Algebra and Logic, Vol. 46, No. 3, 2007

THE p-RANK OF $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ IN CERTAIN MODELS OF ZFC

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UDC 510.223

Keywords: *theory ZFC, supercompact cardinal, strong limit cardinal, torsion-free Abelian group, almost free Abelian group.*

We prove that if the existence of a supercompact cardinal is consistent with ZFC, then it is consistent with ZFC that the p-rank of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ is as large as possible for every prime p and for any torsion-free Abelian group G. Moreover, given an uncountable strong limit cardinal μ of countable cofinality and a partition of Π (the set of primes) into two disjoint subsets Π_0 and Π_1, we show that in some model which is very close to ZFC, there is an almost free Abelian group G of size $2^\mu = \mu^+$ such that the p-rank of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ equals $2^\mu = \mu^+$ for every $p \in \Pi_0$ and 0 otherwise, that is, for $p \in \Pi_1$.

1. PRELIMINARIES

In [1, 2], the well-known Whitehead problem was solved by showing that it is undecidable in ordinary set theory ZFC whether or not every Abelian group G satisfying $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z}) = \{0\}$ has to be free. However, this did not clarify the structure of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ for torsion-free Abelian groups — a problem which has received much attention since then. Easy arguments show that $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ is always a divisible group for every torsion-free group G. Hence it is of the form

$$\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z}) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0, $p \in \Pi$, which are uniquely determined. This brings up the natural question as to which sequences $(\nu_0, \nu_p : p \in \Pi)$ of cardinals can appear as the cardinal invariants of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ for some (which?) torsion-free Abelian group. Obviously, the trivial sequence consisting of zero entries only can be realized by any free Abelian group. However, the solution of the Whitehead problem shows that it is undecidable in ZFC if these are the sole ones. There are a few results on possible sequences $(\nu_0, \nu_p : p \in \Pi)$ provable in ZFC. On the other hand, assuming Gödel's constructible universe ($V = L$) plus there being no weakly compact cardinal, a complete characterization of the cardinal invariants of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ for torsion-free Abelian groups G has recently been completed (see [3-12] for references). In fact, it turned out that

*Number 874 in Shelah's list of publications. Supported by the German–Israeli Foundation for Scientific Research & Development project No. I-706-54.6/2001.

**Supported by a grant from the German Research Foundation DFG.

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almost all divisible groups D may be realized as $\mathrm{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ for some torsion-free Abelian group G of almost any given size.

In this paper we shall hold the opposite point of view. There is a theorem for ZFC stating that every sequence $(\nu_0, \nu_p : p \in \Pi)$ of cardinals such that $\nu_0 = 2^{\lambda_0}$, for some infinite λ_0, and $\nu_p \leqslant \nu_0$ is either finite or of the form 2^{λ_p}, for some infinite λ_p, can arise as the cardinal invariants of $\mathrm{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ for some torsion-free G. Our prime goal is to show that this result is as best as possible by constructing a model of ZFC in which the only realizable sequences are of just this kind. We shall therefore assume the consistency of the existence of a supercompact cardinal (see [13]). This is a strong additional set-theoretic assumption which makes the model we are working in be far from ZFC.

On the other hand, we will also work with models very close to ZFC assuming only the existence of certain ladder systems on successors of strong limit cardinals of cofinality $\aleph_0$. Although this model is close to ZFC, it allows us to construct almost free torsion-free Abelian groups G such that, for instance, $\mathrm{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$, is torsion free, that is, G is coseparable. Also this can be considered as a result at the borderline of what is provable in models close to ZFC since the existence of non-free coseparable groups is independent of ZFC (see [14, Chap. XII; 13]).

Our notation is standard; all groups under consideration are Abelian and are written additively. We shall abbreviate $\mathrm{Ext}(-,-)$ to $\mathrm{Ext}_{\mathbb{Z}}(-,-)$, and Π will denote the set of all primes. A Whitehead group is a torsion-free group G such that $\mathrm{Ext}_{\mathbb{Z}}(G,\mathbb{Z}) = 0$. If H is a pure subgroup of the Abelian group G, then we write $H \subseteq_* G$. We expect the reader to have sufficient knowledge about forcing, large cardinals, and prediction principles like weak diamond, etc., as, for example, in [14-16]. Also reasonable knowledge is assumed about Abelian groups, as, for instance, in [17]. However, we have tried to make the paper as accessible as possible to both algebraists and set theorists.

2. THE STRUCTURE OF $\mathrm{Ext}(G,\mathbb{Z})$

In this section we recall the basic results on the structure of $\mathrm{Ext}(G,\mathbb{Z})$ for torsion-free groups G. Let G be a torsion-free Abelian group. It is easy to see that $\mathrm{Ext}(G,\mathbb{Z})$ is divisible; hence it is of the form

$$\mathrm{Ext}(G,\mathbb{Z}) = \bigoplus_{p\in\Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for certain cardinals ν_p, ν_0, $p \in \Pi$. Since the cardinals ν_p, $p \in \Pi$, and ν_0 completely determine the structure of $\mathrm{Ext}(G,\mathbb{Z})$, we introduce the following terminology. Let $\mathrm{Ext}_p(G,\mathbb{Z})$ be the p-torsion part of $\mathrm{Ext}(G,\mathbb{Z})$ for $p \in \Pi$. Denote by $r_0^e(G)$ the *torsion-free rank* ν_0 of $\mathrm{Ext}(G,\mathbb{Z})$, which is the dimension of $\mathbb{Q} \otimes \mathrm{Ext}(G,\mathbb{Z})$, and by $r_p^e(G)$ the p-rank ν_p of $\mathrm{Ext}(G,\mathbb{Z})$, which is the dimension of $\mathrm{Ext}(G,\mathbb{Z})[p]$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$ for any prime p. There are only a few results provable in ZFC for the case where G is uncountable; yet, under additional set-theoretic assumptions, we can gain a better understanding of the structure of $\mathrm{Ext}(G,\mathbb{Z})$. For instance, a complete characterization has been obtained for Gödel's universe under the assumption that there is no weakly compact cardinal. The aim of this paper is to go to the borderline of the characterization. On the one hand, we show that the p-ranks of $\mathrm{Ext}(G,\mathbb{Z})$ can be made as large as possible for every torsion-free Abelian group G in a model of ZFC with strong additional axioms (the existence of large cardinals). On the other hand, we deal with a model which is very close to ZFC but still allows us to construct uncountable torsion-free groups G such that $\mathrm{Ext}(G,\mathbb{Z})$ is torsion free.

We first justify our restriction to torsion-free G. Let A be any Abelian group and $t(A)$ its torsion subgroup. Then $\mathrm{Hom}(t(A),\mathbb{Z}) = 0$, and hence we obtain the short exact sequence

$$0 \to \mathrm{Ext}(A/t(A),\mathbb{Z}) \to \mathrm{Ext}(A,\mathbb{Z}) \to \mathrm{Ext}(t(A),\mathbb{Z}) \to 0,$$

which must split since $\mathrm{Ext}(A/t(A), \mathbb{Z})$ is divisible. Thus

$$\mathrm{Ext}(A, \mathbb{Z}) \cong \mathrm{Ext}(A/t(A), \mathbb{Z}) \oplus \mathrm{Ext}(t(A), \mathbb{Z}).$$

Since the structure of $\mathrm{Ext}(t(A), \mathbb{Z}) \cong \prod_{p\in\Pi} \mathrm{Hom}(A, \mathbb{Z}(p^\infty))$ in ZFC is well known (see [17]), it is reasonable to assume that A is torsion free and, of course, non-free. Using Pontryagin's theorem yields

LEMMA 2.1 [14, Thm. XII 4.1]. Suppose G is a countable torsion-free group which is not free. Then $r_0^e(G) = 2^{\aleph_0}$.

Similarly, for the p-ranks of G we have

LEMMA 2.2 [14, Thm. XII 4.7]. If G is a countable torsion-free group, then, for any prime p, either $r_p^e(G)$ is finite or $r_p^e(G) = 2^{\aleph_0}$.

This sheds light on the structure of $\mathrm{Ext}(G, \mathbb{Z})$ for countable torsion-free groups G in ZFC. We now turn our attention to uncountable groups. There is a useful characterization of $r_p^e(G)$ using the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/pZ \to 0.$$

The induced sequence

$$\mathrm{Hom}(G, \mathbb{Z}) \xrightarrow{\varphi^p} \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Ext}(G, \mathbb{Z}) \xrightarrow{p_*} \mathrm{Ext}(G, \mathbb{Z})$$

shows that the dimension of

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G, \mathbb{Z})\varphi^p$$

as a vector space over $\mathbb{Z}/p\mathbb{Z}$ is exactly $r_p^e(G)$.

The next result deals with the case where $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

LEMMA 2.3 [7, Thm. 3(b)]. For any cardinal ν_0 of the form $\nu_0 = 2^{\mu_0}$ for some infinite μ_0 and any sequence of cardinals $(\nu_p : p \in \Pi)$ less than or equal to ν_0 such that each ν_p is either finite or of the form 2^{μ_p} for some infinite μ_p, there is a torsion-free group G of cardinality μ_0 such that $\mathrm{Hom}(G, \mathbb{Z}) = 0$, $r_0^e(G) = \nu_0$, and $r_p^e(G) = \nu_p$ for all primes $p \in \Pi$.

The following lemma helps us reach the borderline of what is provable in ZFC.

