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On partitioning the triples of a topological space

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Abstract

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We prove that it is consistent with GCH (and in fact true in L) that there is a 0-dimensional T_2 topological space X of cardinality $\aleph_3$ such that every partition of the triples of X into countably many pieces has a nondiscrete homogeneous set.

Keywords: Partitions, nondiscrete homogeneous set.

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In [1] it was shown to be consistent with GCH that for every T_2 space X there is a partition of the triples of X into countably many pieces such that every countable homogeneous subset of X is discrete, moreover Weiss has shown in [3] that, under similar assumptions, the quadruples of X may be colored with countably many colors so that *every* homogeneous set be discrete. In fact, no additional set-theoretic assumptions beyond GCH are needed if $|X| \leqslant \aleph_\omega$.

In this paper we propose to show that both of these results are sharp because we show the consistency with GCH (and the assumptions of these results) of the existence of a 0-dimensional T_2 space X of cardinality $\aleph_3$ such that every partition of the triples of X into countably many pieces does have a nondiscrete homogeneous set. Of course, such a set must in general be uncountable!

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We shall produce this space in a generic extension of a ground model V satisfying GCH. Our notion of forcing P will be ω_2-closed and will have the ω_3-CC, hence cardinals will be preserved, moreover we have $|P|=\omega_3$ and this implies that GCH will also be preserved.

Forcing with P will yield a generic map $F:\omega_2\times\omega_3\to 2$ that determines a 0-dimensional T_2 topology τ_F on ω_3, namely the topology generated by the sets A^i_ν with $\nu\in\omega_2$ and $i\in 2$, where

$$A^i_\nu=\{\alpha\in\omega_3\colon F(\nu,\alpha)=i\}.$$

Thus a base for the topology τ_F is formed by the sets

$$B_\varepsilon=\bigcap\{A^{\varepsilon(\nu)}_\nu\colon \nu\in D(\varepsilon)\},$$

where $\varepsilon\in\mathrm{Fn}(\omega_2,2)$, i.e., ε is any finite function from ω_2 to 2.

The elements of P will be approximations to F of size $\leqslant\aleph_1$ with certain "side conditions" that intend to fix limit points of certain sets in the eventual topology τ_F. For convenience, we choose the approximations to be of the form $f:\nu\times A\to 2$ where $\nu\in\omega_2$ and $A\in[\omega_3]^{\leqslant\omega_1}$. We denote by Q the set of all such f. Note that since f is the trace of F on $\nu\times A$, we "know" the sets $A^i_\mu\cap A$ for $\mu\in\nu$ and therefore the sets $B_\varepsilon\cap A=B^f_\varepsilon$ for $\varepsilon\in\mathrm{Fn}(\nu,2)$.

In what follows we denote by $[A]^{\omega_1+1}$ the collection of all subsets of A of order type ω_1+1, for any $A\subset\omega_3$. Moreover, for $s\in[A]^{\omega_1+1}$ we let $\alpha(s)$ denote the maximal (i.e., ω_1st) element of s.

Now, the notion of forcing P is defined as the set of all pairs $\langle f,\Gamma\rangle$, where $f\in Q$, $D(f)=\nu\times A$, $\Gamma\subset[A]^{\omega_1+1}$ with $|\Gamma|\leqslant\aleph_1$, satisfying the following condition (*):

$$\text{For each } s\in\Gamma \text{ and } \varepsilon\in\mathrm{Fn}(\nu,2),\text{ if } \alpha(s)\in B^f_\varepsilon,\text{ then } |s\cap B^f_\varepsilon|=\aleph_1. \tag{$*$}$$

Of course, we set $\langle f,\Gamma\rangle\leqslant\langle f',\Gamma'\rangle$ iff $f\supset f'$ and $\Gamma\supset\Gamma'$. We can now formulate our result.

Theorem. *Let $V\models$ GCH, G be P-generic over V, and put in $V[G]$*

$$F=\bigcup\{f\colon \exists\Gamma\,(\langle f,\Gamma\rangle\in G)\}.$$

Then $F:\omega_2\times\omega_3\to 2$ and the topology τ_F on ω_3 is 0-dimensional T_2 with the property that every coloring of $[\omega_3]^3$ with countably many colors has a homogeneous set that is not discrete in τ_F.

