Sh:107

Annals of Pure and Applied Logic 25 (1983) 183–212
North-Holland

MODELS WITH SECOND ORDER PROPERTIES IV. A GENERAL METHOD AND ELIMINATING DIAMONDS

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Communicated by M. Rabin
Received 1 June 1979; revised 10 June 1983

We show how to build various models of first-order theories, which also have properties like: tree with only definable branches, atomic Boolean algebras or ordered fields with only definable automorphisms.

For this we use a set-theoretic assertion, which may be interesting by itself on the existence of quite generic subsets of suitable partial orders of power λ^+, which follows from $\diamondsuit_\lambda$ and even weaker hypotheses (e.g., $\lambda = \aleph_0$, or λ strongly inaccessible). For a related assertion, which is equivalent to the morass see Shelah and Stanley [16].

The various specific constructions serve also as examples of how to use this set-theoretic lemma. We apply the method to construct rigid ordered fields, rigid atomic Boolean algebras, trees with only definable branches; all in successors of regular cardinals under appropriate set-theoretic assumptions. So we are able to answer (under suitable set-theoretic assumptions) the following algebraic question.

Saltzman's Question. Is there a rigid real closed field, which is not a subfield of the reals?

0. Introduction

We continue here [8] and [9]. In fact, Sections 1, 3, 4 and half of Section 2 were done together with [8] and [9] (but not the use of $(D\ell)_\lambda$); Sections 2 and 5 were done later.

We thank Wilfred Hodges for rewriting Sections 1 and 2 (which the author somewhat revised mainly adding 2.2–2.7).

We try here to axiomatize the building of a model of power λ^+ by an increasing chain of models of power λ, using terminology of forcing (note the close affinity of genericity and omitting types). See [14] (for $\lambda = \aleph_1$) and Giorgetta and Shelah [3] (on $2^{\aleph_0}$). Let us review the paper.

1. Γ-big formulas

Γ will tell us which formulas $\varphi(x)$ (over M) are big, so that the non Γ-big formulas over M form an ideal; we shall describe how to build generically an elementary extension of M in which a distinguished element realizes a Γ-big type over M. In our case $\|M\| \leq \lambda$, and the extension is built by approximations of power $<\lambda$.

2. *Iteration and an existence theorem*

We describe how to build a model of power λ^+ by a continuous chain M_α ($\alpha < \lambda^+$), each M_α of power $\leqslant \lambda$, where $M_{\alpha+1}$ is built as in Section 1 (over M_α). We describe the corresponding forcing. A closely related work is Bruce [7].

The problem is how for limit $\delta < \lambda^+$ of cofinality $<\lambda$ to continue. We state a theorem (Theorem 2.12) saying we could do it, but we delay the proof of that theorem to our next paper [13].

Later we state a set-theoretic lemma, Theorem 2.17, which follows from the diamond for λ ($\diamondsuit_\lambda$). This lemma says that quite generic subsets exist for λ-complete partial orders $\boldsymbol{P}$ of power λ^+, (which satisfy the λ^+-chain condition, and more conditions). For a companion lemma, with stronger set-theoretic hypothesis (a suitable morass) but having a weaker condition on $\boldsymbol{P}$, see Shelah and Stanley [16].

Theorem 2.17 implies that the construction in 2.12 works.

Problem. In Theorem 2.17 can we replace the hypothesis $(\mathrm{D}\ell)_\lambda$ by $\lambda = \lambda^{<\lambda}$?

Remarks. We assume $\aleph_0 + |T| < \lambda$. If we want to deal with the case $|T| = \lambda$, we can do this, but sometimes there are problems. We cannot use [8, 1.9.1] (nor reprove it for λ-compact models). This seems crucial in 4.3 (and 5.2, 5.3). In 4.9, 5.1 there are no problems (see below). In [13] we shall deal with this, and of course with $\lambda = \aleph_1$.

However if "$\varphi(x, \bar{a})$ is Γ-big in M" is determined by the L_φ-type that $\bar{a}$ realizes in M, $L_\varphi \subseteq L$, $|L_\varphi| < \lambda$, our proofs work with no essential change.

3. *The basic lemmas of the* $(\mathrm{D}\ell)_\lambda$*-free method*

In [8, §1, §2] we use a method which does not require $(\mathrm{D}\ell)_\lambda$, (only $\lambda = \lambda^{<\lambda}$ was required), and by it we originally proved some of the theorems here. But at present it is doubtful whether it is worthwhile to work for replacing $(\mathrm{D}\ell)_\lambda$ by $\lambda = \lambda^{<\lambda}$ as

(a) Given GCH this adds only one cardinal: $\aleph_1$ (see [10]).

(b) Even this gain is doubtful as the problem above is open.

Here we represent the basic lemmas for general Γ which make the older method of [8] work. For understanding their value you should look there.

Note that the discussion above does not invalidate their value, as they are helpful even when there is no λ such that $\lambda = \lambda^{<\lambda} > \aleph_0$ (see [9].)

4. *Applications with diamond*

We represent here some applications assuming always $(\mathrm{D}\ell)_\lambda$ and assuming $\diamondsuit_{\{\delta<\lambda^+ : \mathrm{cf}\,\delta = \lambda\}}$.

In 4.1 we see (with the diamond) that our general construction suffices to do [11]; the cases mentioned in [11] for $\lambda > \aleph_0$ are left to the reader as exercises.

In 4.3 we prove that there is a model of T such that e.g., if A is an atomic Boolean algebra definable in the model, then any automorphism of A is definable in the model by a first-order formula with parameters. So here we improve the result of [8, §1]: λ is not longer required to be strongly inaccessible.

In 4.9–4.11 we prove a similar result for ordered fields.

5. More applications: trees with no branches and the strong independence property revisited under GCH

In 5.1 we prove that if T is a complete theory in $L(Q)$, with (Qx) interpreted as "there are $>\lambda$ x's", then (if $(D\ell)_\lambda$ holds) T has a model such that any definable tree whose set of levels has cofinality λ^+ has no undefinable branches.

In 5.2 we reprove 4.3 under the weaker hypothesis $(D\ell)_\lambda + 2^\lambda = \lambda^+$, and in 5.3 we reprove 4.3 under the hypothesis $(D\ell)_\lambda + (\exists\mu)$ $(\mu < \lambda \leq 2^\mu)$.

Concluding remarks. (1) In order to really understand the method presented here in Sections 1 and 2 the reader has to read the applications. Note also that instead of λ^+ we can use other orders of power λ^+, preferably λ^+-like, and 2.17 can still handle them. I think the method should be applicable to most problems of the form: is there a model of T with no expansions satisfying some T', $T \subseteq T'$ (when λ-saturation does not make a harm.)

(2) A serious problem is to get similar results from ZFC alone.

(3) Salzman has aked: Is every rigid ordered field embeddable into the reals? The answer is negative (under suitable set theoretic hypotheses). This follows from 4.11 and the observation that if there are parameters $\bar{a}$ in a real-closed field K such that some formula $\phi(x, y, \bar{a})$ defines a non-trivial automorphism of K, then (by completeness of the theory of real-closed fields) the same must hold for the reals, which is impossible.

(4) There is some parallelism between this paper and [12]. There we interpret 'complicated theory' by unstable, and prove that T is unstable iff it has the independence property or the strict order property. Here we prove results on existence of models with only definable automorphisms for essentially two cases—the strong independence property and ordered field (the field structure is not essential—just used to carry a definition of an automorphism from an interval to the whole order).

This parallelism is not totally incidential; look at the following property of T:

(Pr) For any theory T_1, $T \subseteq T_1$, and any cardinal $\lambda > |T_1|$ such that $(D\ell)_\lambda$ and $\diamondsuit_{\{\delta<\lambda^+ : \mathrm{cf}\,\delta=\lambda\}}$ hold, T_1 has a model M which satisfies:

(i) M has cardinality λ^+.

(ii) M is λ-saturated, moreover any formula which is not algebraic, is realized by $\|M\|$ elements.

(iii) Every automorphism of $M \restriction L(T)$ is definable (by an $L(T_1)$ formula with parameters).

By the methods of [12, Ch. VI, §5], it should be clear to the reader that for stable T (Pr) fails badly.[1] In fact there is T_1 such that for any $|T|^+$-saturated model of T_1, $M \restriction L(T)$ is saturated, provided that any infinite definable set has cardinality $\|M\|$.

From another point of view we can ask when the notion Γ-big has cases of some interest. If Δ is a finite set of formulas: $\{\varphi_l(\bar{x}; \bar{y}_l) : l < n\}$, we call $p(\bar{x})$ Γ_Δ-big if $R^m(p, \Delta, \aleph_0) \geq \omega$.

This is nonempty iff $R^m(\bar{x} = \bar{x}, \Delta, \aleph_0) \geq \omega$; this occurs for some Γ_Δ iff T is unstable.

Notice that any Γ_Δ-big type p has two contradictory Γ_Δ-big extensions and then call p not Γ_Δ-isolated. Clearly if any Γ-big type p has a Γ-big type p' extending it which is Γ-isolated, then Γ-bigness is totally uninteresting from our point of view. At last notice that if (for Γ a bigness notion) p is Γ-big but has no Γ-isolated extension, then T is unstable.

(5) In all cases, we could have defined 2^{λ^+} models, non-isomorphic and even no e.g. Boolean algebra defined in one is embeddable into a Boolean algebra defined in another.

Notation. Standard, see e.g. [12]. Note that $\varphi^{\text{if(statement)}}$ denotes φ if the statement is true and $\neg\varphi$ if the statement is false.

We let λ, κ be fixed regular cardinals, usually $\lambda = \lambda^{<\kappa} > \aleph_0$.

We let L denote a vocabulary of power $\leq\lambda$ (i.e. set of predicates, each with finite number of places. We treat function symbols and functions as predicates and relations in the usual way.)

We reserve L itself for a fixed vocabulary and let L with indexes denote extensions of it (still of power $\leq\lambda$).

We let T be a fixed complete theory in $L_{\omega,\omega}$ (i.e. first-order logic on L), always we shall assume that L is the vocabulary of the theory T.

For a model M, let L^M be its vocabulary and we call M an L^M-model. For L^1 a vocabulary, let $M^{L^1} = M \restriction L^1$ be the $(L^1 \cap L^M)$-reduct of M. We say M is an expanded model of T if M^L is a model of T. We say that $N = N_1 \vee N_2$ if $|N| = |N_1| \cup |N_2|$, $L^N = L^{N_1} \cup L^{N_2}$, and $R^N = R^{N_1} \cup R^{N_2}$ where for $R \in L^N - L^{N_\epsilon}$, we stipulate $R^{N_{3-\epsilon}} = \emptyset$. We say that N is a st. L-el. extension [i.e. strict L-elementary extension] of M if N^L is an elementary extension of M^L and $N = N^L \vee M$. We let $L^1(M)$ be L^1 extended by an individual constant for each member of M (so if $L^1 = L^M$, M has a canonical expansion to an $L^1(M)$-model M^1; we do not distinguish between M and M^1). Note $L^M \neq L(M)$.

The letter $\mathscr{L}$ denotes a fragment of some $L^1_{\lambda^+,\kappa}$ i.e. a set of $\leq\lambda$ formulas from $L^1_{\lambda^+,\kappa}$ each with $<\kappa$ free variables, containing the formulas of $L^1_{\omega,\omega}$, closed under subformulas, and if $\alpha < \kappa$, $\beta < \kappa$ and $\varphi_i \in L$ $(i < \beta)$ then $\neg\varphi_0$, $\bigwedge_{i<\beta} \varphi_i$,

[1] By [12, Lemma II 3.10] it is enough that M is $|T|^+$-saturated (or even $F^a_{\kappa_r(T)}$-saturated) and that every infinite indiscernible set I has dimension $\|M\|$. If T_1 is rich enough there is a definable set indiscernible for $L(T)$, which include infinitely many members of I (because M_1 is $|T|^+$-saturated). This set has power $\|M\|$, so we finish.

$(\exists\cdots x_j\cdots)_{j<\alpha}\varphi_0$ are in $\mathscr{L}$. We let $\mathscr{L}(M)$ be the set of $\varphi\in\mathscr{L}$ when we substitute members of M for some of the free variables.

For a set of formulas Φ, X a set of free variables let $\Phi\restriction X=\{\varphi\in\Phi:$ all free variables of φ belong to $X\}$. $\mathrm{FV}(\Phi)$ is the set of free variables of Φ.

1. Γ-big formulas

1.1. Description of the construction. Let M be an expanded model of T of cardinality λ and $\mathscr{L}$ a fragment of $L^M_{\lambda^+,\lambda}$.

We shall construct a st. L-el. extension N of M by building a continuous increasing chain $(\Phi_i)_{i<\lambda}$ of sets of formulas[2] of $L(M)$, so that $\Phi_\lambda=\bigcup_{i<\lambda}\Phi_i$ is the complete diagram of N^L. In Section 2 below we shall iterate this construction λ^+ times to produce a model of cardinality λ^+.

The chain $(\Phi_i)_{i<\lambda}$ is built by induction on i, and each Φ_i has cardinality $<\lambda$. As we build the chain we shall aim to perform three separate tasks, as follows:

Task I. The union Φ_λ must be the complete diagaam of an elementary extension N^L of M^L.

Task II. N^L must have an element (labelled x) whose type over M^L is 'big' in a sense made precise below.

Task III. Every sentence of $\mathscr{L}(N)$ which is satisfied by N must be 'forced' by some Φ_i $(i<\lambda)$.

1.1A. Remark. Instead of x we can work with a sequence of $\leqslant\lambda$ variables $\bar{x}$ and this does not change Sections 1 and 2 at all.

1.2. Definition (of bigness). (a) A *functional* (for L) is a sentence, possibly infinitary, whose symbols are symbols of L together with one new 1-ary relation symbol P. If Γ is a functional for L and $\varphi(x,\bar{y})$ is a formula (note necessarily from L), we write $\Gamma(\varphi)$ or $Qx\,\varphi(x,\bar{y})$ for the formula which results if we put $\varphi(z,\bar{y})$ in place of each subformula Pz of Γ, avoiding clash of variables. Note that the operation $\Gamma(\ )$ commutes with substituting terms for free variables, so that the expression $Qx\,\varphi(x,\bar{a})$ is unambiguous.

(b) For K a class of L-models, a *notion of bigness* (for K) is a function Γ for L such that any $M\in K$ satisfies the following axioms, for all first-order formulas $\varphi(x,\bar{y})$ and $\psi(x,\bar{y})$:

(1) $\quad\forall\bar{y}(\forall x(\varphi\to\psi)\wedge Qx\,\varphi\to Qx\,\psi)$,

(2) $\quad\forall\bar{y}(Qx(\varphi\vee\psi)\to Qx\,\varphi\vee Qx\,\psi)$,

(3) $\quad\forall\bar{y}(Qx\,\varphi\to\exists^{\geqslant 2}x\,\varphi)$,

(4) $\quad Qx\,x=x$.

