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Successors of singular cardinals and coloring theorems I

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Abstract. We investigate the existence of strong colorings on successors of singular cardinals. This work continues Section 2 of [1], but now our emphasis is on finding colorings of pairs of ordinals, rather than colorings of finite sets of ordinals.

1. Introduction

The theme of this paper is that strong coloring theorems hold at successors of singular cardinals of uncountable cofinality, except possibly in the case where the singular cardinal is a limit of regular cardinals that are Jonsson in a strong sense.

Our general framework is that $\lambda = \mu^+$, where μ is singular of uncountable cofinality. We will be searching for colorings of pairs of ordinals $< \lambda$ that exhibit quite complicated behaviour. The following definition (taken from [2]) explains what "complicated" means in the previous sentence.

Definition 1.1. Let λ be an infinite cardinal, and suppose $\kappa + \theta \leq \mu \leq \lambda$. $\mathrm{Pr}_1(\lambda, \mu, \kappa, \theta)$ means that there is a symmetric two–place function c from λ to κ such that if $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $< \lambda$ with all $\alpha_{i,\zeta}$'s distinct, then for every $\gamma < \kappa$ there are $i < j < \mu$ such that

$$\zeta_1 < \xi \text{ and } \zeta_2 < \xi \Longrightarrow c(\alpha_{i,\zeta_1}, \alpha_{i,\zeta_2}) = \gamma. \tag{1.1}$$

Just as in [1], one of our main tools is a game that measures how "Jonsson" a given cardinal is.

Recall that a cardinal λ is a Jonsson cardinal if for every $c : [\lambda]^{<\omega} \to \lambda$, we can find a subset $I \subseteq \lambda$ of cardinality λ such that the range of $c \restriction I$ is a proper subset of λ. A reader seeking more background should investigate [4] and [3] in [5].

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Definition 1.2. Assume $\mu \leq \lambda$ are cardinals, γ is an ordinal, $n \leq \omega$, and J is an ideal on λ. We define the game $\mathrm{Gm}^n_J[\lambda, \mu, \gamma]$ as follows:

A play lasts γ moves.

In the α^{th} move, the first player chooses a function $F_\alpha : [\lambda]^{<n} \to \mu$, and the second player responds by choosing (if possible) a subset $A_\alpha \subseteq \lambda$ such that

- $A_\alpha \subseteq \bigcap_{\beta<\alpha} A_\beta$
- $A_\alpha \in J^+$
- $\mathrm{ran}(F_\alpha \restriction [A_\alpha]^{<n})$ is a proper subset of μ.

The second player loses if he has no legal move for some $\alpha < \gamma$, and he wins otherwise.

In the previous definition, if $J = J^{\mathrm{bd}}_\lambda$ then we may omit it. Note that it causes no harm if we use a set E of cardinality λ instead of λ itself; in this case, we write $\mathrm{Gm}^n_J[E, \mu, \gamma]$.

Note that λ is a Jonsson cardinal if and only if Player I does not have a winning strategy in the game $\mathrm{Gm}^\omega[\lambda, \lambda, 1]$. One may view the lack of a winning strategy for Player I in games of longer length as a strong version of Jonsson-ness or a weak version of measurability — if λ is measurable, then Player II can make sure her moves are elements of some λ–complete ultrafilter.

The following claim investigates how the existence of winning strategies is affected by modifications to the game; the proof is left to the reader.

Claim 1.3.

(1) If $\mu' \leq \mu$ and the first player has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu, \gamma]$, then she has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu', \gamma]$.

(2) Suppose we weaken the demand on the second player to

$$\text{“}(\exists \zeta < \lambda)[\mathrm{ran}(F_\alpha \restriction [A_\alpha \setminus \zeta]^{<n}) \text{is a proper subset of } \mu]\text{.”} \tag{1.2}$$

If $\mathrm{cf}(\lambda) \geq \gamma$ and $J \supseteq J^{\mathrm{bd}}_\lambda$, then the first player has a winning strategy in the revised game if and only if she has a winning strategy in the original game.

(3) If J is γ–complete, then the same applies to the case where we weaken the demand on the second player to

$$\text{“}(\exists Y \in J)[\mathrm{ran}(F_\alpha \restriction [A_\alpha \setminus Y]^{<n}) \text{ is a proper subset of } \mu]\text{.”} \tag{1.3}$$

(4) We can allow the second player to pass, i.e., to let $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$ (even if this is not a legal move) as long as we declare that the second player loses if the order–type of the set of moves where he did not pass is $< \gamma$.

(5) If Player I has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu, \gamma]$ for every $\mu < \mu^*$ where μ^* is singular and $\gamma > \mathrm{cf}(\mu^*)$ is regular, then Player I has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu^*, \gamma]$. We can weaken the requirement that γ is regular and instead require that $\mathrm{cf}(\gamma) > \mathrm{cf}(\mu^*)$ and $\omega^\gamma = \gamma$.

In Section 2 of [1], the existence of winning strategies for Player I in variants of the game is investigated. We will prove one such result here; the reader should look in [1] for others.

Claim 1.4. If $2^\chi < \lambda < \beth_{(2^\chi)^+}(\chi)$ then Player I has a winning strategy in $\mathrm{Gm}^\omega[\lambda, \chi, (2^\chi)^+]$.

Proof. At a stage i, Player I will select a function $F_i : [\lambda]^{<\omega} \to \chi$ coding the Skolem functions of some model M_i.

For the initial move, we let the model M_0 have universe λ, and include in our language all relations on λ and all functions from λ to λ of any finite arity that are first order definable in the structure $\langle H(\lambda^+), \in, <^*_{\lambda^+}\rangle$ with the parameters χ and λ.

For subsequent moves, M_i is an expansion of M_0 with universe λ that has all relations on λ and all functions from λ to λ of any finite arity that are first order definable in the structure $\langle H(\lambda^+), \in, <^*_{\lambda^+}\rangle$ from the parameters χ, λ, M_0, and $\langle A_j : j < i\rangle$.

To obtain the function F_i, we let $\langle F^i_n : n < \omega\rangle$ list the Skolem functions of M_i in such a way that F^i_n has $m_i(n) \le n$ places. Let $h : \omega \to \omega$ be such that for all n, $h(n) \le n$ and $h^{-1}(\{n\})$ is infinite. We then define

$$F_i(u) = \begin{cases} F^i_{h(|u|)}(\{\alpha \in u : |u \cap \alpha| < m_i(n)\}) & \text{if this is } < \chi \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

The point of doing this is that whenever Player II chooses A_i, we know that $\mathrm{ran}(F_i \restriction [A_i]^{<\omega})$ will look like the result of intersecting an elementary submodel of M_i with χ; in particular, this range will be closed under the functions from M_i.

Note that M_0 (and all expansions of it) has definable Skolem functions and so for any i and $A \subseteq \lambda$, the Skolem hull of A in M_i (denoted by $\mathrm{Sk}_{M_i}(A)$) is well–defined.

Let $\langle (F_i, A_i) : i < (2^\chi)^+\rangle$ be a play of the game in which Player I uses this strategy (with M_i the model corresponding to F_i). For each i, define

$$\alpha_i = \min\{\alpha : \left|\mathrm{Sk}_{M_0}(A_i) \cap \beth_\alpha(\chi)\right| > \chi\}. \tag{1.5}$$

By the choice of M_0 and M_i, clearly $\alpha(i)$ is a successor ordinal or a limit ordinal of cofinality χ^+, and

$$|\,\mathrm{Sk}_{M_0}(A_i) \cap \beth_{\alpha_i}(\chi)| \le 2^\chi. \tag{1.6}$$

Since $A_i \subseteq A_j$ for $i > j$, we know the sequence $\langle \alpha_i : i < (2^\chi)^+\rangle$ is non–decreasing. Furthermore, for each i we know

$$\alpha_i < \min\{\beta : \lambda \le \beth_\beta(\chi)\} < (2^\chi)^+. \tag{1.7}$$

This means that the sequence $\langle \alpha_i : i < (2^\chi)^+\rangle$ is eventually constant, say with value α^*. Let i^* be the least ordinal $< (2^\chi)^+$ such that $\alpha_i = \alpha^*$ for $i \ge i^*$.

Proposition 1.5. *If* $i^* \le i < (2^\chi)^+$, *then* $\mathrm{Sk}_{M_0}(A_{i+1}) \cap \beth_{\alpha^*}(\chi)$ *is a proper subset of* $\mathrm{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi)$.

Sh:535

Proof. Note that i^*, α^*, and $\beth_{\alpha^*}(\chi)$ are all elements of M_{i+1} as they are definable in $\langle H(\lambda^+), \in, <_{\lambda^+}\rangle$ from the parameters M_0 and $\langle A_j : j \leq i\rangle$. Furthermore,

$$\gamma^* := \min\{\gamma < \lambda : |\operatorname{Sk}_{M_0}(A_i) \cap \gamma| = \chi\} \tag{1.8}$$

is also definable in M_{i+1} (and $< (2^\chi)^+$). Thus the language of M_{i+1} includes a bijection between $\operatorname{Sk}_{M_0}(A_i) \cap \gamma^*$ and χ.