Proof. If $p\in P$ is a condition, we are going to write $p=\langle f^p,\Gamma^p\rangle$ and $D(f^p)=\nu_p\times A^p$. Also, instead of $B^{f^p}_\varepsilon$ we just write B^p_ε.

Let us start by showing that P is ω_2-closed. Indeed, let $\{\langle f_\xi,\Gamma_\xi\rangle=p_\xi\colon \xi\in\omega_1\}$ be a collection of conditions with $p_\xi\leqslant p_\zeta$ for $\xi>\zeta$ and put

$$f=\bigcup\{f_\xi\colon \xi\in\omega_1\},\qquad \Gamma=\bigcup\{\Gamma_\xi\colon \xi\in\omega_1\}.$$

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We only have to show that $\langle f, \Gamma\rangle \in P$, i.e., satisfies $(*)$. But if $\varepsilon \in \mathrm{Fn}(\nu, 2)$ (where $\nu = \bigcup_{\xi\in\omega_1} \nu_\xi$) and $s \in \Gamma$, then there is a $\xi \in \omega_1$ with both $\varepsilon \in \mathrm{Fn}(\nu_\xi, 2)$ and $s \in \Gamma_\xi$. Now if $\alpha(s) \in B^f_\varepsilon$, then $\alpha(s) \in B^{f_\xi}_\varepsilon$ as well, hence by $B^f_\varepsilon \supset B^{f_\xi}_\varepsilon$, $\langle f_\xi, \Gamma_\xi\rangle \in P$ implies

$$|s \cap B^f_\varepsilon| = |s \cap B^{f_\xi}_\varepsilon| = \aleph_1.$$

Next we show that P has the ω_3-CC. Indeed, it follows from GCH by a standard Δ-system and counting argument that among any $\aleph_3$ elements of P we can always find two of the form $\langle f, \Gamma\rangle$ and $\langle f', \Gamma'\rangle$ with $D(f) = \nu \times A$ and $D(f') = \nu \times A'$, such that $f\restriction \nu \times (A \cap A') = f'\restriction \nu \times (A \cap A')$, and $A \cap A' < A\setminus A' < A'\setminus A$. In this case $\langle f \cup f', \Gamma \cup \Gamma'\rangle$ is a common extension of $\langle f, \Gamma\rangle$ and $\langle f', \Gamma'\rangle$, because then $s \in \Gamma \cup \Gamma'$ and $\alpha(s) \in A \cap A'$ implies $s \subset A \cap A'$ and this clearly assures that $\langle f \cup f', \Gamma \cup \Gamma'\rangle$ satisfy $(*)$.

Next fix $\langle \mu, \alpha\rangle \in \omega_2 \times \omega_3$ and put

$$D_{\mu,\alpha} = \{p \in P: \langle \mu, \alpha\rangle \in D(f^p)\}.$$

We claim that $D_{\mu,\alpha}$ is dense in P. Indeed, it is clear first that for any $p \in P$ if $\alpha \notin A^p$, then for any extension g of f^p from $\nu^p \times A^p$ to $\nu^p \times (A^p \cup \{\alpha\})$ we shall have

$$\langle g, \Gamma^p\rangle \leq \langle f^p, \Gamma^p\rangle.$$

Thus it suffices to show that every $p \in P$ has extensions q with arbitrarily large $\nu^q \in \omega_2$. To see this we define by transfinite induction on $\nu \in \omega_2\setminus \nu^p$ conditions $q_\nu = \langle g_\nu, \Gamma^p\rangle$ with $D(g_\nu) = \nu \times A^p$ and $g_\mu \subset g_\nu$ for $\mu < \nu$ as follows.

Let $q_{\nu^p} = p$; if $\mu \in \omega_2\setminus(\nu^p)$ is limit then put $g_\mu = \bigcup\{g_\sigma: \sigma \in \mu\setminus \nu^p\}$, then $\langle g_\mu, \Gamma^p\rangle \in P$ as in the proof of ω_2-closedness. Finally, for $\mu + 1$ we can put

$$g_{\mu+1}(\mu, \alpha) = 0$$

for all $\alpha \in A^p$.