[2] First-order logic.

We write $\tilde{Q}x\,\varphi$ for $\neg Qx\neg\varphi$. Note that Axiom 1 implies $\forall\bar{y}(\forall x(\varphi\to\psi)\wedge\tilde{Q}x\,\varphi\to\tilde{Q}x\,\psi)$, and Axiom 2 implies $\forall\bar{y}(\tilde{Q}x\,\varphi\wedge\tilde{Q}x\,\psi\to\tilde{Q}x\,(\varphi\wedge\psi))$.

When K is the class of models of T [of power λ] we say Γ is a notion of bigness for T [for (T,λ)].

(c) Let M be a model of K, $\bar{a}$ a sequence of elements of M and Γ a notion of bigness for K. We say that $\varphi(x,\bar{a})$ is *Γ-big* (in M) if $Qx\,\varphi(x,\bar{a})$ holds in M. A formula which is not Γ-big (in M) is *Γ-small* (in M). A Γ-big type $p(x)$ is a set of formulas $\psi(x,\bar{a})$, any conjunction of which is Γ-big

(d) *Example.* Let χ be an infinite cardinal, and let $Qx\,\varphi(x,\bar{y})$ be the formula which says "At least χ elements x satisfy $\varphi(x,\bar{y})$". This defines a notion of bigness.

1.3. Definition (of conditions). Let Γ be a notion of bigness for T and M a model of T. A *condition* (strictly, a *Γ-condition over M*) is a set Ψ of $<\lambda$ formulas of $L(M)$ whose free variables are among x and y_i $(i<\lambda)$, such that for every conjunction $\psi(x,\bar{y})$ of finitely many formulas from Ψ, $M\vDash Qx\,\exists\bar{y}\,\psi$.

Really, we should have said a (Γ,λ)-condition, and note that for $\lambda<\lambda_1$, any (Γ,λ)-condition is a (Γ,λ_1)-condition.

1.4. Lemmas (on conditions). *Γ and M are as in* 1.3.

(a) *There exist conditions.* (**Proof.** Axiom 4 in Definition 1.2.)

(b) *A subset of a condition is a condition, and the union of a chain of conditions is a condition provided it has cardinality* $<\lambda$.

(c) *If Ψ is a condition and $\varphi(x,\bar{y})$ is a formula of $L(M)$, then either $\Psi\cup\{\varphi\}$ or $\Psi\cup\{\neg\varphi\}$ is a condition.* (**Proof.** If $\Psi\cup\{\varphi\}$ is not a condition, then there is a finite part ψ of Ψ such that $M\vDash\neg Qx\,\exists\bar{y}\,(\psi\wedge\varphi)$. But by Axiom 1, $M\vDash Qx\,\exists\bar{y}\,(\psi\wedge(\varphi\vee\neg\varphi))$, and so by Axiom 2 either $M\vDash Qx\,\exists\bar{y}\,(\psi\wedge\varphi)$ or $M\vDash Qx\,\exists\bar{y}\,(\psi\wedge\neg\varphi)$. Hence $M\vDash Qx\,\exists\bar{y}\,(\psi\wedge\neg\varphi)$, and the same argument applies if we add to ψ any other formulas from Ψ.)

(d) *Let Ψ be a condition and $\Phi(x,\bar{y},z)$ a set of formulas such that for every conjunction $\varphi(x,\bar{y},z)$ of a finite number of formulas from Φ, Ψ contains $\exists z\,\varphi(x,\bar{y},z)$. Then for some variable y_i, $\Psi\cup\Phi(x,\bar{y},y_i)$ is a condition.* (**Proof.** If not, then there are finite parts $\psi(x,\bar{y})$ and $\varphi(x,\bar{y},z)$ of Ψ and Φ respectively such that $M\vDash\neg Qx\,\exists\bar{y}y_i\,(\psi\wedge\varphi(x,\bar{y},y_i))$, contradicting the assumption that $M\vDash Qx\,\exists\bar{y}\,(\psi\wedge\exists x\varphi)$. Here y_i is any variable not in Ψ or Φ.)

(e) *Task* I *can be performed. More precisely let x be any set of successor ordinals which is cofinal in λ, and suppose a continuous chain of conditions $(\Phi_i)_{i<\lambda}$ is being constructed by induction on i. Then by suitable choices of Φ_i $(i\in X)$ we can ensure that the union Φ_λ is the complete diagram of a reduct of an elementary extension of M^L, regardless of how Φ_j are chosen at successor ordinals $j\notin X$.* (**Proof.** First note that by (a) and (b), a continuous chain of conditions $(\Phi_i)_{i<\lambda}$ can be constructed. Now since λ is infinite, we can list the formulas of $L(M)$ as φ_i $(i\in X)$. Then using (c) and (d), we can choose each Φ_i $(i\in X)$ so that either φ_i or $\neg\varphi_i$ is in Φ_i, and if φ_i

is in Φ_i and has form $\exists z\, \theta$, then Φ_i also contains some formula $\theta(y_k)$. By Axiom 3, Φ_λ is consistent. So Φ_λ is a maximal consistent set with witnesses for existentially quantified formulas. Finally if φ is in the complete diagram of M, then by Axioms 1 and 3, $\neg\varphi$ is not in Φ_λ, so $\varphi \in \Phi_\lambda$.)

(f) *If $\lambda = \lambda^{<\kappa}$, then in* (e) *we can also ensure that the elementary extension of M^L is κ-compact.* (N is *κ-compact* if every 1-type over N with cardinality $<\kappa$ is realised in N. Use (d).)

1.5. Definition. Let Γ be a notion of bigness for T.

(1) We say that Γ is *invariant* if for every $\varphi(x, \bar{y}) \in L_{\omega,\omega}$, $\bar{a} \in M$, $M \prec N$ (models of T) $\varphi(x, \bar{a})$ is Γ-big in M if $\varphi(\bar{x}, \bar{a})$ is Γ-big in N (i.e. only the type of $\bar{a}$ matters).

(2) We say that Γ is *strong* if for every $\varphi(x, \bar{y})$ there is $L_\varphi \subseteq L$, $|L_\varphi| < \lambda$ such that the truth of "$\varphi(x, \bar{a})$ is Γ-big in M" depends only on the type which $\bar{a}$ realizes in $M \restriction L_\varphi$.

(3) We say that Γ is *simple* if the sentence Γ is a conjunction of first-order sentences.

(4) We say that Γ is *very simple* if for every formula $\varphi(x, \bar{y})$ for some formula $\psi(\bar{y})$,

$$T \vdash (\forall \bar{y}) [(Qx)\, \varphi(x, \bar{y}) \equiv \psi(\bar{y})].$$

Example. Let $Qx\,\varphi$ say "There are infinitely many x such that φ". Then this notion of bigness is simple; it is strong if $\lambda > \omega$.

1.6. Lemma. *Let Γ be a notion of bigness for T and M a model of T. Then Tasks* I *and* II *can be performed. More precisely, let X be any set of successor ordinals which is cofinal in λ, and suppose a continuous chain of conditions $(\Phi_i)_{i<\lambda}$ is being constructed by induction on i. Then by suitable choices of Φ_i $(i \in X)$ we can ensure that the unions Φ_λ is the complete diagram of an elementary extension N^L of M^L, and that if x names the element a, then the type of a over M contains only Γ-big formulas (and in particular $a \notin M$).*

Proof. Combine Lemma 1.4(e) with Definition 1.3. Note that if $b \in M$, then by Axiom 3 the formula $x = b$ is Γ-small, and so $a \notin M$.

1.7. Definition (of forcing). Let Γ be a notion of bigness for T and M a model of T. We assume that the formulas of $L_{\lambda^+,\lambda}(M)$ have $<\lambda$ free variables, all these variables are from x and y_i $(i < \lambda)$, and the truth-functions and quantifiers of the formulas are just $\neg$, $\bigwedge$ and $\forall$. Let Φ be a condition (i.e. Γ-condition over M) and ψ a formula of $L_{\lambda^+,\lambda}(M)$. We define the relation $\Phi \Vdash \psi$ ("*Φ forces ψ*") by induction on the structure of ψ:

(1) If ψ is atomic, then $\Phi \Vdash \psi$ iff for some finite conjunction $\varphi(x, \bar{y})$ of formulas

in Φ, $M \models \tilde{Q}x\, \forall \bar{y}\, (\varphi \rightarrow \psi)$. (Equivalently, iff for every condition $\Psi \supseteq \Phi$, $\Psi \cup \{\psi\}$ is a condition.)

(2) $\Phi \Vdash \bigwedge_{j<\xi} \psi_j$ iff $\Phi \Vdash \psi_j$ for all $j<\xi$.

(3) $\Phi \Vdash \forall \bar{z}\, \psi(x, \bar{y}, \bar{z})$ iff $\Phi \Vdash \psi(x, \bar{y}, \bar{y}')$ for some sequence $\bar{y}'$ of variables distinct from each other and from all occurring free in Φ or $\forall \bar{z}\, \psi(x, \bar{y}, \bar{z})$.

(4) $\Phi \Vdash \neg\psi$ iff for all conditions $\Psi \supseteq \Phi$, $\Psi \nVdash \psi$.

1.8. Lemmas (on forcing). (a) *If* $\Phi \Vdash \psi$ *and* Ψ *is a condition* $\supseteq \Phi$, *then* $\Psi \Vdash \psi$.

(b) *If* $\Phi \Vdash \forall \bar{z} \psi(x, \bar{y}, \bar{z})$, *then* $\Phi \Vdash \psi(x, \bar{y}, \bar{y}')$ *for all sequences* $\bar{y}'$, *of variables.*

(c) *If* $\Phi \Vdash \psi$, *then* $\Phi \nVdash \neg\psi$.

(d) *If* $\Phi \nVdash \psi$, *then* $\Psi \Vdash \neg\psi$ *for some condition* $\Psi \supseteq \Phi$. (**Proof.** By induction on the structure of ψ.)

(e) *If* ψ *is an atomic sentence of* $L(M)$ *and* Φ *is a condition, then* $\Phi \Vdash \psi$ *iff* $M \models \psi$. (**Proof.** Suppose $M \models \psi$. Then $M \models \neg\psi \rightarrow \psi \wedge \neg\psi$, so by Axiom 1, $M \models Qx\, \neg\psi$ would imply $M \models Qx\, (\psi \wedge \neg\psi)$, contradicting Axiom 3. Therefore $M \models \tilde{Q}x\, \psi$, and so every condition forces ψ. Conversely suppose $M \models \neg\psi$ and $M \models Qx\, \exists \bar{y}\, \varphi$. Then $M \models \forall x\ \ \forall \bar{y}\ \ (\varphi \rightarrow \varphi \wedge \neg\psi)$, so by Axiom 1, $M \models Qx\, \exists \bar{y}\, (\varphi \wedge \neg\psi)$, whence $M \models \neg\tilde{Q}x\, \forall \bar{y}\, (\varphi \rightarrow \psi)$; so no condition forces ψ.)

(f) *Let* Φ *be a condition and* ψ *a formula of* $L_{\lambda^+,\lambda}$. *Let* Φ^* *be the set of all formulas* $\exists \bar{z}\, \varphi$ *where* φ *is the conjunction of a finite number of formulas from* Φ, *and* $\bar{z}$ *are the y-variables which occur free in* φ *but not in* ψ. *Then* Φ^* *is a condition, and* $\Phi \Vdash \psi$ *iff* $\Phi^* \Vdash \psi$. (**Proof.** By induction on the structure of ψ.)

(g) *Suppose* $\lambda = \lambda^{<\lambda}$ *and that the notion of bigness is a formula of* $L_{\lambda^+,\lambda}(P)$. *Let* $\Phi(x, \bar{y}, \bar{w})$ *and* $\psi(x, \bar{y}, \bar{w})$ *be respectively a set of* $<\lambda$ *first-order formulas of* L *and a formula of* $L_{\lambda^+,\lambda}$. *Then there is a formula* $\theta(\bar{w})$ *of* $L_{\lambda^+,\lambda}$ *such that for every sequence* $\bar{a}$ *of elements of any model* M *of* T,

$$M \models \theta(\bar{a}) \quad \textit{iff} \quad \Phi(x, \bar{y}, \bar{a}) \textit{ is a condition which forces } \psi(x, \bar{y}, \bar{a}).$$

(**Proof.** By induction on the structure of ψ.)

Cf. also Lemma 1.10(f) below.

1.9. Definition (of $\mathscr{L}$-generic sequences). Let Γ be a notion of bigness for T, M an expanded model of T and $\mathscr{L}$ a fragment of $L^M_{\lambda^+,\kappa}$.

An *$\mathscr{L}$-generic sequence* (over M) is a continuous chain $(\Phi_i)_{i<\lambda}$ of conditions such that:

(1) For each formula ψ of $\mathscr{L}$(M) there is $i<\lambda$ such that either $\Phi_i \Vdash \psi$ or $\Phi_i \Vdash \neg\psi$. (Of course, we restrict ourselves to the case $\mathrm{FV}(\Psi) \subseteq \{x, y_\alpha : \alpha < \lambda\}$.)

(2) For each formula $\bigwedge_{j<\xi} \psi_j$ of $\mathscr{L}(M)$ there is $i<\lambda$ such that either $\Phi_i \Vdash \bigwedge_{j<\xi} \psi_j$ or $\Phi_i \Vdash \neg\psi_j$ for some $j<\xi$.

(3) For each formula $\forall \bar{z}\, \psi(x, \bar{y}, \bar{z})$ of $\mathscr{L}(M)$ there is $i<\lambda$ such that either $\Phi_i \Vdash \forall \bar{z}\, \psi(x, \bar{y}, \bar{z})$ or $\Phi_i \Vdash \neg\psi(x, \bar{y}, \bar{y}')$ for some $\bar{y}'$.

An *$\mathscr{L}$-generic set* is the union of an $\mathscr{L}$-generic sequence. We say that the set Φ *forces* ψ if some subset of Φ of cardinality $<\lambda$ is a condition which forces ψ.

We shall say that N is an *$\mathscr{L}$-generic extension of M with distinguished element a* if for some $\mathscr{L}$-generic set Φ, Φ is the complete diagram of N, a is the element named by the variable x, the type of a over M is Γ-big, and for every formula ψ of $\mathscr{L}(M)$, $N \models \psi$ implies that Φ forces ψ. If we construct an $\mathscr{L}$-generic extension of M, we perform Tasks I–III of 1.1 above.

If $\mathscr{L}$ is constant we suppress this prefix.

