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VIVE LA DIFFÉRENCE III

BY

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ABSTRACT

We show that, consistently, there is an ultrafilter $\mathcal{F}$ on ω such that if $N^\ell_n = (P^\ell_n \cup Q^\ell_n, P^\ell_n, Q^\ell_n, R^\ell_n)$ (for $\ell = 1, 2,\ n < \omega$), $P^\ell_n \cup Q^\ell_n \subseteq \omega$, and $\prod_{n<\omega} N^1_n/\mathcal{F} \equiv \prod_{n<\omega} N^2_n/\mathcal{F}$ are models of the canonical theory $t^{\rm ind}$ of the strong independence property, then every isomorphism from $\prod_{n<\omega} N^1_n/\mathcal{F}$ onto $\prod_{n<\omega} N^2_n/\mathcal{F}$ is a product isomorphism.

0. Introduction

In a previous paper [Sh326] we gave two constructions of models of set theory in which the following isomorphism principle fails in various strong respects:

(Iso 1): If M, N are countable elementarily equivalent structures and $\mathcal{F}$ is a non-principal ultrafilter on ω, then the ultrapowers M^*, N^* of M, N with respect to $\mathcal{F}$ are isomorphic.

As is well-known, this principle is a consequence of the Continuum Hypothesis. Recall that Keisler celebrated theorem (from [Ke67]) says that, if $2^\lambda = \lambda^+$ then

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two models, M, N of cardinality at most λ^+ (and vocabulary of cardinality $\le \lambda$) are elementarily equivalent iff for some ultrafilter $\mathcal{F}$ on λ, the ultrapowers $M^\lambda/\mathcal{F}, N^\lambda/\mathcal{F}$ are isomorphic. This has given an algebraic characterization of elementary equivalence.

In [Sh405] our aim originally was to give a related example in connection with the well-known isomorphism theorem of Ax and Kochen. In its general formulation, that result states that a fairly broad class of Henselian fields of characteristic zero satisfying a completeness (or saturation) condition are classified up to isomorphism by the structure of their residue fields and their value groups. That is, the statement that interested us in the second paper in this series [Sh405], was:

(Iso 2): If $\mathcal{F}$ is a nonprincipal ultrafilter on ω, then the ultraproducts $\prod_p \mathbb{Z}_p/\mathcal{F}$ and $\prod_p \mathbb{F}_p[[t]]/\mathcal{F}$ are isomorphic.

The answer we got was, more generally,

THEOREM 0.1 (See [Sh405]): *It is consistent with the axioms of set theory that there is a nonprincipal ultrafilter $\mathcal{F}$ on ω such that for any two sequences of discrete rank 1 valuation rings $(R^i_n)_{n=1,2,\dots}$ $(i = 1, 2)$ having countable residue fields, any isomorphism $F : \prod_n R^1_n/\mathcal{F} \longrightarrow \prod_n R^2_n/\mathcal{F}$ is an ultraproduct of isomorphisms $F_n : R^1_n \longrightarrow R^2_n$ (for a set of n's contained in $\mathcal{F}$). In particular, for $\mathcal{F}$-majority of the n, the valuation rings R^1_n, R^2_n are isomorphic.*

In the case of the rings $\mathbb{F}_p[[t]]$ and $\mathbb{Z}_p$, we see that (Iso 2) fails. For this our main work was to show the following statement which actually from model theoretic point of view is more basic and interesting.

THEOREM 0.2 (See [Sh405]): *It is consistent with the axioms of set theory that there is a nonprincipal ultrafilter $\mathcal{F}$ on ω such that for any two sequences of countable trees $(T^i_n)_{n=1,2,\dots}$ for $i = 1, 2$, with each tree T^i_n countable with ω levels, and with each node having at least two immediate successors, if $\mathcal{T}^i = \prod_n T^i_n/\mathcal{F}$, then for any isomorphism $F : \mathcal{T}^1 \xrightarrow{\simeq} \mathcal{T}^2$ there is an element $a \in \mathcal{T}^1$ such that the restriction of F to the cone above a is the restriction of an ultraproduct of maps $F_n : T^1_n \longrightarrow T^2_n$.*

From a model theoretic point of view this is still not the right level of generality for a problem of this type. There are two natural ways to pose the problem. From now on

CONVENTION 0.3: In the rest of §0 and §2,§3 models are countable with countable vocabulary if not said otherwise, and we use M, N to denote models. If we say a model may be uncountable we assume its vocabulary is countable if not said otherwise.

PROBLEM 1: Characterize the pairs of countable models M, N which are pseudo-isomorphic, where

Definition 0.4: We say that the countable models M, N are pseudo-isomorphic if:

(a) if $\mathcal{F}$ is a nonprincipal ultrafilter over ω, then $M^\omega/\mathcal{F}, N^\omega/\mathcal{F}$ are isomorphic, and

(b) clause (a) holds, even after forcing by any (set) forcing .

Of course, this is not an isomorphism (see below on models of a stable theory). A related problem is

PROBLEM 2: Characterize the pairs of countable models M, N with nonisomorphic ultrapowers $M^\omega/\mathcal{F}, N^\omega/\mathcal{F}$ modulo any nonprincipal ultrafilter $\mathcal{F}$, in some forcing extension. (I.e., the negation is: for every forcing extension for some nonprincipal ultrafilter $\mathcal{F}$ on ω, we have $M^\omega/\mathcal{F} \simeq N^\omega/\mathcal{F}$.)

There are two variants of the second problem: the ultrapowers may be formed either by using one ultrafilter twice (called 2(A)), or by using any two ultrafilters (called 2(B)), see below. Since the continuum hypothesis holds is too easy ask:

PROBLEM 3: Characterize the pairs M, N of countable models such that in some forcing extension failing in continuum hypothesis, for every nonprincipal ultrafilter $\mathcal{F}$ on ω, $M^\omega/\mathcal{F} \cong N^\omega/\mathcal{F}$

PROBLEM 4: Let us write $M \leq N$ whenever in every forcing extension, if $\mathcal{F}$ is an ultrafilter on ω such that $N^\omega/\mathcal{F}$ is saturated, then $M^\omega/\mathcal{F}$ is also saturated. Characterize this relation.

This is related to the Keisler order (see Keisler [Ke67], or [Sh:a], or [Sh:c, Chapter VI]), but does not depend on the fact that the ultrafilter is regular, so some of the results there apply to Problem 4, this in turn implies results on Problem 2(A). By [Sh:c, VI] we know the following. Let $\mathcal{D}$ be a nonprinciple ultrafilter on ω, and M (countable) model (with countable vocabulary). If

Th(M) is stable then $M^\omega/\mathcal{D}$ is saturated. We can replace $\aleph_0$ here by any cardinal κ satisfying $\kappa^{<\kappa} = \kappa$ using regular ultrafilter on κ.

Now, by [Sh13], there is an ultrafilter $\mathcal{D}$ on $2^{\aleph_0}$ such that for countable models (with countable vocabulary) M, N

$$M \equiv N \quad \Rightarrow \quad M^{2^{\aleph_0}}/\mathcal{D} \cong N^{2^{\aleph_0}}/\mathcal{D}.$$

and we can add "$M^\omega/\mathcal{D}$ is κ-saturated" for every κ such that $2^{<\kappa} = 2^{2^{\aleph_0}}$. Also, if $2^{\aleph_0} = \aleph_1$, $\mathcal{F}$ is a nonprincipal ultrafilter on ω and $M_1 \equiv M_2$ are countable, then $M_1^\omega/\mathcal{F} \cong M_2^\omega/\mathcal{F}$ (as they are saturated); similarly if M_n^ℓ are countable models (for $\ell = 1, 2$, $n < \omega$), $M_\ell = \prod_{n<\omega} M_n^\ell/\mathcal{F}_\ell$, and $\mathcal{F}_\ell$ are nonprincipal ultrafilters on ω, then $M_1 \equiv M_2 \ \Rightarrow \ M_1 \cong M_2$. On the other hand, if $2^{\aleph_0} > \aleph_1$, then by [Sh:c, Ch VI] for every regular cardinal θ, $\aleph_1 \le \theta < 2^{\aleph_0}$ there is a nonprincipal ultrafilter $\mathcal{F}_\theta$ on ω such that the downward cofinality of $(\omega, <)^\omega/\mathcal{F}_\theta$ above ω is θ so $\theta_1 \neq \theta_2 \ \Rightarrow \ (\omega, <)^\omega/\mathcal{F}_{\theta_1} \ncong (\omega, <)^\omega/\mathcal{F}_{\theta_2}$. This gives negative results on Problem 2(B) above. If Th(M) is unstable then some such $\mathcal{D}, M^\omega/\mathcal{D}$ are not $\aleph_2$-saturated. Why? We can choose $\varphi(\bar{x}, \bar{y})$ which has the order property, $\lg(\bar{x}) = m$ and let $\bar{a}_{n,i} \in {}^m M (i < n < \omega)$ be such that $M \models \varphi[\bar{a}_{n,i}, \bar{a}_{n,j}]$ iff $i < j < n$. Let $P_n = \{\bar{a}_{n,i} : i < \omega\}$, $<_n = \{(\bar{a}_{n,i}, \bar{a}_{n,j}) : i < j < n\}$. Consider $(N, P) := \prod_{n<\omega}(M, P_n, <_n) \setminus \mathcal{D}$, now use a "cut" of $\prod_{n<\omega}(P_n, <_n)/\mathcal{D}$ with cofinality $(\aleph_0, \aleph_1)$. So for Problem 4, the stable theories are minimal.

A more general problem is

Problem 5: For which quadruples (M_1, N_1, M_2, N_2) of countable models such that $M_1 \equiv M_2$ and $N_1 \equiv N_2$, in some forcing extension for some ultrafilter $\mathcal{F}$ on ω, $M_1^\omega/\mathcal{F} \cong N_1^\omega/\mathcal{F}$ but $M_2^\omega/\mathcal{F} \ncong N_2^\omega/\mathcal{F}$? (and other variants as above).

We can also replace above the countable model M by the first order theory Th(M) e.g. we can define: $T_1 \le T_2$ iff (T_1, T_2 are countable theories such that) for every countable model M_1 of T_1 there is a countable model M_2 of T_2 such that $M_1 \le M_2$. The present paper is dedicated to sheding some further light.

Problem 6: We may be more interested in the ultrafilter, so define the order on the family of ultrafilters on ω but here our focus is on model theory. More specifically, we may ask to investigate $\le_{uf}$ where $\mathcal{F}_1 \le_{uf} \mathcal{F}_2$ iff $\mathcal{F}_1, \mathcal{F}_2$ are nonprincipal ultrafilter on ω such that for every countable model M, if $M^\omega/\mathcal{F}_1$ is saturated then $M^\omega/\mathcal{F}_2$ is saturated.

Working on [Sh405] we had hoped to continue it sometime. However, we actually began only after M. Jarden asked:

(∗) Suppose that F^ℓ_n are finite fields (for $n < \omega$, $\ell = 1, 2$) satisfying $F^1_n \not\cong F^2_n$. Can we have (a universe and) an ultrafilter $\mathcal{F}$ on ω such that $\prod_{n<\omega} F^1_n/\mathcal{F}$ and $\prod_{n<\omega} F^2_n/\mathcal{F}$ are elementarily equivalent but not isomorphic?

That was not an arbitrary question: he knew that many such pairs of ultraproducts are elementarily equivalent, because the first order theory of a field F which is isomorphic to an ultraproduct of finite fields is determined by its characteristic and its subfield of algebraic elements. Hence we can find an equivalence relation E_k on the family of finite fields for $k < \omega$ each with finitely many equivalence classes of the form: an equation from Δ_n has a solution in one iff it has a solution in the other with Δ_n finite, and such that if F^1_n, F^2_n are finite fields for $n < \omega$ and $\mathcal{F}$ is a nonprincipal ultrafilter on ω and for each k the set $\{n < \omega : (F^1_n)E_k(F^2_n)\}$ belongs to $\mathcal{F}$ then the respective ultraproducts are elementarily equivalent.

This raises the question whether such theorem of fields has the strong independence property. The following reference to the strong independence property for finite fields was given to me by Gregory Cherlin: Duret [Du80, pp. 136–157].

Here we continue [Sh326, §3], [Sh405, §1] to give an affirmative answer to (∗), we show that after adding $\aleph_3$ Cohen reals to a suitable ground model, one gets a universe with an ultrafilter $\mathcal{F}$ on ω and a sequence of models $\langle M_n : n < \omega\rangle$ on ω such that

(∗∗) if $N^\ell_n = (P^\ell_n \cup Q^\ell_n, P^\ell_n, Q^\ell_n, R^\ell_n)$ (for $\ell = 1, 2$, $n < \omega$), $P^\ell_n \cup Q^\ell_n \subseteq \omega$, and $\prod_{n<\omega} N^1_n/\mathcal{F} \equiv \prod_{n<\omega} N^2_n/\mathcal{F}$ are models of the canonical theory t^{ind} of the strong independence property (see Definition 1.5 below), **then**:
- ⊡ every isomorphism from $\prod_{n<\omega} N^1_n/\mathcal{F}$ onto $\prod_{n<\omega} N^2_n/\mathcal{F}$ is (first order) definable in $\prod_{n<\omega} M_n/\mathcal{F}$ for some models M_n with universe ω or what is equivalent but hopefully more transparent
- ⊡′ if F is an isomorphism from $N^1 = \prod_{n<\omega} N^1_n/\mathcal{F}$ onto $N^2 = \prod_{n<\omega} N^2_n/\mathcal{F}$ then we can find unary functions F_n from N^1_n into N^2_n for every $n < \omega$ such that the set of n for which F_n is an isomorphism from N^1_n onto N^2_n belongs to the ultrafilter and $\prod_{n<\omega}(N^1_n, N^2_n, F_n)/\mathcal{F}$ is (N^1, N^2, F).

Our forcing is adding $\aleph_3$ Cohen reals, but we need that our model of set theory, i.e., the universe over which we force, satisfies some conditions. There are two ways to get a "suitable" ground model. The first way involves taking any ground model which satisfies a relevant portion of the GCH, and extending it by an appropriate preliminary forcing, which generically adds the **name** for an ultrafilter which will appear after addition of the Cohen reals. The alternative approach, which we consider more model–theoretic, is to start with an **L**–like ground model and use instances of diamond (or related weaker principles) to prove that a sufficiently generic name already exists in the ground model. We will fully present the first approach — the second one should be then an easy modification of the arguments presented in [Sh405, §1].

Our presentation is somewhat more general than needed for (**). By allowing more what we call "bigness" properties to be involved in the definition of $\mathbb{A}$pp, we leave room for getting analogs of (**) for more classes of models (getting the conclusion for all of them at once, or possibly only for some) — as long as the respective bigness notions are as in Definition 1.4. This, we hope, would be helpful in connection with the problems above (particularly, Problems 2 and 5). For the problem on fields, only the case associated with the strong independence property is needed; general bigness notions appear for possible general treatment.

Let us comment on our general point of view. In this paper we try to advance in Problems 1 and 2(A) and for this, it seemed, we could take the maximal $\underset{\sim}{\Gamma}$, i.e., allow all $\aleph_0$-bigness notions. However, concerning Problem 4 (investigating the partial order $\leq$ on models), for showing $M \not\leq N$, the construction causes $N^\omega/\mathcal{F}$ to be almost always non-$\aleph_3$–saturated. We need finer tools for them, e.g., using some bigness notion but not others.

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We continue those investigations in [Sh800].

Notation 0.5: Our notation is standard and compatible with that of classical textbooks (like Hodges [Ho93], Chang and Keisler [CK] and Jech [J]). In forcing we keep the older convention that *a stronger condition is the larger one.*

(1) We will use two forcing notions denoted by $\mathbb{C}_{\aleph_3}$ and $\mathbb{A}$pp (see Definitions 2.1 and 2.4, respectively). Conditions in these forcing notions will be called p, q, r (with possible sub/super-scripts). Note that the product $\mathbb{A}\mathrm{pp} \times \mathbb{C}_{\aleph_3}$ is a dense subset of the composition $\mathbb{A}\mathrm{pp} * \mathbb{C}_{\aleph_3}$

(2) All names for objects in forcing extensions will be denoted with a tilde below (e.g., $\underset{\sim}{a}$, $\underset{\sim}{\mathfrak{p}}$).

(3) The letter τ (with possible sub/super-scripts) stands for a vocabulary of a first order language; we may also write $\tau(M)$, $\tau(T)$ for a model M or theory T with the obvious meaning. We will use the letters $\mathfrak{p}, \mathfrak{q}$ (with sub/super-scripts) to denote types.

(4) The universe of a model M will be denoted $|M|$, but we will often abuse this notation and write, e.g., $a \in M$. The cardinality of a set A will be denoted $|A|$, and, for a model M, $\|M\|$ will stand for the cardinality of its universe.

Comment: Why the $\aleph_3$? We like to have a preliminary forcing notion $\mathbb{A}$pp which for some κ, is κ-complete, κ^+-c.c., $\kappa^{<\kappa} = \kappa$; so that every cardinal is preserved. But for $\kappa = \aleph_1, A \subseteq \kappa^+$ countable the number of conditions with this domains (i.e., the number of names of ultrafilters on ω as above) is more than κ hence in the natural choice the κ-c.c may fail, we may remedy this but it is easier to use a cardinal κ such that $\mu < \kappa \Rightarrow \mu^{\aleph_0} < \kappa$.

1. Bigness notions

In this section we will quote relevant definitions and results from [Sh:e, Chapters X, XI] (=[Sh:384], see history there, and [Sh:482]), but we somewhat restrict ourselves here. The reader interested in the field case only and/or finding Definition 1.1 obscure, may jump directly to Definition 1.5.

Definition 1.1 (See [Sh:e, Chapter XI, §1]): Let T be a complete first order theory (in a vocabulary τ), and $\mathcal{K} = \mathcal{K}_T$ be a class of models of T (normally: all models of T) partially ordered by the relation $\prec$ of being elementary submodel. Also let t be a first order theory with a countable vocabulary $\tau(t)$ (including equality, treating function symbols as predicates).

(1) We say that $\mathcal{K}'$ is an A–place in $\mathcal{K}$ if

(a) $\mathcal{K}' \subseteq \mathcal{K}$;

(b) if $M \in \mathcal{K}'$, then $A \subseteq M$;

(c) if $M \prec N$ are from $\mathcal{K}$ and $A \subseteq M$, then $(M \in \mathcal{K}') \Leftrightarrow (N \in \mathcal{K}')$;

(d) if $M \in \mathcal{K}$ and $A \subseteq N \in \mathcal{K}$ and M, N are isomorphic over A, then $M \in \mathcal{K}' \Leftrightarrow N \in \mathcal{K}'$.

(1A) A place is an A-place for some A (alternatively use only $M \prec \mathfrak{C}$ of cardinality $< \bar{\kappa}$, where $\mathfrak{C}$ is $\bar{\kappa}$-saturated model of T, as in [Sh:c]).

(2) For $A \subseteq M \in \mathcal{K}$ we let $\mathcal{K}' = \mathcal{K}_{A,M}$ be the class

$$\{N \in \mathcal{K} : A \subseteq N \text{ and } \bar{a} \in {}^{\omega >}A \Rightarrow \operatorname{tp}(\bar{a}, \emptyset, M) = \operatorname{tp}(\bar{a}, \emptyset, N)\ \}.$$

We call it **the** (A, M)**-place**.

(3) **A local bigness notion** Γ **for** $\mathcal{K}$ (without parameters, in one variable x) is a function with domain $\mathcal{K}$ which for every model $M \in \mathcal{K}$ gives

$$\Gamma^-_M = \Gamma^-(M) \subseteq \{\varphi(x, \bar a) : \varphi \in \mathcal{L}(\tau) \ \& \ \bar a \subseteq M\},$$
$$\Gamma^+_M = \Gamma^+(M) = \{\varphi(x, \bar a) : \varphi \in \mathcal{L}(\tau) \ \& \ \bar a \subseteq M\} \setminus \Gamma^-_M$$

such that

(a) Γ^-_M is preserved by automorphisms of M,

(b) Γ^-_M is a proper ideal, i.e., $\Gamma^+_M \neq \emptyset$ and

(α) if $M \models (\forall x)(\varphi(x, \bar a) \to \psi(x, \bar b))$ and $\psi(x, \bar b) \in \Gamma^-_M$, then $\varphi(x, \bar a) \in \Gamma^-_M$,

(β) if $\varphi_1(x, \bar a_1), \varphi_2(x, \bar a_2) \in \Gamma^-_M$, then $\varphi_1(x, \bar a_1) \vee \varphi(x, \bar a_2) \in \Gamma^-_M$.