If Player I has not won the game at this stage, after Player I selects A_{i+1} we will be able to find an ordinal $\beta < \chi$ such that $\beta \notin \operatorname{ran}(F_{i+1} \restriction [A_{i+1}]^{<\omega})$. By definition of h, we know $\beta' := h^{-1}(\beta)$ is an element of $\operatorname{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi)$. However, β' is not an element of $\operatorname{Sk}_{M_{i+1}}(A_{i+1})$ – since F_{i+1} codes the Skolem functions of M_{i+1}, the range of $F_{i+1} \restriction [A_{i+1}]^{<\omega}$ is $\operatorname{Sk}_{M_{i+1}}(A_{i+1}) \cap \chi$. Since $\operatorname{Sk}_{M_{i+1}}(A_{i+1})$ is closed under h, this contradicts our choice of β. Since $\operatorname{Sk}_{M_0}(A_{i+1}) \subseteq \operatorname{Sk}_{M_{i+1}}(A_{i+1})$, we have established the proposition. $\square$

Note that the preceding proposition finishes the proof of the claim — if play of the game continues for all $(2^\chi)^+$ steps, then $\langle \operatorname{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi) : i < (2^\chi)^+\rangle$ is a strictly decreasing family of subsets of $\operatorname{Sk}_{M_0}(A_{i^*})$, contradicting (1.6). $\square$

2. Club–guessing technology

In this section, we prove that if $\lambda = \mu^+$, where μ is singular, then under certain circumstances we can find a complicated "library" of colorings of smaller cardinals. In the next section, we will use this library of colorings to get a complicated coloring of λ.

The basics of club–guessing are explained in [4], but we will take a few minutes to recall some of the definitions.

Let us recall that if S is a stationary subset of λ, then an S–club system is a sequence $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that for (limit) $\delta \in S$, C_δ is closed unbounded in δ.

In this section, we will be concerned with the case where λ is the successor of a singular cardinal, i.e., $\lambda = \mu^+$ where $\operatorname{cf}(\mu) < \mu$. In this context, if $\bar{C}$ is an S–club system, then for $\delta \in S$ we define an ideal $J_\delta^{b[\mu]}$ on C_δ by $A \in J_\delta^{b[\mu]}$ if and only if $A \subseteq C_\delta$, and for some $\theta < \mu$ and $\gamma < \delta$,

$$\beta \in A \cap \operatorname{nacc}(C_\delta) \Rightarrow \left[\beta < \gamma \text{ or } \operatorname{cf}(\beta) < \theta\right].$$

Note that it is a bit easier to understand the definition of $J_\delta^{b[\mu]}$ by looking at the contrapositive — a subset A of C_δ is "large", i.e., not in $J_\delta^{b[\mu]}$, if and only if $A \cap \operatorname{nacc}(C_\delta)$ is cofinal in δ, and the cofinalities of members of any end segment of $A \cap \operatorname{nacc}(C_\delta)$ are unbounded below μ.

Claim 2.1. Let $\lambda = \mu^+$, where μ is a singular cardinal of cofinality $\kappa < \mu$. Let $S \subseteq \lambda$ be stationary, and assume that $\sup\{\operatorname{cf}(\delta) : \delta \in S\} = \mu^* < \mu$. Let $\bar{C}$ be an S–club system, and for each $\delta \in S$, let J_δ be the ideal $J_\delta^{b[\mu]}$. Let $\langle \kappa_i : i < \kappa\rangle$ be a non–decreasing sequence of cardinals such that

$$\kappa^* = \sum_{i<\kappa} \kappa_i \leq \mu, \tag{2.1}$$

and let $\gamma^* < \mu$.

Sh:535

Assume we are given a λ–club system $\bar{e}$ and a sequence of ideals $\bar{I} = \langle I_\alpha : \alpha < \lambda \rangle$ such that

(1) I_α is an ideal on e_α extending $J^{\mathrm{bd}}_{e_\alpha}$
(2) if $\delta \in S$, then for each $i < \kappa$,

$$\{\alpha \in \mathrm{nacc}(C_\delta) : \text{ Player I wins } \mathrm{Gm}^{\omega}_{I_\alpha}[e_\alpha, \kappa_i, \gamma^*]\} = \mathrm{nacc}(C_\delta) \mod J_\delta$$

(3) for any club $E \subseteq \lambda$, for stationarily many $\delta \in S$,

$$\{\alpha \in \mathrm{nacc}(C_\delta) : B_0[E, e_\alpha] \notin I_\alpha\} \notin J_\delta,$$

where

$$B_0[E, e^*_\alpha] = \{\beta \in \mathrm{nacc}(e_\alpha) : E \text{ meets the interval } (\sup(\beta \cap e_\alpha), \beta)\}.$$

Then there is a function $h : \lambda \to (\kappa + 1)$ and a sequence

$$\bar{F} = \langle F_\delta : \delta < \lambda,\ \delta \text{ a limit } \rangle$$

such that

$$\circledast_1\ F_\delta : [e_\delta]^{<\omega} \longrightarrow \kappa_{h(\delta)} (\text{where } \kappa^* := \kappa_\kappa)$$

and

$\circledast_2$ for every club $E \subseteq \lambda$, for each $i < \kappa$ there are stationarily many $\delta \in S$ such that the set of $\beta \in \mathrm{nacc}(C_\delta)$ satisfying the following
- $h(\beta) \geq i$
- $B_0[E, e_\beta] \notin I_\beta$
- for all $\gamma < \beta$, $\kappa_{h(\beta)} \subseteq \mathrm{ran}(F_\beta \restriction [B_0[E, e_\beta] \setminus \gamma]^{<\omega})$

is *not* in J_δ.

Now admittedly the previous claim is quite a lot to digest, so we will take a little time to illuminate the basic situation we have in mind.

Claim 2.2. The assumptions of Claim 2.1 are satisfied if

(1) $\lambda = \mu^+$ where $\kappa = \mathrm{cf}(\mu) < \mu$
(2) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$
(3) $\delta \in S \to |\delta| = \mu$ (i.e., $S \subseteq \lambda \setminus \mu$)
(4) $\bar{C}$ is an S–club system
(5) $\bar{J} = \langle J_\delta : \delta \in S \rangle$ where $J_\delta = J^{b[\mu]}_{C_\delta}$
(6) $\mathrm{id}_p(\bar{C}, \bar{J})$ is a proper ideal
(7) $\langle \kappa_i : i < \kappa \rangle$ is a non–decreasing sequence of cardinals with supremum $\kappa^* \leq \mu$
(8) $\gamma^* < \mu$, and for each $i < \kappa$, Player I wins the game $\mathrm{Gm}^{\omega}[\theta, \kappa_i, \gamma^*]$ for all large enough regular $\theta < \mu$
(9) $\bar{e}$ is a λ–club system such that $|e_\beta| < \mu$
(10) for $\alpha < \lambda$, $I_\alpha = J^{\mathrm{bd}}_{e_\alpha}$

Proof of Claim 2.2. We need only check items (2) and (3) in the statement of Claim 2.1 — everything else is trivially satisfied. Concerning (2), given $\delta \in S$ and $i < \kappa$, we need to show

$$\{\alpha \in \mathrm{nacc}(C_\delta) : \text{ Player I wins } \mathrm{Gm}^\omega[e_\alpha, \kappa_i, \gamma^*]\} = \mathrm{nacc}(C_\delta) \mod J_\delta.$$

Let A consist of those $\alpha \in \mathrm{nacc}(C_\delta)$ for which Player I does not win the game $\mathrm{Gm}^\omega[e_\alpha, \kappa_i, \gamma^*]$. By our assumptions, there is a $\theta < \mu$ such that $|e_\alpha| < \theta$ for all $\alpha \in A$, and therefore A is in the ideal $J^{b[\mu]}_{C_\delta} = J_\delta$ and we have what we need.

Concerning (3), given $E \subseteq \lambda$ club, we must find stationarily many $\delta \in S$ such that

$$\{\alpha \in \mathrm{nacc}(C_\delta) : B_0[E, e_\alpha] \notin I_\alpha\} \notin J_\delta.$$

Let $E' = \{\xi \in E : \mathrm{otp}(E \cap \xi) = \xi \text{and } \mu \text{ divides } \xi\}$. Clearly E' is a closed unbounded subset of E, and since $\mathrm{id}_p(\bar{C}, \bar{J})$ is a proper ideal, the set

$$S^* := \{\delta \in S \cap E' : E' \cap \mathrm{nacc}(C_\delta) \notin J_\delta\}$$

is stationary.

Fix $\delta \in S^*$, and suppose we are given $\theta < \mu$ and $\xi < \delta$. Since $E' \cap \mathrm{nacc}(C_\delta) \notin J_\delta$, we can find $\alpha \in E' \cap \mathrm{nacc}(C_\delta)$ such that $\alpha > \max\{\xi, \mu\}$ and $\mathrm{cf}(\alpha) > \theta$. Since the order–type of $E \cap \alpha$ is $\alpha \geq \mu > |e_\alpha|$, we know that $B_0[E, e_\alpha]$ is unbounded in e_α hence a member of I_α. This shows that the set of such α is in J^+_δ, as required. □

Now we return to the proof of Claim 2.1.