1.10. Lemmas (on $\mathscr{L}$-generic sequences). (a) *If Φ is an $\mathscr{L}$-generic set and $\Phi = \bigcup_{i<\lambda} \Psi_i$ where $(\Psi_i)_{i<\lambda}$ is a continuous chain of sets of cardinality $<\lambda$, then $(\Psi_i)_{i<\lambda}$ is an $\mathscr{L}$-generic sequence.*

(b) *Let Φ be a generic set and Δ the set of formulas of $\mathscr{L}(M)$ which are forced by Φ. Then Δ is the complete $\mathscr{L}(M)$-diagram of an extension of M.* (**Proof.** By Lemma 1.8(c) and Definition 1.9, Δ is a maximal consistent set of formulas of $\mathscr{L}(M)$ with witnesses for existentially quantified formulas, so it is the complete $\mathscr{L}(M)$-diagram of some structure N. By Lemma 1.8(e) we can choose $N \supseteq M$.)

(c) *Suppose Φ, Δ are as in* (b), *and Φ is the complete diagram of some structure. Then $\Phi = \Delta \cap L_{\omega,\omega}(M)$.* (**Proof.** It suffices to show that if ψ is an atomic formula $\in\Phi$, then $\Phi \Vdash \psi$. But this holds, because by Axioms 1 and 3, $M \models \tilde{Q}x\, \forall \bar{y}\, (\psi \to \psi)$.)

(d) *Assume $\lambda = \lambda^{<\kappa}$ and let Γ be a notion of bigness for T. Then Tasks* I–III *can be performed. More precisely, suppose M is an expanded a model of T of cardinality $\leq\lambda$ and Ψ is a Γ-condition over M. Then there exists an $\mathscr{L}$-generic extension of M which satisfies Ψ.* (**Proof.** Build $(\Phi_i)_{i<\lambda}$ as in the proof of Lemma 1.6, putting $\Phi_0 = \Psi$. That proof allows, say, the conditions Φ_i for odd-numbered i to be chosen arbitrarily. Since $\lambda = \lambda^{<\kappa}$, we can list in order-type λ the tasks which have to be performed in order to satisfy (1)–(3) of Definition 1.9. Lemma 1.8(b, d) shows that suitable choices of the Φ_i for odd i will do the job.)

(e) *Let Φ be a condition and ψ a (first-order) formula of $L(M)$. Then there is a condition $\Psi \supseteq \Phi$ containing ψ if and only if there is a condition $\Psi \supseteq \Phi$ which forces ψ.* (**Proof.** As in the proof of (d), Φ can be extended to a set which forces exactly those first-order formulas which it contains; cf. (c).)

(f) *Let Φ be a condition and $\psi(x, \bar{y})$ a formula of $L(M)$. Then $\Phi \Vdash \psi$ iff for some conjunction $\varphi(x, \bar{y})$ of a finite part of Φ, $M \models \tilde{Q}x\, \forall \bar{y}\, (\varphi \to \psi)$.* (**Proof.** If $M \models Qx\, \exists \bar{y}\, (\varphi \wedge \neg\psi)$ for every finite part φ of Φ, then $\Phi \cup \{\neg\psi\}$ is a condition and hence by (e) some condition extending Φ forces $\neg\psi$; so $\Phi \nVdash \psi$. If $M \models \tilde{Q}x\, \forall \bar{y}\, (\varphi \to \psi)$ for some finite part φ of Φ, then no condition extending Φ contains $\neg\psi$, and so by (e) no condition extending Φ forces $\neg\psi$; hence $\Phi \Vdash \psi$.)

(g) *For any M, $\Vdash$ "the model is κ-compact"* (*when* $\lambda = \lambda^{<\kappa}$).

1.11. The empty model. Formally we consider the empty model to be a model of T and an elemenary submodel of every other model of T. The definitions and constructions above make sense when M is the empty model: an $\mathscr{L}$-generic extension of M will be a model of T which is described by a 'generic set' (defined in a natural way) and has a distinguished element whose type is big. This is a purely formal device which makes the book-keeping easier in the next section.

1.12. Remark. Really except in 3.x we can demand Γ to be a notion of bigness for $K = \{M : M\ a\ \kappa\text{-compact model of } T \text{ or the empty model}\}$.

On the other hand we use only Γ which are invariant and notions of bigness for T.

2. Iteration and an existence theorem

T is a complete first-order theory with infinite models in a language of cardinality $\leq\lambda$, and Γ is a notion of bigness for T. We shall construct a model M^* of T in cardinality λ^+, as the union of a continuous elementary chain $(M_\alpha^L)_{\alpha<\lambda^+}$ of models of cardinality $\leq\lambda$. The construction will be carried out so that whenever $\beta<\lambda^+$, $M_{\beta+1}^L$ is derived from M_β just as N^L was derived from M in 1.1 above; moreover, extra relations will be added to the language and the structures at each stage of the construction.

Fragments $\mathscr{L}_\alpha$ of $(L^\alpha)_{\lambda^+,\lambda}$ are defined, $L^\alpha = L^{M_\alpha}$ is increasing, $M_\beta^{L^\alpha} = M_\beta^L \vee M_\alpha$, (the role of the $L^\alpha - L$, and $\mathscr{L}_\alpha$ is in the genericity demands).

A set-theoretic assumption $(\mathrm{D}\ell)_\lambda$ (cf. 2.10 below) is used in the construction. Then we shall state a combinatorial lemma (cf. 2.17 below) which generalises this construction.

2.1. Definition (of languages). (a) Let $J \cup \lambda^+$ be a linear order extending the order $(\lambda^+, <)$, 0 minimal in it, $\{\alpha : \alpha<\lambda^+\}$ is increasing continuous and unbounded in it, let $[\alpha, \beta]_J = \{t \in J; \alpha \leq t < \beta\}$ and assume $[0, \alpha)_J$ has power $<\lambda^+$ for $\alpha<\lambda^+$. We may let α, β, γ vary on $J\cup\lambda^+$. We write J^+ when $[\alpha, \alpha+1)_J$ is λ-saturated, for every α, and $\alpha \in J \Leftrightarrow \mathrm{cf}\,\alpha = \lambda$ when $\alpha\in\lambda^+$.

(b) For each $\alpha<\lambda^+$ let Π_α be a set of $\leq\lambda$ relation symbols. We assume that when $\alpha<\beta$, Π_α is disjoint from Π_β and from the set of symbols of L. Π_0 is empty.

(c) For each $\alpha<\lambda^+$ we introduce distinct new variables x_t and y_i^t $(i<\lambda)$ for $t\in[\alpha, \alpha+1]_J$; we call them the *α-variables*. When $\alpha<\beta<\lambda^+$, $L^{\beta,\alpha}$ is the first-order language got by adding to L the relation symbols in $\bigcup_{\xi\leq\alpha}\Pi_\xi$ and taking as variables the γ-variables for all $\gamma<\beta$. We write L^β for $\bigcup_{\alpha<\beta} L^{\beta,\alpha}$. We write X_α for the set of $(<\alpha)$-variables. (The elements of X_α will name the elements of M_α.)

$$X_t = \{x_s, y_i^s : i<\lambda, s<t, s\in J\} \quad\text{and}\quad X^t = \{x_t, y_i^t : i<\lambda\}.$$

2.2. Definition. Assume that Γ is invariant or that $J=\lambda^+$. For $s<t\in J\cup\lambda^+$ and an expanded model M of T and a fragment $\mathscr{L}$ of $L_{\lambda^+,\lambda}^M$ we define what are: (a) a (t, s)-condition over M (we suppress Γ, which is constant). (b) $\Vdash$ (i.e. $\Vdash_M^{(t,s)}$) and (c)

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an $\mathscr{L}$-generic sequence, $\mathscr{L}$-generic and $\mathscr{L}$-generic structure, all three with respect to (t, s, M).

If Γ is not invariant the definition is by induction on t (remember that then $J=\lambda^+$).

(I) A (t, s)-*condition* is a set Ψ of $<\lambda$ formulas of $L(M)$ (not $L^M(M)$!) whose free variables are $X_t - X_s$ such that:

(i) If v satisfies $s<v<t$, $\varphi_1, \ldots, \varphi_n$ belong to Ψ and $\bar{z}$ include all variables from $X_t - X_v$ which occur freely in some φ_ℓ $(\ell=1, n)$, *then*

$$(\exists \bar{z})\left(\bigwedge_{\ell=1}^{n} \varphi_\ell\right) \in \Psi \quad \text{or at least} \quad \Psi \restriction X_v \vdash (\exists \bar{z})\left(\bigwedge_{\ell=1}^{n} \varphi_\ell\right)$$

[if we forget this then $\{x_{\beta+2}=x_{\beta+2} \rightarrow x_\beta = x_{\beta+1}\}$ would be a $(\beta, \beta+3)$-condition.]

(ii) If $J \neq \lambda^+$ (hence Γ is invariant), then for every v, $s<v<t$, and any elementary extension N of M^L realizing $\Psi \restriction X_v$, $\Psi \restriction (X_v \cup X^v)$ is a Γ-condition over N (after suitable changes of names).

(iii) If $J=\lambda^+$, then

(α) $\Psi \restriction X_v$ is a (v, s)-condition for $v<t$;

(β) $\Psi \restriction (X_s \cup X^s)$ is a Γ-condition over M except that we use the variables x_α, y_i^α instead x, y_i;

(γ) $\Psi \restriction X_\gamma \Vdash_\mu^{(\alpha, \gamma)}$"$\Psi \restriction X_{\gamma+1}$ is a Γ-condition over the model we get from the (β, γ)-conditions (when the variables in $X_\gamma - X_\alpha$ become elements".

(II) For a (t, s)-condition Ψ, $\theta \in L^M_{\lambda^+,\lambda}$, $\Psi \Vdash_M^{(t,s)} \theta$ is defined just like 1.7.

(III) For $\mathscr{L} \subseteq L^M_{\lambda^+,\lambda}$ and '*$\mathscr{L}$-generic sequence for* (t, s, M)' and '*$\mathscr{L}$-generic model over* (t, s, M)' and an '*$\mathscr{L}$-generic st. L-el. extension for* (t, s, M) with distinguished elements a_v $(s<v<t, v \in J)$ are defined as in 1.9.

(IV) An (t, s)-condition is called *complete* if it is a complete type (in L) over its set of parameters.

2.3. Lemma. (1) *The parallel of* 1.4 *holds for* (β, α)-*conditions* (*with* $x, \bar{y}$ *replaced by variables from* $X_\beta - X_\alpha$).

(2) *The parallel of* 1.8, 1.10 *holds for* (β, α)-*conditions.*

(3) *If* Φ *is an* (γ, α)-*condition over* M, Ψ *a* (β, α) *condition over* M, $\alpha<\beta<\gamma$, $\Phi \restriction X_\beta \subseteq \Psi$, *then* $\Phi \cup \Psi$ *is included in a* (γ, α)-*condition over* M.

2.3A. Remark. However the parallel of the lemma $V^{P*Q}=(V^P)^Q$ in real forcing is problematic here, as we have to have genericity for the formulas θ indicated in 1.8(g). But for e.g. invariant Γ, such problems disappear in proving 2.12.

If $J \neq \lambda^+$, and moreover $J=J^+$, those problems disappear, and we do not see a loss for our application. Now 2.4–2.7 deal with the case $J=\lambda^+$ (which we will not use). Note that 2.6 becomes trivial, $\sim_\lambda^\xi$ is replaced by $\sim_\lambda^0$ for λ-saturated J.

2.4. Definition. We call Γ a *super notion of bigness* if for every $\alpha<\beta$, and an expanded model M of T, and an (β, α)-condition Ψ over M, and an increasing function from $(\alpha, \beta)_J$ into $(\alpha^1, \beta^1)_J$,

$$h(\Psi)=\{\varphi(\ldots, x_{h(\gamma)}, \ldots, y_i^{h(\varepsilon)}, \ldots): \varphi(\ldots, x_\gamma, \ldots, y_i^\xi, \ldots)\in\Psi$$

is an (β^1, α^1)-condition over M (the problem is in Definition 2.2 I(iii)(γ) only).

2.5. Lemma. *If Γ is strong, then Γ is a super notion of bigness.*

Proof. Easy.

2.6. Lemma. *Let $\alpha<\beta_1$, β_2, h an order-preserving partial function from $[\alpha, \beta_1]$ into $[\alpha, \beta_1]$, $h(\beta_1)=\beta_2$, $h(\alpha)=\alpha$, $\lambda>|\mathrm{Dom}\, h|$ and M be an expanded model of T. Assume Γ is a strong bigness notion.*

If $\theta(h)$ is a formula from L^M, with free variables from $X_{\beta_1}-X_\alpha$ (and $<\lambda$ free variables) and Φ is a (β_1, α)-condition, then

$$[\Phi \Vdash_M^{(\beta_1,\alpha)} \theta] \quad\Leftrightarrow\quad [h(\theta)\Vdash_M^{(\beta_2,\alpha)} h(\theta)]$$

provided that $(\beta_1-\alpha, <, i)_{i\in\mathrm{Dom}\,h}$, $(\beta_2-\alpha, <, h(i))_{i\in\mathrm{Dom}\,h}$ are $\sim_\lambda^\zeta$-equivalent (which means that they satisfy the same $L_{\infty,\lambda}$-sentences of quantifier depth $<\zeta$ in their vocabulary) where ζ is Max{*depth of θ, quantifier depth of Γ*}. ($\beta_l-\alpha$ means the set difference.)

Remark. This lemma explains the connection between 2.12 and 2.17.

Proof. Easy, by induction on θ.

2.7. Claim. *Suppose for $\ell=0$, 1, $M_\ell=(\alpha^\ell, <, \beta_0^\ell, \ldots, \beta_i^\ell, \ldots)_{i<i(0)}$, $\beta_0=0$, β_i increasing continuous and we let $\beta_{i(0)}^\ell=\alpha^\ell$. A sufficient condition for $M_0\sim_\lambda^\zeta M_1$ is: for every $i<i(0)$, for some $\gamma<\lambda^\zeta$, $\beta_{i+1}^\ell-\beta_i^\ell=\delta^\ell\lambda^\zeta+\gamma$ for some δ^0, δ^1 and:* $\delta^0=0\Leftrightarrow\delta^1=0$; $\mathrm{cf}\,\delta^0<\lambda\Leftrightarrow\mathrm{cf}\,\delta^1<\lambda$; $\mathrm{cf}\,\delta^0<\lambda\Rightarrow\mathrm{cf}\,\delta^0=\mathrm{cf}\,\delta^1$.

Proof. See Kino [6].

2.8. Definition (of the game $G(T,\Gamma)$). *The game $G(T,\Gamma)$* is played by two players, the Random Player (I) and the Generic Player (II). There are λ^+ stages to the game.

(a) At the beginning of the game, M_0 is chosen to be the empty model (cf. 1.11 above).

