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GENTLY KILLING S-SPACES

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ABSTRACT

We produce a model of ZFC in which there are no locally compact first countable S-spaces, and in which $2^{\aleph_0} < 2^{\aleph_1}$. A consequence of this is that in this model there are no locally compact, separable, hereditarily normal spaces of size $\aleph_1$, answering a question of the second author [9].

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1. Introduction and notation

In Problem 9 of [9], Nyikos asks if there is a ZFC example of a separable, hereditarily normal, locally compact space of cardinality $\aleph_1$. He notes there that for a negative answer, it suffices to produce a model of set theory in which there are neither Q-sets nor locally compact, locally countable, hereditarily normal S-spaces.

We provide such a model in this paper. In fact, in our model $2^{\aleph_0} < 2^{\aleph_1}$ (so in particular there are no Q-sets) and there are no locally compact, first countable S-spaces at all (hence no locally compact, locally countable, hereditarily normal S-spaces).

In fact, we obtain something even more general. Recall that an S-space is a regular, hereditarily separable space which is not hereditarily Lindelöf. By switching the "separable" and "Lindelöf" we get the definition of an L-space. A simultaneous generalization of hereditarily separable and hereditarily Lindelöf spaces is the class of spaces of countable spread—those spaces in which every discrete subspace is countable. One of the basic facts in this little corner of set-theoretic topology is that if a regular space of countable spread is not hereditarily separable, it contains an L-space, and if it is not hereditarily Lindelöf it contains an S-space [10].

In our model, every locally compact 1st countable space of countable spread is hereditarily Lindelof; consequently, there are no S-spaces in locally compact 1st countable spaces of countable spread. This result, reminiscent of one half of a celebrated 1978 result of Szentmiklóssy [12], will be discussed further at the end of the paper in connection with a fifty-year-old problem of M. Katětov [7].[1]

These concepts and results have elegant translations in terms of Boolean algebras via Stone duality. The Stone space $\mathcal{S}(A)$ of a Boolean algebra A is hereditarily Lindelöf iff every ideal of A is countably generated, and first countable iff every maximal ideal is countably generated. Let us recall that a set D is a minimal set of generators for an ideal if it generates the ideal, but no member of D is a member of the ideal generated by the remaining members. Not every ideal will have a minimal set of generators, but it is true that $\mathcal{S}(A)$ is of countable spread if and only if whenever an ideal has a minimal set of generators, then that set is countable.

Hence we now know that $2^{\aleph_0} < 2^{\aleph_1}$ is consistent with the following statement: if a Boolean algebra A has the property that every minimal set of generators for

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1 Larson and Todorcevic have solved Katětov's problem in the time since the research in this paper was done. See [8] for the proof.

an ideal is countable, and every maximal ideal of A is countably generated, then every ideal of A is countably generated. On the other hand, this statement has long been known to be incompatible with CH.

Note that there are restrictions on such models. In [6] it is shown that CH implies the existence of a locally compact first countable S-space, and in Chapter 2 of [13] this is shown to follow from the weaker axiom $\mathfrak{b} = \aleph_1$. Thus the fact that our model satisfies $\mathfrak{b} = \aleph_2$ is no accident of the proof — something along these lines is required.

As far as background goes, we will assume a reasonable familiarity with topological notions such as filters of closed sets and free sequences. We also use a lot of set theory — we will assume that the reader is used to working with proper notions of forcing.

Our main tool is the use of totally proper notions of forcing that satisfy the $\aleph_2$-p.i.c. (properness isomorphism condition). We will take a moment to recall the needed definitions.

Definition 1.1:

(1) Let P be a notion of forcing, and N a countable elementary submodel of $H(\lambda)$ for some large regular λ with $P \in N$. An (N, P)**-generic sequence** is a decreasing sequence of conditions $\{p_n : n \in \omega\} \subseteq N \cap P$ such that for every dense open $D \subseteq P$ in N, there is an n with $p_n \in D$.

(2) A notion of forcing P is said to be totally proper if for every N as above and $p \in N \cap P$, there is an (N, P)-generic sequence $\{p_n : n \in \omega\}$ with $p_0 = p$ that has a lower bound.

We should mention that totally proper forcings are also sometimes called NNR proper in the literature (NNR standing for "no new reals") — see [11], for example.

The following claim summarizes the properties of totally proper notions of forcing that we will need. The proofs are not difficult, and they are explicitly worked out in [3] and [4].

Claim 1.2: *Let P be a totally proper notion of forcing.*

(1) *P adds no new reals; in fact, forcing with P adds no new countable sequences of elements from the ground model.*

(2) *If $G \subseteq P$ is generic, then G is countably closed. In fact, every countable subset of G has a lower bound in G.*

The following definition is from Chapter VIII of [11].

Definition 1.1: P satisfies the $\aleph_2$-p.i.c. provided the following holds (for λ a large enough regular cardinal): If

(1) $i < j < \aleph_2$,
(2) N_i and N_j are countable elementary submodels of $H(\lambda)$,
(3) $i \in N_i$, $j \in N_j$,
(4) $N_i \cap \aleph_2 \subseteq j$,
(5) $N_i \cap i = N_j \cap j$,
(6) h is an isomorphism from N_i onto N_j,
(7) $h(i) = j$,
(8) h is the identity map on $N_i \cap N_j$,
(9) $P \in N_i \cap N_j$,
(10) $p \in N_i \cap P$,

then (letting $\dot{G}$ be the P-name for the generic set) there is a $q \in P$ such that:

(11) $q \Vdash$ "$(\forall r \in N_i \cap P)[r \in \dot{G} \iff h(r) \in \dot{G}]$",
(12) $q \Vdash$ "$p \in \dot{G}$",
(13) q is (N_i, P)-generic.

Notice that if N_i and N_j are as in the above definition, then N_i and N_j contain the same hereditarily countable sets. This follows because h is an isomorphism. In particular, $N_i \cap \omega_1$ and $N_j \cap \omega_1$ are the same ordinal. We also note that in both of the previous two definitions, it does not matter if we require that the models under consideration contain a fixed parameter $x \in H(\lambda)$. Also note that $\aleph_2$ is an element of any relevant model N_i — in the more general case dealing with the κ-p.i.c. for arbitrary κ one must require that $\kappa \in N_i \cap N_j$.

The properties of $\aleph_2$-p.i.c. forcings that we utilize will be spelled out in detail in the last section of the paper when we construct our model. What we use is that forcing with an $\aleph_2$-p.i.c. notion of forcing over a model of CH preserves CH, and that in iterations of length $\leq \omega_2$ where each iterand satisfies the $\aleph_2$-p.i.c., the limit forcing satisfies the (weaker) $\aleph_2$ chain condition.

2. Handling P-ideals

Definition 2.1: A P-ideal in $[\omega_1]^{\aleph_0}$ (the set of all countable subsets of ω_1) is a set $\mathcal{I} \subseteq [\omega_1]^{\aleph_0}$ such that

- if A and B are in $\mathcal{I}$, then so is $A \cup B$,
- if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,
- if $A \in \mathcal{I}$ and $B =^* A$, then $B \in \mathcal{I}$,

- if $A_n \in \mathcal{I}$ for each $n \in \omega$, then there is an $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for each n.

In the preceding, we use the familiar convention that $A \subseteq^* B$ means $A \setminus B$ is finite, and $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$.

Definition 1.1: Let $\mathcal{I}$ be a P-ideal in $[\omega_1]^{\aleph_0}$ generated by a set of size $\aleph_1$. A generating sequence for $\mathcal{I}$ is a sequence $\{A_\alpha: \alpha < \omega_1\}$ such that

- $A_\alpha \subseteq \alpha$,
- if $\alpha < \beta$ then $A_\alpha \subseteq^* A_\beta$,
- if $A \in \mathcal{I}$, then there is an α with $A \subseteq A_\alpha$.

Clearly every such $\mathcal{I}$ has a generating sequence.

Our goal in this section is (assuming CH holds) to define a notion of forcing (which we call $P(\mathbb{I})$) with the property that for every P-ideal $\mathcal{I} \subseteq [\omega_1]^{\aleph_0}$ in the ground model there is an uncountable set A in the extension satisfying $[A]^{\aleph_0} \subseteq \mathcal{I}$ or $[A]^{\aleph_0} \cap \mathcal{I} = \emptyset$. The partial order we use is a modification of one of the posets from [2], itself a modification of the notion of forcing used in [1].

Assume CH, and let $\mathbb{I} = \langle \mathcal{I}_\xi: \xi < \kappa\rangle$ be a sequence of P-ideals in $[\omega_1]^{\aleph_0}$. Let $\{A^\xi_\alpha: \alpha < \omega_1\}$ be a generating sequence for $\mathcal{I}_\xi$ (such a sequence exists because CH holds). The notion of forcing we define depends on our choice of generating sequences, but we abuse notation and call the notion of forcing $P(\mathbb{I})$.

Definition 2.3: A promise is a function f such that

- $\operatorname{dom} f$ is an uncountable subset of ω_1,
- $f(\alpha)$ is a finite subset of α.

Definition 2.4: A condition $p \in P(\mathbb{I})$ is a pair (a_p, Φ_p) such that

(1) a_p is a function,
(2) $\operatorname{dom} a_p$ is a countable subset of $\kappa \times \omega_1$,
(3) $\operatorname{ran} a_p \subseteq 2$,
(4) for $\xi < \kappa$, $[p]_\xi := \{\zeta < \omega_1: a_p(\xi, \zeta) = 1\}$ is in $\mathcal{I}_\xi$ (so $[p]_\xi = \emptyset$ for all but countably many ξ),
(5) Φ_p is a countable collection of pairs (v, f), where $v \subseteq \kappa$ is finite and f is a promise.

A condition q extends p if

(6) $a_q \supseteq a_p$, $\Phi_q \supseteq \Phi_p$,
(7) for $(v, f) \in \Phi_p$,

$$Y(v, f, q, p) = \{\alpha \in \operatorname{dom} f: (\forall \xi \in v)([q]_\xi \setminus [p]_\xi \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable, and

$$(v, f \restriction Y(v, f, q, p)) \in \Phi_q.$$

The intent of $P(\mathbb{I})$ is to attempt to adjoin for each $\xi < \kappa$ an uncountable set A_ξ with $[A_\xi]^{\aleph_0}$ contained in $\mathcal{I}_\xi$. A condition gives us an approximation to A_ξ for countably many ξ, as well as some constraints on future growth of these approximations. A pair $(v, f) \in \Phi_p$ puts limits on how our approximation to A_ξ can grow for the finitely many $\xi \in v$. It may be that the forcing fails to produce an uncountable A_ξ for some ξ, but we show that we can do so in every situation where we need it.

