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IDEALS WITHOUT CCC

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Abstract. Let I be an ideal of subsets of a Polish space X, containing all singletons and possessing a Borel basis. Assuming that I does not satisfy ccc, we consider the following conditions (B), (M) and (D). Condition (B) states that there is a disjoint family $F \subseteq P(X)$ of size $\mathfrak{c}$, consisting of Borel sets which are not in I. Condition (M) states that there is a Borel function $f : X \to X$ with $f^{-1}[\{x\}] \notin I$ for each $x \in X$. Provided that X is a group and I is invariant, condition (D) states that there exist a Borel set $B \notin I$ and a perfect set $P \subseteq X$ for which the family $\{B + x : x \in P\}$ is disjoint. The aim of the paper is to study whether the reverse implications in the chain (D) $\Rightarrow$ (M) $\Rightarrow$ (B) $\Rightarrow$ not-ccc can hold. We build a σ-ideal on the Cantor group witnessing (M) & $\neg$(D) (Section 2). A modified version of that σ-ideal contains the whole space (Section 3). Some consistency results on deriving (M) from (B) for "nicely" defined ideals are established (Sections 4 and 5). We show that both ccc and (M) can fail (Theorems 1.3 and 5.6). Finally, some sharp versions of (M) for invariant ideals on Polish groups are investigated (Section 6).

§1. Introduction. An ideal on a space X is a family I of subsets of X closed under finite unions and subsets (i.e., $A, B \in I \Rightarrow A \cup B \in I$ and $A \subseteq B, B \in I \Rightarrow A \in I$); σ-ideals are closed under countable unions. All ideals we consider are assumed to be non trivial, they do not contain the whole space X. Moreover we want them to contain all singletons $\{x\}$ $(x \in X)$. The ideal I on a Polish space X is called Borel if it has a Borel basis (i.e., if for each set $A \in I$ there is a Borel subset B of X such that $A \subseteq B$ and $B \in I$).

The most popular Borel σ-ideals (e.g., the ideal of meager sets or the ideal of Lebesgue null sets) satisfy the countable chain condition (ccc). This condition says that the quotient Boolean algebra of Borel subsets of the space modulo the σ-ideal is ccc (i.e., every family of disjoint Borel sets which do not belong to the ideal is countable). In this paper we are interested in ideals which do not satisfy this condition. The question that arises here is what can be the reasons for failing ccc. The properties (M) and (D) defined below imply that the ideal does not satisfy ccc (and actually even more, see (B) below).

DEFINITION 1.1. Let I be an ideal on an uncountable Polish space X.

(1) We say that I has *property* (M) if and only if there is a Borel measurable function $f : X \to X$ with $f^{-1}[\{x\}] \notin I$ for each $x \in X$.
(2) Provided that X is a Polish Abelian group and I is invariant (i.e., $A \in I$ and $x \in X$ imply $A + x =_{\text{def}} \{a + x : a \in A\} \in I$), we say that I has *property*

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(D) if and only if there are a Borel set $B \notin I$ and a perfect (non-void) set $P \subseteq X$ such that $(B + x) \cap (B + y) = \emptyset$ for any distinct $x, y \in P$.

Properties (M) and (D) were introduced and investigated in [1]. It was observed that (D) $\Rightarrow$ (M), if I is invariant in the group X. Of course, (M) implies the following condition:

(B) there is a family $F \subseteq P(X)$ of cardinality $\mathfrak{c}$ (the size of the continuum) of pairwise disjoint Borel subsets of X that are not in I.

In [1] Fremlin's theorem stating the consistency of $\neg((B) \Rightarrow (M))$ is shown. However, it is unclear how his proof could be applied to the invariant case. The following questions arise (3 and 4 are posed in [1]):

PROBLEMS WE ADDRESS 1.2.

(1) *Suppose I is a Borel (invariant) ideal on a Polish space (group) X, for which the ccc fails. Does I satisfy* (B)*?*
(2) (Remains open). *Is* $\neg((B) \Rightarrow (M))$ *consistent, for some invariant ideal (σ-ideal)?*
(3) (Remains open). *Is* $\neg((B) \Rightarrow (M))$ *provable in ZFC, for some ideal (σ-ideal)?*
(4) *Does* (M) $\Rightarrow$ (D) *hold for every invariant ideal (σ-ideal)*?

The present paper considers these questions. We mostly restrict ourselves to the Cantor group 2^ω with the coordinatewise addition modulo 2 (denoted further by $\oplus$, or simply, by $+$).

At first, let us show that Question 1 of 1.2 can have the negative answer, if we do not require any additional properties of a σ-ideal I.

THEOREM 1.3. *For each cardinal κ, $\omega < \kappa < \mathfrak{c}$, there exists a Borel σ-ideal I on 2^ω such that κ is the maximal cardinal for which one can find a disjoint family of size κ of Borel sets in 2^ω that are not in I. Consequently, I does not satisfy both ccc and* (B)*, and it satisfies κ^+-cc.*

PROOF. Pick pairwise disjoint nonempty perfect sets $P_\alpha \subseteq 2^\omega$, $\alpha < \kappa$. Define I as follows:

$E \subseteq 2^\omega$ belongs to I if and only if there is a Borel set $B \subseteq 2^\omega$ such that $E \subseteq B$ and for each $\alpha < \kappa$ the intersection $B \cap P_\alpha$ is meager in P_α.

Obviously, I is a Borel σ-ideal on 2^ω and it does not satisfy ccc since each set P_α is not in I. Suppose that F is a family of pairwise disjoint I-positive Borel sets and $|F| = \kappa^+$. Then there is an $\alpha < \kappa$ such that

$$|\{E \in F : E \cap P_\alpha \text{ is non-meager in } P_\alpha\}| = \kappa^+.$$

This is impossible since the ideal of meager sets in P_α satisfies ccc. $\dashv$

A similar idea but in a much more special form will be used to produce the negative answer of a modified version of Problem 1.2(1), where (B) is replaced by (M) and I is a σ-ideal on 2^ω with Π^1_2 definition (Conclusion 5.6). We still do not know what can happen if the invariance of I is assumed.

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§2. The minimal σ-ideal without property (D). In this section we are going to answer question 4 of 1.2 in negative. It is stated in [1] that Bukovský has reformulated the question about (M) $\Rightarrow$ (D) by considering the σ-ideal I_0 generated by the family

$$F_0 = \{B \subseteq 2^\omega : \quad B \text{ is Borel and there is a perfect set } P \subseteq 2^\omega \text{ such that } \{B \oplus x : x \in P\} \text{ is a disjoint family } \}.$$

Then I_0 is the minimal invariant σ-ideal without property (D). Observe that I_0 is not trivial since F_0 is contained in the ideal of measure zero sets (as well as in the ideal of meager sets).

THEOREM 2.1. *There is a continuous function $f : 2^\omega \to 2^\omega$ such that*

$$(\forall x \in 2^\omega)(f^{-1}[\{x\}] \notin I_0).$$

Consequently, the σ-ideal I_0 has property (M) *and does not have property* (D).

The rest of this section will be devoted to the proof of the above theorem. We will break it into several steps presented in consecutive subsections, some of these steps may be interesting *per se*. For simplicity, we shall write $+$ for the addition in 2^ω; also $+$ will be used for the addition of finite sequences of zeros and ones.

2.1. A combinatorial lemma. We start with defining the function f which existence is postulated in Theorem 2.1. Its construction is very simple and based on the following (essentially elementary) observation.

LEMMA 2.2. *For each $n \in \omega$ there are $N \in \omega$ and a subset $C \subseteq 2^N$ such that every n translates of C have non-empty intersection, and likewise for $C' = 2^N \setminus C$.*

PROOF. Let $n \geq 1$. Since $\log_2(2^x - 1) < x$ for each $x > 0$, we can choose ε such that

$$\max_{g \in [1,2^n]} \frac{\log_2(2^g - 1)}{g} < \varepsilon < 1.$$

Then $2^g - 1 < 2^{g\varepsilon}$ for every $g \in [1, 2^n]$. Next, take an integer $N \geq n$ such that

$$\oplus \qquad nN + 1 < 2^N(1 - \varepsilon)$$

(this is possible since $1 - \varepsilon > 0$). We claim that this N is good for our n. To show that there exists a suitable set $C \subseteq 2^N$ we will estimate the number of all "bad" sets. Fix for a moment a sequence $\langle s_0, \dots, s_{n-1}\rangle \subseteq 2^N$. We want to give an upper bound for the number of all subsets D of 2^N such that

$$\otimes \qquad \text{either} \quad \bigcap_{k<n} D + s_k = \emptyset \quad \text{or} \quad \bigcap_{k<n} (2^N \setminus D) + s_k = \emptyset.$$

Let G be the subgroup of $(2^N, +)$ generated by $\{s_0, \dots, s_{n-1}\}$, $g = |G|$. Clearly $1 \leq g \leq 2^n$ and hence, by the choice of ε, $2^g - 1 < 2^{g\varepsilon}$. Suppose that a set $D \subseteq 2^N$

is such $\bigcap_{k<n} D + s_k = \emptyset$. Then for each $s \in 2^N$ there is $k < n$ such that $s_k + s \notin D$. Hence, for every $s \in 2^N$,

$$D \cap (G + s) \neq \{s_0 + s, \dots, s_{n-1} + s\} \subseteq G + s.$$

Consequently, as $\{G + s : s \in 2^N\}$ is a partition of 2^N into $2^N/g$ sets, the number of all $D \subseteq 2^N$ satisfying the condition $\otimes$ is not greater than

$$2 \cdot (2^g - 1)^{2^N/g} < 2 \cdot 2^{g\varepsilon 2^N/g} = 2^{\varepsilon 2^N + 1}.$$

There are $(2^N)^n$ sequences $\langle s_0, \dots, s_{n-1}\rangle \subseteq 2^N$ and each of them eliminates less than $2^{\varepsilon 2^N+1}$ subsets of 2^N. Hence there are at most $2^{nN} \cdot 2^{\varepsilon 2^N + 1} = 2^{\varepsilon 2^N + nN + 1}$ "bad" sets $D \subseteq 2^N$. By $\oplus$ we have that $\varepsilon 2^N + nN + 1 < 2^N$ so there is a set $C \subseteq 2^N$ which is "good". The lemma is proved. $\dashv$

Applying Lemma 2.2 inductively, choose integers n_i and sets C_i (for $i \in \omega$) such that

(α) $0 = n_0 < n_1 < n_2 < \cdots < \omega$, $\quad C_i \subseteq 2^{[n_i, n_{i+1})}$,

(β) if $s_0, \dots, s_{n_i} \in 2^{[n_i, n_{i+1})}$ then both $\bigcap_{k \le n_i} C_i + s_k$ and $\bigcap_{k \le n_i} (2^{[n_i,n_{i+1})} \setminus C_i) + s_k$ are non-empty.

