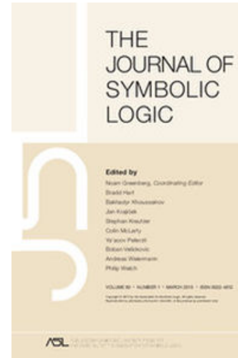
Sh:967

The Journal of Symbolic Logic

http://journals.cambridge.org/JSL

Additional services for ***The Journal of Symbolic Logic:***

Email alerts: Click here
Subscriptions: Click here
Commercial reprints: Click here
Terms of use : Click here



The minimal cofinality of an ultrapower of ω and the cofinality of the symmetric group can be larger than $\mathfrak{b}^+$

Heike Mildenberger and Saharon Shelah

The Journal of Symbolic Logic / Volume 76 / Issue 04 / December 2011, pp 1322 - 1340
DOI: 10.2178/jsl/1318338852, Published online: 12 March 2014

Link to this article: http://journals.cambridge.org/abstract_S0022481200001407

How to cite this article:
Heike Mildenberger and Saharon Shelah (2011). The minimal cofinality of an ultrapower of ω and the cofinality of the symmetric group can be larger than $\mathfrak{b}^+$. The Journal of Symbolic Logic, 76, pp 1322-1340 doi:10.2178/jsl/1318338852

Request Permissions : Click here

CAMBRIDGE JOURNALS

Downloaded from http://journals.cambridge.org/JSL, IP address: 129.93.24.73 on 17 Mar 2015

Sh:967

The Journal of Symbolic Logic
Volume 76, Number 4, Dec. 2011

THE MINIMAL COFINALITY OF AN ULTRAPOWER OF ω AND THE COFINALITY OF THE SYMMETRIC GROUP CAN BE LARGER THAN $\mathfrak{b}^+$

HEIKE MILDENBERGER AND SAHARON SHELAH

Abstract. We prove the statement in the title.

§1. Introduction. We show that $\mathfrak{b}^+$ is neither an upper bound on $\mathfrak{mcf}$ nor on $\mathrm{cf}(\mathrm{Sym}(\omega))$. In all models known formerly the two cardinals were bounded by $\mathfrak{b}^+$ and since the related cardinal $\mathfrak{g}$ is bounded by $\mathfrak{b}^+$ in ZFC the possibility that also these two cardinals be bounded by $\mathfrak{b}^+$ was not excluded before our research. We provide forcing constructions to increase these two cardinal characteristics.

We recall the definitions:

Definition 1.1. *By ultrapower we mean the usual modeltheoretic ultrapower:* $(\omega, <)^\omega/\mathcal{U}$ *is the structure with domain* $\{[f]_\mathcal{U} : f \in {}^\omega\omega\}$ *where* $[f]_\mathcal{U} = \{g \in {}^\omega\omega : \{n : f(n) = g(n)\} \in \mathcal{U}\}$ *and* $[f]_\mathcal{U} \leq_\mathcal{U} [g]_\mathcal{U}$ *iff* $\{n : f(n) \leq g(n)\} \in \mathcal{U}$. *The* minimal cofinality of an ultrapower of ω, $\mathfrak{mcf}$, *is defined as the*

$$\mathfrak{mcf} = \min\{\mathrm{cf}((\omega, <)^\omega)/\mathcal{U}) : \mathcal{U} \text{ non-principal ultrafilter on } \omega\}.$$

Definition 1.2. $\mathrm{Sym}(\omega)$ *is the group of all permutations of* ω. *If* $\mathrm{Sym}(\omega) = \bigcup_{i<\kappa} G_i$ *and* $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\langle G_i : i < \kappa\rangle$ *is strictly increasing,* G_i *is a proper subgroup of* $\mathrm{Sym}(\omega)$, *we call* $\langle G_i : i < \kappa\rangle$ *a* decomposition. *We call the minimal such* κ *the* cofinality of the symmetric group, *and denote it* $\mathrm{cf}(\mathrm{Sym}(\omega))$.

We recall some related cardinal characteristics and some estimates: For $f, g \in {}^\omega\omega$ we write $f \leq^* g$ and say g eventually dominates (bounds) f if $(\exists n)(\forall k \geq n)(f(k) \leq g(k))$. A set $B \subseteq {}^\omega\omega$ is called *unbounded* if there is no g that dominates all members of B. The *bounding number* $\mathfrak{b}$ is the minimal cardinality of an unbounded set. A set $D \subseteq {}^\omega\omega$ is called *dominating* if for every $f \in {}^\omega\omega$ there is a $g \in D$ such that $f \leq^* g$. The minimal cardinal of a dominating set is called the *dominating number*, $\mathfrak{d}$. A set $\mathcal{G} \subseteq [\omega]^\omega$ is called *groupwise dense* if it is closed under almost subsets and for every strictly increasing sequence π_i, $i \in \omega$ there is $A \in [\omega]^\omega$

Received May 20, 2010.
2010 *Mathematics Subject Classification.* 03E05, 03E17, 03E35.
Key words and phrases. forcing technique, minimum cofinality of an ultrapower, cofinality of the symmetric group.
The first author is supported by Marie Curie grant PIEF-2008-219292 of the European Union. The second author's research was partially supported by the United States-Israel Binational Science Foundation (Grant no. 2002323). This is the second author's publication no. 967.

© 2011, Association for Symbolic Logic
0022-4812/11/7604-0012/$2.90

such that $\bigcup_{i\in A}[\pi_i,\pi_{i+1}) \in \mathcal{G}$. A groupwise dense ideal is a groupwise dense set that is additionally closed under finite unions. The *groupwise density number* $\mathfrak{g}$ (groupwise density number for filters $\mathfrak{g}_f$) is the minimal size of a collection of groupwise dense sets (ideals) whose intersection is empty. A set $D \subseteq {}^\omega\omega$ is called *finitely dominating* if for every $f \in {}^\omega\omega$ there is $k \in \omega$ and there are g_i, $i < k$, $g_i \in D$ such that $f \leq^* \max\{g_i : i < k\}$, where the maximum is taken pointwise. The cardinal invariant $\mathrm{cov}(\mathscr{D}_{\mathrm{fin}})$ is the smallest cardinality of a collection of non finitely dominating sets whose union is dominating. An equivalent definition of $\mathrm{cov}(\mathscr{D}_{\mathrm{fin}})$ (see [14]) is the smallest κ such that there are non-principal ultrafilters $\mathcal{U}_\alpha$ on ω, $\alpha < \kappa$, and sequences $g_{\alpha,\beta}$, $\beta < \kappa$ for $\alpha < \kappa$ such that for every $f \in {}^\omega\omega$ there are $\alpha, \beta < \kappa$ such that $f \leq_{\mathcal{U}_\alpha} g_{\alpha,\beta}$.

Obviously $\mathfrak{mcf} \geq \mathfrak{b}$. By Canjar [6], $\mathrm{cf}(\mathfrak{d}) \geq \mathfrak{mcf}$. ZFC also implies $\mathfrak{mcf} \geq \mathfrak{g}$ [4, Theorem 3.1] and $\mathfrak{mcf} \geq \mathfrak{g}_f$ (with the same proof) and $\mathfrak{mcf} \geq \mathrm{cov}(\mathscr{D}_{\mathrm{fin}}) \geq \mathfrak{g}_f$ [11]. There is it shown with an oracle c.c. forcing that $\mathfrak{mcf} = \mathrm{cov}(\mathscr{D}_{\mathrm{fin}}) = \mathfrak{b}^+ = \aleph_2 > \max(\mathfrak{b},\mathfrak{g}) = \aleph_1$ is consistent. A model of $\mathfrak{mcf} = \mathrm{cov}(\mathscr{D}_{\mathrm{fin}}) = \aleph_2 > \max(\mathfrak{g}_f,\mathfrak{u}) = \aleph_1$ is given in [10] ($\mathfrak{u}$ is the minimal character of a non-principal ultrafilter on ω.) Shelah [13] showed that $\mathfrak{g}_f \leq \mathfrak{b}^+$ in ZFC. This consequence of ZFC lead to the question:

QUESTION 1.3. *Are there cardinal invariants "slightly" above $\mathfrak{g}_f$ that still are bounded by $\mathfrak{b}^+$?*

Here we show that there is no such upper bound on $\mathfrak{mcf}$. A similar proof works for $\mathrm{cov}(\mathscr{D}_{\mathrm{fin}})$.

THEOREM 1.4. *Suppose that $\aleph_1 \leq \partial = \mathrm{cf}(\partial) \leq \theta = \mathrm{cf}(\theta) < \kappa = \mathrm{cf}(\kappa) < \lambda$ and GCH holds up to λ. Then there is a notion of forcing $\mathbb{P}$ of size λ that preserves cardinalities and cofinalities and that forces $\mathrm{MA}_{<\partial}$ and $\mathfrak{b} = \theta$ and $\mathfrak{mcf} \geq \kappa$ and $\mathfrak{c} = \lambda$.*

We write the proof here for $\mu^+ = \lambda$ and $\mu^{\aleph_0} < \lambda$. The cardinal preserving forcing $\mathbb{P}$ from the proof of the theorem gives a model of $\kappa \leq \mathfrak{mcf}$ and $\mathfrak{c} = \lambda = \mu^+ > \kappa$. Our constructon gives that κ is a successor. With the collapse $\mathrm{Coll}(\kappa,\lambda)$ we can arrange $\kappa = \lambda$ in the end. Since the collapse is $(<\kappa)$-closed it does not destroy the cardinal invariant constellation of ∂, θ and κ. If we want for example that the continuum is a limit afterwards (or even a weakly inaccessible) then we assume the existence of a strong limit cardinal (or of a strongly inaccessibel cardinal) ν, carry out the forcing $\mathbb{P}$ with $\kappa < \mu$, $\lambda < \nu$ as in the theorem and thus ν stays a strong limit cardinal (or strongly inaccessible). Then after the forcing $\mathbb{P}$ we collapse ν to κ with conditions of size $< \kappa$. $\kappa = \mathfrak{c}$ is a limit cardinal afterwards (or weakly inaccessible).

Sharp and Thomas [12] showed that $\mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{b}^+$ is consistent and also $\mathrm{cf}(\mathrm{Sym}(\omega)) < \mathfrak{b}$ is consistent, and Mildenberger and Shelah [9] showed that $\mathfrak{g} = \aleph_1 < \mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{b} = \aleph_2$ is consistent. Brendle and Losada [5] showed that the inequality $\mathfrak{g} \leq \mathrm{cf}(\mathrm{Sym}(\omega))$ follows from ZFC. Simon Thomas [15] showed that $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathrm{cf}^*(\mathrm{Sym}(\omega)) \leq \mathfrak{d}$. For the definition of $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ and more results on this useful intermediate cardinal we refer the reader to [15]. So also $\mathrm{cf}(\mathrm{Sym}(\omega))$ is a candidate for the question above. Again we prove that it is not bounded.

THEOREM 1.5. *Suppose that $\aleph_1 \leq \partial = \mathrm{cf}(\partial) \leq \theta = \mathrm{cf}(\theta) < \kappa = \mathrm{cf}(\kappa) < \lambda$ and GCH holds up to λ. Then there is a notion of forcing $\mathbb{P}$ of size λ that preserves cardinalities and cofinalities and that forces $\mathrm{MA}_{<\partial}$ and $\mathfrak{b} = \theta$ and $\mathfrak{mcf} \geq \kappa$ and $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \kappa$ and $\mathfrak{c} = \lambda$.*

The same remark about using Lévy collapses afterwards apply. The forcing $\mathrm{Coll}(\kappa,\lambda)$ might add new short sequences of subgroups. However, it does not introduce new witnesses decompositions of length $< \kappa$. Our forcing in the proof of Theorem 1.5 uses only the witness to define an iterand destroying the witness and at the same time all decompositions that have this witness. So $\mathrm{Coll}(\kappa,\lambda)$ preserves $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \kappa$.

§2. Forcing arbitrary spread between $\mathfrak{b}$ and $\mathfrak{mcf}$. In this section we prove Theorem 1.4.

For a set of ordinals C, the set of accumulations points is $\mathrm{acc}(C) = \{\delta \in C : \delta = \sup(C \cap \delta)\}$. If C is closed then $\mathrm{acc}(C) \subseteq C$. For a set C of ordinals, $\mathrm{otp}(C)$ denotes its ordertype, the unique ordinal α such that there is an order preserving bijection from $(\alpha, \in)$ onto $(C, \in)$.

Hypothesis 2.1. *GCH holds up to λ, $\aleph_1 \leq \partial = \mathrm{cf}(\partial) \leq \theta = \mathrm{cf}(\theta) < \kappa = \mathrm{cf}(\kappa) < \lambda$, $\mu^+ = \lambda$.*

Lemma 2.2. *By a preliminary forcing of size λ that preserves cofinalities and cardinalities starting from the hypothesis we get a forcing extension with the following situation:*

(a) $\partial = \mathrm{cf}(\partial) < \kappa = \mathrm{cf}(\kappa) \leq \mu < \lambda = \lambda^{<\lambda}$, $\mu^+ = \lambda$, $\mu^{\aleph_0} < \lambda$.
(b) *$\mathcal{A}_\ell$ is a family of size λ of subsets of $[\mu]^{<\kappa}$, $(\forall A \in \mathcal{A}_0)(\forall B \in \mathcal{A}_1)(A \cap B$ is finite).*
(c) *if $\kappa_1 < \kappa$ and (u_0, u_1) is a partition of μ then there is $\ell \in 2$ and there are λ many $A \in \mathcal{A}_\ell$ such that $A \subseteq u_\ell$ and $|A| \geq \kappa_1$.*
(d) *there is a square sequence $\bar{C} = \langle C_\alpha : \alpha \in \lambda, \alpha$ limit$\rangle$ in $\lambda = \mu^+$ that is club guessing, i.e., $\bar{C}$ has the following properties*
   (1) *$C_\alpha \subseteq \alpha$ is cofinal in α and closed in α, i.e., $\mathrm{acc}(C_\alpha) \subseteq C \cup \{\alpha\}$, $\mathrm{otp}(C_\alpha) \leq \mu$,*
   (2) *for $\beta \in \mathrm{acc}(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$,*
   (3) *for every club E in λ there are stationarily many $\alpha \in \lambda$ with $\mathrm{cf}(\alpha) = \mu$ and $C_\alpha \subseteq E$. We call this "$\bar{C}$ is club guessing".*
(e) *There is an $\leq^*$-unbounded sequence $\langle g_\alpha : \alpha < \theta \rangle$ in ${}^\omega\omega$.*

Proof. We first add by forcing an almost disjoint family $\mathcal{A} \subseteq [\mu]^{<\kappa}$ as in Baumgartner's work [3]. We recall some of the main steps of Baumgartner's forcing in Section 6 [3]: Let $\mathcal{A}(\kappa', \lambda, \kappa', \nu)$ be the following statement: There is a family $\mathcal{A}$ of size λ such that each $A \in \mathcal{A}$ is a subset of κ' of size κ' and for $A \neq B \in \mathcal{A}$, the intersection $A \cap B$ is of size less than ν. Let $\bar{F} = \langle F_\alpha : \alpha < \lambda \rangle$ be a sequence of members of $[\kappa']^{\kappa'}$, repetition is allowed. A basic forcing factor is $\mathbb{Q}'(\kappa', \lambda, \nu, \bar{F})$ consisting of conditions $p = f$ that are partial functions $f : \lambda \to \bigcup F_\alpha$, $|\mathrm{dom}(f)| < \nu$, $f(\alpha) \subseteq F_\alpha$, $|f(\alpha)| < \nu$ and $f \leq_{\mathbb{Q}'(\kappa',\lambda,\nu,\bar{F})} g$ iff $f(\alpha) \subseteq g(\alpha)$ for $\alpha \in \mathrm{dom}(f)$ and for all $\alpha \neq \beta \in \mathrm{dom}(f)$, $f(\alpha) \cap f(\beta) = g(\alpha) \cap g(\beta)$.