Definition 2.5: Let p be a condition in $P(\mathbb{I})$, let D be a dense open subset of $P(\mathbb{I})$, and let v be a finite subset of κ. An ordinal α is bad for (v, p, D) if there is an $F_\alpha \in [\alpha]^{<\aleph_0}$ such that there is no $q \leq p$ in D with

$$[q]_\xi \setminus [p]_\xi \subseteq A^\xi_\alpha \setminus F_\alpha$$

for all $\xi \in v$. Let $\mathrm{Bad}(v, p, D)$ be the set of $\alpha < \omega_1$ that are bad for (v, p, D).

PROPOSITION 2.6: *$\mathrm{Bad}(v, p, D)$ is countable.*

Proof: Suppose not. Let f be the function with domain $\mathrm{Bad}(v, p, D)$ that sends α to F_α, so f is a promise. Let r be the condition in $P(\mathbb{I})$ with $a_r = a_p$, and $\Phi_r = \Phi_p \cup \{(v, f)\}$. Clearly r extends p. Now let $q \leq r$ be in D. By definition, there are uncountably many $\alpha \in \mathrm{dom}(f)$ such that if $\xi \in v$ then $[q]_\xi \setminus [r]_\xi$ is a subset of $A^\xi_\alpha \setminus f(\alpha)$. This is a contradiction, as any $\alpha \in \mathrm{dom} f$ is bad for (v, p, D), yet $q \in D$ and

$$[q]_\xi \setminus [p]_\xi \subseteq A^\xi_\alpha \setminus f(\alpha)$$

for all $\xi \in v$. ∎

THEOREM 1: *$P(\mathbb{I})$ satisfies the $\aleph_2$-p.i.c.*

Proof: Let i, j, N_i, N_j, h, and p be as in Definition 1.3. For $r \in N_i \cap P(\mathbb{I})$, we define

$$r \cup h(r) := (a_r \cup h(a_r), \Phi_r \cup h(\Phi_r)).$$

LEMMA 2.7: *Assume that $r \in N_i \cap P(\mathbb{I})$.*

(1) *$r \cup h(r) \in P(\mathbb{I})$.*

(2) *$r \cup h(r)$ extends both r and $h(r)$.*

(3) *If $s \in N_i \cap P(\mathbb{I})$ and $r \leq s$, then $r \cup h(r) \leq s \cup h(s)$.*

Proof: Left to reader. ∎

Now let $\delta = N_i \cap \omega_1 = N_j \cap \omega_1$, and let $\{D_n: n \in \omega\}$ enumerate the dense open subsets of $P(\mathbb{I})$ that are members of N_i. Our goal is to build a decreasing sequence of conditions $\{p_n: n \in \omega\}$ in $N_i \cap P(\mathbb{I})$ such that $p_0 = p$, $p_{n+1} \in N_i \cap D_n$, and such that the sequence $\{p_n \cup h(p_n): n \in \omega\}$ has a lower bound q. The next lemma shows that this will be sufficient.

LEMMA 2.8: *Let $\{p_n: n \in \omega\}$ be an $(N_i, P(\mathbb{I}))$-generic sequence.*

(1) *$\{h(p_n): n \in \omega\}$ is an $(N_j, P(\mathbb{I}))$-generic sequence.*

(2) *If $\{p_n \cup h(p_n): n \in \omega\}$ has a lower bound q, then q satisfies conditions 11 and 13 of Definition 1.3.*

Proof: The first clause follows immediately from the fact that h is an isomorphism mapping N_i onto N_j. For the second clause, note

$$q \Vdash \text{“}r \in N_i \cap \dot{G}\text{”} \iff r \in N_i \text{ and } \exists n(p_n \leq r).$$

This is because for each $r \in N_i \cap P(\mathbb{I})$, the set of conditions that extend r or that are incompatible with r is a dense open subset of $P(\mathbb{I})$ that is in N_i, and hence for some n either p_n extends r or p_n incompatible with r. Similarly, we have

$$q \Vdash \text{“}r \in N_j \cap \dot{G}\text{”} \iff r \in N_j \text{ and } \exists n(h(p_n) \leq r).$$

Now clause 11 of Definition 1.3 follows easily. Clause 13 holds because the p_n's are an $(N_i, P(\mathbb{I}))$-generic sequence. ∎

Recall that $\delta = N_i \cap \omega_1 = N_j \cap \omega_1$, and let $\{\gamma_n: n \in \omega\}$ enumerate $N_i \cap \kappa$. We construct by induction on $n \in \omega$ objects p_n, F_n, q_n and u_n such that

(i) $p_0 = p$, $F_0 = \emptyset$, $u_0 = \emptyset$,

(ii) $q_n = p_n \cup h(p_n)$,

(iii) $p_{n+1} \in N_i \cap D_n$,

(iv) F_n is a finite subset of δ,

(v) u_n is a finite subset of $N_i \cap \kappa$,

(vi) $p_{n+1} \leq p_n$,

(vii) $F_{n+1} \supseteq F_n$,

(viii) $u_{n+1} \supseteq u_n$,

(ix) $\{\gamma_m: m < n\} \subseteq u_n$,

(x) for $\gamma \in u_{n+1} \cup h(u_{n+1})$, $[q_{n+1}]_\gamma \setminus [q_n]_\gamma \subseteq A_\delta^\gamma \setminus F_{n+1}$,

(xi) if $(v, f) \in \Phi_{q_k}$ for some k, then there is a stage $n \geq k$ for which

$$v \subseteq u_{n+1} \cup h(u_{n+1}) \tag{2.1}$$

and

(2.2) $$\{\alpha \in Y(v, f, q_n, q_k)\colon (\forall \xi \in v)(A^\xi_\delta \setminus F_{n+1} \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable.

We assume that we have fixed a bookkeeping system so that at each stage of the induction we are handed a pair (v, f) from some earlier Φ_{q_k} for which we must ensure (xi), and such that every such (v, f) appearing along the way is treated in this manner.

There is nothing to be done at stage 0, so assume we have carried out the induction through stage n. At stage $n+1$, we will be handed p_n, F_n, q_n, and u_n, and our bookkeeping hands us $(v, f) \in \Phi_{q_k}$ for some $k \leq n$.

To start, we choose $u_{n+1} \supseteq u_n \cup \{\gamma_n\}$ satisfying (v), but large enough so that $v \subseteq u_{n+1} \cup h(u_{n+1})$. This means that (v), (viii), and (ix) hold.

Claim 2.9: *If f is a promise, $B \subseteq \operatorname{dom} f$ uncountable, $v \subseteq \kappa$ finite, and $\beta < \omega_1$, then there is a finite $\bar{F} \subseteq \beta$ such that*

$$\{\alpha \in B\colon (\forall \xi \in v)(A^\xi_\beta \setminus \bar{F} \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable.

Proof: Straightforward, by induction on $|v|$. ∎

(Although the preceding claim has a trivial proof, it does not generalize to the context of the next section and in some sense this fact is the reason why the next section is so complicated.)

Now apply the preceding claim to v, f, $Y(v, f, q_n, q_k)$, $u_{n+1} \cup h(u_{n+1})$, and δ to get a finite $\bar{F} \subseteq \delta$ such that

$$\{\alpha \in Y(v, f, q_n, q_k)\colon (\forall \xi \in u_{n+1} \cup h(u_{n+1}))(A^\xi_\delta \setminus \bar{F} \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable. In particular, our choice of u_{n+1} implies

$$\{\alpha \in Y(v, f, q_n, q_k)\colon (\forall \xi \in v)(A^\xi_\delta \setminus \bar{F} \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable. Now let $F_{n+1} = F_n \cup \bar{F}$. Clearly we have satisfied (iv) and (vii).

Next, we choose $\beta < \omega_1$ such that

$$N_i \models \beta \notin \mathrm{Bad}(p_n, D_n).$$

For each $\gamma \in u_{n+1} \cup h(u_{n+1})$ there is a finite $G_\gamma \subseteq \beta$ such that $A^\gamma_\beta \setminus G_\gamma \subseteq A^\gamma_\delta \setminus F_{n+1}$, so there is a finite $G \subseteq \beta$ such that

$$\forall \gamma \in u_{n+1} \cup h(u_{n+1})[A^\gamma_\beta \setminus G \subseteq A^\gamma_\delta \setminus F_{n+1}].$$

Note that both β and G are in $N_i \cap N_j$, and hence are fixed by h. By (2), we can find $p_{n+1} \in N_i$ such that $p_{n+1} \leq p_n$, $p_{n+1} \in D_n$, and

$$N_i \models (\forall \gamma \in u_{n+1})([p_{n+1}]_\gamma \setminus [p_n]_\gamma \subseteq A^\gamma_\beta \setminus G).$$

Applying h, we see that

$$N_j \models (\forall \gamma \in h(u_{n+1}))([h(p_{n+1})]_\gamma \setminus [h(p_n)]_\gamma \subseteq A^\gamma_\beta \setminus G).$$

Thus

$$(\forall \gamma \in u_{n+1} \cup h(u_{n+1}))([q_{n+1}]_\gamma \setminus [q_n]_\gamma \subseteq A^\gamma_\beta \setminus G \subseteq A^\gamma_\delta \setminus F_{n+1}).$$

Our choice of p_{n+1} (and q_{n+1}) satisfies (ii), (iii), (vi), and (x). Since $\bar{F} \subseteq F_{n+1}$, we have that (xi) is satisfied for this particular (v, f).

Now we need to verify that the sequence $\{q_n : n \in \omega\}$ has a lower bound q. To start, we define

$$a_q = \bigcup_{n\in\omega} a_{q_n} \quad \text{and} \quad [q]_\xi = \bigcup_{n\in\omega} [q_n]_\xi. \tag{2.3}$$

CLAIM 2.10:
(1) $a_q : (N_i \cup N_j) \cap \kappa \to 2$
(2) *If* $\xi \in N_i \cap \kappa$, *then* $[q]_\xi = \cup\{[p_n]_\xi : n \in \omega\}$. *If* $\xi \in N_j \cap \kappa$, *then* $[q]_\xi = \cup\{[h(p_n)]_\xi : n \in \omega\}$.
(3) $[q]_\xi \in \mathcal{I}_\xi$ *for* $\xi < \kappa$.

Proof of Claim: Part 1 of the claim follows because the sequence $\{p_n : n \in \omega\}$ (resp. $\{h(p_n) : n \in \omega\}$) meets every dense set in $P(\mathbb{I})$ that is a member of N_i (resp. N_j). Part 2 follows as in the proof of Lemma 2.7. For the last part, if $\xi \notin (N_i \cup N_j) \cap \kappa$ there is nothing to check, so assume $\xi \in (N_i \cup N_j) \cap \kappa$, and fix n such that $\xi \in \{\gamma_n, h(\gamma_n)\}$. Our construction guarantees that $[q]_\xi \subseteq [q_n]_\xi \cup A^\xi_\delta$, and this latter set is in $\mathcal{I}_\xi$. ∎

CLAIM 2.11: *If* $k \in \omega$ *and* $(v, f) \in \Phi_{q_k}$, *then*

$$K(v, f, k) := \{\alpha \in \operatorname{dom} f : (\forall \xi \in v)([q]_\xi \setminus [q_k]_\xi \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable.

