \xi < \zeta \text{ are successive members of } C^*\} \in \mathrm{fil}(q_\varepsilon).$$

Since $|\mathcal{F}_N^{\mathrm{ult}}| = \lambda$, there are no problems with carrying out the construction. It should be clear that at the end the family $\{q_\xi : \xi < 2^\lambda\}$ is linked, big, and it generates a weakly reasonable ultrafilter. □

Note that we may modify the construction in the proof of Theorem 2.10 so that the resulting H is directed. Namely, by an argument similar to the one in the proof of Claim 2.10.1 we may show that if $H_0 \subseteq \mathbb{Q}_\lambda^*(\mathcal{F}_N^{\mathrm{ult}})$ is linked, $|H_0| < \mathrm{cov}(\mathbf{M}_{\lambda,\lambda}^\lambda)$, $p_0, p_1 \in H_0$, then there is $q \in \mathbb{Q}_\lambda^0(\mathcal{F}_N^{\mathrm{ult}})$ such that $q \subseteq \Sigma(p_0) \cap \Sigma(p_1)$ and $H_0 \cup \{q\}$ is linked. With this claim we may modify the inductive choice of $\langle q_\xi : \xi < 2^\lambda \rangle$ so that at the end $\{q_\xi : \xi < 2^\lambda\}$ is directed. However, we do not know how to guarantee the opposite, that the family $\{q_\xi : \xi < 2^\lambda\}$ is not directed or even better, that for *no* directed $H \subseteq \mathbb{Q}_\lambda^0$ do we have $\mathrm{fil}(H) = \mathrm{fil}(\{q_\xi : \xi < 2^\lambda\})$. Thus the following question remains open.

Problem 2.11 *Does "$H \subseteq \mathbb{Q}_\lambda^0$ is linked and big" imply that "H is directed"?*

3 Systems of local pararegular filters

In this section we are interested in filters associated with the full system $\mathcal{F}^{\mathrm{pr}}$ of local pararegular filters on λ and we show their relation to numbers of generators (in standard sense) of some filters on λ.

Definition 3.1 Suppose that $Z \subseteq \lambda$ is an infinite set, $\alpha = \min(Z)$. A *pararegular filter on Z* is a filter F on Z such that for some system $\langle A_u : u \in [\kappa]^{<\omega} \rangle$ of sets from F we have:

1. $|\omega + \alpha| \leq \kappa < \lambda$, and if $u \subseteq v \in [\kappa]^{<\omega}$, then $A_v \subseteq A_u$.

2. If $U \subseteq \kappa$ is infinite, then $\bigcap\{A_{\{\xi\}} : \xi \in U\} = \emptyset$.

3. $\mathcal{F} = \{B \subseteq Z : (\exists u \in [\kappa]^{<\omega})(A_u \subseteq B)\}$.

If the cardinal κ above satisfies $2^{|\omega+\alpha|} \leq \kappa < \lambda$, then we say that the filter F is *strongly pararegular*.

The full system of local pararegular filters on λ will be denoted by $\mathcal{F}^{\mathrm{pr}}$ and the full system of local strongly pararegular filters on λ is denoted by $\mathcal{F}^{\mathrm{spr}}$. (The latter forms a system of local filters if and only if λ is inaccessible and then $\mathcal{F}^{\mathrm{spr}} \subseteq \mathcal{F}^{\mathrm{pr}}$.)

Let us recall the following strong λ^+-chain condition.

Definition 3.2 (See Shelah [5, Definition 1.1] and [7, Definition 7]) Let $\mathbb{Q}$ be a forcing notion, and $\varepsilon < \lambda$ be a limit ordinal.

1. We define a game $\eth_{\varepsilon,\lambda}^{\mathrm{cc}}(\mathbb{Q})$ of two players, Player I and Player II. A play lasts ε steps, and at each stage $\alpha < \varepsilon$ of the play sequences $\bar{p}^\alpha, \bar{q}^\alpha$ and a function φ^α are chosen so that:

   (a) $\bar{p}^0 = \langle \emptyset_{\mathbb{Q}} : i < \lambda^+ \rangle$, $\varphi^0 : \lambda^+ \longrightarrow \lambda^+$, $i \longmapsto 0$.

   (b) If $\alpha > 0$, then Player I picks $\bar{p}^\alpha, \varphi^\alpha$ such that

      (i) $\bar{p}^\alpha = \langle p_i^\alpha : i < \lambda^+ \rangle \subseteq \mathbb{Q}$ satisfies $(\forall \beta < \alpha)(\forall i < \lambda^+)(q_i^\beta \leq p_i^\alpha)$;

      (ii) $\varphi^\alpha : \lambda^+ \longrightarrow \lambda^+$ is regressive, i. e., $(\forall i < \lambda^+)(\varphi^\alpha(i) < 1 + i)$.

   (c) Player II answers choosing a sequence $\bar{q}^\alpha = \langle q_i^\alpha : i < \lambda^+ \rangle \subseteq \mathbb{Q}$ such that $(\forall i < \lambda^+)(p_i^\alpha \leq q_i^\alpha)$.

If at some stage of the game Player I does not have any legal move, then he loses. If the game lasted ε steps, Player I wins a play $\langle \bar{p}^\alpha, \bar{q}^\alpha, \varphi^\alpha : \alpha < \varepsilon \rangle$ if there is a club C of λ^+ such that for each distinct members i, j of C satisfying $\mathrm{cf}(i) = \mathrm{cf}(j) = \lambda$ and $(\forall \alpha < \varepsilon)(\varphi^\alpha(i) = \varphi^\alpha(j))$, the set $\{q_i^\alpha : \alpha < \varepsilon\} \cup \{q_j^\alpha : \alpha < \varepsilon\}$ has an upper bound in $\mathbb{Q}$.

2. The forcing notion $\mathbb{Q}$ *satisfies condition* $(*)_\lambda^\varepsilon$ if Player I has a winning strategy in the game $\eth_{\varepsilon,\lambda}^{\mathrm{cc}}(\mathbb{Q})$.

Proposition 3.3 (See Shelah [5, Iteration Lemma 1.3] and [7, Theorem 35]) *Let $\varepsilon < \lambda$ be a limit ordinal, and let $\lambda = \lambda^{<\lambda}$. Suppose that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ is a $(<\lambda)$-support iteration such that $\Vdash_{\mathbb{P}_\xi}$ "$\mathbb{Q}_\xi$ satisfies $(*)_\lambda^\varepsilon$" for each $\xi < \gamma$. Then $\mathbb{P}_\gamma$ satisfies $(*)_\lambda^\varepsilon$.*

Definition 3.4 Suppose that D is a uniform filter on λ. We define a forcing notion $\mathbb{Q}_D^{\mathrm{pr}}$ by:

1. a *condition* is a tuple $p = (\zeta^p, \langle \alpha_\xi^p : \xi \leq \zeta^p \rangle, \langle Z_\xi^p, F_\xi^p : \xi < \zeta^p \rangle, \mathcal{A}^p)$ such that

   (α) $\mathcal{A}^p \subseteq D$, $|\mathcal{A}^p| < \lambda$, $\zeta^p < \lambda$;

   (β) $\langle \alpha_\xi^p : \xi \leq \zeta^p \rangle$ is an increasing continuous sequence of ordinals below λ;

   (γ) $Z_\xi^p = [\alpha_\xi^p, \alpha_{\xi+1}^p)$ and F_ξ^p is a pararegular filter on Z_ξ^p;

2. *the order* $\leq_{\mathbb{Q}_D^{\mathrm{pr}}} = \leq$ is given by $p \leq q$ if and only if ($p, q \in \mathbb{Q}_D^{\mathrm{pr}}$ and)

   (i) $\mathcal{A}^p \subseteq \mathcal{A}^q$, $\zeta^p \leq \zeta^q$;

   (ii) $\alpha_\xi^q = \alpha_\xi^p$ for $\xi \leq \zeta^p$, and $Z_\xi^p = Z_\xi^q$, $F_\xi^q = F_\xi^p$ for $\xi < \zeta^p$;

   (iii) if $A \in \mathcal{A}^p$ and $\zeta^p \leq \xi < \zeta^q$, then $A \cap Z_\xi^q \in F_\xi^q$.