LEMMA 2.4 [14, Lemma XII 5.2]. If G is torsion free so that $\mathrm{Hom}(G, \mathbb{Z}) = 0$, then, for all primes p, $r_p^e(G)$ is either finite or of the form 2^{μ_p} for some infinite $\mu_p \leqslant |G|$.

Assuming Gödel's axiom of constructibility, we even know of a complete characterization for the case where $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

LEMMA 2.5 [$V = L$] (see [14, Thm. XII 4.4, Cor. XII 4.5]). Suppose G is a torsion-free non-free group and let B be a subgroup of A of minimum cardinality ν such that A/B is free. Then $r_0^e(G) = 2^\nu$. In particular, $r_0^e(G)$ is uncountable and $r_0^e(G) = 2^{|G|}$ if $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

Note that the above lemma is not true in ZFC since ZFC is, in fact, consistent with the following: for any countable divisible group D, there exists an uncountable torsion-free group G with $\mathrm{Ext}(G, \mathbb{Z}) \cong D$; hence $r_0^e(G) = 1$ is possible if we take $D = \mathbb{Q}$ (see [12]).

The following result is a collection of theorems due to Grossberg, Mekler, Roslanowski, Sageev, and the authors. We show that under the assumption on ($V = L$), almost all possibilities for $r_p^e(G)$ can appear if the group is not of weakly compact cardinality, nor of singular cardinality of cofinality $\aleph_0$.

LEMMA 2.6 [$V = L$]. Let ν be an uncountable cardinal and suppose that $(\nu_p : p \in \Pi)$ is a sequence of cardinals such that $0 \leqslant \nu_p \leqslant 2^\nu$ for each p. Moreover, let H be a torsion-free group of cardinality ν. Then the following statements hold:

(i) [8, Thm. 3.7] if ν is regular and less than the first weakly compact cardinal, then there is an almost free group G of cardinality ν such that $r_0^e(G) = 2^\nu$, and for all primes p, $r_p^e(G) = \nu_p$;

(ii) [5, Thm. 1.0] if ν is a singular strong limit cardinal of cofinality ω, then there is no torsion-free group G of cardinality ν such that $r_p^e(G) = \nu$ for any prime p;

(iii) [9, Main Thm.] if ν is weakly compact and $r_p^e(H) \geqslant \nu$ for some prime p, then $r_p^e(H) = 2^\nu$;

(iv) [11] if ν is singular, is less than the first weakly compact cardinal, and is of cofinality $\mathrm{cf}(\nu) > \aleph_0$, then there is a torsion-free group G of cardinality ν such that $r_0^e(G) = 2^\nu$, and for all primes p, $r_p^e(G) = \nu_p$.

The above results show that under the assumption on $(V = L)$ and on the non-existence of weakly compact cardinals, the structure of $\mathrm{Ext}(G, \mathbb{Z})$ for torsion-free groups G of cardinality ν is clarified for all cardinals ν, and almost all sequences $(\nu_0, \nu_p : p \in \Pi)$ can be realized as the cardinal invariants of some torsion-free Abelian group in almost every cardinality. However, if we weaken the set-theoretic assumptions to GCH (the generalized continuum hypothesis), then even more versions are possible which have been excluded by $(V = L)$ before (see Lemma 2.5).

LEMMA 2.7 [14, Thms. XII 5.3 and 5.49]. (i) Assume GCH. For any torsion-free group A of uncountable cardinality ν, if $\mathrm{Hom}(A, \mathbb{Z}) = 0$ and $r_0^e(A) < 2^\nu$, then $r_p^e(A) = 2^\nu$ for each prime p;

(ii) it is consistent with ZFC and GCH that for any cardinal $\rho \leqslant \aleph_1$, there is a torsion-free group G_ρ such that $\mathrm{Hom}(G_\rho, \mathbb{Z}) = 0$, $r_0^e(G_\rho) = \rho$, and for all primes p, $r_p^e(G_\rho) = 2^{\aleph_1}$.

In the next section, we will see that this rich structure of $\mathrm{Ext}(G, \mathbb{Z})$ (G is torsion free), which exists under the assumption on $(V = L)$, does not appear in other models of ZFC. As a motivation, we cite two results from [13] which show that using Cohen forcing we may enlarge the p-rank of $\mathrm{Ext}(G, \mathbb{Z})$ for torsion-free groups G.

LEMMA 2.8 [13, Thm. 8]. Suppose G is contained in the p-adic completion of a free group F and $|G| > |F|$. If $\lambda \geqslant |F|$ and λ Cohen reals are added to the universe, then $|\mathrm{Ext}_p(G, \mathbb{Z})| \geqslant \lambda$. In particular, adding $2^{\aleph_0}$ Cohen reals to the universe implies that for every torsion-free reduced non-free Abelian group G of cardinality less than the continuum, there is a prime p such that $r_p^e(G) > 0$.

If we assume the consistency of large cardinals we can even get more. Recall that a cardinal κ is *compact* if it is uncountable, regular, and satisfies the condition that for every set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S. This is equivalent to saying that for any set A, $|A| \geqslant \kappa$, there exists a fine measure on $P_\kappa(A)$ (the set of all subsets of A of size less than or equal to κ). If we require the measure to satisfy a normality condition, then we arrive at a stronger notion. A fine measure U on $P_\kappa(A)$ is said to be *normal* if, for $f : P_\kappa(A) \to A$ such that $f(P) \in P$ for almost all $P \in P_\kappa(A)$, f is constant on a set in U. A cardinal κ is *supercompact* if, for every set A such that $|A| \geqslant \kappa$, there exists a normal measure on $P_\kappa(A)$ (for more information on supercompact cardinals, see [14, Chap. II.2] or [18, Chap. 6, Sec. 33]).

LEMMA 2.9 [13, Thm. 11]. Suppose that it is consistent that a supercompact cardinal exists. Then it is consistent with either $2^{\aleph_0} = 2^{\aleph_1}$ or $2^{\aleph_0} < 2^{\aleph_1}$ that for any group G, either $\mathrm{Ext}(G, \mathbb{Z})$ is finite or $r_0^e(G) \geqslant 2^{\aleph_0}$.

3. THE FREE (p-)RANK

In this section we introduce the *free (p-)-rank* of a torsion-free group G (p is a prime), which will induce upper bounds for the cardinal invariants of $\mathrm{Ext}(G, \mathbb{Z})$.

Definition 3.1. For a prime $p \in \Pi$, let K_p be the class of all torsion-free groups G such that $G/p^\omega G$ is free. Moreover, let K_0 be the class of all free groups.

Note that for $G \in K_p$ (p is a prime) we have $G = p^\omega G \oplus F$, where F is a free group, and hence $\mathrm{Ext}(G, \mathbb{Z})[p] = 0$ (since $p^\omega G$ is p-divisible). Notice also that $p^\omega G$ is a pure subgroup of G. Thus $r_p^e(G) = 0$ for $G \in K_p$ and for any prime p. Clearly, $r_0^e(G) = 0$ for all $G \in K_0$.

Definition 3.2. Let G be a torsion-free group. We call

$$\text{fr-rk}_0(G) = \min\{\mathrm{rk}(H) : H \subseteq_* G \text{ such that } G/H \in K_0\}$$

the *free rank of* G, and similarly, we call

$$\text{fr-rk}_p(G) = \min\{\text{fr-rk}(H) : H \subseteq_* G/p^\omega G \text{ such that } (G/p^\omega G)/H \in K_p\}$$

the *free p-rank of* G for any prime $p \in \Pi$.

First, we argue for

LEMMA 3.3. Let G be a torsion-free group and $p \in \Pi$ a prime. Then the following hold:

(i) $r_p^e(G) = r_p^e(G/p^\omega G)$;

(ii) $r_p^e(H) \leqslant r_p^e(G)$ and $r_0^e(H) \leqslant r_0^e(G)$, where H is a pure subgroup of G;

(iii) $\text{fr-rk}_0(G) \geqslant \text{fr-rk}_0(G/p^\omega G)$;

(iv) $\text{fr-rk}_p(G) = \text{fr-rk}_p(G/p^\omega G)$;

(v) $\text{fr-rk}_p(G) \leqslant \text{fr-rk}_0(G)$.

Proof. (i) Let p be a prime. Since $p^\omega G$ is pure in G, $p^\omega G$ is p-divisible. Hence

$$0 \to p^\omega G \to G \to G/p^\omega G \to 0$$

induces an exact sequence such as

$$0 \to \mathrm{Hom}(G/p^\omega G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(p^\omega G, \mathbb{Z}/p\mathbb{Z}) = 0,$$

the latter being trivial because $p^\omega G$ is p-divisible. Thus we have

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}(G/p^\omega G, \mathbb{Z}/p\mathbb{Z}),$$

and it follows easily that

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G, \mathbb{Z})\varphi^p \cong \mathrm{Hom}(G/p^\omega G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G/p^\omega G, \mathbb{Z})\varphi^p.$$

Therefore $r_p^e(G) = r_p^e(G/p^\omega G)$.