Next define $f : 2^\omega \to 2^\omega$ by $f(x)(i) = 1$ (respectively 0) if $x{\restriction}[n_i, n_{i+1})$ belongs to C_i (resp. does not belong to C_i).

In the next steps we will show that the function f is as required in Theorem 2.1. Since, obviously, f is continuous, what we have to prove is that for every $y \in 2^\omega$ its pre-image $f^{-1}[\{y\}]$ is not in the ideal I_0. As in the proof we will use the properties of the sets C_i stated in (α), (β) above only, it should be clear from symmetry considerations, that it suffices to show that the set

$$H =_{\text{def}} \{x \in 2^\omega : (\forall i \in \omega)(f(x)(i) = 1)\}$$

is not in I_0. (Note that the set H consists of those sequences x which satisfy $x{\restriction}[n_i, n_{i+1}) \in C_i$ for all i in ω.)

2.2. A Baire topology on H. At this step, for each sequence $\bar{P} = \langle P_n : n \in \omega\rangle$ of perfect subsets of 2^ω, we introduce a topology $\tau = \tau(\bar{P})$ on H. Let the sequence $\bar{P}$ be fixed in this and the next subsections.

Let $P_n^* = P_n + P_n = \{x + y : x, y \in P_n\}$ and $T_n^* = \{x{\restriction}m : x \in P_n^*, m \in \omega\}$ for $n \in \omega$. Note that P_n^* is a perfect set, $T_n^* \subseteq 2^{<\omega}$ is a perfect tree and its body (i.e., the set of all infinite branches through the tree) is $[T_n^*] = P_n^*$.

In order to define the desired topology τ we need to consider tree orderings, say $\prec$, with domain a positive integer n, and compatible with the natural ordering of integers, together with an assignment of integers $\pi(k, \ell)$ to pairs $\{k, \ell\} \in [n]^2$, where k is the immediate predecessor of ℓ relative to $\prec$ (and thus, in particular, $k < \ell$). Note that the tree ordering $\prec$ can be determined from the mapping π alone; π *is undefined* when k is not the immediate predecessor of ℓ. Such a mapping π will be called *a tree mapping with domain n*, and we shall reserve the letter π, with subscripts and/or superscripts to denote tree mappings.

A sequence $s \in 2^{<\omega}$ will be called *acceptable* if it belongs to the tree of H and $\mathrm{dom}(s)$ is some n_i. Thus if s is acceptable and $\mathrm{dom}(s) = n_i$ then for each $j < i$, the restriction $s{\restriction}[n_j, n_{j+1})$ is in the set C_j.

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Let S consist of all sequences $\rho = \langle \pi, s_0, s_1, \dots, s_{n-1} \rangle$ where π is a tree mapping on n, the s_j's are acceptable with the same domain $n_i \geq n$, and $s_k + s_\ell \in T^*_{\pi(k,\ell)}$ for all (k, ℓ) such that $\pi(k, \ell)$ is defined. We also set $n = n(\rho)$ and $i = i(\rho)$.

LEMMA 2.3. *Suppose that* $\rho = \langle \pi, s_0, \dots, s_{n(\rho)-1} \rangle \in S$ *and* $j > i(\rho)$. *Then there are* $t_0, \dots, t_{n(\rho)-1} \in 2^{n_j}$ *such that* $s_0 \lhd t_0, \dots, s_{n(\rho)-1} \lhd t_{n(\rho)-1}$ *and* $\langle \pi, t_0, \dots, t_{n(\rho)-1} \rangle \in S$ *(where* $s \lhd t$ *means that the sequence* t *is a proper extension of* s*). If* j *is sufficiently large,* $t_0, \dots, t_{n(\rho)-1}$ *can be chosen pairwise distinct.*

PROOF. Let $i = i(\rho)$, $n = n(\rho)$. Let $\prec$ be the tree ordering on n determined by the tree mapping π (so ℓ is the immediate $\prec$-successor of k if and only if $\pi(k, \ell)$ is defined). We will consider the case $j = i + 1$ since, for greater numbers j, simple induction works.

What we have to do is to find sequences $r^k \in C_i$ (for $k < n$) such that

if $k < \ell < n$ and ℓ is the immediate $\prec$-successor of k then $(s_k{}^\frown r^k) + (s_\ell{}^\frown r^\ell) \in T^*_{\pi(k,\ell)}$.

For each $k, \ell < n$ such that k is the immediate $\prec$-predecessor of ℓ (so in particular $k < \ell$) we choose sequences $r_{k,l} \in 2^{[n_i, n_{i+1})}$ such that $(s_k + s_\ell){}^\frown r_{k,\ell} \in T^*_{\pi(k,\ell)}$ (possible as $s_k + s_\ell \in T^*_{\pi(k,\ell)}$ and $T^*_{\pi(k,\ell)}$ is a perfect tree). The sequences $r_{k,\ell}$ are our candidates for the sums $r^k + r^\ell$:

(⊛) if we decide what is r^k then we will put $r^\ell = r^k + r_{k,\ell}$.

As $\prec$ is a tree ordering it follows that if we keep the above rule then the choice of the sequence r^0 determines all the sequences $r^1, \dots, r^{n-1}$. Why? Take $k < n$. Then there exists *the unique* sequence $0 = k_0 < k_1 < \dots < k_m = k$ such that k_{i+1} is the immediate $\prec$-successor of k_i (for $i < m$) and therefore, by (⊛),

$$r^k = r^0 + r_{k_0,k_1} + r_{k_1,k_2} + \dots + r_{k_{m-1},k_m}.$$

So choosing r^0 we have to take care of the demand that $r_k \in C_i$ for *all* of the sequences r^k (for $k < n$). Thus we have to find $r^0 \in C_i$ such that

($\triangle$) if $0 = k_0 < \dots < k_m < n$ and k_{i+1} is the immediate $\prec$-successor of k_i (for $i < m$) then $r^0 \in C_i + r_{k_0,k_1} + r_{k_1,k_2} + \dots + r_{k_{m-1},k_m}$.

Why can we find such an r^0? Each positive $k < n$ appears as the largest element in exactly one sequence $k_0, \dots, k_m$ as in ($\triangle$), so we get $n - 1$ translations of C_i in ($\triangle$). Thus, considering one more trivial translation (the identity function) we have $n \leq n_i$ of them. Applying condition (β) of the choice of C_i, n_{i+1} we may find a suitable $r^0 \in 2^{[n_i, n_{i+1})}$.

Now, as we stated before, the choice of r^0 and (⊛) determine all sequences $r^0, r^1, \dots, r^{n-1}$. Moreover

if $0 = k_0 < \dots < k_m = k < n$ is a sequence as in ($\triangle$) then $r^k = r^0 + r_{k_0,k_1} + \dots r_{k_{m-1},k_m} \in C_i$

(we use the fact that the addition and the subtraction in $2^{<\omega}$ coincide).

Define $t_k = s_k{}^\frown r^k$ for $k < n$. We immediately get $\langle \pi, t_0, \dots, t_{n-1} \rangle \in S$.

Finally suppose that $k, \ell < n$ are such that k is the immediate $\prec$-predecessor of ℓ and $\ell_0 < \ell$. Take $j > i$ and $r^*_{k,\ell}, r^{**}_{k,\ell} \in 2^{[n_i, n_j)}$ such that

$(s_k + s_\ell)^\frown r^*_{k,\ell}, (s_k + s_\ell)^\frown r^{**}_{k,\ell} \in T^*_{\pi(k,\ell)} \cap 2^{n_j}$ and
$r^*_{k,\ell}\restriction n_{j-1} = r^{**}_{k,\ell}\restriction n_{j-1}$ but
$r^*_{k,\ell}\restriction[n_{j-1}, n_j) \neq r^{**}_{k,\ell}\restriction[n_{j-1}, n_j)$

(possible as $T^*_{\pi(k,\ell)}$ is perfect). If we now repeat the procedure described earlier choosing $r^*_{k,\ell}\restriction[n_m, n_{m+1})$ as $r_{k,\ell}$ at the stages $m < j-1$ then, extending the sequences from n_{j-1} to n_j we may use either $r^*_{k,\ell}\restriction[n_{j-1}, n_j)$ or $r^{**}_{k,\ell}\restriction[n_{j-1}, n_j)$. Consequently we may make sure that the respective sequence r^ℓ is distinct from r^{ℓ_0}. Repeating this for all pairs $\ell_0 < \ell < n$ we may get that all the final extensions t_ℓ are distinct. The Lemma is proved. ⊣

Note that if $\rho = \langle \pi, s_0, \dots, s_{n-1}\rangle \in S$, $n + 1 \leq n_{i(\rho)}$ and $m < \omega$ then $\langle \pi', s_0, \dots, s_{n-1}, s_0\rangle \in S$, where π' is such that $\pi'\restriction[n]^2 = \pi$, $\pi'(0,n) = m$ and $\pi'(k,\ell)$ is undefined in all remaining cases. (Remember that finite sequences constantly equal to 0 are in T^*_m.) Thus we may "extend" each element of S (to an element of S) getting both longer sequences s_i and the number of these sequences (i.e., $n(\rho)$) larger. It should be remarked here that S is nonempty—it is easy to give examples ρ of elements of S with $n(\rho) = 1$.