Proof of Claim 2.1. Let $\sigma = \mathrm{cf}(\sigma)$ be a regular cardinal $< \mu$ that is greater than μ^* and γ^*. For each limit $\beta < \lambda$, if there is an $i \leq \kappa$ such that Player I wins the version of $\mathrm{Gm}^\omega_{I_\beta}[e_\beta, \kappa_i, \sigma^+]$ where we allow Player II to pass, then we let $h(\beta)$ be the maximal such i — note that i exists by (5) of Claim 1.3 — and let Str_β be a strategy that witnesses this.

Note that since $\gamma^* < \sigma^+$ and $J_\delta = J^{b[\mu]}_\delta$ for $\delta \in S$, we have that for $\delta \in S$ and $i < \kappa$ that

$$\{\beta \in \mathrm{nacc}(C_\delta) : \mathrm{Str}_\beta \text{ is defined and } i \leq h(\beta)\} = \mathrm{nacc}(C_\delta) \mod J_\delta.$$

We will make σ^+ attempts to build $\bar{F}$ witnessing the conclusion. In stage $\zeta < \sigma^+$, we assume that our prior work has furnished us with a decreasing sequence $\langle E_\xi : \xi < \zeta\rangle$ of clubs in λ, and, for each $\beta < \lambda$ where Str_β is defined, an initial segment $\langle F^\xi_\beta, A^\xi_\beta : \xi < \zeta\rangle$ of a play of $\mathrm{Gm}^\omega_{I_\beta}[e_\beta, \kappa^*_{h(\beta)}, \sigma^+]$ in which Player I uses Str_β. (Note that our convention is that if Player II chooses to pass at a stage, we let A^ξ_β be undefined.)

For each such β, let $F^\zeta_\beta : [e_\beta]^{<\omega} \to \kappa_{h(\beta)}$ be given by Str_β, and for those β for which Str_β is undefined, we let F^β_ζ be any such function. Now if $\langle F^\zeta_\beta : \beta < \lambda\rangle :=$

$\bar{F}^\zeta$ is as required then we are done. Otherwise, there is a club $E' \subseteq \lambda$ and $i_\zeta < \kappa$ exemplifying the failure of $\bar{F}^\zeta$, and without loss of generality,

$$(\forall \delta \in S)\big[B_{i_\zeta}[E'_\zeta, C_\delta, \bar{I}, \bar{e}, \bar{F}^\zeta]\big] \in J_\delta. \tag{2.2}$$

Now let $E_\zeta = \operatorname{acc}(E'_\zeta \cap \bigcap_{\xi<\zeta} E_\xi)$. For each β where Str_β is defined, we let Player II respond to F^ζ_β by playing the set $B_0[E_\zeta, e_\beta]$ if it is a legal move, otherwise we let him pass. We then proceed to stage $\zeta + 1$.

Assuming that this construction continues for all σ^+ stages, we will arrive at a contradiction. Let $E = \bigcap_{\zeta<\sigma^+} E_\zeta$. By assumption (3) there is a $\delta(*) \in S$ for which

$$A_1 := \{\beta \in \operatorname{nacc}(C_{\delta(*)}) : B_0[E, e_\beta] \notin I_\beta\} \notin J_{\delta(*)}.$$

By assumption (2), we have

$$A_2 := \{\beta \in A_1 : \mathrm{Str}_\beta \text{ is defined }\} \notin J_{\delta(*)}.$$

For $\beta \in A_2$, look at the play $\langle F^\zeta_\beta, A^\zeta_\beta : \zeta < \sigma^+\rangle$. Since Player I wins, there is a $\zeta_\beta < \sigma^+$ such that Player II passed at stage ζ for all $\zeta \geq \zeta_\beta$. Since $\sigma > \mu^*$ and $J_{\delta(*)}$ is μ^*–based, for some $\zeta^* < \sigma^+$,

$$A_3 = \{\beta \in A_1 : \mathrm{Str}_\beta \text{ is defined and } \zeta_\beta \leq \zeta^*\} \notin J_{\delta(*)}.$$

Now E_{ζ^*} was defined so that for some i_{ζ^*}, for all $\delta \in S$,

$$B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}, \bar{F}^{\zeta^*}] \in J_\delta, \tag{2.3}$$

but (again by assumption (2))

$$A_4 = \{\beta \in A_1 : \mathrm{Str}_\beta \text{ is defined}, \zeta_\beta \leq \zeta^*, \text{ and } i_{\zeta^*} \leq h(\beta)\} \notin J_{\delta(*)}.$$

For $\beta \in A_4$, we know that at stage ζ^* of our play of $\mathrm{Gm}^\omega_{I_\beta}[e_\beta, \kappa_{h(\beta)}, \sigma^+]$ the set $B_0[E_{\zeta^*}, e_\beta]$ was not a legal move. Since our sequence of clubs is decreasing, we know that $B_0[E_{\zeta^*}, e_\beta]$ is a subset of $B_0[E_\xi, e_\beta]$ for all $\xi < \zeta^*]$, so we have

$$B_0[E_{\zeta^*}, e_\beta] \subseteq \bigcap_{\xi<\zeta^*} A^\xi_\beta.$$

Since $\beta \in A_1$, we know that $B_0[E_{\zeta^*}, e_\beta] \notin I_\beta$. Thus the reason for $B_0[E_{\zeta^*}, e_\beta]$ being an illegal move must be that for all $\gamma < \beta$,

$$\kappa^*_{h(\beta)} \subseteq \operatorname{ran}(F^{\zeta^*}_\beta \restriction [B_0[E_{\zeta^*}, e_\beta] \setminus \gamma]^{<\omega}).$$

All of these facts combine to tells us that $\beta \in B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}, \bar{F}^{\zeta^*}]$, and thus

$$A_4 \subseteq B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}^*, \bar{F}^{\zeta^*}] \notin J_{\delta(*)},$$

contradicting (2.3). □

The proofs in this section (and the next) can be considerably simplified if we are willing to restrict ourselves to the case $\kappa^* < \mu$, as we can dispense with the sequence $\langle \kappa_i : i < \kappa\rangle$.

3. Building the Coloring

We now come to the main point of this paper; we dedicate this section and the next to proving the following theorem.

Theorem 1. *Assume $\lambda = \mu^+$, where μ is a singular cardinal of uncountable cofinality, say $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$. Assume $\langle \kappa_i : i < \kappa \rangle$ is non–decreasing with supremum $\kappa^* \leq \mu$, and there is a $\gamma^* < \mu$ such that for each i, for every large enough regular $\theta < \mu$, Player I has a winning strategy in the game* $\mathrm{Gm}^\omega[\theta, \kappa_i, \gamma^*]$. *Then* $\mathrm{Pr}_1(\lambda, \lambda, \kappa^*, \kappa)$ *holds.*

Let $\langle S_i : i < \kappa \rangle$ be a sequence of pairwise disjoint stationary subsets of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$. For $i < \kappa$, let $\bar{C}^i$ be an S_i–club system such that

- $\lambda \notin \mathrm{id}_p(\bar{C}^i, \bar{J}^i)$, where $\bar{J}^i = \langle J^{b[\mu]}_{C^i_\delta} : \delta \in S_i \rangle$
- for $\delta \in S_i$, $\mathrm{otp}(C^i_\delta) = \mathrm{cf}(\delta) = \kappa = \mathrm{cf}(\mu)$

Such ladder systems can be found by Claim 2.6 (and Remark 2.6A (6)) of [2] — for the second statement to hold, we need that μ has uncountable cofinality.

Claim 3.1. There is a λ–club system $\bar{e}$ such that $|e_\beta| \leq \mathrm{cf}(\beta) + \mathrm{cf}(\mu)$, and $\bar{e}$ "swallows" each $\bar{C}^i$, i.e., if $\delta \in S_i \cap (e_\beta \cup \{\beta\})$, then $C^i_\delta \subseteq e_\beta$.

Proof. Let $S = \cup_{i<\kappa} S_i$, and let $\beta < \lambda$ be a limit ordinal. Let e^0_β be a closed cofinal subset of β of order–type $\mathrm{cf}(\beta)$. We will construct the required ladder e_β in ω–stages, with e^n_β denoting the result of the first n stages of our procedure. The construction is straightforward, but it is worthwhile to note that we need to use the fact that each member of S has uncountable cofinality.

Given e^n_β, let us define

$$B_n = S \cap (e^n_\beta \cup \{\beta\}). \tag{3.1}$$

Now we let e^{n+1}_β be the closure in β of

$$e^n_\beta \cup \bigcup \{C_\delta : \delta \in B_n\}. \tag{3.2}$$

Note that $|e^{n+1}| \leq \mathrm{cf}(\mu) + \mathrm{cf}(\beta)$ as $|C_\delta| = \mathrm{cf}(\mu) = \kappa$ for each $\delta \in S$. Finally, we let e_β be the closure of $\cup_{n<\omega} e^n_\beta$ in β.