Now let $K = \{\mu : \nu \leq \nu' \leq \kappa, \mu$ regular cardinal$\}$ and let

$$\begin{aligned}\mathbb{Q}(\kappa', \lambda, \nu, \bar{F}) = \{\langle f_{\nu'} : \nu' \in K \rangle \in \prod_{\nu' \in K} \mathbb{Q}'(\kappa', \lambda, \nu', \bar{F}) :\\ (\forall \nu'' < \nu' \in K)(\mathrm{dom}(f_{\nu''}) \subseteq \mathrm{dom}(f_{\nu'})\\ \wedge (\forall \alpha \in \mathrm{dom}(f_{\nu''}))f_{\nu''}(\alpha) \subseteq f_{\nu'}(\alpha))\}.\end{aligned}$$

This forcing has size λ, forces the desired witness $\mathcal{A}$ of $\mathcal{A}(\kappa', \lambda, \kappa', \nu)$, and it preserves cardinalities and cofinalities by [3, Lemmata 2.2. to 2.6].

Now we let $\kappa'^{+} = \kappa$ in the successor case, and if κ is a limit, take $\kappa' = \kappa$. Forcing with $\mathbb{Q}(\kappa', \lambda, \nu, \bar{F})$ gives a ν-almost disjoint family $\mathcal{A} \subseteq [\kappa']^{<\kappa}$. We take $\nu = \aleph_0$. We fix $\mu \geq \kappa$. Now we show that (c) is true. Let $((u_0^\alpha, u_1^\alpha)) \colon \alpha < \lambda)$ enumerate all partitions of μ such that each pair appears λ times. Let $\{A_\alpha \colon \alpha < \lambda\}$ enumerate $\mathcal{A}$. Then, given the task (u_0^α, u_1^α) we choose $t^\alpha \in 2$ such that $|u^\alpha_{t^\alpha} \cap A_\alpha| = |A_\alpha|$. In the end we let $\mathcal{A}_\ell = \{u_{t^\alpha} \cap A_\alpha \colon \alpha \in \lambda, t^\alpha = \ell\}$. So we have the desired $\mathcal{A}_0$, $\mathcal{A}_1$ and even more: $\mathcal{A}_0 \cup \mathcal{A}_1$ is a family of almost disjoint sets.

Now, in this forcing extension by Baumgartner's forcing we force again, by a μ-distributive (so no new μ sequences are added, and $\lambda = \mu^+$ is preserved) forcing of size λ: This forcing combines the forcing for adding a square sequence by approximations (as in [7, Exercise 23.3]) with a component that makes the sequence club guessing.

A forcing condition has the form $p = ((C_\alpha \colon \alpha \leq \gamma, \mathrm{acc}(\alpha)), \mathcal{C}) = (C^p_\alpha \colon \alpha \leq \gamma(p), \lim(\alpha)), \mathcal{C}^p)$ with the following properties. $C_\alpha \subseteq \alpha$ is club in α, $\mathrm{otp}(C_\alpha) \leq \mu$, $\gamma < \lambda$, for $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$ and $\mathcal{C}^p$ is a set of size μ of clubs in λ. A condition $q = ((D_\alpha \colon \alpha \leq \gamma', \lim(\alpha)), \mathcal{D})$ is stronger than $p = ((C_\alpha \colon \alpha \leq \gamma, \lim(\alpha)), \mathcal{C})$ iff $(D_\alpha \colon \alpha \leq \gamma', \lim(\alpha))$ is an end extension of $(C_\alpha \colon \alpha \leq \gamma, \lim(\alpha))$, $\mathcal{C} \subseteq \mathcal{D}$, and there is $\alpha \in \gamma'$, $D_\alpha \subseteq \bigcap \mathcal{C}$.

By density arguments, the generic G of this forcing gives rise to

$$\bar{C}_G = \bigcup \{\bar{C} \colon \exists \mathcal{C}(\bar{C}, \mathcal{C}) \in G\}$$

a square sequence with built in club guessing.

We now show that the forcing is indeed μ-distributive.

Let f be a name for a function $f \colon \mu \to \mathbf{V}$, $f \in \mathbf{V}[G]$. By induction on $\alpha \leq \mu$ we choose p_α. Let p_0 be any condition. Let $p_{\alpha+1} \geq p_\alpha$ such that $p_{\alpha+1}$ decides $f(\alpha)$ and such that $C_{\gamma(p_{\alpha+1})}$ has order type $< \mu$. Now assume that $\alpha \leq \mu$ is a limit ordinal. Let $\lim_{\beta \to \alpha} \gamma(p_\beta) = \gamma_0$. Now let $\gamma(p_\alpha) = \gamma_0 + \omega \cdot j$ for a sufficiently large $j < \lambda$. We define $\mathcal{C}(p_\alpha) = \bigcup \{\mathcal{C}(p_\beta) \colon \beta < \alpha\}$. The square sequence part $(C^{p_\alpha}_\beta \colon \beta \leq \gamma(p_\alpha))$ of p_α is the union of the $\bar{C}$-parts of the p_β, $\beta < \alpha$, together with the additional elements: $C^{p_\alpha}_{\gamma_0} := \bigcup_{\beta < \alpha} C^{p_\beta}_{\gamma(p_\beta)}$ is of ordertype $\leq \mu$. Then we prolong the $\bar{C}$-part of the condition p_α coherently by some additional elements $C^{p_\alpha}_{\gamma_0 + \omega \cdot i}$, $i \leq j$, so that the last element $C^{p_\alpha}_{\gamma_0 + \omega \cdot j}$ again has ordertype $< \mu$ and such that there is $i \leq j$ with $C^{p_\alpha}_{\gamma_0 + \omega \cdot i} \subseteq \bigcap \mathcal{C}(p_\alpha)$.

Since $\kappa \geq \aleph_2$ in the ground model and since all the forcings so far are $(< \kappa)$-closed, after the Baumgartner forcing and the square with club guessing forcing we still have the CH. Now we extend by an iteration of length θ of Hechler reals (see, e.g., [2, Def. 3.1.9] for Hechler forcing, called $\boldsymbol{D}$ there) and thus get a sequence $\langle g_\alpha \colon \alpha < \theta \rangle$ that is $\leq^*$-unbounded. $\dashv$

Now we assume that we have families $\mathcal{A}_0$, $\mathcal{A}_1$ and a square sequence with built in club guessing $\bar{C}$ and an unbounded sequence $\langle g_\alpha \colon \alpha < \theta \rangle$ as described in the conclusion of Lemma 2.2 in the ground model, and will now describe the final two forcing orders in the proof of Theorem 1.4. For ease of notation, we consider the model after the forcing from the proof of Lemma 2.2 now as the ground model $\mathbf{V}$ and argue over it.

Sh:967

The first step is a forcing $\mathbb{K} = (\mathbf{K}, \leq_{\mathbb{K}})$ of approximations $\mathbf{q} \in \mathbf{K}$, where $\mathbf{K} = \bigcup\{\mathbf{K}_\alpha : \alpha < \lambda\}$ and $\mathbf{K}_\alpha$ is the set of α-approximations. The relation $\leq_{\mathbb{K}}$ denotes prolonging the forcing iteration and taking an end extension of the partition of the iteration length and of $\bar{A}$. Once we have a generic $\mathbf{G}_{\mathbb{K}}$ for this forcing by approximations and end extension, we force with the direct limit

$$\mathbb{P}_{\underset{\sim}{\mathbf{G}}_{\mathbb{K}}} = \bigcup\{\mathbb{P}^{\mathbf{q}} : \mathbf{q} \in \underset{\sim}{\mathbf{G}}_{\mathbb{K}}\}. \tag{2.1}$$

We let

$$\mathbb{P} := \mathbb{K} * \mathbb{P}_{\underset{\sim}{\mathbf{G}}_{\mathbb{K}}}. \tag{2.2}$$

Definition 2.3. *Assume that $\mathcal{A}_\ell$, $\ell = 0, 1$, λ, μ, κ, ∂, $\bar{g}$ and $\bar{C}$ have the properties listed in the conclusion of Lemma 2.2. A finite support iteration together with three disjoint domains and a sequence of subsets of μ, $\mathbf{q} = (\mathbb{P}^{\mathbf{q}}, \mathfrak{U}_0^{\mathbf{q}}, \mathfrak{U}_1^{\mathbf{q}}, \mathfrak{U}_2^{\mathbf{q}}, \bar{A})$, is an element of the set $\mathbf{K}_\alpha$ of α-*approximations *iff it has the following properties*:

(a) $\mathbb{P}^{\mathbf{q}} = \mathbb{P}^{\mathbf{q}}_\alpha$, *where $\bar{\mathbb{Q}}^{\mathbf{q}} = \langle \mathbb{P}^{\mathbf{q}}_\gamma, \underset{\sim}{\mathbb{Q}}^{\mathbf{q}}_\beta : \beta < \alpha(\mathbf{q}), \gamma \leq \alpha(\mathbf{q})\rangle$ is a finite support iteration of c.c.c. forcings of length $\alpha = \alpha(\mathbf{q}) = \lg(\mathbf{q}) < \lambda$.*

(b) *$\mathfrak{U}_0 = \mathfrak{U}_0^{\mathbf{q}}$ are the odd ordinals in α and $\mathfrak{U}_1$, $\mathfrak{U}_2$ is a partition of the even ordinals in α, $\mathfrak{U}_2$ contains only limit ordinals, and $\bar{A} = \langle A_\beta : \beta \in \alpha \cap \mathfrak{U}_2\rangle$.*

(c) *For $\beta \in \mathfrak{U}_0$, $\mathbb{Q}_\beta$ is the Cohen forcing $({}^{\omega>}2, \lhd)$ and we call the generic real $\underset{\sim}{\varrho}_\beta$.*

(d) *For $\beta \in \mathfrak{U}_1$, $\mathbb{Q}_\beta$ is a c.c.c. forcing of size $\partial_\beta < \partial$.*

(e) *For $\beta \in \mathfrak{U}_2$, there is $\underset{\sim}{\bar{\eta}}_\beta = \langle \eta_{\beta,i} : i < \kappa_\beta\rangle$ of length $\kappa_\beta < \kappa$, that is a $\mathbb{P}_\beta$-name for a sequence of functions from ω to ω.*

Moreover there is a sequence $\langle \xi_{\beta,i} : i < \kappa_\beta\rangle =: \bar{\xi}_\beta$ of $\xi_{\beta,i} = \xi(\beta, i) \in \mathfrak{U}_0 \cap \beta$, increasing with i, of Cohen reals relevant for time β, and there are $A_\beta \subseteq \mu$ and a sequence of conditions $\bar{p}_\beta = \langle p_{\beta,i} : i < \kappa_\beta\rangle$, and $t_\beta \in 2$ with the following properties

$$\{\xi_{\beta,i} : i < \kappa_\beta\} \subseteq \{\varepsilon + 1 : \varepsilon \in \operatorname{acc}(C_\beta)\}, \text{ and}$$

$$\begin{aligned}\bigl(A_\beta \in \mathcal{A}_{t_\beta} \wedge A_\beta \notin \{A_\gamma : \gamma \in \beta \cap \mathfrak{U}_2\}\\ \wedge A_\beta \supseteq \bigl\{\operatorname{otp}(\varepsilon \cap \operatorname{acc}(C_\beta)) : (\varepsilon \in \operatorname{acc}(C_\beta)\\ \wedge \varepsilon + 1 \in \{\xi_{\beta,i} : i < \kappa_\beta\})\bigr\}\bigr) \text{ and}\end{aligned} \tag{2.3}$$

$$\underset{\sim}{\eta}_{\beta,i} \text{ is a } \mathbb{P}_{\xi_{\beta,i}}\text{-name, and}$$
$$\bar{p}_\beta = \langle p_{\beta,i} : i < \kappa_\beta\rangle, p_{\beta,i} \in \mathbb{P}'_{\xi(\beta,i+1)}.$$

(f) *With the objects named in (e), for $\beta \in \mathfrak{U}_2$ we define $\mathbb{P}_{\beta+1}$ as follows: We let $p \in \mathbb{P}_{\beta+1}$ iff $p : \beta + 1 \to \mathbf{V}$, $p \restriction \beta \in \mathbb{P}_\beta$ and*

$$\begin{aligned} p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) = (n, f, u)&\\ &\wedge n \in \omega\\ &\wedge f : n \to \omega\\ &\wedge u \subseteq \kappa_\beta \text{ is finite}\\ &\wedge (\forall i \in u)(p_{\beta,i} \in \mathbf{G}(\mathbb{P}_\beta))\\ &\wedge |\{i \in \kappa_\beta : p_{\beta,i} \in \underset{\sim}{\mathbf{G}}(\mathbb{P}_\beta)\}| = \kappa_\beta.\end{aligned}$$

$$p \leq_{\mathbb{P}_{\beta+1}} q \text{ iff}$$

$$\begin{aligned} q \restriction \beta \Vdash_{\mathbb{P}_\beta} n_{p(\beta)} \leq n_{q(\beta)} & \\ & \wedge f_{p(\beta)} \subseteq f_{q(\beta)} \\ & \wedge (\forall n \in [n_{p(\beta)}, n_{q(\beta)}))(\forall i \in u_{p(\beta)}) \\ & (\underset{\sim}{\varrho}_{\xi_{\beta,i}}(n) = t_\beta \to \underset{\sim}{\eta}_{\beta,i}(n) < f_{q(\beta)}(n)). \end{aligned}$$

(g) *For $\beta \leq \alpha$ we define $\mathbb{P}'_\beta$ to be the set of the $p \in \mathbb{P}_\beta$ with the following properties: If $\gamma \in \operatorname{dom}(p)$, then $p(\gamma) \in \mathbf{V}$ (is not just a name) and if $\gamma \in \operatorname{dom}(p) \cap \mathfrak{U}_2$ then*

$$\begin{aligned} p \restriction \gamma \Vdash i \in u_{p(\gamma)} \to \big(p_{\gamma,i} \leq_{\mathbb{P}_\gamma} p \restriction \gamma & \\ & \wedge \xi_{\gamma,i} \in \operatorname{dom}(p) \\ & \wedge p \restriction \xi_{\gamma,i} \text{ forces a value to } \underset{\sim}{\eta}_{\gamma,i} \restriction \lg(p(\xi_{\gamma,i})), \\ & \wedge n_{p(\gamma)} \leq \lg(p(\xi_{\gamma,i}))\big). \end{aligned} \tag{2.4}$$

Remark: We call $(p(\xi_{\gamma,i}), \underset{\sim}{\eta}_{\gamma,i} \restriction \lg(p(\xi_{\gamma,i})))$ in our indiscernibility arguments $h_{p,\gamma,i}$.