Proposition 3.5 *Assume $\lambda^{<\lambda} = \lambda$ and let D be a uniform filter on λ. Then we have:*

1. *$\mathbb{Q}_D^{\mathrm{pr}}$ is a $(<\lambda)$-complete forcing notion of size 2^λ.*

2. *$\mathbb{Q}_D^{\mathrm{pr}}$ satisfies the condition $(*)_\lambda^\varepsilon$ of Definition 3.2 for each limit ordinal $\varepsilon < \lambda$.*

3. *If $\underset{\sim}{r}$ is a $\mathbb{Q}_D^{\mathrm{pr}}$-name such that*

$$\Vdash_{\mathbb{Q}_D^{\mathrm{pr}}} \underset{\sim}{r} = \{(\alpha_\xi^p, Z_\xi^p, F_\xi^p) : \xi < \zeta^p \,\&\, p \in \Gamma_{\mathbb{Q}_D^{\mathrm{pr}}}\},$$

then $\Vdash_{\mathbb{Q}_D^{\mathrm{pr}}}$ "$\underset{\sim}{r} \in \mathbb{Q}_\lambda^0(\mathcal{F}^{\mathrm{pr}})$ and $D \subseteq \mathrm{fil}(\underset{\sim}{r})$".

Proof.

1. Note that if $\alpha < \beta < \lambda$, then there exist $\leq \sum_{\kappa < \lambda} 2^{\kappa \cdot |[\alpha, \beta)|}$ many pararegular filters on $[\alpha, \beta)$. Hence easily $|\mathbb{Q}_D^{\mathrm{pr}}| = 2^\lambda$.

If $\langle p_\alpha : \alpha < \gamma \rangle \subseteq \mathbb{Q}_D^{\mathrm{pr}}$ is $\leq_{\mathbb{Q}_D^{\mathrm{pr}}}$-increasing, $\gamma < \lambda$, then letting

$$\mathcal{A}^q = \bigcup_{\alpha < \gamma} \mathcal{A}^{p_\alpha}, \quad \zeta^q = \sup(\zeta^{p_\alpha} : \alpha < \gamma), \quad \text{and}$$
$$\langle \alpha_\xi^q, Z_\xi^q, F_\xi^q : \xi < \zeta^q \rangle = \bigcup_{\alpha < \gamma} \langle \alpha_\xi^q, Z_\xi^{p_\alpha}, F_\xi^{p_\alpha} : \xi < \zeta^{p_\alpha} \rangle,$$

we get a condition

$$q = (\zeta^q, \langle \alpha_\xi^q : \xi \leq \zeta^q \rangle, \langle Z_\xi^q, F_\xi^q : \xi < \zeta^q \rangle, \mathcal{A}^q) \in \mathbb{Q}_D^{\mathrm{pr}}$$

stronger than all p_α (for $\alpha < \gamma$).

2. Let $\mathcal{X}$ consist of all sequences $\langle Z_\xi, F_\xi \ : \ \xi < \zeta \rangle$ such that

$$\langle Z_\xi, F_\xi \ : \ \xi < \zeta \rangle = \langle Z_\xi^p, F_\xi^p \ : \ \xi < \zeta^p \rangle$$

for some $p \in \mathbb{Q}_D^{\mathrm{pr}}$. By what we said earlier, $|\mathcal{X}| = \lambda$, so we may fix an enumeration $\langle \bar{\sigma}_\alpha \ : \ \alpha < \lambda \rangle$ of $\mathcal{X}$. Let $\mathbf{st}$ be a strategy of Player I in the game $\beth_{\varepsilon,\lambda}^{\mathrm{cc}}(\mathbb{Q}_D^{\mathrm{pr}})$ which, at a stage $\alpha < \varepsilon$ of the play, instructs her to choose a legal inning $\bar{p}^\alpha, \varphi^\alpha$ such that if $\lambda \le i < \lambda^+$, then

$$\langle Z_\xi^{p_i^\alpha}, F_\xi^{p_i^\alpha} \ : \ \xi < \zeta^{p_i^\alpha} \rangle = \bar{\sigma}_{\varphi^\alpha(i)}.$$

(Note that there exist legal innings for Player I by the completeness of the forcing proved in 1. above.) Plainly, if $\langle \bar{p}^\alpha, \bar{q}^\alpha, \varphi^\alpha \ : \ \alpha < \varepsilon \rangle$ is a play of $\beth_{\varepsilon,\lambda}^{\mathrm{cc}}(\mathbb{Q}_D^{\mathrm{pr}})$ in which Player I follows $\mathbf{st}$ and $\lambda \le i < j < \lambda^+$ are such that for all $\alpha < \varepsilon$, $\varphi^\alpha(i) = \varphi^\alpha(j)$, then the family $\{p_i^\alpha, p_j^\alpha \ : \ \alpha < \varepsilon\}$ has an upper bound. Thus $\mathbf{st}$ is a winning strategy for Player I.

3. Suppose $p \in \mathbb{Q}_D^{\mathrm{pr}}$ and let

$$\kappa = |\mathcal{A}^p| + |\omega + \alpha_{\zeta^p}^p|.$$

Fix a sequence $\langle A_\beta \ : \ \beta < \kappa \rangle$ listing all members of $\mathcal{A}^p \cup \{\lambda\}$ (with possible repetitions) and let $\langle u_\gamma \ : \ \gamma < \kappa \rangle$ be an enumeration of $[\kappa]^{<\omega}$. By induction on $\gamma < \kappa$ choose an increasing sequence $\langle \xi_\gamma \ : \ \gamma < \kappa \rangle \subseteq [\alpha_{\zeta^p}^p, \lambda)$ such that $\xi_\gamma \in \bigcap_{\beta \in u_\gamma} A_\beta$. (Remember, D is a uniform filter and $\kappa < \lambda$.) Let

$$\zeta^q = \zeta^p + 1, \quad \alpha_{\zeta^q}^q = \sup(\xi_\gamma \ : \ \gamma < \kappa) + 1,$$

and for $u \in [\kappa]^{<\omega}$ let $B_u = \{\xi_\gamma \ : \ u \subseteq u_\gamma \,\&\, \gamma < \lambda\}$. Then $F_{\zeta^p}^q = \{B \subseteq [\alpha_{\zeta^p}^q, \alpha_{\zeta^q}^q) \ : \ (\exists u \in [\kappa]^{<\omega})(B_u \subseteq B)\}$ is a pararegular filter on $[\alpha_{\zeta^p}^q, \alpha_{\zeta^q}^q)$ and $A \cap [\alpha_{\zeta^p}^q, \alpha_{\zeta^q}^q) \in F_{\zeta^p}^q$ for all $A \in \mathcal{A}^q$. Hence, we may now take a condition $q \in \mathbb{Q}_D^{\mathrm{pr}}$ stronger than p and such that $Z_{\zeta^p}^q = [\zeta^p, \zeta^q)$, $\mathcal{A}^p = \mathcal{A}^q$. Then $q \Vdash (\alpha_{\zeta^p}^q, Z_{\zeta^p}^q, F_{\zeta^p}^q) \in \underset{\sim}{r}$.