Clearly $|e_\beta| \leq \mathrm{cf}(\mu) + \mathrm{cf}(\beta)$. Also, since each element of S has uncountable cofinality, if $\delta \in S \cap e_\beta$, then there is an n such that $\delta \in e^n_\beta$, and therefore

$$C_\delta \subseteq e^{n+1}_\beta \subseteq e_\beta, \tag{3.3}$$

as required. □

For each $i < \kappa$, there are h_i and $\bar{F}^i = \langle F^i_\delta : \delta < \lambda,\ \delta \text{ limit} \rangle$ as in the conclusion of Claim 2.1 applied to $\bar{C}^i$ and $\bar{e}$; note that we satisfy the assumptions of Claim 2.1 by way of Claim 2.2.

Sh:535

Let $\langle \lambda_i : i < \kappa \rangle$ be a strictly increasing sequence of regular cardinals $> \kappa$ and cofinal in μ such that

$$\lambda = \operatorname{tcf}\Big(\prod_{i<\kappa} \lambda_i / J_\kappa^{\mathrm{bd}}\Big), \tag{3.4}$$

and let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify this. Finally, let $h_0^* : \kappa \to \omega$ and $h_1^* : \kappa \to \kappa$ be such that

$$(\forall n)(\forall i < \kappa)(\exists^\kappa j < \kappa)[h_0^*(j) = n \text{ and } h_1^*(j) = i]. \tag{3.5}$$

Before we can define our coloring, we must recall some of the terminology of [2].

Definition 3.2. Let $0 < \alpha < \beta < \lambda$, and define

$$\gamma(\alpha, \beta) = \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

We also define (by induction on ℓ)

$$\gamma_0(\alpha, \beta) = \beta,$$
$$\gamma_{\ell+1}(\alpha, \beta) = \gamma(\alpha, \gamma_\ell(\alpha, \beta)) \text{ (if defined).}$$

We let $k(\alpha, \beta)$ be the first ℓ for which $\gamma_\ell(\alpha, \beta) = \alpha$. The sequence $\langle \gamma_i(\alpha, \beta) : i \leq k(\alpha, \beta) \rangle$ will be referred to as the *walk from β to α along the ladder system $\bar{e}$.*

We now define the coloring c that will witness $\mathrm{Pr}_1(\lambda, \lambda, \kappa^*, \kappa)$. Recall that c must be a symmetric two–place function from λ to κ^*.

Given $\alpha < \beta$, we let $i = i(\alpha, \beta)$ be the maximal $j < \kappa$ such that $f_\beta(j) < f_\alpha(j)$ (if such an j exists). Next, we walk from β down to α along $\bar{e}$ until we reach an ordinal $\nu(\alpha, \beta)$ such that

$$f_\alpha(i) < f_{\nu(\alpha,\beta)}(i),$$

(again, if such an ordinal exists.) After this, we walk along $\bar{e}$ from α toward the ordinal $\max(\alpha \cap e_{\nu(\alpha,\beta)})$ until we reach an ordinal $\eta(\alpha, \beta)$ for which

$$f_{\nu(\alpha,\beta)}(i) < f_{\eta(\alpha,\beta)}(i).$$

The idea now is to look at how the ladders $e_{\nu(\alpha,\beta)}$ and $e_{\eta(\alpha,\beta)}$ intertwine. Let us make a temporary definition by calling an ordinal $\xi \in e_{\nu(\alpha,\beta)}$ *relevant* if $e_{\eta(\alpha,\beta)}$ meets the interval $(\sup(\xi \cap e_{\nu(\alpha,\beta)}), \xi)$.

If it makes sense, we let $w(\alpha, \beta) \subseteq e_{\nu(\alpha,\beta)}$ be the last $h_0^*(i(\alpha, \beta))$ relevant ordinals in $e_{\nu(\alpha,\beta)}$ (so we need that the relevant ordinals have order–type $\gamma + h_0^*(i(\alpha, \beta))$ for some γ).

Finally, we define our coloring by

$$c(\alpha, \beta) = F_{\nu(\alpha,\beta)}^{h_1^*(i(\alpha,\beta))}(w(\alpha, \beta)). \tag{3.6}$$

If the attempt to define $c(\alpha, \beta)$ breaks down at some point for some specific $\alpha < \beta$, then we set $c(\alpha, \beta) = 0$.

We now prove that this coloring works, so suppose $\langle t_\alpha : \alpha < \lambda\rangle$ are pairwise disjoint subsets of λ such that $|t_\alpha| = \theta_1 < \kappa$ and $j^* < \kappa^*$, and without loss of generality $\alpha < \min t_\alpha$ and $\theta_1 \geq \omega$. We need to find δ_0 and δ_1 such that

$$\alpha \in t_{\delta_0} \text{ and } \beta \in t_{\delta_1} \Rightarrow \alpha < \beta \text{ and } c(\alpha, \beta) = j^*. \tag{3.7}$$

Let j_1 be the least j such that $j^* < \kappa_j$, and let S, $\bar{C}$, and $\bar{F}$ denote S_{j_1}, $\bar{C}^{j_1}$, and $\bar{F}^{j_1}$ respectively.

Given $\delta < \lambda$, we define the *envelope of* t_δ (denoted $\mathrm{env}(t_\delta)$) by the formula

$$\mathrm{env}(t_\delta) = \bigcup_{\zeta \in t_\delta} \{\gamma_\ell(\delta, \zeta) : \ell \leq k(\delta, \zeta)\}. \tag{3.8}$$

The envelope of t_δ is the set of all ordinals obtained by walking down to δ from some $\zeta \in t_\delta$ using the ladder system $\bar{e}$. This makes sense as we have arranged that $\delta < \min t_\delta$. Note also that $|\,\mathrm{env}(t_\delta)| \leq |t_\delta| = \theta_1$.

Next we define functions $g_\delta^{\min}$ and $g_\delta^{\max}$ in $\prod_{i<\kappa} \lambda_i$ by

$$g_\delta^{\min}(i) = \min\{f_\gamma(i) : \gamma \in \mathrm{env}(t_\delta)\}, \tag{3.9}$$

and

$$g_\delta^{\max}(i) = \sup\{f_\gamma(i) + 1 : \gamma \in \mathrm{env}(t_\delta)\}. \tag{3.10}$$

Note that $g_\delta^{\max}$ is well–defined as we assume that $\kappa < \min\{\lambda_i : i < \kappa\}$.

The following claim is quite easy, and the proof is left to the reader.

Claim 3.3.

(1) $f_\delta =_{J_\kappa^{\mathrm{bd}}} g_\delta^{\min}$
(2) $g_\delta^{\min}(i) \leq g_\delta^{\max}(i)$ for all $i < \kappa$
(3) There is a $\delta' > \delta$ such that $g_\delta^{\max} \leq_{J_\kappa^{\mathrm{bd}}} g_{\delta'}^{\min}$.

Now let $\chi^* = (2^\lambda)^+$, and let $\langle M_\alpha : \alpha < \lambda\rangle$ be a sequence of elementary submodels of $\langle H(\chi^*), \in, <^*_{\chi^*}\rangle$ that is increasing and continuous in α and such that each $M_\alpha \cap \lambda$ is an ordinal, $\langle M_\beta : \beta \leq \alpha\rangle \in M_{\alpha+1}$, and $\langle f_\alpha : \alpha < \lambda\rangle$, g, c, $\bar{e}$, S, $\bar{C}$, $\langle t_\alpha : \alpha < \lambda\rangle$ all belong to M_0. Note that $\mu + 1 \subseteq M_0$.

The set $E = \{\alpha < \lambda : M_\alpha \cap \lambda = \alpha\}$ is closed unbounded in λ, and furthermore,

$$\alpha < \delta \in E \Rightarrow \sup t_\alpha < \delta. \tag{3.11}$$

By the choice of $\bar{C}$ and $\bar{F}$, for some $\delta \in S \cap E$ we have the set

$$A = \{\beta \in \mathrm{nacc}(C_\delta) : (\forall \gamma < \beta)\, \mathrm{ran}(F_\beta \restriction \big[B_0[E, e_\beta] \setminus \gamma\big]^{<\omega}) \supseteq \kappa_{j_1}\} \tag{3.12}$$

is not in $J_{C_\delta}^{b[\mu]}$.

Note that $A \subseteq \mathrm{acc}(E)$, as $B_0[E, e_\beta]$ is unbounded in β for $\beta \in A$. For $\beta \in t_\delta$, if $\ell < k(\delta, \beta)$ then $e_{\gamma_\ell(\delta,\beta)} \cap \delta$ is bounded in δ, and since it is closed it has a well–defined maximum. Since $|t_\delta| < \kappa = \mathrm{cf}(\delta)$, this means the ordinal

$$\gamma^\otimes := \sup\{\max[e_{\gamma_\ell(\delta,\beta)} \cap \delta] : \beta \in t_\delta \text{ and } \ell < k(\delta, \beta)\}$$

is strictly less than δ.