For $\rho \in S$ we define the basic set $U(\rho)$ as

$$\begin{aligned}\{x_0 \in H :\ & (\exists x_1, \dots, x_{n(\rho)-1} \in H)(s_0 \lhd x_0, s_1 \lhd x_1, \dots, s_{n(\rho)-1} \lhd x_{n(\rho)-1})\\ & \text{and } (\forall j > i(\rho))(\langle \pi, x_0\restriction n_j, \dots, x_{n(\rho)-1}\restriction n_j\rangle \in S)\}\end{aligned}$$

where $\rho = \langle \pi, s_0, \dots, s_{n(\rho)-1}\rangle$. It follows from Lemma 2.3 that each $U(\rho)$ is a non-empty subset of H.

Proposition 2.4. *If $\rho_1, \rho_2 \in S$, $x_0 \in U(\rho_1) \cap U(\rho_2)$ then, for some $\rho \in S$, we have $x_0 \in U(\rho) \subseteq U(\rho_1) \cap U(\rho_2)$.*

Consequently the family $\{U(\rho) : \rho \in S\}$ forms a (countable) basis of a topology on H. [We will denote this topology by $\tau(\bar{P})$ or just τ if $\bar{P}$ is understood.]

Proof. For $j = 0, 1$, let $\rho_j = \langle \pi^j, s^j_0, \dots, s^j_{n^j-1}\rangle$ and let $x^j_1, \dots, x^j_{n^j-1} \in H$ witness that $x_0 \in U(\rho_j)$. We shall define ρ. Put $n(\rho) = n^0 + n^1 - 1$ and $i = i(\rho) = \max\{i(\rho_0), i(\rho_1), n(\rho)\} + 1$. Define a partial mapping π from $[n(\rho)]^2$ to ω as follows

$$\begin{array}{ll}\pi\restriction[n^0]^2 = \pi^0, & \\ \pi(n^0 - 1 + k, n^0 - 1 + \ell) = \pi^1(k,\ell) & \text{if } 0 < k < \ell < n^1,\ \{k,\ell\} \in \mathrm{dom}(\pi^1),\\ \pi(0, n^0 - 1 + \ell) = \pi^1(0,\ell) & \text{if } 0 < \ell < n^1,\ \{0,\ell\} \in \mathrm{dom}(\pi^1),\\ \pi(k,\ell) \text{ is undefined} & \text{in the remaining cases.}\end{array}$$

It should be clear that π is a tree mapping on $n(\rho)$. Finally, put

$$\rho = \langle \pi, x_0\restriction n_i, x^0_1\restriction n_i, \dots, x^0_{n^0-1}\restriction n_i, x^1_1\restriction n_i, \dots, x^1_{n^1-1}\restriction n_i\rangle.$$

By the choice of i and π we easily check that $\rho \in S$ and $x_0 \in U(\rho)$ is witnessed by $x^0_1, \dots, x^0_{n^0-1}, x^1_1, \dots x^1_{n^1-1}$ and that $U(\rho) \subseteq U(\rho_0) \cap U(\rho_1)$.

To conclude the proof of the proposition note that, since each $x_0 \in H$ is in some $U(\rho)$ (take $n(\rho) = 1$ and $s_0 = x_0\restriction n_5$), the family $\{U(\rho) : \rho \in S\}$ is a (countable) basis of a topology on H. ⊣

PROPOSITION 2.5. *The topology τ is stronger than the product topology of 2^ω restricted to H. Consequently, all (ordinary) Borel subsets of H are Borel relative to the topology τ.*

PROOF. Since the sets $[t] = \{x \in 2^\omega : t \lhd x\}$ for $t \in \bigcup_{i>1} 2^{n_i}$ form a basis of the natural topology in 2^ω, it is enough to show that $s \in 2^{n_i}$, $i > 1$ implies $[s] \cap H \in \tau$. But for this observe that if $[s] \cap H \neq \emptyset$ then $n(\rho) = 1$, $i(\rho) = i$ and $s_0 = s$ generate $\rho \in S$ such that $U(\rho) = [s] \cap H$. ⊣

PROPOSITION 2.6. *$\langle H, \tau\rangle$ is a Baire space, actually each basic τ-open set $U(\rho)$ is not τ-meager.*

PROOF. First note that each $U(\rho)$ (for $\rho \in S$) is a projection of a compact subset of $H^{n(\rho)}$ and hence it is a (non-empty) compact set (in the natural topology). Suppose to the contrary that for some $\rho \in S$ we have $U(\rho) = \bigcup_{k\in\omega} N_k$, where each N_k is τ-nowhere dense. Then we may inductively choose $\rho_0, \rho_1, \dots \in S$ such that $\rho_0 = \rho$ and

$$U(\rho_{k+1}) \cap N_k = \emptyset, \quad \text{and } U(\rho_{k+1}) \subseteq U(\rho_k) \qquad \text{for each } k \in \omega.$$

It is possible as if $\rho^* \in S$, N is τ-nowhere dense then there is a non-empty τ-open set $U \subseteq U(\rho^*)\setminus N$ (remember that $U(\rho^*) \neq \emptyset$). But the sets $U(\rho')$ for $\rho' \in S$ constitute the basis of the topology τ, so we find $\rho' \in S$ with $U(\rho') \subseteq U \subseteq U(\rho^*) \setminus N$.
Now, by the compactness of the sets $U(\rho_k)$ we get

$$\emptyset \neq \bigcap_{k\in\omega} U(\rho_k) \subseteq U(\rho) \setminus \bigcup_{k\in\omega} N_k$$

—a contradiction finishing the proof of the proposition. ⊣

2.3. τ-non-meager subsets of H.

PROPOSITION 2.7. *If $B \subseteq H$ is a τ-non-meager set with the Baire property (with respect to τ) then for every $m \in \omega$ there are distinct $x, y \in P_m$ such that the intersection $(B + x) \cap (B + y)$ is non-empty.*

PROOF. Let $m \in \omega$ and $B \subseteq H$ be a τ-non-meager set with the τ-Baire property. Then for some $\rho \in S$ the set $U(\rho) \setminus B$ is τ-meager. Let $U(\rho) \setminus B = \bigcup_{k\in\omega} N_k$ where each N_k is τ-nowhere dense. Let $\rho = \langle \pi, s_0, \dots s_{n(\rho)-1}\rangle$. Put

$$\rho' = \langle \pi', s_0, \dots, s_{n(\rho)-1}, s_0, \dots, s_{n(\rho)-1}\rangle$$

where π' is a tree mapping on $2n(\rho)$ given by

$$\begin{array}{ll} \pi'{\restriction}[n(\rho)]^2 = \pi, & \\ \pi'(n(\rho)+k, n(\rho)+\ell) = \pi(k,\ell) & \text{if } k < \ell < n(\rho),\ \{k,\ell\} \in \mathrm{dom}(\pi), \\ \pi'(0, n(\rho)) = m, & \\ \pi'(k,\ell) \text{ is undefined} & \text{in the remaining cases.} \end{array}$$

Easily, π' is indeed a tree mapping and $\rho' \in S$ (remember that $s_0 + s_0 = \bar{0} \in T^*_m$).
For a partial function π_0 from $[n]^2$ to ω, $n > n(\rho)$ let $(\pi_0)^*$ be defined by

$$\begin{array}{l} \mathrm{dom}((\pi_0)^*) = \{\{\sigma^{-1}(k), \sigma^{-1}(\ell)\} : \{k,\ell\} \in \mathrm{dom}(\pi_0)\} \quad \text{and} \\ (\pi_0)^*(k,\ell) = \pi(\sigma(k), \sigma(\ell)), \end{array}$$

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where $\sigma : n \longrightarrow n$ is a permutation of n given by

$$\sigma(0) = n(\rho), \quad \sigma(i+1) = i \text{ for } 0 \leq i < n(\rho) \quad \text{and } \sigma(i) = i \text{ for } n(\rho) < i < n.$$

Note that if π_0 is a tree mapping on $n > n(\rho)$ such that $\{0, n(\rho)\} \in \mathrm{dom}(\pi_0)$ then $(\pi_0)^*$ is a tree mapping too. Hence, if $\rho_0 = \langle \pi_0, t_0, \dots, t_{n-1} \rangle \in S$, $n(\rho) < n$ and $\pi_0(0, n(\rho))$ is defined then

$$(\rho_0)^* =_{\mathrm{def}} \langle (\pi_0)^*, t_{n(\rho)}, t_0, \dots, t_{n(\rho)-1}, t_{n(\rho)+1}, \dots, t_{n-1} \rangle \text{ is in } S \text{ too.}$$

CLAIM 2.7.1. *Suppose that $\rho_0 = \langle \pi_0, t_0, \dots, t_{n-1} \rangle \in S$ is such that $n > n(\rho)$ and $\pi_0(0, n(\rho))$ is defined. Let $N \subseteq H$ be a τ-nowhere dense set. Then there is $\rho^+ = \langle \pi^+, s_0^+, \dots, s_{k-1}^+ \rangle \in S$ such that*

(1) $U(\rho^+) \subseteq U(\rho_0) \setminus N$, $U((\rho^+)^*) \subseteq U((\rho_0)^*) \setminus N$, *and*
(2) $i(\rho^+) > i(\rho_0)$, $k = n(\rho^+) > n(\rho_0) = n$, $\pi^+ {\restriction} [n(\rho_0)]^2 = \pi_0$, *and*
(3) *if $\ell < n$ then $t_\ell \lhd s_\ell^+$.*

[The operation $(\cdot)^*$ is as defined before.]