The objects whose existence is presupposed in Def. 2.3(e) are free parameters. There is no book-keeping involved, but the forcing $\mathbb{K}$ with the approximations does a similar job: In Lemmata 2.7, 2.8 and 2.10 we invoke density arguments. Since $\lambda^{<\lambda} = \lambda$ is regular and since $\mathbb{P}_{\mathbf{G}_\mathbf{K}}$ is a finite support iteration of c.c.c. forcings, since $\mathbf{K}$ does not add sequences of length $< \lambda$ and since $\kappa \leq \lambda$, each sequence $\langle \eta_i : i < \kappa' \rangle$ of $\kappa' < \kappa$ reals in $\mathbf{V}^{\mathbb{P}}$ has a $\mathbb{P}_\beta$-name for some $\beta < \lambda$. For $\operatorname{cf}(\beta) = \mu$, once $\langle \underset{\sim}{\eta}_{\beta,i} : i < \kappa_\beta \rangle$ is fixed, it is easy to find suitable A_β, $\bar{\xi}_\beta$, $\bar{p}_\beta$, t_β that fulfil (2.3), as we see in the proof of Lemma 2.7.

We outline the purpose of the properties (a) to (g) listed in Def. 2.3: Item (e) is to keep the Cohen part $\{\xi_{\beta,i} : i < \kappa_\beta\}$ of the supports in the definition of the iterand $\mathbb{Q}_\beta$ almost disjoint from that of another iterand $\mathbb{Q}_\zeta$ with $t_\zeta \neq t_\beta$, $\zeta, \beta \in \mathfrak{U}_2$. The sequence $\langle \underset{\sim}{\eta}_{\beta,i} : i < \kappa_\beta \rangle$ is a possible cofinal sequence in a reduced product. We do not name the ultrafilter, just the fact that a Cohen real $\varrho_{\xi(\beta,i)}$ or its complement will be in the ultrafilter $\mathfrak{D}$ will be used to produce a fast growing function f and a collection of domains $d_i = \varrho^{-1}_{\xi(\beta,i)}\{t_\beta\}$, $i \in U_\beta$, U_β cofinal in κ_β, such that f dominates $\eta_{\beta,i}$ on $d_i \in \mathfrak{D}$ for $i \in U_\beta$. So f shows that the sequence $\eta_{\beta,i}$, $i < \kappa_\beta$, is not cofinal in the reduced ordering. Starting with $\mathbf{p} \in \mathbf{K}_\alpha$, $\bar{\eta} \in \mathbf{V}^{\mathbb{P}^\mathbf{p}}$, and a $\mathbb{P}$-name $\underset{\sim}{\mathfrak{D}}$ for a non-principal ultrafilter on ω, there are a $\beta \geq \alpha$ and $\mathbf{q}^+ \geq_\mathbb{K} \mathbf{q} \geq_\mathbb{K} \mathbf{p}$, $\mathbf{q} \in \mathbf{K}_\beta$, $\mathbf{q}^+ \in \mathbf{K}_{\beta+1}$, such that $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}^+}_\beta$ adds a $\leq_{\mathfrak{D}}$-dominator to $\underset{\sim}{\bar{\eta}}^\mathbf{q}_\beta = \bar{\eta}$ (this will be shown in Lemma 2.7). Item (g) together with equation 2.3 will be used in the "negative theory" (Lemma 2.10): $\mathbb{K} * \mathbb{P}_{G_\mathbb{K}}$ does not destroy the unboundedness of the sequence $\langle g_\alpha : \alpha < \theta \rangle$ from the preliminary forcing.

Definition 2.4. *We let $\mathbf{K} = \bigcup\{\mathbf{K}_\alpha : \alpha < \lambda\}$ be the set of approximations. For $\mathbf{q} = (\mathbb{P}_\alpha, \mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \bar{A}) \in \mathbf{K}_\alpha$ and $\beta < \alpha$ we let $\mathbf{q} \restriction \beta = (\mathbb{P}_\beta, \mathfrak{U}_0 \cap \beta, \mathfrak{U}_1 \cap \beta, \mathfrak{U}_2 \cap \beta, \bar{A} \restriction \beta)$. We let the forcing with approximations be $\mathbb{K} = (\mathbf{K}, \leq_\mathbb{K})$ with the following forcing order: $\mathbf{q} \geq_\mathbb{K} \mathbf{q}_0$ iff $\mathbf{q} \restriction \alpha(\mathbf{q}_0) = \mathbf{q}_0$.*

Lemma 2.5. (1) *For $\alpha < \lambda$, each $\mathbf{q} \in \mathbf{K}_\alpha$ has the c.c.c.*

(2) *If $\alpha < \lambda$ and $\mathbf{q} \in \mathbf{K}_\alpha$ and $\beta < \alpha$ then $\mathbf{q} \restriction \beta \in \mathbf{K}_\beta$.*

PROOF. (1) We prove by induction on α that $\mathbb{P}_\alpha$ has the c.c.c. For limit ordinals β, the c.c.c. is preserved because we are iterating with finite support. In the case of $\alpha = \beta + 1$, if we wish to put $\beta \in \mathfrak{U}_0$ or in $\mathfrak{U}_1$ we have the c.c.c. iterand $\mathbb{Q}_\beta$ and $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathbb{Q}_\beta$. If $\alpha = \beta + 1$ and we wish to put $\beta \in \mathfrak{U}_2$ we prove directly that $\mathbb{P}_\alpha$ has the c.c.c. Suppose that $\{p_\gamma : \gamma \in \omega_1\}$ are conditions in $\mathbb{P}_\alpha$. By induction hypothesis we can take a $\mathbb{P}_\beta$-generic filter G such that $A = \{\gamma \in \omega_1 : p_\gamma \restriction \beta \in G\}$ is uncountable. Now by the definition of $\mathbb{P}_\alpha$, there are $n \in \omega$ and $f : n \to \omega$ such that $B = \{\gamma \in A : p_\gamma \restriction \beta \Vdash (n_{p_\gamma(\beta)}, f_{p_\gamma(\beta)}) = (n, f)\}$ is uncountable. Now we take $\gamma \neq \delta \in B$ such that $p_\gamma \restriction \beta \not\perp p_\delta \restriction \beta$. Since $\gamma, \delta \in B$, also $p_\gamma \not\perp p_\delta$. Hence $\mathbb{P}_\alpha$ has the c.c.c. Now $\mathbb{Q}_\beta$ is the $\mathbb{P}_\beta$ name of $\mathbb{P}_\alpha/\mathbb{P}_\beta$. ⊣

LEMMA 2.6. (1) $\mathbb{K} = (\mathbf{K}, \leq_{\mathbb{K}})$ *is a* $(< \lambda)$*-closed partial order.*

(2) $\Vdash_{\mathbb{K}} \mathbb{P}_{\mathbf{G}_{\mathbb{K}}}$ *satisfies the c.c.c.*

(3) *Forcing by* $\mathbb{K} * \mathbb{P}_{\mathbf{G}_{\mathbb{K}}}$ *does not collapse cofinalities nor cardinals and it forces* $2^{\aleph_0} = \lambda = \lambda^{<\lambda}$ *and the power* μ^κ *for* $\mu \geq \lambda$ *does not change.*

LEMMA 2.7. *In the generic extension by* $\mathbb{P} = \mathbb{K} * \mathbb{P}_{\mathbf{G}_{\mathbb{K}}}$, $\mathrm{MA}_{<\theta}$ *holds and* $\mathfrak{mcf} \geq \kappa$.

PROOF. $\mathrm{MA}_{<\theta}$ holds because of the iterands attached to $\mathfrak{U}_1$ and by Lemma 2.6 as $\mathrm{cf}(\lambda) = \lambda$. Now let a $\mathbb{P}$-name for an ultrafilter $\mathcal{D}$ and $\mathbb{P}$-names η_i, $i < \kappa'$, for some $\kappa' < \kappa$, and $(\mathbf{p}, p) \in \mathbb{P}$ be given.

As $\mathbb{P}_{\mathbf{G}_{\mathbb{K}}}$ is c.c.c., and $\mathbb{K}$ is $(< \lambda)$-closed we can assume that η_i is a $\mathbb{P}^{\mathbf{p}}$-name of a member of ${}^\omega\omega$ and $p = p \in \mathbb{P}^{\mathbf{p}}$.

We show that there is a stronger $(\mathbf{q}, p) \geq_{\mathbb{P}} (\mathbf{p}, p)$ that forces that η_i, $i < \kappa'$, is not cofinal in $\omega^\omega/\mathcal{D}$.

We choose $\langle \mathbf{q}_\alpha : \alpha < \lambda \rangle$ continuously increasing in $\leq_{\mathbb{K}}$ such that $\mathbf{q}_0 = \mathbf{p}$ and $\mathbf{q}_{\alpha+1}$ forces a $\mathbb{P}^{\mathbf{q}_{\alpha+1}}_{\lg(\mathbf{q}_{\alpha+1})}$-name to $\mathcal{D} \cap \mathscr{P}(\omega)^{\mathbf{V}^{\mathbb{P}^{\mathbf{q}_\alpha}}}$. For this we use $(\forall \alpha < \lambda)(\alpha^\omega < \lambda)$ and known reflection properties of finite support iterations of c.c.c. iterands of size $< \lambda$. Then $E = \{\lg(\mathbf{q}_\alpha) : \alpha < \lambda\}$ is a club in λ. So by Lemma 2.2, there are $\beta \geq \lg(\mathbf{p})$, $\beta \in E$, $\mathrm{cf}(\beta) = \mu$ and $C_\beta \subseteq E$ and $\mathrm{otp}(C_\beta) \geq \mu$. Let $\mathbf{q}$ be that $\mathbf{q}_\alpha$ with $\lg(\mathbf{q}_\alpha) = \beta$. Let $\{\varepsilon(i) : i < \mu\}$ enumerate the accumulation points of C_β and note that $i \mapsto \mathrm{otp}(\mathrm{acc}(C_\beta) \cap \varepsilon(i))$ is injective and independent of β, by the coherence of the square sequence $\bar{C}$. For $i < \mu$ we choose $t(i) \in 2$, $p_i \in \mathbb{P}^{\mathbf{q}}_\beta$, $p_i \geq p$ such that

$$p_i \Vdash_{\mathbb{P}^{\mathbf{q}}_{\varepsilon(i+1)}} \{n : \varrho_{\varepsilon(i)+1}(n) = t(i)\} \in \mathcal{D}.$$

Since $\kappa_\beta < \kappa \leq \mu$, for some $\mu_0 < \mu$

$$u_\ell = \{i < \mu : t(i) = \ell, (\forall j < \kappa_\beta)(\eta_j \text{ is a } \mathbb{P}^{\mathbf{q}}_{\mu_0}\text{-name})\}, \ell = 0, 1$$

is a partition of $\mu \setminus \mu_0$ into two parts, and hence by conclusion (d) of Lemma 2.1 there is some $t_\beta \in 2$ such that there is $A = A_\beta \in \mathcal{A}_{t_\beta} \setminus \{A_\gamma : \gamma \in \mathfrak{U}^{\mathbf{q}}_2 \cap \beta\}$ such that

$$\{i \in u_{t_\beta} : \mathrm{otp}(\varepsilon(i) \cap \mathrm{acc}(C_\beta)) \in A_\beta\}$$

has size at least κ'.

Now we thin out $\{\varepsilon(i) + 1 : i \in u_{t_\beta}, \mathrm{otp}(\varepsilon(i) \cap \mathrm{acc}(C_\beta)) \in A_\beta\}$, to a sequence $\langle \xi(i) : i < \kappa' \rangle$ such that $\xi(i) \in \{\varepsilon(i) : i \in u_{t_\beta}, \mathrm{otp}(\varepsilon(i) \cap \mathrm{acc}(C_\beta)) \in A_\beta\}$, $\xi(i) > \xi(j)$ for $j < i$, and $\xi(i)$ increasing with i, such that a there is a strengthening $p_{\beta,i} \geq_{\mathbb{P}_\beta} p_i$ with $p_{\beta,i} \in (\mathbb{P}')^{\mathbf{q}}_{\xi(i+1)}$. We define $\mathbf{q}^+ \geq_{\mathbb{K}} \mathbf{q}$ by

(a) $\mathbf{q}^+ \in \mathbf{K}_{\beta+1}$,

(b) $A_\beta^{\mathbf{q}'} = A$,

(c) $\kappa_\beta^{\mathbf{q}^+} = \kappa'$,

(d) $\langle \underset{\sim}{\eta}_{\beta,i}^{\mathbf{q}^+} : i < \kappa_\beta^{\mathbf{q}^+} \rangle = \langle \underset{\sim}{\eta}_i : i < \kappa' \rangle$,

(e) $\langle \xi_{\beta,i}^{\mathbf{q}^+} : i < \kappa_\beta^{\mathbf{q}^+} \rangle = \langle \xi(i) + 1 : i \in \kappa' \rangle$,

(f) $p_{\beta,i}^{\mathbf{q}^+} = p_{\beta,i} \in (\mathbb{P}')^{\mathbf{q}^+}_{\xi(\beta,i+1)}$.

So $\underset{\sim}{\mathbb{Q}}_\beta^{\mathbf{q}^+}$ is defined by (d), (e), (f).