(b) At the $(\alpha+1)$-th stage the model M_α has just been defined and is an L^α-structure; the variables in X_α have been interpreted as constants which name all the elements of M_α. The Random Player now interprets the symbols in Π_α as relations on M_α, thus expanding M_α to an $L^{\alpha+1}$-structure M'_α. Also he chooses a fragment $\mathcal{L}_\alpha$ of $L_{\lambda^+,\kappa}^{M'_\alpha}$ such that $\bigcup_{\beta<\alpha}\mathcal{L}_\beta\supseteq\mathcal{L}_\alpha$. The Generic Player chooses a maximal consistent set $\Phi^{\alpha+1}$ of formulas of L^α (in the α-variables) which contains the complete diagram of M'_α and defines a st. L-el. extension $M_{\alpha+1}$ of M'_α.

(c) At the δ-th stage, where δ is a limit ordinal, M_δ is defined to be the L^δ-structure which is the limit of the M_α $(\alpha<\delta)$; i.e. for each $\beta<\delta$, $M_\delta \restriction L^\beta = \bigcup_{\beta\leq\gamma<\delta} M_\gamma \restriction L^\beta$.

(d) At the end of the game, two sequences of structures $(M_\alpha)_{\alpha<\lambda^+}$, $(M'_\alpha)_{\alpha<\lambda^+}$ have been constructed, and the variables have been interpreted as elements of these structures. The Generic Player wins iff for all $\alpha<\beta<\lambda^+$, $M_\beta \restriction L^{\alpha+1}$ is an $\mathcal{L}_\alpha$-generic extension for the $(\alpha, \beta, M'_\beta)$ forcing.

2.9. Remarks (on the game $G(T, \Gamma)$). Let us review the three tasks which Player II has to perform for each $\alpha<\beta<\lambda^+$ (cf. 1.1 above). First, define $\Phi^\delta = \bigcup_{\alpha<\delta} \Phi^\alpha$ when δ is a limit ordinal. Then $(\Phi^\alpha)_{\alpha<\lambda^+}$ is a continuous increasing chain of sets of formulas. When $\alpha<\beta<\lambda^+$, $\Phi^\beta \restriction L^{\beta,\alpha}$ is the complete diagram of $M_\beta \restriction L^{\alpha+1}$, so the Task I is performed automatically.

Let Player II always construct $\Phi^{\alpha+1} \cap L$ as the union of a continuous chain of Γ-conditions over M'_α, $(\Phi_i^{\alpha+1})_{i<\lambda}$. Lemma 1.4(e), 2.3(1) says that he can do this. Moreover since Γ is a strong notion of bigness, Lemma 1.6 says that he can ensure that the $L^{\alpha+1}$-type of the element x_α over M'_α contains only Γ-big formulas. Whenever $\alpha<\beta<\lambda^+$, $M_\beta \restriction L^{\alpha+1}$ is an elementary extension of $M_{\alpha+1}$, so that x_α has the same $L^{\alpha+1}$-type over M'_α in M_β as it has in $M_{\alpha+1}$. In this way the Generic Player achieves Task II.

There remains Task III. This looks alarmingly difficult when $\alpha<\beta<\lambda^+$ and β is a limit ordinal of cofinality $<\lambda$. How can the Generic Player possibly ensure that if ψ is a formula of $L^{\beta,\alpha}_{\lambda^+,\lambda}$ which is true in M_β, with the variables in X_α interpreted as constants for elements of M'_α, then ψ is forced by some condition which is true in M_β? After all, ψ may contain ξ-variables for cofinally many $\xi<\beta$. The solution is to make the Generic Player predict at each stage $i<\lambda$ what may happen at later stages, so that if necessary he can stop it happening. This seems to need the following set-theoretic assumption.

2.10. Definition of $(\mathrm{D}\ell)_\lambda$. The *principle* $(\mathrm{D}\ell)_\lambda$ states: there is a family $\{\mathcal{P}_\alpha : \alpha<\lambda\}$ of sets such that each $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{P}_\alpha|<\lambda$, and for every set $A\subseteq\lambda$ there is a stationary set of ordinals α such that $A\cap\alpha\in\mathcal{P}_\alpha$.

Remark. If λ is regular, then $(\mathrm{D}\ell)_\lambda$ implies $\lambda=\lambda^{<\lambda}$. We always assume λ regular.

2.11. Claim. *If any one of* (i)–(iv) *holds, then* $(\mathrm{D}\ell)_\lambda$ *holds.*

(i) $\lambda=\aleph_0$ *or* λ *is strongly inaccessible.*

(ii) $\diamondsuit_\lambda$ *holds.*

(iii) *For some* μ, $\lambda=\mu^+=2^\mu$, *and either* $\mu^{\aleph_0}=\mu$ *or* $(\forall\chi<\mu)\chi^{\aleph_1}<\mu$.

(iv) GCH *holds and* λ *is a regular cardinal* $\neq\aleph_1^\lambda$.

Proof. (i), (ii) are clear. For a proof that (iii) imples $\diamondsuit_\lambda$, and the history of this result, see [10]. Finally (iii) implies (iv).

2.12. Theorem. *Let Γ be a strong notion of bigness for the theory T in a language of cardinality $\leq\lambda$, $J=J^+$ and assume $(\mathrm{D}\ell)_\lambda$ and $\lambda>\omega$. Then the Generic Player has a winning strategy for the game $G(T,\Gamma)$.*

We delay the proof of Theorem 2.12 to our next paper [13]. There we shall prove an even more general theorem.

2.12A. Theorem. *We can let in* 2.1 $P_t\in\Pi_\alpha$ *be monadic predicates for* $t\in[\alpha,\alpha+1)_J$ *and let* $L^{\beta,\alpha}=\bigcup_{\xi\leq\alpha}\Pi_\xi\cup\{P_t:t\in[\alpha,\beta)_J\}$, *and in* 2.8 *demand* $P_t^{M'_{\alpha+1}}=X_t$, *formulas of* $\mathscr{L}_\alpha$ *are allowed to have* $<\kappa$ *mondic predicates, for which* P_t*'s can be substituted*; *and still* 2.12 *conclusion holds.*

Remark. Note that 2.12 and 2.12A follow from 2.17.

2.13. Definition. When a model M^* of T of cardinality λ^+ is constructed as the limit of approximations $(M_\alpha)_{\alpha<\lambda^+}$ when Player II uses his winning strategy in the game $G(T,\Gamma)$, we shall call M^L_α a *generic model* of T. If M^* is generic and $\alpha<\beta<\lambda^+$, then as soon as M'_α has been constructed, we know what are the Γ-conditions over M'_α in $L^{\beta,\alpha}$; these will be known as the (β,α)-*conditions.* (So every $L_{\lambda,\lambda}(M'_\alpha)$-property of $M^L_\beta\vee M'_\alpha$ is forced by some (β,α)-condition.) A β-*condition* is a $(\beta,0)$-condition.

2.14. Remark. One can analyse the proof of Theorem 2.12 and extract the combinatorial principle that makes it work. Some of the applications in later sections use the combinatorial principle directly, rather than Theorem 2.12.

The intuitive idea is as follows. Construct a partial order $\boldsymbol{P}$ whose elements are the possible (β,α)-conditions, $\alpha<\beta<\lambda^+$. Each element of $\boldsymbol{P}$ can be thought of as a 'term' $\tau(\bar{x})$ where $\bar{x}$ are the variables occurring free in it. The ordering relation is inclusion. To find M^* is to find a directed subset of $\boldsymbol{P}$ which is closed in certain ways. We can choose $\boldsymbol{P}$ so that Tasks I and II are automatically achieved.

The problem is to perform Task III. Suppose $\psi(\bar{z},\bar{w})$ is a formula of $L^{\beta,\alpha}_{\lambda,\lambda}$ with $(<\alpha)$-variables $\bar{z}$ and $(\geq\alpha)$-variables $\bar{w}$, and $\tau\in\boldsymbol{P}$. We want some (β,α)-condition extending $\tau\restriction L^{\beta,\alpha}$ to force either ψ or $\neg\psi$. Fortunately the question whether a (β,α)-condition σ forces ψ depends on α and the variables of σ and ψ, but not on β. Thus for each $\psi(\bar{u},\bar{v})$ of $L_{\lambda,\lambda}$ we want a function $F^\psi_{\bar{z},\bar{w}}$ which extends terms τ to terms which force either $\psi(\bar{z},\bar{w})$ or $\neg\psi(\bar{z},\bar{w})$; $F^\psi_{\bar{z},\bar{w}}$ depends on $\bar{z}$, $\bar{w}$ in a uniform way. These are the 'appropriate' functions of Definition 2.9 below.

Because we added new relations at each stage, there will be separate families of functions F^ψ for each stage. In fact the family of functions to be considered at the $(\alpha+1)$-th stage can be taken as a function $H_1(M_\alpha)$ of the model M_α constructed at the α-th stage. Also we could have allowed the Random Player to choose the starting condition $\Phi^{\alpha+1}_0$ for each α, again as a function $H_2(M_\alpha)$ of M_α.

2.15. Definition. A partial order $\boldsymbol{P}$ is called λ^+-*simple* if there is a set $\boldsymbol{T}$ of $\leqslant\lambda$ 'terms' τ such that each term τ has $\alpha_\tau<\lambda$ places, and

(1) $\boldsymbol{P}=\{\tau(\bar{x}):\bar{x}$ is an increasing sequence from λ^+ of length $<\lambda\}$.

(2) the ordering relation $\tau(\bar{x})\leqslant\sigma(\bar{y})$ of $\boldsymbol{P}$ depends only on τ, σ and the order relations between the x_i's and the y_j's.

(3) For every $\tau(\bar{x})\in\boldsymbol{P}$ and $y\in\lambda^+$ there is $\sigma(\bar{z})\geqslant\tau(\bar{x})$ with y among the variables $\bar{z}$.

(4) If $\tau_i(x^{-i})$ $(i<\delta)$ is an increasing sequence such that for each limit $\delta^*<\delta$, $\tau^*_\delta(\bar{x}^{\delta^*})$ is the least upper bound of $\{\tau_i(\bar{x}^i):i<\delta^*\}$, then $\{\tau_i(\bar{x}^i):i<\delta\}$ has a least upper bound $\tau(\bar{x})$ with $\bar{x}=\bigcup_i\bar{x}^i$.

(5) For each $\tau(\bar{x})\in\boldsymbol{P}$ and each $\beta<\alpha_\tau$ there exists a term $\tau(\bar{x})\restriction\beta\in\boldsymbol{P}$ whose variables are the first β variables of $\bar{x}$; if β is a limit, then $\tau(\bar{x})\restriction\beta$ is the least upper bound of $\{\tau(\bar{x})\restriction\gamma:\gamma<\beta\}$.

(6) If $\tau(\bar{x})\restriction\beta\leqslant\sigma(\bar{y})$ and all the variables in $\bar{y}$ are less than the β-th variable of $\bar{x}$, then $\tau(\bar{x})$ and $\sigma(\bar{y})$ are compatible.

2.16. Definition. (a) An *appropriate* function for $\boldsymbol{P}$ is a function F which assigns to each increasing sequence $\bar{y}$ of length $<\lambda$ from λ^+ a function $F_{\bar{y}}:\{\tau(\bar{x})\in\boldsymbol{P}:\bar{y}\subseteq\bar{x}\}\to\boldsymbol{T}$, such that $\tau(\bar{x})\leqslant F_{\bar{y}}(\tau(\bar{x}))(\bar{x})$, $\alpha_\tau=\alpha_{F_{\bar{y}}(\tau(x))}$, and $F_{\bar{y}}(\tau(\bar{x}))$ depends only on τ and the truth or otherwise of the equations $x_i=y_i$. We write $F_{\bar{y}}(\tau(\bar{x}))$ for $(F_{\bar{y}}(\tau(x)))(\bar{x})$.

(b) For each $\alpha<\lambda^+$ we write $\boldsymbol{P}_\alpha$ for the set $\boldsymbol{P}\cap\{\tau(\bar{x}):\bar{x}\subseteq\alpha\}$. If $G\subseteq\boldsymbol{P}_\alpha$, we write $\boldsymbol{P}_G$ for the set of $\tau(\bar{y},\bar{x})\in\boldsymbol{P}$ such that $\bar{y}\subseteq\alpha$, $\bar{x}\subseteq\lambda^+-\alpha$ and $\tau(\bar{y},\bar{x})$ is compatible with every $\sigma(\bar{z})\in G$.

2.17. Theorem. *Assume* $(\mathrm{D}\ell)_\lambda$ *and* $\lambda>\omega$. *Let* $\boldsymbol{P}$, $\boldsymbol{T}$ *be as in Definition* 2.15.

(a) *For each* $\alpha<\lambda$ *let* F^α *be an appropriate function for* $\boldsymbol{P}$. *Then there is a directed* $G\subseteq\boldsymbol{P}$ *such that for every* $\alpha<\lambda$ *and every* $\tau(\bar{x})\in G$ *there is* $\tau'(\bar{x}')\geqslant\tau(\bar{x})$ *such that* $F_{\bar{x}}(\tau'(\bar{x}'))\in G$.

(b) *Let* H_1, H_2 *be functions such that for any* $\alpha<\lambda^+$ *and any* $G\subseteq\boldsymbol{P}_\alpha$, $H_1(G)$ *is a set of appropriate functions for* $\boldsymbol{P}_{f/G}$ *and* $H_2(G)$ *is an element of* $\boldsymbol{P}_{G\cap\boldsymbol{P}_\alpha}\cap\boldsymbol{P}_{\alpha+1}$ (*i.e. an element of* $\boldsymbol{P}_{\alpha+1}$ *which is compatible with every element of* $G\cap\boldsymbol{P}_\alpha$). *Then there is a directed* $G\subseteq\boldsymbol{P}$ *such that for each* $\alpha<\lambda^+$,

(1) *if* $F\in H_1(G\cap\boldsymbol{P}_\alpha)$ *and* $\tau(\bar{x})\in G$, *then there is* $\tau'(\bar{x}')\geqslant\tau(\bar{x})$ *such that* $F(\tau'(\bar{x}'))\in G$,

(2) $H_2(G\cap\boldsymbol{P}_\alpha)\in G$.

(c) *As* (b), *but with the elements* $H_2(G)$ *allowed to by any elements of* $\boldsymbol{P}_{G\cap\boldsymbol{P}_\alpha}$, *and with* (2) *holding only for a closed unbounded set of* α.

The proof of (a)–(c) are all essentially the same as that of Theorem 2.12.

2.18. Generalisations. (a) Theorems 2.6 and 2.11 still hold when $\lambda=\omega$, but the proofs are simpler. In fact when $\lambda=\omega$, only ZFC is needed. See [15, §2], and (with $\diamondsuit_{\aleph_1}$) [14].

(b) In Theorem 2.12 the Generic Player can choose his winning strategy so that for each ordinal $\alpha<\lambda^+$, $M_{\alpha+1}$ is λ-compact. This follows from Lemma 1.4(f). Note that M_δ is then λ-compact whenever δ is a limit ordinal $<\lambda^+$ of cofinality λ.