PROOF OF THE CLAIM. Since N is τ-nowhere dense we find ρ_1 in S such that $U(\rho_1) \subseteq U(\rho_0) \setminus N$. Applying the procedure from the proof of Proposition 2.4 (with an arbitrary $x_0 \in U(\rho_1)$) we may find $\rho_2 = \langle \pi_2, s_0^2, \dots, s_{n(\rho_2)-1}^2 \rangle \in S$ such that $U(\rho_2) \subseteq U(\rho_1)$ and $i(\rho_2) > i(\rho_0)$, $n(\rho_2) > n(\rho_0)$, $\pi_2 {\restriction} [n(\rho_0)]^2 = \pi_0$, $t_\ell \lhd s_\ell^2$ for $\ell < n$. Now we look at $(\rho_2)^* \in S$ and we choose $\rho_3 \in S$ such that

$$U(\rho_3) \subseteq U((\rho_2)^*) \setminus N \subseteq U((\rho_0)^*) \setminus N.$$

Next, similarly as ρ_2, we get $\rho_4 = \langle \pi_4, s_0^4, \dots, s_{n(\rho_4)-1}^4 \rangle \in S$ with the corresponding properties with respect to $(\rho_2)^*$ and ρ_3. So, in particular, $\pi_4 {\restriction} [n(\rho_2)]^2 = (\pi_2)^*$, $s_\ell^2 \lhd s_{\ell+1}^4$ for $\ell < n(\rho)$, $s_{n(\rho)}^2 \lhd s_0^4$ and $s_\ell^2 \lhd s_\ell^4$ for $n(\rho) < \ell < n(\rho_2)$. Finally we apply the inverse operation to $(\cdot)^*$ and we get $\rho^+ \in S$ such that $(\rho^+)^* = \rho_4$. (This is possible, as the only $j \in (0, n(\rho))$ for which the value of $\pi_4(0, j)$ is defined, is $j = 1$.) It should be clear that the ρ^+ is as required in the claim.

Now, by induction on $k < \omega$, we choose n_k, s_i^k (for $i < n_k$), π_k and ρ_k such that

(i) $n_0 = 2n(\rho)$, $s_i^0 = s_{n(\rho)+i}^0 = s_i$ (for $i < n(\rho)$), $\pi_0 = \pi'$,
[So $\rho' = \langle \pi_0, s_0^0, \dots, s_{n_0-1}^0 \rangle = \rho_0$.]
(ii) $\rho_k = \langle \pi_k, s_k^0, \dots, s_{n_k-1}^0 \rangle \in S$, $U(\rho_k) \subseteq U(\rho)$, $U((\rho_k)^*) \subseteq U(\rho)$,
[Here, $(\rho_k)^*$ is the element of S obtained from ρ_k by moving $s_{n(\rho)}^k$ to the first place, see the definition of $(\pi_0)^*$, $(\rho_0)^*$ above.]
(iii) $n_k < n_{k+1}$, $i(\rho_k) < i(\rho_{k+1})$, $s_i^k \lhd s_i^{k+1}$ for $i < n_k$, $\pi_k = \pi_{k+1} {\restriction} [n_k]^2$,
(iv) $U(\rho_{k+1}) \cap N_k = U((\rho_{k+1})^*) \cap N_k = \emptyset$ and s_i^{k+1} (for $i < n_{k+1}$) are pairwise distinct.

The first step of the construction is fully described in the demand (i) above (note that then (ii) is satisfied as $U((\rho_0)^*) \subseteq U(\rho)$ and (iii), (iv) are not relevant).

Suppose that we have defined ρ_k etc. Apply Claim 2.7.1 to ρ_k, N_k standing for ρ_0, N there to get ρ_{k+1} (corresponding to ρ^+ there). It should be clear that the requirements (ii)–(iv) are satisfied except perhaps the last demand of (iv)–the sequences s_i^{k+1} do not have to be pairwise distinct. But this is not a problem as by Lemma 2.3 we may take care of this extending them further.

Let $x_i = \bigcup_{k\in\omega} s_i^k$ for $i < \omega$. Then $x_i \in H$ (by (iii)+(ii)), $x_0 \in \bigcap_{k\in\omega} U(\rho_k)$, $x_{n(\rho)} \in \bigcap_{k\in\omega} U((\rho_k)^*)$ (by the definition of $(\rho_k)^*$). Hence, by (ii)+(iv),

$$x_0, x_{n(\rho)} \in U(\rho) \setminus \bigcup_{k\in\omega} N_k \subseteq B$$

and by the last part of (iv) they are distinct. Since $\pi_k(0, n(\rho)) = m$ for each $k < \omega$ (by (iii)) we may apply the definition of S to conclude that $x_0 + x_{n(\rho)} \in [T_m^*]$. So, $x_0 + x_{n(\rho)} = x + y$ for some $x, y \in P_m$. As $x_0 \neq x_{n(\rho)}$, also $x \neq y$. We finish the proof of the proposition noting that $x_0 + x = x_{n(\rho)} + y$ and $x_0 + x \in B + x$, $x_{n(\rho)} + y \in B + y$, as required. ⊣

2.4. Conclusion of the proof of Theorem 2.1. As we stated at the end of the subsection 2.1, to conclude Theorem 2.1 it is enough to show that the set H defined there is not in the ideal I_0. Suppose to the contrary that H can be covered by the union $\bigcup_{n\in\omega} B_n$ of Borel (in the standard topology) subsets B_n of 2^ω, each B_n from the family F_0. Then, for every $n \in \omega$ we may pick up a perfect set $P_n \subseteq 2^\omega$ witnessing "$B_n \in F_0$". Let $\bar{P} = \langle P_n : n \in \omega\rangle$ and let $\tau = \tau(\bar{P})$ be the topology on H determined by $\bar{P}$ in Proposition 2.4. We know that the space H is not meager in this topology (by Proposition 2.6) and therefore one of the sets $B_n \cap H$ is τ-non-meager, say $B_{n_0} \cap H$. But $B_{n_0} \cap H$ is Borel in the standard topology of H, so it has the τ-Baire property. Applying Proposition 2.7 to it we conclude that there are distinct $x, y \in P_{n_0}$ such that the intersection $(B_{n_0} + x) \cap (B_{n_0} + y)$ is non-empty, a contradiction to the choice of P_{n_0}. ⊣

§3. Non-Borel case. Now, let us omit the assumption that the sets of the family F_0 defined in the previous section are Borel. We thus get

$$F_0^* = \{A \subseteq 2^\omega : \text{ there is a perfect set } P \subseteq 2^\omega \text{ such that } \{A \oplus x : x \in P\} \text{ is a disjoint family }\}.$$

Let I_0^* be the σ-ideal generated by F_0^*. It turns out that I_0^* is not a proper ideal.

THEOREM 3.1. $2^\omega \in I_0^*$.

PROOF. Here $+$ and $-$ will stand for the respective operations on ordinals, while the addition and subtraction in 2^ω are denoted by $\oplus, \ominus$, respectively. Let us define an increasing sequence $\langle \gamma_\beta : \beta < \omega + \omega\rangle$ of ordinals as follows:

$$\gamma_0 = 0,\ \gamma_{n+1} = \gamma_n + \mathfrak{c} \quad \text{for } n \in \omega,$$
$$\gamma_\omega = \sup_{n\in\omega} \gamma_n,\ \gamma_{\omega+n+1} = \gamma_{\omega+n} + \mathfrak{c} \quad \text{for } n \in \omega.$$

Let $P \subseteq 2^\omega$ be a perfect set independent in the Cantor group; cf. [12]. Pick pairwise disjoint perfect sets P_n (for $n \in \omega$) such that $\bigcup_{n\in\omega} P_n = P$ and fix enumerations $P_n = \{x_\alpha : \gamma_n \leq \alpha < \gamma_{n+1}\}$ (for $n \in \omega$; so x_α's are distinct). Extend P to H, a maximal independent set called here a Hamel basis. We may assume that $|H \setminus P| = \mathfrak{c}$. Let $H \setminus P = \{x_\alpha : \gamma_\omega \leq \alpha < \gamma_{\omega+\omega}\}$, where x_α's are distinct. Consequently, $H = \{x_\alpha : \alpha < \gamma_{\omega+\omega}\}$.

Now, let us define y_α for $\alpha < \gamma_{\omega+\omega}$, as follows:

- if $\gamma_\omega \leq \alpha < \gamma_{\omega+\omega}$, put $y_\alpha = x_\alpha$,
- if $\gamma_n \leq \alpha < \gamma_{n+1}, n \in \omega$, put $y_\alpha = x_\alpha \oplus x_{\gamma_{\omega+n}+(\alpha-\gamma_n)\omega+1} \oplus \cdots \oplus x_{\gamma_{\omega+n}+(\alpha-\gamma_n)\omega+n}$.

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Then $\{y_\alpha : \alpha < \gamma_{\omega+\omega}\}$ forms a Hamel basis. Each $y \in 2^\omega$ has the unique representation of the form $\sum_{\alpha\in u_y} y_\alpha$ where $u_y \subseteq \gamma_{\omega+\omega}$ is finite. For $m \in \omega$ let

$$A_m = \{y \in 2^\omega : |u_y| = m\}.$$

Obviously, $2^\omega = \bigcup_{m\in\omega} A_m$ and

(1) $A_m \ominus A_m \subseteq \bigcup_{i\leq 2m} A_i$.

The proof will be complete, if we show that

(2) $\{A_m \oplus x : x \in P_{m+1}\}$ is a disjoint family.

To get this, observe first that

(3) if $x, x' \in P_n$, $x \neq x'$ then $x \ominus x' \in A_{2n+2}$.

Indeed, let $x = x_\alpha$ and $x' = x_\beta$ for some α, β such that $\gamma_n \leq \alpha < \gamma_{n+1}$, $\gamma_n \leq \beta < \gamma_{n+1}$ and $\alpha \neq \beta$. Since

$$x_\alpha = y_\alpha \ominus x_{\gamma_{\omega+n}+(\alpha-\gamma_n)\omega+1} \ominus \dots \ominus x_{\gamma_{\omega+n}+(\alpha-\gamma_n)\omega+n}$$
$$x_\beta = y_\beta \ominus x_{\gamma_{\omega+n}+(\beta-\gamma_n)\omega+1} \ominus \dots \ominus x_{\gamma_{\omega+n}+(\beta-\gamma_n)\omega+n},$$

we conclude that $x \ominus x' \in A_{2n+2}$ (remember that $\beta - \gamma_n \neq \alpha - \gamma_n$) and (3) is proved. To show (2), take distinct $x, x' \in P_{m+1}$ and suppose that the intersection $(A_m \oplus x) \cap (A_m \oplus x')$ is non-empty. Then $a \oplus x = b \oplus x'$ for some $a, b \in A_m$. In other words, $a \ominus b = x' \ominus x$. By (1), we have $x' \ominus x \in \bigcup_{i\leq 2m} A_i$ and, by (3), we get $x' \ominus x \in A_{2m+4}$, a contradiction. $\dashv$

REMARK. For the Cantor group, the operations $\oplus$ and $\ominus$ are identical. However, we have distinguished them since the same proof works for any Abelian Polish group admitting a perfect independent set. Note that a number of groups different from 2^ω are good: by [11], each connected Abelian Polish group which has an element of infinite order admits a perfect independent set.