Elements of Γ^-_M are called Γ**-small in** M, members of Γ^+_M are Γ**-big**.

A local bigness notion Γ **for** $\mathcal{K}$ **with parameters**[1] **from** A is defined similarly but $\mathrm{Dom}(\Gamma)$ is an A-place $\mathcal{K}'$ in $\mathcal{K}$ and in clause (a) the automorphisms are over A.

(4) We say that a local bigness notion Γ is **invariant** for $\mathcal{K}$ (or for an A-place $\mathcal{K}'$) if for $M \prec N$ from $\mathcal{K}$ (or from the A-place $\mathcal{K}'$) we have $\Gamma^-_M \subseteq \Gamma^-_N$ and $\Gamma^+_M \subseteq \Gamma^+_N$.

(5) **A** Γ**-big type** $\mathfrak{p}(x)$ **in** M is a set of formulas $\psi(x, \bar a)$ all of whose finite conjunctions are Γ–big in M.

(6) A **pre** t**-bigness notion scheme** Ω is a sentence ψ_Ω (in possibly infinitary logic) in the vocabulary $\tau(t) \cup \{P^*\}$, where P^* is a unary predicate, we may say "using P^*".

(7) **An interpretation with parameters of** t **in a model** $M \in \mathcal{K}$ is $\bar\varphi = \langle \varphi_R(\bar y_R, \bar a_R) : R \in \tau(t)\rangle$, where $\varphi_R \in \mathcal{L}(\tau)$ and $\bar a_R$ is a sequence of appropriate length of elements of M. So a predicate R from $\tau(t)$ is interpreted as

$$\{\bar b : M \models \varphi_R(\bar b, \bar a_R),\ \lg(\bar b) = \lg(\bar y_R) \ (= \text{ the arity of } R)\ \}.$$

The interpreted model is called $M[\bar\varphi]$ or $M^{[\bar\varphi]}$ and we demand that it is a model of t; so in particular $M[\bar\varphi]$ is a $\tau(t)$-model and its universe is $\{b \in M : M \models \varphi_=(b, b, \bar a_=)\}$ defined by $\varphi_=(x, y, \bar a_=)$ which we demand to be an equivalence relation; here usually equality on its domains, so we may write just $\varphi_=(x, \bar a_=)$ or just $\varphi(x, \bar a)$; of course we could use k-tuples for elements and then $\lg(\bar y_R) = k_n$ for R an n-place predicate from $\tau(t)$

[1] Alternatively use the monster model.

(8) For a pre t-bigness notion scheme $\Omega = \psi_\Omega$ and an interpretation $\bar\varphi$ of t in $M \in \mathcal{K}$ with parameters from $A \subseteq M$, we define the $\bar\varphi$-derived **local pre-bigness notion $\Gamma = \Gamma_{\psi,\bar\varphi} = \Gamma_\psi[\bar\varphi]$ with parameters from $A \subseteq M$ (in the A-place $\mathcal{K}_{A,M}$)** as follows:
Given $M' \in \mathcal{K}_{A,M}$, a formula $\vartheta(x,\bar b)$ in $\mathcal{L}(\tau)$ (with parameters from M' of course) is $\Gamma_\psi[\bar\varphi]$–big in M' if for any quite saturated N^*, $M' \prec N^*$, letting

$$P^* = \{a \in N^*[\bar\varphi] : N^* \models \vartheta[a,\bar b]\}$$

we have $(N^*[\bar\varphi], P^*) \models \psi$.

In full we may write $\Gamma = \Gamma_{(t,\psi,\bar\varphi)}$ and even $\Gamma = \Gamma_{(t,\psi,\bar\varphi,M,A)}$.

(9) We say ψ is a t-bigness notion (for T) if for every interpretation $\bar\varphi$ of t in some A-place $\mathcal{K}' \subseteq \mathcal{K}$, $\Gamma_{t,\psi,\bar\varphi}$ is an invariant[2] local bigness notion for our fixed $\mathcal{K}$. If there is no T mentioned or understood we mean "for every T". So it is enough in (8) above if we define $\Gamma_{M'}$ when $M \prec M'$.

Proposition 1.2: (1) *If Γ is a local bigness notion for $\mathcal{K}$ with parameters in A, $M \in \mathcal{K}_{A,M'}$ and $\mathfrak{p}(x)$ is a Γ-big type in M, then it can be extended to Γ-big type $\mathfrak{q}$ in M which is a complete type over M.*

(2) *Assume that $t,\psi,\bar\varphi,M,A$ are as in Definition 1.1(8). The truth value of "$\vartheta(y,\bar a)$ is $\Gamma_{(t,\psi,\bar\varphi)}$-big" depends just on $(M \restriction \tau', \bar a, c)_{c\in A}$ whenever the formulas in $\bar\varphi$ and ϑ belong to $\mathbb{L}(\tau')$.*

Proposition 1.3: *For $T, \mathcal{K} = \mathcal{K}_T$ and t as in 1.1,*

($\boxtimes$) *if $N \prec M$ are from $\mathcal{K}$, and $\bar\varphi = \langle \varphi_R(\bar y_R, \bar a_R) : R \in \tau(t)\rangle$ is an interpretation of t in N, then $\bar\varphi$ is an interpretation of t in M (i.e., $M[\bar\varphi] \models t$) and moreover $N[\bar\varphi] \prec M[\bar\varphi]$.*

The following definition is crucial in our application, the proofs give some amount of definability, "a local version" and we need to deduce from it a global one. This is a good property, criterion for closing the gap which have in fact been used for $t^{\rm ind}$, see more systematically in [Sh800].

Definition 1.4: Let t be a first order theory in a vocabulary $\tau(t)$. Suppose that ψ is a t-bigness notion scheme, using $P \in \tau(t)$, a unary predicate, and $\vartheta(y,x)$ is a $\tau(t)$-formula. We say that ψ is **$(\aleph_2,\aleph_1)$-(P,ϑ)-separative** whenever the

[2] The "invariant" really follows.

following condition $(\circledast)^{P,\vartheta}_{\Gamma}$ holds and for simplicity we assume $\varphi_=(x,y,\bar a_=)$ is equality on its domain[3].

$(\circledast)^{P,\vartheta}_{\Gamma}$ For every $\aleph_2$– compact[4] τ-model M and every interpretation $\bar\varphi = \langle \varphi_R(\bar y_R, \bar a_R) : R \in \tau(t)\rangle$ of t in M and a set $X \subseteq |M|$ of cardinality at most $\aleph_1$, including all parameters of $\bar\varphi$ we have:

if $N \prec M$, $X \subseteq |N|$, $\|N\| \le \aleph_1$, and $\mathfrak{p}(x)$ is a $\Gamma_\psi[\bar\varphi]$-big type over N, $|\mathfrak{p}(x)| \le \aleph_1$, and a_1, a_2 are distinct members of $|M| \setminus |N|$ with (recalling 1.1(5))

$$M \models \varphi_P[a_1, \bar a_P] \ \wedge \varphi_P[a_2, \bar a_P],$$

then the type $\mathfrak{p}(x) \cup \{\vartheta(a_1,x) \leftrightarrow \neg\vartheta(a_2,x)\}$ is $\Gamma_\psi[\bar\varphi]$–big.

We now define the main bigness notion used

Definition 1.5 (See [Sh:e, Definitions 3.4, 3.5, Chapter XI]): (1) $t^{\rm ind} = t^{\rm ind}_0$ is the first order theory in vocabulary $\tau(t^{\rm ind}) = \{P, Q, R\}$, where P, Q are unary predicates and R is a binary predicate, including sentences

$$(\forall x)(\forall y)(x\ R\ y \ \rightarrow\ P(x)\ \wedge\ Q(y)), \quad \text{and} \quad (\forall x)(P(x) \vee Q(x))$$

and saying that for each $n < \omega$ and any pairwise distinct elements $a_1, \ldots, a_{2n} \in P$, there is $c \in Q$ such that

$$a_i\ R\ c \quad \text{if and only if} \quad i \le n.$$

$t^{\rm ind}_1$ is $t^{\rm ind}_0$ plus

$$(\forall x)(\forall y)(\exists z)\big(Q(x)\ \wedge\ Q(y)\ \wedge\ x \ne y\ \rightarrow\ P(z) \wedge (z\ R\ x \equiv \neg z\ R\ y)\big).$$

(2) We define a pre $t^{\rm ind}$-bigness notion scheme $\Gamma^{\rm ind}$ as follows. The sentence $\psi^{\rm ind}$ says that $P^* \subseteq Q$ and (P, Q, R, P^*) satisfies:

for every $n < \omega$, there is a finite set $A \subseteq P$ such that for every distinct $a_1, \ldots, a_{2n} \in P \setminus A$ there is $c \in P^*$ satisfying

$$a_\ell\ R\ c \quad \text{for } \ell \le n, \quad \text{and} \quad \neg(a_\ell\ R\ c) \quad \text{for } n < \ell \le 2n.$$

(So $\psi^{\rm ind}$ is not first order.)

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[3] Otherwise we should inside $(\circledast)^{P,\vartheta}_{\Gamma}$, demand further that for any $c \in N$ we have $M \models \neg\varphi_=(c, a_1, \bar a_=) \wedge \neg\varphi_=(c, a_2, \bar a_=) \wedge \neg\varphi_=(a_1, a_2, \bar a_=)$.

[4] A model M is called κ-compact if every type over it of cardinality $< \kappa$ is realized; if we omit κ we mean the cardinality of the model

(3) We say that a first order theory T has the strong independence property if some[5] formula $\vartheta(x,y)$ defines a two place relation which is a model of $t_1^{\rm ind}$ with P, Q chosen as the whole model i.e., for $M \models T$ define the $\tau_{t_1^{\rm ind}}$-model $M', |M'| = |M| = P^{M'} = Q^{M'}, R^{M'} = \{(a,b) : M \models \vartheta(a,b)\}$

In such case we may also say "$\vartheta(x,y)$ has the strong independence properties (for T)"

Plainly,

PROPOSITION 1.6: (1) *For a model M of $t_1^{\rm ind}$, an automorphism π of M is determined by $\pi \restriction P^M$ (i.e., if $\pi_1, \pi_2 \in {\rm Aut}(M)$ are such that $\pi_1 \restriction P^M = \pi_2 \restriction P^M$, then $\pi_1 = \pi_2$).*

(2) *Moreover, if $\bar\varphi$ is an interpretation of $t_1^{\rm ind}$ in M^*, $M = M^*[\bar\varphi]$, $\pi \in {\rm Aut}(M)$ and $\pi \restriction P^M$ is definable in M^* (with parameters in M^*), then so is π.*

PROPOSITION 1.7 (See [Sh:e, Chapter XI, §3] and [Sh107]): *$\psi^{\rm ind}$ is a $t^{\rm ind}$-bigness notion scheme. It is $(\aleph_2, \aleph_1)$-(P, ϑ)–separative where $P \in \tau(t_0^{\rm ind})$ is given and we choose $\vartheta(y,x) := y\ R\ x$.*

Definition 1.8: A mapping $F : N^1 \longrightarrow N^2$ is **a Δ–embedding from N^1 to** N^2 whenever Δ is a set of formulas in $\mathbb{L}_{\omega,\omega}(\tau(N^1) \cap \tau(N^2))$ and

if $\varphi \in \Delta$ and $N^1 \models \varphi[a_1, \ldots, a_n]$,
then $N^2 \models \varphi[F(a_1), \ldots, F(a_n)]$.

(of course, if Δ is closed under negation, then we have "if and only if".)

2. The forcing notion $\mathbb{A}$pp

As explained in the introduction, we work in a Cohen generic extension of a suitable ground model. In this section we present how that suitable ground model can be obtained: we start with $\mathbf{V} \models$ GCH and we force with the forcing notion $\mathbb{A}$pp from Definition 2.4 below, the $\mathbb{A}$pp comes for approximations, as the members are approximations to a name for an ultrafilter as we desire.

Definition 2.1: (1) The Cohen forcing of adding $\aleph_3$ Cohen reals is denoted by $\mathbb{C}_{\aleph_3}$. Thus a condition p in $\mathbb{C}_{\aleph_3}$ is a finite partial function from $\aleph_3 \times \omega$ to ω, and the order of $\mathbb{C}_{\aleph_3}$ is the natural one. The canonical $\mathbb{C}_{\aleph_3}$-name for β-th Cohen real will be called $\underset{\sim}{x}_\beta$.

[5] Of course $\vartheta(\bar x, \bar y), {\rm lg}(\bar x) = m = {\rm lg}(\bar y)$ can serve as well.

(2) Let $\mathbf{A} \subseteq \aleph_3$. For a condition $p \in \mathbb{C}_{\aleph_3}$, its restriction to $\mathbf{A} \times \omega$ is called $p \restriction \mathbf{A}$, and we let $\mathbb{C}_{\aleph_3} \restriction \mathbf{A} = \mathbb{C}_{\mathbf{A}} = \{p \restriction \mathbf{A} : p \in \mathbb{C}_{\aleph_3}\}$. Also, we let $\underset{\sim}{\omega}^*_{\mathbf{A}} = ({}^{\omega}\omega)^{\mathbf{V}^{\mathbb{C}_{\aleph_3} \restriction \mathbf{A}}}$.

(3) For a sequence $\langle A_n : n < \omega \rangle$ of non-empty sets (and $\mathbf{A} \subseteq \aleph_3$), we define

$$\prod_{n<\omega}^{\mathbf{A}} A_n = \left\{ \begin{array}{l} f \in \mathbf{V}^{\mathbb{C}_{\aleph_3} \restriction \mathbf{A}} : f \text{ is a function with domain } \omega, \\ \qquad\qquad \text{and such that } f(n) \in A_n \text{ for every } n, \end{array} \right\},$$

and similarly for models.

(4) For $\mathbf{A} \subseteq \aleph_3$ and $m < \omega$, let $I^m_{\mathbf{A}}$ be the set of all ω-sequences of canonical $\mathbb{C}_{\mathbf{A}}$-names for subsets of ${}^m\omega$. Let $Q_{\bar{s}}$ (for $\bar{s} \in I^m_{\mathbf{A}}$, $m < \omega$) be an m-ary predicate, $Q_{\bar{s}_0} \neq Q_{\bar{s}_1}$ whenever $\bar{s}_0 \neq \bar{s}_1$ i.e., even when they are forced to be equal they may be different as sequences of names, and let

$$\tau_{\mathbf{A}} = \{Q_{\bar{s}} : \bar{s} \in I^m_{\mathbf{A}} \ \&\ m < \omega\}$$

(so because of the demand "canonical", $|\tau_{\mathbf{A}}| = \aleph_1 \cdot |\mathbf{A}|$). Let $\underset{\sim}{M}^n_{\mathbf{A}}$ be a $\mathbb{C}_{\mathbf{A}}$-name for the $\tau_{\mathbf{A}}$-model with universe ω such that if $\bar{s} = \langle \underset{\sim}{s}_n : n < \omega \rangle \in I^m_{\mathbf{A}}$, then $\Vdash_{\mathbb{C}_{\mathbf{A}}} (Q_{\bar{s}})^{\underset{\sim}{M}^n_{\mathbf{A}}} = \underset{\sim}{s}_n$. So the vocabulary $\tau_{\mathbf{A}}$ is an object in $\mathbf{V}$, not a name.

(5) If $\mathbf{A}_1 \subseteq \mathbf{A}_2$, and for $\ell = 1, 2$ $\underset{\sim}{\mathcal{F}}_\ell$ is a $\mathbb{C}_{\mathbf{A}_\ell}$- name of an ultrafilter on ω then $\underset{\sim}{\mathcal{F}}_1 = \underset{\sim}{\mathcal{F}}_2 \restriction \mathbf{A}_1$ means that $\Vdash_{\mathbb{C} \restriction \mathbf{A}_2} \underset{\sim}{\mathcal{F}}_1 \subseteq \underset{\sim}{\mathcal{F}}_2$, so $\underset{\sim}{\mathcal{F}}_2 \restriction \mathbf{A}_1$ is unique but not always well defined.[6]

In the definition below the reader can restrict himself to the case $t = t^{\text{ind}}$, $\underset{\sim}{\psi} = \psi^{\text{ind}}$, see Definition 1.5 (so later in Definition 2.4 we use only $\Gamma = \Gamma^{\text{ind}}$)

Definition 2.2: (1) A function $\mathbf{G}$ is called **an** $(\aleph_3, \aleph_2)$**-bigness guide** if the domain $\text{Dom}(\mathbf{G})$ of $\mathbf{G}$ is

$$\left\{ \begin{array}{l} (\mathbf{A}, \underset{\sim}{\mathcal{F}}) : \mathbf{A} \subseteq \aleph_3, \ |\mathbf{A}| \leq \aleph_1, \text{ and} \\ \qquad\qquad \underset{\sim}{\mathcal{F}} \text{ is a } \mathbb{C}_{\mathbf{A}}\text{–name of a non principal ultrafilter on } \omega \end{array} \right\},$$

and

(α) $\mathbf{G}(\mathbf{A}, \underset{\sim}{\mathcal{F}})$ is a non-empty set of triples $(t, \underset{\sim}{\psi}, \underset{\sim}{\bar{\varphi}})$, where[7] t is a (countable) first order theory (or just a $\mathbb{C}_{\mathbf{A}}$-name of a (countable) first order theory),

[6] As for $\mathbb{C}_{\mathbf{A}_1}$-name $\underset{\sim}{A}$ of a subset of ω, the truth value of "$\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}_2$" is an $\mathbb{C}_{\mathbf{A}_2}$-name but in general not a $\mathbb{C}_{\mathbf{A}_1}$-name.

[7] Note that our forcing $\mathbb{A}\text{pp}$ will add no reals so since we are considering only countable t, we can use only old ones. As we may consider names in the Cohen forcing, things are different so we allow such names.

$\underset{\sim}{\psi}$ is a $\mathbb{C}_{\mathbf{A}}$-name of t-bigness notion scheme, and $\underset{\sim}{\bar{\varphi}}$ is (a $\mathbb{C}_{\mathbf{A}}$-name for) an interpretation of t in $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}}/\underset{\sim}{\mathcal{F}}$, and $|\mathbf{G}(\mathbf{A}, \underset{\sim}{\mathcal{F}})| \leq \aleph_2$, and

(β) if $(\mathbf{A}^\ell, \underset{\sim}{\mathcal{F}}^\ell) \in \mathrm{Dom}(\mathbf{G})$ for $\ell = 1, 2$, $\mathbf{A}^1 \subseteq \mathbf{A}^2$ and $\Vdash_{\mathbb{C}_{\mathbf{A}_2}} \underset{\sim}{\mathcal{F}}^1 \subseteq \underset{\sim}{\mathcal{F}}^2$, then $\mathbf{G}(\mathbf{A}^1, \underset{\sim}{\mathcal{F}}^1) \subseteq \mathbf{G}(\mathbf{A}^2, \underset{\sim}{\mathcal{F}}^2)$.

(2) An $(\aleph_3, \aleph_2)$-bigness guide $\mathbf{G}$ is **ind-full** if

(γ) for every $(\mathbf{A}, \underset{\sim}{\mathcal{F}}) \in \mathrm{Dom}(\mathbf{G})$ and a canonical $\mathbb{C}_{\mathbf{A}}$-name $\underset{\sim}{\bar{\varphi}}$ for an interpretation of t^{ind} in $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}}/\underset{\sim}{\mathcal{F}}$ we have $(t^{\mathrm{ind}}, \psi^{\mathrm{ind}}, \underset{\sim}{\bar{\varphi}}) \in \mathbf{G}(\mathbf{A}, \underset{\sim}{\mathcal{F}})$.

(3) We say that $\mathbf{G}$ is **full** whenever the following condition holds.