Proof: Let $n \geq k$ be such that our bookkeeping handed us the promise (v, f) at stage $n + 1$ of the construction. The actions we took at stage $n + 1$ ensure that

$$A := \{\alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in v)(A^\xi_\delta \setminus F_{n+1} \subseteq A^\xi_\alpha \setminus f(\alpha))\}$$

is uncountable. We claim that $A \subseteq K(v, f, k)$; to see this fix $\alpha \in A$, and let $\xi \in v$ be arbitrary. We must verify that $[q]_\xi \setminus [q_k]_\xi$ is a subset of $A^\xi_\alpha \setminus f(\alpha)$.

$$\begin{aligned}
[q]_\xi \setminus [q_k]_\xi =& ([q]_\xi \setminus [q_n]_\xi) \cup ([q_n]_\xi \setminus [q_k]_\xi) \\
\subseteq& (\bigcup_{m \geq n} [q_m]_\xi \setminus [q_n]_\xi) \cup A^\xi_\alpha \setminus f(\alpha) \\
\subseteq& A^\xi_\delta \setminus F_{n+1} \cup A^\xi_\alpha \setminus f(\alpha) \\
\subseteq& A^\xi_\alpha \setminus f(\alpha).
\end{aligned}$$

Notice that in obtaining the second line, we used that $\alpha \in Y(v, f, q_n, q_k)$, and to obtain the third line we used requirement (x) of our construction and the fact that $v \subseteq u_{n+1} \cup h(u_{n+1})$. ∎

Now we define

$$\Phi_q = \bigcup_{n \in \omega} \Phi_{q_n} \cup \bigcup_{n \in \omega} \{(v, f \restriction K(v, f, n)) \colon (v, f) \in \Phi_{q_n}\}$$

and $q = (u_q, x_q, \Phi_q)$ is a lower bound for the sequence $\{q_n \colon n \in \omega\}$ as desired. ∎

Notice that in our proof, the only relevant properties of h were that it is an isomorphism from N_i onto N_j that is the identity on $N_i \cap N_j$ — the other requirements from Definition 1.3 were not used. In particular, our proof goes through in the case that h is actually the identity map (so $N_i = N_j$). Thus we obtain the following.

THEOREM 2: *$P(\mathbb{I})$ is totally proper.*

We are still not through, however, as we have not yet verified that $P(\mathbb{I})$ lives up to its billing.

Definition 2.12: Let f be a promise and $v \subseteq \kappa$ finite. For $\xi \in v$, we define a set $\mathrm{Ban}_\xi(v, f)$ by $\beta \in \mathrm{Ban}_\xi(v, f)$ if and only if

$$\{\alpha \in \operatorname{dom} f \colon \beta \in A^\xi_\alpha \setminus f(\alpha)\} \text{ is countable.}$$

If $\xi \notin v$ then let $\mathrm{Ban}_\xi(v, f) = \emptyset$.

PROPOSITION 2.13: *If $\xi < \kappa$, and there is no uncountable $A \subseteq \omega_1$ with $[A]^{\aleph_0} \cap \mathcal{I}_\xi = \emptyset$, then $\mathrm{Ban}_\xi(v, f)$ is countable.*

Proof: We can assume that $\xi \in v$ as otherwise there is nothing to prove. By way of contradiction, suppose that $\mathrm{Ban}_\xi(v, f)$ is uncountable. Our assumption on $\mathcal{I}_\xi$ means that there is an infinite $B \subseteq \mathrm{Ban}_\xi(v, f)$ with $B \in \mathcal{I}_\xi$. For each $\alpha \in \operatorname{dom} f$, there is a finite set F_α for which $B \setminus F_\alpha \subseteq A^\xi_\alpha \setminus f(\alpha)$. Thus there is a single finite F for which

$$\{\alpha \in \operatorname{dom} f\colon B \setminus F \subseteq A^\xi_\alpha \setminus f(\alpha)\}$$

is uncountable. Therefore any member of $B \setminus F$ is not in $\mathrm{Ban}_\xi(v, f)$, a contradiction. ∎

Proposition 2.14: *If $\xi < \kappa$ and there is no uncountable $A \subseteq \omega_1$ with $[A]^{\aleph_0} \cap \mathcal{I}_\xi = \emptyset$, then for each $\gamma < \omega_1$, the set of conditions p for which $[p]_\xi \setminus \gamma$ is non-empty is dense in $P(\mathbb{I})$.*

Proof: Let ξ and γ be as in the assumption, and let $p \in P(\mathbb{I})$ be arbitrary. By the previous proposition,

$$\bigcup\{\underset{\xi}{\mathrm{Ban}}(v, f)\colon (v, f) \in \Phi_p\}$$

is countable (as Φ_p is countable), hence there is an $\alpha > \gamma$ not in $\mathrm{Ban}_\xi(v, f)$ for any $(v, f) \in \Phi_p$. It is straightforward to see that there is a $q \leq p$ with $\alpha \in [q]_\xi$. ∎

Conclusion 1: Assume CH, and let $\mathbb{I} = \langle I_\xi\colon \xi < \kappa\rangle$ be a list of P-ideals in $[\omega_1]^{\aleph_0}$. Then there is a totally proper notion of forcing $P(\mathbb{I})$, satisfying the $\aleph_2$-p.i.c., so that in the generic extension, for each $\xi < \kappa$ there is an uncountable $A_\xi \subseteq \omega_1$ for which either $[A_\xi]^{\aleph_0} \subseteq \mathcal{I}_\xi$ or $[A_\xi]^{\aleph_0} \cap \mathcal{I}_\xi = \emptyset$.

Proof: We have all the ingredients of the proof already. By Theorems 1 and 2, we know $P(\mathbb{I})$ is totally proper and satisfies the $\aleph_2$-p.i.c. Fix $\xi < \kappa$, assume that $G \subseteq P(\mathbb{I})$ is generic over V, and work for a moment in $V[G]$.

If in V there is an uncountable A_ξ with $[A_\xi]^{\aleph_0} \cap \mathcal{I}_\xi = \emptyset$, then A_ξ still has this property in $V[G]$. (Note that since $P(\mathbb{I})$ is totally proper, no new countable subsets of ω_1 are added, so $\mathcal{I}_\xi$ is unchanged by passing to $V[G]$.) If no such set exists in V, then the set

$$A_\xi\colon = \bigcup_{p \in G} [p]_\xi$$

is uncountable by the previous proposition, and $[A_\xi]^{\aleph_0} \subseteq \mathcal{I}_\xi$ by definition of our forcing notion. ∎

3. Handling relevant spaces

Our goal in this section is to build, assuming that CH holds, a totally proper notion of forcing having the $\aleph_2$-p.i.c. that destroys all first countable, countably compact, non-compact S-spaces in the ground model. In fact, we do a little better than this — if X is a first countable, countably compact, non-compact regular space with no uncountable free sequences, then after we force with our poset, X acquires an uncountable free sequence. The partial order we use is a modification of that used in [4], although things do not work as smoothly as they did in the last section.

Let us call a space X **relevant** if X is first countable, countably compact, non-compact, regular, $|X| = \aleph_1$, and X has no uncountable free sequences. For each relevant X, we fix a maximal filter of closed sets $\mathcal{H}_X$ that is not fixed. These filters lie at the heart of the work that follows.

Definition 3.1: If $\mathcal{H}$ if a filter of closed subsets of X, we say that $Y \subseteq X$ is $\mathcal{H}$-large if $Y \cap A \neq \emptyset$ for every $A \in \mathcal{H}$. We say that $Y \subseteq X$ **diagonalizes** $\mathcal{H}$ if Y is $\mathcal{H}$-large and $Y \setminus A$ is countable for every set $A \in \mathcal{H}$.

Notice that if $\mathcal{H}$ is countably complete and $\mathcal{H}$ is generated by a set of size at most $\aleph_1$, then every $\mathcal{H}$-large set Y has a subset Z that diagonalizes $\mathcal{H}$. If in addition $\mathcal{H}$ is not fixed, then every uncountable subset of Z will diagonalize $\mathcal{H}$ as well.

PROPOSITION 3.2: *Suppose $\mathcal{H}$ is a countably complete filter of closed subsets of the space X, and suppose $Z \subseteq X$ is $\mathcal{H}$-large. If the closure of any countable subset of Z is disjoint to a set in $\mathcal{H}$, then there is an uncountable $F \subseteq Z$ that forms a free sequence in X.*

Proof: We construct F by in induction of length ω_1. At a stage α, we will be choosing $x_\alpha \in Z$ as well as a set $A_\alpha \in \mathcal{H}$ in such a way that

- $x_\alpha \in \bigcap_{\beta<\alpha} A_\beta$, and
- $A_\alpha \cap \mathrm{cl}\{x_\beta \colon \beta \leq \alpha\} = \emptyset$.

At stage α, we can find a suitable x_α because the filter $\mathcal{H}$ is countably complete and Z meets every set in $\mathcal{H}$. A suitable A_α exists because of our other hypothesis on the set Z. Thus the induction carries on through ω_1 stages, and it is routine to verify that the set constructed is actually a free sequence in X. ∎

Note that in the preceding proposition we do not assume that $\mathcal{H}$ is a maximal filter. Also note that as a corollary, we see that $\mathcal{H}$ is generated by separable sets if X has no uncountable free sequences.

COROLLARY 3.3: *If CH holds and X is a relevant space, then the filter $\mathcal{H}_X$ is generated by a family of size $\aleph_1$.*

Proof: Since $|X| = \aleph_1$, we know X has at most $\aleph_1^{\aleph_0}$ separable subsets. By CH, we know that $\aleph_1^{\aleph_0} = \aleph_1$. Since X has no uncountable free sequences, we know $\mathcal{H}_X$ is generated by separable sets. ∎

COROLLARY 3.4: *If CH holds and X is a relevant space, then there is a set $Y_X \subseteq X$ of size $\aleph_1$ that diagonalizes $\mathcal{H}_X$.*

Proof: We know that $\mathcal{H}_X$ is generated by $\aleph_1$ sets. Since $\mathcal{H}_X$ is countably complete, we can fix a decreasing family $\{A_\alpha\colon \alpha < \omega_1\} \subseteq \mathcal{H}_X$ that generates $\mathcal{H}_X$. To build Y_X, we simply choose a for each α a point $x_\alpha \in A_\alpha$ in such a way that $x_\alpha \neq x_\beta$ for $\beta < \alpha$. Since the set $\{A_\alpha\colon \alpha < \omega_1\}$ generates $\mathcal{H}_X$, the family $Y_X = \{x_\alpha\colon \alpha < \omega_1\}$ diagonalizes $\mathcal{H}_X$. ∎

Since we are assuming that CH holds, let us choose for each relevant X a subspace Y_X of size $\aleph_1$ that diagonalizes $\mathcal{H}_X$. By passing to a subset if necessary, we may assume that Y_X is right-separated in type ω_1.