(ii) We consider the exact sequence

$$0 \to H \to G \to G/H \to 0,$$

which induces an exact sequence such as

$$\ldots \to \mathrm{Ext}(G/H, \mathbb{Z}) \xrightarrow{\alpha} \mathrm{Ext}(G, \mathbb{Z}) \to \mathrm{Ext}(H, \mathbb{Z}) \to 0.$$

Since G/H is torsion free, we conclude that $\mathrm{Ext}(G/H, \mathbb{Z})$ is divisible, and hence $\mathrm{Im}(\alpha)$ is as well. Thus $\mathrm{Ext}(G, \mathbb{Z}) = \mathrm{Ext}(H, \mathbb{Z}) \oplus \mathrm{Im}(\alpha)$, and so $r_0^e(H) \leqslant r_0^e(G)$ and $r_p^e(H) \leqslant r_p^e(G)$ for every prime p.

(iii) We need only observe the following: whenever $G = H \oplus F$ for some free group F, $p^\omega G \subseteq H$ for every prime p; hence $G/p^\omega G = H/p^\omega G \oplus F$.

(iv) Note that $p^\omega(G/p^\omega G) = \{0\}$, and so $\text{fr-rk}_p(G) = \text{fr-rk}_p(G/p^\omega G)$.

(v) Keeping in mind that the class K_0 is contained in the class K_p for every prime p, we need only appeal to (iii), (iv) and the definitions of $\text{fr-rk}_p(G)$ and $\text{fr-rk}_0(G)$. □

Remark 3.4. If G is a torsion-free group and p is a prime, then it follows from Lemma 3.4(i), (iv) that, regarding the free p-rank of G, we may assume without loss of generality that G is p-reduced. This is also justified by the following fact:

$$\text{fr-rk}_p(G) = \{H \subseteq_* G \text{ such that } G/H \in K_p\},$$

whose proof is an easy exercise.

To simplify the notation, in what follows we put $\Pi_0 = \Pi \cup \{0\}$.

LEMMA 3.5. Let G be a torsion-free group. Then the following hold:

(i) $r_0^e(G) \leqslant 2^\lambda$, where $\lambda = \max\{\aleph_0, \text{fr-rk}_0(G)\}$; in particular, $r_0^e(G) \leqslant 2^{\text{fr-rk}_0(G)}$ if $\text{fr-rk}_0(G)$ is infinite;

(ii) $r_p^e(G) \leqslant p^{\text{fr-rk}_p(G)}$ for all $p \in \Pi$.

Proof. (i) Choose a subgroup $H \subseteq G$ such that $\text{rk}(H) = \text{fr-rk}_0(G)$ and $G/H \in K_0$. It follows that $G = H \oplus F$ for some free group F, and hence $\text{Ext}(G, \mathbb{Z}) = \text{Ext}(H, \mathbb{Z})$. Therefore $r_0^e(G) = r_0^e(H) \leqslant 2^\lambda$, where $\lambda = \max\{\aleph_0, \text{rk}(H)\} = \max\{\aleph_0, \text{fr-rk}_0(G)\}$.

(ii) Let p be a prime; then $r_p^e(G) = r_p^e(G/p^\omega G)$ and $\text{fr-rk}_p(G) = \text{fr-rk}_p(G/p^\omega H)$ by Lemma 3.3(i), (iv). Hence there is no loss of generality in assuming that $p^\omega G = \{0\}$. Let $H \subseteq G$ be such that $\text{fr-rk}_0(H) = \text{fr-rk}_p(G)$ and $G/H \in K_p$. Then $G/H = D \oplus F$ for some free group F and some p-divisible group D. As in the proof of Lemma 3.3(i), it follows that $r_p^e(G) = r_p^e(H)$. Now, we let $H = H' \oplus F'$ for some free group F' such that $\text{rk}(H') = \text{fr-rk}_0(H) = \text{fr-rk}_p(G)$. Hence $\text{Ext}(H, \mathbb{Z}) = \text{Ext}(H', \mathbb{Z})$, and therefore $r_p^e(G) = r_p^e(H) = r_p^e(H')$. Consequently $r_p^e(G) = r_p^e(H') \leqslant p^{\text{rk}(H')} = p^{\text{fr-rk}_p(G)}$. □

Note that, for instance, in $(V = L)$, for any torsion-free group G, $2^{\text{fr-rk}_0(G)}$ is the actual value of $r_0^e(G)$ by Lemma 2.5. The next lemma justifies the fact that, as far as the free p-rank of a torsion-free group is concerned, we may also assume that $\text{fr-rk}_p(G) = \text{rk}(G)$ if $p \in \Pi_0$.

LEMMA 3.6. Let G be a torsion-free group and $p \in \Pi_0$ and $H \subseteq_* G$ be such that:

(i) $G/(H \oplus F) \in K_p$ for some free group F;

(ii) $\text{rk}(H) = \text{fr-rk}_p(G)$.

Then $\text{fr-rk}_p(H) = \text{rk}(H)$ and $r_p^e(G) = r_p^e(H)$.

Proof. Let G, H, and p be given. If $p = 0$, then the claim is trivially true. Hence we assume that $p \in \Pi$ and $\text{rk}(H) = \text{fr-rk}_p(G)$. Then there is a free group F such that $H' = H \oplus F$ is a pure subgroup of G satisfying $G/H' \in K_p$. Thus $\text{fr-rk}_p(G) = \text{rk}(H) = \text{fr-rk}_0(H')$. Without loss of generality, we may assume that G/H' is p-divisible by splitting the free part.

By way of contradiction, suppose that $\text{fr-rk}_p(H) < \text{rk}(H)$. Let $H_1 \subseteq_* H$, $H/H_1 \in K_p$, and $\text{fr-rk}_0(H_1) = \text{fr-rk}_p(H) < \text{rk}(H)$. Then there are a free group F_1 and a p-divisible group D for which

$$H/H_1 = D \oplus F_1.$$

Choose a pure subgroup $H_1 \subseteq_* H_2 \subseteq_* H$ such that $H_2/H_1 \cong D$. Thus $H/H_2 \cong F_1$, and so $H \cong H_2 \oplus F_1$. Without loss of generality, we may assume that $H = H_2 \oplus F_1$. Consequently $\text{rk}(H_2) = \text{rk}(H)$ (since $\text{rk}(H) = \text{fr-rk}_p(G)$). Let $H_3 = H_1 \oplus F_1 \oplus F$. Then

$$\text{fr-rk}_0(H_3) = \text{fr-rk}_0(H_1) < \text{rk}(H) = \text{fr-rk}_p(G).$$

Moreover,

$$G/H' \cong (G/H_3)\,/\,(H'/H_3)$$

is p-divisible. Since $H'/H_3 \cong H_2/H_1$ is also p-divisible and all groups under consideration are torsion free, we conclude that G/H_3 is p-divisible. Hence $\text{fr-rk}_p(G) \leqslant \text{fr-rk}_0(H_3) < \text{fr-rk}_p(G)$, which is a contradiction. Finally, $r_p^e(G) = r_p^e(H)$ derives as in the proof of Lemma 3.5. □

We now show how to calculate $\text{fr-rk}_p(G)$ explicitly for torsion-free groups G of finite rank and for $p \in \Pi$ (note that $\text{fr-rk}_0(G)$ can be easily calculated). Recall that a torsion-free group G of finite rank is *almost free* if every subgroup H of G of smaller rank than the rank of G is free. Obviously, we have

LEMMA 3.7. Let G be a non-free torsion-free group of finite rank n and $p \in \Pi$. Then we can calculate $\text{fr-rk}_p(G)$ as follows.

(i) If G is almost free, then let $H \subseteq \mathbb{Q}$ be the outer type of G. Then:

(a) $\text{fr-rk}_p(G) = n$ if H is not p-divisible;

(b) $\text{fr-rk}_p(G) = 0$ if H is p-divisible.

(ii) If G is not almost free, then choose a filtration $\{0\} = G_0 \subseteq_* G_1 \subseteq_* \ldots \subseteq_* G_m \subseteq_* G$ with G_{k+1}/G_k almost free. Then

$$\text{fr-rk}_p(G) = \sum_{k<m} \text{fr-rk}_p(G_{k+1}/G_k).$$

In order to prove our main Theorem 4.3 in Sec. 4, we need a further result on the class K_p, $p \in \Pi$.

LEMMA 3.8. Let p be a prime and G a torsion-free group of infinite rank. Then the following hold:

(i) if G is of singular cardinality, then $G \in K_p$ iff every pure subgroup H of G of smaller cardinality than is one of G satisfies $H \in K_p$;

(ii) $G \notin K_p$ iff $r_p^e(G) > 0$ whenever we add $|G|$ Cohen reals to the universe;

(iii) if $\text{rk}(G) \geqslant \aleph_0$, then adding $|G|$ Cohen reals to the universe adds a new member to $\text{Ext}_p(G, \mathbb{Z})$ while preserving the old ones.

Proof. (i) We need only apply the Singular Compactness Theorem given in [19].

(ii) One of the implications is trivial, and hence we only verify the second. Let $G \notin K_p$. By Lemma 3.3(ii), we may assume that G does not have any pure subgroup of smaller rank than is one of G which would satisfy (ii). It is easy to see that the rank $\delta = \text{rk}(G)$ of G must be uncountable. Thus $\delta > \aleph_0$ should be regular by (i). Let $G = \bigcup_{\alpha<\delta} G_\alpha$ be a filtration of G by pure subgroups G_α, $\alpha < \delta$, of G. The claim now follows by repeating the argument in [13, Thms. 9 and 10] (cf. also Lemma 2.8). The only difference is that in our situation the group G is not almost free; hence we require in [13, Thm. 10] that, for every α, the stationary set E contains an element $a \in G_{\alpha+1} \backslash (G_\alpha + p^\omega G_{\alpha+1})$ which belongs to the p-adic closure of $G_\alpha + p^\omega G_{\alpha+1}$. This makes only a minor change in the proof of [13, Thm. 10].