Now $\mathbb{P}^{\mathbf{q}^+}$ has the c.c.c., hence there is $p' \geq_{\mathbb{P}^{\mathbf{q}^+}_{\beta+1}} p$,

$$p' \Vdash_{\mathbb{P}^{\mathbf{q}^+}} \text{“}W = \{i < \kappa' : p_i \in \mathbf{G}(\mathbb{P}^{\mathbf{q}})\} \text{ has cardinality } \kappa'\text{”}.$$

So $(\mathbf{q}^+, p')$ forces for the $\underset{\sim}{\mathbb{Q}}_\beta^{\mathbf{q}^+}$-generic real $\underset{\sim}{g}_\beta$ that

$$i \in W \to \underset{\sim}{\eta}_i \restriction \varrho^{-1}_{\xi_{\beta,i}}\{t_\beta\} \leq^* \underset{\sim}{g}_\beta \restriction \varrho^{-1}_{\xi_{\beta,i}}\{t_\beta\}. \qquad \dashv$$

Since $i \mapsto \mathrm{otp}(\mathrm{acc}(C_\beta) \cap \varepsilon(i))$ is by the coherence of the square sequence independent of β and injective, equation (2.3) has an important consequence: If $t_{\beta_0} \neq t_{\beta_1}$, then $\{(i,j) \in \kappa_{\beta_0} \times \kappa_{\beta_1} : \xi_{\beta_0,i} = \xi_{\beta_1,j}\} \subseteq \{(i,j) : \mathrm{otp}(\xi_{\beta,i} \cap \mathrm{acc}(C_{\beta_0})) \in A_{\beta_0} \wedge \mathrm{otp}(\xi_{\beta_1,j} \cap \mathrm{acc}(C_{\beta_1})) \in A_{\beta_1}\}$, and this is finite, since $A_{\beta_\ell} \in \mathcal{A}_{\beta_\ell}$, $\ell = 0, 1$. This finiteness will enter in Claim 2.11 part (2).

Now in the remainder we prove that in the generic extension $\mathfrak{b} = \theta$.

Lemma 2.8. *If $\mathbf{q} \in \mathbf{K}_\alpha$ and $\beta \leq \alpha$ then $\mathbb{P}'_\beta = (\mathbb{P}')^{\mathbf{q}}_\beta$ is a dense subset of $\mathbb{P}_\beta = \mathbb{P}^{\mathbf{q}}_\beta$.*

Proof. Let for $\beta_1 < \beta_2 \leq \alpha$, $\mathbb{P}'_{\beta_1,\beta_2} = \{p \in \mathbb{P}_{\beta_2} :$ the demands from Definition 2.3(g) hold for $\gamma \in \mathrm{dom}(p) \setminus \beta_1$ for all $i \in u_{p(\gamma)} \setminus \beta_1$, and if $i \in u_{p(\gamma)} \cap \beta_1$ then we only demand $p_{\gamma,i} \leq p \restriction \gamma$ and $\xi_{\gamma,i} \in \mathrm{dom}(p)\}$.

So we prove by induction on $\beta_1 \leq \alpha$ for every $\beta_2 \in [\beta_1, \alpha)$ for every $p \in \mathbb{P}'_{\beta_1,\beta_2}$ there is $q \in \mathbb{P}'_{\beta_2}$ such that $p \leq_{\mathbb{P}_{\beta_2}} q$ and $p \restriction [\beta_1, \beta_2) = q \restriction [\beta_1, \beta_2)$.

Case 1: $\beta_1 = 0$. Since $\mathbb{P}'_{\beta_2} = \mathbb{P}'_{\beta_1,\beta_2}$ we can take $p = q$.

Case 2: β_1 is a limit ordinal. We let $\beta_0 = \sup(\mathrm{dom}(p) \cap \beta_1)) < \beta_1$ and use the induction hypothesis for $\beta_0 + 1$.

Case 3: $\beta_1 = \beta_0 + 1$ and $\beta_0 \in \mathfrak{U}_0^{\mathbf{q}}$. If $\beta_0 \notin \mathrm{dom}(p)$ we use the induction hypothesis. If $\beta_0 \in \mathrm{dom}(p)$ we let $v = \{\gamma \in \beta_2 : \gamma \in \mathfrak{U}_2 \cap \mathrm{dom}(p_1) \setminus \beta_1$ and for some $i < \kappa_{\beta_0}$, $\beta_0 = \xi_{\gamma,i}\}$. For $\gamma \in v$ let $i(\gamma)$ witness it. Let $n_* = \sup\{n_{p(\gamma)} : \gamma \in v\}$. Let $q_0 \in \mathbb{P}_{\beta_0}$, $q_0 \geq p \restriction \beta_0$ and force a value to $p(\beta_0)$, a Cohen condition. As usual w.l.o.g., $\lg(p(\beta_0)) \geq n_*$. Now $\{\underset{\sim}{\eta}_{\gamma,i(\gamma)} : \gamma \in v\}$ is a finite set of $\mathbb{P}_{\beta_0}$-names so some $q_1 \in \mathbb{P}_{\beta_0}$, $q_1 \geq q_0$ forces a value to $\underset{\sim}{\eta}_{\gamma,i(\gamma)} \restriction \lg(p(\beta_0))$ for $\gamma \in v$. W.l.o.g. $p \restriction \beta_0 = q_1$ and we are done.

Case 4: $\beta_1 = \beta_0 + 1$ and $\beta_0 \in \mathfrak{U}_1^{\mathbf{q}}$. If $\beta_0 \notin \mathrm{dom}(p)$ we use the induction hypothesis. If $\beta_0 \in \mathrm{dom}(p)$, $p(\beta_0)$ is a $\mathbb{P}_{\beta_0}$-name of a member of $\mathbb{Q}_\beta$, i.e., an ordinal $< \partial_\beta$. Now $p \restriction \beta_0 \in \mathbb{P}_{\beta_0}$ hence there is $q_1 \in \mathbb{P}_{\beta_0}$ as in the induction hypothesis and such that $p \restriction \beta_0 \leq q_1$ and q_1 forces a value to $p(\beta_0)$. Now let $\mathrm{dom}(q) = \mathrm{dom}(q_1) \cup \mathrm{dom}(p)$, $q \restriction \beta_0 = q_1$ and $q(\beta_0) = \zeta$ and $q \restriction [\beta_0 + 1, \beta_2) = p \restriction [\beta_0 + 1, \beta_2)$. Now easily q is as required.

Case 5: $\beta_1 = \beta_0+1$ and $\beta_0 \in \mathfrak{U}_2^{\mathbf{q}}$. If $\beta_0 \notin \mathrm{dom}(p)$ we use the induction hypothesis. If $\beta_0 \in \mathrm{dom}(p)$, $p(\beta_0)$ is a $\mathbb{P}_{\beta_0}$-name of a member of $\mathbb{Q}_\beta$, and by strengthening $p \restriction \beta_0$ we can assume that $p \restriction \beta_0$ forces a value to $p(\beta_0)$, say (n, f, u). Since $\beta_0 \in \mathfrak{U}_2$ it is a limit ordinal.

Choose $q_1 \in \mathbb{P}_{\beta_0}$ such that $(p \restriction \beta_0) \leq q_1$ and for every $i \in u_{p(\beta_0)}$, $\xi_{\beta_0,i} \in \mathrm{dom}(q_1)$ and $q_1 \geq p \restriction \beta_0$ and $q_1 \restriction \beta_0 \geq p_{\beta_0,i}$. W.l.o.g., $q_1 = p \restriction \beta_0$ and $p(\beta_0) = (n, f, u)$. Let $\beta_* = \sup(\mathrm{dom}(p) \cap \beta_0) + 1$. Now apply the induction hypothesis to p and β_*. ⊣

Definition 2.9. *Let a and b be finite sets of ordinals and $|a| = |b|$. By $\mathrm{OP}(a,b)$ we denote the unique order preserving bijection from a onto b.*

Lemma 2.10. *Let $\bar{g} = \langle g_\varepsilon : \varepsilon < \theta\rangle$ be a $\leq^*$-increasing sequence in $\mathbf{V}$ that does not have an upper bound, $\partial \leq \theta < \kappa$. Then, for every $\alpha < \lambda$ and $\mathbf{q} \in \mathbf{K}_\alpha$, after forcing with $\mathbb{P}^{\mathbf{q}}$ the sequence $\bar{g}$ is still unbounded.*

Corollary 2.11. *After forcing with $\mathbb{P}$, $\bar{g}$ is unbounded.*

Proof of the lemma. Towards a contradiction assume that $\mathbf{q} \in \mathbf{K}_\alpha$ and there is $p_* \in \mathbb{P}^{\mathbf{q}}$ and there is a $\mathbb{P}^{\mathbf{q}}$-name g such that $p_* \Vdash_{\mathbb{P}^{\mathbf{q}}} (\forall\varepsilon < \theta)(g_\varepsilon \leq^* g)$.

Hence we can choose for $\varepsilon < \theta$, $(p_\varepsilon, n_\varepsilon)$ with the following properties: $p_\varepsilon \in (\mathbb{P}')^{\mathbf{q}}$, $p_* \leq_{\mathbb{P}^{\mathbf{q}}} p_\varepsilon$, $n_\varepsilon \in \omega$ and $p_\varepsilon \Vdash n \in [n_\varepsilon, \omega) \to g_\varepsilon(n) \leq g(n)$. We let $p_\varepsilon(\gamma) = (n_{\varepsilon,\gamma}, f_{\varepsilon,\gamma}, u_{\varepsilon,\gamma})$ for $\gamma \in \mathrm{dom}(p_\varepsilon) \cap \mathfrak{U}_2$. We let $u_\varepsilon = \bigcup\{u_{\varepsilon,\gamma} : \gamma \in \mathrm{dom}(p_\varepsilon) \cap \mathfrak{U}_2\}$.

Now by the Δ-system lemma and by Fodor's lemma there is a stationary $S \subseteq \theta$ and there are

$$\left(n_*, m_*, m_2^*, v_*, u_*, (n_\gamma, \hat{f}_\gamma)_{\gamma \in v_* \cap \mathfrak{U}_2}, (p_\gamma^{**})_{\gamma \in v_* \cap (\mathfrak{U}_0 \cup \mathfrak{U}_1)}\right)$$

with the following homogeneity properties:

(1) For $\varepsilon \in S$, $|\mathrm{dom}(p_\varepsilon)| = m_*$ and $n_\varepsilon = n_*$ and $|u_\varepsilon| = m_2^*$.
(2) For $\varepsilon \in S$, $\beta_0 \neq \beta_1 \in \mathrm{dom}(p_\varepsilon) \cap \mathfrak{U}_2$ with $t_{\beta_0} \neq t_{\beta_1}$, the finite set $\{\xi_{\beta_0,i_0} : i_0 < \kappa_{\beta_0}\} \cap \{\xi_{\beta_1,i_1} : i_1 < \kappa_{\beta_1}\}$ is independent of ε, just dependent on the position of β_0 and β_1 in $\mathrm{dom}(p_\varepsilon)$.
(3) For $\varepsilon \neq \zeta \in S$, $\mathrm{dom}(p_\varepsilon) \cap \mathrm{dom}(p_\zeta) = v_*$ and $u_\varepsilon \cap u_\zeta = u_*$.
(4) For $\varepsilon, \zeta \in S$ the function $\mathrm{OP}(\mathrm{dom}(p_\varepsilon), \mathrm{dom}(p_\zeta))$ maps v_* to itself and $(\beta_0, \xi_{\beta_0,i_0})$ to $(\beta_1, \xi_{\beta_1,i_1})$, that means: if $i_0 \in u_{p_\varepsilon(\beta_0)}$, then $i_1 \in u_{p_\zeta(\beta_1)}$ and $h_{p_\varepsilon,\beta_0,i_0} = h_{p_\zeta,\beta_1,i_1}$.
(5) For $\varepsilon \in S$, if $\gamma \in v_* \cap \mathfrak{U}_2$, then $n_{\varepsilon,\gamma} = n_\gamma$ and $f_{\varepsilon,\gamma} = \hat{f}_\gamma$.
(6) For $\varepsilon \in S$, if $\gamma \in v_* \cap (\mathfrak{U}_0 \cup \mathfrak{U}_1)$ then $p_\varepsilon(\gamma) = p_\gamma^{**}$.

We fix $\bar{\varepsilon} = \langle \varepsilon(k) : k \in \omega\rangle$ with the following properties: The sequence $\langle \varepsilon(k) : k \in \omega\rangle$ is increasing $\varepsilon(k) \in S$ and there is $n \geq n_*, n_\gamma$ $\gamma \in v_*$, such that $p_{\varepsilon(k)} \Vdash g_{\varepsilon(k)}(n) \geq k$ for every k. It is important that n is indendent of k. Since $\langle g_\varepsilon : \varepsilon \in S\rangle$ is $\leq^*$-unbounded, there is such a countable subsequence that has such an n.

Now take $q \in \mathbb{P}'_\alpha$, $q \geq p_{\varepsilon(0)}$ such that $q \Vdash g(n) = \imath$ for some $\imath \in \omega$.

Since $\mathrm{dom}(p_\varepsilon)$, $\varepsilon \in S$, is a Δ-system with root v_* there is $k(*) > \imath$ such that $\mathrm{dom}(p_{\varepsilon(k(*))}) \cap \mathrm{dom}(q) \subseteq v_*$, w.l.o.g., $= v_*$ and $u_q = \bigcup\{u_{q(\gamma)} : \gamma \in \mathrm{dom}(q) \cap \mathfrak{U}_2\}$.

Now here is the critical claim, leading to a contradiction:

Claim 2.12. *The conditions $p_{\varepsilon(k(*))}$ and q are compatible in $\mathbb{P}^{\mathbf{q}}$.*

Proof. The obvious candidate for a condition witnessing compatibility is r with

(a) $\mathrm{dom}(r) = \mathrm{dom}(q) \cup \mathrm{dom}(p_{\varepsilon(k(*))})$,
(b) for $\beta \in \mathrm{dom}(q) \setminus \mathrm{dom}(p_{\varepsilon(k(*))})$, $r(\beta) = q(\beta)$,

(c) for $\beta \in \mathrm{dom}(p_{\varepsilon(k(*))}) \setminus \mathrm{dom}(q)$, $r(\beta) = p_{\varepsilon(k(*))}(\beta)$,
(d) for $\beta \in v_* \cap (\mathfrak{U}_0^{\mathfrak{q}} \cup \mathfrak{U}_1^{\mathfrak{q}})$, $r(\beta) = q(\beta) = p_{\varepsilon(k(*))}(\beta)$,
(e) for $\beta \in v_* \cap \mathfrak{U}_2^{\mathfrak{q}}$, $r(\beta) = (n_{q(\beta)}, f_{q(\beta)}, u_{q(\beta)} \cup u_{p_{\varepsilon(k(*))}(\beta)})$.

Does r belong to $\mathbb{P}_\alpha$? Is it $\geq q, p_{\varepsilon(k(*))}$? The critical case is $r \geq p_{\varepsilon(k(*))}$, and herein the critical case is

$$(\forall \beta \in v_* \cap \mathfrak{U}_2)(\forall i \in u_{p_{\varepsilon(k(*))}(\beta)})(\forall n \in [n_\beta, \lg(f_{q(\beta)})))$$
$$r \restriction \beta \Vdash \underset{\sim}{\varrho}_{\xi_{\beta,i}}(n) = t_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i}(n). \tag{2.5}$$

Fix $\beta' \in v_* \cap \mathfrak{U}_2$. Let $i' \in u_{p_{\varepsilon(k(*))}(\beta')}$. Let $\xi = \xi_{\beta',i'} \in \mathrm{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_0 \setminus \mathrm{dom}(q)$. We consider

$$w_\xi = \{\beta \in v_* \cap \mathfrak{U}_2^{\mathfrak{q}} : (\exists i)(\xi_{\beta,i} = \xi)\}.$$

There is $t^*_\xi \in 2$ such that $\beta \in w_\xi \to t_\beta = t^*_\xi$. Why?