§4. Getting (M) from "not ccc".

In this section we try to conclude (consistently) the property (M) from the property (B) for nicely defined Borel ideals. The results here are complementary, in a sense, to Fremlin's theorem mentioned in the Introduction. (But note that here we deal with ideals with simple definitions, while the ideal constructed by Fremlin is very complicated.)

THEOREM 4.1. *Let $n \geq 2$. The following statement is consistent with ZFC$+\mathfrak{c} = \omega_n$:*

if $B \subseteq 2^\omega \times 2^\omega$ is a Σ^1_3 set such that, for some set $A \subseteq 2^\omega$ of size ω_2, the sections B_x, $x \in A$ are nonempty pairwise disjoint (where $B_x = \{y \in 2^\omega : \langle x, y\rangle \in B\}$) then *for some perfect set $P \subseteq 2^\omega$, the sections B_x, $x \in P$ are nonempty pairwise disjoint.*

PROOF. Start with the universe $\mathbf{V}$ satisfying CH.

Let $\mathbb{C}_{\omega_n}$ be the (finite support) product of ω_n copies of the Cohen forcing notion $\mathbb{C}$ and let $\langle c_\alpha : \alpha < \omega_n\rangle$ be the sequence of Cohen reals, $\mathbb{C}_{\omega_n}$-generic over $\mathbf{V}$. Work in $\mathbf{V}[c_\alpha : \alpha < \omega_n]$.

Clearly $\mathfrak{c} = \omega_n$. Suppose that $B \subseteq 2^\omega \times 2^\omega$ is a Σ^1_3 set and $\langle x_\zeta : \zeta < \omega_2\rangle$ is a sequence of reals such that $\zeta_1 < \zeta_2 < \omega_2$ implies that the sections $B_{x_{\zeta_1}}$, $B_{x_{\zeta_2}}$ are nonempty disjoint. Let $U \in \mathbf{V}\cap[\omega_n]^\omega$ be such that the parameters of the Σ^1_3 formula $\Phi(x, y)$ defining B are in $\mathbf{V}[c_\alpha : \alpha \in U]$. Next choose a sequence $\langle U_\zeta : \zeta < \omega_2\rangle \in \mathbf{V}$

of countable subsets of $\omega_n \setminus U$ such that each real x_ζ is in $\mathbf{V}[c_\alpha : \alpha \in U \cup U_\zeta]$. Moreover, we demand that

$$\mathbf{V}[c_\alpha : \alpha \in U \cup U_\zeta] \models (\exists z)\Phi(x_\zeta, z)$$

(remember that Φ is Σ^1_3; use Shoenfield's absoluteness). By CH in $\mathbf{V}$ and the Δ-lemma we find $A \in [\omega_2]^{\omega_2}$ and a countable set $U^* \supseteq U$ such that $\{U_\zeta \setminus U^* : \zeta \in A\}$ is a disjoint family. We may assume that all sets $U_\zeta \setminus U^*$ are infinite (by adding more members). Let $\mathbf{V}' = \mathbf{V}[c_\alpha : \alpha \in U^*]$.

Each sequence $\langle c_\alpha : \alpha \in U_\zeta \setminus U^* \rangle$ is essentially one Cohen real; denote this real by d_ζ. Note that if $\zeta_0, \zeta_1, \zeta_2 \in A$ are distinct then $\langle d_{\zeta_0}, d_{\zeta_1}, d_{\zeta_2} \rangle$ is $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$-generic over $\mathbf{V}'$. A real $x \in 2^\omega$ in the one Cohen real extension is the value of a Borel function $f : 2^\omega \to 2^\omega$ from the ground model at this Cohen real. Consequently, we find a sequence $\langle f_\zeta : \zeta \in A \rangle \in \mathbf{V}'$ of Borel functions from 2^ω into 2^ω such that

$$\mathbf{V}[c_\alpha : \alpha < \omega_n] \models x_\zeta = f_\zeta(d_\zeta).$$

By CH in $\mathbf{V}'$ we find $A^* \in [A]^{\omega_2} \cap \mathbf{V}'$ and a Borel function $f : 2^\omega \to 2^\omega$, $f \in \mathbf{V}'$ such that $f_\zeta = f$ for $\zeta \in A^*$. By Shoenfield's absoluteness we have that for distinct $\zeta_0, \zeta_1, \zeta_2 \in A^*$:

$$\mathbf{V}'[d_{\zeta_0}, d_{\zeta_1}, d_{\zeta_2}] \models$$
$$\text{“}\neg(\exists z)[\Phi(f(d_{\zeta_0}), z) \,\&\, \Phi(f(d_{\zeta_1}), z)] \,\&\, (\exists z)\Phi(f(d_{\zeta_0}), z) \,\&\, (\exists z)\Phi(f(d_{\zeta_1}), z)\text{”}$$

(remember the choice of U and U_ζ: the witness for $(\exists z)(\Phi(x_\zeta, z))$ is in $\mathbf{V}[c_\alpha : \alpha \in U \cup U_\zeta]$ already).

As the d_ζ's are Cohen reals over $\mathbf{V}'$ and their supports are disjoint, by density argument we get that in $\mathbf{V}'$:

$$(*) \quad \Vdash_{\mathbb{C}\times\mathbb{C}\times\mathbb{C}}$$
$$\text{“}\neg(\exists z)[\Phi(f(\dot{c}_0), z) \,\&\, \Phi(f(\dot{c}_1), z)] \,\&\, (\exists z)\Phi(f(\dot{c}_0), z) \,\&\, (\exists z)\Phi(f(\dot{c}_1), z)\text{”},$$

where $\langle \dot{c}_0, \dot{c}_1, \dot{c}_2 \rangle$ is the canonical $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$-name for the generic triple of Cohen reals. In $\mathbf{V}[c_\alpha : \alpha < \omega_n]$ take a perfect set $P \subseteq 2^\omega$ such that $P \times P \setminus \Delta \subseteq O$ for every open dense subset O of $2^\omega \times 2^\omega$ coded in $\mathbf{V}'$ (Δ stands for the diagonal $\{\langle x, x \rangle : x \in 2^\omega\}$). Then $\langle x, y \rangle$ is $\mathbb{C} \times \mathbb{C}$-generic over $\mathbf{V}'$ for each distinct $x, y \in P$. (Such a perfect set is added by one Cohen real; the property that it is a perfect set of mutually Cohen reals over $\mathbf{V}'$ is preserved in passing to an extension.) We claim that for distinct $x, y \in P \cap \mathbf{V}[c_\alpha : \alpha < \omega_n]$:

$$\mathbf{V}[c_\alpha : \alpha < \omega_n]$$
$$\models \text{“ } (\exists t)\Phi(f(x), t) \,\&\, (\exists t)\Phi(f(y), t) \,\&\, \neg(\exists t)[\Phi(f(x), t) \,\&\, \Phi(f(y), t)] \text{ ”}.$$

Why? As $\langle x, y \rangle$ is $\mathbb{C} \times \mathbb{C}$-generic, by upward absoluteness for Σ^1_3 formulas and $(*)$ we get

$$\mathbf{V}[c_\alpha : \alpha < \omega_n] \models (\exists t)\Phi(f(x), t) \,\&\, (\exists t)\Phi(f(y), t).$$

Assume that

$$\mathbf{V}[c_\alpha : \alpha < \omega_n] \models (\exists t)(\Phi(f(x), t) \,\&\, \Phi(f(y), t)).$$

The formula here is Σ^1_3, so we have a real $z \in 2^\omega \cap \mathbf{V}[c_\alpha : \alpha < \omega_1]$ such that (by Π^1_2 absoluteness)

$$(**) \qquad \mathbf{V}'[x, y, z] \models (\exists t)[\Phi(f(x), t) \,\&\, \Phi(f(y), t)].$$

This real is added by one Cohen real over $\mathbf{V}'[x, y]$. So we may choose z to be a Cohen real over $\mathbf{V}'[x, y]$ (Σ^1_3-upward absoluteness again). Then $\langle x, y, z\rangle$ is $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$-generic over $\mathbf{V}'$ and $(**)$ contradicts $(*)$.

Now, working in $\mathbf{V}[c_\alpha : \alpha < \omega_n]$, we see that for distinct $x, y \in P$ the sections $B_{f(x)}, B_{f(y)}$ are disjoint and nonempty. Hence the function f is one-to-one on the perfect P and we can easily get a required perfect P'. ⊣

Definition 4.2. We say that a Borel ideal I on 2^ω has a Π^1_n definition if there is a Π^1_n-formula $\Phi(x)$ such that

$$\Phi(a) \equiv \text{“}a \text{ is a Borel code and the set } \#a \text{ coded by } a \text{ belongs to } I\text{”}.$$

We say that I has a projective definition if it has a Π^1_n definition for some n.

Corollary 4.3. *Let $n \geq 2$. It is consistent with ZFC $+\mathfrak{c} = \omega_n$ that:*

for each Borel ideal I on 2^ω with a Π^1_3 definition, if there exists a sequence $\langle B_\alpha : \alpha < \omega_2\rangle$ of disjoint Borel sets not belonging to I then I satisfies (M).

In particular the statement

“(B) $\Rightarrow$ (M) *for Borel ideals which are* Π^1_3”

is consistent.

Proof. Work in the model of Theorem 4.1 (so after adding ω_n Cohen reals to a model of CH). Suppose that I is a Borel ideal with Π^1_3 definition and let $\Psi(x)$ be the Π^1_3 formula witnessing it. Let

$$\Phi(x, y) \equiv \text{“}x \text{ is a Borel code } \&\neg\Psi(x) \,\&\, y \in \#x\text{”}.$$

This is a Σ^1_3 formula defining a Σ^1_3 subset B of $2^\omega \times 2^\omega$. If there are ω_2 pairwise disjoint I-positive Borel sets then they determine ω_2 pairwise disjoint nonempty sections of the set B. By Theorem 4.1 we can find a perfect set $P \subseteq 2^\omega$ such that $\{B_x : x \in P\}$ is a family of disjoint nonempty sets. Define a function $f : 2^\omega \to P$ by:

for $y \in 2^\omega$, if there is $x \in P$ such that $(x, y) \in B$ then $f(y)$ is this (unique) x, otherwise $f(y) = x_0$ where x_0 is a fixed element of P.