Since $\mathcal{H}_X$ is a maximal filter of closed sets, this means that Y_X is a sub-Ostaszewski subspace of X, i.e., every closed subset of Y_X is either countable or co-countable. This tells us immediately that every uncountable subset of Y_X is $\mathcal{H}_X$-large, and the filter $\mathcal{H}_X$ is reconstructible from Y_X as the set of all closed subsets of X that meet Y_X uncountably often.

We assume that each Y_X has ω_1 as an underlying set, and that this correspondence is set up so that initial segments are open. Thus given a collection of relevant spaces, a countable ordinal α is viewed as a point in each of the spaces.

We also fix a function $\mathcal{B}$ so that for each relevant space X and ordinal $\alpha < \omega_1$, $\{\mathcal{B}(X, \alpha, n)\colon n \in \omega\}$ is a decreasing neighborhood base for α as a point in X. We will need one more definition before defining our notion of forcing.

Definition 3.5: A promise f is a function whose domain is an uncountable subset of ω_1 and whose range is a subset of ω.

Until said otherwise, $\mathcal{X} = \{X_\xi\colon \xi < \kappa\}$ is a collection of relevant spaces, and CH holds. To save a bit on notation, let us declare that $\mathcal{H}_\xi = \mathcal{H}_{X_\xi}$, and $Y_\xi = Y_{X_\xi}$.

Definition 3.6: A condition $p \in P(\mathcal{X})$ is a pair (a_p, Φ_p) such that

(1) a_p is a function,

(2) $\operatorname{dom} a_p$ is a countable subset of $\{(\xi, x)\colon \xi < \kappa \text{ and } x \in X_\xi\}$,

(3) $\operatorname{ran} a_p \subseteq 2$,

(4) for each $\xi < \kappa$, $[p]_\xi\colon = \{x \in X_\xi\colon a_p(\xi, x) = 1\}$ satisfies $\mathrm{cl}_{X_\xi}[p]_\xi \notin \mathcal{H}_\xi$,

(5) Φ_p is a countable set of pairs (v, f) where $v \subseteq \kappa$ is finite and f is a promise.

A condition q extends p if

(6) $a_q \supseteq a_p$, $\Phi_q \supseteq \Phi_p$,

(7) for $(v, f) \in \Phi_p$,

$$Y(v, f, q, p)\colon = \{\alpha \in \operatorname{dom} f\colon (\forall \xi \in v)[[q]_\xi \setminus [p]_\xi \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable, and

$$(v, f \restriction Y(v, f, q, p)) \in \Phi_q.$$

The notion of forcing we have described (seemingly) need not be proper. If, however, we put restrictions on the family $\mathcal{X}$ we get a proper notion of forcing. We will need some notation to express the necessary ideas.

Definition 3.7: Let $v \subseteq \kappa$ be finite. We define

$$X_v = \prod_{\xi \in v} X_\xi,$$

and we let $\mathcal{H}_v$ be the filter of closed subsets of X_v that is generated by sets of the form $\prod_{\xi \in v} A_\xi$, where $A_\xi \in \mathcal{H}_\xi$.

Note that $\mathcal{H}_v$ will be countably complete and generated by $\leq \aleph_1$ sets because each $\mathcal{H}_\xi$ is.

Definition 3.8: Let $v \subseteq \kappa$ be finite, and let f be a promise. A point $(x_\xi\colon \xi \in v) \in X_v$ is banned by (v, f) if

$$\{\alpha \in \operatorname{dom} f\colon (\forall \xi \in v)[x_\xi \in \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is countable. We let $\mathrm{Ban}(v, f)$ be the collection of all points in X_v that are banned by (v, f). We may abuse notation and write things like $\mathrm{Ban}(\{X\}, f)$ in the sequel — all such expressions have the obvious meanings.

Definition 3.9: Let $v \subseteq \kappa$ be finite. We say v is **dangerous** if there is a promise f such that $\mathrm{Ban}(v, f)$ is $\mathcal{H}_v$-large. $\mathcal{X}$ is **safe** if no finite $v \subseteq \kappa$ is dangerous.

Our definition of "safe" was formulated so that the proof of the following theorem goes through — the proof of Claim 3.13 is the place where we really need it.

THEOREM 3: *If $\mathcal{X} = \{X_\xi\colon \xi < \kappa\}$ is safe, then $P(\mathcal{X})$ is totally proper.*

Before we commence with the proof of this theorem, we need a definition and lemma.

Definition 3.10: Let $v \subseteq \kappa$ be finite, $p \in P(\mathcal{X})$, and let $D \subseteq P(\mathcal{X})$ be dense. An ordinal $\gamma < \omega_1$ is said to be **bad for** (v, p, D) if there is an n such that there is no $q \leq p$ in D such that for all $\xi \in v$,

$$[q]_\xi \setminus [p]_\xi \subseteq \mathcal{B}(X_\xi, \gamma, n).$$

We let $\mathrm{Bad}(v, p, D)$ be the collection of all $\gamma < \omega_1$ that are bad for (v, p, D).

So $\gamma \notin \mathrm{Bad}(v, p, D)$ means for every n, we can find a $q \leq p$ in D such that $[q]_\xi \setminus [p]_\xi \subseteq \mathcal{B}(X_\xi, \gamma, n)$ for all $\xi \in v$.

LEMMA 3.11: *$\mathrm{Bad}(v, p, D)$ is countable.*

Proof: Suppose not. The function f with domain $\mathrm{Bad}(v, p, D)$ that sends γ to the n that witnesses $\gamma \in \mathrm{Bad}(v, p, D)$ is a promise. Now we define $r = (a_p, \Phi_p \cup \{(v, f)\})$. Clearly $r \leq p$ in $P(\mathcal{X})$, and since D is dense there is a $q \leq r$ in D. Now $Y(v, f, q, r)$ is uncountable, and for $\gamma \in Y(v, f, q, r)$ and $\xi \in v$ we have

$$[q]_\xi \setminus [p]_\xi = [q]_\xi \setminus [r]_\xi \subseteq \mathcal{B}(X_\xi, \gamma, f(\gamma))$$

and this contradicts the definition of f. ∎

LEMMA 3.12: *Let (v, f) be a promise, and suppose $(x_\xi \colon \xi \in v)$ is not in* $\mathrm{Ban}(v, f)$. *Then there is $(U_\xi \colon \xi \in v)$ such that U_ξ is a neighborhood of $x_\xi \in X_\xi$ and*

$$\{\alpha \in \mathrm{dom}\, f \colon (\forall \xi \in v)[U_\xi \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable. In particular, $\mathrm{Ban}(v, f)$ *is a closed subset of X_v.*

Proof: Let $\{V_n \colon n \in \omega\}$ be a neighborhood base for $(x_\xi \colon \xi \in v)$ in the (first countable) space X_v, and define

$$A = \{\alpha \in \mathrm{dom}\, f \colon (\forall \xi \in v)[x_\xi \in \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}.$$

By assumption, A is uncountable, and for each $\alpha \in A$ there is an n for which

$$V_n \subseteq \prod_{\xi \in v} \mathcal{B}(X_\xi, \alpha, f(\alpha)).$$

Thus there is a single n for which

$$\{\alpha \in A \colon V_n \subseteq \prod_{\xi \in v} \mathcal{B}(X_\xi, \alpha, f(\alpha))\}$$

is uncountable. The definition of the product topology then gives us the U_ξ's that we need. ∎

Proof of Theorem 3: Let $N \prec H(\lambda)$ be countable with $P(\mathcal{X}) \in N$. Let $p \in N \cap P(\mathcal{X})$ be arbitrary, and let $\{D_n : n \in \omega\}$ list the dense open subsets of $P(\mathcal{X})$ that are members of N. Let $\delta = N \cap \omega_1$, and let $\{\gamma_n : n < \omega\}$ enumerate $N \cap \kappa$.

Since all the spaces in $\mathcal{X}$ are countably compact and N is countable, there is a sequence $\{\delta_n : n \in \omega\}$ increasing and cofinal in δ such that for every $\xi \in N \cap \kappa$, the sequence $\{\delta_n : n \in \omega\}$ converges in X_ξ to a point z_ξ.

CLAIM 3.13: *If $v = \{\xi_0, \ldots, \xi_{n-1}\} \subseteq N \cap \kappa$ and $f \in N$ is a promise, then $(z_{\xi_0}, \ldots, z_{\xi_{n-1}})$ is not banned by (v, f).*

Proof: Since $\mathcal{X}$ is safe and $(v, f) \in N$, there are sets $A_i \in \mathcal{H}_{\xi_i} \cap N$ for $i < n$ such that $A_0 \times \cdots \times A_{n-1}$ is disjoint to $\mathrm{Ban}(v, f)$. Since $A_i \cap \omega_1$ is co-countable, for all sufficiently large ℓ we have $\delta_\ell \in A_i$. Since this holds for each i, for all sufficiently large ℓ the n-tuple $(\delta_\ell, \ldots, \delta_\ell)$ is in $A_0 \times \cdots \times A_{n-1}$. Since this latter set is closed, we have that $(z_{\xi_0}, \ldots, z_{\xi_{n-1}})$ is in $A_0 \times \cdots \times A_{n-1}$, hence $(z_{\xi_0}, \ldots, z_{\xi_{n-1}})$ is not banned by (v, f). ∎

Let $\{V(z_\xi, n) : n \in \omega\}$ be a decreasing neighborhood base for z_ξ in X_ξ, with $\mathrm{cl}_{X_\xi} V(z_\xi, 0) \notin \mathcal{H}_\xi$; this uses the fact that each X_ξ is regular.

We define $p_n \in P(\mathcal{X})$, $u_n \subseteq \kappa$, and a function $g : \omega \to \omega_1$ such that

(1) $p_0 = p$, $u_0 = \emptyset$, $g(0) = 0$,

(2) $p_{n+1} \leq p_n$,

(3) $p_{n+1} \in N \cap D_n$,

(4) u_n is finite,

(5) $u_{n+1} \supseteq u_n$,

(6) $g(n+1) > g(n)$,

(7) $\{\gamma_m : m < n\} \subseteq u_n$,

(8) for $\gamma \in u_{n+1}$, $[p_{n+1}]_\gamma \setminus [p_n]_\gamma \subseteq V(z_\gamma, g(n+1))$,

(9) if (v, f) appears in Φ_{p_k} for some k, then there is an $n \geq k$ for which $v \subseteq u_{n+1}$ and

$$\{\alpha \in Y(v, f, p_n, p_k) : (\forall \xi \in u_{n+1})[V(z_\xi, g(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable.

Assume that a suitable bookkeeping procedure has been set up so that at each stage $n+1$ we are handed a (v, f) in Φ_{p_k} for some earlier k for the purposes of ensuring condition 9, and in such a way that every such (v, f) so appears.

There is nothing to be done at stage 0. At stage $n+1$ we will be handed p_n, u_n, and $g \restriction n+1$, and our bookkeeping hands us a $(v, f) \in \Phi_{p_k}$ for some $k \leq n$.