Sh:535

For $\beta \in t_\delta$, let us define

$$A_\beta := \{\beta' \in A : (\exists \ell \leq k(\beta, \delta))[\mathrm{cf}(\beta') \leq |e_{\gamma_\ell(\delta,\beta)}|]\}. \tag{3.13}$$

Since the cardinality of each ladder in $\bar{e}$ is less than μ, each set A_β is an element of $J^{b[\mu]}_{C_\delta}$. The ideal $J^{b[\mu]}_{C_\delta}$ is κ–complete, so the fact that $|t_\delta| < \kappa$ and $k(\beta, \delta)$ is finite for each $\beta \in t_\delta$ together imply that

$$\bigcup_{\beta \in t_\delta} A_\beta \in J^{b[\mu]}_{C_\delta}. \tag{3.14}$$

By the definition of A and our choice of δ, this means it is possible to choose $\beta^* \in A \setminus (\gamma^\otimes + 1)$ that is not in any A_β, i.e.,

$$\beta \in t_\delta \text{ and } \ell < k(\delta, \beta) \Longrightarrow \mathrm{cf}(\beta^*) > |e_{\gamma_\ell(\delta,\beta)}|. \tag{3.15}$$

Claim 3.4.

(1) If $\epsilon \in t_\delta$, and $\ell = k(\delta, \epsilon) - 1$, then $\beta^* \in \mathrm{nacc}(e_{\gamma_\ell(\delta,\epsilon)})$.
(2) If $\epsilon \in t_\delta$ and $\gamma^\otimes < \gamma' \leq \beta^*$, then
- $\gamma_\ell(\delta, \epsilon) = \gamma_\ell(\gamma', \epsilon)$ for $\ell < k(\delta, \epsilon)$, and
- $\gamma_{k(\delta,\epsilon)}(\gamma', \epsilon) = \beta^*$

Proof. For the first clause, note that δ is an element of $e_{\gamma_\ell(\delta,\epsilon)}$ and hence by our choice of $\bar{e}$, $C_\delta \subseteq e_{\gamma_\ell(\delta,\epsilon)}$. Thus $\beta^* \in e_{\gamma_\ell(\delta,\epsilon)}$, and since $\mathrm{cf}(\beta^*) > |e_{\gamma_\ell(\delta,\epsilon)}|$, we know that β^* cannot be an accumulation point of $e_{\gamma_\ell(\delta,\epsilon)}$.

The first part of the second statement follows because of the definition of $\gamma^\otimes$. As far as the second part of the second statement goes, it is best visualized as follows:

We walk down the ladder system $\bar{e}$ from ϵ to γ', we eventually hit a ladder that contains δ — this happens at stage $k(\delta, \epsilon) - 1$. Since C_δ is a subset of this ladder, the next step in our walk from ϵ to γ' must be down to β^* because $\gamma^\otimes < \gamma' < \beta^*$. □

We can visualize the preceding claim in the following manner: β^* is chosen so that for all sufficiently large $\gamma' < \beta^*$, all the walks from some element of t_δ to γ' are funnelled through β^* — β^* acts as a bottleneck. This will be key when want to prove that our coloring works.

Since $\beta^* \in A$, we can choose a finite increasing sequence $\xi_0 < \xi_1 < \cdots < \xi_n$ of ordinals in $\mathrm{acc}(E) \cap \mathrm{nacc}(e_{\beta^*}) \setminus (\gamma^\otimes + 1)$ such that $F^{j_1}_{\beta^*}(\{\xi_0, \ldots, \xi_n\}) = j^*$, the color we are aiming for.

For each $\ell \leq n$, we can find $\zeta_\ell \in E \setminus (\gamma^\otimes + 1)$ such that

$$\sup(e_{\beta^*} \cap \xi_\ell) < \zeta_\ell < \xi_\ell.$$

Now we let $\phi(x_0, y_0, x_1, y_1, \ldots, x_n, y_n, z_0, z_1)$ be the formula (with parameters $\gamma^\otimes$, $\bar{f}$, $\langle \lambda_i : i < \kappa \rangle$, $\bar{C}$, $\bar{e}$, $\langle t_\alpha : \alpha < \lambda \rangle$, h, h_0, j^*) that describes our current situation with x_ℓ, y_ℓ standing for ζ_ℓ, ξ_ℓ, and z_0, z_1 standing for β^*, δ , i.e., ϕ states

- $\gamma^\otimes < x_0 < y_0 < \cdots < x_n < y_n < z_0 < z_1$ are ordinals $< \lambda$
- $z_1 \in S$ and $z_0 \in \mathrm{nacc}(C_{z_1})$

- $\gamma^{\otimes} = \sup\{\max[e_{\gamma_\ell(z_1,\zeta)} \cap z_1] : \ell < k(z_1,\zeta) \text{ and } \zeta \in t_{z_1}\}$
- $z_0 \in \mathrm{nacc}(e_{\gamma_{k(z_1,\epsilon)}(z_1,\epsilon)})$ for all $\epsilon \in t_{z_1}$
- $F^{j_1}_{z_0}(\{y_0, \dots, y_n\}) = j^*$

Now clearly we have

$$H(\chi) \models \phi[\zeta_0, \xi_0, \dots, \zeta_n, \xi_n, \beta^*, \delta]. \tag{3.16}$$

Recall that all the parameters needed in ϕ are in M_0, except possibly for $\gamma^{\otimes}$, so the model $M_{\gamma^{\otimes}+1}$ contains all the parameters we need. Also, $\{\zeta_0, \xi_0, \dots, \zeta_n, \xi_n\} \in M_{\beta^*}$, $\beta^* \in M_\delta \setminus M_{\beta^*}$, and since $\delta \in \lambda \setminus M_\delta$, we have (recalling that $\exists^* z < \lambda$ means "for unboundedly many $z < \lambda$)

$$M_\delta \models (\exists^* z_1 < \lambda)\phi(\zeta_0, \xi_0, \dots, \zeta_n, \xi_n, \beta^*, z_1). \tag{3.17}$$

Therefore, this formula is true in $H(\chi)$ because of elementarity. Similarly, we have

$$H(\chi) \models (\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)\phi(\zeta_0, \xi_0, \dots, \zeta_n, \xi_n, z_0, z_1).$$

Now each of the intervals $[\gamma^{\otimes}+1, \zeta_0)$, $[\zeta_0, \xi_0)$, $\dots$, contains a member of E, so (by the definition of E) similar considerations give us

$$H(\chi) \models (\exists^* x_0 < \lambda) \dots (\exists^* y_n < \lambda)(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)\phi(x_0, y_0, \dots, z_0, z_1).$$

Now we can choose (in order)

$$\zeta_0^a < \zeta_0^b < \xi_0^a < \zeta_1^a < \xi_0^b < \zeta_1^b < \dots < \zeta_n^a < \xi_{n-1}^b < \zeta_n^b < \xi_n^a \tag{3.18}$$

such that

$$(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)[\phi(\zeta_0^a, \dots, \xi_{n-1}^a, \zeta_n^a, \xi_n^a, z_0, z_1)], \tag{3.19}$$

and

$$(\exists^* y_n < \lambda)(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)[\phi(\zeta_0^b, \dots, \xi_{n-1}^b, \zeta_n^b, y_n, z_0, z_1)], \tag{3.20}$$

Our goal is to show that for all sufficiently large $i < \kappa$, it is possible to choose objects β^a, δ^a, ξ_n^b, β^b, and δ^b such that

Table 1

(1) $\zeta_n^b < \beta^a < \delta^a < \min(t_{\delta^a}) \le \max(t_{\delta^a}) < \xi_n^b < \beta^b < \delta^b$
(2) $\phi(\zeta_0^a, \dots, \xi_n^a, \beta^a, \delta^a)$
(3) $\phi(\zeta_0^b, \dots, \xi_n^b, \beta^b, \delta^b)$
(4) for all $\epsilon \in \mathrm{env}(t_{\delta^a})$, $g^{\min}_{\delta^a} \restriction [i,\kappa) \le f_\epsilon \restriction [i,\kappa) \le g^{\max}_{\delta^a} \restriction [i,\kappa)$
(5) for all $\epsilon \in \mathrm{env}(t_{\delta^b})$, $g^{\min}_{\delta^b} \restriction [i,\kappa) \le f_\epsilon \restriction [i,\kappa) \le g^{\max}_{\delta^b} \restriction [i,\kappa)$
(6) $g^{\max}_{\delta^b}(i) < g^{\min}_{\delta^a}(i) \le g^{\max}_{\delta^a}(i) < f_{\beta^b}(i) < f_{\beta^a}(i)$
(7) $g^{\max}_{\delta^a} \restriction [i+1,\kappa) < g^{\min}_{\delta^b} \restriction [i+1,\kappa)$

Claim 3.5. If for all sufficiently large $i < \kappa$ it is possible to find objects satisfying the requirements of Table 1, then we can find $\delta^a < \delta^b$ such that $c(\epsilon^a, \epsilon^b) = j^*$ for all $\epsilon^a \in t_{\delta^a}$ and $\epsilon^b \in t_{\delta^b}$.

Proof. Let us choose $i^* < \kappa$ such that

- suitable objects (as above) can be found, and
- $h_1^*(i^*) = j_1$ and $h_0^*(i^*) = n$

Choose $\epsilon^a \in t_{\delta^a}$ and $\epsilon^b \in t_{\delta^b}$; we verify that $c(\epsilon^a, \epsilon^b) = j^*$.

Subclaim 1. $i(\epsilon^a, \epsilon^b) = i^*$.

