Sh:196

ANOTHER FORM OF A CRITERION FOR THE EXISTENCE OF TRANSVERSALS

R. AHARONI, C. ST. J. A. NASH-WILLIAMS AND S. SHELAH

ABSTRACT

In [2] we proved a necessary and sufficient condition for a family of sets to possess a transversal. We now prove a slightly more concrete version of this result, using the function q of [4].

1. *Introduction and definitions*

In [2] we established a criterion for deciding whether any family of sets possesses a transversal. The purpose of the present paper is to prove a variant of this theorem which is in some sense more explicit. As in [2], we replace the terminology of 'transversals' by that of 'marriage in societies', which we now describe.

If F is a set of ordered pairs, a is any element and A is any set, then $F\langle a\rangle$ denotes $\{y:(a,y)\in F\}$, $F(a)$ denotes the element of $F\langle a\rangle$ if $|F\langle a\rangle| = 1$, $F[A]$ denotes $\bigcup\{F\langle a\rangle : a\in A\}$, $\operatorname{dom} F$ (the *domain* of F) denotes $\{a : F\langle a\rangle \neq \emptyset\}$, $F^{-1} = \{(y,x):(x,y)\in F\}$, $\operatorname{rge} F$ (the *range* of F) denotes $\operatorname{dom} F^{-1}$ and $F\restriction A$ denotes $F\cap(A\times \operatorname{rge} F)$ (that is, the restriction of F to A). We say that F is a *function* if $|F\langle a\rangle| = 1$ for every $a\in\operatorname{dom} F$, and that F is *injective* if $|F^{-1}\langle x\rangle| = 1$ for every $x\in\operatorname{rge} F$. An *injection of A into* a set B is an injective function F such that $\operatorname{dom} F = A$ and $\operatorname{rge} F\subseteq B$.

A *society* is a triple (M, W, K) where M, W are disjoint sets and $K\subseteq M\times W$. Elements of M and W are *men* and *women* of the society respectively. A man m and woman w are said to *know* each other if $(m,w)\in K$. For a society Λ we denote by $M_\Lambda, W_\Lambda, K_\Lambda$ the sets such that $\Lambda = (M_\Lambda, W_\Lambda, K_\Lambda)$. Throughout this paper, the symbol Γ will denote a society and the symbols M, W, K will denote $M_\Gamma, W_\Gamma, K_\Gamma$ respectively.

If $X\subseteq W$ then $D(X)$ denotes $\{a\in M : K\langle a\rangle\subseteq X\}$. If we want to specify that $D(X)$ is taken in a society Δ, we shall write $D_\Delta(X)$ in place of $D(X)$: thus $D_\Delta(X) = \{a\in M_\Delta : K_\Delta\langle a\rangle\subseteq X\}$ when $X\subseteq W_\Delta$. When no subscript is attached to D, it will be understood that $D(X)$ means $D_\Gamma(X)$, that is, $D(X)$ is interpreted in whichever society is denoted by the symbol Γ unless a subscript is used to indicate the contrary. For $A\subseteq M$ and $X\subseteq W$, $\Gamma[A,X]$ denotes the society $(A, X, K\cap(A\times X))$, and $\Gamma - A$, $\Gamma - u$ (where $u\in W$), $\Gamma - A - X$ denote, respectively, $\Gamma[M\setminus A, W]$, $\Gamma[M, W\setminus\{u\}]$, $\Gamma[M\setminus A, W\setminus X]$. A society Γ' is called a *subsociety* of Γ if $\Gamma' = \Gamma[A,X]$ for some $A\subseteq M$, $X\subseteq W$. The society $\Gamma - A - X$ is denoted in this case by Γ/Γ'. A subsociety Π of Γ is said to be *saturated* if $K_\Gamma[M_\Pi]\subseteq W_\Pi$. We write $\Pi\lhd\Gamma$ for 'Π is a saturated subsociety of Γ'. If $\bar\Pi$ denotes a family $((M_i, W_i, K_i) : i\in I)$ of subsocieties of Γ then the *union* $\bigcup\bar\Pi$ of these subsocieties is the society $\left(\bigcup_{i\in I} M_i, \bigcup_{i\in I} W_i, \bigcup_{i\in I} K_i\right)$, and their *join* $\bigvee\bar\Pi$ is

Received 10 January, 1983.

J. London Math. Soc. (2), 29 (1984), 193–203

Sh:196

$\Gamma\left[\bigcup_{i\in I} M_i, \bigcup_{i\in I} W_i\right]$. The *intersection* $\Pi \cap \Sigma$ of two subsocieties Π, Σ of Γ is the society $(M_\Pi \cap M_\Sigma, W_\Pi \cap W_\Sigma, K_\Pi \cap K_\Sigma)$. A society which contains just one woman and no man is said to be *maidenly*. The society $(\varnothing, \varnothing, \varnothing)$, which contains no men and no women, is said to be *empty*.