(iii) Is similar to (ii) using the proof of [13, Thm. 11]. □

Finally, we consider the *p-closure* of a pure subgroup H of some torsion-free Abelian group G, which will be needed in the proof of Theorem 4.3.

Definition 3.9. Let G be torsion free and H be a pure subgroup of G. For every prime $p \in \Pi$, the set

$$\text{cl}_p(G, H) = \{x \in G : \text{ for all } n \in \mathbb{N} \text{ there is } y_n \in H \text{ such that } x - y_n \in p^n G\}$$

is called the *p-closure of H*.

LEMMA 3.10. Let G be torsion free and H be a pure subgroup of G. For all primes $p \in \Pi$, the following statements hold:

(i) $H \subseteq \text{cl}_p(G, H)$;

(ii) $\text{cl}_p(G, H)$ is a pure subgroup of G;

(iii) $\mathrm{cl}_p(G,H)/H$ is p-divisible.

Proof. We fix a prime $p \in \Pi$.

(i) Is trivial.

(ii) Assume that $mx \in \mathrm{cl}_p(G,H)$ for some $m \in \mathbb{N}$ and $x \in G$. Then, for every $n \in \mathbb{N}$, there is $y_n \in H$ such that $mx - y_n \in p^n G$, say, $y_n = p^n g_n$ for some $g_n \in G$. Without loss of generality, we may assume that $(m,p) = 1$. Hence $1 = km + lp^n$ for some $k, l \in \mathbb{Z}$. Thus

$$x = kmx + lp^n x = kp^n g_n + ky_n + lp^n x,$$

and hence $x - ky_n \in p^n G$ with $ky_n \in H$. Therefore $x \in \mathrm{cl}_p(G,H)$.

(iii) Follows easily from (ii). □

4. SUPERCOMPACT CARDINALS AND LARGE p-RANKS

In this section we shall assume that the existence of a supercompact cardinal is consistent with ZFC. We shall then determine the cardinal invariants $(r_0^e(G), r_p^e(G) : p \in \Pi)$ of $\mathrm{Ext}(G,\mathbb{Z})$ for every torsion-free Abelian group in this model and show that they are as large as possible. We start with a theorem from [13] (see also [2]). Recall that, for cardinals μ, γ, and δ, we can define a partially ordered set

$$Fn(\mu,\gamma,\delta) = \{f : \mathrm{dom}(f) \to \gamma : \mathrm{dom}(f) \subseteq \mu, |\mathrm{dom}(f)| < \delta\}.$$

The partial order is given by $f \leqslant g$ iff $g \subseteq f$ treated as functions.

LEMMA 4.1 [13, Thm. 19]. Suppose κ is a supercompact cardinal, V is a model of ZFC which satisfies $2^{\aleph_0} = \aleph_1$ and $\mathcal{P} = Fn(\mu, 2, \aleph_1) \times Fn(\rho, 2, \aleph_0)$, where $\mu, \rho > \aleph_1$. Then $\mathcal{P}$ forces every κ-free group to be free.

As a consequence we obtain

LEMMA 4.2 [13, Cor. 20]. If it is consistent with ZFC that a supercompact cardinal exists, then both of the statements

— every $2^{\aleph_0}$-free group is free and $2^{\aleph_0} < 2^{\aleph_1}$, and

— every $2^{\aleph_0}$-free group is free and $2^{\aleph_0} = 2^{\aleph_1}$

are consistent with ZFC. Furthermore, if it is consistent that there is a supercompact cardinal then it is consistent that there is a cardinal $\kappa < 2^{\aleph_1}$ such that if κ Cohen reals are added to the universe then every κ-free group is free.

We are now ready to prove the main theorem of this section working in the model from Lemma 4.2. Assume that the existence of a supercompact cardinal is consistent with ZFC. Let V be any model in which there exists a supercompact cardinal κ such that the weak diamond principle $\diamondsuit^*_{\lambda^+}$ holds for all regular cardinals $\lambda \geqslant \kappa$. Now, we use Cohen forcing to add κ Cohen reals to V to obtain a new model $\mathcal{V}$. Thus, in $\mathcal{V}$, we still have $\diamondsuit^*_{\lambda^+}$ for all regular $\lambda \geqslant \kappa$, and also $\diamondsuit_\kappa$ holds. Moreover, we have $2^{\aleph_0} = \kappa$ and every κ-free group (of arbitrary cardinality) is free by [13].

THEOREM 4.3. In any model $\mathcal{V}$ described as above, for every non-free torsion-free Abelian group G and every prime $p \in \Pi$, the following hold:

(i) $r_0^e(G) = 2^{\max\{\aleph_0, \text{fr-rk}_0(G)\}}$;

(ii) if $\text{fr-rk}_p(G)$ is finite, then $r_p^e(G) = \text{fr-rk}_p(G)$;

(iii) if $\text{fr-rk}_p(G)$ is infinite, then $r_p^e(G) = 2^{\text{fr-rk}_p(G)}$.

We first note that the above theorem shows that $r_p^e(G)$, $p \in \Pi_0$, is as large as possible for every torsion-free Abelian group G in the model $\mathcal{V}$. Moreover, by Lemma 3.6, every sequence of cardinals $(\nu_p : p \in \Pi_0)$ not excluded by Theorem 4.3 may be realized as the cardinal invariants of $\mathrm{Ext}(H, \mathbb{Z})$ for some torsion-free group H.

Proof. Let $p \in \Pi_0$ be fixed. By Lemma 3.3, we may assume that $p^\omega G = 0$ if $p \in \Pi$. Moreover, in view of Lemma 3.6, there is no loss of generality in assuming that $\text{fr-rk}_p(G) = \mathrm{rk}(G)$. We use induction on the rank $\lambda = \mathrm{rk}(G) = \text{fr-rk}_p(G)$.

Case A: λ is finite. We may assume without loss of generality that $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$ (since G is of finite rank). If $p = 0$, then $r_0^e(G) = 2^{\aleph_0} = \kappa$ follows from Lemma 2.1, since G is not free. Further, let $p > 0$. Then $r_p^e(G)$ is the dimension of $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Since $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ is dual to the vector space G/pG, it follows that $r_p^e(G) = \mathrm{rk}(G) = \text{fr-rk}_p(G)$. Note that G is p-reduced by assumption.

Case B: $\lambda = \aleph_0$. For a group G of countable rank, as is well known, there exists a decomposition $G = G' \oplus F$, where F is a free group and G' satisfies $\mathrm{Hom}(G', \mathbb{Z}) = \{0\}$. Moreover, $\text{fr-rk}_p(G) = \lambda = \aleph_0$ by assumption, and hence $\mathrm{rk}(G') = \text{fr-rk}_p(G') = \aleph_0$. Therefore it follows from Lemma 2.1 that $r_0^e(G) = r_0^e(G') = 2^{\aleph_0} = \kappa$. If $p > 0$ then we conclude that $\mathrm{Hom}(G', \mathbb{Z}/p\mathbb{Z})$ has cardinality $2^{\aleph_0}$. In view of $\mathrm{Hom}(G', \mathbb{Z}) = \{0\}$, by Lemma 3.5(ii),

$$2^{\aleph_0} \leqslant r_p^e(G') = r_p^e(G) \leqslant 2^{\aleph_0},$$

and hence $r_p^e(G) = 2^{\aleph_0} = \kappa$ for $p \in \Pi_0$.

Case C: $\aleph_0 < \lambda < \kappa$. Note that $\kappa = 2^{\aleph_0} = 2^\lambda$, and so $r_0^e(G) = 2^{\aleph_0} = \kappa$ follows from Lemma 2.5. In fact, by the induction hypothesis, every Whitehead group H of size less than λ has to be free (because $r_p^e(H) = \text{fr-rk}_0(H)$), which suffices for Lemma 2.5 to apply. Now, assume that $p \in \Pi$. By Lemma 3.5(ii), we deduce

$$r_p^e(G) \leqslant 2^{\text{fr-rk}_p(G)} \leqslant 2^\lambda = 2^{\aleph_0},$$

and hence it remains to prove that $r_p^e(G) \geqslant 2^{\aleph_0}$. The proof is very similar to that of [13, Thm. 8] (see also Lemma 2.8), and so we only briefly recall it.