So we easily conclude that indeed $\Vdash_{\mathbb{Q}_D^{\mathrm{pr}}}$ "$\underset{\sim}{r} \in \mathbb{Q}_\lambda^0(\mathcal{F}^{\mathrm{pr}})$ and $D \subseteq \mathrm{fil}(\underset{\sim}{r})$" (remember the definition of the order on $\mathbb{Q}_D^{\mathrm{pr}}$, specifically Definition 3.4, 2.(iii)). □

Corollary 3.6 *Assume* $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$, $2^{\lambda^+} = \lambda^{++}$. *Then there is a* $(< \lambda)$-*complete* λ^+-*cc forcing notion* $\mathbb{P}$ *such that*

$$\Vdash_\mathbb{P} \text{``}2^\lambda = \lambda^{++} \text{ and if } D \text{ is a uniform filter on } \lambda \text{ generated by less than } \lambda^{++} \text{ elements,}$$
$$\text{then } D \subseteq \mathrm{fil}(r) \text{ for some } r \in \mathbb{Q}_\lambda^0(\mathcal{F}^{\mathrm{pr}})\text{''}.$$

P r o o f. Using a standard bookkeeping argument build a $(< \lambda)$-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi \ : \ \xi < \lambda^{++} \rangle$ such that

1. for each $\xi < \lambda^{++}$ we have that $\Vdash_{\mathbb{P}_\xi} \underset{\sim}{\mathbb{Q}}_\xi = \mathbb{Q}_{\underset{\sim}{D}}^{\mathrm{pr}}$ for some $\mathbb{P}_\xi$-name $\underset{\sim}{D}$ for a uniform filter on λ;

2. if $\langle \underset{\sim}{A}_\beta \ : \ \beta < \lambda^+ \rangle$ is a sequence of $\mathbb{P}_{\lambda^{++}}$-names for subsets of λ, then for some $\xi < \lambda^{++}$ such that every $\underset{\sim}{A}_\beta$ is a $\mathbb{P}_\xi$-name, we have

$$\Vdash_{\mathbb{P}_\xi} \text{``if } \langle \underset{\sim}{A}_\beta \ : \ \beta < \lambda^+ \rangle \text{ generates a uniform filter } D \text{ on } \lambda, \text{ then } \underset{\sim}{\mathbb{Q}}_\xi = \mathbb{Q}_D^{\mathrm{pr}}\text{''}.$$

Now look at the limit $\mathbb{P}_{\lambda^{++}} = \lim(\bar{\mathbb{Q}})$ (and remember Propositions 3.5, 3.3). □

Proposition 3.7 *Assume that* $2^\lambda = \lambda^+$. *Then there is a uniform ultrafilter* D *on* λ *which contains no* $\mathrm{fil}(p)$ *for* $p \in \mathbb{Q}_\lambda^*(\mathcal{F}^{\mathrm{pr}})$.

P r o o f. First note that if F is a pararegular filter on Z, then for each β we have $Z \setminus \{\beta\} \in F$. Consequently, if $\mathcal{A} \subseteq [\lambda]^\lambda$ is a family with the finite intersection property, $\{[\alpha, \lambda) \ : \ \alpha < \lambda\} \subseteq \mathcal{A}$, $|\mathcal{A}| \le \lambda$, and $p \in \mathbb{Q}_\lambda^*(\mathcal{F}^{\mathrm{pr}})$, then we may choose $A \subseteq \lambda$ such that

1. $\mathcal{A} \cup \{A\}$ has the finite intersection property;

2. for each $(\alpha, Z, F) \in p$ we have $|Z \cap A| \le 1$ so also $Z \setminus A \in F$.

Hence, by induction on $\xi < \lambda^+$, we may choose a sequence $\langle A_\xi : \xi < \lambda^+ \rangle$ of unbounded subsets of λ such that

1. for $\xi < \lambda$, $A_\xi = [\xi, \lambda)$;
2. $\{A_\xi : \xi < \lambda^+\}$ has the finite intersection property;
3. for every $A \subseteq \lambda$ there is $\xi < \lambda^+$ such that either $A_\xi \subseteq A$ or $A_\xi \cap A = \emptyset$;
4. for every $p \in \mathbb{Q}^*_\lambda(\mathcal{F}^{\mathrm{pr}})$ there is $\xi < \lambda^+$ such that $\lambda \setminus A_\xi \in \mathrm{fil}(p)$.

Then $D = \{A \subseteq \lambda : A_{\xi_0} \cap \cdots \cap A_{\xi_n} \subseteq A$ for some $\xi_0, \ldots, \xi_n < \lambda^+, n < \omega\}$ is an ultrafilter as required. $\qquad\square$

Proposition 3.8 *Assume that*

(a) *there exists a λ-Kurepa tree with 2^λ λ-branches;*

(b) *D is a uniform filter on λ;*

(c) *$p \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{pr}})$ is such that $\mathrm{fil}(p) \subseteq D$;*

(d) *if λ is a limit cardinal, then it is strongly inaccessible and $p \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{spr}})$.*

Then the filter D cannot be generated by less than 2^λ sets, i. e., for every family $\mathcal{X} \subseteq D$ of size less than 2^λ there is a set $A \in D$ such that $|X \setminus A| = \lambda$ for all $X \in \mathcal{X}$.

Proof. Let T be a λ-Kurepa tree with 2^λ λ-branches (so every level in T is of size $< \lambda$). For $\xi < \lambda$ let T_ξ be the ξth level of T. Now choose an increasing continuous sequence $\langle \alpha_\xi : \xi < \lambda \rangle$ such that if $(\alpha, Z, F) \in p$ and $\alpha_\xi \leq \alpha < \alpha_{\xi+1}$, then $Z \subseteq \alpha_{\xi+1}$ and there is a system $\langle A^\alpha_u : u \in [\kappa_\alpha]^{<\omega} \rangle$ of sets from F which witnesses that F is pararegular (strongly pararegular if λ is inaccessible) with κ_α satisfying $|T_\xi| \leq \kappa_\alpha$. For every $\xi < \lambda$ and $(\alpha, Z, F) \in p$ such that $\alpha_\xi \leq \alpha < \alpha_{\xi+1}$, let us fix an injection $\pi^\alpha_\xi : T_\xi \xrightarrow{1\text{-}1} \kappa_\alpha$. Next for every λ-branch η through T let us choose a set $A_\eta \in D$ so that if $\xi < \lambda$, $\nu \in T_\xi \cap \eta$, $(\alpha, Z, F) \in p$, $\alpha_\xi \leq \alpha < \alpha_{\xi+1}$, then we have that $A_\eta \cap Z = A^\alpha_{\{\pi^\alpha_\xi(\nu)\}}$. For our conclusion, it is enough to show that if $B \in D$, then there exist at most finitely many λ-branches η through T such that $|B \setminus A_\eta| < \lambda$. So suppose towards contradiction $\eta_0, \eta_1, \eta_2, \ldots$ are distinct λ-branches through T, $B \in D$, and $|B \setminus A_{\eta_n}| < \lambda$ for each $n < \omega$. The set $\{(\alpha, Z, F) \in p : B \cap Z \in F^+\}$ is of cardinality λ, so we may find $\xi < \lambda$ and $\nu_n \in T_\xi$ (for $n < \omega$) such that

1. $\eta_n \cap T_\xi = \{\nu_n\}$ and $\nu_n \neq \nu_m$ for distinct n, m;
2. $B \cap Z^* \in (F^*)^+$ for some $(\alpha^*, Z^*, F^*) \in p$ satisfying $\alpha_\xi \leq \alpha^* < \alpha_{\xi+1}$;
3. $B \setminus \alpha_\xi \subseteq A_{\eta_n}$ for all $n < \omega$.