If $\beta_0 \neq \beta_1 \in w_\xi$ and $t_{\beta_0} \neq t_{\beta_1}$, then $\{\xi_{\beta_0,i} : i < \kappa_{\beta_0}\} \cap \{\xi_{\beta_1,i} : i < \kappa_{\beta_1}\} = F$ is finite and non-empty and by item (2) independent of $\varepsilon \in S$. Since v_* is the heart of the Δ-system $\{\mathrm{dom}(p_\varepsilon) : \varepsilon \in S\}$, there is $\varepsilon \in S$ such that $\mathrm{dom}(p_\varepsilon) \setminus v_*$ is disjoint from F. By the indiscernibility (2) also $\mathrm{dom}(p_{\varepsilon(k(*))}) \setminus v_*$ is disjoint from F, in contradiction to the choice of $\xi \in \mathrm{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_0 \setminus \mathrm{dom}(q) \subseteq \mathrm{dom}(p_{\varepsilon(k(*))}) \setminus v_*$ and $\xi \in F$.

Well, equation 2.5 is not quite correct. We correct r to a stronger condition r^+ by letting for $\xi \in \mathrm{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_0 \setminus \mathrm{dom}(q)$,

$$r^+(\xi) = r(\xi)^\frown\langle 1 - t^*_\xi, 1 - t^*_\xi, \dots \rangle,$$

and otherwise $r^+(\xi) = r(\xi)$. Now $r^+ \geq q, p_{\varepsilon(k(*))}$. We prove

$$(\forall \beta \in v_* \cap \mathfrak{U}_2)(\forall i \in u_{p_{\varepsilon(k(*))}(\beta)})(\forall n \in [n_\beta, \lg(f_{q(\beta)})))$$
$$r^+ \restriction \beta \Vdash \underset{\sim}{\varrho}_{\xi_{\beta,i}}(n) = t_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i}(n). \tag{2.6}$$

First case: $n \in [n_\beta, \lg(p_{\varepsilon(k(*))}(\xi)))$. Then $f_{q(\beta)}$ is big enough as demanded in the definition of $p_{\varepsilon(k(*))}(\beta) \leq r(\beta)$. Why? The point is that we look at $\xi_0 = \mathrm{OP}(\mathrm{dom}(p_{\varepsilon(k(*))}), \mathrm{dom}(p_{\varepsilon(0)}))(\xi)$ and recall we we have the same h and that $p \restriction \xi_{\gamma,i}$ forces a value to $\underset{\sim}{\eta}_{\gamma,i} \restriction \lg(p(\xi_{\gamma,i}))$. Since $\beta \in v_*$, and $i \in u_{p_{\varepsilon(k(*))}(\beta)}$, $\xi_0 = \xi_{\beta,i'}$ for some $i' \in u_{p_{\varepsilon(0)}(\beta)}$. So we have from $q \geq p_{\varepsilon(0)}$ that

$$q \restriction \beta \Vdash \underset{\sim}{\varrho}_{\xi_0}(n) = t^{\mathfrak{q}}_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i'}(n).$$

Now since $n < \lg(p_{\varepsilon(k(*))}(\xi)) = \lg(p_{\varepsilon(0)}(\xi_0))$, already $p_{\varepsilon(0)}$ forces this:

$$p_{\varepsilon(0)} \restriction \beta \Vdash \underset{\sim}{\varrho}_{\xi_0}(n) = t^{\mathfrak{q}}_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i'}(n).$$

Now from the requirement (d) about the same h we get

$$p_{\varepsilon(k(*)} \restriction \beta \Vdash \underset{\sim}{\varrho}_\xi(n) = t^{\mathfrak{q}}_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i}(n)$$

and hence

$$r \restriction \beta \Vdash \underset{\sim}{\varrho}_\xi(n) = t^{\mathfrak{q}}_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i}(n)).$$

Second case: Now we look at $\lg(p_{\varepsilon(k(*))}(\xi)) \leq n < \lg(f_{q(\beta)}$. We show that $f_{q(\beta)}$ is big enough as demanded in the definition of $p_{\varepsilon(k(*))}(\beta) \leq r(\beta)$. Now by

Sh:967

our thinning out procedure by the requirements we imposed on OP, $p_{\varepsilon(0)}(\xi) = p_{\varepsilon(k(*))}(\xi)$.

Now $\xi \notin \mathrm{dom}(q)$ and hence $r(\xi) = p_{\varepsilon(0)}(\xi) = p_{\varepsilon(k(*))}(\xi)$. So for any $\beta \in w_\xi$ we get $t^*_\xi = t^{\mathbf{q}}_\beta$ and

$$r^+ \restriction \beta \Vdash \underset{\sim}{\varrho}_\xi = p_{\varepsilon(0)}(\xi)^\frown\langle 1 - t^{\mathbf{q}}_\beta, 1 - t^{\mathbf{q}}_\beta, \dots \rangle,$$

and since $p_{\varepsilon(0)}(\xi) = p_{\varepsilon(k(*))}(\xi)$ we get

$$r^+ \restriction \beta \Vdash \underset{\sim}{\varrho}_{\xi_{\beta,i}}(n) = t^{\mathbf{q}}_\beta \to f_{q(\beta)}(n) > \underset{\sim}{\eta}_{\beta,i}(n). \qquad \dashv$$

§3. Increasing cf(Sym(ω)) at the same time. In this section we prove Theorem 1.5.

DEFINITION 3.1. (1) *For* $h \in \mathrm{Sym}(\omega)$, *let* $\mathrm{supp}(h) = \{n : h(n) \neq n\}$.
(2) *For* $u \subseteq \omega$ *let* $H_u = \{f \in \mathrm{Sym}(\omega) :\ \mathrm{supp}(f) \subseteq u\}$.
(3) *Let* $w_i = \{k \in \omega : k \equiv i \mod 3\}$.
(4) *Let* $u_i = \{k \in \omega : k \not\equiv i \mod 3\}$.

DEFINITION 3.2. (1) *We say* $\bar{e}$ *is a* witness for the decomposition $\bar{G} = \langle G_i : i < \kappa\rangle$ *iff* $\bar{e} = \langle e_i : i < \kappa\rangle$ *and* $e_i \in G_{i+1} \setminus G_i$ *and* e_i *is of order* 2 *and* $e_i \in H_{w_1}$.
(2) $\bar{e}$ *is a* witness *iff there is a decomposition* $\bar{G}$ *such that* $\bar{e}$ *is a witness for* $\bar{G}$.

Since there are only countably many recursive permutations and since all decompositions have uncountable lengths [8], we have: If there is a decomposition $\bar{G}$ then there is a decomposition $\bar{G}'$ with the same length such that all recursive permutations are in G'_0. So for increasing cf(Sym(ω)) by forcing it is sufficient to show that there are no short decompositions with all recursive permutations in the first subgroup.

LEMMA 3.3. *Every decomposition* $\bar{G}$ *such that all recursive permutations are in* G_0 *has a witness.*

PROOF. We first show that $\bigcup_{i<3} H_{u_i}$ generates Sym(ω). Let $f \in \mathrm{Sym}(\omega)$ be arbitrary. There is $\ell \in 3$ such that $v_0 = \{n : n \equiv 0 \mod 3 \wedge f(n) \equiv \ell \mod 3\}$ is infinite. We take $\ell_1 \in 3 \setminus \{0, \ell\}$. There is $g_1 \in H_{u_{\ell_1}}$ such that $\forall n \in v_0, g_1 \circ f(n) = n$. There is $g_2 \in H_{u_2}$, g_2 maps v_0 onto w_0 and $g_2 \restriction \{n : n \equiv 2 \mod 3\} = \mathrm{id}$, so $g_2 \in H_{u_2}$. So $f_2 = g_2 \circ g_1^{-1} \circ f \circ g_2^{-1}$ is the identity on $\{n : n \equiv 0 \mod 3\}$, so $f_2 \in H_{u_0}$. So f is a composition of permutations in $\bigcup_{i<3} H_{u_i}$.

Now let $\langle G_i : i < \kappa\rangle$ be a decomposition such that all recursive permutations are in G_0. Since $\bigcup_{i<3} H_{u_i}$ generates Sym(ω), for every $\alpha < \kappa$ there is $i(\alpha)$ such that that there is $g_\alpha \in (G_{\alpha+1} \setminus G_\alpha) \cap H_{u_{i(\alpha)}} \neq \emptyset$. Now since $\mathrm{supp}(g_\alpha) \subseteq u_{i(\alpha)}$ there is a recursive $g_{\alpha,0}$ of order 2 such that $g'_\alpha = g_{\alpha,0} \circ g_\alpha \circ g_{\alpha,0} \in H_{\{6n+1 :\, n \in \omega\}} \cap (G_{\alpha+1} \setminus G_\alpha)$: $g_{\alpha,0}$ maps $u_{i(\alpha)}$ bijectively to $\{6n + 1 : n \in \omega\}$ and $g'_\alpha \in G_{\alpha+1} \setminus G_\alpha$ imitates g_α after this bijection. Now there $e_{\alpha,1}, e_{\alpha,2} \in G_{\alpha+1} \cap H_{w_1}$ of order 2 such that $g'_\alpha = e_{\alpha,1} \circ e_{\alpha,2}$: $e_{\alpha,1}(6n + 1) = g'_\alpha(6n + 1) + 3$, $e_{\alpha,1}(3n + i) = 3n + i$ for $i = 0, 2$, $e_{\alpha,1}(6n + 4) = (g'_\alpha)^{-1}(6n + 1)$. $e_{\alpha,2}(6n + 1) = 6n + 4$, $e_{\alpha,2}(3n + i) = 3n + i$ for $i = 0, 2$, $e_{\alpha,2}(6n + 4) = 6n + 1$. So $e_{\alpha,1} \in (G_{\alpha+1} \setminus G_\alpha) \cap H_{w_1}$ is of order 2, and put it into the witness. $\dashv$

We explain why we work with permutations of order 2. At the very end of the proof we will use the following:

LEMMA 3.4. *Suppose e, f are permutations of order 2 and $\mathrm{supp}(e) \subseteq w_1$ and $\mathrm{supp}(f) \subseteq w_0$ and both supports are infinite. Then there is g of order 2, $\mathrm{supp}(g) \subseteq u_2$ such that*

$$e = g \circ f \circ g.$$

PROOF. $\mathrm{supp}(e)$ is the union over a collection of pairs $\{i, e(i)\}$ for i from a set called E_0. Note that $i \neq e(i)$. $\mathrm{supp}(f)$ is the union of a collection of pairs $\{i, f(i)\}$ for i in a set called F_0. Both E_0 and F_0 are infinite and $\omega \setminus (E_0 \cup F_0)$ is infinite. Let $g \colon E_0 \cup e''E_0 \cup F_0 \cup f''F_0 \to \omega$ be defined such that for every $i \in E_0$, $g(i) = j$ iff $g(e(i)) = f(j)$, and for every $j \in F_0$, $g(j) = i$ iff $g(f(j)) = e(i)$. Such a g exists, since there is a bijection from $\{(i, e(i)) \colon i \in E_0\}$ to $\{(i, f(i)) \colon i \in F_0\}$ and both e and f are of order 2. Let g identity on $\omega \setminus E_0 \cup e''E_0 \cup F_0 \cup f''F_0$. ⊣

We have a preliminary forcing similar to the one from the proof of Theorem 1.4. This time the preliminary forcing establishes a little more almost disjointness in the family $\mathcal{A}$. This family $\mathcal{A}$ will be used as previously to find the Cohen supports in the history for the iterands adding $\leq_{\mathfrak{D}}$-dominating reals, and now as well to find (disjoint from the former ones) Cohen support in the history for a new kind of iterands that destroys a given decomposition of length $< \kappa$.

LEMMA 3.5. *By a preliminary forcing of size λ that preserves cofinalities and cardinalities starting from the premises of Theorem 1.5 we get a forcing extension with the following situation:*

(a) $\partial = \mathrm{cf}(\partial) < \kappa \leq \mu < \lambda = \lambda^{<\lambda} = \mathrm{cf}(\lambda)$, $\mu^+ = \lambda$, $\mu^{\aleph_0} < \lambda$,
(b) *$\mathcal{A}$ is a family of almost disjoint subsets of $[\mu]^{<\kappa}$,*
(c) *if (u_0, u_1) is a partition of μ, then there are $\ell \in 2$ and λ many sets $A \in \mathcal{A}$ such that $A \subseteq u_\ell$,*
(d) *there is a square sequence $\bar{C} = \langle C_\alpha \colon \alpha \in \lambda, \lim(\alpha)\rangle$ in $\lambda = \mu^+$ that is club guessing (so as in Lemma 2.2),*
(e) *there is an $\leq^*$-unbounded sequence $\langle g_\alpha \colon \alpha < \theta\rangle$ in ${}^\omega\omega$.*

PROOF. We do the Baumgartner forcing first, as in Lemma 2.2. However, then we do not water down the resulting almost disjoint family $\mathcal{A} \subseteq [\kappa']^{\kappa'}$ as we did in the proof of Theorem 1.4. Let $\kappa'^{+} = \kappa$. How do we modify $\mathcal{A}$ in order to get item (c)? Let $\mathcal{A}$ be $\{A_\alpha \colon \alpha < \lambda\}$. Enumerate by $\{(u_0^\alpha, u_1^\alpha) \colon \alpha < \lambda\}$ all partitions of μ into two parts, each of them appearing λ times. Then we choose $t^\alpha \in 2$ such that $|A_\alpha \cap u_{t^\alpha}| = |A_\alpha|$. We set $A'_\alpha = A_\alpha \cap u_{t^\alpha}$. Now $\mathcal{A}' = \{A'_\alpha \colon \alpha < \lambda\}$ has also property (c). The rest of the proof is like in Lemma 2.2. ⊣

Now we use the forcing framework as described in equations (2.1), (2.2) and we use the same letters as there. However, we define a richer notion of α-approximation, $\mathbf{K}_\alpha$.

Fix a bijection $\mathbf{h} \colon {}^{\omega>}2 \to \{3n \colon n \in \omega\}$, e.g., $h'(\eta) = \sum\{3 \cdot 2^n \colon \eta(n) = 1, n < \lg(\eta)\}$, and $\mathbf{h}(\eta) = b(\lg(\eta), h'(\eta))$ for some bijection $b \colon \omega \times \{3n \colon n \in \omega\} \to \{3n \colon n \in \omega\}$. The purpose of this bijection is to interpret one Cohen real as 2^ω almost disjoint Cohen reals that operate on branches of the tree ${}^{\omega>}2$.