An *espousal* of Γ is an injective function E such that $\operatorname{dom} E = M$ and $E \subseteq K$. A society is *espousable* if it has an espousal (if, informally speaking, all its men can be assigned distinct wives in such a way that each man marries a woman whom he knows), and *inespousable* if not. A *partial espousal* of Γ is an injective function E such that $E \subseteq K$. A society Γ is *critical* if it is espousable and $\operatorname{rge} E = W$ for every espousal E of Γ.

In this paper, lower case Greek letters will denote ordinals, and in particular κ, μ will denote cardinals. A *κ-subset* of a set S is a subset of S with cardinality κ.

Let κ be a regular uncountable cardinal. A subset Ω of κ is *closed* (in κ) if $\sup \Xi \in \Omega \cup \{\kappa\}$ for every non-empty subset Ξ of Ω, and is *unbounded* (in κ) if $\sup \Omega = \kappa$. A subset Φ of κ is *stationary* (or *κ-stationary*) if $\Phi \cap \Omega \neq \varnothing$ for every closed unbounded subset Ω of κ. A function $f : \Phi \to \kappa$ is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in \Phi \setminus \{0\}$.

We require the following property of stationary sets.

LEMMA 1 (Fodor's Lemma: see, for example, [3, Theorem 22]). *Let κ be a regular uncountable cardinal. If Φ is κ-stationary and $f : \Phi \to \kappa$ is a regressive function then there exist a κ-stationary subset Ψ of Φ and an ordinal $\beta < \kappa$ such that $f[\Psi] = \{\beta\}$: in particular, $|f^{-1}\langle\beta\rangle| = \kappa$.*

In this paper, the word 'sequence' means 'transfinite sequence', that is, a function whose domain is an ordinal number or, equivalently, a family of the form $(x_\alpha : \alpha < \zeta)$ indexed by the ordinals less than some ordinal ζ. (These definitions are equivalent since we understand a 'family' $(x_i : i \in I)$ to be the same thing as the function $\{(i, x_i) : i \in I\}$.) We call x_α the *α-th term* of a sequence $(x_\alpha : \alpha < \zeta)$. If s denotes this sequence and $\theta \leqslant \zeta$ then s_θ will denote the sequence $(x_\alpha : \alpha < \theta)$ or, equivalently, $s \restriction \theta$ and will be called an *initial segment* of s. We write $t \prec s$ for 't is an initial segment of s'. An injective transfinite sequence will be called a *string*. A string s will be called a string *in* a set A if $\operatorname{rge} s \subseteq A$, and will be called a string *on* A if $\operatorname{rge} s = A$. The sequence whose domain is the ordinal 0 will be denoted by $\square$. In fact, since the ordinal 0 is the empty set, it follows that $\square$ is also the empty set, but we denote the empty set by $\square$ and not $\varnothing$ when it plays the rôle of a sequence. The string $\{(0, x)\}$ will be denoted by $[x]$: in other words, $[x]$ is the string s such that $\operatorname{dom} s = 1 = \{0\}$ and $s(0) = x$. If s, t are strings with disjoint ranges and $\operatorname{dom} s = \alpha$, $\operatorname{dom} t = \beta$ then the *concatenation* $s * t$ of s and t is the string u with domain $\alpha + \beta$ such that $u(\theta) = s(\theta)$ for every $\theta < \alpha$ and $u(\alpha + \phi) = t(\phi)$ for every $\phi < \beta$. If s is a string and $\alpha < \beta \leqslant \operatorname{dom} s$ then $s_{[\alpha, \beta)}$ will denote the string such that $s_\beta = s_\alpha * s_{[\alpha, \beta)}$.

A sequence of subsocieties of Γ may often be denoted by a Greek capital letter with a bar above it, and then the α-th term of this sequence will be denoted by the same Greek capital letter, unbarred, with subscript α. For example, if $\bar{\Lambda}$ is a sequence of subsocieties of Γ then Λ_α is its α-th term. Moreover, if $\bar{\Lambda} = (\Lambda_\alpha : \alpha < \zeta)$ and $\theta \leqslant \zeta$ then $\bar{\Lambda}_\theta$ will denote the sequence $(\bar{\Lambda})_\theta = (\Lambda_\alpha : \alpha < \theta)$. The sequence $\bar{\Lambda}$ will be called *non-descending* if Λ_α is a subsociety of Λ_β whenever $\alpha < \beta < \zeta$; and $\bar{\Lambda}$ will be called *continuous* if it is non-descending and $\bigcup \bar{\Lambda}_\theta = \Lambda_\theta$ for every limit ordinal $\theta < \zeta$. A

Sh:196

ζ-tower in Γ is a continuous non-descending sequence $(\Pi_\alpha : \alpha \leqslant \zeta)$ of saturated subsocieties of Γ such that Π_0 is empty. A *ζ-ladder* in Γ is a sequence $\bar{\Lambda} = (\Lambda_\alpha : \alpha < \zeta)$ of subsocieties of Γ such that $\Lambda_\alpha \cap \Lambda_\beta$ is empty whenever $\alpha < \beta < \zeta$ and $\bigvee \bar{\Lambda}_\alpha^\wedge \lhd \Gamma$ for every $\alpha \leqslant \zeta$. A sequence of subsocieties of Γ is a *tower* (*ladder*) if it is a ζ-tower (ζ-ladder) for some ordinal ζ.