This of course immediately implies that every $\langle \mu, \alpha\rangle \in \omega_2 \times \omega_3$ will be in the domain of F.

To see that τ_F is T_2 we have to show that for $\alpha \neq \beta$ in ω_3 there is some $\mu \in \omega_2$ with $F(\mu, \alpha) \neq F(\mu, \beta)$, so assume that $p \in P$ is arbitrary with $\alpha, \beta \in A^p$. If there is no $\mu \in \nu = \nu^p$ with $f^p(\mu, \alpha) \neq f^p(\mu, \beta)$, then let us consider the collection $\mathscr{C} = \{B^p_\varepsilon: \varepsilon \in \mathrm{Fn}(\nu, 2)$ and $|B^p_\varepsilon| = \aleph_1\}$ and fix a partition $A^p = C_0 \cup C_1$ of A^p such that $\alpha \in C_0$, $\beta \in C_1$ and $|B^p_\varepsilon \cap C_i| = \aleph_1$ for each $B^p_\varepsilon \in \mathscr{C}$ and $i \in 2$. Then if g extends f^p to $(\nu + 1) \times A^p$ by the stipulation

$$g(\nu, \gamma) = i \quad \text{iff} \quad \gamma \in C_i,$$

then clearly $\langle g, \Gamma^p\rangle$ is an extension of p in P that forces $F(\nu, \alpha) \neq F(\nu, \beta)$, i.e., $g(\nu, \alpha) \neq g(\nu, \beta)$.

Now let $p = \langle f, \Gamma\rangle \in P$ and $s \in \Gamma$. We claim that p forces $\alpha(s)$ to be a limit point of $s\setminus\{\alpha(s)\}$. Indeed, assume that $p \in G$ is generic, $\varepsilon \in \mathrm{Fn}(\omega_2, 2)$ and B_ε is a basic open set for τ_F with $\alpha(s) \in B_\varepsilon$. Since the $D_{\nu,\alpha}$ are dense, we can find an extension

q of p in G such that $\varepsilon \in \mathrm{Fn}(\nu^q, 2)$, but then we have

$$B_\varepsilon \cap (s \backslash \{\alpha(s)\}) \supset B_\varepsilon^q \cap (s \backslash \{\alpha(s)\}) \neq \emptyset$$

by (*) for q, hence indeed $\alpha(s)$ is in the closure of $s \backslash \{\alpha(s)\}$.

Finally let us turn to proving that, in $V[G]$, every partition of $[\omega_3]^3$ into countably many pieces has a nondiscrete homogeneous set. Thus assume that

$$1 \Vdash \dot{t} : [\omega_3]^3 \to \omega$$

for a P-name $\dot{t}$. It will clearly suffice to show that every $p \in P$ has an extension q such that for some $s \in \Gamma^q$ we have

$$\mathrm{q} \Vdash \text{“}s \text{ is } \dot{t}\text{-homogeneous”},$$

because q also forces that s is nondiscrete!

Let us start the proof of this with the following observation: For every condition $p \in P$ there is an extension q and a map $t_q : [A^q]^3 \to \omega$ (in V!) such that

$$q \Vdash \dot{t} \restriction [A^q]^3 = t_q. \tag{**}$$

Indeed, this follows easily from our assumption about $\dot{t}$ and that P is ω_2-closed, using a standard closure argument.

Let us denote by D the set of such q with a t_q satisfying (**), then D is dense in P, moreover let T be the map on D such that $T(q) = t_q$ for each $q \in D$.

Now fix a condition $p \in P$ and choose a large enough regular cardinal λ such that, P, $\dot{t}$ and all other relevant objects belong to $H(\lambda)$ (clearly, $\lambda = \aleph_5$ will do). Then we take an elementary submodel $N \prec H(\lambda)$ such that

(i) $\omega_2, P, \dot{t}, D, T, p \in N$;

(ii) $|N| = \aleph_2$, $N^{\omega_1} \subset N$ and if $\alpha \in N \cap \omega_3$, then $\alpha \subset N$ as well.