Note that the set $\{(x, y) \in B : x \in P\}$ is Borel and has the property that its projection onto y's axes is one-to-one. Consequently the set

$$\{y \in 2^\omega : (\exists x \in P)((x, y) \in B)\}$$

is Borel and hence the function f is Borel. Thus f witnesses (M) for the ideal I. (Note that no harm is done that the function is onto P instead of 2^ω.) ⊣

Remark 1. (1) If we start with a model for CH and add simultaneously ω_n random reals over $\mathbf{V}$ (by the measure algebra on 2^{ω_n}) then in the resulting model we will have a corresponding property for Σ^1_2 subsets of the plane $2^\omega \times 2^\omega$ and the ideals with Π^1_2 definitions. The proof is essentially the same as in Theorem 4.1. The only difference is that the perfect set P is a perfect set of “sufficiently random” reals. For a countable elementary submodel $N \in \mathbf{V}'$ of $\mathscr{H}(\chi)^{\mathbf{V}'}$ we choose by the

Mycielski theorem (cf. [13]) a perfect set $P \in \mathbf{V}[r_\alpha : \alpha < \omega_2]$ such that each two distinct members of P are mutually random over N.

(2) Note that the demand that the ideal I in Corollary 4.3 has to admit ω_2 disjoint Borel I-positive sets is not an accident. By Conclusion 5.6, the failure of ccc is not enough.

(3) In the presence of large cardinals we may get more, see Theorem 4.4 below.

THEOREM 4.4. *Assume that κ is a weakly compact cardinal and $\theta \geq \kappa^+$ is a cardinal such that $\theta^{<\kappa} = \theta$. Then there exists a κ-cc forcing notion $\mathbb{P}$ which forces*

" $\mathfrak{c} = \theta$ *and*
if I is a Borel ideal with a projective definition such that there is a sequence $\langle B_\alpha : \alpha < \omega_2 \rangle$ of disjoint Borel sets not belonging to I then I satisfies (M)"

PROOF. The forcing notion $\mathbb{P}$ is the limit $\mathbb{P}_\theta$ of the finite support iteration $\langle \mathbb{P}_0, \dot{\mathbb{Q}}_\alpha : \alpha < \theta \rangle$ such that $\mathbb{P}_0 = \mathrm{Coll}(\omega, {<}\kappa)$ and each $\dot{\mathbb{Q}}_\alpha$ is the Cohen forcing notion. Now repeat the arguments of Theorem 4.1 with $\mathbf{V}^{\mathbb{P}_0}$ as the ground model, to conclude that in $\mathbf{V}^{\mathbb{P}_\theta}$

"if $B \subseteq 2^\omega \times 2^\omega$ is a projective set which has ω_2 disjoint non-empty sections then B has a perfect set of disjoint non-empty sections."

The point is that projective formulas are absolute between intermediate models containing $\mathbf{V}^{\mathbb{P}_0}$ (see e.g., the explanations to [8, 6.5]). We finish as in Corollary 4.3. $\dashv$

THEOREM 4.5. *Assume that I is a Borel ideal with Π^1_2 definition. Suppose that there exists a Borel function $f : 2^\omega \to 2^\omega$ such that $f^{-1}[\{x_\alpha\}] \notin I$ for some distinct points $x_\alpha, \alpha < \omega_2$. Then there is a perfect set $P \subseteq 2^\omega$ such that*

$$(\forall x \in P)(f^{-1}[\{x\}] \notin I).$$

Consequently, the ideal I has property (M).

PROOF. Let $\Phi(x)$ be the Π^1_2 definition of the ideal I. Consider the relation E on 2^ω:

$$xEy \iff (x = y \text{ or } (\Phi(f^{-1}[\{x\}]) \,\&\, \Phi(f^{-1}[\{y\}]))).$$

This is an equivalence Π^1_2 relation with at least ω_2 classes. As E remains an equivalence relation after adding a Cohen real we may apply [6] (or [15]) to get a perfect set of pairwise nonequivalent elements. Removing at most one point from this set we find the desired one. $\dashv$

REMARK. In the above theorem, if the ideal I has a Π^1_1 definition then it is enough to assume that there are ω_1 respective points x_α for f.

§5. An ideal on trees. Here we present a Borel σ-ideal I^* with Π^1_2 definition which (in ZFC) satisfies the ω_2-cc but does not have the ccc. Thus if CH fails then this ideal cannot have the property (M). But we can conclude this even under CH, if we have enough Cohen reals (see Conclusion 5.6).

DEFINITION 5.1. Let $\mathbf{Tr}$ be the set of all subtrees T of $\omega^{<\omega}$ and let

$$\mathbf{Trw} = \{T \in \mathbf{Tr} : T \text{ is well founded }\}.$$

For a tree $T \in \mathbf{Trw}$ let $h_T : T \to \omega_1$ be the canonical rank function. For $\alpha < \omega_1$ put

$$A_\alpha = \{T \in \mathbf{Trw} : h_T(\langle\rangle) = \alpha\}.$$

Clearly $\mathbf{Tr}$ is a closed subset of the product space $2^{(\omega^{<\omega})}$, so it is a Polish space with the respective topology. The set $\mathbf{Trw}$ is a Π^1_1-subset of it and the sets A_α are Borel subsets of $\mathbf{Tr}$.

The ideal I^* will live on the space $\mathbf{Tr}$. To define it we introduce topologies τ_α on the sets A_α (for $\alpha < \omega_1$). Fix $\alpha < \omega_1$ for a moment. The basis of the topology τ_α consists of all sets of the form

$$U(n, T) = \{T' \in A_\alpha : T' \cap n^{\leq n} = T \cap n^{\leq n} \;\&\; h_{T'}\restriction n^{\leq n} = h_T \restriction n^{\leq n}\}$$

for $T \in A_\alpha$, $n \in \omega$. It should be clear that this is a (countable) basis of a topology.

LEMMA 5.2. (1) *$\langle A_\alpha, \tau_\alpha\rangle$ is a Baire space.*
(2) *If $\alpha > 0$ then there is no isolated point in τ_α.*
(3) *Each Borel (in the standard topology) subset of A_α is τ_α-Borel.*

PROOF. (1) Suppose that $O_k \subseteq A_\alpha$ are τ_α-open dense (for $k < \omega$) and that $T \in A_\alpha$, $n \in \omega$. For each limit $\beta \leq \alpha$ fix an increasing sequence $\beta(m) \to \beta$. Next define inductively sequences $\langle n_k : k \in \omega\rangle$, $\langle T_k : k \in \omega\rangle$ such that

(α) $n_k < n_{k+1} < \omega$, $T_k \in A_\alpha$, $n_0 = n$, $T_0 = T$,
(β) $U(n_{k+1}, T_{k+1}) \subseteq U(n_k, T_k) \cap O_k$,
(γ) for each $\nu \in T_k \cap n_k^{\leq n_k}$, if $\beta = h_{T_k}(\nu)$ is a limit ordinal then for each $\ell \leq k$ there is $\nu' \in T_{k+1} \cap n_{k+1}^{\leq n_{k+1}}$ such that $\beta(\ell) \leq h_{T_{k+1}}(\nu')$ and $\nu \lhd \nu'$, and if $\beta = h_{T_k}(\nu)$ is a successor ordinal then there is $\nu' \in T_{k+1} \cap n_{k+1}^{\leq n_{k+1}}$ extending ν and such that $h_{T_{k+1}}(\nu) = \beta - 1$.

Put $T^* = \bigcup_{k\in\omega} T_k \cap n_k^{\leq n_k}$, $h^* = \bigcup_{k\in\omega} h_{T_k}\restriction n_k^{\leq n_k}$. Then $h^* : T^* \to \alpha + 1$ is a rank function (so T^* is well founded). Moreover, the condition (γ) guarantees that it is the canonical rank function on T^* and hence $T^* \in \bigcap_{k\in\omega} O_k \cap U(n, T)$.
The assertions (2) and (3) should be clear. ⊣

DEFINITION 5.3. The ideal I^* consists of subsets of Borel sets $B \subseteq \mathbf{Tr}$ such that

$$(\forall 0 < \alpha < \omega_1)(B \cap A_\alpha \text{ is } \tau_\alpha\text{-meager}).$$

PROPOSITION 5.4. *I^* is a non-trivial Borel σ-ideal on* $\mathbf{Tr}$ *which does not satisfy the ccc but satisfies the ω_2-cc.*

PROOF. As $\{A_\alpha : \alpha < \omega_1\}$ is a partition of $\mathbf{Trw}$ and each A_α is not τ_α-meager (by Lemma 5.2), we get that I^* is a non-trivial σ-ideal on $\mathbf{Tr}$. Since each set A_α is not in I^*, the ccc fails for I^*. If $\{B_\zeta : \zeta \in \omega_2\}$ is a family of I^*-almost disjoint I^*-positive Borel sets then for some $\alpha < \omega_1$ and a set $Z \in [\omega_2]^{\omega_2}$, $\{B_\zeta \cap A_\alpha : \zeta \in Z\}$ is a family of τ_α-Borel τ_α-non-meager (τ_α-meager)-almost disjoint sets. This contradicts the fact that the topology τ_α has a countable basis (and each basic set is not τ_α-meager). ⊣

Now, we want to estimate the complexity of I^*.