If $\mathscr{T}$ is the set of towers in Γ and $\mathscr{L}$ is the set of ladders in Γ then there is an obvious bijection $l : \mathscr{T} \to \mathscr{L}$ such that

(i) if $\bar{\Pi}$ is a ζ-tower in Γ then $l(\bar{\Pi})$ is the ζ-ladder $(\Pi_{\alpha+1}/\Pi_\alpha : \alpha < \zeta)$,

(ii) if $\bar{\Lambda}$ is a ζ-ladder in Γ then $l^{-1}(\bar{\Lambda})$ is the ζ-tower $(\bigvee \bar{\Lambda}_\alpha : \alpha \leqslant \zeta)$.

We shall call $l(\bar{\Pi})$ the *ladder of* $\bar{\Pi}$.

Let Υ be the class of all cardinals κ such that either $0 < \kappa < \aleph_0$ or κ is regular. We shall define by induction on κ what is meant by saying that a subsociety of Γ is a 'κ-obstruction' in Γ, for every $\kappa \in \Upsilon$. First, when $0 < \kappa \leqslant \aleph_0$, a subsociety Π of Γ is a *κ-obstruction in* Γ if it is saturated and $\Pi - L$ is critical for some κ-subset L of M_Π. Suppose now that κ is regular and uncountable and that 'μ-obstruction' has been defined for every $\mu \in \Upsilon$ such that $\mu < \kappa$. Let a subsociety of Γ be called a *$(<\kappa)$-obstruction in* Γ if it is a μ-obstruction in Γ for some $\mu \in \Upsilon$ such that $\mu < \kappa$. If $\bar{\Pi}$ is a κ-tower in Γ whose ladder is $\bar{\Lambda}$, let $\Psi(\bar{\Pi})$ denote the set of all $\alpha < \kappa$ such that Λ_α is a $(<\kappa)$-obstruction in Γ/Π_α. A κ-tower $\bar{\Pi}$ in Γ whose ladder is $\bar{\Lambda}$ will be said to be *obstructive* (in Γ) if

(a) for each $\alpha < \kappa$, Λ_α is either a $(<\kappa)$-obstruction in Γ/Π_α or maidenly, and

(b) $\Psi(\bar{\Pi})$ is κ-stationary (that is, the first alternative of (a) occurs for a 'reasonably large' set of values of α).

A subsociety Π of Γ is a *κ-obstruction in* Γ if $\Pi = \bigcup \bar{\Pi}$ for some obstructive κ-tower $\bar{\Pi}$ in Γ.

The following theorem was proved in [2].

THEOREM 1. *A society Γ is inespousable if and only if, for some $\kappa \in \Upsilon$, there exists a κ-obstruction in Γ.*

The set of integers will be denoted by $\mathbb{Z}$, and $\mathbb{Z}^*$ will denote $\mathbb{Z} \cup \{-\infty, \infty\}$, that is a set whose elements are the integers and two further 'numbers' ∞ and $-\infty$. Elements of $\mathbb{Z}^*$ will be called *quasi-integers*. The *size* $\|A\|$ of a set A is defined to be its cardinality $|A|$ if A is finite and to be ∞ if A is infinite: thus $\|A\| \in \mathbb{Z}^*$ for every set A. The *sum* $a_1 + \ldots + a_n$ of n quasi-integers $a_1, \ldots, a_n$ has its usual meaning if the a_i are all integers, is defined to be ∞ if at least one a_i is ∞, and is defined to be $-\infty$ if no a_i is ∞ but at least one is $-\infty$. The *difference* $a - b$ of two quasi-integers is the sum of a and $-b$; and likewise the sum of the quasi-integers $a, -b, c$ may be denoted by $a - b + c$, etc. For our purposes, the most important distinctive feature of these definitions is that $\infty - \infty$ is defined to be ∞, since we wish to think of $\infty - \infty$ as the *largest possible* value of $\|A \setminus B\|$ for sets A, B such that $B \subseteq A$ and $\|A\| = \|B\| = \infty$. *Inequalities* between quasi-integers are defined in the obvious way. The *infimum* $\inf S$ of a non-empty subset S of $\mathbb{Z}^*$ is the greatest quasi-integer a such that $a \leqslant s$ for every $s \in S$, and the *supremum* $\sup S$ is analogously defined. If λ is a limit ordinal and

$a_\theta \in \mathbb{Z}^*$ for every $\theta < \lambda$, we define $\liminf_{\theta \to \lambda} a_\theta$ to be $\sup\{i_\phi : \phi < \lambda\}$, where i_ϕ denotes $\inf\{a_\theta : \phi \leqslant \theta < \lambda\}$.