Proof. Immediate by (4)–(7) in the table. □

Subclaim 2. $\nu(\epsilon^a, \epsilon^b) = \beta^b$.

Proof. Note that $\gamma^\otimes < \epsilon^a < \beta^b$. Clause (3) of the table implies that the assumptions of Claim 3.4 hold. Thus by Claim 3.4, for $\ell < k(\delta^b, \epsilon^b)$ we have

$$\gamma_\ell(\epsilon^a, \epsilon^b) = \gamma_\ell(\delta^b, \epsilon^b),$$

hence $\gamma_\ell(\epsilon^a, \epsilon^b) \in \mathrm{env}(t_{\delta^b})$ and (by (6) of the table and the definitions involved)

$$f_{\gamma_\ell(\epsilon^a,\epsilon^b)}(i^*) \leq g_{\delta^b}^{\max}(i^*) < g_{\delta^a}^{\min}(i^*) \leq f_{\epsilon^a}(i^*). \tag{3.21}$$

For $\ell = k(\delta^b, \epsilon^b)$, Claim 3.4 tells us

$$\gamma_\ell(\epsilon^a, \epsilon^b) = \beta^b,$$

and we have arranged that

$$f_{\epsilon^a}(i^*) \leq g_{\delta^a}^{\max}(i^*) < f_{\beta^b}(i^*). \tag{3.22}$$

This establishes $\beta^b = \nu(\epsilon^a, \epsilon^b)$. □

Subclaim 3. $\eta(\epsilon^a, \epsilon^b) = \beta^a$.

Proof. Let $\alpha = \max(e_{\beta^b} \cap \epsilon^a)$. We have arranged that

$$\zeta_n^b < \beta^a < \delta^a < \epsilon^a < \xi_n^b$$

and $\gamma^\otimes < \max(e_{\beta^b} \cap \delta^a)$, hence $\gamma^\otimes < \alpha < \beta^a$. For $\ell < k(\delta^a, \epsilon^a)$, Claim 3.4 implies

$$\gamma_\ell(\alpha, \epsilon^a) = \gamma_\ell(\delta^a, \epsilon^a) \in \mathrm{env}(t_{\delta^a}).$$

By our choice of i^*, we have

$$f_{\gamma_\ell(\alpha,\epsilon^a)}(i^*) \leq g_{\delta^a}^{\max}(i^*) < f_{\beta^b}(i^*). \tag{3.23}$$

For $\ell = k(\delta^a, \epsilon^a)$, Claim 3.4 implies $\gamma_\ell(\alpha, \epsilon^a) = \beta^a$, and we have ensured

$$f_{\beta^b}(i^*) < f_{\beta^a}(i^*). \tag{3.24}$$

Thus β^a is the first ordinal η in the walk from ϵ^a to $\max(e_{\beta^b} \cap \epsilon^a)$ for which $f_\eta(i^*) > f_{\beta^b}(i^*)$, and therefore $\eta(\epsilon^a, \epsilon^b) = \beta^a$. □

Sh:535

Subclaim 4. $w(\epsilon^a, \epsilon^b) = \{\xi_0^b, \dots \xi_n^b\}$.

Proof. Our previous subclaims imply that an ordinal $\xi \in e_{\beta^b}$ is relevant if and only if the ladder e_{β^a} meets the interval $(\sup(e_{\beta^b} \cap \xi), \xi)$. Since $h_0^*(i^*) = n + 1$, we know that $w(\epsilon^a, \epsilon^b)$ consists of the last $n + 1$ relevant ordinals in e_{β^b}.

For $i \le n$, clearly $\xi_i^b \in e_{\beta^b}$ and $\sup(\xi_i^b \cap e_{\beta^b}) \le \zeta_n^b$. We have made sure that $e_{\beta^a} \cap (\zeta_i^b, \xi_i^b) \ne \emptyset$ (for example, ξ_i^a is an element in this intersection) and so each ξ_i^b is relevant.

Since $\beta^a < \xi_n^b$, it is clear that there are no relevant ordinals larger than ξ_n^b.

Given $i < n$, if $\xi \in e_{\beta^b} \cap (\xi_i^b, \xi_{i+1}^b)$, then

$$\xi_i^b \le \sup(\xi \cap e_{\beta^b}) \le \xi \le \zeta_{i+1}^b.$$

Since $\zeta_{i+1}^a < \xi_i^b < \zeta_{i+1}^b < \xi_{i+1}^a$, it follows that

$$[\sup(\xi \cap e_{\beta^b}), \xi) \subseteq [\zeta_{i+1}^a, \xi_{i+1}^a),$$

and so ξ is not relevant. Thus $\{\xi_0^b, \dots, \xi_n^b\}$ are the last $n + 1$ relevant elements of e_{β^b}, as was required. □

To finish the proof of Claim 3.5, we note that as $h_1^*(i^*) = j^*$, we have

$$c(\epsilon^a, \epsilon^b) = F_{\beta^b}^{j_1}(\{\xi_0^b, \dots, \xi_n^b\}) = j^*. \tag{3.25}$$

□

4. Finding the required ordinals

The whole of this section will be occupied with showing that for all sufficiently large $i < \kappa$, it is possible to find objects satisfying the requirements of Table 1.

We begin with some notation intended to simplify the presentation.

- $\phi^a(z_0, z_1)$ abbreviates the formula $\phi(\zeta_0^a, \dots, \xi_n^a, z_0, z_1)$
- $\phi^b(y_n, z_0, z_1)$ abbreviates the formula $\phi(\zeta_0^b, \zeta_b^n, y_n, z_0, z_1)$
- For $i < \kappa$, $\psi(i, z_1)$ abbreviates the formula

$$(\forall \epsilon \in \mathrm{env}(t_{z_1}))[g_{z_1}^{\min} \restriction [i, \kappa) \le f_\epsilon \restriction [i, \kappa) \le g_{z_1}^{\max} \restriction [i, \kappa)] \tag{4.1}$$

We have arranged things so that the sentence

$$(\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)$$
$$(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\phi^a(z_0^a, z_1^a) \wedge \phi^b(y_n^b, z_0^b, z_1^b)] \tag{4.2}$$

holds.

There are far too many alternations of quantifiers in the above formula for most people to deal with comfortably; the best way to view them is as a single quantifier that asserts the existence of a tree of 5–tuples with the property that every node of

Sh:535

the tree has λ successors, and every branch through the tree gives us five objects satisfying $\phi^a \wedge \phi^b$.

Let $\Phi(i, z_0^a, \ldots, z_1^b)$ abbreviate the formula

$$\phi^a(z_0^a, z_1^a) \wedge \phi^b(y_n^b, z_0^b, z_1^b) \wedge \psi(i, z_1^a) \wedge \psi(i, z_1^b) \\ \wedge \left(g_{z_1^a}^{\max}\restriction[i+1, \kappa) < g_{z_1^b}^{\min}\restriction[i+1, \kappa)\right).$$

By pruning the tree so that every branch through it is a strictly increasing 5–tuple, we get

$$(\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)(\forall^* i < \kappa)[\Phi(i, z_0^a, \ldots, z_1^b)]. \quad (4.3)$$

We now make a rather *ad hoc* definition of another quantifier in an attempt to make the arguments that follow a little bit clearer. Given $i < \kappa$, let the quantifier $\exists^{*,i} z_0^b < \lambda$ mean that not only are there unboundedly many z_0^b's below λ satisfying whatever property, but also that for each $\alpha < \lambda_i$, we can find unboundedly many suitable z_0^b's for which $f_{z_0^b}(i)$ is greater than α.

Claim 4.1. If we choose $\beta^a < \delta^a < \xi_n^b$ such that

$$(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)(\forall^* i < \kappa)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)], \quad (4.4)$$

then

$$(\forall^* i < \kappa)(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \quad (4.5)$$

Proof. Suppose that we have $\beta^a < \delta^a < \xi_n^b$ such that (4.4) holds but (4.5) fails. Then there is an unbounded $I \subseteq \kappa$ such that for each $i \in I$,

$$\neg(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \quad (4.6)$$

In (4.4), we can move the quantifier "$\forall^* i < \kappa$" past the quantifiers to its left, i.e.,

$$(\forall^* i < \kappa)(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)], \quad (4.7)$$

so without loss of generality, for all $i \in I$,

$$(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \quad (4.8)$$

Since (4.6) holds for all $i \in I$, it must be the case that for each $i \in I$, there is a value $g(i) < \lambda_i$ such that for all sufficiently large $\beta < \lambda$, if

$$(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, \beta, z_1^b)], \quad (4.9)$$

then

$$f_\beta(i) \leq g(i). \quad (4.10)$$

Sh:535

Since $\{f_\alpha : \alpha < \lambda\}$ witnesses that the true cofinality of $\prod_{i<\kappa} \lambda_i$ is λ, we know

$$(\forall^* x < \lambda)(\forall^* i \in I)[g(i) < f_x(i)]. \tag{4.11}$$

When we combine this with (4.4), we see that it is possible to choose $\beta^b < \lambda$ such that

$$(\forall^* i \in I)[g(i) < f_{\beta^b}(i)], \tag{4.12}$$

and

$$(\exists^* z_1^b < \lambda)(\forall^* j < \kappa)[\Phi(j, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)]. \tag{4.13}$$