PROPOSITION 5.5. *The ideal I^* has a Π^1_2 definition.*

PROOF. Fix $\alpha < \omega_1$. A basic open subset of A_α (in the topology τ_α) is determined by a pair $\langle F, h\rangle$, where $F \subseteq \omega^{<\omega}$ is a finite tree and $h : F \to \alpha + 1$ is a rank function such that $h(\langle\rangle) = \alpha$. The basic sets $U(F, h)$ are Borel subsets of A_α (in the standard topology). Let

$$\begin{aligned}\Phi_{\rm nwd}(a, \alpha) \equiv\ & \text{“}a \text{ is a Borel code}\\ & \&(\forall\langle F, h\rangle)(\exists\langle F', h'\rangle)[F \subseteq F' \ \&\ h \subseteq h' \ \&\ U(F', h') \cap \#a = \emptyset]\text{”}.\end{aligned}$$

It should be clear that $\Phi_{\rm nwd}(\cdot, \alpha)$ is a Π^1_1-formula. Consequently, the formula “a is a Borel code and the set $\#a \cap A_\alpha$ is τ_α-nowhere dense” is Π^1_1. Let

$$\begin{aligned}\Phi_{\rm meager}(a, \alpha) \equiv\ & \text{“}a \text{ is a Borel code}\\ & \&(\exists\langle c_n : n \in \omega\rangle)[(\forall n)\Phi_{\rm nwd}(c_n, \alpha) \ \&\ \#a \cap A_\alpha \subseteq \textstyle\bigcup_{n\in\omega} \#c_n]\text{”}.\end{aligned}$$

This formula says that the intersection $\#a \cap A_\alpha$ is τ_α-meager; it is Σ^1_2. On the other hand, its negation is equivalent to

$$\begin{aligned}&\Phi_{\neg\,\rm meager}(a, \alpha) \equiv \text{“}a \text{ is not a Borel code or}\\ &\quad(\exists\langle F, h\rangle)(\exists\langle c_n : n \in \omega\rangle)[(\forall n)\Phi_{\rm nwd}(c_n, \alpha) \ \&\ U(F, h) \subseteq \#a \cup \textstyle\bigcup_{n\in\omega} \#c_n]\text{”}\end{aligned}$$

which is Σ^1_2 too. Consequently,

$$\text{“}a \text{ is a Borel code such that } \#a \cap A_\alpha \text{ is } \tau_\alpha\text{-meager”}$$

is absolute between all transitive models of (a large enough part of) ZFC containing $a, \alpha, \ldots$. Now, we can give a Π^1_2-definition of the ideal I^*:

$$\begin{aligned}\Phi(a) \equiv\ & a \text{ is a Borel code}\\ & \&(\forall E \subseteq \omega \times \omega)[E \text{ is not well founded or } \langle\omega, E\rangle \not\models \Theta^* \text{ or}\\ & a \text{ is not encoded in } \langle\omega, E\rangle \text{ or } \langle\omega, E\rangle \not\models \mathbf{V} = \mathbf{L}[a] \text{ or}\\ & \langle\omega, E\rangle \models \text{“}(\forall\alpha \in \omega_1)(\Phi_{\rm meager}(a, \alpha))\text{”}].\end{aligned}$$

The Θ^* above is a sentence carrying a large part of information on ZFC, in particular for each transitive set M, $M \models \Theta^*$ should imply that M is A-adequate for all $A \in M$ and the absoluteness of Σ^1_1, Π^1_1 formulas holds for M (see [7, Ch. 2, §15]). Thus, if E is a well founded relation on ω, a is a real encoded in E (which means that a belongs to the transitive collapse of E) and $\langle\omega, E\rangle \models \Theta^* + \mathbf{V} = \mathbf{L}[a]$ then the transitive collapse of E is $\mathbf{L}_\alpha[a]$ for some $\alpha < \omega_1$. Now, if $\Phi(a)$ holds then for all $\alpha < \omega_1$ such that $\mathbf{L}_\alpha[a]$ is a-adequate we have

$$\mathbf{L}_\alpha[a] \models \text{“}(\forall\beta < \omega_1)\Phi_{\rm meager}(a, \beta)\text{”}.$$

The absoluteness implies that if $\mathbf{L}_\alpha[a] \models \Phi_{\rm meager}(a, \beta)$ then $\Phi_{\rm meager}(a, \beta)$. Consequently, $\Phi(a)$ implies $(\forall\beta < \omega_1)\Phi_{\rm meager}(a, \beta)$. Suppose now that a is a Borel code for which $\Phi(a)$ fails. Then for some $\alpha < \omega_1$ (suitably closed) we have

$$\mathbf{L}_\alpha[a] \models (\exists\beta)(\neg\Phi_{\rm meager}(a, \beta))$$

and hence for some $\beta < \alpha$ we get

$$\mathbf{L}_\alpha[a] \models \Phi_{\neg\,\rm meager}(a, \beta).$$

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Once again the absoluteness implies that $\Phi_{\neg\,\text{meager}}(a,\beta)$ holds. Consequently, $\neg\Phi(a)$ implies that there is $\beta<\omega_1$ such that $\Phi_{\neg\,\text{meager}}(a,\beta)$ and finally,

$$\Phi(a) \iff a \text{ is a Borel code } \& \#a \in I^*.$$

It should be clear that $\Phi(x)$ is a Π^1_2 formula. ⊣

CONCLUSION 5.6. Assume that either CH fails or every Δ^1_2 set of reals has the Baire property. Then the ideal I^* does not have the property (M) [and: it satisfies the ω_2-cc, it does not satisfy the ccc and has a Π^1_2 definition].

PROOF. If we are in the situation of $\neg$CH then (M) cannot hold because of ω_2-cc. So assume that all Δ^1_2 sets of reals have Baire property.

Suppose that $f: 2^\omega \to 2^\omega$ is a Borel function (of course, it is coded by a real). Let $\langle c_\alpha : \alpha<\omega_2\rangle$ be a $\mathbb{C}_{\omega_2}$-generic sequence of Cohen reals. In $\mathbf{V}[c_\alpha:\alpha<\omega_2]$ we have that for sufficiently large α, $f^{-1}[\{c_\alpha\}]\in I^*$. Since the definition of I^* is absolute (and it involves no parameters),

$$\mathbf{V}[c_\alpha:\alpha<\omega_2]\models f^{-1}[\{c_\beta\}]\in I^* \quad\text{implies}\quad \mathbf{L}[f][c_\beta]\models f^{-1}[\{c_\beta\}]\in I^*.$$

By density arguments and the ccc of $\mathbb{C}_{\omega_2}$ we conclude that

$$\mathbf{L}[f]\models \text{“} \Vdash_{\mathbb{C}} f^{-1}[\{\dot c\}]\in I^*\text{”}$$

where $\dot c$ is the canonical $\mathbb{C}$-name for the generic Cohen real. Our assumption that Δ^1_2 sets of reals have the Baire property is equivalent to the statement

“for each real r there exists a Cohen real over $\mathbf{L}[r]$”

(see e.g., [9]). Let $c\in\mathbf{V}$ be a Cohen real over $\mathbf{L}[f]$. Then

$$\mathbf{L}[f][c]\models f^{-1}[\{c\}]\in I^*.$$

By Shoenfield absoluteness we conclude $\mathbf{V}\models f^{-1}[\{c\}]\in I^*$. Consequently, f cannot witness the property (M) for I^*. ⊣

§6. Other variants of (M).

The following natural question concerning the hereditary behavior of our conditions (B), (M) and (D) arises:

> If an ideal I on X satisfies one of those conditions and $E\notin I$ is a Borel subset of X, is it true that $I\cap P(E)$ satisfies the same condition (on E; at this moment we consider the ideal on the space E and modify respectively the sense of (B), (M) and (D)))?

When we do not assume a group structure on X and the invariance of I, one can easily construct I which makes the answer negative. If the invariance of I is supposed, the problem becomes less trivial.

EXAMPLE. Let I be the ideal of null sets with respect to 1-dimensional Hausdorff measure (cf. [5]) on $\mathbf{R}^2$ treated as an additive group. Clearly, I is invariant. It satisfies (M) since the fibers of the continuous function given by $f(x,y)=x$ are the lines $\{x\}\times\mathbf{R}$, for $x\in\mathbf{R}$, which are not in I. However, 1-dimensional Hausdorff measure on $\mathbf{R}$ coincides with the linear Lebesgue measure, so $I\cap P(\{x\}\times\mathbf{R})$ does not satisfy (B) since ccc works there.

Further, we shall concentrate on some hereditary versions of (M) connected with open sets.

DEFINITION 6.1. Assume that I is an ideal on an uncountable Polish space X and that each open nonempty subset of X is not in I.

(1) We say that I has *property* (M) *relatively to a set* $E \subseteq X$ (in short, I has (M) rel E) if and only if there is a Borel function $f : E \to X$ whose all fibers $f^{-1}[\{x\}]$, $x \in X$, are not in I.

(2) We say that I has *property* (M$'$) if and only if it has property (M) rel U for each nonempty open set $U \subseteq X$.

(3) Let $x \in X$. We say that I has *property* (M'_x) if and only if it has property (M) rel U for each open neighborhood U of x.

(4) We say that I has *property* (M*) if and only if there is a Borel function $f : X \to X$ such that $f^{-1}[\{x\}] \cap U \notin I$ for any $x \in X$ and open non-void $U \subseteq X$.

REMARKS.

(1) Plainly, (M) in the sense of Definition 1.1 is (M) rel X. Moreover, if E is Borel, (M) rel E implies (M) rel X since one can consider any Borel extension of the respective Borel function $f : E \to X$ to the whole X.

(2) Obviously, I has (M$'$) iff it has (M'_x) for each x from a fixed dense set in X. In some cases (M'_x) satisfied for one point x implies (M$'$). This holds if x is taken from a fixed dense set whose any two points s, t have homeomorphic open neighborhoods U_s, U_t and the homeomorphism preserves I. In particular, if X is a group, $x \in X$ is fixed, Q is a dense set in X and I is Q-invariant (i.e., $E \in I$ and $t \in Q$ imply $E + t \in I$) then (M'_x) guarantees (M$'$).

(3) Studies of (M*) were initiated in [2] where some examples are given. Condition (M*) for the ideal of nowhere dense sets means the existence of a Borel function f from X onto X with dense fibers. For $X = (0, 1)$ such functions are known as being strongly Darboux (cf. [4]). From Mauldin's proof in [10] it follows that the σ-ideal generated by closed Lebesgue null sets satisfies (M*) (for details, see [2]).