If f is a string in W, then $D(f)$, $\Gamma[f]$, Γ/f will denote $D(\mathrm{rge} f)$, $\Gamma[D(f)\,\mathrm{rge} f]$ and $\Gamma/\Gamma[f]$ respectively. With any string f in W, we associate a quasi-integer $q(f)$, defined as follows. Define $q(\square)$ to be $-\|D(\varnothing)\|$. If now $\mathrm{dom}\, f$ is an ordinal $\sigma > 0$ and $q(f')$ has been defined for every string f' whose domain is less than σ, then define $q(f)$ to be

(i) $q(f_\rho)+1-\|D(f)\setminus D(f_\rho)\|$ if σ is a successor ordinal $\rho+1$,

(ii) $\liminf_{\theta \to \sigma} q(f_\theta) - \left\| D(f) \setminus \bigcup_{\theta < \sigma} D(f_\theta) \right\|$ if σ is a limit ordinal.

If we wish to indicate that $D(f)$ or $q(f)$ are to be interpreted in a society Δ, we shall write $D_\Delta(f)$ or $q_\Delta(f)$ respectively. When no subscript is attached to D or q, it will be understood that $D(f)$ and $q(f)$ mean $D_\Gamma(f)$ and $q_\Gamma(f)$ respectively, that is, $D(f)$ and $q(f)$ are then interpreted in whichever society is denoted by the symbol Γ. For example, the equation in Lemma 4 below means $q_{\Gamma-A}(f) = q_\Gamma(f)+\|A \cap D_\Gamma(f)\|$.

A society $\Gamma = (M, W, K)$ is *q-admissible* if $q(f) \geqslant 0$ for every string f in W, and is *packed* if $q(f) = 0$ for some string f **on** W. Roughly speaking, as explained in [**4**], $q(f)$ is an upper bound for the number of women whom we could hope to leave unmarried in $\mathrm{rge} f$ after working along the sequence f term by term, trying at each stage to ensure that wives have been found for all the men who demand them from amongst the set of women so far considered. Thus an espousable society must necessarily be q-admissible: more precisely, this follows from Lemma 2 below. In [**4**] it was shown that every q-admissible society with countably many men is espousable (and a somewhat stronger version of this result was proved in [**5**]). The following result, which was Corollary 3.6 of [**1**], is more relevant for our present purposes.

THEOREM 2. *A society is critical if and only if it is packed and q-admissible.*

Since Theorem 2 gives a good understanding of the structure of critical societies, it is reasonable to use these societies in defining the notion of 'obstruction' which is the crucial feature of Theorem 1. However, it seems plausible that the power of a critical subsociety Δ of Γ to help to obstruct espousability of Γ lies in Δ being packed and not in its being q-admissible. Thus arguably it might be appropriate to replace 'critical' by 'packed' subsocieties in the definition of 'obstruction' and show that Theorem 1 still remains true. This is the purpose of the present paper.

These remarks suggest the following modification of the definition of 'κ-obstruction'. We shall define by induction on κ what is meant by saying that a subsociety of Γ is a 'κ-hindrance' in Γ, for every $\kappa \in \Upsilon$. First, when $0 < \kappa \leqslant \aleph_0$, a subsociety Π of Γ is a *κ-hindrance in* Γ if it is saturated and $\Pi - L$ is packed for some κ-subset L of M_Π. Suppose now that κ is regular and uncountable and that 'μ-hindrance' has been defined for every $\mu \in \Upsilon$ such that $\mu < \kappa$. Let a subsociety of Γ be called a *$(<\kappa)$-hindrance in* Γ if it is a μ-hindrance in Γ for some $\mu \in \Upsilon$ with $\mu < \kappa$. If $\bar{\Pi}$ is a κ-tower in Γ whose ladder is $\bar{\Lambda}$, let $\Phi(\bar{\Pi})$ denote the set of all $\alpha < \kappa$ such that Λ_α is a $(<\kappa)$-hindrance in Γ/Π_α. A κ-tower $\bar{\Pi}$ in Γ whose ladder is $\bar{\Lambda}$ will be said to be *hindering* (in Γ) if

(a) for each $\alpha < \kappa$, Λ_α is either a $(<\kappa)$-hindrance in Γ/Π_α or maidenly, and

(b) $\Phi(\bar{\Pi})$ is κ-stationary.

A subsociety Π of Γ is a *κ-hindrance in* Γ if $\Pi = \bigcup\bar{\Pi}$ for some hindering κ-tower $\bar{\Pi}$ in Γ.

The purpose of this paper is to prove the following theorem.

THEOREM 3. *A society Γ is inespousable if and only if, for some $\kappa \in \Upsilon$, there exists a κ-hindrance in Γ.*

An advantage of Theorem 3 (as compared with Theorem 1) might be that it avoids any suggestion of characterizing espousable societies in terms of espousable societies. The notion of espousability occurs in the definition of critical societies, which in turn are mentioned in the definition of a κ-obstruction, but the definition of a κ-hindrance contains no direct or indirect reference to espousability. Theorem 3 also gives a somewhat clearer impression of 'how to construct all possible inespousable societies'.