Let V be the ground model and P be the Cohen forcing, that is, $P = P(\kappa \times \omega, 2, \omega) = \{h : \mathrm{Dom}(h) \to \{0, 1\} : \mathrm{Dom}(h)$ is a finite subset of $\kappa \times \omega\}$. If $\mathbb{G}$ is a P-generic filter over V, we let $\tilde{h} = \bigcup_{g \in \mathbb{G}} g$. For the notational reasons, we may also write $\mathcal{V} = V[\mathbb{G}] = V[\tilde{h}]$ for the extension model determined by the generic filter $\mathbb{G}$. Choose $A \subseteq [\kappa]^{\leqslant\lambda}$ so that G belongs to $V[\tilde{h}\restriction_{A \times \omega}]$. Without loss of generality, we may assume that $\alpha \in A$ iff $\beta \in A$ whenever $\alpha + \lambda = \beta + \lambda$. We shall prove the claim by splitting the forcing. For each α such that $\lambda\alpha \in \kappa \backslash A$, let

$$\tilde{f}_\alpha \in V[A \times \omega \cup [\lambda\alpha, \lambda\alpha + \lambda) \times \omega]$$

be a member of $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ computed by $\tilde{h}\restriction_{[\lambda\alpha, \lambda\alpha+\lambda)\times\omega}$. Note that $\tilde{f}_\alpha$ exists by Prop. 3.8(iii). Then $\tilde{f}_\alpha$ is also computed from $\tilde{h}\restriction_{[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}$ over $V[\tilde{h}\restriction_{\kappa\backslash[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}]$, and so $\tilde{f}_\alpha$ is not equivalent to any $f' \in V[\tilde{h}\restriction_{\kappa\backslash[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}]$ modulo $\mathrm{Hom}(G, \mathbb{Z})\varphi^p$. Thus the set $\{\tilde{f}_\alpha : \lambda\alpha < \kappa, \lambda\alpha \notin A\}$ of homomorphisms exemplifies that $r_p^e(G) \geqslant \kappa = 2^{\aleph_0}$, and hence

$$2^\lambda = 2^{\aleph_0} = \kappa \leqslant r_p^e(G) \leqslant 2^\lambda,$$

which shows that $r_p^e(G) = 2^\lambda = 2^{\text{fr-rk}_p(G)}$.

Case D: $\lambda \geqslant \kappa$. Let $p \in \Pi_0$. We distinguish two subcases. Note that, for $p \in \Pi$, the assumption that $\text{fr-rk}_p(G) = \mathrm{rk}(G)$ also implies that $\text{fr-rk}_0(G) = \mathrm{rk}(G)$ by Lemma 3.3(v).

Case D1: $\lambda \geqslant \kappa$ and λ is regular. The case $p = 0$ is treated as in [14, Thm. XII 4.4]. Let $\langle G_\alpha : \alpha < \lambda \rangle$ be a filtration of G into pure subgroups G_α, $\alpha < \lambda$, so that if G/G_α is not λ-free, then $G_{\alpha+1}/G_\alpha$ is not free. Using [3, Lemma 2.4], we choose an associate free resolution of G, that is, a free resolution

$$0 \to K \xrightarrow{\Phi} F \to G \to 0$$

of G for which $F = \bigoplus_{\alpha<\lambda} F_\alpha$ and $K = \bigoplus_{\alpha<\lambda} K_\alpha$ are free groups such that $|F_\alpha| < \lambda$ and $|K_\alpha| < \lambda$ for all $\alpha < \lambda$, and the induced sequences

$$0 \to \bigoplus_{\beta<\alpha} K_\beta \to \bigoplus_{\beta<\alpha} F_\beta \to G_\alpha \to 0$$

are exact for every $\alpha < \lambda$.

Since $\text{fr-rk}_0(G) = \lambda$, the set $E = \{\alpha < \lambda : G_{\alpha+1}/G_\alpha \text{ is not free}\}$ is stationary. For any subset $E' \subseteq E$, let $K(E') = \bigoplus_{\alpha\in E'} K_\alpha$ and $G(E') = F/\Phi(K(E'))$. Then $\Gamma(G(E')) \geq \tilde{E}'$, where $\Gamma(G(E'))$ is the Γ-invariant of $G(E')$ (see [14] for details on the Γ-invariant). Now, $\diamondsuit^*_\lambda$ by assumption; hence we may decompose E into λ disjoint stationary sets E'_α, each of which is non-small, that is, $\diamondsuit^*_\lambda(E'_\alpha)$ holds. Hence $G(E'_\alpha)$ is not free since $\tilde{E'_\alpha} \leq \Gamma(G(E'_\alpha))$ for every $\alpha < \lambda$. By [13], we conclude that $G(E'_\alpha)$ is not κ-free and, therefore, has a non-free pure subgroup H_α of rank less than κ. By the induction hypothesis, $\text{Ext}(H_\alpha, \mathbb{Z}) \neq 0$, and hence also $\text{Ext}(G(E'_\alpha), \mathbb{Z}) \neq 0$. As in [3, Lemma 1.1] (see also [14, Lemma XII 4.2]), there is an epimorphism

$$\text{Ext}(G, \mathbb{Z}) \to \prod_{\alpha<\lambda} \text{Ext}(G(E'_\alpha), \mathbb{Z}) \to 0,$$

which yields $r^e_0(G) \geqslant 2^\lambda$, and hence $r^e_0(G) = 2^\lambda$ (cf. [14, Lemma 4.3]).

Now assume that $p > 0$. Again, let $\langle G_\alpha : \alpha < \lambda \rangle$ be a filtration of G into pure subgroups G_α, $\alpha < \lambda$, such that $\text{Ext}_p(G_{\alpha+1}/G_\alpha, \mathbb{Z}) \neq 0$ iff $\text{Ext}_p(G_\beta/G_\alpha, \mathbb{Z}) \neq 0$ for some $\beta > \alpha$. Fix $\alpha < \lambda$. We claim that $G/\text{cl}_p(G, G_\alpha)$ is not free. Conversely, assume that $G/\text{cl}_p(G, G_\alpha)$ is free. Hence $G = \text{cl}_p(G, G_\alpha) \oplus F$ for some free group F. Therefore $G/G_\alpha = (\text{cl}_p(G, G_\alpha) \oplus F)/G_\alpha = \text{cl}_p(G, G_\alpha)/G_\alpha \oplus F$ is a direct sum of a p-divisible group and a free group by Lemma 3.5(iii). It follows that $\text{fr-rk}_p(G) \leqslant \text{rk}(G_\alpha) < \lambda$, contradicting the fact that $\text{fr-rk}_p(G) = \lambda$. By [13], we conclude that $G/\text{cl}_p(G, G_\alpha)$ is not κ-free since we are working in the model $\mathcal{V}$. Let $G'/\text{cl}_p(G, G_\alpha) \subseteq_* G/\text{cl}_p(G, G_\alpha)$ be a non-free pure subgroup of $G/\text{cl}_p(G, G_\alpha)$ of size less than κ. Then there exists $\alpha \leqslant \beta < \lambda$ such that $G' \subseteq_* G_\beta$. By purity, it follows that $\text{cl}_p(G, G_\alpha) \cap G_\beta = \text{cl}_p(G_\beta, G_\alpha)$. Hence

$$G_\beta/\text{cl}_p(G_\beta, G_\alpha) = (G_\beta + \text{cl}_p(G, G_\alpha))/\text{cl}_p(G, G_\alpha)$$

is torsion free but not free. Without loss of generality, we may assume that $\beta = \alpha + 1$. Hence we can suppose that, for all $\alpha < \lambda$, the quotient $G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha)$ is a torsion-free non-free group. Note that $G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha)$ is also p-reduced since $\text{cl}_p(G_{\alpha+1}, G_\alpha)$ is the p-closure of G_α inside $G_{\alpha+1}$.

Since the cardinality of $G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha)$ is less than λ, the induction hypothesis applies. Hence $r^e_p(G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha)) = 2^{\text{fr-rk}_p(G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha))}$. We claim that $\text{Ext}_p(G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha), \mathbb{Z}) \neq \{0\}$ stationarily often. If not, then there is a cube $C \subseteq \lambda$ such that for all $\alpha < \beta \in C$ we have $\text{Ext}_p(G_{\alpha+1}/\text{cl}_p(G_{\alpha+1}, G_\alpha), \mathbb{Z}) = \{0\}$, and equivalently, $\text{Ext}_p(G_\beta/G_\alpha, \mathbb{Z}) = 0$; hence $\text{fr-rk}_p(G_\beta) = \text{fr-rk}_p(G_\alpha)$ by the induction hypothesis. As in [14, Prop. XII 1.5], it follows that $\text{Ext}_p(G/G_\alpha, \mathbb{Z}) = 0$ for all $\alpha \in C$, which contradicts the fact that $\text{fr-rk}_p(G) = \lambda$.

Without loss of generality, we may assume that, for every $\alpha < \lambda$, there exist homomorphisms $h^0_\alpha, h^1_\alpha \in \text{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$ such that:

(1) $h^0_\alpha \restriction_{\mathrm{cl}_p(G_{\alpha+1},G_\alpha)} = h^1_\alpha \restriction_{\mathrm{cl}_p(G_{\alpha+1},G_\alpha)}$;

(2) there are no homomorphisms $g^0_\alpha, g^1_\alpha \in \mathrm{Hom}(G_{\alpha+1},\mathbb{Z})$ for which

(a) $g^0_\alpha\restriction_{\mathrm{cl}_p(G_{\alpha+1},G_\alpha)} = g^1_\alpha\restriction_{\mathrm{cl}_p(G_{\alpha+1},G_\alpha)}$ (or, which is equivalent, $g^0_\alpha\restriction_{G_\alpha} = g^1_\alpha\restriction_{G_\alpha}$), and

(b) $h^0_\alpha = g^0_\alpha\varphi^p$ and $h^1_\alpha = g^1_\alpha\varphi^p$.