(Note that we have quietly used the fact that $|I| < \lambda = \mathrm{cf}(\lambda)$ to get a β^b that is "large enough" so that (4.9) implies (4.10) for all $i \in I$ for this particular β^b.) This last equation implies

$$(\forall^* j < \kappa)(\exists^* z_1^b < \lambda)[\Phi(j, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)],$$

so it is possible to choose $i \in I$ large enough so that

$$g(i) < f_{\beta^b}(i)$$

and

$$(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)].$$

This is a contradiction, as (4.9) holds for our choice of i and $\beta = \beta^b$, yet (4.10) fails. $\square$

Notice that an immediate corollary of the preceding claim is

$$\begin{aligned}&(\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\forall^* i < \kappa)\\ &\qquad (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)].\end{aligned} \tag{4.14}$$

Claim 4.2. If $\beta^a < \lambda$ is chosen so that

$$\begin{aligned}&(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\forall^* i < \kappa)\\ &\qquad (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)],\end{aligned} \tag{4.15}$$

then

$$(\forall^* i < \kappa)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda)[\psi' \wedge \psi'']$$

where

$$\psi' := g_{z_1^a}^{\max}(i) < v,$$

and

$$\begin{aligned}\psi'' := &(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\\ &\left[v < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\right].\end{aligned}$$

Proof. In (4.15), we can move the quantifier "$(\forall^* i < \kappa)$" past the other quantifiers to its left, so

$$(\forall^* i < \kappa)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \quad (4.16)$$

holds. The claim will be established if we show that for each $i < \kappa$ for which

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) \\ (\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \quad (4.17)$$

holds, it is possible to find $v < \lambda_i$ such that

$$(\exists^* z_1^a < \lambda)\Big[g_{z_1^a}^{\max}(i) < v \text{ and} \\ (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\Big[v < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\Big]\Big]. \quad (4.18)$$

Despite the lengths of the formulas involved, this is not that hard to accomplish. Since $\lambda_i < \lambda = \mathrm{cf}(\lambda)$, we can find $v < \lambda_i$ such that

$$(\exists^* z_1^a < \lambda)\big[g_{z_1^a}^{\max}(i) < v \text{ and} \\ (\exists^* y_n^b < \lambda)(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big],$$

and now the result follows from of the definition of "$\exists^{*,i} z_1^b < \lambda$". $\square$

Thus there are unboundedly many $z_0^a < \lambda$ for which there is a function $g \in \prod_{i<\kappa} \lambda_i$ such that for all sufficiently large $i < \kappa$,

$$(\exists^* z_1^a < \lambda)\Big[g_{z_1^a}^{\max}(i) \leq g(i) \text{ and} \\ (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g(i) < f_{z_0^b}(i) \\ \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, z_0^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big]\Big]. \quad (4.19)$$

Now this is logically equivalent to the statement

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \\ \big[g_{z_1^a}^{\max}(i) \leq g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, z_0^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big]. \quad (4.20)$$

Suppose we are given a particular $z_0^a < \lambda$ for which a function g as above can be found, and let us fix $i < \kappa$ "large enough" so that (4.19) holds. Also fix ordinals $\delta^a < \lambda$ and $\xi_n^b < \lambda$ that serve as suitable z_1^a and y_n^b. Just to be clear, this means that for these choices we have

$$(\exists^* z_0^b < \lambda)\big[g_{\delta^a}^{\max}(i) \leq g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]\big].$$

Sh:535

Since $\lambda_i < \lambda = \mathrm{cf}(\lambda)$, there must be some value w satisfying

$$(\exists^* z_0^b < \lambda)\big[g(i) < f_{z_0^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]\big].$$

This implies for our particular β^a, g, i, δ^a, and ξ_n^b that

$$\begin{aligned}&(\forall^* w < \lambda_i)(\exists^* z_0^b < \lambda)\big[g_{\delta^a}^{\max}(i) \leq g(i) < f_{z_1^b}(i) < w \text{ and}\\ &(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, y_n^b, z_0^b, z_1^b)]\big].\end{aligned} \tag{4.21}$$

Since $\lambda_i < \lambda = \mathrm{cf}(\lambda)$, the quantifier $(\forall^* w < \lambda_i)$ can move to the left past the quantifiers $(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)$. This tells us that for our β^a and g,

$$\begin{aligned}&(\forall^* i < \kappa)(\forall^* w < \lambda_i)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\\ &\big[g_{z_1^a}^{\max}(i) \leq g(i) < f_{z_0^b}(i) < w \text{ and}\\ &(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].\end{aligned} \tag{4.22}$$

When we put all this together, we end up with the statement

$$\begin{aligned}&(\exists^* z_0^a < \lambda)(\forall^* i < \kappa)(\exists v < \lambda_i)(\forall^* w < \lambda_i)(\exists^* z_1^a < \lambda)\\ &(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g_{z_1^a}^{\max}(i) \leq v < f_{z_0^b}(i) < w\\ &\qquad \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].\end{aligned} \tag{4.23}$$

Since both κ and λ_i are less than $\lambda = \mathrm{cf}(\lambda)$, we can move some quantifiers around and achieve

$$\begin{aligned}&(\forall^* i < \kappa)(\forall^* w < \lambda_i)(\exists^* z_0^a < \lambda)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda)\\ &(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g_{z_1^a}^{\max}(i) \leq v < f_{z_0^b}(i) < w\\ &\qquad \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].\end{aligned} \tag{4.24}$$

Thus there is a function $h \in \prod_{i<\kappa} \lambda_i$ such that

$$\begin{aligned}&(\forall^* i < \kappa)(\exists^* z_0^a < \lambda)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda)\\ &(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g_{z_1^a}^{\max}(i) \leq v < f_{z_0^b}(i) < h(i)\\ &\qquad \text{and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].\end{aligned} \tag{4.25}$$

After all this work, it is finally time to prove that we can select objects $\beta^a < \delta^a < \xi_n^b < \beta^b < \delta^b$ that satisfy all of our requirements.

Clearly, for every unbounded $\Lambda \subseteq \lambda$,

$$(\exists i < \kappa)(\exists^* x \in \Lambda)(h \upharpoonright [i, \kappa) < f_x \upharpoonright [i, \kappa).$$

Thus we can choose $i^* < \kappa$ such that $h_1^*(i^*) = j_1$ and $h_0^*(i^*) = n$, and

$$\begin{aligned}&(\exists^* z_0^a < \lambda)\Big[h \upharpoonright [i^*, \kappa) < f^{z_0^a} \upharpoonright [i^*, \kappa) \text{ and } (\exists v < \lambda_i)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)\\ &(\exists^* z_0^b < \lambda)\big[g_{z_1^a}^{\max}(i^*) \leq v < f_{z_0^b}(i^*) < h(i^*) \text{ and}\\ &(\exists^* z_1^b < \lambda)[\Phi(i^*, z_0^a, \ldots, z_1^b)]\big]\Big].\end{aligned}$$

So now we choose β^a such that $h(i^*) < f_{\beta^a}(i^*)$ and for some $\alpha < \lambda_{i^*}$,

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g^{\max}_{z_1^a}(i^*) \leq \alpha < f_{z_0^b}(i^*) < h(i^*) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i^*, z_0^a, \dots, z_1^b)]\big].$$

Now we choose δ^a, ξ_n^b, β^b, and δ^b such that

- $\beta^a < \delta^a < \xi_n^b < \beta^b$
- $g^{\max}_{\delta^a}(i^*) \leq \alpha < f_{\beta^b}(i^*) < h(i^*) < f_{\beta^a}(i^*)$
- $\Phi(i^*, \beta^a, \delta^a, \xi_n^b, \beta^b, \delta^b)$

It is straightforward to check that these objects satisfy all the requirements listed in Table 1, so by Claim 3.5, we are done.

5. Conclusions

In this final section, we will deduce some conclusions in a few concrete cases.

Theorem 2. *If μ is a singular cardinal of uncountable cofinality that is not a limit of regular Jonsson cardinals, then* $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ *holds.*

Proof. The proof of this theorem occurs in two stages—we first show that $\mathrm{Pr}_1(\mu^+, \mu^+, \mu, \mathrm{cf}(\mu))$ holds, and then we show that this result can be upgraded to obtain $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu)$.

Let μ be as hypothesized, and let us define $\lambda = \mu^+$ and $\kappa = \mathrm{cf}(\mu)$.

Claim 5.1. $\mathrm{Pr}_1(\lambda, \lambda, \mu, \kappa)$ holds.

Proof. Let $\langle \kappa_i : i < \kappa \rangle$ be a strictly increasing continuous sequence cofinal in μ. Let $S \subseteq \{\delta \in [\mu, \lambda) : \mathrm{cf}(\delta) = \kappa\}$ be stationary. Standard club–guessing results tell us that there is an S–club system $\bar{C}$ such that $\mathrm{id}_p(\bar{C}, \bar{J})$ is a proper ideal, where J_δ is the ideal $J^{b[\mu]}_{C_\delta}$ for $\delta \in S$, and furthermore, satisfying $|C_\delta| = \kappa$. (Note that this last requires that $\kappa = \mathrm{cf}(\mu)$ is uncountable.)