2. *Proof of Theorem* 3

Lemmas 2–5 below are (in essence) Lemmas 1.1, 2.8 and 2.13 and Corollary 2.7 of [**4**], where proofs may be found. They were in fact stated only for countable strings in [**4**], but their proofs do not depend on the countability of the strings.

LEMMA 2. *If E is an espousal of Γ and f is a string in W then* $\|(\mathrm{rge}\, f)\setminus E[D(f)]\| \leqslant q(f)$.

LEMMA 3. *If f, g are strings in W with disjoint ranges then* $D_{\Gamma/f}(g) = D(f * g)\setminus D(f)$.

LEMMA 4. *If A is a finite subset of M and f is a string in W then* $q_{\Gamma - A}(f) = q(f) + \|A \cap D(f)\|$.

LEMMA 5. *If $V \subseteq W$ and $\Pi = \Gamma[D(V), V]$, and if f is a string in V then* $q_\Pi(f) = q(f)$.

LEMMA 6. *If $u \in W$ and f is a string in $W\setminus\{u\}$ and $q(f * [u]) \in \mathbb{Z}$ then* $q(f * [u]) = q_{\Gamma - u}(f) + 1$.

Proof. Let $A = D(f * [u])\setminus D(f)$. Since $q(f) + 1 - \|A\| = q(f * [u]) \in \mathbb{Z}$, it follows that A is finite. Let $\Pi = \Gamma[D(f * [u]), \mathrm{rge}\, f]$. Then

$$\Pi = (\Gamma - u)[D_{\Gamma - u}(\mathrm{rge}\, f), \mathrm{rge}\, f] \qquad \text{and} \qquad \Pi - A = \Gamma[D(\mathrm{rge}\, f), \mathrm{rge}\, f].$$

Therefore, by Lemma 5, $q_{\Gamma - u}(f) = q_\Pi(f)$ and $q(f) = q_{\Pi - A}(f)$. Hence, using

Lemma 4,

$$q(f) = q_{\Pi-A}(f) = q_{\Pi}(f)+\|A \cap D_{\Pi}(f)\| = q_{\Gamma-u}(f)+\|A\|$$

(since $A \subseteq M_{\Pi} = D_{\Pi}(f)$), and so (since A is finite)

$$q_{\Gamma-u}(f)+1 = q(f)+1-\|A\| = q(f * [u]).$$

LEMMA 7. *If f, g are strings in W and $q(f) \neq -\infty$ and* $(\operatorname{rge} f) \cap (\operatorname{rge} g) = \varnothing$ *then*

$$q(f * g) = q(f)+q_{\Gamma/f}(g). \tag{1}$$

Proof. Observe first that

$$D_{\Gamma/f}(\varnothing) = \{m \in M_{\Gamma/f} : K_{\Gamma/f}\langle m\rangle = \varnothing\} = \{m \in M \setminus D(f) : K\langle m\rangle \setminus \operatorname{rge} f = \varnothing\}$$

$$= \{m \in M \setminus D(\operatorname{rge} f) : K\langle m\rangle \subseteq \operatorname{rge} f\} = \varnothing$$

by the definition of $D(\operatorname{rge} f)$. Therefore $q_{\Gamma/f}(\square) = 0$, and so (1) is true when $\operatorname{dom} g = 0$.

Now suppose that $\operatorname{dom} g = \sigma > 0$, and assume the inductive hypothesis that $q(f * h) = q(f)+q_{\Gamma/f}(h)$ for every string h in $W \setminus \operatorname{rge} f$ with $\operatorname{dom} h < \sigma$.

Obviously, if s and t denote strings in W, then

$$s \prec t \Rightarrow D(s) \subseteq D(t). \tag{2}$$

If σ is a successor ordinal $\rho+1$ then, by the inductive hypothesis, Lemma 3 and (2),

$$q(f)+q_{\Gamma/f}(g) = q(f)+q_{\Gamma/f}(g_\rho)+1-\|D_{\Gamma/f}(g)\setminus D_{\Gamma/f}(g_\rho)\|$$

$$= q(f * g_\rho)+1-\|(D(f * g)\setminus D(f))\setminus(D(f * g_\rho)\setminus D(f))\|$$

$$= q(f * g_\rho)+1-\|D(f * g)\setminus D(f * g_\rho)\| = q(f * g).$$

Now suppose that σ is a limit ordinal. Let $\operatorname{dom} f = \delta$. The hypothesis that $q(f) \neq -\infty$ and the inductive hypothesis imply that

$$q(f) + \liminf_{\theta \to \sigma} q_{\Gamma/f}(g_\theta) = \liminf_{\theta \to \sigma} (q(f)+q_{\Gamma/f}(g_\theta))$$

$$= \liminf_{\theta \to \sigma} q(f * g_\theta) = \liminf_{\theta \to \delta+\sigma} q((f * g)_\theta).$$