(c) Again in Theorem 2.12 the Generic Player can choose his winning strategy so that whenever $\alpha<\beta<\lambda^+$, $M_\beta \restriction L^{\alpha+1}$ is generic over M'_α for a family of λ formulas of $L^{\beta;\alpha}_{\lambda^+,\lambda}$. This family of formulas can be chosen as soon as M'_α has been constructed, and the Generic Player can do (b) at the same time. The proof is the same as that of Theorem 2.12. Without loss of generality one should assume that the family of formulas is closed under subformulas and under first-order operations.

(d) One can weaken clause (4) of Definition 2.9 to:

(4′) If $\tau_i(\bar{x}^i)$ $(i<\delta)$ is an increasing sequence, then $\{\tau_i(\bar{x}^i): i<\delta\}$ has an upper bound $\tau(\bar{x})$ with $\bar{x}=\bigcup_i \bar{x}^i$.

(e) Theorem 2.12 adapts to the case where L is replaced by $L(Q)$ and $Qx\,\varphi$ means "For at least λ elements x, φ holds". [10] was written in terms of this case.

(f) If in 2.17 the terms have finitely many variables (so λ-completeness holds when we fix the set of variables), then the theorem holds even if we assume just $\lambda=\lambda^{<\lambda}$.

3. The basic lemmas of the $(\mathrm{D}\ell)_\lambda$-free method

We here revisit the methods of [8], in a more general fashion. They were used in previous proofs of some theorems, but [10] and (Section 2) gave simpler proofs, with a slightly stronger hypothesis. So we decided to omit those proofs, but the lemmas may be useful for proofs when e.g. $\lambda^{<\lambda}>\lambda$ (as in [9, Theorem 12] where we prove that any T has a model in which no tree has a non-definable branch).

3.1. Lemma. *Suppose M is λ-compact, p is a Γ-big type over M, Γ any invariant notion of bigness for T, $|p|<\lambda$, and the set of parameters appearing in p is A. Let $\boldsymbol{I}=\{\bar{a}_i: i<\alpha\}\subseteq M$ be an infinite indiscernible sequence over A, then we can find $q\in S^m(A\cup\boldsymbol{I})$, such that $p\subseteq q$, q is Γ-big, and if $\bar{a}$ realizes q, $\boldsymbol{I}$ is an indiscernible sequence over $A\cup\bar{a}$.*

Remark. We can do this to $\boldsymbol{I}$ whose index-set is any infinite ordered set. If a suitable partition theorem holds, we can even let $\boldsymbol{I}$ be partially ordered in some more general way, cf. [12. VII §2].

Proof. We can replace M by any λ-compact elementary extension, and similarly α can be increased. So w.l.o.g. $\alpha=\mu$, $\mu=\beth_{(2^\chi)^+}$, $\chi=\lambda+|T|$. We can extend p to some Γ-big $p_1\in S^m(A\cup\boldsymbol{I})$ and assume $\bar{a}\in M$ realizes p_1. Expand M to N by making all elements of $A\cup\bar{a}$ into individual constants, and making the set $\boldsymbol{I}$ and the order $<=\{\langle\bar{a}_i,\bar{a}_j\rangle: i<j<\mu\}$ into relations of M. The fact that $\bar{a}$ realizes over

$A\cup \boldsymbol{I}=A\cup R^N$ a Γ-big complete L-type, can be expressed by omitting some types.

By Morley theorem on the Hanf number of omitting types and a generalization of Chang, (see [17]), there is a model N', elementarily equivalent to N and omitting all the types than N omits, such that in $R^{N'}$ there is an infinite indiscernible sequence $\boldsymbol{J}$ (in the language of N') and $\bar{a}$ realizes a Γ-big complete L-type over $A\cup \boldsymbol{J}$. Now we can find q from $\mathrm{tp}(\bar{a}, A\cup \boldsymbol{J})$ in the L-reduct of N'.

3.2. Lemma. *Suppose*

(a) *p is a Γ-big type over A in M, and $\bar{a}\in M$.*

(b) *$\bar{a}_i\in A$ for each $i\in I$, and D is a filter over I.*

(c) *For any formula $\varphi(\bar{x}, \bar{b})$ with parameters from A, if $\{i\in I: M\models\varphi[\bar{a}_i, \bar{b})\}\in D$, then $\models\varphi[\bar{a}, \bar{b}]$.*

(d) *Γ is a simple notion of bigness.*

Then we can extend p to a Γ-big type $q\in S^m(A\cup\bar{a})$ such that for any formula $\varphi(\bar{x}, \bar{y}, \bar{b})$, if $\bar{b}\in A$, and $\{i\in I:\varphi(\bar{x}, \bar{a}_i, \bar{b})\in q\}\in D$, then $\{i\in I:\varphi(\bar{x}, \bar{a}, \bar{b})\in q\}\in D$

Proof. W.l.o.g. $p\in S^m(A)$, and now we define q_1, by

$$q_1=\{\varphi(\bar{x}, \bar{a}, \bar{b}):\bar{b}\in A, \{i\in I:\varphi(\bar{x}, \bar{a}_i, \bar{b})\in p\}\in D\}.$$

By the hypothesis on $\bar{a}$, q_1 is consistent, and as Γ is simple also Γ-big. Obviously it extends p, and satisfies the conclusion of the lemma, so we finish as we can extend q_1 to a Γ-big $q\in S^m(A)$.

3.3. Lemma. *In* 3.2 *we can waive the simplicity of Γ, if D is $|T|^+$-complete, and Γ is invariant.*

3.4. Theorem. *For any $\lambda=\lambda^{<\lambda}$ the Generic Player* II *can still win the game from* 2.8, *if we weaken the demands for his winning a play to*

(i) *For any $\alpha<\lambda^+$, $\mathrm{cf}\,\alpha=\lambda\vee\alpha=0\vee(\exists\gamma)\alpha=\gamma+1)$ and set Φ of $<\kappa$ formulas from $L_{\omega,\omega}(\bigcup_{\beta<\lambda^+}M^L_\beta)$ if Φ is finitely satisfiable in M_α, then Φ is satisfiable in M_α.*

(ii) *If $\langle\bar{a}^n_i: i<\lambda, n<\omega\rangle$ is a sequence of sequences from M_α, $|T|<\lambda$ and*

$(*)_{M^L_\alpha}$: *for every $\bar{c}\in M_\alpha$ for a closed unbounded set of $\delta<\lambda$, $\langle\bar{a}^n_\delta: n<\omega\rangle$ is an indiscernible sequence over $\bar{c}$,*

and player I *includes the sentence saying $(*)_{M\restriction L}$ in $L_{\alpha+1}$, then $(*)_{\bigcup_\beta M^L_\beta}$ holds*

Proof. Clear, just like [8, §1, §2] using 3.1, 3.2 resp. for (i), (ii).

We can phrase those results in a more general way.

3.5. Definition. Let Γ_1, Γ_2 be two notions of bigness for T, for the sequences of variable $\bar{x}^1$, $\bar{x}^2$ resp. (maybe infinite). We say that Γ_1, Γ_2 are *orthogonal* (or Γ_1 is orthogonal to Γ_2) *if* for any model M of T, $A\subseteq M$, and sequences $\bar{a}^1, \bar{a}^2\in M$ of length $l(\bar{x}^1)$, $l(\bar{x}^2)$ resp. such that $\mathrm{tp}(\bar{a}^l, A)$ is Γ_l-big for $l=1,2$, *there are* an elementary extension N of M, and sequences $\bar{b}^1, \bar{b}^2 \in M$ of length $l(\bar{x}^1)$, $l(\bar{x}^2)$ resp. such that for $l = 1, 2$ $\bar{b}^l$ realizes $\operatorname{tp}(\bar{a}^l, A)$ and $\operatorname{tp}(\bar{b}^l, A \cup \bar{b}^{3-l})$ is Γ_l-big.

3.6. Theorem. *Let* $|T| \leq \lambda$, λ *regular,* $S_\alpha \subseteq \lambda$ $(\alpha < \lambda^+)$ *stationary,* $[\alpha \neq \beta \Rightarrow |S_\alpha \cap S_\beta| < \lambda]$, *and also* Γ *a bigness notion for* T. *Then Player* II *(the Generic Player) wins in the following game with* λ^+ *moves:*

(a) M_0 *is the empty model (see* 1.11 *above).*

(b) *In the* $(\alpha+1)$-th *stage the model* M_α *has already been defined (it is a model of* T *of power* λ*). The Random Player chooses* λ *sequences* $\langle \Gamma_i^\alpha, \langle \bar{a}_{i,\xi}^\alpha : \xi < \lambda \rangle \rangle$ *for* $i < \lambda$, *such that* Γ_i^α *is a bigness notion (for* T*) orthogonal to* Γ, $\bar{a}_{i,\xi}^\alpha$ *a sequence from* M_α *(of the length of* $\bar{x}_i^\alpha$, *the sequence of distinguished variables of* Γ_i^α*) such that, for each* i

$(*)_\alpha^i$: *for any finite* $\bar{c} \subseteq M_\alpha$ *for a closed unbounded set*[3] *of* $\xi \in S_{\lambda\alpha+i}$, $\operatorname{tp}(\bar{a}_{i,\xi}^\alpha, \bar{c})$ *is* Γ_i^α*-big.*

Then the Generic Player II *chooses a model* $M_{\alpha+1}$, *which is an elementary extension of* M_α *of power* λ *in which* $\bar{a}^\alpha \subseteq M_\alpha + 1$ *realizes over* M_α *a* Γ*-big type,* M_α *is* κ*-compact whenever* $\lambda = \lambda^{<\kappa}$.

(c) *At the* δ*-th stage,* M_δ *is defined as* $\bigcup_{\beta<\alpha} M_\beta$.

(d) *At the end, the Generic Player* II *wins the play if for any* $\alpha < \beta < \lambda^+$, $i < \lambda$:

$(*)_{\alpha,\beta}^i$: *for any finite* $\bar{c} \subseteq M_\alpha$, *for a closed unbounded set of* $\xi \in S_{\alpha\lambda+i}$, $\operatorname{tp}(a_{i,\xi}^\alpha, \bar{c})$ *is* Γ_i^α*-big.*

Remarks. (1) So part of the hypothesis is "the closed unbounded filter on λ is not λ^+-saturated" (saturated in the set-theoretic sense); this is not so strong—every successor $\lambda > \aleph_1$ satisfies it.

Also we can omit it but then the proof becomes somewhat more complicated (as in [8, §2]); and in $(*)_\alpha^i$ we replace "$\xi \in S_{\lambda\alpha+i}$" by "$\xi < \lambda$" and in $(*)_{\alpha,\beta}^i$ we omit "closed".

(2) Theorem 3.5 is a particular case of 3.7—Lemmas 3.1, 3.2 just say that suitable Γ_i^α are orthogonal to Γ.

(3) We can in 3.7 replace the choice of $M_{\alpha+1}$ (by the Generic Player) by a play of length λ where the players choose an increasing chain $\langle \Phi_\alpha : i < \lambda \rangle$ as in I (Φ a Γ-big set of formulas of M_α, $\operatorname{FV}(\Phi_\alpha) \subseteq \{x_\alpha, y_j^\alpha : j < \lambda\}$ and Φ_α has power $<\lambda$, and player II has the choice of Φ_α for a closed unbounded set of $\alpha \in \bigcup_{\gamma<\lambda^+} S_\gamma$ (e.g. a closed unbounded set of α's).

4. Applications with diamonds

4.1. Theorem (CH). *There is an uncountable Boolean algebra* B *with no uncountable chain nor an uncountable pie* (= *a set of pairwise incomparable elements*).

[3] This means that for some closed unbounded $C \subseteq \lambda$, for every $\zeta \in C \cap S_{\lambda\alpha+i} \cdots$.

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Moreover there is some countable $B_0 \subseteq B$ *such that for every uncountable* $I \subseteq B$ *the following holds.*

$(f)_1$: *there is a* B_0*-partition* $b_0, \ldots, b_n$ $(n>0)$ *of* 1 (*i.e.*, $b_l \in B_0$, $\bigcup_{l=0}^n b_l = 1$, $l \neq m \Rightarrow b_l \cap b_m = 0$) *and* $c \in B$, $c \leqslant b_0$ *such that for every* $b_l' < b_l'' \leqslant b_l$ *in* B_0 $(l = 1, \ldots, n)$ *there is* $x \in I$ *such that* $x \cap b_0 = c$, $b_l' < x \cap b_l < b_l''$ (*for* $l = 1, \ldots, n$).

Proof. Let B_0 be a countable atomless Boolean algebra. We use the context of 2.15 for $\lambda = \aleph_0$ (and see 2.18(a)). We define $\boldsymbol{T}$ and $\boldsymbol{P}$ as follows: $\boldsymbol{T}$ is the set of all conjunctions $\tau(x_{i(0)}, \ldots, x_{i(n-1)})$ of form $\bigwedge_{l<n} [(x_{i(l)} \cap b_l) = c_l]$ where $c_l \leqslant b_l < 1$ are members of B_0. We put $\tau(\bar{x}) \geqslant \sigma(\bar{y})$ if $\tau(\bar{x})$ implies $\sigma(\bar{y})$. We use Theorem 2.17 for the following F's and H_1.

(a) For every $a \in B_0$, $F^a_{x_\alpha}(\tau(x))$ is such that it has a conjunction $x_\alpha \cap b = c$ such that $a \leqslant b$ or $a \cup b = 1$.

Now this assures us that if $G \subseteq \boldsymbol{P}_\alpha$ is generic for the F^a's, then for each $\beta < \alpha$, $I_\beta = \{b$: for some c $(x_\beta \cap b = c) \in G\}$ is a maximal ideal, and there is a unique Boolean algebra B generated by $B_0 \cup \{x_\beta : \beta < \alpha\}$ satisfying G. We let $B_\alpha \overset{\text{df}}{=} B_\alpha(G) \overset{\text{df}}{=}$ the Boolean algebra freely generated by $B_0 \cup \{x_\beta : \beta < \alpha\}$, except that it satisfies the equations in G and the equations which B_0 satisfies.

(b) By CH we can list as $\{A_\alpha : \alpha < \omega_1\}$ all countable sets of Boolean terms in members of $B_0 \cup \{x_i : i < \aleph_1\}$; these terms will serve as names for the elements of B_{ω_1}, and w.l.o.g.[4] $A_i \subseteq B_i$. We define $H_1(G)$ (for $G \subseteq P_\alpha \cdots$) as follows: if A_α satisfies $(f)_1$, then $F^*_{\bar{y}}(\tau(\bar{x}))$ is $\tau(\bar{x})$ when it should be defined. Suppose A_α does not satisfy $(f)_1$. Suppose

$$A_\alpha = \{\sigma(x_{j(0)}, \ldots, x_{j(m-1)}, x_{j(m,\beta)}, \ldots, x_{j(n-1,\beta)}, \bar{c}) : \beta < \beta_0\}$$

for some $\beta_0 < \aleph_1$, $\bar{c} \in B_0$, σ a Boolean term, $j(0) < \cdots < j(m-1) < j(m, \beta) < \cdots < j(n-1, \beta)$, $[\gamma < \beta \Rightarrow j(n-1, \gamma) < j(m, \beta)]$. Suppose further $-b_0^*, -b_m^*, \ldots, -b_{n-1}^*$ is a B_0-partition of 1, $c_l^* \leqslant b_l^*, c^* \in B_0$, $x_{j(l)} \cap b_l^* = c_l^*$ for $l = 1$, $m-1$, and $x_{j(l,\beta)} \cap b_l^* = c_l^*$ for $m \leqslant l < n$.