Then $\emptyset \neq B \cap Z^* \subseteq \bigcap\{A^\alpha_{\{\pi^\alpha_\xi(\nu_n)\}} : n < \omega\}$, a contradiction. $\qquad\square$

4 Forcing a very reasonable ultrafilter

Our goal here is to show that the inaccessibility of λ in the assumptions of [8, Proposition 1.6(1)] is needed. This answers the request of the referee of [8] and fulfills the promise stated in [8, Remark 1.7]. Assuming that κ is strongly inaccessible, we will construct a countable support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \kappa \rangle$ of proper forcing notions such that

$$\Vdash_{\mathbb{P}_\kappa} \text{ ``there is a } (\leq \omega_1)\text{-directed family } H \subseteq \mathbb{Q}^0_{\omega_1} \text{ such that } \mathrm{fil}(H) \text{ is a weakly reasonable}$$
$$\text{ultrafilter on } \omega_1 \text{ and yet Odd has a winning strategy in } \partial_{\mathrm{fil}(H)}\text{''}.$$

Let us recall the following definition.

Definition 4.1 (Shelah [8, Definition 1.4]) Let D be a uniform ultrafilter on λ. We define a game ∂_D between two players, Odd and Even, as follows. A play of ∂_D lasts λ steps and during a play an increasing continuous sequence $\bar{\alpha} = \langle \alpha_i : i < \lambda \rangle \subseteq \lambda$ is constructed. The terms of $\bar{\alpha}$ are chosen successively by the two players so that Even chooses the α_i for even i (including limit stages i, where she has no free choice) and Odd chooses α_i for odd i. Even wins the play if and only if $\bigcup\{[\alpha_{2i+1}, \alpha_{2i+2}) : i < \lambda\} \in D$.

The following result was shown in [8, Proposition 1.6].

Proposition 4.2 *Assume D is a uniform ultrafilter on λ.*

1. *If λ is strongly inaccessible and Odd has a winning strategy in ∂_D, then D is not weakly reasonable.*

2. *If D is not weakly reasonable, then Odd has a winning strategy in the game ∂_D.*

Before we define our iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \,:\, \alpha < \kappa \rangle$ let us introduce two main ingredients used in the construction.

Sealing the branches: At each stage of the iteration we will first use forcing notions that have been introduced by Shelah in [6, Chapter XVII, §2].

For a tree $T \subseteq {}^{<\omega_1}\omega_1$, the set of all ω_1-branches through T will be denoted by $\lim(T)$. Thus

$$\lim(T) = \{\eta \in {}^{\omega_1}\omega_1 \,:\, (\forall \alpha < \omega_1)(\eta \restriction \alpha \in T)\}.$$

Lemma 4.3 (Shelah [6, Chapter XVII, Fact 2.2]) *Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a tree of height ω_1. Let $\mathbb{C}$ be the Cohen forcing and $\mathbb{L}$ be a $\mathbb{C}$-name for the Levy collapse of $2^{\aleph_2}$ to $\aleph_1$ (with countable conditions, so it is a σ-closed forcing notion). Then $\Vdash_{\mathbb{C}*\mathbb{L}} \lim(T) = (\lim(T))^{\mathbf{V}}$.*

Definition 4.4 (Shelah [6, Chapter XVII, Definition 2.3]) Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a tree of height ω_1 such that $|T| = \aleph_1$ and $|\lim(T)| \leq \aleph_1$. Let $\langle B_i \,:\, i < \omega_1 \rangle$ list all members of $\lim(T)$ (with possible repetitions), and let $\langle y_i \,:\, i < \omega_1 \rangle$ list all elements of T so that $y_j \lhd y_i$ implies $j < i$. For $j < \omega_1$ we define

$$B_j^* = \begin{cases} B_i & \text{if } j = 2i, \\ \{y_i\} & \text{if } j = 2i+1 \end{cases} \quad \text{and} \quad B_j' = B_j^* \setminus \bigcup_{i<j} B_i^*.$$

Let $w = \{j < \omega_1 \,:\, B_j' \neq \emptyset\}$ and for $j \in w$ let $x_j = \min(B_j')$. Finally, we put $A = \{x_i \,:\, i \in w\}$. We define a forcing notion $\mathbb{P}_T$ for *sealing the branches of T*:

1. a *condition p in $\mathbb{P}_T$* is a finite function from $\mathrm{dom}(p) \subseteq A$ into ω such that if $\varrho, \nu \in \mathrm{dom}(p)$ and $\varrho \lhd \nu$, then $p(\eta) \neq p(\nu)$;

2. *the order $\leq_{\mathbb{P}_T}$ of $\mathbb{P}_T$* is the inclusion, i. e., $p \leq q$ if and only if $p, q \in \mathbb{P}_T$ and $p \subseteq q$.

Lemma 4.5 (Shelah [6, Chapter XVII, Lemma 2.4]) *Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a tree of height ω_1 such that $|T| = \aleph_1$ and $|\lim(T)| \leq \aleph_1$, and $\mathbb{P}_T$ is the forcing notion for sealing the branches of T.*

(a) *$\mathbb{P}_T$ satisfies the ccc.*

(b) *If $G \subseteq \mathbb{P}_T$ is generic over $\mathbf{V}$ and $\mathbf{V}^*$ is a universe of* ZFC *extending $\mathbf{V}[G]$ and $(\aleph_1)^{\mathbf{V}^*} = \aleph_1^{\mathbf{V}} (= (\aleph_1)^{\mathbf{V}[G]})$, then $\mathbf{V}^* \vDash \lim(T) = (\lim(T))^{\mathbf{V}}$.*

Adding a bound to $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$ and a family $\mathcal{U} \subseteq \mathcal{P}(\omega_1)$: After sealing the branches of a tree, we will force a new member r^* of our family $H \subseteq \mathbb{Q}^0_{\omega_1}$ at the same time making sure that some family $\mathcal{U}$ of subsets of ω_1 is included in $\mathrm{fil}(r^*)$.

Definition 4.6 Suppose that $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$ and $\mathcal{U} \subseteq \mathcal{P}(\omega_1)$ are such that

(a) $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$ is $\leq^*$-directed;

(b) $U_0 \cap \cdots \cap U_n \in (\mathrm{fil}(\mathcal{G}))^+$ for every $U_0, \ldots, U_n \in \mathcal{U}$, $n < \omega$.

We define a forcing notion $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})$ as follows:

1. a *condition p in $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})$* is a triple $(r^p, \mathbf{G}^p, \mathbf{U}^p)$ such that $r^p \subseteq \mathcal{F}^{\mathrm{ult}}_{\omega_1}$ is countable and strongly disjoint (i. e., it satisfies the demands of Definition 1.2, 2.), $\mathbf{G}^p \subseteq \mathcal{G}$ is countable, and $\mathbf{U}^p \subseteq \mathcal{U}$ is countable;

2. *the order $\leq = \leq_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})}$* is defined by $p \leq q$ if and only if $(p, q \in \mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})$ and) $\mathbf{U}^p \subseteq \mathbf{U}^q$, $\mathbf{G}^p \subseteq \mathbf{G}^q$, $r^p \subseteq r^q$, and for every $(\alpha, Z, d) \in r^q \setminus r^p$ we have that

(i) $(\forall(\alpha', Z', d') \in r^p)(Z' \subseteq \alpha)$;

(ii) $(\forall r \in \mathbf{G}^p)((\alpha, Z, d) \in \Sigma(r))$ ($\Sigma(r)$ was defined in Definition 2.5);

(iii) $(\forall U \in \mathbf{U}^p)(U \cap Z \in d)$.