($\boxplus$) Assume $(\mathbf{A}, \underset{\sim}{\mathcal{F}}) \in \mathrm{Dom}(\mathbf{G})$ and $\underset{\sim}{t}$ is a canonical $\mathbb{C}_{\mathbf{A}}$-name of a (countable) first order theory in the vocabulary $\tau(\underset{\sim}{t}) \in \mathcal{H}(\aleph_1)$, $\underset{\sim}{\psi}$ is a canonical $\mathbb{C}_{\mathbf{A}}$-name for a pre $\underset{\sim}{t}$-bigness notion scheme, $\underset{\sim}{\psi} \in \mathbb{L}_{\aleph_1,\aleph_1}(\tau(\underset{\sim}{t}) \cup \{P^*\})$. Let $\underset{\sim}{\bar{\varphi}}$ be a $\mathbb{C}_{\mathbf{A}}$-name for an interpretation of $\underset{\sim}{t}$ in $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}}/\underset{\sim}{\mathcal{F}}$; no need for parameters as all elements are interpretation of an individual constant. Suppose $(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar{\varphi}})$ is forced to define a bigness notion[8] $\Gamma = \Gamma_{(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar{\varphi}})}$. Then $(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar{\varphi}}) \in \mathbf{G}(\mathbf{A}, \underset{\sim}{\mathcal{F}})$.

The clause 2.2(2) is added for our particular application. It can be replaced by the use of a family of bigness notions relevant to your interest.

PROPOSITION 2.3: (1) *There is a full* $(\aleph_3, \aleph_2)$*-bigness guide* $\mathbf{G}$.
(2) *If a bigness guide* $\mathbf{G}$ *is full, then it is ind-full.*
(3) *Full and even just ind-full implies non-emptiness, i.e.,* $\mathbf{G}(\mathbf{A}, \underset{\sim}{\mathcal{F}}) \neq \emptyset$ *when defined.*

Proof. Trivial. ∎

Definition 2.4: Let $\mathbf{G}$ be an $(\aleph_3, \aleph_2)$-bigness guide. We define the forcing notion $\mathbb{A}\mathrm{pp} = \mathbb{A}\mathrm{pp}_{\mathbf{G}}$. (When $\mathbf{G}$ is fixed, as typically in the present paper, we may and usually will not mention it.)

(1) **A condition** q **in** $\mathbb{A}\mathrm{pp}$ is a triple $q = (\mathbf{A}, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\bar{\Gamma}}) = (\mathbf{A}^q, \underset{\sim}{\mathcal{F}}^q, \underset{\sim}{\bar{\Gamma}}^q)$ such that:

(a) $\mathbf{A}$ is a subset of $\aleph_3$ of cardinality $\leq \aleph_1$;

(b) $\underset{\sim}{\mathcal{F}}$ is a canonical $\mathbb{C}_{\mathbf{A}}$-name of a nonprincipal ultrafilter on ω, such that for $\beta < \aleph_3$,

$$\underset{\sim}{\mathcal{F}} \restriction (\mathbf{A} \cap \beta) \stackrel{\mathrm{def}}{=} \underset{\sim}{\mathcal{F}} \cap \{\underset{\sim}{a} : \underset{\sim}{a} \text{ is a } \mathbb{C}_{\mathbf{A}\cap\beta}\text{-name of a subset of } \omega\}$$

is a $\mathbb{C}_{\mathbf{A}\cap\beta}$-name (of an ultrafilter on ω);

[8] We can fix a $\mathbb{C}_{\aleph_3}$-name of countable first order theory.

Why "canonical"? for the same reasons as in 2.1(4)

(c) $\bar{\underset{\sim}{\Gamma}} = \langle \underset{\sim}{\Gamma}_\beta : \beta \in \mathbf{A} \ \&\ \mathrm{cf}(\beta) = \aleph_2 \rangle$, where each $\underset{\sim}{\Gamma}_\beta$ is a local bigness notion $\Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar{\varphi}}]$ for some $(t, \underset{\sim}{\psi}, \underset{\sim}{\bar{\varphi}}) \in \mathbf{G}(\mathbf{A} \cap \beta, \underset{\sim}{\mathcal{F}} \restriction (\mathbf{A} \cap \beta))$;

(d) If $\mathrm{cf}(\beta) = \aleph_2$, $\beta \in \mathbf{A}$, then it is forced (i.e., $\Vdash_{\mathbb{C}_{\aleph_3}}$ equivalently $\Vdash_{\mathbb{C}_{\mathbf{A}}}$) that:

the type realized by the element $\underset{\sim}{x}_\beta$ in the model $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}\cap\beta}/\underset{\sim}{\mathcal{F}}$ over the model $\prod^{\mathbf{A}\cap\beta}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}\cap\beta}/\big(\underset{\sim}{\mathcal{F}} \restriction (\mathbf{A} \cap \beta)\big)$ (so it is a type in the vocabulary $\tau_{\mathbf{A}\cap\beta}$) is $\underset{\sim}{\Gamma}_\beta$-big and complete of course, and moreover this type is a $\mathbb{C}_{\mathbf{A}\cap\beta}$-name; actually we should say "by the element $\underset{\sim}{x}_\beta/(\underset{\sim}{\mathcal{F}} \restriction \mathbf{A})$". We call it "the type induced by $\underset{\sim}{x}_\beta$ according to q".

(2) **The order** $\leq_{\mathbb{A}\mathrm{pp}} = \leq$ **of** $\mathbb{A}\mathrm{pp} = \mathbb{A}\mathrm{pp}_{\mathbf{G}}$ is the natural one: $q_1 \leq q_2$ if and only if $\mathbf{A}^{q_1} \subseteq \mathbf{A}^{q_2}$, $\Vdash_{\mathbb{C}_{\mathbf{A}^{q_2}}} \underset{\sim}{\mathcal{F}}^{q_1} \subseteq \underset{\sim}{\mathcal{F}}^{q_2}$, and $\bar{\underset{\sim}{\Gamma}}^{q_2} \restriction \mathbf{A}^{q_1} = \bar{\underset{\sim}{\Gamma}}^{q_1}$.

(3) We say that $q_2 \in \mathbb{A}\mathrm{pp}$ is **an end extension** of $q_1 \in \mathbb{A}\mathrm{pp}$, and we write $q_1 \leq_{\mathrm{end}} q_2$, if $q_1 \leq q_2$, $q_1 = q_2 \restriction \beta$ and $\sup(\mathbf{A}^{q_1}) \leq \beta = \min(\mathbf{A}^{q_2} \setminus \mathbf{A}^{q_1})$.

(4) For a condition $q \in \mathbb{A}\mathrm{pp}$ and an ordinal $\beta \in \aleph_3$ we define $q \restriction \beta = \big(\mathbf{A}^q \cap \beta, \underset{\sim}{\mathcal{F}}^q \restriction (\mathbf{A}^q \cap \beta), \bar{\underset{\sim}{\Gamma}}^q \restriction (\mathbf{A}^q \cap \beta)\big)$.

(5) For $\beta < \aleph_3$ we let $\mathbb{A}\mathrm{pp} \restriction \beta = \mathbb{A}\mathrm{pp}_\beta = \{q \in \mathbb{A}\mathrm{pp} : \mathbf{A}^q \subseteq \beta\}$ with inherited order. If $G \subseteq \mathbb{A}\mathrm{pp}$ is generic over $\mathbf{V}$, then we let $G \restriction \beta = G \cap (\mathbb{A}\mathrm{pp} \restriction \beta)$.

One easily checks that

PROPOSITION 2.5: (1) *If $q \in \mathbb{A}\mathrm{pp}$, $\beta < \aleph_3$, then $q \restriction \beta \in \mathbb{A}\mathrm{pp}$ and $q \restriction \beta \leq_{\mathrm{end}} q$.* (2) *Both $\leq_{\mathbb{A}\mathrm{pp}}$ and $\leq_{\mathrm{end}}$ are partial orders, (pedantically quasi orders) on $\mathbb{A}\mathrm{pp}$.*

LEMMA 2.6: *If $\langle q_\zeta : \zeta < \xi \rangle$ is an increasing sequence of members of $\mathbb{A}\mathrm{pp}$, $\xi \leq \aleph_1$, and $q_{\zeta_1} \leq_{\mathrm{end}} q_{\zeta_2}$ for $\zeta_1 < \zeta_2$, then there is $q \in \mathbb{A}\mathrm{pp}$ such that $\mathbf{A}^q = \bigcup_{\zeta<\xi} \mathbf{A}^{q_\zeta}$ and $q_\zeta \leq_{\mathrm{end}} q$ for all $\zeta < \xi$.*

Proof. We can assume that $\xi > 0$ is a limit ordinal. If $\mathrm{cf}(\xi) > \aleph_0$, then we let $\mathbf{A}^q = \bigcup_{\zeta<\xi} \mathbf{A}^{q_\zeta}$, $\underset{\sim}{\mathcal{F}}^q = \bigcup_{\zeta<\xi} \underset{\sim}{\mathcal{F}}^{q_\zeta}$ and $\bar{\underset{\sim}{\Gamma}}^q = \bigcup_{\zeta<\xi} \bar{\underset{\sim}{\Gamma}}^{q_\zeta}$. If $\mathrm{cf}(\xi) = \aleph_0$, then additionally we have to extend $\bigcup_{\zeta<\xi} \underset{\sim}{\mathcal{F}}^{q_\zeta}$ to a $\mathbb{C}_{\mathbf{A}^q}$-name of an ultrafilter on ω, which is no problem. ∎

LEMMA 2.7: *Suppose that $q \in \mathbb{A}\mathrm{pp}$, $\mathbf{A}^q \subseteq \gamma \in \aleph_3$, and $\underset{\sim}{\mathfrak{p}}$ is a $\mathbb{C}_{\mathbf{A}^q}$-name of a type over the model $\prod^{\mathbf{A}^q}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^q$ (so the type $\underset{\sim}{\mathfrak{p}} = \underset{\sim}{\mathfrak{p}}(x)$ is in the vocabulary $\tau_{\mathbf{A}^q}$, finitely satisfiable in $\prod^{\mathbf{A}^q}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^q$). Then:*

(1) *If* $\mathrm{cf}(\gamma) < \aleph_2$, *then there is a condition* $r \in \mathbb{A}\mathrm{pp}$ *stronger than* q *such that* $\mathbf{A}^r = \mathbf{A}^q \cup \{\gamma\}$, *and*

$$\Vdash_{\mathbb{C}_{\mathbf{A}^r}} \text{“}\underset{\sim}{x}_\gamma/\underset{\sim}{\mathcal{F}}^r \text{ realizes } \underset{\sim}{\mathfrak{p}} \text{ in } \prod_{n<\omega}^{\mathbf{A}^r} \underset{\sim}{M}^n_{\mathbf{A}^r}/\underset{\sim}{\mathcal{F}}^r\text{”}.$$

(2) *If* $\mathrm{cf}(\gamma) = \aleph_2$, $(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar\varphi}) \in \mathbf{G}(\mathbf{A}^q, \underset{\sim}{\mathcal{F}}^q)$ *and the type* $\underset{\sim}{\mathfrak{p}}$ *is (forced to be)* $\Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar\varphi}]$-*big, then there is a condition* $r \in \mathbb{A}\mathrm{pp}$ *as in (1) and such that* $\underset{\sim}{\Gamma}^r_\gamma = \Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar\varphi}]$.

Proof. 1) Extend $\underset{\sim}{\mathcal{F}}^q$ to $\underset{\sim}{\mathcal{F}}^r$ so that $\underset{\sim}{x}_\gamma/\underset{\sim}{\mathcal{F}}^r$ realizes the required type, (using "$\underset{\sim}{x}_\gamma$ is Cohen over $\mathbf{V}^{\mathbb{C}\restriction\mathbf{A}}$").

2) Note that every $\Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar\varphi}]$-big type can be extended to a complete $\Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar\varphi}]$- big one by 1.2 and the proof of 2.8(1) below. ∎

Lemma 2.8: (1) *Suppose* $q_0, q_1, q_2 \in \mathbb{A}\mathrm{pp}$, $q_0 = q_2 \restriction \beta$, $q_0 \le q_1$, $\mathbf{A}^{q_1} \subseteq \beta$. *Suppose further that* $\mathbf{A}^{q_2} \setminus \mathbf{A}^{q_0} = \{\beta\}$ *and* $\mathrm{cf}(\beta) = \aleph_2$. *Assume further that* $\underset{\sim}{\mathfrak{p}}_1$ *is a* $\mathbb{C}_{\mathbf{A}^{q_1}}$-*name for a complete* $\underset{\sim}{\Gamma}^{q_2}_\beta$-*big type over* $(\prod^{\mathbf{A}^{q_1}}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{q_1}}/\underset{\sim}{\mathcal{F}}^{q_1})$ *such that* $\underset{\sim}{\mathfrak{p}}_1$ *contains the type* $\underset{\sim}{\mathfrak{p}}_0$ *induced by* $\underset{\sim}{x}_\beta$ *according to* q_2 *(such* $\mathfrak{p}_1$ *necessarily exists, by the properties of bigness). Then there is* $q_3 \ge q_1, q_2$ *with* $\mathbf{A}^{q_3} = \mathbf{A}^{q_1} \cup \{\beta\}$, *such that* $\underset{\sim}{x}_\beta$ *induces* $\underset{\sim}{\mathfrak{p}}_1$ *on* $(\prod^{\mathbf{A}^{q_1}}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{q_1}}/\underset{\sim}{\mathcal{F}}^{q_1})$ *(according to* q_3*).*

(2) *Assume* $q_0, q_1, q_2 \in \mathbb{A}\mathrm{pp}$, $q_0 = q_2 \restriction \beta$, $q_0 \le q_1$ *and* $\mathbf{A}^{q_1} \subseteq \beta$. *If* $\mathbf{A}^{q_2} \setminus \mathbf{A}^{q_0} = \{\beta\}$ *and* $\mathrm{cf}(\beta) < \aleph_2$, *then there is* $q_3 \in \mathbb{A}\mathrm{pp}$, $q_3 \ge q_1, q_2$ *such that* $\mathbf{A}^{q_3} = \mathbf{A}^{q_1} \cup \mathbf{A}^{q_2}$. *This clause is like the first one except the cofinality.*

(3) *Assume that* δ_1, $\delta_2 < \aleph_2$, *and* $\langle \beta_j : j < \delta_2 \rangle$ *is a non-decreasing sequence of ordinals below* $\aleph_3$. *Let* $\langle p_i : i < \delta_1 \rangle$ *be an* $\le$-*increasing sequence from* $\mathbb{A}\mathrm{pp}$. *Suppose that* $q_j \in \mathbb{A}\mathrm{pp} \restriction \beta_j$ *(for* $j < \delta_2$*) are such that:*

$$p_i \restriction \beta_j \le q_j \quad \text{for } i < \delta_1,\ j < \delta_2,$$

$$q_j \le_{\mathrm{end}} q_{j'} \quad \text{for } j < j' < \delta_2.$$

Then there is an $r \in \mathbb{A}\mathrm{pp}$ *with* $p_i \le r$ *and* $q_j \le_{\mathrm{end}} r$ *for all* $i < \delta_1$ *and* $j < \delta_2$.

(4) *If* $\bar p = \langle p_i : i < \delta_1 \rangle$ *an increasing sequence in* $\mathbb{A}\mathrm{pp}$, $\delta_1 < \aleph_2$, *then* $\bar p$ *has an upper bound in* $\mathbb{A}\mathrm{pp}$. *If* $\mathrm{cf}(\delta_1) = \aleph_1$ *we use the (naturally defined) union.*

(5) *Assume*

(a) γ *is a limit ordinal of cofinality* $\aleph_0$

(b) $p \in \mathbb{A}\mathrm{pp}_\gamma$ *and* $\underset{\sim}{\mathfrak{p}}$ *is a* $\mathbb{C}_{\mathbf{A}^p}$-*name of a finitely satisfiable set of formulas in one free variable* x *over* $\prod^{\mathbf{A}^p}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^p}/\underset{\sim}{\mathcal{F}}^p$

(c) $\gamma_n \in \gamma \setminus \mathbf{A}^p, \gamma_n < \gamma_{n+1}$ *and* $\gamma = \bigcup\{\gamma_n : n < \omega\}$

Then there is q *such that*

(α) $p \le q \in \mathbb{A}\mathrm{pp}_\gamma$
(β) $\mathbf{A}^q = \mathbf{A}^p \cup \{\gamma_n : n < \omega\}$
(γ) $\Vdash_{\mathbb{C}_{\mathbf{A}^q}}$ "$\underset{\sim}{\mathfrak{p}}$ *is realized in* $\prod^{\mathbf{A}^q}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^q$"

Proof. 1) Note that this is a strong form of the $\aleph_2$-c.c., see the proof of 2.9 below. Let $\mathbf{A}_i = \mathbf{A}^{q_i}$ and let $\underset{\sim}{\mathcal{F}}_i = \underset{\sim}{\mathcal{F}}^{q_i}$ for $i < 3$, and $\mathbf{A}_3 = \mathbf{A}_1 \cup \mathbf{A}_2 = \mathbf{A}_1 \cup \{\beta\}$. Possibly the only not clear part is to show that, in $\mathbf{V}^{\mathbb{C}_{\mathbf{A}_3}}$, there is an ultrafilter extending $\underset{\sim}{\mathcal{F}}_1 \cup \underset{\sim}{\mathcal{F}}_2$ which contains $\underset{\sim}{\mathcal{F}}'$, the family of all the sets

$$\{n < \omega : \underset{\sim}{M}^n_{\mathbf{A}_3} \models \underset{\sim}{\varphi}[\underset{\sim}{x}_\beta(n), \underset{\sim}{\bar{a}}(n)]\}$$

for $\underset{\sim}{\varphi}(x, \bar{y}) \in \underset{\sim}{\mathfrak{p}}_1$, $\ell g(\bar{y}) = m$, and a $\mathbb{C}_{\mathbf{A}_1}$-name $\underset{\sim}{\bar{a}}$ of an m-tuple from $\underset{\sim}{\omega}^*_{\mathbf{A}_1}$ (and in our notation above $\underset{\sim}{\bar{a}}(n)$ is a $\mathbb{C}_{\mathbf{A}_1}$-name for an m-tuple of elements of ω, so pedantically we define $\underset{\sim}{\bar{a}} = \langle \underset{\sim}{a}_\ell : \ell < m \rangle$, $\underset{\sim}{a}_\ell = \langle \underset{\sim}{a}_\ell(n) : n < \omega \rangle$ where $\underset{\sim}{a}_\ell(n)$ is a $(\mathbb{C} \restriction \mathbf{A})$-name of a natural number and $\underset{\sim}{\bar{a}}(n) = \langle \underset{\sim}{a}_\ell(n) : \ell < m \rangle$ and we should use below $\langle \underset{\sim}{a}_\ell/\underset{\sim}{\mathcal{F}} : \ell < m \rangle$ instead $\underset{\sim}{\bar{a}}$). As $\underset{\sim}{\mathcal{F}}_1, \underset{\sim}{\mathcal{F}}_2, \underset{\sim}{\mathcal{F}}'$ are (forced, i.e., $\Vdash_{\mathbb{C}_{\mathbf{A}_3}}$) to be closed under intersections (of two, and hence of finitely many), clearly if this fails, then (as $\underset{\sim}{\mathcal{F}}_0$ is forced to be a non-principal ultrafilter on ω so $m < \omega$ implies $\Vdash [m, \omega) \in \underset{\sim}{\mathcal{F}}_0$) there are a condition $p \in \mathbb{C}_{\mathbf{A}_3}$, a $\mathbb{C}_{\mathbf{A}_1}$-name $\underset{\sim}{\mathbf{a}}$ of a member of $\underset{\sim}{\mathcal{F}}_1$, a $\mathbb{C}_{\mathbf{A}_2}$-name $\underset{\sim}{\mathbf{b}}$ of a member of $\underset{\sim}{\mathcal{F}}_2$, a ($\mathbb{C}_{\mathbf{A}_1}$-name for a) $\tau_{\mathbf{A}_1}$-formula $\underset{\sim}{\varphi}$ and a $\mathbb{C}_{\mathbf{A}_1}$-name for an m-tuple $\underset{\sim}{\bar{a}}$ from $\underset{\sim}{\omega}^*_{\mathbf{A}_1}$ such that

$$p \restriction \mathbf{A}_1 \Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“}\underset{\sim}{\varphi}(x, \underset{\sim}{\bar{a}}) \in \underset{\sim}{\mathfrak{p}}_1\text{”} \quad \text{and} \quad p \Vdash_{\mathbb{C}_{\mathbf{A}_3}} \text{“}\underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} \cap \underset{\sim}{\mathbf{c}} = \emptyset\text{”},$$

where

$$\underset{\sim}{\mathbf{c}} = \{n : \underset{\sim}{M}^n_{\mathbf{A}_3} \models \underset{\sim}{\varphi}[\underset{\sim}{x}_\beta(n), \underset{\sim}{\bar{a}}(n)]\}.$$