Indeed, there is a homomorphism $\varphi : G_{\alpha+1}/\mathrm{cl}_p(G_{\alpha+1},G_\alpha) \to \mathbb{Z}/p\mathbb{Z}$ which cannot be factored by φ_p since $\mathrm{Ext}_p(G_{\alpha+1}/\mathrm{cl}_p(G_{\alpha+1},G_\alpha),\mathbb{Z}) \neq \{0\}$. Let $h^0_\alpha = 0$ and h^1_α be given by

$$h^1_\alpha : G_{\alpha+1} \to G_{\alpha+1}/\mathrm{cl}_p(G_{\alpha+1},G_\alpha) \xrightarrow{\varphi} \mathbb{Z}/p\mathbb{Z}.$$

It is then easy to check that h^0_α and h^1_α are as required. In particular, we may assume that $h^0_\alpha = 0$ for every $\alpha < \lambda$. An immediate consequence is the following property:

(U) Let $f : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$ and $g : G_\alpha \to \mathbb{Z}$ so that $g\varphi_p = f$. Then there exists $\tilde{f} : G_{\alpha+1} \to \mathbb{Z}/p\mathbb{Z}$ such that $\tilde{f}\restriction_{G_\alpha} = f$ and there is no homomorphism $\tilde{g} : G_{\alpha+1} \to \mathbb{Z}$ satisfying both $\tilde{g}\restriction_{G_\alpha} = g$ and $\tilde{g}\varphi_p = \tilde{f}$.

Indeed, let $\hat{f} : G_{\alpha+1} \to \mathbb{Z}/p\mathbb{Z}$ be any extension of f, which exists by the pure injectivity of $\mathbb{Z}/p\mathbb{Z}$. If $\hat{f}$ is as required, we let $\tilde{f} = \hat{f}$. Otherwise, choose $\hat{g} : G_{\alpha+1} \to \mathbb{Z}$ so that $\hat{g}\restriction_{G_\alpha} = g$ and $\hat{g}\varphi_p = \hat{f}$. Put $\tilde{f} = \hat{f} - h^1_\alpha$. Then $\tilde{f}\restriction_{G_\alpha} = f$. Assume that there exists $\tilde{g} : G_{\alpha+1} \to \mathbb{Z}$ such that $\tilde{g}\restriction_{G_\alpha} = g$ and $\tilde{g}\varphi_p = \tilde{f}$. Choosing $g^1_\alpha = \tilde{g} - \hat{g}$ we conclude that $-g^1_\alpha\varphi_p = h^1_\alpha$, which is a contradiction with (2). Note that $h^0_\alpha = 0$.

We now proceed exactly as in [7, Prop. 1] to show that $r^e_p(G) = 2^{\mathrm{fr\text{-}rk}_p(G)} = 2^\lambda$. We therefore outline the proof only briefly, and for simplicity, we even suppose that $\diamondsuit_\lambda$ holds. It is an easy exercise (and so left to the reader) to prove the result assuming the weak diamond principle only. Suppose that $r^e_p(G) = \sigma < 2^\lambda$ and let $L = \{f^\alpha : \alpha < \sigma\}$ be a complete list of representatives of elements in $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G,\mathbb{Z})\varphi_p$. Without loss of generality, let $\{g_\alpha : G_\alpha \to \mathbb{Z} : \alpha < \lambda\}$ be the Jensen functions given by $\diamondsuit_\lambda$; hence for every homomorphism $g : G \to Z$, there exists α such that $g\restriction_{G_\alpha} = g_\alpha$. We now define a sequence $\{f^*_\alpha : G_\alpha \to \mathbb{Z}/p\mathbb{Z} : \alpha < \lambda\}$ of homomorphisms so that:

(1) $f^*_0 = f^0$;

(2) $f^*_\alpha\restriction_{G_\beta} = f^*_\beta$ for all $\beta < \alpha$;

(3) if $f^* = \bigcup\limits_{\alpha<\lambda} f^*_\alpha$, then $f^* - f^\alpha$ is an element of $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ but not of $\mathrm{Hom}(G,\mathbb{Z})\varphi_p$.

Suppose that f^*_β has been defined for all $\beta < \alpha$. If α is a limit ordinal, then we let $f^*_\alpha = \bigcup\limits_{\beta<\alpha} f^*_\beta$, which is a well-defined homomorphism by (2). If $\alpha = \beta + 1$ is a successor ordinal, then we distinguish two cases. If $f^*_\beta - f^\beta\restriction_{G_\beta} \neq g_\beta\varphi_p$, we let $f^*_\alpha : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$ be any extension of f^*_β, which exists since $\mathbb{Z}/p\mathbb{Z}$ is pure injective and $G_\beta \subseteq_* G_\alpha$. If $f^*_\beta - f^\beta\restriction_{G_\beta} = g_\beta\varphi_p$, then (U) shows that there is a homomorphism $\tilde{f} : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$ extending $f^*_\beta - f^\beta\restriction_{G_\beta}$ such that there is no $\tilde{g} : G_{\beta+1} \to \mathbb{Z}$ with both extending g_β and $\tilde{g}\varphi_p = \tilde{f}$. Lastly, put $f^*_\alpha = \tilde{f} + f^\alpha\restriction_{G_\alpha}$ and $f^* = \bigcup\limits_{\alpha<\lambda} f^*_\alpha$. It is now straightforward to see that f^* satisfies (3), and hence f^* contradicts the maximality of the list L.

Case D2: $\lambda \geqslant \kappa$ and λ is singular. First, note that $\mathrm{fr\text{-}rk}_p(G) > \kappa$ since $\kappa = 2^{\aleph_0}$ is regular. By induction on $\alpha < \lambda$, we choose subgroups K_α of G such that:

(1) K_α is a pure non-free subgroup of G;

(2) $|K_\alpha| < \kappa$;

(3) $K_\alpha \cap \sum\limits_{\beta<\alpha} K_\beta = \{0\}$;

(4) $\sum\limits_{\beta<\alpha} K_\beta$ is a pure subgroup of G.

Assume that we have succeeded in constructing the groups K_α, $\alpha < \lambda$. Then

$$K = \sum_{\beta<\lambda} K_\beta = \bigoplus_{\beta<\lambda} K_\beta$$

is a pure subgroup of G, and hence $r_p^e(G) \geqslant r_p^e(K)$ by Lemma 3.3(ii). If $\mathrm{Ext}_p(K_\alpha, \mathbb{Z}) = 0$, then $K_\alpha \in K_p$ follows by induction. Since G is p-reduced, we obtain $K_\alpha \in K_0$, contradicting (1). Thus $\mathrm{Ext}_p(K_\alpha, \mathbb{Z}) \neq \{0\}$ for every $\alpha < \lambda$, which implies $r_p^e(K) \geqslant 2^\lambda$ (since $\mathrm{Ext}(K, \mathbb{Z}) \cong \prod\limits_{\alpha<\lambda} \mathrm{Ext}(K_\alpha, \mathbb{Z})$). Hence we need only complete the construction of the groups K_α, $\alpha < \lambda$. Assume that K_β, $\beta < \alpha$, have been constructed. Let $\mu = (\kappa + |\alpha|)^{<\kappa}$, which is a cardinal less than λ. Let H_α be such that:

(i) $H_\alpha \subseteq_* G$;

(ii) $\sum\limits_{\beta<\alpha} K_\beta \subseteq H_\alpha$;

(iii) $|H_\alpha| = \mu$;

(iv) if $K \subseteq G$ is of cardinality less than κ, then there is a subgroup $K' \subseteq_* G$ such that $H_\alpha \cap K \subseteq K'$ and K and K' are isomorphic over $K \cap H_\alpha$, i.e., there exists an isomorphism $\psi : K \to K'$, which is the identity if restricted to $K \cap H_\alpha$.

It is easy to see that H_α exists. Now, G/H_α is a non-free group since $\text{fr-rk}_0(G) = \lambda$ and $p^\omega(G/H_\alpha) = \{0\}$. Hence [13] implies that there is $K'_\alpha \subseteq G$ such that $(K'_\alpha + H_\alpha)/H_\alpha$ is not free and $|K'_\alpha| < \kappa$. Let $K^0_\alpha \subseteq_* H_\alpha$ be as in (5), that is, $K'_\alpha \cap H_\alpha \subseteq K^0_\alpha$ and there is an isomorphism $\psi_\alpha : K'_\alpha \to K^0_\alpha$, which is the identity on $K'_\alpha \cap H_\alpha$. Let $K_\alpha = \{x - x\psi_\alpha : x \in K'_\alpha\}$. Then K_α is as required. For instance,

$$K_\alpha \cong K^0_\alpha / (K'_\alpha \cap H_\alpha) \cong K'_\alpha / (K'_\alpha \cap H_\alpha)$$

shows that K_α is not free. □

COROLLARY 4.4. In the model $\mathcal{V}$, let $\langle \mu_p : p \in \Pi_0 \rangle$ be a sequence of cardinals. Then there exists a torsion-free non-free Abelian group G such that $r_p^e(G) = \mu_p$ for all $p \in \Pi_0$ if and only if the following hold:

(i) $\mu_0 = 2^{\lambda_0}$ for some infinite cardinal λ_0;

(ii) $\mu_p \leqslant \mu_0$ for all $p \in \Pi$;

(iii) μ_p is either finite or of the form 2^{λ_p} for some infinite cardinal λ_p.

The **proof** follows easily from Lemma 2.3 and Theorem 4.3. □

COROLLARY 4.5. In the model $\mathcal{V}$, let $\langle \mu_p : p \in \Pi_0 \rangle$ be a sequence of cardinals. Then there exists a non-free $\aleph_1$-free Abelian group G such that $r_p^e(G) = \mu_p$ for all $p \in \Pi_0$ if and only if $\mu_p \leqslant \mu_0$ and $\mu_p = 2^{\lambda_p}$ for some infinite cardinal λ_p for every $p \in \Pi_0$.