At this point, we have satisfied all of the assumptions of Claim 2.2 except possibly for clause (8). It suffices to show that for each $i < \kappa$, for all sufficiently large regular $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega[\theta, \kappa_i, 1]$. Since μ is not a limit of regular Jonsson cardinals, it follows that for all sufficiently large regular $\theta < \mu$, Player I has a winning strategy in $\mathrm{Gm}^\omega[\theta, \theta, 1]$. This implies, by Lemma 1.3 (1), that for all sufficiently large regular θ, Player I has a winning strategy in $\mathrm{Gm}^\omega[\theta, \kappa_i, 1]$, and so clause (8) of Claim 2.2 is satisfied. $\square$

To finish the proof of Theorem 2, it remains to show that we can increase the number of colors from μ to $\lambda = \mu^+$ — we need $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \kappa)$ instead of $\mathrm{Pr}_1(\lambda, \lambda, \mu, \kappa)$.

Lemma 5.2. *There is a coloring $c_1 : [\lambda]^2 \to \lambda$ such that whenever we are given*

- $\theta < \kappa$,
- $\langle t_\alpha : \alpha < \lambda \rangle$ *a sequence of pairwise disjoint elements of* $[\lambda]^\theta$,

- *$\zeta_\alpha \in t_\alpha$ for $\alpha < \lambda$, and*
- *$\Upsilon < \lambda$,*

we can find $\alpha < \beta$ such that $t_\alpha \subseteq \min(t_\beta)$ and

$$(\forall \zeta \in t_\alpha)[c_1(\zeta, \zeta_\beta) = \Upsilon]. \tag{5.1}$$

Proof. Let $c : [\lambda]^2 \to \mu$ be a coloring that witnesses $\Pr_1(\lambda, \lambda, \mu, \kappa)$. For each $\alpha < \lambda$, let g_α be a one–to–one function from α into μ. We define

$$c_1(\alpha, \beta) = g_\beta^{-1}(c(\alpha, \beta)). \tag{5.2}$$

Suppose now that we are given objects θ, $\langle t_\alpha : \alpha < \lambda\rangle$, $\langle \zeta_\alpha : \alpha < \lambda\rangle$, and Υ as in the statement of the lemma. Clearly we may assume that $\min(t_\alpha) > \alpha$.

For $i < \mu$, we define $X_i := \{\alpha \in [\gamma, \lambda) : g_{\zeta_\alpha}(\Upsilon) = i\}$. Since λ is a regular cardinal, it is clear that there is $i^* < \mu$ for which $|X_{i^*}| = \lambda$. Since c exemplifies $\Pr_1(\lambda, \lambda, \mu, \kappa)$, for some $\alpha < \beta$ in X_{i^*} we have $t_\alpha \subseteq \min(t_\beta)$ and

$$(\forall \zeta \in t_\alpha)[c(\zeta, \zeta_\beta) = i^*]. \tag{5.3}$$

By definition, this means

$$(\forall \zeta \in t)[c_1(\zeta, \zeta_\beta) = g^{-1}(c(\alpha, \beta)) = g^{-1}(i^*) = \Upsilon], \tag{5.4}$$

hence α and β are as required. □

To continue the proof of Theorem 2, we define a coloring $c_2 : [\lambda]^2 \to \lambda$ by

$$c_2(\alpha, \beta) = c_1(\alpha, \nu(\alpha, \beta)), \tag{5.5}$$

where $\nu(\alpha, \beta)$ is as in the proof of Theorem 1.

It remains to check that c_2 witnesses $\Pr_1(\lambda, \lambda, \lambda, \kappa)$. Toward this end, suppose we are given $\theta < \kappa$, $\langle t_\alpha : \alpha < \lambda\rangle$ a sequence of pairwise disjoint members of $[\lambda]^\theta$, and $\Upsilon < \lambda$. We need to find δ^a and δ^b less than λ such that

$$\epsilon^a \in t_{\delta^a} \wedge \epsilon^b \in t_{\delta^b} \implies c_2(\epsilon^a, \epsilon^b) = \Upsilon. \tag{5.6}$$

Lemma 5.3. *There is a stationary set of $\gamma_1 < \lambda$ such that for some $\gamma_0 < \gamma_1$ and $\beta \in [\gamma_1, \lambda)$, if $\gamma_0 \le \alpha < \gamma_1$, then the function ν is constant on $t_\alpha \times t_\beta$.*

Proof. Let E be an arbitrary closed unbounded subset of λ, and let W be the set of ordinals $< \lambda$ satisfying the properties of γ_1. In the proof of Theorem 1, without loss of generality we can have $E \in M_0$. This means that the ordinal β^* found in the course of that proof will be in E, so we finish by observing that $\beta^* \in W$. □

An application of Fodor's Lemma gives us a single ordinal γ_0 and a stationary $W' \subseteq W$ such that for all $\gamma \in W'$, there is a $\beta_\gamma \in [\gamma, \lambda)$ such that for all $\alpha \in [\gamma_0, \gamma)$, $\nu\restriction(t_\alpha \times t_\beta)$ is constant.

Using properties of the coloring c_1, we can find α and γ such that

- $\gamma_0 \le \alpha < \lambda$
- $\gamma \in W' \setminus (\sup(t_\alpha) + 1)$, and

Sh:535

- $\zeta \in t_\alpha \Longrightarrow c_1(\zeta, \gamma) = \Upsilon$.

Now given $\epsilon^a \in t_\alpha$ and $\epsilon^b \in t_{\beta_\gamma}$, we find

$$c_2(\epsilon^a, \epsilon^b) = c_1(\epsilon^a, \gamma) = \Upsilon, \tag{5.7}$$

and therefore c_2 exemplifies $\Pr(\lambda, \lambda, \lambda, \kappa)$. □

Theorem 2 strengthens results in [1] as clearly $\Pr_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ implies that μ^+ has a Jonsson algebra (i.e., μ^+ is not a Jonsson cardinal). The question of whether the successor of a singular cardinal can be a Jonsson cardinal is a well–known open question.

We note that many of the results from Section 2 of [1] dealing with the existence of winning strategies for Player I in $\mathrm{Gm}^\omega[\lambda, \mu, \gamma]$ can be combined with Theorem 1 to give new results. For example, we have the following result from [1].

Proposition 5.4. *If $\tau \leq 2^\kappa$ but $(\forall\theta < \kappa)[2^\theta < \tau]$, then Player I has a winning strategy in the game* $\mathrm{Gm}^\omega(\tau, \kappa, \kappa^+)$.

Proof. See Claim 2.3(1) and Claim 2.4(1) of [1]. □

Armed with this, the following claim is straightforward.

Claim 5.5. Let μ be a singular cardinal of uncountable cofinality. Further assume that χ is a cardinal such that $2^{<\chi} \leq \mu < 2^\chi$. Then $\Pr_1(\mu^+, \mu^+, \chi, \mathrm{cf}(\mu))$ holds.

Proof. If $2^{<\chi} < \mu$, then Claims 2.3(1) and 2.4(1) of [1] imply that for every sufficiently large $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega(\theta, \chi, \chi^+)$.

If $\mu = 2^{<\chi}$, then $\mathrm{cf}(\mu) = \mathrm{cf}(\chi)$. Let $\langle \kappa_i : i < \mathrm{cf}(\mu)\rangle$ be a strictly increasing continuous sequence of cardinals cofinal in χ. Given $i < \mathrm{cf}(\mu)$, we claim that for all sufficiently large regular $\tau < \mu$, Player I has a winning strategy in $\mathrm{Gm}^\omega(\tau, \kappa_i, \chi)$. Once we have established this, $\Pr_1(\mu^+, \mu^+, \chi, \mathrm{cf}(\mu))$ follows by Theorem 1.

Given $\tau = \mathrm{cf}(\tau)$ satisfying $2^{\kappa_i} < \tau < \mu$, let η be the least cardinal such that $\tau \leq 2^\eta$. Clearly $\kappa_i < \eta < \chi$. By Proposition 5.4, Player I wins the game $\mathrm{Gm}^\omega(\tau, \eta, \eta^+)$. This implies (since $\eta^+ < \chi$ and $\kappa_i < \eta$) that Player I wins the game $\mathrm{Gm}^\omega(\tau, \kappa_i, \chi)$ as required. □

We can also use Claim 1.4 to prove similar results. For example we have the following.

Claim 5.6. Let μ be a singular cardinal of uncountable cofinality. Further assume that $\chi < \mu$ satisfies $2^\chi < \mu < \beth_{(2^\chi)^+}(\chi)$. Then $\Pr_1(\mu^+, \mu^+, \chi, \mathrm{cf}(\mu))$ holds.

Proof. Again, the main point is that for all sufficiently large regular $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega[\theta, \chi, (2^\chi)^+]$. This follows immediately from Claim 1.4. Since $(2^\chi)^+ < \mu$, Theorem 1 is applicable. □

In a sequel to this paper, we will address the situation where λ is the successor of a singular cardinal of countable cofinality. Similar results hold, but the combinatorics involved are trickier.

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