We may easily eliminate parameters, so we may assume that we have $\underset{\sim}{\varphi}[\underset{\sim}{x}_\beta(n)]$ only (remember the definition of $\tau_{\mathbf{A}_1}$). Let $p_i = p \restriction \mathbf{A}_i$ for $i = 0, 1, 2$, and let $H^0 \subseteq \mathbb{C}_{\mathbf{A}_0}$ be generic over $\mathbf{V}$ such that $p_0 \in H^0$. For $n < \omega$ let $\underset{\sim}{A}^*_n$ be a $\mathbb{C}_{\mathbf{A}_0}$-name such that

$$\underset{\sim}{A}^*_n[H^0] = \left\{ \begin{array}{c} y \in \underset{\sim}{M}^n_{\mathbf{A}_2} \text{ :there is } p'_2 \in \mathbb{C}_{\mathbf{A}_2} \text{ such that } p_2 \le p'_2,\ p'_2 \restriction \mathbf{A}_0 \in H^0 \\ \text{and } p'_2 \Vdash \text{“ } \underset{\sim}{x}_\beta(n) = y \text{ and } n \in \underset{\sim}{\mathbf{b}}\text{”} \end{array} \right\}$$

(recall $y \in \underset{\sim}{M}^n_{\mathbf{A}_2}$ means $y \in \omega$). Let $\underset{\sim}{A}^* = \prod^{\mathbf{A}_0}{}_{n<\omega} \underset{\sim}{A}^*_n/\underset{\sim}{\mathcal{F}}_0$. So $\underset{\sim}{A}^*[H^0]$ is (the interpretation of) an unary predicate from $\tau_{\mathbf{A}_0}$; in fact $Q_{\langle \underset{\sim}{A}^*_n : n<\omega \rangle}$ is such a predicate, but we shall write $\underset{\sim}{A}^*(x)$ instead $Q_{\langle \underset{\sim}{A}_n : n<\omega \rangle}(x)$. Thus, in $\mathbf{V}[H^0]$, either $\underset{\sim}{A}^*(x) \in \underset{\sim}{\mathfrak{p}}_0$ or $\neg \underset{\sim}{A}^*(x) \in \underset{\sim}{\mathfrak{p}}_0$. The latter is impossible by the choice of

$\underset{\sim}{A}^*$, so necessarily $\underset{\sim}{A}^*(x) \in \underset{\sim}{\mathfrak{p}}_0$. As also $p \restriction A_1 \Vdash_{\mathbb{C}_{\mathbf{A}_1}}$ "$\underset{\sim}{\varphi}(y) \in \underset{\sim}{\mathfrak{p}}_1$", clearly if $H^1 \subseteq \mathbb{C}_{\mathbf{A}_1}$ is generic over $\mathbf{V}$ and $H^0 \cup \{p_1\} \subseteq H^1$, then in $\mathbf{V}[H^1]$ we have

$$\{n \in \omega : \underset{\sim}{M}^n_{\mathbf{A}_1} \models (\exists y)\big(\underset{\sim}{A}^*(y) \;\&\; \underset{\sim}{\varphi}(y)\big)\} \in \underset{\sim}{\mathcal{F}}_1[H^1]$$

(remember $\underset{\sim}{\mathfrak{p}}_1$ is a type over $\prod^{\mathbf{A}^{q_1}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{q_1}}/\underset{\sim}{\mathcal{F}}_1$ extending $\underset{\sim}{\mathfrak{p}}_0$). Consequently, we may find a condition $p_1' \in H^1 \subseteq \mathbb{C}_{\mathbf{A}_1}$ stronger than p_1, an integer $n < \omega$, and an element $y \in \underset{\sim}{M}^n_{\mathbf{A}_1}$ (so $y \in \omega$) such that

$$p_1' \restriction \mathbf{A}_0 \in H^0, \quad \text{and} \quad p_1' \Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“}\underset{\sim}{M}^n_{\mathbf{A}_1} \models \big(\underset{\sim}{A}^*(y) \;\&\; \underset{\sim}{\varphi}(y)\big) \text{ and } n \in \underset{\sim}{\mathbf{a}}\text{”}.$$

As $\underset{\sim}{A}^*_n$ is a $\mathbb{C}_{\mathbf{A}_0}$-name, we really have $y \in \underset{\sim}{A}^*_n[H^0]$, and hence (by its definition) for some $p_2' \in \mathbb{C}_{\mathbf{A}_2}$ we have

$$p_2 \leq p_2', \quad p_2' \restriction \mathbf{A}_0 \in H^0, \quad \text{and} \quad p_2' \Vdash \text{“}y = \underset{\sim}{x}_\beta(n) \text{ and } n \in \underset{\sim}{\mathbf{b}}\text{”}.$$

Now for our n we can force $n \in \underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} \cap \underset{\sim}{\mathbf{c}}$ by amalgamating the corresponding conditions p_1', p_2', getting a contradiction. As said above this finishes the proof of the existence of q_3.

2) The proof is essentially contained in the previous one (use the very trivial bigness notion: $\varphi(x, \bar{a})$ is big in M if and only if $M \models (\exists x)\varphi(x, \bar{a})$, so we may use a $\underset{\sim}{\mathfrak{p}}_1$). See also the end of the proof of (3).

3) We will prove by induction on $\gamma \in \aleph_3$ that if all $\beta_j \leq \gamma$ and all p_i belong to $\mathbb{A}\mathrm{pp} \restriction \gamma$, then the assertion in (3) holds for some $r \in \mathbb{A}\mathrm{pp} \restriction \gamma$.

We may assume that $\delta_1 > 0$ (otherwise apply 2.6) and $\delta_2 > 0$ (otherwise let $\delta_2' = 1$, $\beta_0 = 0$, $q_0' \in \mathbb{A}\mathrm{pp} \restriction 0$ be above $p_i \restriction 0$ for $i < \delta_1$; so it just means $\underset{\sim}{\mathcal{F}}^{q_0'}$ is an ultrafilter extending $\underset{\sim}{\mathcal{F}}^{p_i \restriction 0}$ for $i < \delta_1$; now if $\gamma = 0$, then $r = q_0'$ is as required and otherwise we have reduced the case $\delta_2 = 0$ to the case $\delta_2 = 1$).

We may assume that $\beta_j = \sup\{\alpha + 1 : \alpha \in \mathbf{A}^{q_j}\}$ (for $j < \delta_2$), and also that the sequence $\langle \beta_j : j < \delta_2 \rangle$ is strictly increasing. Let $\beta = \sup_{j<\delta_2} \beta_j$ and let $q = (\bigcup_{j<\delta_2} \mathbf{A}^{q_j}, \bigcup_{j<\delta_2} \underset{\sim}{\mathcal{F}}^{q_j}, \bigcup_{j<\delta_2} \underset{\sim}{\bar{\Gamma}}^{q_j})$, this triple is not necessarily a member of $\mathbb{A}\mathrm{pp}$.

We first deal with

Case 1: $\mathrm{cf}(\gamma) \neq \aleph_0$.

If $\gamma = \beta$, then $q \in \mathbb{A}\mathrm{pp}$ and we may take $r = q$. So let us assume $\beta < \gamma$. If δ_2 is a successor ordinal, or a limit ordinal of uncountable cofinality, then we let $q^* = q$ (clearly $q^* \in \mathbb{A}\mathrm{pp} \restriction \beta$). If $\mathrm{cf}(\delta_2) = \aleph_0$, then we may first apply the inductive hypothesis to $\langle p_i \restriction \beta : i < \delta_1 \rangle$ (and $\langle \beta_j, q_j : j < \delta_2 \rangle$) to get a condition

$q^* \in \mathbb{A}\mathrm{pp} \restriction \beta$ which is stronger than all $p_i \restriction \beta$ and which end-extends all q_j. So in all these cases, we have a condition $q^* \in \mathbb{A}\mathrm{pp} \restriction \beta$ end extending all q_j for $j < \delta_2$ and stronger than all $p_i \restriction \beta$ for $i < \delta_1$ (and we are looking for an end-extension of it which is a bound to all $p_i \restriction \beta$). The following three subcases suffice as we have already dealt with the possibility $\gamma = 0$.

THE SUBCASE 1A. $\gamma = \gamma_0 + 1$ is a successor. In this case our inductive hypotheses applies to the $p_i \restriction \gamma_0, q^*$, and γ_0, yielding r_0 in $\mathbb{A}\mathrm{pp} \restriction \gamma_0$ with $p_i \restriction \gamma_0 \le r_0$ for $i < \delta_1$ and $q^* \le_{\mathrm{end}} r_0$. What remains to be done is an amalgamation of r_0 with all of the p_i, where $\mathbf{A}^{p_i} \subseteq \mathbf{A}^{r_0} \cup \{\gamma_0\}$, and where one may as well suppose that γ_0 is in $\mathbf{A}^{p_i}$ for all i. This is a slight variation on (1) or (2). For instance, suppose $\mathrm{cf}(\gamma_0) = \aleph_2$. We let

- $\mathbf{A}_2 = \bigcup_{i<\delta_1} \mathbf{A}^{p_i}$, $\mathbf{A}_0 = \mathbf{A}_2 \setminus \{\gamma_0\}$, $\mathbf{A}_1 = \mathbf{A}^{r_0}$, $\mathbf{A}_3 = \mathbf{A}_2 \cup \mathbf{A}_1$.
- $\underset{\sim}{\mathcal{F}}_1 = \underset{\sim}{\mathcal{F}}^{r_0}$, $\underset{\sim}{\mathcal{F}}_2 = \bigcup_{i<\delta_1} \underset{\sim}{\mathcal{F}}^{p_i}$. (The latter might be only a $\mathbb{C}_{\mathbf{A}_2}$-name of a filter).
- For $i < \delta_1$ let $\underset{\sim}{\mathfrak{p}}^i$ be the $\mathbb{C}_{\mathbf{A}^{p_i} \cap \gamma_0}$-name for the $\underset{\sim}{\Gamma}^{p_i}_{\gamma_0}$-big type induced by $\underset{\sim}{x}_{\gamma_0}$ over the model $\prod^{\mathbf{A}^{p_i} \cap \gamma_0}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{p_i} \cap \gamma_0} / \underset{\sim}{\mathcal{F}}^{p_i \restriction \gamma_0}$. Then let $\underset{\sim}{\mathfrak{p}}_0 = \bigcup_{i<\delta_1} \underset{\sim}{\mathfrak{p}}^i$, and note that it is a $\mathbb{C}_{\mathbf{A}_0}$-name for a $\underset{\sim}{\Gamma}^{p_i}_{\gamma_0}$-big type over the model $\prod^{\mathbf{A}_0}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}_0} / \underset{\sim}{\mathcal{F}}_0$.
- Let $\underset{\sim}{\mathfrak{p}}_1$ be (a $\mathbb{C}_{\mathbf{A}_1}$-name for) a complete $\underset{\sim}{\Gamma}^{p_i}_{\gamma_0}$-big type over $\prod^{\mathbf{A}_1}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}_1} / \underset{\sim}{\mathcal{F}}_0$ extending $\underset{\sim}{\mathfrak{p}}_0$. (Exists by 1.2; the role of $\underset{\sim}{\mathfrak{p}}_1$ is to be the type which x_{γ_0} realizes over $\prod^{\mathbf{A}_1}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}_1} / \mathcal{F}^{r_0}$ according to a condition r which we will choose below so necessarily it extends $\bigcup_{i<\delta_1} \underset{\sim}{\mathfrak{p}}^i$).

Now, in $\mathbf{V}^{\mathbb{C}_{\mathbf{A}_3}}$, we would like to extend $\underset{\sim}{\mathcal{F}}_1 \cup \underset{\sim}{\mathcal{F}}_2$ to an ultrafilter $\mathcal{F}'$ containing the sets of the form $\{n < \omega : \underset{\sim}{M}^n_{\mathbf{A}_3} \models \underset{\sim}{\varphi}[\underset{\sim}{x}_{\gamma_0}(n)]\}$ for all $\underset{\sim}{\varphi}(x) \in \underset{\sim}{\mathfrak{p}}_1$. If this fails, then as

$$\Vdash_{\mathbb{C}_{\mathbf{A}_1}} \text{“}\langle \underset{\sim}{\mathcal{F}}^{p_i} : i < \delta_1 \rangle \text{ is increasing”}$$

we find a condition $p \in \mathbb{C}_{\mathbf{A}_3}$, a $\mathbb{C}_{\mathbf{A}_1}$-name $\underset{\sim}{\mathbf{a}}$ of a member of $\underset{\sim}{\mathcal{F}}_1$, and $i < \delta_1$, and a $\mathbb{C}_{\mathbf{A}_2}$-name $\underset{\sim}{\mathbf{b}}$ for a member of $\underset{\sim}{\mathcal{F}}_i$, and $\underset{\sim}{\varphi}$ such that

$$p \restriction \mathbf{A}_1 \Vdash \text{“}\underset{\sim}{\varphi}(x) \in \underset{\sim}{\mathfrak{p}}^i \subseteq \underset{\sim}{\mathfrak{p}}_1\text{”} \quad \text{and} \quad p \Vdash_{\mathbb{C}_{\mathbf{A}_3}} \text{“}\underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} \cap \{n : M^n_{\mathbf{A}_3} \models \underset{\sim}{\varphi}[x_\beta(n)]\} = \emptyset\text{”}.$$

Next, we continue exactly as in the proof of (1).

THE SUBCASE 1B. γ is a limit ordinal of cofinality $\aleph_2$.
Since $\delta_1 < \aleph_2$ there is some $\gamma_0 < \gamma$ such that all p_i lie in $\mathbb{A}\mathrm{pp} \restriction \gamma_0$ and $\beta < \gamma_0$, and the induction hypothesis then yields the claim.

THE SUBCASE 1C. γ is a limit ordinal of cofinality $\aleph_1$.
Choose a strictly increasing and continuous sequence $\langle \gamma_j : j < \aleph_1 \rangle$ with supremum γ, starting with $\gamma_0 = \beta$. By induction on j choose $r_j \in \mathbb{A}\mathrm{pp} \restriction \gamma_j$ (for $j < \aleph_1$) such that:

- $r_0 = q^*$;
- $r_j \leq_{\mathrm{end}} r_{j'}$ for $j < j' < \aleph_1$;
- $p_i \restriction \gamma_j \leq r_j$ for $i < \delta_1$ and $j < \aleph_1$.

(Thus, at a successor stage $j+1$, the inductive hypothesis is applied to $p_i \restriction \gamma_{j+1}, r_j, \gamma_j$, and γ_{j+1}. At a limit stage j, we apply the inductive hypothesis to $p_i \restriction \gamma_j$ for $i < \delta_1$, $r_{j'}$ for $j' < j$, $\gamma_{j'}$ for $j' < j$, and γ_j.) Finally, we let $r = (\bigcup_{j<\aleph_1} \mathbf{A}^{r_j}, \bigcup_{j<\aleph_1} \underset{\sim}{\mathcal{F}}^{r_j}, \bigcup_{j<\aleph_1} \underset{\sim}{\bar{\Gamma}}^{r_j})$. Clearly $r \in \mathbb{A}\mathrm{pp}$ is as required.

Now we are going to consider the remaining case:

THE CASE 2. γ is a limit ordinal of cofinality $\aleph_0$.

If $\beta < \gamma$ (where β is as defined at the beginning of the proof), then we first pick a strictly increasing sequence $\langle \gamma_j : j < \aleph_0 \rangle$ of ordinals such that $\beta \leq \gamma_0$ and $\sup_{j<\aleph_0} \gamma_j = \gamma$. Then we apply repeatedly the inductive hypothesis to build a sequence $\langle q'_j : j < \aleph_0 \rangle$ such that $q'_j \in \mathbb{A}\mathrm{pp} \restriction \gamma_j$, $q'_{j_0} \leq_{\mathrm{end}} q'_{j_1}$ for $j_0 < j_1$, $q_j \leq_{\mathrm{end}} q'_0$ (for all $j < \delta_2$), and $p_i \restriction \gamma_j \leq q'_j$ (for all $i < \delta_1$, $j < \aleph_0$). Thus we have reduced this sub-case to the only one remaining: $\beta = \gamma$. Now if for some $j < \delta_2$ we have $\beta_j = \gamma$, then $r = q_j$ is as required, so without loss of generality $(\forall j < \delta_2)(\beta_j < \gamma)$. Then necessarily $\mathrm{cf}(\delta_2) = \aleph_0$ and we may equally well assume that $\delta_2 = \aleph_0$.

We take q as defined earlier (so it is the "union" of all q_j), but it does not have to be a condition in $\mathbb{A}\mathrm{pp}$: the filter $\bigcup_{j<\aleph_0} \underset{\sim}{\mathcal{F}}^{q_j}$ does not have to be an ultrafilter, and we need to extend it to one that contains also $\bigcup_{i<\delta_1} \underset{\sim}{\mathcal{F}}^{p_i}$. Note that $\mathbf{A}^* \stackrel{\mathrm{def}}{=} \bigcup_{i<\delta_1} \mathbf{A}^{p_i} \subseteq \bigcup_{j<\aleph_0} \mathbf{A}^{q_j} \stackrel{\mathrm{def}}{=} \mathbf{A}^+$, but there might be $\mathbb{C}_{\mathbf{A}^*}$-names for elements of $\bigcup_{i<\delta_1} \underset{\sim}{\mathcal{F}}^{p_i}$ that are not $\mathbb{C}_{\mathbf{A}^{q_j}}$-names for any $j < \aleph_0$, so seemingly it could happen that one name like that is forced to be disjoint from some element of $\underset{\sim}{\mathcal{F}}^{q_j}$. Still, also here $\bigcup_{j<\aleph_0} \underset{\sim}{\mathcal{F}}^{q_j}$ is closed under finite intersection and similarly $\bigcup_{i<\delta_1} \underset{\sim}{\mathcal{F}}^{p_i}$. So assume toward contradiction, that there are a condition $p \in \mathbb{C}_{\mathbf{A}^+}$, ordinals $i < \delta_1$ and $j < \aleph_0$, a $\mathbb{C}_{\mathbf{A}^{p_i}}$-name $\underset{\sim}{\mathbf{a}}$, and a $\mathbb{C}_{\mathbf{A}^{q_j}}$-name $\underset{\sim}{\mathbf{b}}$ such that

$$p \Vdash_{\mathbb{C}_{\mathbf{A}^+}} \text{"}\underset{\sim}{\mathbf{a}} \in \underset{\sim}{\mathcal{F}}^{p_i} \quad \& \quad \underset{\sim}{\mathbf{b}} \in \underset{\sim}{\mathcal{F}}^{q_j} \quad \& \quad \underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} = \emptyset\text{"}.$$

Increasing j if necessary, we may also assume that $p \in \mathbb{C}_{\mathbf{A}^{q_j}}$ so $\mathrm{Dom}(p) \subseteq \beta_j \times \omega$. Let $H^0 \subseteq \mathbb{C}_{\mathbf{A}^{p_i} \cap \beta_j}$ be generic over $\mathbf{V}$ such that $p \restriction \mathbf{A}^{p_i} \in H^0$, and let

$$\mathbf{c} = \left\{ \begin{array}{c} n \in \omega : \text{there is a condition } p' \in \mathbb{C}_{\mathbf{A}^{p_i}} \text{ stronger than } p \restriction \mathbf{A}^{p_i} \text{ and} \\ \text{such that } p' \restriction (\mathbf{A}^{p_i} \cap \beta_j) \in H^0 \text{ and } p' \Vdash_{\mathbb{C}_{\mathbf{A}^{p_i}}} \text{“} n \in \underset{\sim}{\mathbf{a}} \text{”} \end{array} \right\}.$$

Clearly, $\mathbf{c} \in \mathbf{V}[H^0]$ is a set from $\big(\underset{\sim}{\mathcal{F}}^{p_i} \restriction (\mathbf{A}^{p_i} \cap \beta_j)\big)[H^0]$. Since $p_i \restriction \beta_j \le q_j$, we find a condition $p'' \in \mathbb{C}_{\mathbf{A}^{q_j}}$ and $n \in \mathbf{c}$ such that

$$p \le p'' \quad \& \quad p'' \restriction (\mathbf{A}^{p_i} \cap \beta_j) \in H^0 \quad \& \quad p'' \Vdash_{\mathbb{C}_{\mathbf{A}^{q_j}}} \text{“} n \in \underset{\sim}{\mathbf{b}} \text{”}.$$

For this n we find $p' \in \mathbb{C}_{\mathbf{A}^{p_i}}$ witnessing that $n \in \mathbf{c}$ (i.e. $p' \restriction (A^{p_i} \cap \beta_j) \in H^0$ and $p' \Vdash_{\mathbb{C}_{\mathbf{A}^{p_i}}}$ “$n \in \underset{\sim}{\mathbf{a}}$”) and next we let $p^* = p' \cup p''$. Clearly $p^* \Vdash n \in \underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}}$, a contradiction.