Note that any uncountable $A \subseteq B_{\omega_1}[G]$ will contain such a countable subset. Then we define F as follows: if $\bar{y} = \langle x_{i(m)}, \ldots, x_{i(n-1)} \rangle$,

$$\tau(\bar{x}) = \bigwedge_{l=m}^{n-1} x_{i(l)} \cap b_l = c_l \wedge \sigma(\bar{z})$$

(where $\bar{y}$, $\bar{z}$ are disjoint, $\sigma(z) \in \boldsymbol{P}$) and there are b_l', $c_l' \in B_0$ such that $c_l' \leqslant b_l' < 1$, $b_l \leqslant b_l'$, $b_l - c_l \leqslant b_l' - c_l'$, and for no $\beta < \beta_0$ does

$$B_\alpha \vDash \bigwedge_{l=m}^{n-1} x_{j(l,\beta)} \cap b_l' = c_l' \wedge \sigma(\bar{z})$$

then for some such b_l', c_l'

$$F^*_{\bar{y}}(\tau(\bar{x})) = \bigwedge_{l=m}^{n-1} x_{i(l)} \cap b_l' = c_l' \wedge \sigma(\bar{z}).$$

[4] B_i is the set of terms in $B_0 \cup \{x_\alpha : \alpha < i\}$. We do not strictly distinguish between B_i and $B_i(G)$.

If there are no such b_l^i, c_l^i, then $F_{\bar{y}}^*(\tau(\bar{x})) = \tau(\bar{x})$. We leave the rest to the reader (he can consult the relevant parts of [11]). □

Let us consider another example. See Rubin and Shelah [7] and [8].

4.2. Definition. *Let P, Q be one place predicates, R a two place predicate and suppose the theory T contains the formula $(\forall xy)\,[xRy \to Px \wedge Qy]$.*

We say R (*or P, Q, R*)) *has the strong independence property* (for the theory T) if for every $n<\omega$, $M \models T$, distinct $a_1, \ldots, a_{2n} \in P^M$ there is $c \in Q^M$ such that $a_i R c$ iff $i \leqslant n$. We assume

$$\forall xy\, \exists z\, \{Q(x) \wedge Q(y) \wedge x \neq y \to P(z) \wedge (zRx \equiv \neg zRy)\}.$$

4.3. Theorem. *Suppose $(\mathrm{D}\ell)_\lambda$, $|T| < \lambda$ and $\diamondsuit_{\{\delta<\kappa^+ : \mathrm{cf}\,\delta=\lambda\}}$ (hence $\lambda = \lambda^{<\lambda} > \aleph_0$). Then T has a λ-saturated model M of power λ^+ such that:*

(1) *If (P, Q, R) has the strong independence property for T, then every automorphism of the structure $(P^M \cup Q^M, Q^M, R^M)$ is definable in M (as a set of pairs, by a first order formula with parameters)*

(2) *The same holds for any $(\varphi(x, \bar{c}), \psi(x, \bar{c}), \theta(x, y, \bar{c}))$.*

Remark. We shall prove (1) only. We shall later work on weakening the set-theoretic hypothesis.

4.4. Example. (1) T = theory of $(\omega, +, \times, 0, 1)$.

Px: x is prime,
xRy: x divides y, and x is prime.

(2) Px: $x = x$.
xRy: y codes a sequence in which x appears.

(3) T = infinite atomic B.A.
PX: x an atom,
Qx: $x = x$,
xRy: $x \leqslant y$.

4.5. Definition. Suppose R has the strong independence property. Γ_R is defined so that $\varphi(x, \bar{a})$ is Γ_R-big if for every $n<\omega$ it is Γ_R-n-big, i.e. there is a finite $A \subseteq P^M$ so that for any distinct $a_1, \ldots, a_{2n} \in P^M - A$, $\varphi(x, \bar{a}) \wedge \bigwedge_{i \leqslant 2n} [a_i Rx]^{\mathrm{if}\,(i \leqslant n)}$ is satisfiable. In this case we say that A is a Γ_R-n-witness. Let Γ_R-n-small means just not Γ_R-n-big.

4.6. Claim. *Γ_R-big is an invariant notion of bigness (hence by 2.5 a super notion of bigness).*

Proof. (1) Clearly being Γ_R-big depends on tp($\bar{a}$), φ only, hence Γ_R is invariant.

(2) If $\varphi_1 \to \varphi_2$, φ_1 Γ_R-n-big and A exemplifies this, then A will work for φ_2 also.

(3) So Axiom 1 (Definition 1.2) holds if φ_1 and φ_2 are small. How about $\varphi_1 \vee \varphi_2$? Suppose φ_1 is not Γ_R-n-big, φ_2 is not Γ_R-m-big and A is a Γ_R-$(n+m)$-witness for $\varphi_1 \vee \varphi_2$.

As φ_1 is Γ_R-n-small there are $a_1, \ldots, a_{2n} \in P^M - A$ such that

$$\neg(\exists x)\,(\varphi(x, \bar{a}) \wedge \bigwedge_{i \leq 2n} [a_i R x]^{\text{if}(i \leq n)}.$$

Now $A' = A \cup \{a_1, \ldots, a_{2n}\}$ is not a Γ_R-m-witness for φ_2. So there are $b_1, \ldots, b_{2m} \in P^M - A'$ such that

$$\neg(\exists x)\left(\varphi(x, \bar{a}) \wedge \bigwedge_{i \leq 2m} [b_i R x]^{\text{if}(i \leq m)}\right).$$

Clearly $a_1, \ldots, a_{2n}, b_1, \ldots, b_{2m} \in P^M - A$ are distinct and

$$\neg(\exists x)\left(\varphi_1 \vee \varphi_2 \wedge \bigwedge_{i \leq 2m} [a_i R X]^{\text{if}(i \leq n)} \wedge \bigwedge_{i \leq 2m} [b_i R x]^{\text{if}(i \leq m)}\right).$$

So A is not a Γ-$(n+m)$-witness for $\varphi_1 \vee \varphi_2$.

Proof of 4.3. We apply 2.12A. So there is a model $M = \bigcup_{\alpha < \lambda^+} M_\alpha$, which we get as a result of the game, when player II wins, with $\Gamma = \Gamma_R$. We want to show

4.7. Claim. *In M, $(P^M \cup Q^M, P^M, R^M)$ has only automorphism defined in M, if player II plays in the right way.*

Suppose F is such an automorphism, $\beta \leq \alpha < \lambda^+$ and in his α-th move player I chose $F \restriction M_\beta$ as one of the relations in M'_α, cf $\beta =$ cf $\alpha = \lambda$, and $F \restriction M_\beta$ maps $P^{M_\beta} \cup Q^{M_\beta}$ onto itself. By diamond we can assume $\alpha = \beta$, (by using just $2^\lambda = \lambda^+$ we could only assume that there are such $\alpha \leq \beta$).

There is an element $d = F(x_\alpha)$. So in M,

$$(\forall z \in M_\beta)\,(P(z) \to [zRx_\alpha \leftrightarrow F(z) R d]) \tag{4.8}$$

The use of $F(z)$ is legitimate by player I's decision.

We can choose γ, $\alpha < \gamma < \lambda^+$ such that $d \in M_\gamma$ but $M_\gamma \restriction L^{\alpha+1}$ is $\mathscr{L}_{\alpha+1}$-generic for $(\gamma, \alpha, M'_\alpha)$ so some (γ, α)-condition Φ forces (4.8) and is satisfied in the model for some variable $y \in X_\gamma - X_\alpha$, $(y = d)$.

We can assume the set of parameters of Φ is $\subseteq M^* \subseteq M_\alpha$, $\|M^*\| < \lambda$, and $M_\beta \cap M^*$ is closed under F, F', (as $|\Phi| < \lambda$, $|T| < \lambda$, λ regular this is possible). Let

$$K = \{(b, c) : b, c \in P^{M_\beta} - M^*,\ \Phi \Vdash^{(\gamma,\alpha)}_{M_\alpha} \text{“}bRx_\alpha \leftrightarrow cRy\text{'' } or\ b \in M^* \cap P^{M_\beta},\ F(b) = c\};$$

whether or not $(b, c) \in K$ is determined by $\mathrm{tp}(\langle b, c\rangle, |M^*|)$ (by the definition of (γ, α)-conditions etc.).

Let us show K is the graph of $F \restriction P^{M_\beta}$. Clearly the graph is included in K, otherwise for some $b, c \in M_\beta$, $F(b) = c$, $\Phi' \supseteq \Phi$, $\Phi' \Vdash \neg bRx \leftrightarrow cRy$ contradicts the choice of Φ.

Suppose (b, c) is in K but not in the graph. Clearly $b, c \in M_\beta - M^*$. So there is $b' \in P^{M_\beta} - M^*$, $F(b') = c$ so $b' \neq b$. So $\Phi \Vdash bRx_\alpha \leftrightarrow cRy$ as $(b, c) \in K$, and $\Phi \Vdash b'Rx_\alpha \leftrightarrow cRy$ as $F(b') = c$, and the graph of F^{M_β} is included in K. So $\Phi \Vdash$ "$bRx_\alpha \leftrightarrow b'Rx_\alpha$", $b' \neq b \in M_\beta - M^*$. We want to show that $\Phi \cup \{bRx_\alpha, \neg b'Rx_\alpha\}$ is included in a (γ, α)-condition. By 2.3(3) it suffices to show that $\Phi \cup \{bRx_\alpha, \neg b'Rx_\alpha\}$ is included in a $(\alpha+1, \alpha)$-condition i.e. is Γ_R-big.

It suffices to show Γ_R-n-bigness of $\varphi(x_\alpha, \bar{y}, \bar{a}) \cup \{bRx_\alpha, \neg b'Rx_\alpha\}$ ($\bar{a}$ from M^*) for any $n < \omega$ and $\varphi(x_\alpha, \bar{y}, \bar{a}) \in (\Phi \restriction X_{\alpha+1})$. Now $\varphi(x_\alpha, y, \bar{a})$ has a finite Γ_R-n-witness set $A \subseteq M_\alpha$. We can prove we can choose $A \subseteq M^*$ as "there is a Γ_R-n-witness of power $|A|$" is a first-order property (and by an even simpler way, we could have chosen here M^* like this).

So the graph of $F \restriction P^{M_\alpha}$ is defined in M_α in the prescribed way, i.e., by a sentence of $L_{\infty,\lambda}$ in the original language i.e., $a, b \in P^{M_\beta}$, $F(a) = b$ *iff* $a, b \in P^{M_\beta}$, $M_\alpha \models \varphi[a, b, \bar{c}]$, (for some $\bar{c} \in M_\alpha$ of length $< \lambda$). By [8, 1.9.1, p. 64] this implies, if $\alpha = \beta$, that $F \restriction M_\beta$ is first-order definable there. As this argument can be repeated for stationarily many sets of β's, we can use Fodors theorem to get one definition. $\square$

4.9. Theorem (On dense linear ordering with no undefinable automorphism). *Suppose* $(\mathrm{D}\ell)_\lambda$, $\lambda > |T|$ *and* $\diamondsuit_{\{\delta<\lambda^+ : \mathrm{cf}\,\delta = \lambda\}}$. *Then* T *has a* λ-*saturated model of power* λ^+ *such that:*

(1) *If* $T \vdash$ "$<$ *is a dense linear ordering*", *then every automorphism* F *of* $(P^M, <)$ *is locally definable* (*i.e. every interval has a subinterval in which* F *is definable*).

(2) *For any formula* $\varphi(x, y, \bar{z})$, *and* $\bar{c} \in M$ *if* $\models$ "$\varphi(x, y, \bar{c})$ *is a dense linear ordering of* $\{x : (\exists y)\, \varphi(x, y, \bar{c})\}$", *then every automorphism of* $(\{a E M : M \models (\exists y)\, \varphi(a, y, \bar{c})\}, \varphi(x, y, \bar{c})\}$ *is locally definable in* M.

Proof. We concentrate on the proof of (1).

For this proof we define Γ by: $\varphi(x, \bar{a})$ is Γ-big if there are $b < c \in P^M$ such that $\varphi(M, \bar{a})$ is dense in (b, c) [small here = nowhere dense]; this is clearly a bigness notion, very simple and invariant.

Again we can find $M = \bigcup_{\alpha<\lambda^+} M_\alpha$, the result of a game won by II (using 2.12). Let F be an automorphism of $(P^M, <^M)$ and assume $\diamondsuit_{\{\delta<\lambda^+ : \mathrm{cf}\,\delta=\lambda\}}$. So we can assume that for stationarily many $\delta < \lambda^+$, $\mathrm{cf}\,\delta = \lambda$, $F \restriction M_\delta$ is in M'_δ. We try to see what occurs to $d = F(x_\delta)$, $(\forall z \in M_\delta)\,(P(z) \rightarrow (z < x_\delta \leftrightarrow F(z) < d)$. W.l.o.g. d is a variable $y \in X_\gamma = X_\delta$. So some (γ, δ)-condition Φ, $\gamma > \delta$, Φ is realized and forces this with y for d. There is an interval (b_0, c_0) of P^{M_δ} such that for every $b_0 < b < c < c_0$, $\Phi \cup \{b < x_\delta < c\}$ is a (γ, δ)-condition, and assume $(b_0 < x_\delta < c_0) \in \Phi$.

By hypothesis we have $\Phi \cup \{b < x_\delta\} \Vdash F(b) < y$ for any $b \in (b_0, c_0)^{M_\delta}$. Let

$$K = \{(b, c) \in P^{M_\delta} : b_0 < b < c_0,\ F(b_0) < c < F(c_0),\ \Phi \cup \{b < x_\delta\} \Vdash c < F(x_0)\}.$$

Clearly $F(b) \geq c \Rightarrow (b, c) \in K$ for b, c in the right intervals. As in the previous proof we can show the inverse. So this gives us a definition of K, hence of $F \restriction P^{M_\delta}$ in (b_0, c_0). So for stationarily many $\delta < \lambda^+$, cf $\delta = \lambda$, we get a $L_{\lambda,\lambda}$ definition in M_δ^L of $F \restriction P^{M_\delta}$ in some interval. By Fodors theorem, for stationarily many δ's the definition and the interval are the same. By λ-saturatedness there is *one* formula defining it.