Choose $u_{n+1} \subseteq N \cap \kappa$ finite with $u_n \cup v \cup \{\gamma_n\} \subseteq u_{n+1}$. Clearly u_{n+1} satisfies 4, 5, and 7.

Let f' be the promise $f \restriction Y(v, f, p_n, p_k)$. Clearly f' is in N. By Claim 3.13, we know that $(z_\xi \colon \xi \in u_{n+1})$ is not banned by (u_{n+1}, f'). Thus by an application of Lemma 3.12 we can choose a value for $g(n+1) > g(n)$ large enough so that

$$\{\alpha \in \operatorname{dom} f' \colon (\forall \xi \in u_{n+1})[V(z_\xi, g(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable. Now we choose $\ell < \omega$ large enough so that $\delta_\ell \notin \mathrm{Bad}(u_{n+1}, p_n, D_n)$ and

$$(\forall \xi \in u_{n+1})[\delta_\ell \in V(z_\xi, h(n+1))].$$

Next choose m large enough so that

$$(\forall \xi \in u_{n+1})[\mathcal{B}(X_\xi, \delta_\ell, m) \subseteq V(z_\xi, h(n+1))].$$

Since $\mathcal{B} \in N$, we can apply the definition of $\delta_\ell \notin \mathrm{Bad}(u_{n+1}, p_n, D_n)$ to get $p_{n+1} \leq p_n$ in $N \cap D_n$ such that

$$(\forall \xi \in u_{n+1})[[p_{n+1}]_\xi \setminus [p_n]_\xi \subseteq \mathcal{B}(X_\xi, \delta_\ell, m) \subseteq V(z_\xi, h(n+1))].$$

Now why does the sequence $\{p_n \colon n \in \omega\}$ have a lower bound?

Define $a_q = \bigcup_{n \in \omega} a_{p_n}$ Note that a_q is a function satisfying requirements 1–3 of Definition 3.6, and $[a_q]_\xi \neq \emptyset$ only if $\xi \in N \cap \kappa$. If $\xi \in N \cap \kappa$, then $\xi = \gamma_m$ for some $m \in \omega$, and our construction guarantees that

$$[a_q]_\xi \subseteq [p_m]_\xi \cup V(z_\xi, 0)$$

and so $\mathrm{cl}_{X_\xi}[a_q]_\xi \notin \mathcal{H}_\xi$.

Now suppose $(v, f) \in \Phi_{p_k}$ for some $k \in \omega$. Define

$$K(v, f, k) = \{\alpha \in \operatorname{dom} f \colon (\forall \xi \in v)[[x_q]_\xi \setminus [p_k]_\xi \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}.$$

Claim 3.14: *$K(v, f, k)$ is uncountable.*

Proof: Let $n \geq k$ be as in condition 9 for (v, f), so

$$A \colon = \{\alpha \in Y(v, f, p_n, p_k) \colon (\forall \xi \in v)[V(z_\xi, h(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable. For $\alpha \in A$ and $\xi \in v$, we have

$$\begin{aligned} [a_q]_\xi \setminus [p_k]_\xi &= \bigcup_{m \geq n} [p_m]_\xi \setminus [p_n]_\xi \cup [p_n]_\xi \setminus [p_k]_\xi \\ &\subseteq \bigcup_{m \geq n} [p_m]_\xi \setminus [p_n]_\xi \cup \mathcal{B}(X_\xi, \alpha, f(\alpha)) \quad (\text{as } A \subseteq Y(v, f, p_n, p_k)) \\ &\subseteq V(z_\xi, h(n+1)) \cup \mathcal{B}(X_\xi, \alpha, f(\alpha)) \quad (\text{by 8 of our construction}) \\ &\subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha)) \quad (\text{as } \alpha \in A). \end{aligned}$$

Thus $A \subseteq K(v, f, k)$. ∎

So if we define

$$\Phi_q = \bigcup_{n \in \omega} \Phi_{p_n} \cup \bigcup_{n \in \omega} \{(v, f \restriction K(v, f, n)) \colon (v, f) \in \Phi_{p_n}\}$$

we have $q = (a_q, \Phi_q)$ is a lower bound for $\{p_n \colon n \in \omega\}$. ∎

Proposition 3.15: *A singleton is safe, so if $\mathcal{X} = \{X\}$ then $P(\mathcal{X})$ is totally proper.*

Proof: Suppose $(\{X\}, f)$ form a counterexample. Then $\mathrm{Ban}(\{X\}, f)$ is a $\mathcal{H}_X$-large subset of X. Since X has no uncountable free sequences, there is a countable $A = \{x_n \colon n \in \omega\} \subseteq \mathrm{Ban}(\{X\}, f)$ such that $\mathrm{cl}_X A \in \mathcal{H}_X$ and hence

$$B \colon = \mathrm{dom} f \cap \mathrm{cl}_X A$$

is uncountable. If $\alpha \in B$, then there is an $n \in \omega$ with $x_n \in \mathcal{B}(X, \alpha, f(\alpha))$. Thus there is a single n for which the set of $\alpha \in B$ with $x_n \in \mathcal{B}(X, \alpha, f(\alpha))$ is uncountable, and this contradicts the fact that $x_n \in \mathrm{Ban}(v, f)$. ∎

Since the union of an increasing chain of safe collections is itself safe, we know that maximal safe collections of relevant spaces exist.

Proposition 3.16: *Assume $\mathcal{X} = \{X_\xi \colon \xi < \kappa\}$ is safe, $u \subseteq \kappa$ is finite, and $p \in P(\mathcal{X})$. There is a set $A \in \mathcal{H}_u$ such that for any $(x_\xi \colon \xi \in u) \in A$, there is a $q \leq p$ such that $x_\xi \in [q]_\xi$ for all $\xi \in u$.*

Proof: For each $\xi \in u$ we define a set $A_\xi \in \mathcal{H}_\xi$ as follows:

Let $\{(v_n, f_n) \colon n \in \omega\}$ list all members of Φ_p with $\xi \in v_n$ (the assumption that this set is infinite is purely for notational convenience). For each $n \in \omega$ there is a set

$$B_n \colon = \prod_{\zeta \in v_n} B^n_\zeta \in \mathcal{H}_{v_n}$$

that is disjoint to $\mathrm{Ban}(v_n, f_n)$. Note that this means that for every $w \subseteq v_n$ and $(x_\zeta : \zeta \in w) \in \prod_{\zeta \in w} B^n_\zeta$, the set

$$\{\alpha \in \operatorname{dom} f_n : (\forall \zeta \in w)[x_\zeta \in \mathcal{B}(X_\zeta, \alpha, f(\alpha))]\}$$

is uncountable.

We let $A_\xi = \bigcup_{n \in \omega} B^n_\xi$, and we check that $A = \prod_{\xi \in u} A_\xi$ is as required.

So suppose $x_\xi \in A_\xi$ for $\xi \in u$, and define

$$a_q = a_p \cup \{\langle \xi, x_\xi, 1 \rangle : \xi \in u\}.$$

We want to show that for $(v, f) \in \Phi_p$ the set

$$K(v, f, p) = \{\alpha \in \operatorname{dom} f : (\forall \xi \in v)[[a_q]_\xi \setminus [p]_\xi \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable. Note that this reduces to showing

$$\{\alpha \in \operatorname{dom} f : (\forall \xi \in u \cap v)[x_\xi \in \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable, and this follows easily from the fact that the set in (3) is uncountable.

Thus if we define

$$\Phi_q = \Phi_p \cup \{(v, f \upharpoonright K(v, f, p)) : (v, f) \in \Phi_p\},$$

then $q = (a_q, \Phi_q)$ is the desired extension of p. ∎

Corollary 3.17: *If $v \subseteq \kappa$ is finite, $Z \subseteq X_v$ is $\mathcal{H}_v$-large, and $p \in P(\mathcal{X})$, then there is a $q \leq p$ and $(x_\xi : \xi \in v) \in Z$ such that $x_\xi \in [q]_\xi$ for all $\xi \in v$.*

Theorem 4: *Suppose $\mathcal{X}$ is a maximal safe family, and let X be an arbitrary relevant space. If $G \subseteq P(\mathcal{X})$ is generic, then*

$$V[G] \models \text{"}X \text{ has an uncountable free sequence"}.$$

Proof:

CASE 1: $X \in \mathcal{X}$

In this case $X = X_\xi$ for some $\xi < \kappa$. The filter $\mathcal{H}_\xi$ generates a countably complete filter of closed subsets of X_ξ in the extension; we will abuse notation a little bit and call this filter $\mathcal{H}_\xi$ as well. Note that a set is $\mathcal{H}_\xi$-large in $V[G]$ if and only if it meets every set in $\mathcal{H}_\xi \cap V$.

Now let $A = \bigcup_{p \in G}[p]_\xi$. Clearly A is a subset of X_ξ in the extension, and since G is countably closed, if we are given a countable $A_0 \subseteq A$ there is a $p \in G$ with $A_0 \subseteq [p]_\xi$. The closure of $[p]_\xi$ is the same whether computed in V or $V[G]$, and in V we know that it misses some set in $\mathcal{H}_\xi$. This tells us that the closure of any countable subset of A is disjoint to a set in $\mathcal{H}_\xi$ in $V[G]$.

Now given a set $Z \in \mathcal{H}_\xi$, we can apply Corollary 3.17 with $v = \{\xi\}$ to conclude that $A \cap Z$ is non-empty. Thus in $V[G]$ the set A is $\mathcal{H}_\xi$-large. By Proposition 3.2, X_ξ has an uncountable free sequence.

CASE 2: $X \notin \mathcal{X}$

In this case, by the maximality of $\mathcal{X}$ there is a finite $v \subseteq \kappa$ such that $\{X_\xi \colon \xi \in v\} \cup \{X\}$ is dangerous. To save ourselves from notational headaches, we assume that $v = n$, and we will refer to X as X_n. We will also let w stand for $n + 1$ so the notation $\mathcal{H}_w$ and X_w will have the obvious meaning.

Let f be a promise witnessing that $\{X_i \colon i \leq n\}$ is dangerous. In $V[G]$, for $i < n$ we let $A_i = \bigcup_{r \in G}[r]_i$ be the subset of X_i obtained from the generic filter.

By a density argument, there is a $p \in G$ such that $(v, f) \in \Phi_p$. Thus if $q \leq p$ in $P(\mathcal{X})$ the set

$$Y(v, f, q, p) = \{\alpha \in \operatorname{dom} f \colon (\forall i < n)[[q]_i \setminus [p]_i \subseteq \mathcal{B}(X_i, \alpha, f(\alpha))]\}$$

is uncountable.

Claim 3.18: *In $V[G]$, if A_i' is a countable subset of $A_i \setminus [p]_i$ for each $i < n$, then*

$$\{\alpha \in \operatorname{dom} f \colon (\forall i < n)[A_i' \subseteq \mathcal{B}(X_i, \alpha, f(\alpha))]\}$$

is uncountable.