4) Follows, i.e., it is the case $\delta_2 = 0$ of part (3).

5) We choose $q_n \in \mathbb{A}\mathrm{pp}_{\gamma_n}$ for $n < \omega$ such that $\mathbf{A}^{q_n} := \mathbf{A}^p \cup \{\gamma_\ell : \ell < n\}, p \restriction \gamma_n \le q_n$ and $q_n \le_{\mathrm{end}} q_{n+1}$ for $n < \omega$ and let $\mathbf{A} = \bigcup\{\mathbf{A}^{q_n} : n < \omega\}$

This is possible as for $n = 0$ let $q_n = p \restriction \gamma_{n+1}$, for $n = k+1$, let $q'_n \in \mathbb{A}\mathrm{pp}$ be such that $\mathbf{A}^{q'_n} = \mathbf{A}^{q_k} \cup \{\gamma_n\}$ and $q_k \le_{\mathrm{end}} q'_n$, exists by 2.7, and then q_n as required exists by 2.8(1).

Let $\underset{\sim}{x}$ be the following $\mathbb{C}_{\mathbf{A}}$-name of an ω-sequence:

$$\underset{\sim}{x} = \langle \underset{\sim}{x}_{\gamma_n}(n) : n < \omega \rangle.$$

Now we shall choose q such that $\mathbf{A}^q = \mathbf{A} = \bigcup\{\mathbf{A}^{q_n} : n < \omega\} = \mathbf{A}^p \cup \{\gamma_n : n < \omega\}$, $n < \omega \Rightarrow q_n \le_{\mathrm{end}} q$ and $p \le q$ and $\Vdash_{\mathbb{C}_{\mathbf{A}}}$ “$\underset{\sim}{x}$ realizes $\underset{\sim}{\mathfrak{p}}$”.

Again the only problem is to find a $\mathbb{C}_{\mathbf{A}}$-name of an ultrafilter on ω which include

$$\underset{\sim}{\mathcal{F}}^p \cup \bigcup\{\underset{\sim}{\mathcal{F}}^{q_n} : n < \omega\} \cup \{\{n : \underset{\sim}{M}^n_{\mathbf{A}^p} \models \varphi(\underset{\sim}{x}(n))\} : \varphi(x) \in \underset{\sim}{\mathfrak{p}}\}$$

Since without loss of generality $\underset{\sim}{\mathfrak{p}}$ is closed under conjunction it is enough to show that:

⊛ if $\underset{\sim}{\mathbf{a}}$ is a $\mathbb{C}_{\mathbf{A}^p}$-name of a member of $\underset{\sim}{\mathcal{F}}^p$, $n < \omega$, $\underset{\sim}{\mathbf{b}}$ is a $\mathbb{C}_{\mathbf{A}^{q_n}}$-name of a member of $\underset{\sim}{\mathcal{F}}^{q_n}$ $\underset{\sim}{\varphi}(x)$ is a $\mathbb{C}_{\mathbf{A}^p}$-name of a formula from $\mathfrak{p}$
then $\Vdash_{\mathbb{C}_{\mathbf{A}}}$ “$\underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} \cap \{n : M_{\mathbf{A}^p} \models \underset{\sim}{\varphi}(\underset{\sim}{x}(n))\} \neq \emptyset$”. As in previous cases this is easy. ∎

LEMMA 2.9: *Assume* $\mathbf{V} \models$ GCH. *The forcing notion* $\mathbb{A}\mathrm{pp}$ *satisfies the* $\aleph_3$*-chain condition, it is* $\aleph_2$*-complete,* $|\mathbb{A}\mathrm{pp}| = \aleph_3$ *and* $|\mathbb{A}\mathrm{pp} \restriction \gamma| \le \aleph_2$ *for every* $\gamma \in \aleph_3$.

Consequently, the forcing with $\mathbb{A}\mathrm{pp}$ *does not collapse cardinals nor changes cofinalities, and* $\Vdash_{\mathbb{A}\mathrm{pp}}$ GCH.

Proof. Perhaps the only unclear part is the chain condition. Suppose towards a contradiction that we have an antichain $\{q_\alpha : \alpha \in \aleph_3 \ \&\ \mathrm{cf}(\alpha) = \aleph_2\} \subseteq \mathbb{A}\mathrm{pp}$ (the index α is taken to vary over ordinals of cofinality $\aleph_2$ just for convenience). An important point is that $\mathbf{G}$ can "offer" at most $\aleph_2$ candidates for the bigness notion at $\delta < \aleph_3$, $\mathrm{cf}(\delta) = \aleph_2$, hence for each $\gamma \in \aleph_3$ the restricted forcing $\mathbb{A}\mathrm{pp} \restriction \gamma$ has cardinality $\leq \aleph_2$. Applying Fodor's lemma twice, we find a stationary set $S \subseteq \{\alpha \in \aleph_3 : \mathrm{cf}(\alpha) = \aleph_2\}$ and a condition $q^* \in \mathbb{A}\mathrm{pp}$ such that $(\forall \alpha \in S)(q_\alpha \restriction \alpha = q^*)$. Pick $\alpha_1, \alpha_2 \in S$ such that $\sup(\mathbf{A}^{q_{\alpha_1}}) < \alpha_2$; it follows from Lemma 2.8(3) that the conditions $q_{\alpha_1}, q_{\alpha_2}$ are compatible, a contradiction. ∎

PROPOSITION 2.10: (1) *For each* $p \in \mathbb{A}\mathrm{pp}$ *and* $\alpha \in \aleph_3$, *there is a condition* $q \in \mathbb{A}\mathrm{pp}$ *stronger than* p *and such that* $\alpha \in \mathbf{A}^q$.

(2) $\underset{\sim}{\mathcal{F}} \stackrel{\mathrm{def}}{=} \bigcup\{\underset{\sim}{\mathcal{F}}^r : r \in \underset{\sim}{G}_{\mathbb{A}\mathrm{pp}}\}$ *is an* $\mathbb{A}\mathrm{pp}$*-name of a* $\mathbb{C}_{\aleph_3}$*-name for a non-principal ultrafilter on* ω. *Also, for each* $r \in \underset{\sim}{G}_{\mathbb{A}\mathrm{pp}}$ *we have:* $\underset{\sim}{\mathcal{F}} \cap \mathcal{P}(\omega)^{(\mathbf{V}[\underset{\sim}{G}_{\mathbb{A}\mathrm{pp}}])^{\mathbb{C}_{\mathbf{A}^r}}} = \underset{\sim}{\mathcal{F}}^r$.

Proof. Should be clear (for (1) use 2.7 + 2.8(3); then (2) follows). ∎

Definition 2.11: (1) Suppose $G_{\mathbb{A}\mathrm{pp}} \subseteq \mathbb{A}\mathrm{pp}$ is generic over $\mathbf{V}$, $\mathbf{V}^* = \mathbf{V}[G_{\mathbb{A}\mathrm{pp}}]$. For $\alpha \leq \aleph_3$ we let $G_\alpha = G_{\mathbb{A}\mathrm{pp}} \cap (\mathbb{A}\mathrm{pp} \restriction \alpha)$. It is a generic subset of $\mathbb{A}\mathrm{pp} \restriction \alpha$; let $\underset{\sim}{\mathcal{F}}^\alpha$ be the $(\mathbb{A}\mathrm{pp} \restriction \alpha)$-name of the $\mathbb{C}_\alpha$-name $\bigcup\{\underset{\sim}{\mathcal{F}}^q : q \in G_\alpha\}$. Note: $\underset{\sim}{\mathcal{F}}^q$ being a $\mathbb{C}_{\mathbf{A}^q}$-name is a $\mathbb{C}_\alpha$-name when $\mathbf{A}^q \subseteq \alpha$. So in $\mathbf{V}^*$ the sequence $\langle \underset{\sim}{\mathcal{F}}^\alpha : \alpha < \aleph_3 \rangle$ is forced (i.e. $\Vdash_{\mathbb{C}}$) to be increasing, let $\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{F}}^{\aleph_3}$ so $\underset{\sim}{\mathcal{F}}^\alpha$ is the $\mathbb{C}_\alpha$-name for the restriction $\underset{\sim}{\mathcal{F}} \restriction \alpha$ of the ultrafilter $\underset{\sim}{\mathcal{F}}$ to the sets from the universe $(\mathbf{V}^*)^{\mathbb{C}_\alpha}$.

(2) We define an $\mathbb{A}\mathrm{pp}$-name $\underset{\sim}{\Gamma}_\delta$ of a $\mathbb{C}_\delta$-name as Γ^p_δ for every $p \in \underset{\sim}{G}_{\mathbb{A}\mathrm{pp}}$ such that $\delta \in \mathbf{A}^p$. (So it is an $\mathbb{A}\mathrm{pp} * \mathbb{C}_\delta$-name.)

LEMMA 2.12: (1) *Suppose that* $G_{\mathbb{A}\mathrm{pp}} \subseteq \mathbb{A}\mathrm{pp}$ *is generic over* $\mathbf{V}$, $\mathbf{V}^* = \mathbf{V}[G_{\mathbb{A}\mathrm{pp}}]$, *and* $\delta < \aleph_3$, $\mathrm{cf}(\delta) = \aleph_2$, *and* $H^\delta \subseteq \mathbb{C}_\delta$ *is generic over* $\mathbf{V}^*$. *Then, in* $\mathbf{V}[G_{\mathbb{A}\mathrm{pp}} \cap (\mathbb{A}\mathrm{pp} \restriction \delta)][H^\delta]$, *we have*[9]: $\prod_{n<\omega} \underset{\sim}{M}^n_\delta / \underset{\sim}{\mathcal{F}}^\delta[H^\delta]$ *is* $\aleph_2$*-compact.*

(2) *Also if* $H \subseteq \mathbb{C}_{\aleph_3}$ *is generic over* $\mathbf{V}^*$, $H \supseteq H^\delta$, *then in* $\mathbf{V}^*[H]$:

(a) $\prod_{n<\omega} \underset{\sim}{M}^n_{\aleph_3} / \underset{\sim}{\mathcal{F}}[H]$ *is* $\aleph_2$*–compact,*

[9] Note: $\underset{\sim}{M}^n_\delta$ is $\underset{\sim}{M}^n_{\mathbf{A}}$ for $\mathbf{A} = \delta$.

(b) $\underset{\sim}{x}_\delta[H]/\mathcal{F}[H] \in \prod_{n<\omega} \underset{\sim}{M}^n_{\aleph_3}/\mathcal{F}[H]$ *realizes a* $\underset{\sim}{\Gamma}_\delta[G][H^\delta]$*-big type over* $\prod^\delta{}_{n<\omega} \underset{\sim}{M}^n_\delta/\underset{\sim}{\mathcal{F}}^\delta[H^\delta]$.

Proof. By 2.7(1)+2.7(2). We can use some $\underset{\sim}{x}_\beta$ with β of cofinality less than $\aleph_2$ to realize each type. ∎

3. Definability

HYPOTHESIS 3.1: In this section we assume that $\mathbf{G}$ is an $(\aleph_3, \aleph_2)$-bigness guide, $\mathbb{A}\mathrm{pp} = \mathbb{A}\mathrm{pp}_{\mathbf{G}}$, $G^* \subseteq \mathbb{A}\mathrm{pp}$ is a generic filter over $\mathbf{V}$, and $\mathbf{V}^* = \mathbf{V}[G^*]$. For an ordinal $\alpha < \aleph_3$, we let $G^*_\alpha = G^* \cap (\mathbb{A}\mathrm{pp} \restriction \alpha)$. Also, $\underset{\sim}{H}$, $\underset{\sim}{H}^\alpha$ are the canonical $\mathbb{C}_{\aleph_3}$-and $\mathbb{C}_\alpha$-names of the generic subsets of $\mathbb{C}_{\aleph_3}$ and $\mathbb{C}_\alpha$, respectively. We work mostly in $\mathbf{V}^*$.

(Note that, by Lemma 2.9, $\mathbf{V}^* \models \mathrm{GCH}$.)

Definition 3.2: (1) We say that $\mathbf{m}$ is **an** $(\aleph_3, \aleph_2)$**-isomorphism candidate** (or just **an isomorphism candidate**, in $\mathbf{V}$ or in $\mathbf{V}^*$, see below) if;

(i) $\mathbf{m}$ consists of $\mathbf{A}^* = \mathbf{A}^*[\mathbf{m}] \in [\aleph_3]^{<\aleph_2}$, $p^* = p^*[\mathbf{m}]$, $\underset{\sim}{N}^\ell_n = \underset{\sim}{N}^\ell_n[\mathbf{m}]$, $\underset{\sim}{t}^\ell_n$ (for $n < \omega$, $\ell \in \{1,2\}$), $\underset{\sim}{F} = \underset{\sim}{F}[\mathbf{m}]$, $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}[\mathbf{m}]$ and $(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar\varphi}, \underset{\sim}{\Delta}) = (\underset{\sim}{t}[\mathbf{m}], \underset{\sim}{\psi}[\mathbf{m}], \underset{\sim}{\bar\varphi}[\mathbf{m}], \underset{\sim}{\Delta}[\mathbf{m}])$,

(ii) $\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar\varphi}$ are $\mathbb{C}_{\mathbf{A}^*}$-names as in 2.2(1), $\underset{\sim}{\Delta} \subseteq \mathbb{L}(\tau(\underset{\sim}{t}))$ is a $\mathbb{C}_{\mathbf{A}^*}$-name, equality belongs to it, and $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_{(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar\varphi},)}$ is a bigness notion as there, $\tau(\underset{\sim}{t})$ is countable; we can assume $\tau(\underset{\sim}{t})$ is an object (not a name) by adding for each $m, \aleph_0$ predicates with m places said (by $\underset{\sim}{t}$) to be empty.

(iii) $\underset{\sim}{N}^\ell_n$, for $n < \omega$ and $\ell \in \{1,2\}$, are $\mathbb{C}_{\mathbf{A}^*}$-names for countable models of a (countable) theory $\underset{\sim}{t}^\ell_n$, and the universes $|\underset{\sim}{N}^\ell_n|$ are subsets of ω and with vocabulary $\tau(\underset{\sim}{t})$. Also it is forced (i.e., $\Vdash_{\mathbb{C}_{\aleph_3}}$) that $\underset{\sim}{t} \subseteq \mathrm{Th}\Big(\prod_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}\Big) = \mathrm{Th}\Big(\prod_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}}\Big)$, where the $\prod_{n<\omega}$ is $\prod^{\aleph_3}{}_{n<\omega}$. Note that we cannot require that $\underset{\sim}{t}^\ell_n = \underset{\sim}{t}$, as $\underset{\sim}{t}$ may be infinite, (e.g. t^{ind}_0 is) and no N^ℓ_n is a model of $\underset{\sim}{t}$.

(iv) We have predicates $Q^\ell_R \in \tau_{\mathbf{A}^*}$ (for $R \in \tau(\underset{\sim}{t})$) such that $\underset{\sim}{\bar\varphi}^\ell = \langle Q^\ell_R : R \in \tau(t)\rangle$ is the interpretation of $\tau(\underset{\sim}{t})$ in $\prod^{\mathbf{A}^*}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^*}/\underset{\sim}{\mathcal{F}}$ giving $\prod_{n<\omega} \underset{\sim}{N}^\ell_n/\underset{\sim}{\mathcal{F}}$ and $\bar\varphi = \bar\varphi^1$. (Remember 2.1(4), 1.4(1); so by the choice of $\tau_{\mathbf{A}^*}$ actually $\underset{\sim}{\bar\varphi}^\ell = \bar\varphi^\ell$.)

(v) $\underset{\sim}{F}$ is a $\mathbb{C}_{\aleph_3}$-name (more accurately an $\mathbb{A}\mathrm{pp}$-name of such name, but we sometimes write $\underset{\sim}{F}$ instead of $\underset{\sim}{F}[G^*]$ as when G^* is constant) and

$p^* \in \mathbb{C}_{\aleph_3}$ is a condition such that:

$$p^* \Vdash_{\mathbb{C}_{\aleph_3}} \text{“}\underset{\sim}{F} \text{ is a map from } \prod_{n<\omega} \underset{\sim}{N}^1_n \text{ into } \prod_{n<\omega} \underset{\sim}{N}^2_n\text{”}$$
$$p^* \Vdash_{\mathbb{C}_{\aleph_3}} \text{“}\underset{\sim}{F} \text{ represents a } \underset{\sim}{\Delta}\text{–embedding modulo } \underset{\sim}{\mathcal{F}}\text{”}.$$

(If **m** is clear from the context we may omit it.)

Remark 3.3: (1) In **m**, note that $\underset{\sim}{\Delta}$ tells us which first order formulas in the vocabulary $\tau(\underset{\sim}{t})$ does the function $\underset{\sim}{F}$ preserve. In our main case those are the atomic and negation of atomic formulas in τ^{ind}

(2) Of course, **m** gives us two interpretations of t in the ultraproduct: one for $\ell = 1$ and another for $\ell = 2$, and the interpreting formulas define N^ℓ_n in the nth coordinate. Without loss of generality the universe of $\underset{\sim}{N}^\ell_n$ is nonempty for every $n < \omega$ (and $\ell = 1, 2$).

Definition 3.4: For **m** as in 3.2 let $\mathbf{m}^- = \langle \underset{\sim}{t}, \underset{\sim}{\psi}, \bar{\underset{\sim}{\varphi}}, \underset{\sim}{\Delta}, \langle \underset{\sim}{N}^\ell_n : n < \omega,\ \ell = 1, 2\rangle\rangle$, those names involve countably many of the Cohens $\underset{\sim}{x}_\beta$. Also note that as $\mathbb{A}\mathrm{pp}$ is $\aleph_2$-complete, this forcing does not add new $\mathbf{m}^-$, i.e., $\mathbf{V}$ and $\mathbf{V}^*$ have the same set of $\mathbf{m}^-$, though we have an $\mathbb{A}\mathrm{pp}$-name $\underset{\sim}{\mathbf{m}}$ of such object.

Observation 3.5: *Assume, in $\mathbf{V}^*$, that* **m** *is an $(\aleph_3, \aleph_2)$-isomorphism candidate, $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}[\mathbf{m}] = \Gamma_{(\underset{\sim}{t}, \underset{\sim}{\bar{\varphi}}, \underset{\sim}{\psi})}$. Then there is a stationary set of ordinals $\delta < \aleph_3$ such that:*

- $(\mathrm{a})_\delta$ $\mathbf{A}^* = \mathbf{A}^*[\mathbf{m}] \subseteq \delta \cap \mathbf{A}^q, \mathrm{cf}(\delta) = \aleph_2$, *and* $p^* = p^*[\mathbf{m}] \in \mathbb{C}_\delta$, *and for some* $q \in G^*$ *we have that* $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}^q_\delta$ *is* $\Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar{\varphi}}]$ *(for* $(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar{\varphi}})$ *from 2.2), so* $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_\delta$ *see 2.11(2)*
- $(\mathrm{b})_\delta$ *for every* $\mathbb{C}_{\aleph_3} \restriction \delta$*-name* $\underset{\sim}{x}$ *for an element of* $\prod_{n<\omega} \underset{\sim}{N}^1_n$, $\underset{\sim}{F}(\underset{\sim}{x})$ *is a* $(\mathbb{C}_{\aleph_3} \restriction \delta)$*-name, (recall* $\mathbb{A}\mathrm{pp}$ *satisfies the* $\aleph_3$*–c.c)*
- $(\mathrm{c})_\delta$ *similarly for* $\underset{\sim}{F}^{-1}$ *and for “*$y \in \mathrm{Rang}(\underset{\sim}{F})$*”,*
- $(\mathrm{d})_\delta$ $\Vdash_{\mathbb{C}_{\aleph_3}}$ *“*$\{n < \omega : \underset{\sim}{x}_\delta(n) \in \underset{\sim}{N}^1_n\} \in \underset{\sim}{\mathcal{F}}$ *(so* $\underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}} \in \prod_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}$*)”.*

For such δ, we let $\underset{\sim}{y}^* = \underset{\sim}{y}^*_\delta = \underset{\sim}{y}^*_{\delta,\underset{\sim}{F}} = y^*_{\delta,\mathbf{m}}$ be $\underset{\sim}{F}(\underset{\sim}{x}_\delta) \in \prod_{n<\omega} \underset{\sim}{N}^2_n$.