So we need the parallel of [8, 1.9.1].

4.10. Claim. *Suppose M is a λ-saturated model of T, and T 'says' "$<$densely linearly orders $\{x : P(x)\}$ and $\{x : Q(x)\}$" and F is an isomorphism from $(P^M, <)$ onto $(Q^M, <)$ defined by a $L_{\lambda,\lambda}$ formula $\varphi = \varphi(x, y)$ (possibly with parameters). Then we can find such a first-order φ, which defines F in some interval.*

Proof. Let A be the set of parameters appearing in φ. We can find $p \in S(A)$ such that the set of $a \in M$ realizing p is dense in the interval (b, c) (for some $b < c$ in P^M). (See 1.4(c), (e).)

Let $b < a < c$, a realizes p. As in [8, p. 64] there is a first-order formula $\psi(x, y)$ with parameters in A, such that $\psi[a, F(a)]$ and $\forall x\, \exists^{\leq 1} y\, \psi(x, y)$, $\forall y\, \exists^{\leq 1} x\, \varphi(x, y)$. As in [8, 1.9.1A], the same holds for any $b < a' < c$ realizing p. Look at

$$(p(x_1) \cup p(x_2) \cup (\exists y_1 y_2)\, [\psi(x_1, y_1) \wedge \psi(x_2, y_2) \wedge \neg(x_1 < x_2 \equiv y_1 > y_2)] \\ \cup \{b < x_1 < c \wedge b < x_2 < c\}.$$

Clearly it is not realized (as F is an isomorphism). As M is λ-saturated, there is a finite subset which is not realzed, and we can assume we get it by replacing $p(x_1)$, $p(x_2)$ by $\theta(x_1)$, $\theta(x_2)$ respectively. We can assume also

$$M \vDash (\forall x)\, [\theta(x) \wedge b < x < c \rightarrow (\exists y)\, \psi(x, y)].$$

So $\psi(x, y)$ defines $F \restriction \{a \in M : \vDash b < a < c \wedge \theta\{a\}\}$. As F is onto Q and by the choice of p,

$$\psi^*(x, y) = (\forall z)\, (\exists u, v)\, (b < z < c \wedge P(z) \wedge z \neq x \rightarrow P(u) \wedge Q(v) \wedge b < u < c \\ \wedge (z < u < x \vee x < u < z) \wedge \psi(u, v) \wedge \theta(u) \wedge [u < x \equiv v < y])$$

defines $F \restriction \{x \in P^M : b < x < c\}$. So we proved 4.10. $\square$

Now suppose $T \vdash$ "$(P, <, +, \times)$ is an ordered field", then as before, in M, every automorphism F of $(P, <, +, \times)$, being an automorphism of $(P, <)$, is definable in some (b_0, c_0). Now, there is a 1–1 map from (b_0, c_0) onto P defined in M: $x \mapsto (x(b_0+c_0)/2)/b_0 - c_0$ and so we can define F. So

4.11. Conclusion. *If T is first-order, $\diamondsuit_{\{\delta<\lambda^+:\mathrm{cf}\,\delta=\lambda\}}$, $(\mathrm{D}\ell)_\lambda$ holds, $|T|<\lambda$, then there is a λ-saturated model M of T, $\|M\|=\lambda^+$, such that for any ordered field defined in M, every automorphism is definable.*

5. More applications: trees with no branches and the strong independence property revisted under GCH

5.1. Theorem (On trees with only definable branches). *Suppose $(\mathrm{D}\ell)_\lambda$ holds. Let T^* be a complete consistent theory in $L(Q)$, $|T^*|\leq\lambda$ (Q here is not as in* 1.2!). *Suppose further that in every model M of T^*, $(|M|, <^M)$ is a tree (i.e. a partial order, and $\{y : y<x\}$ is linearly ordered) and $(\forall x)(\neg Qy)(y<x), (Qy)(y=y) \in T^*$.*

Then T^* has a model in the λ^+-interpretation such that every branch of $(|M|, <^M)$ (i.e. a maximal linearly ordered set) of power λ^+ is definable.

Remarks. (1) Note that Q is the well-known quantifier: syntactically if φ is a formula so is $(Qx)\varphi$; in the λ-interpretation $M\vDash(Qx)\,\varphi(x, \bar a)$ iff $\{b\in M : M\vDash\varphi[b, \bar a]\}$ has power $\geq\lambda$. The standard interpretation is the $\aleph_1$-interpretation. (So consistent means in the $\aleph_1$-interpretation (see Keilser [4]). If $\lambda=\lambda^{<\lambda}$, $\psi\in L(Q)$ has a model in the $\aleph_1$-interpretation iff it has a model in the λ^+-interpretation, and in the λ^+-interpretation, $L(Q)$ is λ-compact (see e.g. [17]).

(2) Keisler [5] had dealt with this problem assuming $\diamondsuit_{\{\delta<\lambda^+:\mathrm{cf}\,\delta=\lambda\}}$ and $\lambda=\lambda^{<\lambda}$. In [9] we deal with such problems for $T\subseteq L_{\omega,\omega}$ and for $\lambda=\aleph_1$ for $T=L(Q)$.

(3) We could have deal simultaneously with all definable trees.

(4) We can waive the requirement $\forall x\,\neg Qy\,(y<x)$, but then restrict ourselves to branches of cofinality λ^+. In the definition of closed we should also demand that for $y\in\mathrm{FV}(\Phi)\cap(X_{\alpha+1}-X_\alpha)$, $y<z\in\Phi$ for some $z\in X_\alpha$, or in no extension of Φ this occurs, and in defining $Y(\Phi)$ restrict ourselves to such y's.

Proof. W.l.o.g. in T^*, every formula is equivalent to a predicate, and let $T=T^*\cap L$, so T is a complete first-order theory in L.

We restrict ourselves to the case of models of T^* with no definable branches and $|T|<\lambda$ (for simplicity only). Now we define $\Gamma : \varphi(x, \bar a)$ is Γ-big in M (M a model of T) if $M\vDash R_{\varphi(x,\bar y)}(\bar a)$, where $R\in L$ is the predicate (or a predicate) such that $(\forall\bar y)\,[(Qx)\,\varphi(x, \bar y)\equiv R(\bar y)]\in T^*$. Clearly this is a notion of bigness (really the prototype).

We call a Γ-condition Φ *closed* if for every $\alpha<\lambda^+$, $\Phi\restriction X_\alpha$ is the complete diagram of a model, and if $y_i^\alpha\in\mathrm{FV}(\Phi)$, then either for some y_j^β, $\beta<\alpha$, $y_j^\beta=y_i^\alpha\in\Phi$ *or* Φ has no Γ-condition extending it in which this occurs. Let $Y(\Phi)$ be the set of variables for which the second case occurs (obviously $x_\alpha\in\mathrm{FV}(\Phi)\Rightarrow x_\alpha\in Y(\Phi)$).

5.1A. Fact. *Every $(\beta, 0)$-condition can be extended to a closed one.*

Let I be a λ-saturated dense linear order of power λ. Now we shall prove the theorem using 2.17, so we have to define $\boldsymbol{P}$. $\boldsymbol{P}$ will be the set of pairs (Φ, f) satisfying: Φ *is a closed* $(\lambda^+, 0)$-condition, f is a function from $Y(\Phi)$ to I such that $(x<t) \in \Phi$, $x \in Y(\Phi)$, $y \in Y(\Phi)$ implies $I \vDash f(x) < f(y)$.

Note that

5.1B. Fact. *If $\Phi \subseteq \Psi$ are closed $(\lambda^+, 0)$-conditions $(\Phi, f) \in \boldsymbol{P}$, then for some g, $f \subseteq g$, $(\Psi, g) \in \boldsymbol{P}$ (remember I is a λ-saturated dense linear order).*

The main point is

5.1C. Fact. *If $(\Phi, f) \in \boldsymbol{P}$, $(\Phi, f) \upharpoonright X_\alpha \leqslant (\Psi, g) \in \boldsymbol{P}$ Ψ a closed $(\alpha, 0)$-condition, then (Φ, f), (Ψ, g) are compatible in $\boldsymbol{P}$.*

Clearly w.l.o.g. Φ is an $(\alpha+1, 0)$-condition.

We know that Φ, Ψ are compatible as $(\lambda^+, 0)$-conditions. It is enough to find an $(\alpha+1, 0)$-condition Φ_1, $\Phi \cup \Psi \subseteq \Phi_1$ such that for every $y \in Y(\Phi) - X_\alpha$, $z \in Y(\Psi) - Y(\Phi)$: either (i) $\neg(z \leqslant y) \in \Phi_1$ *or* (ii) for some $v \in Y(\Phi) \cap Y(\Psi)$, $(v \leqslant y) \in \Phi$, $(z \leqslant v) \in \Psi$. As Γ is very simple it is enough to prove that for $n < \omega$, pairs (y_l, z_l) as above failing (ii) for $l = 1, n$, $\Phi \cup \Psi \cup \{\neg z_l \leqslant y_l \colon l < n\}$ is an $(\alpha+1, 0)$-condition, w.l.o.g. $y_l \neq x_\alpha$.

Let M_α be an $(\alpha, 0)$-generic model realizing Ψ, and let $\varphi(x_\alpha, y_0, \ldots, y_{n-1}) \in \Phi$ (suppressing the appearance of members of $X_\alpha = |M_\alpha|$) and we shall prove that

$$\varphi(x_\alpha, y_0, \ldots, y_{n-1}) \wedge \bigwedge_{l<n} \neg(z_l \leqslant y_l)$$

is Γ-big over M_α; this clearly suffices. Remember that as Φ is a closed $(\alpha+1, 0)$-condition. $\varphi(x_\alpha, y_0, \ldots, y_{n-1})$ is Γ-big over M but for $l<n$, $c \in M$, $\varphi(x_\alpha, y_0, \ldots) \wedge y_l = c$ is not Γ-big. As Γ is very simple the set (of n-tuples)

$$A = \{\langle c_0, \ldots, c_{n-1}\rangle \in M \colon \varphi(x_\alpha, y_l, \ldots, y_{n-1}) \wedge \bigwedge_{l<n} \neg(c_l \leqslant y_l) \text{ is not } \Gamma\text{-big}\}$$

is definable in M_α, and by the choice of φ its set of parameters comes from $\mathrm{FV}(\Phi) \cap X_\alpha$, which is the universe of an elementary submodel N of M. Clearly in A there are no $(n+1)$ n-tuples $(c_0^i, \ldots, c_{n-1}^i) \in M$ $(i \leqslant n)$ such that

$$\bigwedge_{i \neq j} \bigwedge_{l} [c_l^i, c_l^j \text{ are incomparable}]$$

(as then we can choose them in N; for every i for $l(i) < n$ $[c_{l(i)}^i \leqslant y_{l(i)}] \in \Phi$, hence for some $i \neq j$, $l(i) = l(j)$, hence Φ 'says' $c_{l(i)}^i$ and $c_{l(j)}^j$ are comparable as $<$ is a tree). Now every countable subset of T has a model in the $\aleph_1$-interpretation having no $\aleph_1$-branches, hence a model in which the tree is special (see [9]), so (by the definition of A) this gives a contradiction.

We still are left with the case T says there are denable Γ-big branches. We then should use the device of the proof of Theorem 6 in [9]. □

Now we revisit 4.3; (we proved that if R has the strong independence property (for P, Q, in the theory T), then for the model M constructed there, $(P^M \cup Q^M, R^M, P^M)$ has only automorphisms which are defined in M (by first order formulas with parameters).) We have used the hypothesis $\diamondsuit_{\{\delta<\lambda^+;\mathrm{cf}\,\delta=\lambda\}}$ (in addition to $(\mathrm{D}\ell)_\lambda$.)

5.2. Theorem. *We assume* $(\mathrm{D}\ell)_\lambda$ *and* $2^\lambda=\lambda^+$.

(1) *Any first order theory* T, $|T|<\lambda$ *has a* λ*-compact model* M *of power* λ^+, *in which if* R *has the strong independence property (for* T, P, Q*), then every automorphism of* $(P^M \cup Q^M, R^M, P^M)$ *is definable in* M *(i.e. by a formula with parameters).*

(2) *We can in* (1) *assume* $|T|\leq\lambda$, *and get that if* $\langle\varphi_l(\bar{x},\bar{a}),\ \psi_l(\bar{y},\bar{a}),\ \theta_l(\bar{x},\bar{y},\bar{a})\rangle$ $(\bar{a}\in M,\ \psi_l,\ \theta_l,\ \varphi_l\in L(T))$ *has the strong independence property, then any one-to-one function* F *from* $\{b\in M:\Vdash\varphi_0[b,\bar{a}]\}\vee\phi_0[b,\bar{a}]\}$ *onto* $\{c\in M:\Vdash\varphi_1(c,\bar{a})\vee\psi_0(c,\bar{a})\}$ *such that* $\theta_0[b,c,\bar{a}]\Leftrightarrow\theta_1[F(b),F(c),\bar{a})]$ *is definable in* M.

Proof. Similar to the proof of 4.3 the proof of which we follow.

We concentrate on (1).

5.2a. Definition. We redefine what is a Γ_R-condition Φ over M: it is a set of $<\lambda$ formulas, $\mathrm{FV}(\Phi)\subseteq\{x\}\cup\{y_i:i<\lambda\}$ and for every finite conjunction $\varphi=\varphi(x,y_{i_1},\ldots,y_{i_m})$ of members of Φ and every $n<\omega$ for some finite $A^n_\varphi\subseteq M$ for every distinct $b_0,\ldots,b_{2n-1}\in P^M-A$,

$$\{\varphi(x^l,y^l_{i_1},\ldots,y^l_{i_m});\ l<n\}\cup\{[b_kRx^l]^{\mathrm{if}[k<n]}:k<2n,\ l<n\}$$
$$\cup\{y^l_{i_k}\neq y^{l(1)}_{i_{k(1)}}:l,\ l(1)<n \text{ and } k,\ k(1)<m \text{ and } (l,k)\neq(l(1),k(1))\}$$

is satisfiable in M. (This is the change indicated in (1) but we use the variables y_i instead of more x's).

Now all of Sections 1 and 2 holds (only be more careful in 2.4).

We call a $(\lambda^+,0)$-condition Φ closed if $\Phi\restriction X_\alpha$ is a complete diagram of a model, for each α. Clearly we can extend every Φ to a closed one, s.t. $\{\alpha:\mathrm{FV}(\Phi\cap(X_{\alpha+1}-X_\alpha)\neq0\}$ does not change.