Moreover, by Lemma 3 and (2),

$$D_{\Gamma/f}(g)\setminus \bigcup_{\theta<\sigma} D_{\Gamma/f}(g_\theta) = (D(f * g)\setminus D(f))\setminus \bigcup_{\theta<\sigma} (D(f * g_\theta)\setminus D(f))$$

$$= D(f * g)\setminus \bigcup_{\theta<\sigma} D(f * g_\theta) = D(f * g)\setminus \bigcup_{\theta<\delta+\sigma} D((f * g)_\theta).$$

Sh:196

Hence

$$q(f)+q_{\Gamma/f}(g) = q(f) + \liminf_{\theta \to \sigma} q_{\Gamma/f}(g_\theta) - \left\| D_{\Gamma/f}(g) \setminus \bigcup_{\theta < \sigma} D_{\Gamma/f}(g_\theta) \right\|$$

$$= \liminf_{\theta \to \delta+\sigma} q\big((f * g)_\theta\big) - \left\| D(f * g) \setminus \bigcup_{\theta < \delta+\sigma} D((f * g)_\theta) \right\| = q(f * g).$$

LEMMA 8. *Let ξ be a limit ordinal, let $(\Pi_\alpha : \alpha < \xi)$ be a non-descending sequence of saturated subsocieties of Γ and let $\Pi_{\alpha+1}/\Pi_\alpha = (M_\alpha, W_\alpha, K_\alpha)$ for each $\alpha < \xi$. Let E be a partial espousal of Γ and let Q be a subset of W. Let $E_\alpha = E \cap (M_\alpha \times W_\alpha)$, $A_\alpha = M_\alpha \setminus \operatorname{dom} E_\alpha$, $X_\alpha = W_\alpha \setminus \operatorname{rge} E_\alpha$ for each $\alpha < \xi$. Suppose that $W = \bigcup\{W_\alpha : \alpha < \xi\}$ and $|X_\alpha \setminus Q| \leqslant |A_\alpha|$ for every $\alpha < \xi$. Then*

$$|W \setminus \operatorname{rge} E| \leqslant |M \setminus \operatorname{dom} E| + |Q|.$$

Proof. Observe first that, for each $\alpha < \xi$,

$$E[M_\alpha] \subseteq K[M_\alpha] \subseteq \bigcup\{W_\theta : \theta \leqslant \alpha\} \tag{3}$$

since $\Pi_{\alpha+1} \lhd \Gamma$. For each $\alpha < \xi$, select an injection I_α of $X_\alpha \setminus Q$ into A_α and let $I = \bigcup\{I_\alpha : \alpha < \xi\}$. Let $\mathscr{D}$ be a directed graph whose set of vertices is $M \cup W$ and whose edges are a directed edge from a to $E(a)$ for each $a \in \operatorname{dom} E$ and a directed edge from u to $I(u)$ for each $u \in \operatorname{dom} I$. Then each vertex of $\mathscr{D}$ has invalency 0 or 1 and outvalency 0 or 1, and hence each connected component of $\mathscr{D}$ is a directed path or directed circuit. Consider any $w \in W \setminus \operatorname{rge} E$. Since w has invalency 0, it is the initial vertex of a directed path $\mathscr{P}_w$ which is a component of $\mathscr{D}$. Let the vertices of $\mathscr{P}_w$, in order of occurrence as $\mathscr{P}_w$ is described starting from w, be $w_1(= w)$, $m_1, w_2, m_2, w_3, m_3, \ldots$ (where this sequence terminates if $\mathscr{P}_w$ is finite). Let $w_i \in W_{\alpha(i)}$ for each w_i in $\mathscr{P}_w$. For each m_i in $\mathscr{P}_w$ we have $(w_i, m_i) \in I \subseteq \bigcup\{X_\alpha \times A_\alpha : \alpha < \xi\}$ and $w_i \in W_{\alpha(i)}$, and therefore $w_i \in X_{\alpha(i)}$ and

$$m_i \in A_{\alpha(i)} = M_{\alpha(i)} \setminus \operatorname{dom}\big(E \cap (M_{\alpha(i)} \times W_{\alpha(i)})\big). \tag{4}$$

If i is such that w_i, w_{i+1} both exist, then $w_{i+1} = E(m_i) \notin W_{\alpha(i)}$ by (4) but, by (4) and (3),

$$w_{i+1} = E(m_i) \in E[M_{\alpha(i)}] \subseteq \bigcup\{W_\theta : \theta \leqslant \alpha(i)\}. \tag{5}$$