DEFINITION 3.6. $\mathbf{q} = (\mathbb{P}, \bar{\mathbb{Q}}, (\mathcal{U}_\ell)_{\ell<5}, \bar{A}, \bar{\mathbf{w}}) = (\mathbb{P}^{\mathbf{q}}, \bar{\mathbb{Q}}^{\mathbf{q}}, (\mathcal{U}^{\mathbf{q}}_\ell)_{\ell<5}, \bar{A}^{\mathbf{q}}, \bar{\mathbf{w}}^{\mathbf{q}}) \in H(\lambda)$ *is an α-*approximation *iff*

(α) $\mathbb{P}^{\mathbf{q}} = \mathbb{P}^{\mathbf{q}}_\alpha$, *where $\bar{\mathbb{Q}}^{\mathbf{q}} = \langle \mathbb{P}^{\mathbf{q}}_\gamma, \underset{\sim}{\mathbb{Q}}^{\mathbf{q}}_\beta \colon \beta < \alpha(\mathbf{q}), \gamma \leq \alpha(\mathbf{q})\rangle$ is a finite support iteration of c.c.c. forcings of length $\alpha(\mathbf{q}) = \lg(\mathbf{q}) < \lambda$.*

(β) $(\mathfrak{U}_\ell)_{\ell<5}$ *is a partition of* $\lg(\mathbf{q})$.

(γ) $\mathfrak{U}_0 \cup \mathfrak{U}_3$ *is the set of odd ordinals below* $\lg(\mathbf{q})$, $\mathfrak{U}_2 \cup \mathfrak{U}_4$ *is a subset of the limit ordinals.*

(δ) *Clauses* (c) *to* (f) *from Definition* 2.3 *hold with* $\mathcal{A}$ *instead of* $\mathcal{A}_0 \cup \mathcal{A}_1$.

(ε) *If* $\beta \in \mathfrak{U}_3$ *then* $\mathbb{Q}_\beta$ *is actually a Cohen forcing but interpreted a bit differently.* $p \in \mathbb{Q}_\beta$ *iff*

(a) $p = (n, g, b, \varrho) = (n_p, g_p, b_p, \varrho_p)$,

(b) $b \subseteq \{3k : k \in \omega\}$ *is finite,* $n \in \omega$, $\varrho \in {}^n 2$,

(c) $\{\mathbf{h}^{-1}(m) : m \in b\} \subseteq \{\nu : \nu \trianglelefteq \varrho\}$,

(d) g *is a permutation of* $\mathrm{dom}(g) = \max(n+1, \max(b)+1)$,

(e) g *is the identity on* $\mathrm{dom}(g) \setminus (b \cup w_1)$, *remember* $w_1 = \{k : k \equiv 1 \mod 3\}$,

(f) g *has order* 2,

(g) g *interchanges* $(n+1) \cap w_1$ *and* b,

(h) $p \leq q$ *if* $n_p \subseteq n_q$ *and* $b_p \subseteq b_q$ *and* $g_p \subseteq g_q$ *and* $\varrho_p \trianglelefteq \varrho_q$.

(ζ) $\bar{\mathbf{w}} = \langle \mathbf{w}_\beta : \beta \in \mathfrak{U}_4 \cap \alpha \rangle$ *is string such that for* $\beta \in \mathfrak{U}_4 \cap \alpha$,

$$\mathbf{w}_\beta = (\underset{\sim}{\kappa_\beta}, \underset{\sim}{\bar{G}_\beta}, \underset{\sim}{\bar{\xi}_\beta}, \underset{\sim}{\bar{e}_\beta}, \underset{\sim}{\bar{j}_\beta}, \underset{\sim}{\bar{p}_\beta})$$

has the following properties:

(a) $\underset{\sim}{\kappa_\beta} = \mathrm{cf}(\kappa_\beta) \in [\aleph_1, \kappa)$,

(b) $\underset{\sim}{\bar{G}_\beta}$ *is a* $\mathbb{P}_\beta$*-name,*

(c) $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\bar{G}_\beta} = \langle \underset{\sim}{G}_{\beta,i} : i < \kappa_\beta \rangle$ *is a* κ_β *decomposition*"

(d) *there is a string* $\{\xi_{\beta,i} : i < \kappa_\beta\} \subseteq \mathfrak{U}_3 \cap \{\varepsilon + 1 : \varepsilon \in \mathrm{acc}(C_\beta)\}$ (*the latter has size* μ *by induction hypothesis*) *and*

$$\begin{aligned}\big(A_\beta \in \mathcal{A} \wedge A_\beta \notin \{A_\gamma : \gamma \in \beta \cap (\mathfrak{U}_2 \cup \mathfrak{U}_4)\} \\ \wedge A_\beta \supseteq \big\{\mathrm{otp}(\varepsilon \cap \mathrm{acc}(C_\beta)) : (\varepsilon \in C_\beta \wedge \varepsilon + 1 \in \{\xi_{\beta,i} : i < \kappa_\beta\})\big\}\big).\end{aligned} \tag{3.1}$$

(e) $\Vdash_{\mathbb{P}_{\min(C_\beta)}}$ "$\underset{\sim}{\bar{e}_\beta}$ *is a witness for* $\underset{\sim}{\bar{G}_\beta}$.". *So* $\underset{\sim}{\bar{e}_\beta} = \langle \underset{\sim}{e}_{\beta,i} : i < \kappa_\beta \rangle$

(f) $\bar{p}_\beta = \langle p_{\beta,i} : i < \kappa_\beta \rangle$, $p_{\beta,i} \in \mathbb{P}'_{\xi_{\beta,i+1}}$, $\bar{p}_\beta$ *is a* Δ*-system, see later for* $\mathbb{P}'_i$,

(g) $\bar{j}_\beta = \langle j_{\beta,i} : i < \kappa_\beta \rangle$ *is increasing,* $j_{\beta,i} < \kappa_\beta$,

(h) $p_{\beta,i} \Vdash \underset{\sim}{g}_{\xi(\beta,i)} \in \underset{\sim}{G}_{\beta,j_{\beta,i}}$.

(η) *For* $\beta \in \mathfrak{U}_4$ *we define* $\mathbb{P}_{\beta+1}$ *as follows: First we have* $\mathbf{w}_\beta$ *as in item* (ζ). *We let* $p \in \mathbb{P}_{\beta+1}$ *iff* $p : \beta + 1 \to \mathbf{V}$, $p \restriction \beta \in \mathbb{P}_\beta$ *and*

$$\begin{aligned} p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) = (n, f, u) & \\ & \wedge n \in \omega \\ & \wedge f : n \to \omega \\ & \wedge u \subseteq \kappa_\beta \text{ is finite} \\ & \wedge (\forall i \in u)(p_{\beta,i} \in \underset{\sim}{\mathbf{G}}(\mathbb{P}_\beta)) \\ & \wedge |\{i \in \kappa_\beta : p_{\beta,i} \in \underset{\sim}{\mathbf{G}}(\mathbb{P}_\beta)\}| = \kappa_\beta \\ & \wedge f \text{ is a permutation of order } 2 \\ & \wedge \forall m \in n \setminus \big(w_1 \cup \bigcup\{b_{p(\xi_{\beta,i})} : i \in u\}\big) f(m) = m \\ & \wedge \underset{\sim}{\varrho}_{\xi(\beta,i)} \restriction n, i \in u, \text{ are pairwise different.} \end{aligned}$$

$$p \leq_{\mathbb{P}_{\beta+1}} q \textit{ iff}$$

$$\begin{aligned} q \restriction \beta \Vdash_{\mathbb{P}_\beta} n_{p(\beta)} \leq n_{q(\beta)} & \\ & \wedge f_{p(\beta)} \subseteq f_{q(\beta)} \\ & \wedge (\forall i \in u_{p(\beta)})(\forall n \in [n_{p(\beta)}, n_{q(\beta)}) \cap b_{p(\xi_{\beta,i})}) \\ & ((\underset{\sim}{g}_{\xi_{\beta,i}} \circ \underset{\sim}{e}_{\beta,i} \circ \underset{\sim}{g}_{\xi_{\beta,i}})(n) = f_{q(\beta)}(n)). \end{aligned}$$

(θ) *For $\alpha \leq \lg(\mathbf{q})$ we let $\mathbb{P}'_\alpha = (\mathbb{P}')^{\mathbf{q}}_\alpha$ be those $p \in \mathbb{P}_\alpha$ such that for $\beta \in \operatorname{dom}(p)$ $p(\beta)$ is an object from* $\mathbf{V}$ *and not just a name and for $\gamma \in \operatorname{dom}(p) \cap \mathfrak{U}_2$ the requirements for $\mathbb{P}'_\alpha$ from Definition 2.3 (g) hold and for $\gamma \in \operatorname{dom}(p) \cap \mathfrak{U}_4$*

$$\begin{aligned} p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} i \in u_{p(\gamma)} \rightarrow \big(p_{\gamma,i} \leq_{\mathbb{P}_\gamma} p \restriction \gamma & \\ & \wedge \xi_{\gamma,i} \in \operatorname{dom}(p) \\ & \wedge p \restriction \xi_{\gamma,i} \textit{ forces a value to } \underset{\sim}{e}_{\gamma,j_{\gamma,i}} \restriction \lg(p(\xi_{\gamma,i})), \\ & \wedge n_{p(\gamma)} \leq \lg(p(\xi_{\gamma,i}))\big). \end{aligned} \tag{3.2}$$

Again we call $(p(\xi_{\gamma,i}), \underset{\sim}{e}_{\gamma,j_{\gamma,i}} \restriction \lg(p(\xi_{\gamma,i})))$ in our indiscernibility arguments $h_{p,\gamma,i}$.

NOTATION/OBSERVATION 3.7. *For $\xi \in \mathfrak{U}_3$ we get the generic objects $(\underset{\sim}{\varrho}, g, \underset{\sim}{B}) = (\underset{\sim}{\varrho}_\xi, \underset{\sim}{g}_\xi, \underset{\sim}{B}_\xi) := (\bigcup\{\mathbf{h}^{-1}(n) \colon n \in b_p, p \in \mathbf{G}(\mathbf{q}_\xi)\}, \bigcup\{g_p \colon p \in \underset{\sim}{\mathbf{G}}(\mathbb{Q}_\xi)\}, \bigcup\{b_p \colon p \in \underset{\sim}{\mathbf{G}}(\mathbb{Q}_\xi)\}) \in {}^\omega 2 \times \operatorname{Sym}(\omega) \times \mathscr{P}(\omega)$ and $\underset{\sim}{B}_\xi$ is an infinite subset of $\{n < \omega \colon \mathbf{h}^{-1}(n) \trianglelefteq \underset{\sim}{\varrho}_\xi\}$, it is considered as a branch by the identification* $\mathbf{h}$.

NOTATION 3.8. *For $\beta \in \mathfrak{U}_4$, Let $(\underset{\sim}{U}_\beta, \underset{\sim}{f}_\beta) = (\bigcup\{u_p \colon p \in \mathbf{G}(\mathbb{Q}_\beta)\}, \bigcup\{f_p \colon p \in \mathbf{G}(\underset{\sim}{\mathbb{Q}}_\beta)\})$,*

Now we show that the forcing $\mathbb{P}$ is as desired.

LEMMA 3.9. *For $\mathbf{q} \in \mathbf{K}_\alpha$, $\mathbb{P}'_\alpha$ is dense in $\mathbb{P}_\alpha$.*

PROOF. Like in Lemma 2.8.

LEMMA 3.10. *For $\beta \in \mathfrak{U}_4$, if $\langle \eta_{\beta,i} \colon i < \kappa_\beta\rangle$ is such that κ_β is not cofinal in β there are $t_\beta \in 2$, $A_\beta \in \mathcal{A}_{t_\beta}$ and $\{\xi_{\beta,i} \colon i < \kappa_\beta\}$ such that Equation (2.3) in the Definition of $\mathbb{P}_{\beta+1}$ are true. Then $\mathbb{Q}_\beta \neq \emptyset$ and for every $n \in \omega$, $i_0 \in \kappa_\beta$, the $q \in \mathbb{P}_{\beta+1}$ with $n_{q(\beta)} \geq n$ and $\exists i \in u_{q(\beta)} \cap [i_0, \kappa_\beta)$ are dense in $\mathbb{P}_{\beta+1}$.*

PROOF. The first statement follows from Definition 2.2(c) and (d), applied to $u_1 = \{\operatorname{otp}(\varepsilon \cup \operatorname{acc}(C_\beta)) \colon \langle \eta_{\beta,i} \colon i < \kappa_\beta\rangle$ is a $\mathbb{P}_{\varepsilon+1}$-name$\}$ and $\mu \setminus u_1 = u_0$: Since $|u_0| < \mu$ there are $\bar{\xi}_\beta$, $\bar{p}_\beta$, t_β and A_β for u_1 as in Def. 3.6(ζ). This is shown as in the proof of Lemma 2.7. So we can define $\mathbb{P}_{\beta+1}$. Now for the density argument: Let $p \in \mathbb{P}'_{\beta+1}$ be given. We assume $n > n_{p(\beta)}$ and $u_{p(\beta)} < i_0$. We show that there is $q \geq p$ such that for $n_p \leq m < n_q$ for $i \in u_{p(\beta)}$, if $m \in b_{p(\xi_{\beta,i})}$, then

$$q \Vdash f_{q(\beta)}(m) = g_{\xi_{\beta,i}}(m) \circ e_{\beta,j_{\beta,i}} \circ g_{\xi_{\beta,i}}(m)$$

and such that $f_{q(\beta)}$ is a permutation of $n_{q(\beta)}$ and such that it is the identity on $n_{q(\beta)} \setminus (u_1 \cup \bigcup\{b_{q(\xi_{\beta,i})} \colon i \in u_{p(\beta)}\})$. Now $u_{p(\beta)}$ is finite. Fix for a moment a $\mathbb{P}_{\beta+1}$ generic $\mathbf{G}$ with $p \in \mathbf{G}$. First choose $i \in \{i \colon p_{\beta,i} \in \mathbf{G}\} \setminus i_0$. Since $\mathbb{P}_\beta$ has the c.c.c. and since $p_{\beta,i} \in \mathbb{P}_\beta$ such an i exists. We let $u_{q(\beta)} = u_{p(\beta)} \cup \{i\}$. $\mathbf{h}^{-1}(m) \trianglerighteq \varrho_{\xi_{\beta,i}} \restriction n_p$ for just one $i \in u_{p(\beta)}$, since $\varrho_{\xi_{\beta,i}} \restriction n_{p(\beta)}$ for $i \in u_{p(\beta)}$ are pairwise different. We can choose $f_q(m)$ so that the equation is true. The c.c.c. for $\mathbb{Q}_\beta$ is proved by induction on $\lg(\mathbf{q})$ as in the proof of Lemma 2.5. $\dashv$

Sh:967

Remark 3.11. In Section 2 finding a bound g for many $\{\eta_{\beta,i} : i < \kappa_\beta\}$ is easier than showing that for $\beta \in \mathfrak{U}_4$, $\bar{G}_\beta$ is not a decomposition since we have to put together permutations on the almost disjoint (by the last clause in Def. 3.6(η)) sets $\langle B_{\xi_{\beta,i}} : i \in U_\beta\rangle$. The set U_β is not all of $\{i < \kappa_\beta : p_{\beta,i} \in \mathbf{G}(\mathbb{P}_\beta)\}$ but as in Section 2, an unbounded subset of κ_β suffices.