Let M, M_α $(\alpha<\lambda^+)$ be the models we obtained from Theorem 2.12A for the specification mentioned above and let G be an automorphisms of $(P^M\cup Q^M,R^M)$. Let

$$C=\{\beta<\lambda^+:M_\beta \text{ is closed under } G,G^{-1}\}.$$

Clearly C is closed unbounded. Now as in the proof of 4.3, for every $\beta \in C$ ($\mathrm{cf}\,\beta = \lambda$) there are $\alpha, \gamma, \beta \leq \alpha \leq \gamma$ and a closed (γ, α)-condition over M_α Φ satisfied by M (whose set of parameters is $\subseteq M^* \subseteq M_\alpha$, $\|M^*\| < \lambda$) and $y \in X_\gamma - X_\alpha$ such that the graph of $G \restriction P^{M_\beta}$ is:

$$K = \{(b, c)\colon \text{either } b, c \in P^{M_\beta} - M^*,\ \Phi \Vdash^{(\gamma,\alpha)}_{M_\alpha} bRx_\alpha \equiv cRy_\alpha, \\ \text{or } b \in P^{M_\beta} \cap M^*,\ F(b) = c\}.$$

Note that from K, K_1 = the graph of $G \restriction M_\beta$ is easily defined. Hence there is an (α, β)-condition Ψ, satisfied by M, which forces that K induces the graph of an automorphism G of $(P^{M_\beta} \cup Q^{M_\beta}, R^{M_\beta}, Q^{M_\beta})$ (provided that in applying 2.17 we use more appropriate F's). Let the set of parameters of Ψ be $\subseteq N < M_\beta$, $|N| < \lambda$, and also $M^* \cap M_\beta \subseteq N$. Assume:

5.2b. Definition. Call $c \in M_\beta$ a Ψ-possible value of $G(b)$ if there is a (β, α)-condition Ψ', $\Psi \subseteq \Psi'$, and $\Psi \Vdash$ "$(b, c) \in K$". We define similarly when b is a possible value of $F^{-1}(c)$.

5.2c. Fact. *The relation c [resp. b] is a Ψ-possible value of $G(b)$ (resp. of $G^{-1}(c)$] is preserved by automorphisms of M_β over N,*

Suppose $\delta < \beta$, $\mathrm{cf}\,\delta = \lambda$, M_δ is closed under G, G^{-1} and moreover Ψ forces that K is like that. Notice that if c is a Ψ-possible value of $G(b)$, then $b \in M_\delta \Leftrightarrow c \in M_\delta$.

5.2d. Fact. *For some A^*, $N \cap M_\delta \subseteq A^* \subseteq M_\delta$, $|A^*| < \lambda$ and $\xi < \delta$ for every $b, c \in M_\delta - M_\xi$, if c is a Ψ-possible value of $G(b)$, then $c \in \mathrm{acl}(A^* \cup \{b\})$.*

Proof of 5.2d. Suppose the conclusion fails. We use the genericity of the set of (β, δ)-conditions satisfied by M_β. Let X be $\subseteq X_\beta - X_\delta$, $N - M_\delta$ included in its set of interpretations. The proposition "$c \in M_\beta$ is a Ψ-possible value of $G(b)$ (in M_β)" is expressible by a formula from $L_{\lambda^+,\lambda}$, but the number of such formulas is λ, so we can assume genericity for them.

So suppose Ψ_1 is a (β, δ)-condition, forcing the failure of 5.2d, its set of variables is $Y_1 \supseteq N - M_\delta$, its set of parameters $N_1 \supseteq N \cap M_\delta$. Let $\xi < \delta$ be such that $N_1 \subseteq M_\xi$ (ξ exists as $\mathrm{cf}\,\delta = \lambda$). As Ψ_1 forces the failure of 5.2d (for $A^* \overset{\mathrm{df}}{=} N_1$), there is a (β, δ)-condition Ψ_2, $\Psi_1 \subseteq \Psi_2$, and $b, c \in M_\delta - M_\xi$ such that:

(1) Ψ_2 forces that c is a Ψ-possible value of $G(b)$, i.e., it determines the type of $\langle b, c\rangle$ over N is the right way (equivalently determine the type of Y over $N_1 \cup \{b, c\}$ in the right way.

(2) $c \notin \mathrm{acl}(N_1 \cup \{b\})$.

Let the set of parameters of Ψ_2 be N_2, it set of variables Y_2, and let

$$Y_2 \cap \{y_i^\delta : i < \lambda\} \subseteq \{y_i^\delta : i < i(0)\} \quad \text{where } i(0) < \lambda.$$

Let

$$A = \mathrm{acl}(N_1 \cup \{b\}), \qquad N_2 - A = \{a_i : i < i(1)\}, \quad i(1) < \lambda, \quad a_0 = c.$$

Let

$$\Psi_2' = \{\varphi(\bar z, y_{i(0)+j_1}, \ldots, y_{i(0)+j_k}, \bar a) : \bar a \in A, \quad \varphi \in L, \quad j_1 \cdots j_k < i(1),$$
$$\bar z \in Y_2 \quad \text{and} \quad \varphi(\bar z, a_{j_1}, \ldots, a_{j_k}, \bar a) \in \Psi_2\}.$$

Trivially $\Psi_1 \subseteq \Psi_2' \in \boldsymbol{P}$ but Ψ_2' forces that

(1') $y^{\delta}_{i(0)}$ is a Ψ-possible value of $G(b)$ (but $b \in M_\delta$, $y_{i(0)} \notin M_\delta$), as the relevant information is preserved. But this contradicts the choice of Ψ_1.

So we have proved fact 5.2d and we can w.l.o.g. assume M_ξ is closed under G, G^{-1}. Note also that we could have started with $\beta' > \beta$ instead β, and choose β as δ: and the proof above could be applied as well for G^{-1}. We can also increase N to include A^*, so

5.2e. Fact. *There is $\xi < \beta$, such that for every $b, c \in M_\beta - M_\xi$, if c is a Ψ-possible value of $G(b)$, then $b \in \operatorname{acl}(N \cup \{c\})$, $c \in \operatorname{acl}(N \cup \{b\})$.*

5.2f. Fact. *If b is a Ψ-possible value of $G(c)$, $b, c \in P^{M_\beta} - M_\xi$, then c is definable over $N \cup \{b\}$ in a unique way (and b over $N \cup \{c\}$) (maybe after replacing ξ by some $\xi' < \beta$, and Ψ by $\Psi' \supseteq \Psi$).*

Proof. Suppose for every $\xi_0 < \beta$ there are $b, c \in P^{M_\beta} - M_{\xi_0}$ such that c is a Ψ-possible value of $G(b)$ but c is not definable over $N \cup \{b\}$.

Let $\mu = |N| + |T| < \lambda$, and let $\kappa = \min\{\kappa : \mu^\kappa > \mu\}$, so $\kappa \leq \mu < \lambda$ $\mu^{<\kappa} = \mu$. We now define by induction on $\gamma \leq \kappa$, a (β, α)-condition Ψ_η $(\eta \in {}^\gamma\mu)$ and elements $a^l_{\eta,i,j} \in P^{M_\beta} - M_\xi$ $(\eta \in {}^{\gamma >}\mu,\ i \neq j,\ i < \mu,\ j < \mu,\ l = 0, 1, 2)$ such that:

(0) $\Psi_{\langle\rangle} = \Psi$, $|\Psi_\eta| \leq \mu$.

(1) If ν is an initial segment of η, *then* $\Psi_\nu \subseteq \Psi_\eta$.

(2) $a^{l(1)}_{\eta_1,i_1,j_1} = a^{l(2)}_{\eta_2,i_2,j_2}$ implies $\eta_1 = \eta_2$, $i_1 = i_2$, $j_1 = j_2$ and $l(1) = l(2) \vee \{l(1), l(2)\} = \{0, 1\}$.

(3) $\quad \Psi_{\eta^\frown\langle i\rangle} \Vdash \underset{\sim}{G}(a^0_{\eta,i,j}) = a^1_{\eta,i,j}, \qquad \Psi_{\eta^\frown\langle j\rangle} \Vdash \underset{\sim}{G}(a^0_{\eta,i,j}) = a^2_{\eta,i,j}.$

For $\alpha = 0$, and α limit there are no problems. For $\gamma + 1$, $\eta \in {}^\gamma\lambda$ we define together $\Psi_{\eta^\frown\langle i\rangle}$, $a^l_{\eta,i,j}$ $(i \neq j < \mu,\ 1 < 3)$; (we do it by induction on η by an arbitrary well ordering of ${}^\gamma\mu$). For this let $\{\langle i, j\rangle : i \neq j < \mu\} = \{\langle i_\sigma, j_\sigma\rangle : \sigma < \mu\}$, and we define by induction on $\sigma \leq \mu$, $\Psi_{\eta,i,\sigma}$, an (α, β)-condition increasing with σ, $\Psi_{\eta,i,0} = \Psi_\eta$, $\Psi_{\eta,i,\mu} = \Psi_{\eta^\frown\langle i\rangle}$ and

$$\Psi_{\eta,i_\sigma+1,\sigma+1} \Vdash G(a^0_{\eta,i_\sigma,j_\sigma}) = a^1_{\eta,i_\sigma,j_\sigma}, \qquad \Psi_{\eta,j_\sigma+1,\sigma+1} \Vdash \underset{\sim}{G}(a^0_{\eta,i_\sigma,j_\sigma}) = a^2_{\eta,i_\sigma,j_\sigma}$$

and $a^0_{\eta,i,j} \neq a^2_{\eta,i,j}$, $a^1_{\eta,i,j} \neq a^2_{\eta,i,j}$ and they are distinct from all previously defined a^l_{ν,i_1,j_1}'s. The atomic step is done by the assumption that 5.2f fails, and we can keep $|\Psi_{\eta,j,\sigma}| \leq \mu$ by 5.2c. Let

$$q = \{a^1_{\eta,i,j} R x \wedge \neg a^2_{\eta,i,j} R x : \eta \in {}^\gamma\mu,\ \gamma < \kappa\}$$

be consistent (by the strong independence property). Hence there is $c \in Q^{M_\beta} - M_\xi$ realizing it, and for each $\eta \in {}^\kappa\mu$, there is a Ψ_η-possibe value of $G^{-1}(c)$, b_η. But trivially any Ψ-possible value of $G^{-1}(c)$ belongs to $\mathrm{acl}(N \cup \{c\})$ hence (as $\mu^\kappa > \kappa$) there are $\eta \neq \nu$ in ${}^\kappa\mu$ such that $b_\eta = b_\nu$. Choose $\gamma < \kappa$, $\eta \restriction \gamma = \nu \restriction \gamma$, $\eta(\gamma) \neq \nu(\gamma)$; let $i = \eta(\gamma)$, $j = \nu(\gamma)$. But $M_\beta \models a^1_{\eta,i,j} Rc$, Ψ_η forces $G(a^0_{\eta,i,j}) = a^1_{\eta,i,j}$, (as $\Psi_{\eta \restriction (\gamma+1)}$ does) and b_η is a Ψ_η-possible value of $G^{-1}(c)$. Hence

$$M_\beta \models a^0_{\eta,i,j} R b_\eta.$$

Similarly $M_\beta \models \neg a^0_{\eta,i,j} R b_\nu$, contradicting $b_\eta = b_\nu$.

We can conclude that for some $\xi_0 < \beta$ for every $b, c \in P^{M_\beta} - M_{\xi_0}$ if c is a possible value of $G(b)$, then c is definable in a unique way over $N \cup \{b\}$. Applying the same proof to G^{-1} we get also that b is definable in a unique way over $N \cup \{b\}$.

We can assume M_ξ is closed under G, G^{-1}. Now if $b, c \in M_\xi$, c a Ψ-possible value of $G(b)$, $b, c \notin N$ (=acl N), *then* by what we already have proved some time ago, there are $b', c' \in M_\beta - M_\xi$, such that $\mathrm{tp}(\langle b, c\rangle, N) = \mathrm{tp}(\langle b', c'\rangle, N)$ and this implies c is a Ψ-possible value of $G(b')$. We can conclude $F \restriction P^{M_\beta}$ is defined (by an $L_{\lambda,\lambda}$ formula with parameters). By [8, 1.9.1] it is definable (by an L-formula with parameters). As β was any member of a closed unbounded set $C \subseteq \lambda^{+}$ which was cofinality λ, by Fodor's theorem we know this holds in $M = M_{\lambda^+}$, so we finish.

5.2g. Concluding remarks. (1) We can ask whether 5.2(2) causes any problems. Dealing with more triples. (P, Q, R), just makes us redefine $\boldsymbol{P}$, so that if $x_\beta \in \bar{x}$, $\tau(\bar{x}) \in \boldsymbol{P}$, then $\tau(x)$ chooses such triple which is defined in M_β, and let $\bar{x}_\beta$ be "Γ_R-big" for this R.

(2) If we have $\aleph_0 < |T| = \lambda$, we should replace λ-saturated by λ-compact, and note in Fact 5.2c that if $L^* \subseteq L$, has cardinality $< \lambda$, and P, Q, R, and the non-logical-symbols appearing in Ψ are in it, any automorphism of the L^*-reduction of M_β over N will do, then we have to proceed accordingly.

The case $\lambda = \aleph_0$ will be discussed elsewhere.

5.3. Theorem. *Suppose* $(\mathrm{D}\ell)_\lambda$ *holds, and* λ *is not a strong limit cardinal. Then the conclusion of* 5.2 *holds.*

Proof. Suppose $\mu < \lambda \leq 2^\mu$, T. P. Q, R as there.

For a model M a set of formulas p in the variables x^α_i $(i < \mu)$, $y^\alpha_{i,j}$ $(i < \mu, j < \lambda)$, A a set of parameters from M, $|A| < \lambda$, M λ-saturated, is called Γ^*_R-big if there are a^α_i, $b^\alpha_{i,j}$ in M (for $i < \mu, j < \lambda$) realizing Γ such that

(1) For each i, $\mathrm{tp}(\langle a^\alpha_i, b^\alpha_{i,j} : j < \lambda\rangle, A \cup \bigcup\{a^\alpha_\xi, b^\alpha_{\xi,j} : \xi < i, j < \lambda\})$ is as required in 5.2a.

(1) For every $b_2 \neq b_1 \in P^M \cap \mathrm{acl}\, A$ for some $i < \mu$, $b_1 R a^\alpha_i \wedge \neg b_2 R a^\alpha_i$ (note that the truth value of this statment is determined by $\mathrm{tp}(\langle a^\alpha_i : i < \mu\rangle, A)$).

Now we repeat the proof of 5.2, but replacing Γ_R-big by Γ^*_R-big, and not using

H_1 (there we used $2^\lambda = \lambda^+$ to produce all relations on any M_β). Instead we use the following observation.

If G is an automorphism of $(P^M \cup Q^M, R^M, P^M)$, M closed under G, G^{-1}, *then* from $\langle \boldsymbol{x}_i^\beta, G(\boldsymbol{x}_i^\beta) : i \langle \mu \rangle$ we can find $G \restriction M_\beta$ (by condition (2) above).

Acknowledgement

The author wishes to thank the United States–Israel Binational Science Foundation, for partially supporting this research.

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