We also define a $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})$-name $\underline{r}$ by $\Vdash_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})} \underline{r} = \bigcup\{r^p \,:\, p \in \Gamma_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})}\}$.

Lemma 4.7 *Assume $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$, $\mathcal{U} \subseteq \mathcal{P}(\omega_1)$ satisfy demands* (a), (b) *of Definition* 4.6. *Then*

1. $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})$ *is a σ-closed forcing notion;*

2. $\Vdash_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G}, \mathcal{U})}$ "*$\underline{r} \in \mathbb{Q}^0_{\omega_1}$ and $(\forall r \in \mathcal{G})(r \leq^* \underline{r})$ and $\mathcal{U} \subseteq \mathrm{fil}(\underline{r})$*".

P r o o f.

1. Straightforward.
2. To argue that $\Vdash_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})} \underline{r} \in \mathbb{Q}^0_{\omega_1}$, suppose $p \in \mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})$. Let

$$\{r_n : n < \omega\} = \mathbf{G}^p, \quad \{U_n : n < \omega\} = \mathbf{U}^p$$

(we allow repetitions). Choose inductively $(\alpha_m, Z_m, d_m) \in \mathcal{F}^{\mathrm{ult}}_{\omega_1}$ such that for $m < \omega$ we have

(a) $(\forall(\alpha', Z', d') \in r^p)(Z' \subseteq \alpha_0), Z_m \subseteq \alpha_{m+1}$;
(b) $(\alpha_m, Z_m, d_m) \in \Sigma(r_0) \cap \cdots \cap \Sigma(r_m)$;
(c) $U_0 \cap \cdots \cap U_m \cap Z_m \in d_m$.

[Why is the choice possible? Since $\mathcal{G}$ is directed, we may first choose $s \in \mathcal{G}$ such that $r_0, \ldots, r_m \leq^* s$. Then for some $\beta < \omega_1$, if $(\alpha, Z, d) \in s$ and $\beta \leq \alpha$, then $(\alpha, Z, d) \in \Sigma(r_0) \cap \cdots \cap \Sigma(r_m)$. By the assumption of Definition 4.6(b) on $\mathcal{U}$ we know that $U_0 \cap \cdots \cap U_m \cap Z \in d$ for ω_1 many $(\alpha, Z, d) \in s$, so $(\alpha_m, Z_m, d_m) \in s$ may be chosen as required.]

After the above construction is carried out, pick any uniform ultrafilter e on ω and put

$$\alpha = \alpha_0, \quad Z = \bigcup_{m<\omega} Z_m, \quad \text{and} \quad d = \bigoplus^e_{m<\omega} d_m.$$

Then $q = (r^p \cup \{(\alpha, Z, d)\}, \mathbf{G}^p, \mathbf{U}^p) \in \mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})$ is a condition stronger than p. Thus by an easy density argument we see that $\Vdash_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})} |\underline{r}| = \omega_1$. The rest should be clear. $\qquad\square$

Let us recall that a very reasonable ultrafilter on λ is a weakly reasonable ultrafilter D such that $D = \mathrm{fil}(H)$ for some $(< \lambda^+)$-directed family $H \subseteq \mathbb{Q}^0_\lambda$ (see [8, Definition 2.5(5)]). Now we may state and prove our result.

Theorem 4.8 *Assume that κ is a strongly inaccessible cardinal. Then there is a κ-cc proper forcing notion $\mathbb{P}$ such that*

$$\Vdash_{\mathbb{P}} \text{ ``there is a } \leq^*\text{-increasing sequence } \langle r_\xi : \xi < \omega_2 \rangle \subseteq \mathbb{Q}^0_{\omega_1} \text{ such that } \mathrm{fil}(\{r_\xi : \xi < \omega_2\}) \text{ is a}$$
$$\text{very reasonable ultrafilter on } \omega_1 \text{ but Odd has a winning strategy in the game } \eth_{\{r_\xi : \xi < \omega_2\}} \text{''}.$$

P r o o f. The forcing notion $\mathbb{P}$ will be obtained as the limit of a countable support iteration of proper forcing notions

$$\langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \kappa \rangle.$$

The iteration will be built so that for each $\xi < \kappa$,

$$\Vdash_{\mathbb{P}_\xi} \text{ ``} \mathbb{Q}_\xi \text{ is a proper forcing notion of size } < \kappa \text{''},$$

so we will be sure that the intermediate stages $\mathbb{P}_\xi$ and the limit $\mathbb{P}_\kappa$ will be proper and each $\mathbb{P}_\xi$ (for $\xi < \kappa$) will have a dense subset of cardinality $< \kappa$. Thus $\mathbb{P}_\kappa$ will satisfy κ-cc (and κ will not be collapsed). Since in the process of iteration we will also collapse to $\aleph_1$ all uncountable cardinals below κ, we will know that

$$\Vdash_{\mathbb{P}_\kappa} \aleph_1 = (\aleph_1)^{\mathbf{V}} \,\&\, 2^{\aleph_1} = \aleph_2 = \kappa.$$

Thus we may set up a bookkeeping device that gives us a list $\langle \underline{C}_\zeta, \underline{A}_\zeta, \underline{\varrho}_\zeta : \zeta < \kappa \rangle$ such that

1. $\underline{C}_\zeta$ is a $\mathbb{P}_\zeta$-name for a club of ω_1;
2. $\underline{A}_\zeta$ is a $\mathbb{P}_\zeta$-name for a subset of ω_1;
3. $\underline{\varrho}_\zeta$ is a $\mathbb{P}_\zeta$-name for a function from ω_1 to ω_1;
4. for each $\mathbb{P}_\kappa$-name $\underline{C}$ for a club of ω_1, for some $\zeta < \kappa$ we have $\Vdash_{\mathbb{P}_\kappa} \underline{C} = \underline{C}_\zeta$, and similarly for names $\underline{A}$ for subsets of ω_1 and names $\underline{\varrho}$ for elements of $^{\omega_1}\omega_1$.

Before continuing let us set some terminology used later. *A partial strategy* is a function σ such that

1. $\mathrm{dom}(\sigma) \subseteq \{\eta \in {}^{<\omega_1}\omega_1 : \mathrm{lh}(\eta) \text{ is an odd ordinal}\}$;
2. $(\forall\nu \in \mathrm{dom}(\sigma))(\sigma(\nu) \in \omega_1 \setminus (\sup(\nu) + 1))$.

We say that a sequence $\eta \in {}^{\leq \omega_1}\omega_1$ *is played according to a partial strategy* σ if

 1. the sequence η is increasing continuous, and
 2. for every odd ordinal $\alpha < \mathrm{lh}(\eta)$ we have $\eta \restriction \alpha \in \mathrm{dom}(\sigma)$ and $\eta(\alpha) = \sigma(\eta \restriction \alpha)$.

If $\varrho, \eta \in {}^{\omega_1}\omega_1$ and η is played according to σ, then we say that $\eta = \sigma[\varrho]$ if

$$\eta(0) = \varrho(0) \qquad \text{and} \qquad \eta(2\alpha + 2) = \eta(2\alpha + 1) + \varrho(1 + \alpha) + 1 \quad \text{for each } \alpha < \lambda.$$

Also, for an increasing sequence $\eta \in {}^{\omega_1}\omega_1$ let

$$U_\eta = \bigcup \{ [\eta(2\alpha), \eta(2\alpha + 1)) \,:\, \alpha < \omega_1 \}.$$

Now we will inductively choose $\mathbb{Q}_\xi$ and $\underset{\sim}{T}_\xi, \underset{\sim}{\sigma}_\xi, \underset{\sim}{r}_\xi$ so that for each $\xi < \kappa$ the following demands are satisfied.