Remark: Notice also that the clauses $(\mathrm{b})_\delta$, $(\mathrm{c})_\delta$ of 3.5 above say that $\underset{\sim}{F}^\delta[G^*]$ is really a $\mathbb{C}_\delta$-name for a function from $(\prod_{n<\omega} \underset{\sim}{N}^1_n)^{(\mathbf{V}^*)^{\mathbb{C}_\delta}}$ into $(\prod_{n<\omega} \underset{\sim}{N}^2_n)^{(\mathbf{V}^*)^{\mathbb{C}_\delta}}$ preserving $\underset{\sim}{\Delta}$-formulas; in the main case it is “onto”.

The Main Isomorphism Theorem 3.6: *Assume that* **m** *is an $(\aleph_3, \aleph_2)$–isomorphism candidate as in 3.2, and $\delta < \aleph_3$ is as in Observation 3.5. Then there are $q_\delta, \underset{\sim}{\Gamma}, \underset{\sim}{y}$ such that*

(a) $q_\delta \in \mathbb{A}\mathrm{pp}$, *moreover* $q_\delta \in G^*$, *and* $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}^{q_\delta}_\delta$ *is* $\Gamma_{\underset{\sim}{\psi}}[\underset{\sim}{\bar\varphi}]$ *for* $(\underset{\sim}{t}, \underset{\sim}{\psi}, \underset{\sim}{\bar\varphi})$ *from 2.2 (the set of choices of* q_δ *is dense and quite closed)*

(b) $q_\delta \Vdash_{\mathbb{A}\mathrm{pp}} p^* \Vdash_{\mathbb{C}_{\aleph_3}}$ *"*$\underset{\sim}{F}(x_\delta) = \underset{\sim}{y}^*$*", where* $\underset{\sim}{y}^*$ *is a* $\mathbb{C}_{\mathbf{A}^{q_\delta}}$*-name of a member of* ${}^\omega\omega$,

(c) $\mathbf{A}^* \subseteq \mathbf{A}^{q_\delta}$, $\mathbf{A}_\delta \overset{\mathrm{def}}{=} \mathbf{A}^{q_\delta} \cap \delta$,

(d) *in* $\mathbf{V}[G^*_\delta][\underset{\sim}{H}^\delta]$ *we have:*

(i) $\mathcal{F}_\delta = \underset{\sim}{\mathcal{F}}_\delta[G^*_\delta][\underset{\sim}{H}^\delta]$ *is a non-principal ultrafilter on* ω.

(ii) *The model* $M_\delta = \prod^\delta{}_{n<\omega} M^n_\delta/\mathcal{F}_\delta$ *with the vocabulary* τ_δ *is* $\aleph_2$*-compact where* $M^n_\delta = \underset{\sim}{M}^n_\delta[G^*_\delta][H^\delta]$ *and* $N^\ell_n = \underset{\sim}{N}^\ell_n[G^*_\delta][\underset{\sim}{H}^\delta]$.

(iii) *The vocabulary* $\tau_{\mathbf{A}_\delta} \subseteq \tau_\delta$ *is of cardinality* $\leq \aleph_1$.

(iv) $M_{\mathbf{A}_\delta} = \prod^{\mathbf{A}_\delta}{}_{n<\omega} M^n_{\mathbf{A}_\delta}/\mathcal{F}^{q_\delta \restriction \delta}[\underset{\sim}{H}^\delta] \prec M_\delta \restriction \tau_{\mathbf{A}_\delta}$.

(v) $p^* \Vdash_{\mathbb{C}_\delta}$ *"*$\underset{\sim}{F}_\delta = (\underset{\sim}{F} \restriction \delta)[\underset{\sim}{H}^\delta] = \big((\underset{\sim}{F} \restriction \delta)[G^* \cap (\mathbb{A}\mathrm{pp} \restriction \delta)]\big)[\underset{\sim}{H}^\delta]$ *is a* $\underset{\sim}{\Delta}$*–embedding from the model* $\prod^\delta{}_{n<\omega} N^1_n/\mathcal{F}_\delta$ *into* $\prod^\delta{}_{n<\omega} N^2_n/\mathcal{F}_\delta$*", recalling* $p^* = p^*[\mathbf{m}]$

(vi) *Let* $\underset{\sim}{p}_\delta = \underset{\sim}{p}_\delta(x)$ *be the (*$\mathbb{C}_{\mathbf{A}_\delta}$*-name of the) 1-type in the vocabulary* $\tau_{\mathbf{A}_\delta}$ *such that* $q_\delta \Vdash_{\mathbb{A}\mathrm{pp}} p^* \Vdash_{\mathbb{C}_\delta}$ *"*$\underset{\sim}{p}_\delta(x)$ *is the type realized by* $\underset{\sim}{x}_\delta$ *over* $M_{\mathbf{A}_\delta}$ *in* $\prod_{n<\omega} M^n_{\mathbf{A}^{q_\delta}}/\underset{\sim}{\mathcal{F}}^{q_\delta}$*". [Clearly it is a* $\mathbb{C}_{\mathbf{A}^{q_\delta}}$*-name, or an* $\mathbb{A}\mathrm{pp} * \mathbb{C}_{\mathbf{A}^{q_\delta}}$*-name; see clause (d) of Definition 2.4(1).]*
Clearly $q_\delta \Vdash_{\mathbb{A}\mathrm{pp}} p^* \Vdash_{\mathbb{C}_\delta}$ *"*$\underset{\sim}{p}_\delta$ *is* $\underset{\sim}{\Gamma}$*–big".*

(vii) *For* $\ell = 1, 2$ *let* $\underset{\sim}{N}^\ell_\delta = \prod^\delta{}_{n<\omega} \underset{\sim}{N}^\ell_n/\underset{\sim}{\mathcal{F}}_\delta$ *(they are in* $\mathbf{V}^*[\underset{\sim}{H}^\delta]$*, even in* $\mathbf{V}[G^*_\delta][\underset{\sim}{H}^\delta]$*). We define* $R_{\delta,m} \subseteq (\underset{\sim}{N}^1_\delta)^m \times (\underset{\sim}{N}^2_\delta)^m$ *for* $m < \omega$ *so that they are* $(\mathbb{A}\mathrm{pp} \restriction \delta) * \mathbb{C}_\delta$*- names and* $(q_\delta \restriction \mathbf{A}_\delta, p^*)$ *forces*

$(\circledast)_1$ $R_{\delta,m}$ *includes the graph of* F_δ*, i.e., if* $\bar a$ *is an* m*-tuple from* N^1_δ*, then* $(\bar a, F_\delta(\bar a)) \in R_{\delta,m}$,

$(\circledast)_2$ *the truth value of* $(\bar a, \bar b) \in R_{\delta,m}$ *depends only on* $\mathbb{L}_{\omega,\omega}(\tau_{\mathbf{A}_\delta})$*-type realized by* $(\bar a, \bar b)$ *over* $M_{\mathbf{A}_\delta}$ *in* M_δ,

$(\circledast)_3$ $R_{\delta,m}$ *is minimal such that* $(\circledast)_1$ *and* $(\circledast)_2$ *hold.*

(viii) *The relations* $R_{\delta,m}$ *mentioned above satisfy (i.e.* $(q_\delta \restriction \mathbf{A}_\delta, p^*)$ *forces):*

$(\oplus)_1$ *if* $\bar a_1, \bar a_2$ *are finite sequences of the same length* m *of members of* N^1_δ*, and* $p_\delta \cup \{\vartheta^{N^1_\delta}(x, \bar a_1), \neg\vartheta^{N^1_\delta}(x, \bar a_2)\}$ *is a* Γ*-big type over* M_δ*, and* $\vartheta, \neg\vartheta \in \underset{\sim}{\Delta}[\mathbf{m}]$*, where* $\vartheta^{N^1_\delta}$ *is* ϑ *as interpreted in the interpretation* $\bar\varphi^1$*, then* $(\bar a_1, F_\delta(\bar a_2)) \notin R_{\delta,m}$.

$(\oplus)_2$ *Above, we may replace* $\vartheta, \neg\vartheta$ *by any pair* ϑ_0, ϑ_1 *of contradictory formulas from* $\underset{\sim}{\Delta}[\mathbf{m}]$.

(ix) *Note that also*

$(*)^{p^*}_{\underset{\sim}{y}^*,\delta}$ $p^* \Vdash_{\mathbb{C}_{\aleph_3}}$

"the $\underset{\sim}{\Delta}$*–type which* $\underset{\sim}{y}^*$ *realizes over* $\underset{\sim}{N}^2_\delta = (\prod_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3}\restriction\delta}}$ *in the model* $\underset{\sim}{N}^2 = (\prod_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3}}}$ *includes the image under* $\underset{\sim}{F}$ *of the* $\underset{\sim}{\Delta}$*–type which* $\underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}}$ *realizes over* $\underset{\sim}{N}^1_\delta = (\prod_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3}\restriction\delta}}$ *in the model* $\underset{\sim}{N}^1 = (\prod_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}})^{(\mathbf{V}^*)^{\mathbb{C}_{\aleph_3}}}$ *".*

THE PROOF OF THE MAIN ISOMORPHISM THEOREM 3.6. Note that we use the countability of $\underset{\sim}{t}$.

Take a condition $q_\delta \in G^*$ such that

$(A)^{q_\delta}$ $\mathbf{A}^* \subseteq \mathbf{A}^{q_\delta}$ recalling that $\mathbf{m}$ determine $\mathbf{A}^*$, $\underset{\sim}{x}_\delta, \underset{\sim}{y}^* = \underset{\sim}{F}(\underset{\sim}{x}_\delta)$ are $\mathbb{C}_{\mathbf{A}^{q_\delta}}$-names (so $\delta \in \mathbf{A}^{q_\delta}$), and $p^* \in \mathbb{C}_{\mathbf{A}^{q_\delta}\cap\delta}$, and

$(B)^{q_\delta}$ the condition q_δ forces (in $\mathbb{A}$pp) that clauses $(b)_\delta$, $(c)_\delta$ and $(d)_\delta$ from 3.5 hold true (so in particular q_δ forces that $\underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}} \in \prod_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}$, $\underset{\sim}{y}^* \in \prod_{n<\omega} \underset{\sim}{N}^2_n$ and $(*)^{p^*}_{\underset{\sim}{y}^*,\delta}$ from clause (ix) of 3.6 holds as $\underset{\sim}{F}$ is (forced to be) a Δ-embedding), and

$(C)^{q_\delta}$ **if** $\underset{\sim}{x}$ is a $\mathbb{C}_{\mathbf{A}^{q_\delta}}$-name for a member of ${\prod^{\mathbf{A}^{q_\delta}}}_{n<\omega} \underset{\sim}{N}^1_n$ (of ${\prod^{\mathbf{A}^{q_\delta}}}_{n<\omega} \underset{\sim}{N}^2_n$, respectively), **then** $\underset{\sim}{F}(\underset{\sim}{x})$ ($\underset{\sim}{F}^{-1}(\underset{\sim}{x})$, respectively) is also a $\mathbb{C}_{\mathbf{A}^{q_\delta}}$-name.

Before we continue with the proof of 3.6, let us note the following.

LEMMA 3.7: *Let* $\delta < \aleph_3$, $q_\delta \in \mathbb{A}\mathrm{pp}$ *and* $\underset{\sim}{y}^*, p^*$ *be as above. Suppose that*

$$q_\delta \restriction \delta = q \le q' \in G^* \cap (\mathbb{A}\mathrm{pp} \restriction \delta).$$

Let $\underset{\sim}{\vartheta}^*$ *be a* $\mathbb{C}_{\mathbf{A}^*}$*-name of a* $\tau(\underset{\sim}{t})$*-formula. Assume further that* $\underset{\sim}{x}'$, $\underset{\sim}{x}''$ *and* $\underset{\sim}{y}'$, $\underset{\sim}{y}''$ *are* $\mathbb{C}_{\mathbf{A}^{q'}}$*-names, and* $p^* \le p \in \mathbb{C}_{\mathbf{A}^{q'}}$*, and the condition* p *forces (in* $\mathbb{C}_{\mathbf{A}^{q'}}$*) that*

(α) $\underset{\sim}{x}', \underset{\sim}{x}'' \in \prod_{n<\omega} \underset{\sim}{N}^1_n$*, and* $\underset{\sim}{y}', \underset{\sim}{y}'' \in \prod_{n<\omega} \underset{\sim}{N}^2_n$*, and*

(β) *the types of* $(\underset{\sim}{x}', \underset{\sim}{y}')$ *and of* $(\underset{\sim}{x}'', \underset{\sim}{y}'')$ *over* ${\prod^{\mathbf{A}^q}}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^q$ *in the model* ${\prod^{\mathbf{A}^{q'}}}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^{q'}$ *(i.e., the vocabulary and the* ω *structures are from* $\mathbf{V}[G^*_\delta][\underset{\sim}{H} \cap \mathbb{C}_{\mathbf{A}^q}]$*, the ultraproduct is taken in* $\mathbf{V}[G^*_\delta][\underset{\sim}{H} \cap \mathbb{C}_{\mathbf{A}^{q'}}]$*) are equal.*

Then the following conditions are equivalent.

(A) *There is* $r^0 \in \mathbb{A}\mathrm{pp}$ *such that* $q_\delta, q' \leq r^0$, $r^0 \restriction \delta \in G^* \cap (\mathbb{A}\mathrm{pp} \restriction \delta)$, *and*

$$p \Vdash_{\mathbb{C}_{\mathbf{A}^{r^0}}} \text{“}\prod^{\mathbf{A}^{r^0}}{}_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}^{r^0} \models \vartheta^*[\underset{\sim}{x}'/\underset{\sim}{\mathcal{F}}^{r^0}, \underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}}^{r^0}] \ \text{ and } \prod^{\mathbf{A}^{r^0}}{}_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}}^{r^0} \models \neg\vartheta^*[\underset{\sim}{y}'/\underset{\sim}{\mathcal{F}}^{r^0}, \underset{\sim}{y}^*/\underset{\sim}{\mathcal{F}}^{r^0}]\text{”}.$$

(B) *There is* $r^1 \in \mathbb{A}\mathrm{pp}$ *such that* $q_\delta, q' \leq r^1$, $r^1 \restriction \delta \in G^* \cap (\mathbb{A}\mathrm{pp} \restriction \delta)$ *and*

$$p \Vdash_{\mathbb{C}_{\mathbf{A}^{r^1}}} \text{“}\prod^{\mathbf{A}^{r^1}}{}_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}^{r^1} \models \underset{\sim}{\vartheta}^*[\underset{\sim}{x}''/\underset{\sim}{\mathcal{F}}^{r^1}, \underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}}^{r^1}] \ \text{ and } \prod^{\mathbf{A}^{r^1}}{}_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}}^{r^1} \models \neg\underset{\sim}{\vartheta}^*[\underset{\sim}{y}''/\underset{\sim}{\mathcal{F}}^{r^1}, \underset{\sim}{y}^*/\underset{\sim}{\mathcal{F}}^{r^1}]\text{ ”}.$$

Remark: Note that $\underset{\sim}{y}^*$ is not necessarily a $\mathbb{C} \restriction (\mathbf{A}^{q_\delta} \cap (\delta+1))$-name (though $\underset{\sim}{x}_\delta$ is), this somewhat complicates the proof.

Proof. By symmetry it suffices to show that (A) implies (B). So suppose that r^0 is as in (A). By 3.10 and 3.11 below we are done. ∎

Proof. Continuation of the proof of 3.6: We define some $\mathbb{C}_\delta$-names; recall $\underset{\sim}{H}^\delta \subseteq \mathbb{C}_{\aleph_3} \restriction \delta$ is generic over $\mathbf{V}^*$, $\underset{\sim}{\mathcal{F}}_\delta[\underset{\sim}{H}^\delta] = \bigcup\{\underset{\sim}{\mathcal{F}}^{r'}[\underset{\sim}{H}^\delta] : r' \in G_\delta\}$, and

$$\underset{\sim}{M}^*_\delta = \prod_{n<\omega}^{\delta} \underset{\sim}{M}^n_\delta/\underset{\sim}{\mathcal{F}}_\delta, \quad \text{and} \quad \underset{\sim}{N}^\ell_\delta = \prod_{n<\omega}^{\delta} \underset{\sim}{N}^\ell_n/\underset{\sim}{\mathcal{F}}_\delta \quad (\text{for } \ell = 1, 2).$$

Let

$$\begin{aligned} \underset{\sim}{Z}^1_\delta[\underset{\sim}{H}^\delta] = \Big\{ & (\underset{\sim}{x}/\underset{\sim}{\mathcal{F}}_\delta, \underset{\sim}{y}/\underset{\sim}{\mathcal{F}}_\delta) \in \underset{\sim}{N}^1_\delta \times \underset{\sim}{N}^2_\delta : \text{ there are a } \tau(\underset{\sim}{t})\text{–formula } \vartheta \in \underset{\sim}{\Delta} \text{ and} \\ & \text{conditions } p \in \mathbb{C}_{\aleph_3} \text{ and } r^0 \in \mathbb{A}\mathrm{pp} \text{ such that } p^* \leq p,\ p \restriction \delta \in H^\delta, \\ & \underset{\sim}{x}, \underset{\sim}{y} \text{ are } \mathbb{C}_{\mathbf{A}^{r^0} \cap \delta}\text{-names, and } q_\delta \leq r^0,\ r^0 \restriction \delta \in G^* \cap (\mathbb{A}\mathrm{pp} \restriction \delta), \text{ and} \\ & p \Vdash_{\mathbb{C}_{\mathbf{A}^{r^0}}} \text{“} \prod^{\mathbf{A}^{r^0}}{}_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}^{r^0} \models \vartheta[\underset{\sim}{x}/\underset{\sim}{\mathcal{F}}^{r^0}, \underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}}^{r^0}] \text{ and} \\ & \prod^{\mathbf{A}^{r^0}}{}_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}}^{r^0} \models \neg\vartheta[\underset{\sim}{y}/\underset{\sim}{\mathcal{F}}^{r^0}, \underset{\sim}{y}^*/\underset{\sim}{\mathcal{F}}^{r^0}] \text{”} \Big\}, \\ \underset{\sim}{Z}^0_\delta[\underset{\sim}{H}^\delta] = {} & (\underset{\sim}{N}^1_\delta \times \underset{\sim}{N}^2_\delta) \setminus \underset{\sim}{Z}^1_\delta. \end{aligned}$$

Now, it follows from 3.7 (and 2.8) that

$(\boxdot)_\delta$ in $\mathbf{V}[G^* \cap (\mathbb{A}\mathrm{pp} \restriction \delta)][\underset{\sim}{H}^\delta]$, **if** the types realized by $(\underset{\sim}{x}'/\underset{\sim}{\mathcal{F}}_\delta, \underset{\sim}{y}'/\underset{\sim}{\mathcal{F}}_\delta)$ and $(\underset{\sim}{x}''/\underset{\sim}{\mathcal{F}}_\delta, \underset{\sim}{y}''/\underset{\sim}{\mathcal{F}}_\delta)$ over the model $\prod^{\mathbf{A}^{q_\delta} \cap \delta}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{q_\delta} \cap \delta}/\underset{\sim}{\mathcal{F}}^{q_\delta \restriction \delta}$ in the model $\prod^\delta{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{q_\delta} \cap \delta}/\underset{\sim}{\mathcal{F}}_\delta$ are equal, **then**

$$(\underset{\sim}{x}'/\underset{\sim}{\mathcal{F}}_\delta, \underset{\sim}{y}'/\underset{\sim}{\mathcal{F}}_\delta) \in \underset{\sim}{Z}^0_\delta \quad \text{if and only if} \quad (\underset{\sim}{x}''/\underset{\sim}{\mathcal{F}}_\delta, \underset{\sim}{y}''/\underset{\sim}{\mathcal{F}}_\delta) \in \underset{\sim}{Z}^0_\delta.$$

Now, most clauses of 3.6 should be clear; we say more on (d)(vii,viii), for notational simplicity for $m = 1$.