Now we take the framework as in the previous section 2.4, 2.5, 2.6. We let $\mathbb{P} = \mathbb{K} * \mathbb{P}_{\mathbf{G}_\mathbb{K}}$, now with the α-approximations from Definition 3.6.

LEMMA 3.12. *In the generic extension by* $\mathbb{P} = \mathbb{K} * \mathbb{P}_{\mathbf{G}_\mathbb{K}}$, $\mathrm{MA}_{<\vartheta}$ *holds and* $\mathfrak{mcf} \geq \kappa$ *and* $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \kappa$.

PROOF. $\mathrm{MA}_{<\vartheta}$ and $\mathfrak{mcf} \geq \kappa$ are shown as in Lemma 2.7. Now let a $\mathbb{P}$-name for a decomposition $\bar{G} = \langle \underset{\sim}{G}_i : i < \kappa'\rangle$ of length $\kappa' < \kappa$ and a $\mathbb{P}$-name $\langle \underset{\sim}{e}_i : i < \kappa'\rangle$ for a witness for $\bar{G}$, and $(\mathbf{p}, p) \in \mathbb{P}$ be given. As $\mathbb{P}_{\mathbf{G}_\mathbb{K}}$ is c.c.c. and $\mathbb{K}$ is $(<\lambda)$-closed we can assume that $\underset{\sim}{p} = p \in \mathbb{P}^\mathbf{p}$. We show that there is a stronger $(\mathbf{q}, \underset{\sim}{p}) \geq_\mathbb{P} (\mathbf{p}, \underset{\sim}{p})$ that forces that $\bar{G}$ is not a decomposition.

We choose $\langle \mathbf{q}_\alpha : \alpha < \lambda\rangle$ continuously increasing in $\leq_\mathbb{K}$ such that $\mathbf{q}_0 = \mathbf{p}$ and and $\mathbf{q}_{\alpha+1}$ forces a $\mathbb{P}^{\mathbf{q}_{\alpha+1}}_{\lg(\mathbf{q}_{\alpha+1})}$-name to $\underset{\sim}{G}_i \cap \mathscr{P}(\omega)^{\mathbf{V}^{\mathbb{P}^{\mathbf{q}_\alpha}}}$ and a $\mathbb{P}^{\mathbf{q}_{\alpha+1}}_{\lg(\mathbf{q}_{\alpha+1})}$-name to $\underset{\sim}{e}_i \in ({}^\omega\omega)^{\mathbf{V}^{\mathbb{P}^{\mathbf{q}_\alpha}}}$ for each $i < \kappa'$.

For this we use $2^\omega = \theta < \lambda$ and known reflection properties of finite support iterations of c.c.c. iterands of size $< \lambda$. Then $E = \{\lg(\mathbf{q}_\alpha) : \alpha < \lambda\}$ is a club in λ. So by clubguessing property of $\bar{C}$, there are $\beta \geq \lg(\mathbf{p})$, $\beta \in E$, $\mathrm{cf}(\beta) = \mu$ and $C_\beta \subseteq E$ and $\mathrm{otp}(C_\beta) \geq \kappa'$. Let $\mathbf{q}$ be that $\mathbf{q}_\alpha$ with $\lg(\mathbf{q}_\alpha) = \beta$. Let $\{\varepsilon(i) : i < \mu\}$ enumerate limits of C_β, and note that $i \mapsto \mathrm{otp}(\mathrm{acc}(C_\beta) \cap \varepsilon(i))$ is injective. We choose $\{\varepsilon'(i) : i < \kappa'\} \subseteq \mathfrak{U}_3^\mathbf{q} \cap \{\varepsilon(i) + 1 : i < \mu\}$ and we choose $A = A^\mathbf{q}_\beta$ such that

$$\begin{aligned}(A \in \mathcal{A} \wedge A \notin \{A^\mathbf{q}_\gamma : \gamma \in \beta \cap (\mathfrak{U}^\mathbf{q}_2 \cup \mathfrak{U}^\mathbf{q}_4)\}\\ \wedge A_\beta \supseteq \{\mathrm{otp}(\varepsilon \cap C_\beta) : (\varepsilon \in C_\beta \wedge \varepsilon + 1 \in \{\varepsilon'(i) : i < \kappa^\mathbf{q}_\beta\})\}). \quad (3.3)\end{aligned}$$

Now we thin out $\langle \varepsilon'(i) : i < \kappa_\beta\rangle$ to a continuous sequence $\langle \xi(i) : i < \kappa_\beta\rangle$ such that there are $\bar{p}_\beta = \langle p_i : i < \kappa_\beta\rangle$, $p_i \in \mathbb{P}'_{\xi(i+1)}$, $\bar{p}_\beta$ is a Δ-system, and $\bar{j}_\beta = \langle j_{\beta,i} : i < \kappa_\beta\rangle$ is increasing, $j_{\beta,i} < \kappa_\beta$, $p_i \Vdash_{\mathbb{P}^\mathbf{q}_{\xi(i+1)}} \underset{\sim}{g}_{\xi(\beta,i)} \in \underset{\sim}{G}_{\beta,j_{\beta,i}}$.

Now we define $\mathbf{q}^+ \geq_\mathbb{K} \mathbf{q}$ and $\mathbf{w}_\beta$

(a) $\mathbf{q}^+ \in \mathbf{K}_{\beta+1}$, $\bar{j}_\beta$ as above,

(b) $A^{\mathbf{q}^+}_\beta = A$,

(c) $\kappa^{\mathbf{q}^+}_\beta = \kappa'$,

(d) $\langle \underset{\sim}{G}^{\mathbf{q}^+}_{\beta,i} : i < \kappa^{\mathbf{q}^+}_\beta\rangle = \langle \underset{\sim}{G}_i : i < \kappa'\rangle$,

(e) $\langle \underset{\sim}{e}^{\mathbf{q}^+}_{\beta,i} : i < \kappa^{\mathbf{q}^+}_\beta\rangle = \langle \underset{\sim}{e}_i : i < \kappa'\rangle$,

(f) $\xi^{\mathbf{q}^+}_{\beta,i} = \xi^{\mathbf{q}^+}(\beta, i) = \xi(i) + 1$ as above,

(g) $p^{\mathbf{q}^+}_{\beta,i} = p_i \in \mathbb{P}^{\mathbf{q}^+}_{\xi(\beta,i+1)}$.

So $\underset{\sim}{\mathbb{Q}}^{\mathbf{q}^+}_\beta$ is defined by (a) to (g).

Now $\mathbb{P}^{\mathbf{q}^+}$ has the c.c.c., hence there is $p' \geq_{\mathbb{P}^{\mathbf{q}^+}} p$,

$$p' \Vdash_{\mathbb{P}^{\mathbf{q}^+}} \text{“}\underset{\sim}{U} = \{i < \kappa' : p_i \in \mathbf{G}(\mathbb{P}^\mathbf{q}), i \in U_\beta\} \text{ has cardinality } \kappa'\text{”}.$$

Sh:967

So $(\mathbf{q}^+, p')$ forces for the $\mathbb{Q}^{\mathbf{q}'}_{\beta}$-generic real $\underset{\sim}{f}_\beta$ that (by Def. 3.6(η)(b))

$$(\forall i \in \underset{\sim}{U})(\forall^* m \in B_{\xi_{\beta,i}})(\underset{\sim}{f}_\beta(m) = (\underset{\sim}{g}_{\xi_{\beta,i}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i}} \circ g_{\xi_{\beta,i}})(m).$$

We take a $\mathbb{P}^{\mathbf{q}'}$-generic filter $\mathbf{G}$ with $p' \in \mathbf{G}$ and let $\underset{\sim}{x}[\mathbf{G}] = x$. We can invert the composition of permutations and together with $g''_{\xi_{\beta,i}} \underset{\sim}{B}_{\xi_{\beta,i}} = w_1$ we get

$$e_{\beta,j_{\beta,i}}(n) = (g_{\xi_{\beta,i}} \circ f_\beta \circ g_{\xi_{\beta,i}})(n)$$

for all $n \in w_1$ but finitely many. Since outside w_1, $e_{\beta,j_{\beta,i}}$ and the righthand side are the identity and w_1 is recursive, we so have that $e_{\beta,j(\beta,i)}$ is in the step of the decomposition as the righthand side. Note that by Def. 3.6(ζ)(h), $g_{\beta,i} \in G_{\beta,j_{\beta,i}}$. Now U_β is cofinal in κ_β and $j_{\beta,i}$ is cofinal in κ_β and $\bar{G}_\beta$ is a decomposition. Hence there is $i \in U_\beta$ such that $f_\beta \in G_{\beta,j(i,\beta)}$. A permutation with finite support making up for the finitely many mistakes is in $G_{\beta,0}$. So also $e_{\beta,j_{\beta,i}} \in G_{\beta,j_{\beta,i}}$. So f_β shows that $\langle e_{\beta,i} : i < \kappa_\beta \rangle$ is not a witness for the decomposition $\bar{G}_\beta$.

How did we refer to $\bar{G}_\beta$? Only $j(\beta, \cdot) \colon \kappa_\beta \to \kappa_\beta$ entered the forcing $\mathbb{Q}_\beta$. So if an iteration covers all possible j and all witnesses, then it covers all short decompositions. This argument is used for the remark from the end of the introduction, that $\mathrm{Coll}(\kappa, \lambda)$ does not destroy the achievement of Theorem 1.5. $\dashv$

Now in the remainder we prove that in the generic extension $\mathfrak{b} = \theta$.

LEMMA 3.13. *Let $\bar{g} = \langle g_\varepsilon : \varepsilon < \theta \rangle$ be a $\leq^*$-increasing sequence in* $\mathbf{V}$ *that does not have an upper bound, $\partial \leq \theta < \kappa$. Then, for every $\alpha < \lambda$, after forcing with $\mathbb{P}^{\mathbf{q}}$ for $\mathbf{q} \in \mathbf{K}_\alpha$, the sequence $\bar{g}$ is still unbounded.*

COROLLARY 3.14. *After forcing with $\mathbb{P}$, $\bar{g}$ is unbounded.*

Proof of the lemma. Towards a contradiction assume that $\mathbf{q} \in \mathbf{K}_\alpha$ and there is $p_* \in \mathbb{P}^{\mathbf{q}}$ and there is a $\mathbb{P}^{\mathbf{q}}$-name g such that $p_* \Vdash_{\mathbb{P}^{\mathbf{q}}} (\forall \varepsilon < \theta)(g_\varepsilon \leq^* g)$.

Hence we can choose for $\varepsilon < \theta$, $(p_\varepsilon, n_\varepsilon)$ with the following properties: $p_\varepsilon \in (\mathbb{P}')^{\mathbf{q}}_\alpha$, $p_* \leq_{\mathbb{P}^{\mathbf{q}}} p_\varepsilon$, $n_\varepsilon \in \omega$ and $p_\varepsilon \Vdash_{\mathbb{P}^{\mathbf{q}}_\alpha} n \in [n_\varepsilon, \omega) \to g_\varepsilon(n) \leq \underset{\sim}{g}(n)$ and let $p_\varepsilon(\gamma) = (n_{\varepsilon,\gamma}, f_{\varepsilon,\gamma}, u_{\varepsilon,\gamma})$ for $\gamma \in \mathrm{dom}(p_\varepsilon) \cap (\mathfrak{U}_2 \cup \mathfrak{U}_4)$. We let $u_\varepsilon = \bigcup \{u_{\varepsilon,\gamma} : \gamma \in \mathrm{dom}(p_\varepsilon) \cap (\mathfrak{U}_2 \cup \mathfrak{U}_4)\}$.

Now by the Δ-system lemma and by Fodor's lemma there is a stationary $S \subseteq \theta$ and there are

$$\big(n_*, m_*, m_2^*, v_*, u_*, (n_\gamma, \hat{f}_\gamma)_{\gamma \in v_* \cap (\mathfrak{U}_2 \cap \mathfrak{U}_4)}, (p_\gamma^{**})_{\gamma \in v_* \cap (\mathfrak{U}_0 \cup \mathfrak{U}_1 \cup \mathfrak{U}_3)}\big)$$

with the following homogeneity properties:

(1) For $\varepsilon \in S$, $|\mathrm{dom}(p_\varepsilon)| = m_*$ and $n_\varepsilon = n_*$ and $|u_\varepsilon| = m_2^*$.

(2) For $\varepsilon \in S$, $\beta_0 \neq \beta_1 \in \mathrm{dom}(p_\varepsilon) \cap (\mathfrak{U}_2 \cup \mathfrak{U}_4)$ the finite set $\{\xi_{\beta_0,i_0} : i_0 < \kappa_{\beta_0}\} \cap \{\xi_{\beta_1,i_1} : i < \kappa_{\beta_1}\}$ (as in equation (3.1), that together with Definition 3.6(d) ensures the claimed finiteness) is independent of ε, just dependent on the position of β_0 and β_1 in $\mathrm{dom}(p_\varepsilon)$.

(3) For $\varepsilon \neq \zeta \in S$, $\mathrm{dom}(p_\varepsilon) \cap \mathrm{dom}(p_\zeta) = v_*$ and $u_\varepsilon \cap u_\zeta = u_*$.

(4) For $\varepsilon, \zeta \in S$ the function $\mathrm{OP}(\mathrm{dom}(p_\varepsilon), \mathrm{dom}(p_\zeta))$ maps v_* to itself and $(\beta_0, \xi_{\beta_0,i_0})$ to $(\beta_1, \xi_{\beta_1,i_1})$, that means: if $i_0 \in u_{p_\varepsilon(\beta_0)}$, then $i_1 \in u_{p_\zeta(\beta_1)}$, and if $\beta \in \mathfrak{U}_2 \cup \mathfrak{U}_4$ and $i \in u_{p_{\varepsilon(\beta_0)}}$, then $h_{p_\varepsilon,\beta_0,i_0} = h_{p_\zeta,\beta_1,i_1}$. $\mathrm{OP}(\mathrm{dom}(p_\varepsilon), \mathrm{dom}(p_\zeta))$ preserves the predicates $\mathfrak{U}_i$.

(5) For $\varepsilon \in S$, if $\gamma \in v_* \cap (\mathfrak{U}_2 \cup \mathfrak{U}_4)$, then $n_{\varepsilon,\gamma} = n_\gamma$ and $f_{\varepsilon,\gamma} = \hat{f}_\gamma$.

(6) For $\varepsilon \in S$, if $\gamma \in v_* \cap (\mathfrak{U}_0 \cup \mathfrak{U}_1 \cup \mathfrak{U}_3)$ then $p_\varepsilon(\gamma) = p_\gamma^{**}$.