$(\boxplus)_1$ $\underset{\sim}{r}_\xi$ is a $\mathbb{P}_{\xi+1}$-name for a member of $\mathbb{Q}^0_{\omega_1}$ and $\Vdash_{\mathbb{P}_{\xi+1}} (\forall \zeta < \xi)(\underset{\sim}{r}_\zeta \leq^* \underset{\sim}{r}_\xi)$.

$(\boxplus)_2$ $\underset{\sim}{T}_\xi$ is a $\mathbb{P}_\xi$-name for a subtree of ${}^{<\omega_1}\omega_1$ of height ω_1 (with no maximal nodes).

$(\boxplus)_3$ $\underset{\sim}{\sigma}_\xi$ is a $\mathbb{P}_\xi$-name for a partial strategy with domain $\{\eta \in \underset{\sim}{T}_\xi \,:\, \mathrm{lh}(\eta) \text{ is odd}\}$, and all nodes of the tree $\underset{\sim}{T}_\xi$ are played according to $\underset{\sim}{\sigma}_\xi$.

$(\boxplus)_4$ $\Vdash_{\mathbb{P}_{\xi+1}} (\exists \eta \in \lim(\underset{\sim}{T}_{\xi+1}))(\eta = \underset{\sim}{\sigma}_{\xi+1}[\varrho_\xi])$.

$(\boxplus)_5$ $\Vdash_{\mathbb{P}_\xi} (\forall \zeta < \xi)(\underset{\sim}{T}_\zeta \subseteq \underset{\sim}{T}_\xi \,\&\, \underset{\sim}{\sigma}_\zeta \subseteq \underset{\sim}{\sigma}_\xi)$ and

$$\Vdash_{\mathbb{P}_{\xi+1}} \text{``if } \nu \in {}^{<\omega_1}\omega_1 \text{ is increasing continuous and such that } \mathrm{lh}(\nu) = \gamma + 1 \text{ for a limit } \gamma$$
$$\text{and } (\forall \alpha < \gamma)(\nu \restriction \alpha \in \underset{\sim}{T}_\xi) \text{ but } \nu \restriction \gamma \notin \underset{\sim}{T}_\xi, \text{ then } \nu \in \underset{\sim}{T}_{\xi+1} \text{''}.$$

$(\boxplus)_6$ $\Vdash_{\mathbb{P}_\xi} (\forall \eta_0, \ldots, \eta_n \in \lim(\underset{\sim}{T}_\xi))(\forall \zeta < \xi)(U_{\eta_0} \cap \cdots \cap U_{\eta_n} \in (\mathrm{fil}(\underset{\sim}{r}_\zeta))^+)$ for each $n < \omega$, and

$$\Vdash_{\mathbb{P}_{\xi+1}} (\forall \eta \in \lim(\underset{\sim}{T}_\xi))(U_\eta \in \mathrm{fil}(\underset{\sim}{r}_\xi)).$$

$(\boxplus)_7$ $\Vdash_{\mathbb{P}_{\xi+1}}$ "$\underset{\sim}{A}_\xi \in \mathrm{fil}(\underset{\sim}{r}_\xi)$ or $\omega_1 \setminus \underset{\sim}{A}_\xi \in \mathrm{fil}(\underset{\sim}{r}_\xi)$" and

$$\Vdash_{\mathbb{P}_{\xi+1}} \text{``if } \langle \delta_\alpha \,:\, \alpha < \lambda \rangle \text{ is the increasing enumeration of } \underset{\sim}{C}_\xi, \text{ then for some club } C^* \subseteq \omega_1$$
$$\text{we have } \omega_1 \setminus \bigcup \{ [\delta_\alpha, \delta_{\alpha+1}) \,:\, \alpha \in C^* \} \in \mathrm{fil}(\underset{\sim}{r}_\xi) \text{''}.$$

$(\boxplus)_8$ For $\xi > 0$, $\mathbb{Q}_\xi$ is the $\mathbb{P}_\xi$-name for the composition

$$\mathbb{C} * \underset{\sim}{\mathbb{L}} * \underset{\sim}{\mathbb{P}}_{T_\xi} * \underset{\sim}{\mathbb{Q}}^{\mathrm{bd}}(\{\underset{\sim}{r}_\zeta \,:\, \zeta < \xi\}, \{U_\eta \,:\, \eta \in \lim(\underset{\sim}{T}_\xi)\})$$

(see Lemma 4.3, Definitions 4.4, 4.6). Hence we know that also

$(\boxplus)_9$ for every $\mathbb{P}_{\xi+1}$-name $\mathbb{Q}$ for a proper forcing notion, $\Vdash_{\mathbb{P}_{\xi+1} * \mathbb{Q}} \lim(\underset{\sim}{T}_\xi) = (\lim(\underset{\sim}{T}_\xi))^{\mathbf{V}^{\mathbb{P}_\xi}}$.

To start, we let r_{-1} be any fixed element of $\mathbb{Q}^0_{\omega_1}$. We choose $\sigma' : {}^{<\omega_1}\omega_1 \longrightarrow \omega_1$ so that for every $\eta \in {}^{<\omega_1}\omega_1$ there is $(\alpha, Z, d) \in r$ such that $\sup(\eta) < \alpha$ and $Z \subseteq \sigma'(\eta)$, and we let

$$\underset{\sim}{T}_0 = T_0 = \{\sigma'[\varrho_0] \restriction \alpha \,:\, \alpha < \omega_1\} \subseteq {}^{<\omega_1}\omega_1.$$

(So T_0 is a tree with $\lim(T_0) = \{\sigma'[\varrho_0]\}$.) Finally $\underset{\sim}{\sigma}_0 = \sigma_0 = \sigma' \restriction \{\nu \in T_0 \,:\, \mathrm{lh}(\nu) \text{ is odd}\}$. Now the forcing notion $\mathbb{Q}_0$ is $\mathbb{C} * \underset{\sim}{\mathbb{L}} * \underset{\sim}{\mathbb{P}}_{T_0} * \underset{\sim}{\mathbb{Q}}^{\mathrm{bd}}(\{r_{-1}\}, \{U_{\sigma_0[\varrho_0]}\})$. Clearly, the families $\{r_{-1}\}, \{U_{\sigma_0[\varrho_0]}\}$ satisfy the demands (a) and (b) of Definition 4.6.