Proof. In view of Theorem 4.3, we need only prove the existence claim of the corollary. It suffices to construct $\aleph_1$-free groups G_p for $p \in \Pi_0$ so that $r_p^e(G) = r_0^e(G) = 2^{\aleph_0} = \kappa$ and $r_q^e(G) = 0$ for all $p \neq q \in \Pi$. Then $B = G_0^{(\lambda_0)} \oplus \bigoplus\limits_{p\in\Pi} G_p^{(\lambda_p)}$ will be as required (see, e.g., the proof of [7, Thm. 3(b)]). Fix $p \in \Pi_0$. From [14, Thm. XII 4.10] or [10], it follows that there exists an $\aleph_1$-free non-free group G_p of size $2^{\aleph_0}$ such that $r_p^e(G_p) = 2^{\aleph_0} = \kappa$ if $p \in \Pi$. In [14, Thm. XII 4.10], it is then assumed that $2^{\aleph_0} = \aleph_1$ to show that $r_0^e(G_p) = \kappa$. However, since we work in the model $\mathcal{V}$, and G_p is not free, Theorem 4.3 implies that $\text{fr-rk}_0(G_p) \geqslant \aleph_1$, and hence $r_p^0(G_p) = 2^{\text{fr-rk}_0(G_p)} = \kappa$. □

Recall that a reduced torsion-free group G is said to be *coseparable* if $\mathrm{Ext}(G, \mathbb{Z})$ is torsion free. By [13], it is consistent that all coseparable groups are free. By [14, Thm. XII 4.10], yet, there exist coseparable groups which are not free if $2^{\aleph_0} = \aleph_1$. Note that the groups constructed in Lemma 2.3 are not reduced and, hence, do not provide examples of coseparable groups.

COROLLARY 4.6. In the model $\mathcal{V}$, there exist non-free coseparable groups.

The **proof** follows from Corollary 4.5 letting $\mu_0 = 2^{\aleph_0}$ and $\mu_p = 0$ for all $p \in \Pi$. □

5. A MODEL CLOSE TO ZFC

In this section we shall construct a coseparable group which is not free in a model of ZFC and is very close to ZFC. As mentioned in Sec. 4, the question if all coseparable groups are free is undecidable in ZFC.

Let $\aleph_0 < \lambda$ be a regular cardinal and S be a stationary subset of λ consisting of limit ordinals of cofinality ω. We recall the definition of a ladder system on S (see, e.g., [14, p. 405]).

Definition 5.1. A *ladder system* $\bar{\eta}$ *on* S is a family of functions $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ such that $\eta_\delta : \omega \to \delta$ is strictly increasing with $\sup(\mathrm{rg}(\eta_\delta)) = \delta$, where $\mathrm{rg}(\eta_\delta)$ denotes the range of η_δ. We say that the ladder system is *tree-like* if, for all $\delta, \nu \in S$ and for every $n, n \in \omega$, $\eta_\delta(n) = \eta_\nu(m)$ implies $n = m$ and $\eta_\delta(k) = \eta_\nu(k)$ for all $k \leqslant n$.

One way to construct almost free groups is to use κ-free ladder systems.

Definition 5.2. Let κ be an uncountable regular cardinal. The ladder system $\bar{\eta}$ is said to be *κ-free* if, for every subset $X \subseteq S$ of cardinality less than κ, there is a sequence of natural numbers $\langle n_\delta : \delta \in X \rangle$ such that

$$\langle \{ \eta_\delta \restriction_l : n_\delta < l < \omega \} : \delta \in X \rangle$$

is a sequence of pairwise disjoint sets.

Finally, recall that a stationary set $S \subseteq \lambda$ with λ uncountable regular is *non-reflecting* if $S \cap \kappa$ is not stationary in κ for every $\kappa < \lambda$ with $\mathrm{cf}(\kappa) > \aleph_0$.

THEOREM 5.3. Let μ be an uncountable strong limit cardinal such that $\mathrm{cf}(\mu) = \omega$ and $2^\mu = \mu^+$. Put $\lambda = \mu^+$ and assume that there exists a λ-free tree-like ladder system on a non-reflecting stationary subset $S \subseteq \lambda$. If $\Pi = \Pi_0 \cup \Pi_1$ is a partition of Π into disjoint subsets Π_0 and Π_1, then there exists an almost free group G of size λ such that:

(i) $r_0^e(G) = 2^\lambda$;

(ii) $r_p^e(G) = 2^\lambda$ if $p \in \Pi_0$;

(iii) $r_p^e(G) = 0$ if $p \in \Pi_1$.

Proof. Let $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ be the λ-free ladder system, where S is a stationary non-reflecting subset of λ consisting of ordinals less than λ of cofinality ω. Without loss of generality, we may assume that $S = \lambda$. Let $\mathrm{pr} : \mu^2 \to \mu$ be a pairing function; then pr is bijective. And if $\alpha \in \mu$ then we denote by $(\mathrm{pr}_1(\alpha), \mathrm{pr}_2(\alpha))$ the unique pair $(\beta, \gamma) \in \mu^2$ such that $\mathrm{pr}(\mathrm{pr}_1(\alpha), \mathrm{pr}_2(\alpha)) = \alpha$. Let

$$L = \bigoplus_{\alpha < \mu} \mathbb{Z} x_\alpha$$

be the free Abelian group generated by independent elements x_α, $\alpha < \mu$.

To simplify the notation, we may assume that $\Pi_1 \neq \varnothing$ and let $\langle (p_\beta, f_\beta) : \beta < \lambda \rangle$ be a listing of all pairs (p, f) with $p \in \Pi_1$ and $f \in \mathrm{Hom}(L, \mathbb{Z}/p\mathbb{Z})$. Recall that $\lambda = 2^\mu$. By induction on $\beta < \lambda$, we choose triples $(g_\beta, \nu_\beta, \rho_\beta)$ for which the following conditions hold:

(1) $g_\beta \in \mathrm{Hom}(L, \mathbb{Z})$;

(2) $f_\beta = g_\beta \varphi_p$, where $\varphi_p : \mathrm{Hom}(L, \mathbb{Z}) \to \mathrm{Hom}(L, \mathbb{Z}/p\mathbb{Z})$ is the canonical map;

(3) for $\nu_\beta, \rho_\beta : \omega \to \mu$, $\eta_\beta(n) = \mathrm{pr}_1(\nu_\beta(n)) = \mathrm{pr}_1(\rho_\beta(n))$;

(4) for all $\delta \leqslant \beta$, there exist $n = n(\delta, \beta) \in \omega$ such that $g_\delta(x_{\nu_\beta(m)}) = g_\delta(x_{\rho_\beta(m)})$ for all $m \geqslant n$;

(5) for all $\delta < \beta$, there exist $n = n(\delta, \beta) \in \omega$ such that, for some sequence $\langle b_m^{\delta,\beta} : m \in [n, \omega) \rangle$ of natural numbers, $\left(\prod_{p \in \Pi_1 \cap m} p \right) b_{m+1}^{\delta,\beta} = b_m^{\delta,\beta} + g_\beta(x_{\nu_\delta(m)}) - g_\beta(x_{\rho_\delta(m)})$ for all $m \geqslant n$;

(6) $\nu_\beta(m) \neq \rho_\beta(m)$ for all $m \in \omega$.

Fix $\beta < \lambda$ and assume that we have constructed $(g_\delta, \nu_\delta, \rho_\delta)$ for all $\delta < \beta$. Choose a function $h_\beta : \beta \to \omega$ so that $h_\beta(\delta) > p_\delta$ for all $\delta < \beta$, and

$$\langle \{\eta_\delta \restriction_l : l \in [h_\beta(\delta), \omega)\} : \delta < \beta \rangle \tag{5.1}$$

is a sequence of pairwise disjoint sets. Note that such a choice is possible since the ladder system $\bar{\eta}$ is λ-free by assumption. Moreover, by (3), the pairing function pr also ensures that

$$\langle \{\nu_\delta \restriction_l, \rho \restriction_l : l \in [h_\beta(\delta), \omega)\} : \delta < \beta \rangle \tag{5.2}$$

is a sequence of pairwise disjoint sets. Now, we choose the function g_β for which (2) and (5) hold. For $\delta < \beta$, let $n = n(\delta, \beta) = h_\beta(\delta)$. Since L is free, first we choose $g_\beta(x_\alpha)$ satisfying $g_\beta(x_\alpha) + p_\beta\mathbb{Z} = f_\beta(x_\alpha)$ for every α such that $\mathrm{pr}_1(\alpha) \neq \eta_\delta(l)$ for all $\delta < \beta$ and $l \geqslant n(\delta, \beta)$, that is to say, for those α for which x_α does not appear in (5). Second, for $\delta < \beta$, by induction on $m \geqslant n(\delta, \beta)$, we choose integers $b^{\delta,\beta}_{m+1}$ so that

$$0 + p_\beta\mathbb{Z} = b^{\delta,\beta}_{m+1} + f_\beta(x_{\nu_\delta(m)}) - f_\beta(x_{\rho_\delta(m)}) + p_\beta\mathbb{Z},$$

and then choose $g_\beta(x_{\nu_\delta(m)})$ and $g_\beta(x_{\rho_\delta(m)})$ such that (5) holds for δ. This inductive process, note, is possible by the choice of h_β and condition (5.1).