Therefore $w_{i+1} \in W_\theta$ for some $\theta < \alpha(i)$, that is $\alpha(i+1) < \alpha(i)$. Since there cannot be an infinite decreasing sequence of ordinals $\alpha(1) > \alpha(2) > \ldots$, it follows that $\mathscr{P}_w$ has a terminal vertex $g(w)$. For any i such that w_i exists, either $i = 1$ and $w_i = w \notin \operatorname{rge} E$, or $i > 1$ and $w_i \in E[M_{\alpha(i-1)}]$ by (5) which, since $\alpha(i) < \alpha(i-1)$, implies that $w_i \notin E[M_{\alpha(i)}]$. In both cases it follows that $w_i \notin \operatorname{rge} E_{\alpha(i)}$ and so $w_i \in X_{\alpha(i)} \subseteq Q \cup \operatorname{dom} I$. Therefore $w_i \in Q$ if $w_i = g(w)$. If $m_i = g(w)$ then clearly $m_i \in M \setminus \operatorname{dom} E$; and so $g(w) \in Q \cup (M \setminus \operatorname{dom} E)$. Hence $\{(w, g(w)) : w \in W \setminus \operatorname{rge} E\}$ is an injection of $W \setminus \operatorname{rge} E$ into $Q \cup (M \setminus \operatorname{dom} E)$, which proves the lemma.

LEMMA 9. *If f is a string on W and $q(f) \in \mathbb{Z}$ and $A \subseteq M$, $X \subseteq W$ and $\Gamma - A - X$ is espousable then $|X| \leqslant |A| + q(f)$ (with the natural convention that $|A| + q(f)$ means $|A|$ if $|A|$ is infinite and $q(f)$ is a negative integer).*

Proof. The proof will be by induction on $\operatorname{dom} f$. If $\operatorname{dom} f = 0$ then $\varnothing = \operatorname{rge} f = W$ and so $D(\varnothing) = M$ and therefore $-\|M\| = -\|D(\varnothing)\| = q(f) \in \mathbb{Z}$, which shows that M is finite. Moreover, since $X \subseteq W = \varnothing$ and $\Gamma - A - X$ is espousable, it follows that $A = M$ and $|X| = 0 = |M| - |M| = |A| + q(f)$.

Now suppose that $\operatorname{dom} f > 0$, and assume the inductive hypothesis that if Δ is a society and g is a string on W_Δ and $q_\Delta(g) \in \mathbb{Z}$ and $B \subseteq M_\Delta$, $Y \subseteq W_\Delta$ and $\Delta - B - Y$ is espousable and $\operatorname{dom} g < \operatorname{dom} f$ then $|Y| \leqslant |B| + q_\Delta(g)$.

Let E be an espousal of $\Gamma - A - X$.

Suppose first that A is finite. Since E is an espousal of $\Gamma - A$, it follows by Lemma 2 that

$$\|(\operatorname{rge} f) \setminus E[D_{\Gamma - A}(f)]\| \leqslant q_{\Gamma - A}(f). \tag{6}$$

Since $\operatorname{rge} f = W$, it follows that

$$X \subseteq (\operatorname{rge} f) \setminus E[D_{\Gamma - A}(f)] \tag{7}$$

and that $D(f) = M$, so that $A \cap D(f) = A$ and therefore, by Lemma 4,

$$q_{\Gamma - A}(f) = \|A\| + q(f). \tag{8}$$

By (7), (6) and (8), $\|X\| \leqslant \|A\| + q(f)$, which implies that $|X| \leqslant |A| + q(f)$ since A is finite and $q(f) \in \mathbb{Z}$.

Suppose now that A is infinite and $\operatorname{dom} f$ is a successor ordinal. Then $f = g * [u]$ for some string g and some woman u. Since E is an espousal of $\Gamma - A - X$, it follows that $E \restriction (M \setminus E^{-1}\langle u \rangle)$ is an espousal of $(\Gamma - u) - (A \cup E^{-1}\langle u \rangle) - (X \setminus \{u\})$. Furthermore, g is a string on $W_{\Gamma - u}$ and $q_{\Gamma - u}(g) \in \mathbb{Z}$ by Lemma 6 (since $q(g * [u]) = q(f) \in \mathbb{Z}$). Therefore, by the inductive hypothesis, $|X \setminus \{u\}| \leqslant |A \cup E^{-1}\langle u \rangle| + q_{\Gamma - u}(g)$, which implies that $|X| \leqslant |A| + q(f)$ since A is infinite and $q_{\Gamma - u}(g)$, $q(f) \in \mathbb{Z}$.