(4) Evidently, (M*) $\Rightarrow$ (M$'$) $\Rightarrow$ (M). It is interesting to know whether those implications can be reversed. A simple method producing ideals (σ-ideals) with property (M) and without (M$'$) follows from the example given in [2]. If one part of an ideal I, defined on a Borel set $B \subseteq X$ has (M) rel B, and the remaining part, defined on $X \setminus B$ with $\mathrm{int}(X \setminus B) \neq \emptyset$ has not (M) rel $X \setminus B$ (for instance, ccc holds there) then the ideal is as desired. If we want I to be a σ-ideal invariant in the group X then the situation is different.

THEOREM 6.2. *Suppose that I is an invariant σ-ideal on* $\mathbf{R}$. *If the ideal I has the property* (M) *then it has the property* (M*).

PROOF. We start with two claims of a general character.

CLAIM 6.2.1. *Suppose that $f : \mathbf{R} \longrightarrow \mathbf{R}$ is a Borel function. Then there exists a perfect set $P \subseteq \mathbf{R}$ such that $f^{-1}[P]$ is both meager and Lebesgue null.*

PROOF OF THE CLAIM. It follows directly from the fact that each perfect set can be divided into continuum disjoint perfect sets and both the ideal of meager sets and the ideal of null sets satisfy ccc.

CLAIM 6.2.2. *Suppose that $U \subseteq \mathbf{R}$ is an open set and $A \subseteq \mathbf{R}$ is a meager null set. Then there exist disjoint sets $D_k \subseteq U$ and reals x_k (for $k \in \omega$) such that $\bigcup_{k\in\omega} D_k$ is nowhere dense and $A \subseteq \bigcup_{k\in\omega} D_k + x_k$.*

PROOF OF THE CLAIM. First we prove the claim under the assumption that A is nowhere dense (and null). For this choose an interval $J \subseteq U$. Since A is null we can find (open) intervals $\langle J_k : k \in \omega\rangle$ such that

$$A \subseteq \bigcup_{k\in\omega} J_k \qquad \text{and} \qquad \sum_{k\in\omega} |J_k| < |J|$$

(here $|J|$ stands for the length of J). Choose disjoint open intervals $J_k^* \subseteq J$ (for $k \in \omega$) such that $|J_k^*| = |J_k|$ (for $k \in \omega$). Let reals x_k be such that $J_k^* + x_k = J_k$. For $k \in \omega$ we put

$$D_k = (A \cap J_k) - x_k.$$

Clearly $A \subseteq \bigcup_{k\in\omega} D_k + x_k$. Since the sets D_k are nowhere dense and contained in disjoint open intervals we get that their union $\bigcup_{k\in\omega} D_k$ is nowhere dense.

If now A is just a meager null set then we represent it as a union $A = \bigcup_{n\in\omega} A_n$, where each set A_n is nowhere dense and null. Take disjoint open sets $U_n \subseteq U$ (for $n \in \omega$) and apply the previous procedure to each pair $\langle A_n, U_n\rangle$ getting suitable D_k^n, x_k^n. Then $\langle D_k^n, x_k^n : n, k \in \omega\rangle$ is as required for A and U (proving the claim).

Suppose now that I is an invariant σ-ideal on $\mathbf{R}$ and that $f : \mathbf{R} \longrightarrow \mathbf{R}$ is a Borel function witnessing the property (M) for I. By Claim 6.2.1 we find a perfect set $P \subseteq \mathbf{R}$ such that $f^{-1}[P]$ is both meager and null.

Let $\{U_n : n \in \omega\}$ be an enumeration of all rational open intervals in $\mathbf{R}$. As finite union of nowhere dense sets is nowhere dense, we may apply Claim 6.2.2 and choose inductively (by induction on $n \in \omega$) closed sets $D_{n,k}$ and reals $x_{n,k}$ such that

(a) $D_{n,k}$ (for $n, k \in \omega$) are pairwise disjoint

and for each $n \in \omega$:

(b) $\bigcup_{k\in\omega} D_{n,k}$ is a nowhere dense subset of U_n,

(c) $f^{-1}[P] \subseteq \bigcup_{k\in\omega} D_{n,k} + x_{n,k}$.

Now we define a function $f^* : \mathbf{R} \longrightarrow P$ by

$$f^*(x) = \begin{cases} f(x + x_{n,k}) & \text{if } x \in D_{n,k}, \quad n, k \in \omega, \quad \text{and } f(x + x_{n,k}) \in P, \\ y_0 & \text{otherwise,} \end{cases}$$

where y_0 is a fixed element of P. Clearly the function f^* is Borel. We claim that it witnesses the property (M*) for I. Why? Suppose that $U \subseteq \mathbf{R}$ is an open nonempty set and $y \in P$. Take $n \in \omega$ such that $U_n \subseteq U$. We know that $f^{-1}[P] \subseteq \bigcup_{k\in\omega} D_{n,k} + x_{n,k}$ and $f^{-1}[\{y\}] \notin I$. Since I is σ-complete, we find $k \in \omega$ such that

$$f^{-1}[\{y\}] \cap (D_{n,k} + x_{n,k}) \notin I.$$

As I is translation invariant we get

$$(f^{-1}[\{y\}] - x_{n,k}) \cap D_{n,k} \notin I.$$

Clearly $(f^{-1}[\{y\}] - x_{n,k}) \cap D_{n,k} \subseteq (f^*)^{-1}[\{y\}] \cap U$ so the last set does not belong to I. The theorem is proved. $\dashv$

In the above proof the use of Lebesgue measure is important. In Theorem 5.2 we can replace **R** by e.g., 2^ω but we do not know if we can have a corresponding result for all Polish groups. Moreover we do not know if we can omit the assumption of σ-completeness of I (i.e., prove the theorem for (finitely additive) ideals on **R** which do not contain nonempty open sets).

For the question of dependences between (M), (M′) and (M*) the following simple theorem seems useful.

THEOREM 6.3. *Let I be an ideal on X and $\{U_n : n \in \omega\}$ - a base of open sets in X. For $f : X \to X$ define $H_n(f) = \{y \in X : U_n \cap f^{-1}[\{y\}] \notin I\}$.*

1. *Let $x \in X$. Condition (M'_x) holds iff there is a Borel function $f : X \to X$ such that, for each $n \in \omega$ with $x \in U_n$, the set $H_n(f)$ contains a perfect set.*
2. *Condition (M') holds iff there is a Borel function $f : X \to X$ such that each set $H_n(f)$, $n \in \omega$, contains a perfect set.*
3. *Condition (M^*) holds iff there is a Borel function $f : X \to X$ such that $\bigcap_{n\in\omega} H_n(f)$ contains a perfect set.*
4. *Suppose that I is a σ-ideal, $f : X \to X$ is a Borel function such that $f^{-1}[\{x\}] \notin I$ for all $x \in X$, and each set $H_n(f)$, $n \in \omega$, either is countable or contains a perfect set. Then (M'_x) holds for some $x \in X$.*

PROOF. (1) Necessity. Fix $n \in \omega$ such that $x \in U_n$. Then (M) rel U_n holds. It suffices to extend the respective Borel function $g : U_n \to X$ to a Borel function defined on the whole X.

Sufficiency. Fix $n \in \omega$ such that $x \in U_n$. Let $P \subseteq H_n(f)$ be a perfect set and let $B = f^{-1}[P]$. Extend $f{\restriction}B$ to a Borel function $g : U_n \to P$. Consider a Borel function h from P onto X. Then $h \circ g$ witnesses (M) rel U_n.

The proofs of (2) and (3) are analogous.

(4) Suppose it is not the case. Thus for each $x \in X$ choose $n_x \in \omega$ such that I has not (M) rel U_{n_x}. For $T = \{n_x : x \in X\}$ we get $X = \bigcup_{n\in T} U_n$. Since I is a σ-ideal, therefore, by the properties of f, we get $X = \bigcup_{n\in T} H_n(f)$. Hence, by the assumption, there is $n \in T$ such that $H_n(f)$ contains a perfect set. Now, as in the proof of (1), we infer that I has (M) rel U_n, a contradiction. ⊣

EXAMPLE. Let I_2 (I_ω) be a Mycielski ideal (defined in [14]) on 2^ω (ω^ω, respectively) generated by the respective system $\{K_t : t \in 2^{<\omega}\}$ of infinite sets in ω. It is shown in [3] that I_2, I_ω satisfy (M). By Theorem 6.2 the ideal I_2 satisfies (M*). However, it is not clear if I_ω does. Let us modify the proof of (M) for I_ω to get (M) rel $[s]$ where $[s]$, for $s \in \omega^{<\omega}$, is a basic open set in ω^ω. To this end choose K_t such that $K_t \cap lh(s) = \emptyset$. Assume that $\omega \setminus (K_t \cup lh(s)) = \{n_0, n_1, \dots\}$ and define $f : [s] \to [s]$ by

$$f(s^\frown\langle x_0, x_1, \dots\rangle) = s^\frown\langle x_{n_0}, x_{n_1}, \dots\rangle.$$

As in [3] we observe that f realizes (M) rel $[s]$. Consequently, I_ω satisfies (M′).

PROBLEM 6.4. (1) Does Mycielski ideal I_ω on ω^ω satisfy (M*) ?

(2) Does there exist a translation invariant σ-ideal I on $\omega^\omega \equiv \mathbf{Z}^\omega$ which satisfies (M) but not (M*) ?

(3) Is there (necessarily finitely additive) invariant ideal I on **R** with (M) but without (M*) (and such that no nonempty open set is in I)?

REMARK. By assertion (4) of Theorem 6.3, if the axiom of determinacy AD (cf. [7]) is assumed, we get (M) $\iff$ $(\exists x \in X)(\mathrm{M}'_x)$ for any σ-ideal and consequently, (M) $\iff$ (M$'$) for any invariant σ-ideal. The following operation (cf. [3]) $\Phi_I : P(X \times X) \to P(X)$ plays an important role. Namely, for $E \subseteq X \times X$, let $\Phi_I(E) = \{y \in X : E^y \notin I\}$ where $E^y = \{x \in X : \langle x, y \rangle \in E\}$. If Φ_I sends Borel sets into analytic sets (respectively, into projective sets when projective determinacy is assumed) then statement (4) of Theorem 6.3 works.

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