Finally, let $\beta = \bigcup_{n\in\omega} A_n$ be the union of an increasing chain of sets A_n for which $|A_n| < \mu$ (recall that we have assumed without loss of generality that $S = \lambda$, and so β is of cofinality ω). By induction on $n < \omega$, we may now choose $\rho_\beta(n)$ and $\nu_\beta(n)$ as distinct ordinals so that:

$\rho_\beta(n), \nu_\beta(n) \in \mu$;

$\rho_\beta(n), \nu_\beta(n) \notin \{\nu_\beta(m), \rho_\beta(m) : m < n\}$;

$\mathrm{pr}_1(\rho_\beta(n)) = \mathrm{pr}_1(\nu_\beta(n)) = \eta_\beta(n)$;

$\langle g_\delta(x_{\nu_\beta(n)}) : \delta \in A_n \rangle = \langle g_\delta(x_{\rho_\beta(n)}) : \delta \in A_n \rangle$.

Hence (3), (4), and (6) hold, and we have carried on the induction. Now, let G be freely generated by L and $\{y_{\beta,n} : \beta < \lambda, n \in \omega\}$ be subject to the following relations for $\beta < \lambda$ and $n \in \omega$:

$$\left(\prod_{p\in\Pi_1\cap n} p\right) y_{\beta,n+1} = y_{\beta,n} + x_{\nu_\beta(n)} - x_{\rho_\beta(n)}.$$

Then G is a torsion-free Abelian group of size λ. Moreover, since the ladder system $\bar{\eta}$ is λ-free and S is stationary but not reflecting, it follows by standard calculations using (5.2) that G is almost free but not free (see, e.g., [21]).

It remains to prove that (i), (ii), and (iii) of the theorem hold. For $\beta < \lambda$, let

$$G_\beta = \langle L, y_{\delta,n} : \delta < \beta, n \in \omega \rangle_* \subseteq_* G$$

so that $G = \bigcup_{\beta<\lambda} G_\beta$ is the union of the continuous increasing sequence of pure subgroups G_β, $\beta < \lambda$.

(iii) Let $p \in \Pi_1$ and choose $f \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$. By assumption, there is $\beta < \lambda$ such that $(p, f\restriction_L) = (p_\beta, f_\beta)$. Inductively we define an increasing sequence of homomorphisms $g_{\beta,\gamma} : G_\gamma \to \mathbb{Z}$ for $\gamma \geqslant \beta$ such that $g_{\beta,\gamma}\varphi_p = f\restriction_{G_\gamma}$. For $\gamma = \beta$, we choose $n(\delta, \beta)$ and $\langle b^{\delta,\beta}_m : m \in [n(\delta, \beta), \omega) \rangle$, where $\delta < \beta$, as in (5). We let $g_{\beta,\beta}\restriction_L = g_\beta$, where g_β is chosen as in (1). Moreover, put $g_{\beta,\beta}(y_{\delta,m}) = b^{\delta,\beta}_m$ for $m \in [n(\delta, \beta), \omega)$ and $\delta < \beta$. By downward induction, we choose $g_{\beta,\beta}(y_{\delta,m})$ for $m < n(\delta\beta)$, $\delta < \beta$. It is easy to see that $g_{\beta,\beta}$ is as required, that is, satisfies $g_{\beta,\beta}\varphi_p = f\restriction_{G_\beta}$. Now, assume that $\gamma > \beta$. If γ is a limit ordinal, then we let $g_{\beta,\gamma} = \bigcup_{\beta\leqslant\varepsilon<\gamma} g_{\beta,\varepsilon}$. If $\gamma = \varepsilon + 1$, then (4) implies that there is $n(\beta,\varepsilon) < \omega$ such that $g_\beta(x_{\nu_\varepsilon(m)}) = g_\beta(x_{\rho_\varepsilon(m)})$ for all $m \in [n(\beta,\varepsilon),\omega)$. Therefore, putting $g_{\beta,\gamma}\restriction_{G_\varepsilon} = g_{\beta,\varepsilon}$ and $g_{\beta,\gamma}(y_{\varepsilon,m}) = 0$ for $m \in [n(\beta,\varepsilon),\omega)$ and determining $g_{\beta,\gamma}(y_{\varepsilon,m})$ by downward induction on $m < n(\beta,\varepsilon)$, we obtain $g_{\beta,\gamma}$, as required. Lastly, let $g = \bigcup_{\gamma\geqslant\beta} g_{\beta,\gamma}$, which satisfies $g\varphi_p = f$. Since f was chosen arbitrarily, it follows that $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Z})\varphi_p$ for all $p \in \Pi_1$, and hence $r_p^e(G) = 0$ for $p \in \Pi_1$.

(ii) We now turn to the case $p \in \Pi_0$. By the definition of G, it follows that every homomorphism $\psi : L \to \mathbb{Z}$ has at most one extension to a homomorphism $\psi' : G \to \mathbb{Z}$. Thus $|\mathrm{Hom}(G,\mathbb{Z})| \leqslant 2^\mu$. However, for every $\beta < \lambda$, any homomorphism $\psi : G_\beta \to \mathbb{Z}/p\mathbb{Z}$ has more than one extension to a homomorphism $\psi' : G_{\beta+1} \to \mathbb{Z}/p\mathbb{Z}$, and hence $|\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})| = 2^\lambda > 2^\mu$. Consequently $r_p^e(G) = 2^\lambda$.

(i) That $r_0^e(G) = 2^\lambda$ can be shown similarly. □

COROLLARY 5.4. Let μ be an uncountable strong limit cardinal such that $\mathrm{cf}(\mu) = \omega$ and $2^\mu = \mu^+$. Put $\lambda = \mu^+$ and assume that there exists a λ-free ladder system on a stationary subset $S \subseteq \lambda$. Then there exists an almost free non-free coseparable group of size λ.

The **proof** follows from Theorem 5.3 letting $\Pi_0 = \varnothing$ and $\Pi_1 = \Pi$. □

REFERENCES

1. S. Shelah, "Whitehead groups may not be free even assuming *CH*, I," *Isr. J. Math.*, **28**, 193-204 (1977).
2. S. Shelah, "Whitehead groups may not be free even assuming *CH*, II," *Isr. J. Math.*, **35**, 257-285 (1980).
3. P. C. Eklof and M. Huber, "On the rank of Ext," *Math. Z.*, **175**, 159-185 (1980).
4. P. C. Eklof and S. Shelah, "The structure of $\mathrm{Ext}(A,\mathbb{Z})$ and *GCH*: possible co-Moore spaces," *Math. Z.*, **239**, No. 1, 143-157 (2002).
5. R. Grossberg and S. Shelah, "On the structure of $\mathrm{Ext}_p(G,\mathbb{Z})$," *J. Alg.*, **121**, No. 1, 117-128 (1989).
6. R. Grossberg and S. Shelah, "On cardinalities in quotients of inverse limits of groups," *Math. Jap.*, **47**, No. 2, 189-197 (1998).
7. H. Hiller, M. Huber, and S. Shelah, "The structure of $\mathrm{Ext}(A,\mathbb{Z})$ and $V = L$," *Math. Z.*, **162**, 39-50 (1978).
8. A. Mekler, A. Roslanowski, and S. Shelah, "On the p-rank of Ext," *Isr. J. Math.*, **112**, 327-356 (1999).
9. G. Sageev and S. Shelah, "Weak compactness and the structure of $\mathrm{Ext}(G,\mathbb{Z})$," in *Abelian Group Theory, Proc. Oberwolfach Conf., Lect. Notes Math.*, **874**, Springer-Verlag (1981), pp. 87-92.
10. G. Sageev and S. Shelah, "On the structure of $\mathrm{Ext}(A,\mathbb{Z})$ in ZFC^+," *J. Symb. Log.*, **50**, 302-315 (1985).
11. S. Shelah and L. Strüngmann, "A characterization of $\mathrm{Ext}(G,\mathbb{Z})$ assuming $(V = L)$," forthcoming.
12. S. Shelah, "The consistency of $\mathrm{Ext}(G,\mathbb{Z}) = \mathbb{Q}$," *Isr. J. Math.*, **39**, 74-82 (1981).
13. A. Mekler and S. Shelah, "Every coseparable group may be free," *Isr. J. Math.*, **81**, 161-178 (1993).

14. P. C. Eklof and A. Mekler, *Almost Free Modules, Set-Theoretic Methods* (revised ed.), *North-Holland Math. Lib.*, **65**, North-Holland, New York (2002).

15. K. Kunen, *Set Theory. An Introduction to Independence Proofs, Stud. Log. Found. Math.*, **102**, North Holland, New York (1980).

16. S. Shelah, *Proper and Improper Forcing, Persp. Math. Log.*, Springer-Verlag (1998).

17. L. Fuchs, *Infinite Abelian Groups*, Vols. I and II, *Pure Appl. Math.*, **36**, Academic Press, New York (1970 and 1973).

18. T. Jech, *Set Theory, Pure Appl. Math. Ser. Mon. Textbooks*, Academic Press, New York (1978).

19. S. Shelah, "A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals," *Isr. J. Math.*, **21**, 319-349 (1975).

20. S. Ben David, "On Shelah's compactness of cardinals," *Isr. J. Math.*, **31**, 34-56 (1978).

21. P. C. Eklof and S. Shelah, "On Whitehead modules," *J. Alg.*, **142**, No. 2, 492-510 (1991).