Now suppose that we have arrived to a successor stage $\xi = \zeta + 1$ (and that we have already defined $\mathbb{P}_\zeta$ and $\mathbb{P}_\zeta$-names $\underset{\sim}{T}_\zeta, \underset{\sim}{\sigma}_\zeta$, and $\mathbb{P}_{\varepsilon+1}$-names $\underset{\sim}{r}_\varepsilon$ for $\varepsilon < \zeta$ so that the demands of $(\boxplus)_1 - (\boxplus)_6$ hold). From $(\boxplus)_1 + (\boxplus)_6$ it follows that $\mathbb{Q}_\zeta$ is correctly determined by clause $(\boxplus)_8$, so

$$\Vdash_{\mathbb{P}_\zeta} \mathbb{Q}_\zeta = \mathbb{C} * \underset{\sim}{\mathbb{L}} * \underset{\sim}{\mathbb{P}}_{T_\zeta} * \underset{\sim}{\mathbb{Q}}^{\mathrm{bd}}(\{\underset{\sim}{r}_\varepsilon \,:\, \varepsilon < \zeta\}, \{U_\eta \,:\, \eta \in \lim(\underset{\sim}{T}_\zeta)\}).$$

(Remember also that, by $(\boxplus)_9$, all ω_1-branches of $\underset{\sim}{T}_\zeta$ in extensions by proper forcing over $\mathbf{V}^{\mathbb{P}_\zeta * \mathbb{Q}_\zeta}$ are the same as those in $\mathbf{V}^{\mathbb{P}_\zeta}$.) Note that the last factor of $\mathbb{Q}_\zeta$ adds an element $\underset{\sim}{r} \in \mathbb{Q}^0_{\omega_1}$ (see Lemma 4.7, 2.) and we know that

$$\Vdash_{\mathbb{P}_\zeta * \mathbb{Q}_\zeta} \text{``}(\forall \varepsilon < \zeta)(\underset{\sim}{r}_\varepsilon \leq^* \underset{\sim}{r}) \text{ and } \{U_\eta \,:\, \eta \in \lim(\underset{\sim}{T}_\zeta)\} \subseteq \mathrm{fil}(\underset{\sim}{r})\text{''}.$$

In $\mathbf{V}^{\mathbb{P}_\zeta * \mathbb{Q}_\zeta}$, we may choose a thin enough uncountable subset of $\underset{\sim}{r}$, getting $\underset{\sim}{r}' \subseteq \underset{\sim}{r}$ satisfying the demand in $(\boxplus)_7$ and such that

$$(\forall (\alpha, Z, d), (\alpha', Z', d') \in \underset{\sim}{r}')(\alpha < \alpha' \Rightarrow \sup(Z) + \omega < \alpha').$$

Let $\underset{\sim}{\sigma}' : {}^{<\omega_1}\omega_1 \longrightarrow \omega_1$ be such that $\underset{\sim}{\sigma}' \restriction \mathrm{dom}(\underset{\sim}{\sigma}_\zeta) = \underset{\sim}{\sigma}_\zeta$ and for $\nu \in {}^{<\omega_1}\omega_1 \setminus \mathrm{dom}(\underset{\sim}{\sigma}_\zeta)$ we have

$$\underset{\sim}{\sigma}'(\nu) = \min\{\beta < \omega_1 \,:\, (\exists (\alpha, Z, d) \in \underset{\sim}{r}')(\sup(\nu) < \alpha \,\&\, Z \subseteq \beta)\}.$$

Let $\underset{\sim}{\eta}^* = \underset{\sim}{\sigma}'[\underset{\sim}{\varrho}_\zeta]$ and let $\underset{\sim}{r}_\zeta = \{(\alpha, Z, d) \in \underset{\sim}{r}' \,:\, U_{\eta^*} \cap Z \in d\}$. It follows from our choices so far that $\underset{\sim}{r}_\zeta \in \mathbb{Q}^0_{\omega_1}$, and $\underset{\sim}{r}_\varepsilon \leq^* \underset{\sim}{r}_\zeta$ for $\varepsilon < \zeta$, and also for each $i < \omega_1$,

$$\underset{\sim}{\eta}^* \restriction (2i+1) \notin \underset{\sim}{T}_\zeta \Rightarrow (\exists (\alpha, Z, d) \in \underset{\sim}{r}_\zeta)(\underset{\sim}{\eta}^*(2i) < \alpha \,\&\, Z \subseteq \underset{\sim}{\eta}^*(2i+1)).$$

Put

$$\underset{\sim}{T}^*_\zeta = \underset{\sim}{T}_\zeta \cup \{\underset{\sim}{\eta}^* \restriction \alpha \,:\, \alpha < \omega_1\}$$

and define $\underset{\sim}{\sigma}'' : {}^{<\omega_1}\omega_1 \longrightarrow \omega_1$ so that $\underset{\sim}{\sigma}'' \restriction \underset{\sim}{T}^*_\zeta = \underset{\sim}{\sigma}' \restriction \underset{\sim}{T}^*_\zeta$ and for $\nu \in {}^{<\omega_1}\omega_1 \setminus \underset{\sim}{T}^*_\zeta$ we have

$$\underset{\sim}{\sigma}''(\nu) = \min\{\beta < \omega_1 \,:\, (\exists (\alpha, Z, d) \in \underset{\sim}{r}_\zeta)(\sup(\nu) < \alpha \,\&\, Z \subseteq \beta)\}.$$

Let

$$\underset{\sim}{S} = \{\nu \in {}^{<\omega_1}\omega_1 \,:\, \nu \text{ is increasing continuous of length } \mathrm{lh}(\nu) = \gamma + 1 \\ \text{for some limit } \gamma \text{ and } (\forall \alpha < \gamma)(\nu \restriction \alpha \in \underset{\sim}{T}_\zeta) \text{ but } \nu \restriction \gamma \notin \underset{\sim}{T}^*_\zeta\}.$$

For each $\nu \in \underset{\sim}{S}$ let $\eta_\nu \in {}^{\omega_1}\omega_1$ be such that $\nu \lhd \eta_\nu$, η_ν is played according to $\underset{\sim}{\sigma}''$, and $\eta_\nu(\alpha+1) = \eta_\nu(\alpha) + 889$ for every odd ordinal $\alpha \geq \mathrm{lh}(\nu)$. Put

$$\underset{\sim}{T}_{\zeta+1} = \underset{\sim}{T}^*_\zeta \cup \{\eta_\nu \restriction \alpha \,:\, \nu \in \underset{\sim}{S} \,\&\, \alpha < \omega_1\}.$$

Note that (still in $\mathbf{V}^{\mathbb{P}_\zeta * \mathbb{Q}_\zeta}$) we have that $\lim(\underset{\sim}{T}_{\zeta+1}) = \lim(\underset{\sim}{T}_\zeta) \cup \{\eta_\nu \,:\, \nu \in \underset{\sim}{S}\} \cup \{\underset{\sim}{\eta}^*\}$.

It follows from the choice of $\underset{\sim}{r}, \underset{\sim}{r}_\zeta$ that $(\forall \eta \in \lim(\underset{\sim}{T}_\zeta))(U_\eta \in \mathrm{fil}(\underset{\sim}{r}_\zeta))$ and by the definition of $\underset{\sim}{\sigma}''$ we obtain that $(\forall \nu \in \underset{\sim}{S})(U_{\eta_\nu} \in \mathrm{fil}(\underset{\sim}{r}_\zeta))$ (remember the choice of $\underset{\sim}{r}'$). Hence, remembering the definition of $\underset{\sim}{r}_\zeta$, we conclude that $(\forall \eta \in \lim(T_{\zeta+1}))(U_\eta \in \mathrm{fil}(\underset{\sim}{r}_\zeta))$.

Finally we put

$$\underset{\sim}{\sigma}_{\zeta+1} = \underset{\sim}{\sigma}'' \restriction \{\nu \in T_{\zeta+1} \,:\, \mathrm{lh}(\nu) \text{ is odd}\}.$$

One easily verifies that the relevant demands in $(\boxplus)_1 - (\boxplus)_7$ hold for $\underset{\sim}{T}_{\zeta+1}$, $\underset{\sim}{\sigma}_{\zeta+1}$, and $\underset{\sim}{r}_\zeta$. Let us also stress for future reference that

$(\circledast)_1$ if $\nu \in \mathrm{dom}(\underset{\sim}{\sigma}_{\zeta+1}) \setminus \underset{\sim}{T}_\zeta$ is of length $2i+1$, then there is $(\alpha, Z, d) \in \underset{\sim}{r}_\zeta$
       with $\nu(2i) < \alpha$ and $Z \subseteq \underset{\sim}{\sigma}_{\zeta+1}(\nu)$.