We let $\underset{\sim}{R}_{\delta,1} = \underset{\sim}{Z}^0_\delta$, so clause (d)(vii)($\circledast$)$_2$ holds.

Since $\underset{\sim}{F}$ is (an $\mathbb{A}\mathrm{pp} * \mathbb{C}_{\aleph_3}$-name for) a $\underset{\sim}{\Delta}$-embedding from $\prod_{n<\omega} \underset{\sim}{N}^1_n/\underset{\sim}{\mathcal{F}}$ into $\prod_{n<\omega} \underset{\sim}{N}^2_n/\underset{\sim}{\mathcal{F}}$, if $\underset{\sim}{x}/\underset{\sim}{\mathcal{F}}_\delta \in \underset{\sim}{N}^1_\delta$, then $\Vdash_{\mathbb{C}_\delta}$ "$(\underset{\sim}{x}/\underset{\sim}{\mathcal{F}}_\delta, \underset{\sim}{F}(\underset{\sim}{x})/\underset{\sim}{\mathcal{F}}_\delta) \in \underset{\sim}{Z}^0_\delta$". Hence clause (d)(viii)($\oplus$)$_1$ holds.

Thus the proof of 3.6 is completed. ∎

Conclusion 3.8: In $\mathbf{V}[G^*][H^{\aleph_3}]$, for each $\mathbf{m}$, there is a stationary set $S \subseteq \{\delta < \aleph_3 : \mathrm{cf}(\delta) = \aleph_2\}$ and conditions $q, q_\delta \in \mathbb{A}\mathrm{pp}$ for each $\delta \in S$ such that:

- clauses (a)$_\delta$–(d)$_\delta$ of 3.5 are satisfied,
- $q_\delta \in G^*$, $q_\delta \restriction \delta = q$, $q_\delta, \underset{\sim}{y}_\delta$ as in 3.5,
- the conclusion of 3.6 holds,
- for every $\delta_1, \delta_2 \in S$ there is a one-to-one order preserving function $h : \mathbf{A}^{q_{\delta_1}} \xrightarrow{\text{onto}} \mathbf{A}^{q_{\delta_2}}$ (so it is the identity on $\mathbf{A}^q$) which maps $\delta_1, q_{\delta_1}, \underset{\sim}{x}_{\delta_1}$, $\underset{\sim}{F}(\underset{\sim}{x}_{\delta_1}) = \underset{\sim}{y}_{\delta_1}$ onto $\delta_2, q_{\delta_2}, \underset{\sim}{x}_{\delta_2}, \underset{\sim}{F}(\underset{\sim}{x}_{\delta_2}) = \underset{\sim}{y}_{\delta_2}$,

Proof. Straightforward. ∎

We still have some debts, as 3.11,3.10 were used in the proof of 3.6

Definition 3.9: (1) Let $\circledast_{\beta,q,r,s,\underset{\sim}{f}}$ mean that

(a) $q, r, s \in \mathbb{A}\mathrm{pp}_\beta$;

(b) $q \le r$ and $q \le s$;

(c) $\mathbf{A}^r = \mathbf{A}^s$ call it $\mathbf{A}$;

(d) $\underset{\sim}{f}$ is a $\mathbb{C}_\mathbf{A}$-name of a partial (one to one) elementary mapping from $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^r$ into $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^s$ over $\prod^{\mathbf{A}_q}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^q$; i.e.

(α) $\underset{\sim}{f}$ is a subset of $\{(\underset{\sim}{a}, \underset{\sim}{b}) : \underset{\sim}{a}, \underset{\sim}{b}$ are canonical $\mathbb{C}_\mathbf{A}$-names of ω-sequences of natural numbers$\}$,

(β) if $G_\mathbf{A} \subseteq \mathbb{C}_\mathbf{A}$ is generic over $\mathbf{V}$ then in $\mathbf{V}[G_\mathbf{A}]$, the set $\{(\underset{\sim}{a}[G_\mathbf{A}], \underset{\sim}{b}[G_\mathbf{A}]) : (\underset{\sim}{a}, \underset{\sim}{b}) \in \underset{\sim}{f}\}$ is a function and

(γ) if moreover in $\mathbf{V}[G_\mathbf{A}]$ the first order formula $\varphi(x_1, \ldots, x_n)$ is in the vocabulary $\tau_{\mathbf{A}^q}$ and $(\underset{\sim}{a}_\ell, \underset{\sim}{b}_\ell) \in \underset{\sim}{f}$ for $\ell = 1, \ldots, n$ and we let $\mathcal{F}_1 = \underset{\sim}{\mathcal{F}}^r[G_\mathbf{A}]$ and $\mathcal{F}_2 = \underset{\sim}{\mathcal{F}}^s[G_\mathbf{A}]$ then

$$\prod_{n<\omega}^{\mathbf{A}} M^n_{\mathbf{A}^q}/\mathcal{F}_1 \models \varphi[(\underset{\sim}{a}_1[G_\mathbf{A}])/\mathcal{F}_1, \ldots, (\underset{\sim}{a}_n[G_\mathbf{A}])/\mathcal{F}_1]$$

iff

$$\prod_{n<\omega}^{\mathbf{A}} M^n_{\mathbf{A}^q}/\mathcal{F}_2 \models \varphi[(\underset{\sim}{b}_1[G_\mathbf{A}])/\mathcal{F}_2, \ldots, (\underset{\sim}{b}_n[G_\mathbf{A}])/\mathcal{F}_2].$$

(δ) $\underset{\sim}{f}$ include the identity may on $\prod^{\mathbf{A}^q}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\mathcal{F}^q$.

(2) Let $\circledast^+_{\beta,q,r,s,\underset{\sim}{f}}$ means that in part (1) we add: $\underset{\sim}{f}$ is an isomorphism from $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^r$ onto $\prod^{\mathbf{A}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^q}/\underset{\sim}{\mathcal{F}}^s$ (i.e. this is $\Vdash_{\mathbb{C}_{\mathbf{A}}}$).

Observation 3.10: *Assume* $\circledast_{\beta,q,r,s,\underset{\sim}{f}}$

If $\mathrm{cf}(\beta) = \aleph_2$ *or* β *is divisible by* $\aleph_2$ *and has cofinality* $\aleph_0$ *then we can find* r', s', f' *such that* $\circledast^+_{\beta,q,r',s',\underset{\sim}{f}'}$ *and* $r \le r', s \le s'$ *and* $\Vdash_{\mathbb{C}_{\mathbf{A}^{r'}}}$ "$\underset{\sim}{f} \subseteq \underset{\sim}{f}'$"

Proof. By $\aleph_1$ uses of 2.7(1) and 2.8(4) if $\mathrm{cf}(\beta) = \aleph_2$ and by $\aleph_1$ uses of 2.8(5) and 2.8(4) if $\mathrm{cf}(\beta) = \aleph_0$. ∎

Lemma 3.11: *If* $\beta_1 < \beta_2 < \aleph_3$ *are divisible by* $\aleph_2$, $q_2 \in \mathbb{A}\mathrm{pp}_{\beta_2}, q_0 = q_2 \restriction \beta_1, q_0 \le r_0 \in \mathbb{A}\mathrm{pp}_{\beta_1}, q_0 \le s_\sigma \in \mathbb{A}\mathrm{pp}_{\beta_1}, r_0 \le r_1 \in \mathbb{A}\mathrm{pp}_{\beta_2}, q_2 \le r_1 \in \mathbb{A}\mathrm{pp}_{\beta_2}$, *and* $\circledast^+_{\beta_1,q_0,r_0,s_0,\underset{\sim}{f}}$ *(see Definition 3.9)*
then we can find r_2, s_2 *and* $\underset{\sim}{f}'$ *such that:*

(i) $\circledast^+_{\beta_2,q_2,r_2,s_2,\underset{\sim}{f}'}$;
(ii) $r_1 \le r_2$;
(iii) $s \le s_2$ *and* $q_2 \le r_2$
(iv) $\Vdash_{\mathbb{C}_{\mathbf{A}^{r_2}}} \underset{\sim}{f} \subseteq \underset{\sim}{f}'$.

Proof of 3.11. Let $\underset{\sim}{f}_1 = \underset{\sim}{f} \cup (\underset{\sim}{a}, \underset{\sim}{a})$: $\underset{\sim}{a}$ is a canonical $\mathbb{C}_{\mathbf{A}^{q_2}}$-name of an w-sequence of natural numbers. It is enough to find $s' \in \mathbb{A}\mathrm{pp}$ such that letting $r' = r_1$ we have $\mathbf{A}^{s'} = \mathbf{A}^{r'}, p \le s', s_0 \le s'$ and $\circledast_{\beta_2,q_2,r',s',\underset{\sim}{f}_1}$ (i.e., without the $+$), this is enough by observation 3.10. This weaker statement we prove for every $\beta_1 < \beta_2$ (not necessarily divisible by $\aleph_2$). We prove this by induction on β_2.

Case 1: $\beta_2 = 0$.

Empty.

Case 2: $\beta_2 = \beta_1 + \aleph_2$ and $\mathrm{cf}(\beta_1) < \aleph_2$.

Let $\underset{\sim}{f} = \{(\underset{\sim}{a}_\epsilon, \underset{\sim}{b}_\epsilon) : \epsilon < \epsilon^*\}$. So it suffices to find a $\mathbb{C}_{\mathbf{A}^{r'}}$-name of an ultrafilter which is forced to include the following families (recall that number of $\prod^{\mathbf{A}^{q_2}}{}_{n<m} M^n{}_{\mathbf{A}^{q_2}}$ can be represented as individual constants.)

(a) $\mathcal{F}^{q_2}$;
(b) $\mathcal{F}^s$;
(c) the sets of the form $\{n : \underset{\sim}{M}^n_{\mathbf{A}^{r'}} \models \underset{\sim}{\varphi}(\underset{\sim}{b}_{\epsilon_0}(n), \ldots, \underset{\sim}{b}_{\epsilon_{k-1}}(n))\}$: where $\epsilon_0, \ldots, \epsilon_{k-1} < \epsilon^*$ and $\underset{\sim}{\varphi}(x_0, \ldots x_{k-1})$ is a $\mathbb{C}_{\mathbf{A}^{q_2}}$- name of a first order formula

in the vocabulary $\tau_{\mathbf{A}^{q_2}}$ such that

$$\prod_{n<\omega}^{\mathbf{A}^{r_1}} \underset{\sim}{M}^n_{\mathbf{A}^{q_2}}/\underset{\sim}{\mathcal{F}}^{r_1} \models \underset{\sim}{\varphi}[\underset{\sim}{a}_{\epsilon_0}/\underset{\sim}{\mathcal{F}}^{r_1}, \ldots, \underset{\sim}{a}_{\epsilon_{k-1}}/\underset{\sim}{\mathcal{F}}^{r_1}].$$

So it suffices to prove that any finite intersection is not empty, but each of those families is closed under finite intersection, hence it suffices to prove the following

⊛ $p \nVdash_{\mathbb{C}_{\mathbf{A}^{r'}}}$ "$\underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} \cap \underset{\sim}{\mathbf{c}} = \emptyset$" when
- (a) $p \in \mathbb{C}_{\mathbf{A}^r}$;
- (b) $\underset{\sim}{\mathbf{a}}$ is a $\mathbb{C}_{\mathbf{A}^{q_2}}$- name such that $p \restriction \mathbf{A}^{q_2} \Vdash$ "$\underset{\sim}{\mathbf{a}} \in \underset{\sim}{\mathcal{F}}^{q_2}$";
- (c) $\underset{\sim}{\mathbf{b}}$ is a $\mathbb{C}_{\mathbf{A}^s}$-name such that $p \restriction \mathbf{A}^s \Vdash$ "$\underset{\sim}{\mathbf{b}} \in \underset{\sim}{\mathcal{F}}^s$";
- (d) $\underset{\sim}{\mathbf{c}} = \{n : \underset{\sim}{M}^n_{\mathbf{A}^{r_1}} \models \underset{\sim}{\varphi}[\underset{\sim}{b}_{\epsilon_0}(n), \ldots, \underset{\sim}{b}_{\epsilon_{k-1}}(n)]\}$ where $\underset{\sim}{\varphi}$ is a $\mathbb{C}_{\mathbf{A}^{q_2}}$-name of a first order formula in the vocabulary $\tau_{\mathbf{A}^{q_2}}$, without loss of generality a predicate as an atomic formula, such that $p \Vdash_{\mathbb{C}_{\mathbf{A}^{r_1}}}$ "$\prod_{n<\omega}^{\mathbf{A}^{r_1}} \underset{\sim}{M}^n_{\mathbf{A}^{q_2}}/\underset{\sim}{\mathcal{F}}^{r_1} \models \varphi[\underset{\sim}{a}_{\epsilon_0}/\underset{\sim}{\mathcal{F}}^{r_1}, \ldots, \underset{\sim}{a}_{\epsilon_{k-1}}/\underset{\sim}{\mathcal{F}}^{r_1}]$".

Without loss of generality p forces that $\underset{\sim}{\varphi} = Q_{\langle \underset{\sim}{R}^2_n : n<\omega\rangle}$. Let $H \subseteq \mathbb{C}_{\mathbf{A}^{q_0}}$ be generic over $\mathbf{V}$ such that $p \restriction \mathbf{A}^{q_0} \in H$. In $\mathbf{V}[H]$ for each n we define a k-place relation R^0_n on ω $R^0_n = \{(m_0, \ldots, m_{k-1})$: there is p', $p \restriction \mathbf{A}^{q_2} \le p' \in \mathbb{C}_{\mathbf{A}^{q_2}}$, $p' \restriction \mathbf{A}^{q_0} \in H$ such that $p' \Vdash n \in \underset{\sim}{\mathbf{a}}$ and $\langle m_0, \ldots, m_{k-1}\rangle \in \underset{\sim}{R}^2_n\}$

Now

$(*)_0$ $p \Vdash_{\mathbb{C}^{r_1}} \prod^{\mathbf{A}^{r_1}}{}_{n<\omega} M^n_{\mathbf{A}^{q_2}}/\underset{\sim}{\mathcal{F}}^{r_1} \models \varphi[\underset{\sim}{a}_{\epsilon_0}/\underset{\sim}{\mathcal{F}}^{r_1}, \ldots, \underset{\sim}{a}_{\varepsilon_{k-1}}/\underset{\sim}{\mathcal{F}}^r]$
hence

$(*)_1$ $p \Vdash_{\mathbb{C}^{r_1}}$ "$\langle \underset{\sim}{a}_{\epsilon_0}/\underset{\sim}{\mathcal{F}}^{r_1}, \ldots, \underset{\sim}{a}_{\epsilon_{k-1}}/\underset{\sim}{\mathcal{F}}^{r_1}\rangle \in Q_{\langle \underset{\sim}{R}^2_n : n<\omega\rangle}$"
hence

$(*)_2$ $p \restriction \mathbf{A}^{r_1} \Vdash \langle \underset{\sim}{a}_{\epsilon_0}/\underset{\sim}{\mathcal{F}}^{r_1}, \ldots, \underset{\sim}{a}_{\epsilon_{k-1}}/\underset{\sim}{\mathcal{F}}^{r_1}\rangle \in Q_{\langle \underset{\sim}{R}^0_n : n<\omega\rangle}$"
hence

$(*)_3$ $p \restriction \mathbf{A}^{r_0} \Vdash$
"$\langle \underset{\sim}{b}_{\epsilon_0}/\underset{\sim}{\mathcal{F}}^{s_0}, \ldots, \underset{\sim}{b}_{\epsilon_{k-1}}/\underset{\sim}{\mathcal{F}}^{s_0}\rangle$ satisfies $Q_{\langle R^0_n : n<\omega\rangle}$ in $\prod^{\mathbf{A}^{s_0}}{}_{n<\omega} \underset{\sim}{M}^n_{\mathbf{A}^{q_0}}/\underset{\sim}{\mathcal{F}}^{s_0}$"
so

$(*)_4$ in $\mathbf{V}[H]$, we have $\mathbf{b}' \in \underset{\sim}{\mathcal{F}}^{\mathbf{q_0}}$ where $\mathbf{b}' = \{n :$ for some $p', p \restriction \mathbf{A}^{s_0} \le p' \in \mathbb{C}_{\mathbf{A}^{s_0}}$ and $p' \restriction \mathbf{A}^{q_0} \in H$ and $p' \Vdash_{\mathbb{C}_{\mathbf{A}^{s_0}}}$ "$n \in \underset{\sim}{\mathbf{b}}$ and $\langle \underset{\sim}{b}_{\epsilon_0}(n), \ldots, \underset{\sim}{b}_{\epsilon_{k-1}}(n)\rangle \in \underset{\sim}{R}^0_n[H]$"$\}$.

So clearly $\mathbf{b}'$ is a non-empty set of natural numbers, so choose $n \in \mathbf{b}'$. So there is $p_1 \in \mathbb{C}_{\mathbf{A}^s}, p \restriction \mathbf{A}^s \le p_1, p_1 \restriction \mathbf{A}^{q_0} \in H$, $p_1 \Vdash$ "$n \in \underset{\sim}{\mathbf{b}}$ and $\langle \underset{\sim}{b}_{\epsilon_0}(n), \ldots \underset{\sim}{b}_{\epsilon_{k-1}}(n)\rangle \in \underset{\sim}{R}^0_n[H]$". Without loss of generality p_1 forces values to $\underset{\sim}{b}_{\epsilon_0}(n), \ldots, \underset{\sim}{b}_{\epsilon_{k-1}}(n)$, call them $m_0, \dots, m_{k-1}$. So $\langle m_0, \dots, m_{k-1} \rangle \in \underset{\sim}{R}^0_n[H]$, hence by its definition there is p_2 such that $p \restriction \mathbf{A}^{q_2} \le p_2 \in \mathbb{C}_{\mathbf{A}^{q_2}}$, $p_2 \Vdash$ "$n \in \underset{\sim}{\mathbf{a}}$ and $\langle m_0, \dots, m_{k-1} \rangle \in \underset{\sim}{R}^2_n$".

Now $p^* =: p_1 \cup p_2 \in \mathbb{C}_{\mathbf{A}^{r'}}$ is above p, $p^* \restriction \mathbf{A}^{q_0} \in H$, and it forces that $n \in \underset{\sim}{\mathbf{a}} \cap \underset{\sim}{\mathbf{b}} \cap \underset{\sim}{\mathbf{c}}$, which is enough.

Case 3: $\beta_2 = \beta + 1, \beta \ne \beta_1$.

First by the induction hypotheses we can find $r', s', \underset{\sim}{f}'$ such that

$\boxdot$ (a) $r_0 \le r' \in \mathbb{A}\mathrm{pp} \restriction \beta$,

(b) $s_0 \le s' \in \mathbb{A}\mathrm{pp} \restriction \beta$,

(c) $q_2 \restriction \beta_1 \le s'$,

(d) $\circledast^+_{\beta_1, q_2 \restriction \beta, r', s', \underset{\sim}{f}'}$.

Now we continue as in case 2, noting: if $cp(\beta) = \aleph_2$ then the $\underset{\sim}{\Gamma}^{q_2}_{\beta_1}$-bigness of $\underset{\sim}{x}_{\beta_1}$ is automatic.

Case 4: β_2 is a limit ordinal.

Let $\langle \gamma_\epsilon : \epsilon < \mathrm{cf}(\beta_2) \rangle$ be increasing continuous with limit β_2 such that $\gamma_0 = \beta_1, \mathrm{cf}(\gamma_\epsilon) < \aleph_2$ and stipulate $\gamma_{\mathrm{cf}(\beta_2)} = \beta_2$. We choose $(r'_\epsilon, s'_\epsilon, \underset{\sim}{f}_\epsilon)$ by induction on $\epsilon \le \mathrm{cf}(\beta_2)$ such that

$\boxtimes$ (a) $\circledast_{\beta_{\gamma_\epsilon}, q_2 \restriction \gamma_\epsilon, r'_\epsilon, s'_\epsilon, \underset{\sim}{f}_\epsilon}$ holds;

(b) $(r'_0, s'_0, \underset{\sim}{f}_0) = (r_0, s_0, \underset{\sim}{f})$;

(c) $r'_1 \restriction \gamma_\epsilon \le r'_\epsilon$;

(d) if $\zeta < \epsilon$ then $r'_\zeta \le r'_\epsilon, s'_\zeta \le s'_\epsilon$ and $\Vdash_{\mathbb{C}_{\mathbf{A}^{r'_\epsilon}}} \underset{\sim}{f}_\zeta \subseteq \underset{\sim}{f}_\epsilon$.