Proof: Since G is countably closed, there is a $q \leq p$ in G such that $A_i' \subseteq [q]_i \setminus [p]_i$ for all $i < n$. Now we apply the fact that $Y(v, f, q, p)$ is uncountable. ∎

Now back in V, our assumption is that $\mathrm{Ban}(w, f)$ is $\mathcal{H}_w$-large. Since $\mathcal{H}_w$ is $\aleph_1$-complete and generated by $\aleph_1$ sets, we can choose

$$Z \colon = \{(x_i^\xi \colon i \in w) \colon \xi < \omega_1\} \subseteq \mathrm{Ban}(w, f)$$

diagonalizing $\mathcal{H}_w$. By passing to a subsequence, we may assume that

$$\xi_0 \neq \xi_1 \Rightarrow x_i^{\xi_0} \neq x_i^{\xi_1}$$

for all $i \leq n$. Note also that

- $\{(x_i^\xi \colon i < n) \colon \xi < \omega_1\}$ diagonalizes $\mathcal{H}_v$,
- $\{x_n^\xi \colon \xi < \omega_1\}$ diagonalizes $\mathcal{H}_X$.

CLAIM 3.19: *In $V[G]$, $I = \{\xi < \omega_1 : (\forall i < n) x_i^\xi \in A_i\}$ is uncountable.*

Proof: This will follow by an easy density argument in V. Given $\xi_0 < \omega_1$, the set $\{(x_i^\xi : i < n) : \xi \geq \xi_0\}$ still diagonalizes $\mathcal{H}_v$, so in particular it is $\mathcal{H}_v$-large. Now Corollary 3.17 tells us that the set of conditions forcing the existence of a $\xi > \xi_0$ such that $(\forall i < n)[x_i^\xi \in [q]_i]$ is dense in $P(\mathcal{X})$, hence G contains such a condition. ∎

Since I is uncountable, in $V[G]$ the set $\{x_n^\xi : \xi \in I\}$ will diagonalize $\mathcal{H}_X$.

CLAIM 3.20: *In $V[G]$, if $I_0 \subseteq I$ is countable, then $\mathrm{cl}_X\{x_n^\xi : \xi \in I_0\}$ is disjoint to a set in $\mathcal{H}_X$.*

Proof: Suppose this fails, so there is a countable $I_0 \subseteq I$ witnessing it. Note that $\{x_n^\xi : \xi \in I_0\} \in V$ as $P(\mathcal{X})$ is totally proper, and also that the closure of this set is the same whether computed in V or $V[G]$. In particular, by the maximality of $\mathcal{H}_X$ in V, all but countably many $\alpha < \omega_1$ are in $\mathrm{cl}_X\{x_n^\xi : \xi \in I_0\}$. For $i < n$, we define

$$A_i' = \{x_i^\xi : \xi \in I_0\},$$

and by Claim 3.18, the set

$$B = \{\alpha \in \mathrm{dom}\, f : (\forall i < n)[A_i' \subseteq \mathcal{B}(X_i, \alpha, f(\alpha))]\}$$

is uncountable. By throwing away a countable subset of B, we can assume that for all $\alpha \in B$, there is a $\xi \in I_0$ such that $x_n^\xi \in \mathcal{B}(X_n, \alpha, f(\alpha))$. Thus there is a single $\xi \in I_0$ for which

$$\{\alpha \in B : x_n^\xi \in \mathcal{B}(X_n, \alpha, f(\alpha))\}$$

is uncountable. Now this contradicts the fact that $(x_i^\xi : i \leq n)$ is in $\mathrm{Ban}(w, f)$ ∎

We have shown that in $V[G]$, there is a set that diagonalizes $\mathcal{H}_X$ with the property that the closure of every countable subset is disjoint to a set in $\mathcal{H}_X$. Now Proposition 3.2 tells is that X has an uncountable free sequence. ∎

THEOREM 5: *If $\mathcal{X}$ is a safe collection of relevant spaces, then $P(\mathcal{X})$ satisfies the $\aleph_2$-p.i.c.*

Proof: Let i, j, N_i, N_j, h, and p be as in Definition 1.3. Just as in the previous section, if $r \in N_i \cap P(\mathcal{X})$, we define

$$r \cup h(r) := (a_r \cup h(a_r), \Phi_r \cup h(\Phi_r)).$$

LEMMA 3.21: *Assume that $r \in N_i \cap P(\mathcal{X})$.*
(1) *$r \cup h(r) \in P(\mathcal{X})$,*
(2) *$r \cup h(r)$ extends both r and $h(r)$,*
(3) *if $s \in N_i \cap P(\mathcal{X})$ and $r \leq s$, then $r \cup h(r) \leq s \cup h(s)$.*

Proof: The proof is essentially the same as the one for Lemma 2.7. ∎

Just as in the proof of Theorem 1, it suffices to produce an $(N_i, P(\mathcal{X}))$-generic sequence $\{p_n : n \in \omega\}$ (with $p_0 = p$) such that $\{p_n \cup h(p_n) : n \in \omega\}$ has a lower bound.

Let $\{D_n : n \in \omega\}$ list the dense open subsets of $P(\mathcal{X})$ that are members of N_i. Let $\delta = N_i \cap \aleph_1 = N_j \cap \aleph_1$, and let $\{\gamma_n : n < \omega\}$ enumerate $N_i \cap \kappa$. Also fix a sequence $\{\delta_n : n \in \omega\}$ strictly increasing and cofinal in δ such that for each $\xi \in (N_i \cup N_j) \cap \kappa$, the sequence $\{\delta_n : n \in \omega\}$ converges in X_ξ to a point z_ξ.

CLAIM 3.22: *If $v \subseteq N_i \cap \kappa$ is finite and $f \in N_i$ is a promise, then $(z_\xi : \xi \in v)$ is not banned by (v, f). The same holds with N_i replaced by N_j.*

For $\xi \in (N_i \cup N_j) \cap \kappa$, let $\{V(z_\xi, n) : n \in \omega\}$ be a decreasing neighborhood base for z_ξ in X_ξ, with $\mathrm{cl}_{X_\xi} V(z_\xi, 0) \notin \mathcal{H}_\xi$. We will define p_n, q_n, u_n, and $g \in {}^\omega\omega$ such that
(1) $p_0 = p$, $q_0 = p_0 \cup h(p_0)$, $u_0 = \emptyset$, $g(0) = 0$,
(2) $p_{n+1} \leq p_n$,
(3) $p_{n+1} \in N_i \cap D_n$,
(4) $q_n = p_n \cup h(p_n)$,
(5) $u_n \subseteq N_i \cap \kappa$ is finite,
(6) $u_{n+1} \supseteq u_n$,
(7) $\{\gamma_m : m < n\} \subseteq u_n$,
(8) $g(n+1) > g(n)$,
(9) for $\gamma \in u_{n+1} \cup h(u_{n+1})$, $[q_{n+1}]_\gamma \setminus [q_n]_\gamma \subseteq V(z_\gamma, g(n+1))$,
(10) if $(v, f) \in \Phi_{q_k}$ for some k, then there is a stage $n \geq k$ for which

$$v \subseteq u_{n+1} \cup h(u_{n+1})$$

and

$$\{\alpha \in Y(v, f, q_n, q_k) : (\forall \xi \in v)[V(z_\xi, g(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))\}$$

is uncountable.

Fix a bookkeeping procedure as in the proof of Theorem 1. At stage $n+1$ we will be handed p_n, q_n, u_n, $g \restriction n+1$, and $(v, f) \in \Phi_{q_k}$ for some $k \leq n$.

Choose $u_{n+1} \subseteq N_i \cap \kappa$ finite with $u_n \cup \{\gamma_n\} \subseteq u_n$ and $v \subseteq u_{n+1} \cup h(u_{n+1})$. To define $g(n+1)$, we need to split into cases depending on whether (v, f) comes from p_k or $h(p_k)$.

CASE 1: $(v, f) \in N_i$

Note that $Y(v, f, q_n, q_k) = Y(v, f, p_n, p_k)$, so $f' = f \restriction Y(v, f, p_n, p_k)$ is a promise in N_i. We know $(z_\xi \colon \xi \in v)$ is not banned by (v, f'), hence there is a value $g(n+1) > g(n)$ large enough such that

$$\{\alpha \in \operatorname{dom} f' \colon (\forall \xi \in v)[V(z_\xi, g(n+1)) \subseteq \mathcal{B}(X_\xi, \alpha, f(\alpha))]\}$$

is uncountable.

CASE 2: $(v, f) \in N_j \setminus N_i$

This case is analogous — we use the fact that

$$Y(v, f, q_n, q_k) = Y(v, f, h(p_n), h(p_k)) \in N_j.$$

In either case, we have ensured that condition (10) of our construction is satisfied for (v, f).

Now choose $\ell < \omega$ large enough so that

$$\delta_\ell \notin \operatorname{Bad}(u_{n+1}, p_n, D_n)$$

and

$$(\forall \xi \in u_{n+1} \cup h(u_{n+1}))[\delta_\ell \in V(z_\xi, g(n+1))].$$

Choose m large enough so that

$$(\forall u_{n+1} \cup h(u_{n+1}))[\mathcal{B}(X_\xi, \delta_\ell, m) \subseteq V(z_\xi, g(n+1))].$$

In N_i, apply the definition of $\delta_\ell \notin \operatorname{Bad}(u_{n+1}, p_n, D_n)$ to get $p_{n+1} \leq p_n$ in $N_i \cap D_n$ such that

$$(\forall \xi \in u_{n+1})([p_{n+1}]_\xi \setminus [p_n]_\xi \subseteq \mathcal{B}(X_\xi, \delta_\ell, m)).$$

Applying the isomorphism h tells us that

$$(\forall \xi \in h(u_{n+1}))([h(p_{n+1})]_\xi \setminus [h(p_n)]_\xi \subseteq \mathcal{B}(X_\xi, \delta_\ell, m)).$$

The choice of m, together with (3) and (4), tells us

$$(\forall \xi \in u_{n+1} \cup h(u_{n+1}))([q_{n+1}]_\xi \setminus [q_n]_\xi \subseteq V(z_\xi, g(n+1))).$$

Thus we have achieved everything required of us at stage $n+1$. The verification that $\{q_n \colon n \in \omega\}$ has a lower bound proceeds just as in the proof of Theorem 3. ∎

Conclusion 2: Assume CH holds. There is a totally proper notion of forcing $P(\mathcal{X})$, satisfying the $\aleph_2$-p.i.c., such that every relevant space in the ground model acquires an uncountable free sequence in the generic extension.

4. The iteration

We now construct a model of ZFC in which $2^{\aleph_0} < 2^{\aleph_1}$ and there are no locally compact first countable S-spaces. Starting with a ground model V satisfying $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_{17}$, we will do a countable support iteration of length ω_2.