The existence of such an N follows from $2^{\aleph_1} = \aleph_2$.

Let us put $\sigma = \omega_3 \cap N$, clearly $\mathrm{cf}(\sigma) = \omega_2$. Now we are going to define by transfinite induction two sequences of conditions $\langle p_\xi : \xi \in \omega_2 \rangle$ and $\langle q_\xi : \xi \in \omega_2 \rangle$ with p_ξ, $q_\xi \in D$ as follows:

Let p_0 be such that $\sigma \in A^{p_0}$, $p_0 \leq p$ and $p_0 \in D$.

If p_ξ is defined with $p_\xi = \langle f_\xi, \Gamma_\xi \rangle$, $D(f_\xi) = \nu_\xi \times A_\xi$, then put $A'_\xi = N \cap A_\xi = \sigma \cap A_\xi$ and $\Gamma'_\xi = N \cap \Gamma_\xi = [A'_\xi]^{\omega_1 + 1} \cap \Gamma_\xi$ and $p'_\xi = \langle f'_\xi, \Gamma'_\xi \rangle$, where $f'_\xi = f_\xi \restriction \nu_\xi \times A'_\xi$. Then we have $p'_\xi \in N \cap P$, moreover the functions $h_\xi : \nu_\xi \to 2$ and $k_\xi : [A'_\xi]^2 \to \omega$ defined by $h_\xi(\mu) = f_\xi(\mu, \sigma)$ and $k_\xi(\{\alpha, \beta\}) = t_{p_\xi}(\{\alpha, \beta, \sigma\})$, respectively belong to N as well.

Now the following statement is true in $H(\lambda)$: "p'_ξ has an extension q in D such that $\nu^q = \nu_\xi$, $A'_\xi < A^q \backslash A'_\xi$, and if γ is the minimal element of $A^q \backslash A'_\xi$, then $f^q(\mu, \gamma) = h_\xi(\mu)$ for every $\mu \in \nu_\xi$, and for each $\{\alpha, \beta\} \in [A'_\xi]^2$ we have $k_\xi(\{\alpha, \beta\}) = t_q(\{\alpha, \beta, \gamma\})$". Since $N \prec H(\lambda)$ we can then also find a $q_\xi \in N$ with all of these properties. Note that then we have $\nu^{q_\xi} = \nu_\xi = \nu^{p_\xi}$ and $A^{q_\xi} \cap A^{p_\xi} = A'_\xi < A^{q_\xi} \backslash A'_\xi < A^{p_\xi} \backslash A'_\xi$, and clearly $f^{q_\xi} \restriction \nu_\xi \times A'_\xi = f^{p_\xi} \restriction \nu_\xi \times A'$, hence as we remarked earlier p_ξ and q_ξ are compatible. Thus we can define $p_{\xi+1}$ to be any common extension of p_ξ and q_ξ with $p_{\xi+1} \in D$.

Finally, if $\xi \in \omega_2$ is a limit ordinal and p_ξ, q_ξ have been defined for all $\eta \in \xi$ as above and so that $\eta < \zeta < \xi$ imply $p_\zeta \leq p_\eta$, then we can use the argument with which we proved the ω_2-closedness of P to put $p_\xi = \langle f_\xi, \Gamma_\xi \rangle$ where $f_\xi = \bigcup \{f_\eta : \eta \in \xi\}$ and $\Gamma_\xi = \bigcup \{\Gamma_\eta : \eta \in \xi\}$. It is easy to see that, since $p_\eta \in D$ for every $\eta \in \xi$, we shall have $p_\xi \in D$ as well, moreover, we also have $p_\xi \leq p_{\eta+1} \leq q_\eta$ for $\eta \in \xi$. This completes the induction.

Let us denote by γ_ξ the minimal element of $A^{q_\xi} \setminus A'_\xi$, then $\gamma_\xi \in \sigma$ and the sequence $\langle \gamma_\xi : \xi \in \omega_2 \rangle$ is strictly increasing.