Clearly if we succeed we are done with case 4.

For $\epsilon = 0$ this is trivial.

For $\epsilon = \zeta + 1$ first find $r''_\zeta \in \mathbb{A}\mathrm{pp}_{\gamma_\epsilon}$ such that $r_1 \restriction \gamma_\epsilon \le r''_\zeta$ and $r''_\zeta \restriction \gamma_\zeta = r'_\zeta$, possibly by 2.8(3). Second apply the induction hypothesis with $(\gamma_\zeta, \gamma_\epsilon, q_2 \restriction \beta_\zeta, q_2 \restriction \beta_\epsilon, r_\zeta, s'_\zeta, r''_\zeta, \underset{\sim}{f}_\zeta, \underset{\sim}{f}_\epsilon)$ standing for $(\beta_0, \beta_2, q_0, q_2, r_0, s_0, r_1, \underset{\sim}{f}, \underset{\sim}{f}')$.

For ϵ limit of uncountable cofinality take the union (see 2.8(4)).

For ϵ limit of countable cofinality, we first repeat the argument in case 2. Then use 2.8 and then 3.10. ∎

4. Back to Model Theory

In this section we present just enough to solve the problem on finite fields.

Definition 4.1: Let M be a model. Assume $N_1 = M^{[\bar\varphi^1]}$, $N_2 = M^{[\bar\varphi^2]}$ are models of t_0 interpreted in M by the sequences $\bar\varphi^1, \bar\varphi^2$ of formulas with parameters from M, and they have the same vocabulary $\tau^* = \tau(N_1) = \tau(N_2)$. Furthermore, let Γ be an invariant bigness notion in M (over some set A_0 of $< \kappa$ parameters, more exactly in $\mathcal{K}_{(M,A_0)}$), and $\Delta \subseteq \mathbb{L}_{\omega,\omega}(\tau(N_1))$ and $\kappa > \aleph_0$ (for simplicity) and for a formula $\vartheta(\bar x) \in \Delta$ let $\vartheta_{\bar\varphi^\ell}(\bar x)$ be the result of substituting $\bar\varphi^\ell$ in ϑ so $N^\ell \models \vartheta[\bar a]$ iff $\bar a \in {}^{\lg \bar x}(N^\ell)$ and $M \models \vartheta_{\bar\varphi^\ell}[\bar a]$.

(1) We say that (N_1, N_2) **is** (κ, Γ, Δ)**-complicated in** M when:
for every Δ-embedding F of N_1 into N_2, and for every Γ-big type $\mathfrak{p}_0(x)$ inside M of cardinality $< \kappa$ such that $\mathfrak{p}_0(M) \subseteq N_1$, there is a Γ-big type $\mathfrak{p}_1(x)$ inside M of cardinality $< \kappa$ which includes $\mathfrak{p}_0(x)$ and such that, letting $\tau(\mathfrak{p}_1) \subseteq \tau(M)$ consist of those predicates and function symbols mentioned in $\mathfrak{p}_1(x)$ (so $|\tau(\mathfrak{p}_1)| < \kappa$) and $A \subseteq M$ which is the set of parameters of $\mathfrak{p}_0$ union with A_0 so $|A| < \kappa$ and $A_0 \subseteq A$, we have
$(*)_{\mathfrak{p}_1(x)}$ letting

$$R_m \overset{\text{def}}{=} \{(\bar a, \bar b) : \quad \bar a \in {}^m(N_1),\ \bar b \in {}^m(N_2) \text{ and for some } \bar c \in {}^m(N_1) \text{ we have } \operatorname{tp}_{\mathbb{L}_{\omega,\omega}(\tau(\mathfrak{p}_1))}(\bar a {}^\frown \bar b, A, M) = \operatorname{tp}_{\mathbb{L}_{\omega,\omega}(\tau(\mathfrak{p}_1))}(\bar c {}^\frown F(\bar c), A, M)\ \}$$

the parallel of 3.6(vii)+(viii) holds, so

$(\oplus)_1$ if $\bar a_1, \bar a_2$ are finite sequences of the same length m of members of N_1, and $\mathfrak{p}_1 \cup \{\vartheta^{N_1}_{\bar\varphi^1}(x, \bar a_1), \neg\vartheta^{N_1}_{\bar\varphi^1}(x, \bar a_2)\}$ is a Γ-big type over M, and $\vartheta, \neg\vartheta \in \Delta$, then $(\bar a_1, F(\bar a_2)) \notin R_m$.

$(\oplus)_2$ Moreover, in $\oplus_1$ we can replace $\vartheta, \neg\vartheta$ by any pair ϑ_0, ϑ_1 of contradictory formulas from Δ.

(2) In part (1):
  (i) We do not mention Δ if it is the set of quantifier free formulas (of $\mathbb{L}_{\omega,\omega}(\tau(N_1))$).
  (ii) We replace Γ by (t, ψ) if we mean "for all bigness notions of the form $\Gamma = \Gamma_{(t,\psi,\bar\varphi)}$, where $\bar\varphi$ is an interpretation of t in M with $< \kappa$ parameters and $|t| < \kappa$, $\psi \in \mathbb{L}_{\kappa,\omega}$" (i.e., $\psi \in \mathbb{L}_{\mu^+,\omega}$ for some $\mu < \kappa$ and in the vocabulary $\tau(t) \cup \{P^*\}$).
  (iii) We omit Γ if we mean "for all Γ's as in (ii)".
  (iv) We say M **is** κ**-complicated** (or: (κ, Γ, Δ)**-complicated**) and omit N_1, N_2 if this holds for all N_1, N_2 as in our assumptions, but with $|\tau(N_1)| < \kappa$.

Remark 4.2: More on the relation R_n etc., see [Sh800].

THEOREM 4.3: *Let* $\mathbf{G}$ *be a full* $(\aleph_3, \aleph_2)$*-bigness guide (see 2.2; recall there is one by 2.3). Assume that* $G \subseteq \mathbb{A}\mathrm{pp}_{\mathbf{G}}$ *is generic over* $\mathbf{V}$ *and* $H \subseteq \mathbb{C}_{\aleph_3}$ *is generic over* $\mathbf{V}[G]$ *and* $\mathcal{F} = \underset{\sim}{\mathcal{F}}_{\aleph_3}[G][H]$*, and let* $\langle M_n = M^n_{\aleph_3} : n < \omega\rangle$ *be a sequence of models as in 2.1(4), that is each with a countable universe being the set of natural numbers for simplicity, all with the same vocabulary such that for every* k *and a sequence* $\langle R_n : n < \omega\rangle$ *with* R_n *being a* k*-place relation on* M_n *there is a* k*-place predicate in the common vocabulary satisfying* $R^{M_n} = R_n$ *for each* n*. Then*

(1) *in* $\mathbf{V}[G][H]$ *the model* $M = \prod_{n<\omega} M^n_{\aleph_3}/\mathcal{F}$ *is* $\aleph_2$*-complicated and* $\aleph_2$*-compact.*
(2) *We can change the demands on* $\mathbf{G}$ *accordingly to the version of* $\aleph_2$*-complicated we actually used (e.g., not all* Γ*-s, etc.), (so we are using a different* $\mathbf{G}$*).*
(3) *If* N^1, N^2 *are models of* t^{ind}_1 *(which is defined in Definition 1.5), interpreted in* M*, then any isomorphism* π *from* N^1 *onto* N^2 *is definable in* M*.*
(4) *If* $N^\ell = \prod_{n<\omega} N^\ell_n/\mathcal{F}$*, each* N^ℓ_n *is countable, and* N^ℓ *is a model of* t^{ind}_1 *(for* $\ell = 1, 2$*), and* π *is an isomorphism from* N^1 *onto* N^2*, then there are* $A \in \mathcal{F}$ *and isomorphisms* π_n *from* N^1_n *onto* N^2_n *(for* $n \in A$*) such that* $\pi = \prod_{n<\omega} \pi_n/\mathcal{F}$*.*
(5) *Above we may replace : "*N^ℓ *is a model of* t^{ind}_1*" by "some formula* $\phi(x, y)$ *in the vocabulary of* N^1 *which is equal to that of* N^2*, has the strong independence property" (in their common theory* [10]*, see Definition 1.5 on the strong independence property).*
(6) *If* N^ℓ_n *are finite fields (for* $\ell = 1, 2$ *and* $n < \omega$*), and* $\prod_{n<\omega} N^1_n/\mathcal{F}$ *is isomorphic to* $\prod_{n<\omega} N^2_n/\mathcal{F}$*, then the set* $\{n < \omega : N^1_n \cong N^2_n\}$ *belongs to* $\mathcal{F}$*.*

Proof. (1) By 3.8.

(2) The same proof.

(3) By 4.4 below and 1.6(2).

(4) Without loss of generality, the universe of N^ℓ_n is $\alpha^\ell_n \leq \omega$. Now, for $\ell = 1, 2$, we can find $P_\ell \in \tau_M$ such that $(P_\ell)^{M^n_{\aleph_3}} = |N^\ell_n|$ and for $Q_\ell \in \tau(N^\ell_n)$ there is $Q_\ell \in \tau_M$ with $(Q_\ell)^{M^n_{\aleph_3}} = Q^{N^\ell_n}$ and $R_\ell \in \tau_M$ with $(R_\ell)^{M^n_{\aleph_3}} = R^{N^\ell_n}$.

[10] Of course if the strong independence property holds when we restrict ourselves to say a predicate P we get less, but see [Sh800]

Therefore, $N^\ell = \prod_{n<\omega} N^\ell_n/\mathcal{F}$ can be viewed as an interpretation in M by $\bar{\varphi}^\ell$. Now apply part (3) for $\Gamma = \Gamma_{(t_1^{\rm ind},\psi^{\rm ind},\bar{\varphi}^1)}$.

(5) This follows by part (4), as the vocabulary is finite, being an isomorphism is expressibly a first order sentence.

(6) This is a particular case of part (4). Of course without loss of generality the fields N^1, N^2 are infinite. By part (5) it suffices for infinite ultraproducts N^ℓ of finite fields to find a formula $\vartheta(x,y)$ in the vocabulary of fields which has the strong independence property see Definition 1.5. First we deal with the case that the fields are of characteristic > 2. Consider the formula $\vartheta(x,y)$ saying that $x+y$ has a square root in the field.

We rely on a theorem of Duret, [Du80, p. 982, Lemma 10], for the value $p = 2$ the hypothesis of this lemma holds as the field contains all pth roots of the unit (that is $1, -1$). The conclusion says that for n and any pairwise distinct elements $a_1, \dots, a_n, b_1, \dots, b_n$ of the field there is an element c such that $a_m + c$ has a square root and $b_m + c$ does not have a square root for $m = 1, \dots, n$. So the formula $\vartheta_p(x,y) = (\exists z)(z^p = x + y)$ is as required.

Of course, if the characteristic of the field is 2, then we naturally use the same theorem only choosing $p = 3$, so, of course, the field may perhaps fail to have all the pth roots of the unit, however, as Duret does, in this case we consider an algebraic extension of N^ℓ of order 3 by adding a root of $x^3 - 1$ hence all of them getting a new field N^ℓ_*. Now the set of elements of N^ℓ_* can be represented as the set of triples of elements of N^ℓ, and the operations of N^ℓ_* are definable in N^ℓ; so our problem is almost notational. E.g., we can note that recalling $N^\ell = \prod_{n<\omega} N^\ell_n/\mathcal{F}$ then $N^\ell_* = \prod_{n<\omega} N^\ell_{*,n}/\mathcal{F}$ where $N^\ell_{*,n}$ is equal to N^ℓ_n if N^ℓ_n has three 3rd roots of the unit and an algebraic extension of N^ℓ_n of order three which has this property otherwise. Again the first order theory of N^ℓ_* has the strong independence property and for N^1_*, N^2_* (by asking on the existence of cubic roots) we get the desired conclusion; but any isomorphism from N^1 onto N^2 can be extended to an isomorphism from N^1_* onto N^2_* and we can easily finish. (We could have used the "strong independence property for m-types".) ∎

Proposition 4.4: *Assume that M is a κ-complicated κ-compact model. Let N_1, N_2 be interpretations of $t_1^{\rm ind}$ in M. Then for any isomorphism π from N_1 onto N_2, the function π is definable in M by a first order formula (with parameters).*

Proof. Let $N_\ell = M^{[\bar\varphi^\ell]}$ (so $\bar\varphi^\ell$ has parameters in M) for $\ell = 1, 2$ and let F be an isomorphism from N_1 onto N_2.

Let Γ be the bigness notion $\Gamma_{(t^{\rm ind}, \psi^{\rm ind}, \bar\varphi^1,)}$ (so $\psi^{\rm ind} \in \mathbb{L}_{\omega_1,\omega}$). Let $\mathfrak{p}_0(x)$ be the type just saying $x \in Q^{N_1}$, and let $\mathfrak{p}_1$ be the type guaranteed to exists in Definition 4.1(1), without loss of generality closed under conjunctions. Let $A \subseteq M$, $|A| < \kappa$ and $\tau^* \subseteq \tau_M$, $|\tau^*| < \kappa$ be given by the definition of being κ-complicated (applied to F). (Without loss of generality, A includes the parameters of $\bar\varphi^1, \bar\varphi^2$ and is closed under F and F^{-1}, and for every n and for every formula $\varphi(x) \in \mathfrak{p}_1, A$ includes the finite set mentioned in 1.5(2).)

Let R_1 be as in 4.1(1). Clearly, recalling Definition 1.5(2), there are no distinct $a_1, a_2 \in P^{N_1} \setminus A$ and $b \in N_2$ such that $(a_1, b), (a_2, b) \in R_1$, but $a \in P^{N_1} \Rightarrow (a, F(a)) \in R_1$. Hence

$$\{(b,a) : (a,b) \in R_1 \text{ and } a \in P^{N_1}\}$$

is the graph of a partial function from P^{N_2} into P^{N_1} which includes the graph of $F^{-1} \restriction P^{N_2}$. But F is one-to-one and onto. Therefore, $R_1 \restriction (P^{N_1} \times P^{N_2})$ is the graph of $F \restriction P^{N_1}$. But $R_1 \restriction P^{N_1}$ is definable in $(M \restriction \tau^*, c)_{c \in A}$ by a formula from $\mathbb{L}_{\infty,\kappa}$, so also $F \restriction P^{N_1}$ is, and thus if N_1, N_2 are models of $t_1^{\rm ind}$ also F is (by 1.6). Applying [Sh72, 1.9] (or [Sh:e, Ch XI]) we conclude that it is definable by a first order formula with parameters from M, as required. ∎

Similarly we can show the following.

PROPOSITION 4.5: *Assume that Γ is a $(\aleph_2, \aleph_1)$–(P, ϑ)-separative bigness notion, see Definition 1.4. Suppose that N_1, N_2 are interpretations of t in M, and M is κ-compact κ-complicated (or just κ-complicated for Γ), $\kappa > \aleph_0$.*

(a) *If F is an isomorphism from N_1 onto N_2, then*
 $(*)_1$ *$F \restriction P^{N_1}$ is definable in $(M \restriction \tau^*, c)_{c \in A}$ by a formula from $\mathbb{L}_{\infty,\kappa}$, recalling $\tau \subseteq \tau_M$, $|\tau| < \kappa$, $A \subseteq M$, $|A| < \kappa$.*

(b) *If F is an embedding of N_1 into N_2, then*
 $(*)_2$ *there is a partial function f from P^{N_2} into P^{N_1} which extends F^{-1} and is definable in $(M \restriction \tau^*, c)_{c \in A}$ by a formula from $\mathbb{L}_{\infty,\kappa}$, where τ^*, A are as above.*

Remark 4.6: (1) The proposition 4.5 should be the beginning of an analysis of first order theories T. For more in this direction see [Sh503], [Sh800].

(2) As stated in the introduction, we may avoid the preliminary forcing with $\mathbb{A}$pp and construct the name $\underset{\sim}{\mathcal{F}}$ in the ground model $\mathbf{V}$, provided $\mathbf{V}$ is somewhat $\mathbf{L}$-like. Assuming $\diamondsuit_{\{\delta<\aleph_3:\mathrm{cf}(\delta)=\omega_2\}}$ is enough, but we may also use the weaker principle from [HLSh162] and [Sh405, Appendix].

(3) We may vary the cardinals, e.g., we may replace $\aleph_2, \aleph_3$ by κ, λ, respectively, provided $\lambda = \kappa^+$, $\kappa = \kappa^{<\kappa}$ (so an approximation has size $< \kappa$).

Moreover we can replace $\aleph_0$ by $\theta = \theta^{<\theta}$, so in full let us assume that

$$\theta = \theta^{<\theta} < \kappa = \kappa^{<\kappa} < \lambda = \kappa^+.$$

(a) For $\mathbf{A} \subseteq \lambda$ let $\mathbb{C}(\mathbf{A}) = \mathbb{C}_\mathbf{A} = \{p : p$ is a partial function from $\mathrm{Dom}(p) \in [\mathbf{A}]^{<\theta}$ to ${}^{\theta>}2$ $\}$ ordered by

$$p_1 \leq_{\mathbb{C}_\mathbf{A}} p_2 \quad \text{iff} \quad \mathrm{Dom}(p_1) \subseteq \mathrm{Dom}(p_2) \quad \text{and} \quad (\forall \alpha \in \mathrm{Dom}(p_1))(p_1(\alpha) \trianglelefteq p_2(\alpha)).$$

(b) We define $\mathbb{A}\mathrm{pp}_{\bar{\mathbf{G}}}^-$ as the set of $q = (\mathbf{A}^q, \underset{\sim}{\mathcal{F}}^q)$ where $\mathbf{A}^q \in [\lambda]^{<\kappa}$ and $\underset{\sim}{\mathcal{F}}^q$ is a $\mathbb{C}_{\mathbf{A}^q}$-name of a regular ultrafilter on θ such that for each $\alpha < \lambda$, $\underset{\sim}{\mathcal{F}}^q \cap \mathcal{P}(\theta)^{\mathbf{V}^{\mathbb{C}(\mathbf{A}^q \cap \alpha)}}$ is a $\mathbb{C}_{\mathbf{A}^q \cap \alpha}$–name.

(c) For $\alpha \in \mathbf{A} \in [\lambda]^{<\kappa}$, $\underset{\sim}{x}_\alpha$ is the $\mathbb{C}_\mathbf{A}$–name $\bigcup\{p(\alpha) : p \in \underset{\sim}{G}_{\mathbb{C}(\mathbf{A})}\}$ of a member of ${}^\theta\theta$.

(d) We define $\underset{\sim}{M}^\varepsilon_\mathbf{A}$ for $\varepsilon < \theta$, $\mathbf{A} \in [\lambda]^{<\kappa}$ as the following $\mathbb{C}_\mathbf{A}$-name:
it is a model with universe θ,
$\tau_{\underset{\sim}{M}^\varepsilon_\mathbf{A}} = \{P_{\bar{\underset{\sim}{R}}} : \bar{\underset{\sim}{R}} = \langle \underset{\sim}{R}_\varepsilon : \varepsilon < \theta\rangle$, for some m each $\bar{\underset{\sim}{R}}_\varepsilon$ is a $\mathbb{C}_\mathbf{A}$- name of an m–place relation on $\theta\}$,
$(P_{\bar{\underset{\sim}{R}}})^{\underset{\sim}{M}^\varepsilon_\mathbf{A}} = \underset{\sim}{R}_\varepsilon$.
So we may think of $\tau_{\underset{\sim}{M}^\varepsilon_\mathbf{A}}$ to be an old object whose members are indexed as $P_{\bar{\underset{\sim}{R}}}$, where each $\underset{\sim}{R}_\varepsilon$ is a $\mathbb{C}_\mathbf{A}$–name. Or we can consider $\tau_{\underset{\sim}{M}^\varepsilon_\mathbf{A}}$ to be a name and interpret it in $\mathbf{V}[G_{\mathbb{C}(\mathbf{A})}]$.

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