More specifically, let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \colon \alpha < \omega_2 \rangle$ be a countable support iteration defined by

- P_0 is the trivial poset,
- if $\alpha = \beta + 1$, then $V^{P_\alpha} \models \dot{Q}_\alpha$ is Laver forcing,
- if α is a limit ordinal, then $V^{P_\alpha} \models \dot{Q}_\alpha = \dot{P}(\mathbb{I}) * \dot{P}(\mathcal{X})$, where

$$V^{P_\alpha} \models \mathbb{I} \text{ is the collection of all P-ideals in } [\omega_1]^{\aleph_0},$$

and

$$V^{P_\alpha * \dot{P}(\mathbb{I})} \models \dot{\mathcal{X}} \text{ is a maximal safe family of relevant spaces.}$$

We don't actually use much about Laver forcing; the relevant facts we need are that it is proper, assuming CH it satisfies the $\aleph_2$-p.i.c. (Lemma VIII.2.5 of [11]), and it adds a real $r \in {}^\omega\omega$ that eventually majorizes every real in the ground model.

The point of using the partial orders from sections 2 and 3 is that they can handle all "candidates" from a given ground model, instead of just one at a time. This means that in ω_2 stages we can catch our tail, even though there are $\aleph_{17}$ "candidates" to worry about at each stage of the iteration.

Having defined our iteration, we arrive at the main theorem of this paper.

THEOREM 6: *In the model $V[G_{\omega_2}]$, there are no locally compact first countable S-spaces, and $2^{\aleph_0} < 2^{\aleph_1}$. More generally, every locally compact first countable space of countable spread is hereditarily Lindelöf.*

The rest of this section will comprise the proof of this theorem. Standard facts about $\aleph_2$-p.i.c. iterations make it easy to show by induction on $\alpha < \omega_2$ that the following hold:

$$P_\alpha \text{ has the } \aleph_2\text{-p.i.c.},$$
$$V^{P_\alpha} \models \text{CH},$$

and

$$V^{P_\alpha} \models \dot{Q}_\alpha \text{ has the } \aleph_2\text{-p.i.c.}$$

Statement (4) is conclusion 1 of Lemma VIII.2.4 of [11], while statement (4) follows from statement (4) using Claim VIII.2.9 of [11]. Statement (4) follows from our previous work, although we should point out that we need (4) in order for this work to apply.

Conclusion 2 of Lemma VIII.2.4 of [11] together with (4) imply

$$P_{\omega_2} \text{ satisfies the } \aleph_2\text{-chain condition.}$$

We should point out that (4) does not claim that P_{ω_2} satisfies the $\aleph_2$-p.i.c. — the $\aleph_2$-p.i.c. is only preserved for iterations of length $< \omega_2$.

Note that (4) together with the fact that we are adding many Laver reals in the iteration implies

$$V^{P_{\omega_2}} \models \mathfrak{b} = 2^{\aleph_0} = \aleph_2 \quad \text{and} \quad 2^{\aleph_1} = \aleph_{17}.$$

Thus the cardinal arithmetic in $V^{P_{\omega_2}}$ is as advertised, and we need only verify that every locally compact 1st countable space of countable spread is hereditarily Lindelöf in $V[G_{\omega_2}]$. We first reduce our task by showing that it suffices to consider only X with a certain form.

CLAIM 4.1: *If Z is a locally compact space of countable spread which is not hereditarily Lindelöf, then there are X, Y, and $\{U_\alpha: \alpha < \omega_1\}$ such that*

- *X is a locally compact non-Lindelöf subspace of Z,*
- *$Y \subseteq X$ is right separated in type ω_1, witnessed by open sets $\{U_\alpha: \alpha < \omega_1\}$,*
- *$X = \bigcup_{\alpha<\omega_1} U_\alpha$,*
- *$X = \operatorname{cl} Y$,*
- *the Lindelöf degree of X is exactly $\aleph_1$, i.e., $\ell(X) = \aleph_1$.*

Proof: By a basic lemma [10], Z has a right-separated subspace Y of cardinality $\aleph_1$, $\{y_\alpha: \alpha < \omega_1\}$, and any such subspace is hereditarily separable because Z is of countable spread. For each y_α pick an open neighborhood W_α whose closure is compact and misses all the later y_β. Every locally compact space is Tychonoff, so for each α there is a cozero-set neighborhood V_α of y_α inside W_α. Let $V = \bigcup\{V_\alpha: \alpha \in \omega_1\}$. Then V is locally compact, and it is not Lindelöf because each V_α contains only countably many y_α. In fact, $\ell(V) = \aleph_1$ because we carefully took the union of the V_α instead of the union of the W_α, and each V_α is sigma-compact. Now it is clear that $X = \operatorname{cl}_V Y$ is as desired. ∎

We work now in the model $V[G_{\omega_2}]$ and assume for purposes of contradiction that Z is a locally compact first countable space of countable spread which is not Lindelöf. Let X and Y be as in the previous claim. For each $y_\alpha \in Y$, we choose a neighborhood V_α such that $\operatorname{cl} V_\alpha$ is a compact subset of U_α. Let $A_\alpha = V_\alpha \cap Y \in [\omega_1]^{\aleph_0}$.

CLAIM 4.2: *X satisfies Property D, i.e., every countable closed discrete subset of X expands to a discrete collection of open sets.*

Proof: This follows from the general result that every 1st countable regular space X satisfying $\ell(X) < \mathfrak{b}$ satisfies Property D. The proof of this is only a minor modification of the proof of [15, 12.2] which was for $|X| < \mathfrak{b}$ because van Douwen could not find any use for the added generality given by $\ell(X) < \mathfrak{b}$. However, for the sake of self-containment we give the proof of this result here.

Let $\ell(X) < \mathfrak{b}$ and let $D = \{x_n : n \in \omega\}$ be a countable closed discrete subspace of X. Using regularity, let $\{U_n : n \in \omega\}$ be a family of disjoint open sets such that $x_n \in U_m$ if and only if $x_n = x_m$. For each n let $\{B_i^n : i \in \omega\}$ be a decreasing local base at x_n such that $B_0^n \subset U_n$. Let $U = \bigcup\{U_n : n \in \omega\}$ and for each $y \in X \setminus U$ let V_y be an open neighborhood of y whose closure misses D, and let $f_y : \omega \to \omega$ be such that $B^n_{f_y(n)}$ has closure missing V_y for all n. Since $X \setminus U$ has Lindelöf degree $< \mathfrak{b}$, we can find $\{y_\alpha : \alpha < \kappa\}$, $(\kappa < \mathfrak{b})$ such that $\{V_{y_\alpha} : \alpha < \kappa\}$ covers $X \setminus U$. Using the definition of $\mathfrak{b}$, let $f : \omega \to \omega$ be such that $f_{y_\alpha} <^* f$ for all α. In other words, there exists $k \in \omega$ such that $f_{y_\alpha}(n) < f(n)$ for all $n \geq k$. We then have all of $X \setminus U$ covered by open sets each of which meets at most finitely many of the sets $B^n_{f(n)}$, which is thus a locally finite collection of disjoint open sets. Hence it is a discrete open expansion of D, as desired. ∎

Our assumptions on X imply that $|X| \leq \omega_2$ — every point in X is the limit of a sequence from Y. We will assume that in fact $|X| = \aleph_2$ (this is the difficult case) and that the underlying set of X is ω_2, with $Y = \omega_1 \subseteq X$.

Since X is first countable, we have that $w(X) \leq \aleph_2$, so let $\mathcal{B} = \{W_\xi : \xi < \omega_2\}$ be a base for X. For technical reasons, we assume $W_\xi = U_\xi$ for $\xi < \omega_1$ (here U_ξ is as in Claim 4.1) with repetitions allowed in the case $w(X) = \aleph_1$. Let $\dot{\mathcal{B}}$ be a P_{ω_2}-name for $\mathcal{B}$, and let N be an elementary submodel of $V[G_{\omega_2}]$'s version of $H(\lambda)$ satisfying

- $|N| = \aleph_1$,
- X, $\mathbb{P}$, $\mathcal{B}$, $\dot{\mathcal{B}}$, $\{U_\xi : \xi < \omega_1\}$, $\{V_\xi : \xi < \omega_1\}$, and G_{ω_2} are in N,
- $N \cap \omega_2 = \alpha$ for some $\alpha < \omega_2$.

(The set of such N is closed and unbounded in $[H(\lambda)]^{\aleph_1}$.)

For an ordinal $\beta < \omega_2$, define $\mathcal{B}_\beta$: $= \{W_\xi \cap \beta \colon \xi < \beta\}$.

CLAIM 4.3: *With α as above,*

(1) *$\mathcal{B}_\alpha$ is a base for the topology on α as a subspace of X,*

(2) *$\mathcal{B}_\alpha \in V[G_\alpha]$.*

Proof: (1) Suppose $\beta < \alpha$ and $U \subseteq X$ is a neighborhood of β. Since X is first countable and $\beta \in N$, there is a neighborhood U' of β such that $U' \in N$ and $U' \subseteq U$. Now

$$N \models (\exists \gamma < \omega_2)[\beta \in W_\gamma \wedge W_\gamma \subseteq U'].$$

Thus there is such a $\gamma < \alpha$ and we are done.

(2) For each pair $\bar{\beta} = (\beta_0, \beta_1) \in \alpha$, there is a condition $p_{\bar{\beta}} \in G_{\omega_2}$ that decides whether or not $\beta_1 \in W_{\beta_0}$, hence there is such a condition in N. Now the support of $p_{\bar{\beta}}$ is a countable subset of ω_2 that is in N, hence there is a $\gamma < \alpha$ with the support of $p_{\bar{\beta}}$ a subset of γ. This means to decide whether or not β_1 is in W_{β_0}, we need only $\dot{\mathcal{B}}$ and $G_{\omega_2} \restriction P_\gamma = G_\gamma$. Thus $\mathcal{B}_\alpha$ can be recovered from $\dot{\mathcal{B}}$ and the sequence $\langle G_\gamma \colon \gamma < \alpha \rangle$, both of which are in $V[G_\alpha]$. ∎

Now let $\mathfrak{N} = \langle N_\xi \colon \xi < \omega_2 \rangle$ be a continuous, increasing $\in$-chain of elementary submodels of $H(\lambda)$ such that

- each N_ξ is as in the previous discussion,
- $\langle N_\zeta \colon \zeta < \xi \rangle \in N_{\zeta+1}$,
- $[\omega_2]^{\aleph_0} \subseteq \bigcup_{\xi < \omega_2} N_\xi$.

Now we define a function $F \colon \omega_2 \to \omega_2$ by letting $F(\xi)$ equal the least ζ such that

$$V[G_\xi] \cap [\xi]^{\aleph_0} \subseteq N_\zeta$$

and

$$N_\xi \cap [\xi]^{\aleph_0} \subseteq V[G_\zeta].$$

Note that since both $V[G_\xi] \cap [\xi]^{\aleph_0}$ and $N_\xi \cap [\xi]^{\aleph_0}$ have cardinality at most $\aleph_1$, the function F is defined for all $\xi < \omega_2$.