Since $\xi \in \eta \in \omega_2$ implies $p_\eta \leq p_\xi$ we also have then $t_\eta \supset t_\xi$, where, of course, $t_\xi = T(p_\xi)$. Therefore, if we put $A^* = \bigcup \{A_\xi = A_\xi^p : \xi \in \omega_2\}$, then $t^* = \bigcup \{t_\xi : \xi \in \omega_2\}$ is a partition of the triples of A^* into ω pieces, i.e., $t^* : [A^*]^3 \to \omega$. We claim that the sequence $\langle \gamma_\xi : \xi \in \omega_2 \rangle$ is end-homogeneous for t^*. Indeed, this follows by noting that if $\eta \in \xi$, then $\gamma_\eta \in A'_\xi$, hence for $\zeta, \eta \in \xi$ we have $t^*(\{\gamma_\zeta, \gamma_\eta, \gamma_\xi\}) = t_{q_\xi}(\{\gamma_\zeta, \gamma_\eta, \gamma_\xi\}) = k_\xi(\{\gamma_\zeta, \gamma_\eta\}) = t_{p_\xi}(\{\gamma_\zeta, \gamma_\eta, \sigma\}) = t^*(\{\gamma_\zeta, \gamma_\eta, \sigma\})$.

Consequently, if we consider the partition $k : [\omega_2]^2 \to \omega$ defined by

$$k(\{\eta, \xi\}) = t^*(\{\gamma_\eta, \gamma_\xi, \sigma\}),$$

and pick a homogeneous set $S \subset \omega_2$ of order type ω_1 for it, as we know we can do by the Erdös–Rado theorem and $2^{\aleph_0} = \aleph_1$, then

$$s = \{\gamma_\xi : \xi \in S\} \cup \{\sigma\}$$

will have order type $\omega_1 + 1$ and will be homogeneous for t^*.

Let us put $\xi^* = \sup S$, and note that since $s \subset A_{\xi^*}$ and $t_{\xi^*} \subset t^*$, $p_{\xi^*} \Vdash$ "s is homogeneous for $\dot{t}$" because p_{ξ^*} also forces that "$\dot{t} \restriction [A_{\xi^*}]^3 = t_{\xi^*}$".

Consequently, we shall be done if we can show that $p^* = \langle f_{\xi^*}, \Gamma_{\xi^*} \cup \{s\} \rangle \in P$, i.e., satisfies (*) for s, since clearly $p^* \leq p$. To see this note that for every $\mu \in \nu_{\xi^*}$ there is a $\xi_\mu \in S$ such that $\mu \in \nu_{\xi_\mu}$ and then for every $\xi \in S \setminus \xi_\mu$ we have $f_{\xi^*}(\mu, \gamma_\xi) = f_\xi(\mu, \gamma_\xi) = h_\xi(\mu) = h_{\xi^*}(\mu) = f_{\xi^*}(\mu, \sigma)$. In other words this says that if $\sigma \in (A_{\xi^*})^i_\mu$ for $\mu \in \nu_{\xi^*}$ and $i \in 2$, then $s \setminus (A_{\xi^*})^i_\mu$ is countable, consequently for every $\varepsilon \in \mathrm{Fn}(\nu_{\xi^*}, 2)$ we have that if $\sigma \in B_\varepsilon^{f_{\xi^*}}$, then $s \cap B_\varepsilon^{f_{\xi^*}}$ is cocountable in s, and hence uncountable. This completes the proof of our theorem. □

Let us remark that our notion of forcing P and the family of dense sets required to be met by G satisfy the conditions of [2], and therefore the existence of a P-generic F and consequently that of the topology τ_F is implied by an $(\omega_2, 1)$-morass. In particular, the space we constructed also exists in L. This is of interest because the results on countable homogeneous sets being discrete for the triples and arbitrary homogeneous sets being discrete for the quadruples mentioned in the introduction were proven under GCH + some (weak) versions of $\square_\kappa$ for singular κ, so in particular are also valid in L.

We would like to point out that the following problem remains open: Does GCH (or just ZFC?) imply the existence of a T_2 (or T_3?) space X such that every partition of the triples of X into countably many pieces has a nondiscrete homogeneous set?

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