We fix $\bar{\varepsilon} = \langle \varepsilon(k) \colon k \in \omega \rangle$ with the following properties: The sequence $\langle \varepsilon(k) \colon k \in \omega \rangle$ is increasing $\varepsilon(k) \in S$ and there is $n \geq n_*, n_\gamma$ $\gamma \in v_*$, such that $p_{\varepsilon(k)} \Vdash g_{\varepsilon(k)}(n) \geq k$ for every k.

Now take $q \in \mathbb{P}'_\alpha$, $q \geq p_{\varepsilon(0)}$ such that $q \Vdash g(n) = \imath$ for some $\imath \in \omega$.

Since $\operatorname{dom}(p_\varepsilon)$, $\varepsilon \in S$, is a Δ-system with root v_* there is $k(*) > \imath$ such that $\operatorname{dom}(p_{\varepsilon(k(*))}) \cap \operatorname{dom}(q) \subseteq v_*$, w.l.o.g., $= v_*$ and $u_q = \bigcup\{u_{q(\gamma)} \colon \gamma \in \operatorname{dom}(q) \cap \mathfrak{U}_2\}$.

Now here is the critical claim, leading to a contradiction:

Claim 3.15. *The conditions $p_{\varepsilon(k(*))}$ and q are compatible in $\mathbb{P}^{\mathbf{q}}$.*

Proof. The obvious candidate for a condition witnessing compatibility is r with the properties (a) to (e) from the proof of Claim 2.12. As in the proof of Claim 2.12, we let w_ξ for $\xi \in \mathfrak{U}_0$ be defined as there for $\beta \in v_*$, $\xi = \xi_{\beta,i} \in \operatorname{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_0 \setminus \operatorname{dom}(q)$. Since $\mathcal{A}$ consists of almost disjoint sets, the proof in Claim 2.12 shows that w_ξ is a singleton so t^*_ξ is well defined. We correct r by to a stronger condition r^+ by letting, for $\beta \in v_*$, $\xi = \xi_{\beta,i} \in \operatorname{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_0 \setminus \operatorname{dom}(q)$ with $w_\xi \neq \emptyset$,

$$r^+(\xi) = r(\xi)^\frown \langle 1 - t^*_\xi, 1 - t^*_\xi, \dots \rangle,$$

and otherwise $r^+(\xi) = r(\xi)$. Now $r^+ \geq q, p_{\varepsilon(k(*))}$ in the old cases.

Does r^+ belong to $\mathbb{P}_\alpha$? Is it $\geq q, p_{\varepsilon(k(*))}$? The new critical case in $r^+ \geq p_{\varepsilon(k(*))}$ is

$$\begin{aligned} &(\forall \beta \in v_* \cap \mathfrak{U}_4) \\ &r^+ \restriction \beta \Vdash (\forall i \in u_{p_{\varepsilon(k(*))}(\beta)} \setminus u_{q(\beta)}) \\ &(\forall m \in [n_\beta, \lg(f_{q(\beta)})) \cap b_{p_{\varepsilon(k(*)}(\xi_{\beta,i})}) \\ &f_{q(\beta)}(m) = \underset{\sim}{g}_{\xi_{\beta,i}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i}} \circ \underset{\sim}{g}_{\xi_{\beta,i}}(m). \end{aligned} \tag{3.4}$$

Fix $\beta' \in v_* \cap \mathfrak{U}_4$. Let $i' \in u_{p_{\varepsilon(k(*))}(\beta')}$. Let $\xi = \xi_{\beta',i'} \in \operatorname{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_2 \setminus \operatorname{dom}(q)$. (For $\xi \in \operatorname{dom}(q)$, (3.4) is true as q is a condition.) We consider

$$v_\xi = \{\beta \in v_* \cap \mathfrak{U}_4^{\mathbf{q}} \colon (\exists i)(\xi_{\beta,i} = \xi)\}.$$

Since $\mathcal{A}$ is a family of almost disjoint sets, and $\xi = \xi_{\beta',i'} \in \operatorname{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_2 \setminus \operatorname{dom}(q)$, v_ξ is a singleton: If $\beta_0 \neq \beta_1 \in v_\xi$, then by Definition 3.6(d), $\{\xi_{\beta_0,i} \colon i < \kappa_{\beta_0}\} \cap \{\xi_{\beta_1,i} \colon i < \kappa_{\beta_1}\}$ is finite and non-empty and by item (2) independent of $\varepsilon \in S$. Since v_* is the heart of the Δ-system $\operatorname{dom}(p_\varepsilon)$, $\varepsilon \in S$, there is $\varepsilon \in S$ such that $\operatorname{dom}(p_\varepsilon) \setminus v_*$ is disjoint from this finite set. By the indiscernibility (2) also $\operatorname{dom}(p_{\varepsilon(k(*))}) \setminus v_*$ is disjoint from the finite set, in contradiction to the choice of $\xi \in \operatorname{dom}(p_{\varepsilon(k(*))}) \cap \mathfrak{U}_0 \setminus \operatorname{dom}(q) \subseteq \operatorname{dom}(p_{\varepsilon(k(*))}) \setminus v_*$.

First case: $m \in [n_\beta, \lg(p_{\varepsilon(k(*))}(\xi))) \cap b_{p_{\varepsilon(k(*)}(\xi_{\beta,i})}$. Then $f_{q(\beta)}(m)$ is the shift of the witness $e_{\beta,i}$ to the branch $b_{r^+(\xi_{\beta,i})}$ by $g_{\xi_{\beta,i}}$ as required in $p_{\varepsilon(k(*))}(\beta) \leq r(\beta)$. Why? The point is that we look at $\xi_0 = \xi_{\beta,i'} = \operatorname{OP}(\operatorname{dom}(p_{\varepsilon(k(*))}), \operatorname{dom}(p_{\varepsilon(0)}))(\xi)$ and recall we have that $p \restriction \xi_{\gamma,i}$ forces a value to $p(\xi_{\gamma,i})$ and we have the same $p_\cdot(\xi_{\gamma,\cdot})$ for $\gamma \in v_*$. Since $\beta \in v_*$, and $i \in u_{p_{\varepsilon(k(*))}(\beta)}$, $\xi_0 = \xi_{\beta,i'}$ for some $i' \in u_{p_{\varepsilon(0)}(\beta)}$. So we have from $q \geq p_{\varepsilon(0)}$ that

$$\begin{aligned} q \restriction \beta \Vdash\ &(\forall m \in [n_\beta, \lg(f_{p_{\varepsilon(0)}(\beta)})) \cap b_{p_{\varepsilon(k(*)}(\xi_{\beta,i})}) \\ &(f_{q(\beta)}(m) = g_{\xi_{\beta,i'}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i'}} \circ g_{\xi_{\beta,i'}}(m)). \end{aligned}$$

Now since $n < \lg(p_{\varepsilon(k(*))}(\xi)) = \lg(p_{\varepsilon(0)}(\xi_0))$ and $p_{\varepsilon(k(*))}(\xi)) = p_{\varepsilon(0)}(\xi_0)$, already $p_{\varepsilon(0)}$ forces this:

$$p_{\varepsilon(0)} \restriction \beta \Vdash (\forall m \in [n_\beta, \lg(f_{p_{\varepsilon(0)}(\beta)})) \cap b_{p_{\varepsilon(k(*))}(\xi_{\beta,i})}) \\ (f_{q(\beta)}(m) = g_{\xi_{\beta,i'}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i'}} \circ g_{\xi_{\beta,i'}}(m)).$$

Now from the requirement about the same h in item (4) of the homogeneity properties we get

$$p_{\varepsilon(k(*)} \restriction \beta \Vdash (\forall m \in [n_\beta, \lg(f_{p_{\varepsilon(k(*))}(\beta)})) \cap b_{p_{\varepsilon(k(*))}(\xi_{\beta,i})}) \\ (f_{q(\beta)}(m) = g_{\xi_{\beta,i}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i}} \circ g_{\xi_{\beta,i}}(m)),$$

and hence

$$r^+ \restriction \beta \Vdash (\forall m \in [n_\beta, \lg(f_{p_{\varepsilon(k(*))}(\beta)})) \cap b_{p_{\varepsilon(k(*))}(\xi_{\beta,i})}) \\ (f_{q(\beta)}(m) = g_{\xi_{\beta,i}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i}} \circ g_{\xi_{\beta,i}}(m)).$$

Second case: Now we look at $\lg(p_{\varepsilon(k(*))}(\xi)) \leq m < \lg(f_{q(\beta)})$, $m \in b_{p_{\varepsilon(k(*))}(\xi_{\beta,i})}$ and $\xi = \xi_{\beta,i}$. Now we can change neither $f_{q(\beta)}$ nor $e_{\beta,j(\beta,i)}$. However, we can make them conjugated by correcting, i.e., strengthening, our condition r^+ once more to a condition called r^{++}: Note v_ξ is a singleton, and $\xi \in \mathcal{U}_3 \setminus \mathrm{dom}(q)$ and hence $r^+(\xi) = r(\xi) = p_{\varepsilon(0)}(\xi) = p_{\varepsilon(k(*))}(\xi)$.

So for ξ and the unique $\beta \in v_*$ such that $v_\xi = \{\beta\}$, we have that also there is just one i such there is $\beta \in v_\xi$ with $\xi_{\beta,i} = \xi$. We let

$$r^+(\xi) = (n_{r^+(\xi)}, g_{r^+(\xi)}, b_{r^+(\xi)}, \varrho_{r^+(\xi)}),$$

so $\lg(r^+(\xi)) = \lg(p_{\varepsilon(k(*))}(\xi)) = n_{r^+(\xi)}$. We let

$$r^{++}(\xi) = (n_{r^{++}(\xi)}, g_{r^{++}(\xi)}, b_{r^{++}(\xi)}, \varrho_{r^{++}(\xi)})$$

so that for $m \in [\lg(p_{\varepsilon(k(*))}(\xi)), \lg(f_{q(\beta)})) \cap b_{p_{\varepsilon(k(*))}(\xi_{\beta,i})})$,

$$g_{r^{++}(\xi)} \circ e_{\beta,j_{\beta,i}} \circ g_{r^{++}(\xi)}(m) = f_{q(\beta)}(m).$$

Note that such an r^{++} exists by Lemma 3.5, since $\mathrm{supp}(e_{\beta,i}) \subseteq w_0$ and $\mathrm{supp}(f_{q(\beta)}) \subseteq w_1$. So for any $\beta \in v_*$ with $\{\beta\} = v_\xi$ we get

$$r^{++} \restriction \beta \Vdash (\forall m \in [n_\beta, \lg(f_{q(\beta)})) \cap b_{r^{++}(\xi_{\beta,i})}) \\ (f_{q(\beta)}(m) = g_{\xi_{\beta,i}} \circ \underset{\sim}{e}_{\beta,j_{\beta,i}} \circ g_{\xi_{\beta,i}}(m)). \quad \dashv$$

§4. Open questions.

QUESTION 4.1. *Is* $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf}$ *a consequence of* ZFC?

Remark: If there are no Q-points, the answer is positive, even for $\mathrm{cf}^*(\mathrm{Sym}(\omega))$, see [1].

QUESTION 4.2. *Is* $\mathrm{cf}(\mathrm{Sym}(\omega)) \geq \mathfrak{g}_f$ *a consequence of* ZFC?

Remark: The answer is positive for $\mathfrak{g}$, by Brendle and Losada [5].

Sh:967

Acknowledgement. We thank the referee for a detailed and careful report.

REFERENCES

[1] TARAS BANAKH, DUŠAN REPOVŠ, and LYUBOMYR ZDOMSKYY, *On the length of chain of proper subgroups covering a topological group*, ***Archive for Mathematical Logic***, vol. 50 (2011), no. 3-4, pp. 411–421.

[2] TOMEK BARTOSZYŃSKI and HAIM JUDAH, ***Set theory, on the structure of the real line***, A K Peters, 1995.

[3] JAMES BAUMGARTNER, *Almost-disjoint sets, the dense-set problem, and the partition calculus*, ***Annals of Mathematical Logic***, vol. 9 (1976), pp. 401–439.

[4] ANDREAS BLASS and HEIKE MILDENBERGER, *On the cofinality of ultrapowers*, this JOURNAL, vol. 64 (1999), pp. 727–736.

[5] JÖRG BRENDLE and MARIA LOSADA, *The cofinality of the inifinite symmetric group and groupwise density*, this JOURNAL, vol. 68 (2003), no. 4, pp. 1354–1361.

[6] MICHAEL CANJAR, *Cofinalities of countable ultraproducts: The existence theorem*, ***Notre Dame Journal of Formal Logic***, vol. 30 (1989), pp. 309–312.

[7] THOMAS JECH, ***Set theory***, The Third Millenium, revised and expanded ed., Springer, 2003.

[8] DUGALD MACPHERSON and PETER NEUMANN, *Subgroups of the infinite symmetric group*, ***Journal of the London Mathematical Society***, vol. 42 (1990), pp. 64–84.

[9] HEIKE MILDENBERGER and SAHARON SHELAH, *The relative consistency of* $\mathfrak{g} < \mathrm{cf}(\mathrm{sym}(\omega))$, this JOURNAL, vol. 67 (2002), pp. 297–314.

[10] ———, *The principle of near coherence of filters does not imply the filter dichotomy principle*, ***Transactions of the American Mathematical Society***, vol. 361 (2009), pp. 2305–2317, [MdSh:894].

[11] HEIKE MILDENBERGER, SAHARON SHELAH, and BOAZ TSABAN, *Covering the Baire space with meager sets*, ***Annals of Pure and Applied Logic***, vol. 140 (2006), pp. 60–71.

[12] JAMES D. SHARP and SIMON THOMAS, *Unbounded families and the cofinality of the infinite symmetric group*, ***Archive for Mathematical Logic***, vol. 34 (1995), pp. 33–45.

[13] SAHARON SHELAH, *Groupwise density cannot be much bigger than the unbounded number*, ***Mathematical Logic Quarterly***, vol. 54 (2008), pp. 340–344.

[14] SAHARON SHELAH and BOAZ TSABAN, *Critical cardinalities and additivity properties of combinatorial notions of smallness*, ***Journal of Applied Analysis***, vol. 9 (2003), pp. 149–162.

[15] SIMON THOMAS, *Groupwise density and the cofinality of the infinite symmetric group*, ***Archive for Mathematical Logic***, vol. 37 (1998), pp. 483–493.

ABTEILUNG FÜR MATHEMATISCHE LOGIK
MATHEMATISCHES INSTITUT
UNIVERSITÄT FREIBURG
ECKERSTR. 1
79104 FREIBURG IM BREISGAU, GERMANY
E-mail: heike.mildenberger@math.uni-freiburg.de

EINSTEIN INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
EDMOND SAFRA CAMPUS GIVAT RAM
JERUSALEM 91904, ISRAEL
E-mail: shelah@math.huji.ac.il