CLAIM 4.4: *Suppose $\gamma < \omega_2$ has cofinality $\aleph_1$ and is closed under the function F. Then $N_\gamma \cap [\gamma]^{\aleph_0} = V[G_\gamma] \cap [\gamma]^{\aleph_0}$.*

Proof: Suppose first that $A \in [\gamma]^{\aleph_0} \cap V[G_\gamma]$. Then there is a β such that $\sup A < \beta < \gamma$ and $A \in V[G_\beta]$. Now $F(\beta) < \gamma$ and $A \in N_{F(\beta)} \cap [\beta]^{\aleph_0} \subseteq N_\gamma \cap [\gamma]^{\aleph_0}$.

Conversely, suppose $A \in [\gamma]^{\aleph_0} \cap N_\gamma$. Since $\sup A < \gamma$ and γ is a limit ordinal, there is a $\beta > \sup A$ below γ with $A \in N_\beta$. Then $A \in V[G_{F(\beta)}] \cap [\beta]^{\aleph_0} \subseteq V[G_\gamma] \cap [\gamma]^{\aleph_0}$. ∎

Let $\alpha_0 < \omega_2$ be large enough that $\{A_\xi\colon \xi < \omega_1\} \in V[G_{\alpha_0}]$ (the A_ξ's were defined right before Claim 4.2), and let $\alpha < \omega_2$ satisfy

(1) $\alpha > \alpha_0$,

(2) $\mathrm{cf}(\alpha) = \aleph_1$,

(3) $N_\alpha \cap \omega_2 = \alpha$,

(4) $N_\alpha \cap [\alpha]^{\aleph_0} = V[G_\alpha] \cap [\alpha]^{\aleph_0}$.

Such an α can be found by using the preceding claim, as the set of ordinals satisfying (3) is closed unbounded in ω_2.

CLAIM 4.5: *$V[G_\alpha] \models \mathcal{I}\colon = \{B \in [\omega_1]^{\aleph_0}\colon |A_\xi \cap B| < \aleph_0$ for all $\xi < \omega_1\}$ is a P-ideal.*

Proof: Clearly $\mathcal{I}$ is an ideal (and in $V[G_\alpha]$). Let $\{B_n\colon n \in \omega\} \subseteq \mathcal{I}$ be given; without loss of generality the B_n's are pairwise disjoint. Let $h_n\colon \omega \to \omega_1$ be an enumeration of B_n.

Since $\mathrm{cf}(\alpha) = \aleph_1$, there is a β in the interval (α_0, α) such that $\{h_n\colon n \in \omega\} \in V[G_\beta]$. For each $\xi < \omega_1$, define a function $f_\xi \in {}^\omega\omega$ by

$$f_\xi(n) = 1 + \max\{m\colon h_n(m) \in A_\xi\}.$$

Since $\alpha_0 < \beta$, each f_ξ is in $V[G_\beta]$. Now in $V[G_\alpha]$ there is an $r \in {}^\omega\omega$ dominating $\{f_\xi\colon \xi < \omega_1\}$ — r can be taken to be the Laver real added at stage $\beta + 2 < \alpha$. Now let

$$B\colon = \bigcup_{n\in\omega} B_n \setminus \{h_n(m)\colon m \leq r(n)\}.$$

Clearly $B \in \mathcal{I}$ and $B_n \subseteq^* B$ for all $n \in \omega$. ∎

Now let X_α be the topological space with underlying set α and base given by $\mathcal{B}_\alpha$.

Claim 4.3 tells us that $X_\alpha \in V[G_\alpha]$, and that in $V[G_{\omega_2}]$, X_α is a subspace of X. We will use this implicitly throughout the remainder of the section.

CLAIM 4.6:

(1) *If $A \in V[G_\alpha] \cap [X_\alpha]^{\aleph_0}$ has a limit point in X, then A has a limit point in X_α.*

(2) *$V[G_\alpha] \models X_\alpha$ has Property D.*

Proof: (1) Suppose $A \in V[G_\alpha] \cap [X_\alpha]^{\aleph_0}$ has a limit point in X. Our choice of α and Claim 4.4 together imply that $A \in N_\alpha$, and hence there is a limit point of A in N_α. This gives us the required limit point for A in X_α.

(2) Suppose $D = \{x_n: n \in \omega\}$ is a closed discrete subset of X_α in $V[G_\alpha]$. By the first part of the Claim, D is a closed discrete subset of X, and by Claim 4.4 we know that $D \in N_\alpha$. Since X satisfies Property D, D expands to a discrete collection of open sets, without loss of generality members of our fixed base $\mathcal{B}$. Since $D \in N_\alpha$, there is such an expansion in N_α. Now the countable subset of ω_2 that indexes this cover is in $N_\alpha \cap [\alpha]^{\aleph_0}$, hence it is in $V[G_\alpha]$ as well. This gives us the required discrete family of open sets in $V[G_\alpha]$. ∎

Our goal is to show that in $V[G_{\alpha+1}]$, X_α acquires an uncountable discrete subset. Since X_α is a subspace of X in $V[G_{\omega_2}]$, if we attain our goal we will have a contradiction, proving that such a space X does not exist in $V[G_{\omega_2}]$.

We work for a bit in $V[G_\alpha]$. The first thing we do is force with $P(\mathbb{I})$, where $\mathbb{I}$ lists all the P-ideals in $V[G_\alpha]$. If H_0 is a generic subset of $P(\mathbb{I})$, then in $V[G_\alpha][H_0]$, either there is an uncountable $B \subseteq \omega_1$ with $[B]^{\aleph_0} \subseteq \mathcal{I}$, or there is an uncountable $B \subseteq \omega_1$ with $[B]^{\aleph_0} \cap \mathcal{I} = \emptyset$.

Let us suppose the first possibility occurs. This means that every countable subset of B has finite intersection with every A_ξ (in $V[G_\alpha][H_0]$). This continues to hold in $V[G_{\omega_2}]$, so in $V[G_{\omega_2}]$ there is an uncountable $B \subseteq Y$ that meets each V_ξ at most finitely often, i.e., B has no limit points in Y. Thus B is a discrete subspace of $Y \subseteq X_\alpha$, and we achieve our goal and reach a contradiction.

Now suppose the second possibility occurs. This means that in $V[G_\alpha][H_0]$, there is an uncountable B such that every countably infinite subset of B meets some A_ξ in an infinite set.

Claim 4.7: *$V[G_\alpha][H_0] \models Z: = \mathrm{cl}_{X_\alpha} B$ is countably compact and non-compact.*

Proof: First note that any countable subset of Z from $V[G_\alpha][H_0]$ is in $V[G_\alpha]$, as $P(\mathbb{I})$ is totally proper. Given $B_0 \in [B]^{\aleph_0}$, there is a $\xi < \omega_1$ such that $B_1 = B_0 \cap A_\xi$ is infinite.

Now step into the model $V[G_{\omega_2}]$. Since $B_1 \subseteq A_\xi \subseteq V_\xi$ and $\mathrm{cl}\, V_\xi$ is compact, B_0 has a limit point. Since B_0 is in the model $V[G_\alpha]$, our choice of α implies that B_0 has a limit point in X_α.

Now X_α has Property D in $V[G_\alpha]$, and since no new countable subsets of X_α appear in $V[G_\alpha][H_0]$, X_α has Property D in this model as well.

This means any alleged infinite closed discrete subset of $\mathrm{cl}_{X_\alpha} B$ (in $V[G_\alpha][H_0]$) would expand to a discrete collection of open sets, thereby yielding an infinite subset of B with no limit point in X_α. We have already argued that this is impossible. Thus

$$V[G_\alpha][H_0] \models \mathrm{cl}_{X_\alpha} B \text{ is countably compact.}$$

Now the open cover $\{X_\alpha \cap U_\xi : \xi < \omega_1\}$ of X_α is in $V[G_\alpha]$ (here we use another assumption we made about $\mathcal{B}$), and each of these sets meets B at most countably often, and so $\mathrm{cl}_{X_\alpha} B$ is not compact. ∎

If it happens that Z contains an uncountable discrete subset, then we are done, so we may assume this does not happen. In particular, we may assume that Z contains no uncountable free sequence. By virtue of the preceding claim, this means that Z is a relevant space (terminology from the last section) in $V[G_\alpha][H_0]$.

The next thing we do in our iteration is to force with $P(\mathcal{X})$, where

$$V[G_\alpha][H_0] \models \mathcal{X} \text{ is a maximal safe collection of relevant spaces.}$$

The results of the preceding section tell us that Z acquires an uncountable discrete subset after we do this forcing. Thus

$$V[G_{\alpha+1}] \models X_\alpha \text{ has an uncountable discrete subset}$$

and again we have achieved our goal, reaching a contradiction. Thus every first countable locally compact space of countable spread is hereditarily Lindelöf; in particular, there are no locally compact first countable S-spaces in $V[G_{\omega_2}]$ and Theorem 6 is established.

Theorem 6 is reminiscent of the theorem of Szentmiklóssy recounted in [10] that $\mathrm{MA}(\omega_1)$ implies that no compact space of countable tightness can contain an S-space or an L-space.

Every compact space of countable spread is of countable tightness, and if a locally compact space is of countable spread, so is its one-point compactification. So our result may be looked upon as a mild version of one half of Szentmiklóssy's theorem [12] for models of $2^{\aleph_0} < 2^{\aleph_1}$. It would be very nice if we could get even a similarly mild version of the other half—it would settle a famous fifty-year-old problem of Katětov [7]:

Problem: If a compact space has hereditarily normal ("T_5") square, must it be metrizable?

The second author showed that the answer is negative if there is a Q-set, so that in particular $\mathrm{MA}(\omega_1)$ implies a negative answer. Gary Gruenhage showed that CH also implies a negative answer. Proofs appeared in [5] along with a theorem connecting Katětov's problem with the theory of S and L spaces:

THEOREM 7: *If there does not exist a Q-set, and X is a compact nonmetrizable space with T_5 square, then at least one of the following is true:*

(1) *X is an L-space,*
(2) *X^2 is an S-space,*
(3) *X^2 is of countable spread, and contains both an S-space and an L-space.*

Parts (2) and (3) are ruled out in our model because of Katětov's theorem that every compact space with T_5 square is perfectly normal, hence first countable. If it could be shown that there are no compact L-spaces (which are automatically first countable) in our model, then Katětov's fifty-year-old problem would be fully solved. It is not out of the question that first countable compact L-spaces can be gently killed, so that even if some of these spaces exist in this model, we can maybe throw in a few more notions of forcing to explicitly banish them.

There is a tantalizing sort of duality between our model and the model obtained by adding $\aleph_2$ random reals to a model of MA $+\mathfrak{c} = \aleph_2$. There, too, there are no Q-sets (even though $2^{\aleph_0} = 2^{\aleph_1}$); but there, it is L-subspaces of compact spaces of countable spread that do not exist (see [14]), so that (1) and (3) are ruled out there, and it is the status of locally compact first countable S-spaces that is unknown.

If neither of these models works out, it is to be hoped that the techniques we have introduced in this paper will some day produce a model that does settle Katětov's problem.

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