Suppose now that we have arrived to a limit stage $\xi < \kappa$ and we have defined $\mathbb{P}_\zeta$ names $\mathbb{Q}_\zeta$, $\underset{\sim}{T}_\zeta$, $\underset{\sim}{\sigma}_\zeta$, and $\underset{\sim}{r}_\zeta$ for all $\zeta < \xi$. In $\mathbf{V}^{\mathbb{P}_\xi}$ we define $\underset{\sim}{T}_\xi = \bigcup_{\zeta < \xi} \underset{\sim}{T}_\zeta$ and $\underset{\sim}{\sigma}_\xi = \bigcup_{\zeta < \xi} \underset{\sim}{\sigma}_\zeta$. We have to argue that the relevant demands in $(\boxplus)_2 - (\boxplus)_6$ are satisfied, and the only problematic one is the first condition of $(\boxplus)_6$. If $\mathrm{cf}(\xi) = \omega_0$, then

$$\Vdash_{\mathbb{P}_\xi} \lim(\underset{\sim}{T}_\xi) = \bigcup_{\zeta < \xi} \lim(\underset{\sim}{T}_\zeta),$$

so there are no problems. We will show that $(\boxplus)_6$ holds also if $\mathrm{cf}(\xi) \geq \omega_1$ and for this we will argue a contrario.

So suppose towards contradiction that ($\mathrm{cf}(\xi) \geq \omega_1$ and) we have $\mathbb{P}_\xi$-names $\underset{\sim}{\eta}_0, \ldots, \underset{\sim}{\eta}_n$ ($n < \omega$) and a condition $p \in \mathbb{P}_\xi$ such that

$$p \Vdash_{\mathbb{P}_\xi} \text{``} \underset{\sim}{\eta}_0, \ldots, \underset{\sim}{\eta}_n \in \lim(\underset{\sim}{T}_\xi) \text{ and } (\exists \zeta < \xi)(\omega_1 \setminus (U_{\eta_0} \cap \cdots \cap U_{\eta_n}) \in \mathrm{fil}(\underset{\sim}{r}_\zeta)) \text{''}.$$

Remembering that $(\boxplus)_1 + (\boxplus)_6$ hold on earlier stages, we may pass to a stronger condition (if necessary) and assume additionally that for some $\zeta < \xi$, $\gamma < \omega_1$, and pairwise distinct $\nu_0, \ldots, \nu_n \in {}^\gamma\omega_1$ we have

$$p \Vdash_{\mathbb{P}_\xi} \text{``} \eta_0, \ldots, \eta_n \notin \bigcup_{\varepsilon < \xi} \lim(T_\varepsilon) \text{ and } \eta_0 \restriction \gamma = \nu_0, \ldots, \eta_n \restriction \gamma = \nu_n$$
$$\text{and } (\forall(\alpha, Z, d) \in r_\zeta)(\gamma \leq \sup(Z) \Rightarrow U_{\eta_0} \cap \cdots \cap U_{\eta_n} \cap Z \notin d)\text{''}.$$

The forcing notion $\mathbb{P}_\xi$ is proper, so we may choose a countable elementary submodel $N \prec \mathcal{H}(\chi)$ such that

$$\eta_0, \ldots, \eta_n, \nu_0, \ldots, \nu_n, \zeta, \xi, \gamma, p, \ldots \in N$$

and then we may pick an $(N, \mathbb{P}_\xi)$-generic condition $q \geq p$. Let $\gamma^* = N \cap \omega_1$ and $\xi^* = \sup(N \cap \xi)$, and we may assume $q \in \mathbb{P}_{\xi^*}$. Then

$$(\circledast)_2 \qquad q \Vdash_{\mathbb{P}_{\xi^*}} (\forall i \leq n)(\forall \varepsilon < \xi^*)(\exists \delta < \gamma^*)(\eta_i \restriction \delta \notin T_\varepsilon),$$

and hence $q \Vdash_{\mathbb{P}_{\xi^*}} \eta_0 \restriction \gamma^*, \ldots, \eta_n \restriction \gamma^* \notin T_{\xi^*}$. [Why? As for each $\varepsilon \in N \cap \xi$ we have a name $\delta \in N$ for an ordinal below ω_1 such that $p \Vdash \eta_i \restriction \delta \notin T_\varepsilon$, so we may use the genericity of q.]

By a similar argument,

$$(\circledast)_3 \qquad q \Vdash_{\mathbb{P}_{\xi^*}} (\forall i \leq n)(\forall \delta < \gamma^*)(\exists \varepsilon < \xi^*)(\eta_i \restriction \delta \in T_\varepsilon),$$

so also $q \Vdash_{\mathbb{P}_{\xi^*}} (\forall \delta < \gamma^*)(\eta_0 \restriction \delta, \ldots, \eta_n \restriction \delta \in T_{\xi^*})$, and $q \Vdash_{\mathbb{P}_{\xi^*}} \eta_0(\gamma^*) = \cdots = \eta_n(\gamma^*) = \gamma^*$. (Remember that the η_i are increasing continuous.) Now consider a $\mathbb{P}_{\xi^*}$-name q for the following member of $\mathbb{Q}_{\xi^*}$:

$$(\emptyset_{\mathbb{C}}, \emptyset_{\mathbb{L}}, \emptyset_{\mathbb{P}_{T_{\xi^*}}}, (\emptyset, \{r_\zeta\}, \emptyset)).$$

Directly from the definition of the order of the forcing $\mathbb{Q}^{\mathrm{bd}}$ and the choice of r_{ξ^*} we see that

$$q \cup \{(\xi^*, q)\} \Vdash_{\mathbb{P}_{\xi^*+1}} r_{\xi^*} \subseteq \Sigma(r_\zeta).$$

It follows from $(\circledast)_2$, $(\circledast)_3$, and $(\boxplus)_5$ that

$$q \cup \{(\xi^*, q)\} \Vdash_{\mathbb{P}_{\xi^*+1}} \eta_0 \restriction (\gamma^* + 1), \ldots, \eta_n \restriction (\gamma^* + 1) \in T_{\xi^*+1} \setminus T_{\xi^*},$$

so let us look what are the respective values of the partial strategy σ_{ξ^*+1}. By $(\circledast)_1$ we know that

$$q \cup \{(\xi^*, q)\} \Vdash_{\mathbb{P}_{\xi^*+1}} \text{``there exists } (A, Z, d) \in r_{\xi^*} \text{ such that for each } i \leq n,$$
$$\eta_i(\gamma^*) = \gamma^* < \alpha \text{ and } Z \subseteq \eta_i(\gamma^* + 1)\text{''}.$$

Since $\gamma^* > \gamma$ we get a contradiction with the choice of p.

This completes the inductive definition of the iteration and the names T_ξ, σ_ξ and r_ξ. It should be clear that

$$\Vdash_{\mathbb{P}_\kappa} \text{``the sequence } \langle r_\xi : \xi < \kappa \rangle \text{ is } \leq^*\text{-increasing and } \mathrm{fil}(\{r_\xi : \xi < \omega_2\}) \text{ is a very reasonable}$$
$$\text{ultrafilter on } \omega_1 \text{ and } \bigcup_{\xi < \kappa} \sigma_\xi \text{ is a winning strategy for Odd in the game } \partial_{\{r_\xi : \xi < \kappa\}}\text{''}. \qquad \square$$

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