Now suppose that A is infinite and $\operatorname{dom} f$ is a limit ordinal τ. Since

$$\liminf_{\theta \to \tau} q(f_\theta) - \left\| D(f_\tau) \setminus \bigcup_{\theta < \tau} D(f_\theta) \right\| = q(f) \in \mathbb{Z},$$

it follows that $\liminf_{\theta \to \tau} q(f_\theta)$ is an integer m, say. Therefore there exists $\varepsilon < \tau$ such that $q(f_\theta) \geqslant m$ for $\varepsilon < \theta < \tau$. Let $\Xi = \{\theta : \varepsilon < \theta < \tau,\ q(f_\theta) = m\}$. If ρ is a limit ordinal less than τ and $\rho = \sup \Xi'$ for some non-empty subset Ξ' of $\Xi \setminus \{\rho\}$ then $\varepsilon < \rho < \tau$ and therefore $m \leqslant q(f_\rho) \leqslant \liminf_{\theta \to \rho} q(f_\theta) \leqslant m$ since $q(f_\theta) = m$ for every $\theta \in \Xi'$; and therefore $\rho \in \Xi$. Hence Ξ is closed in τ. Moreover $\sup \Xi = \tau$ since $\liminf_{\theta \to \tau} q(f_\theta) = m$, and so we can write $\Xi = \{\phi(\alpha) : 1 \leqslant \alpha < \xi\}$, where ξ is a limit ordinal and $\phi(\alpha) < \phi(\beta)$ whenever $1 \leqslant \alpha < \beta < \xi$. Let Π_0 be the empty society, and let $\Pi_\alpha = \Gamma[f_{\phi(\alpha)}]$ and $f^\alpha = f_{[\phi(\alpha), \phi(\alpha+1))}$ for $1 \leqslant \alpha < \xi$. For every $\alpha < \xi$ let

$$\Pi_{\alpha+1}/\Pi_\alpha = \Lambda_\alpha = (M_\alpha, W_\alpha, K_\alpha), \qquad E_\alpha = E \cap (M_\alpha \times W_\alpha),$$

$$A_\alpha = M_\alpha \setminus \operatorname{dom} E_\alpha, \qquad X_\alpha = W_\alpha \setminus \operatorname{rge} E_\alpha.$$

Sh:196

Then E_α is an espousal of $\Lambda_\alpha - A_\alpha - X_\alpha$. By Lemma 5, $q_{\Lambda_0}(f_{\phi(1)}) = q(f_{\phi(1)}) = m$ and so, substituting $\Delta = \Lambda_0$, $g = f_{\phi(1)}$, $B = A_0$, $Y = X_0$ in our inductive hypothesis, it follows that $|X_0| \leqslant |A_0| + m$. Therefore X_0 has a subset Q such that $|Q| \leqslant \max(0, m)$ and $|X_0 \setminus Q| \leqslant |A_0|$. If $1 \leqslant \alpha < \xi$ then $q_{\Lambda_\alpha}(f^\alpha) = q_{\Gamma/f_{\phi(\alpha)}}(f^\alpha)$ by Lemma 5 and $q(f_{\phi(\alpha+1)}) = q(f_{\phi(\alpha)}) + q_{\Gamma/f_{\phi(\alpha)}}(f^\alpha)$ by Lemma 7 and $q(f_{\phi(\alpha)}) = q(f_{\phi(\alpha+1)}) = m$: therefore $q_{\Lambda_\alpha}(f^\alpha) = 0$ and so, substituting $\Delta = \Lambda_\alpha$, $g = f^\alpha$, $B = A_\alpha$, $Y = X_\alpha$ in our inductive hypothesis, it follows that $|A_\alpha| \geqslant |X_\alpha| = |X_\alpha \setminus Q|$. Moreover $W = \bigcup\{W_\alpha : \alpha < \xi\}$ since $\operatorname{rge} f = W$ and Ξ is closed in τ and $\sup \Xi = \tau$. Hence, by Lemma 8,

$$|X| \leqslant |W \setminus \operatorname{rge} E| \leqslant |M \setminus \operatorname{dom} E| + |Q| = |A| + |Q| = |A| = |A| + q(f)$$

since A is infinite and Q is finite and $q(f) \in \mathbb{Z}$.

LEMMA 10. *If $\kappa \in \Upsilon$ and Π is a κ-hindrance in a society then Π is inespousable.*

Proof. We shall in fact prove that if Π is a κ-hindrance in a society Γ then Π is inespousable and $|W_\Pi \setminus \operatorname{rge} E| \leqslant \max(|M_\Pi \setminus \operatorname{dom} E|, \kappa)$ for every partial espousal E of Π. This will be proved first for $1 \leqslant \kappa \leqslant \aleph_0$, and then induction on κ will be used for regular cardinals $\kappa > \aleph_0$.

Suppose first that $1 \leqslant \kappa \leqslant \aleph_0$. Then there exist a κ-subset L of M_Π and a string f on W_Π such that $q_{\Pi - L}(f) = 0$. Let E be any partial espousal of Π, $A = M_\Pi \setminus (L \cup \operatorname{dom} E)$, $X = W_\Pi \setminus E[M_\Pi \setminus L]$. Since $E \upharpoonright (M_\Pi \setminus L)$ is an espousal of $(\Pi - L) - A - X$ and $q_{\Pi - L}(f) = 0$, it follows by Lemma 9 that $|X| \leqslant |A|$ and so

$$\begin{aligned} |W_\Pi \setminus \operatorname{rge} E| + \kappa &= |W_\Pi \setminus \operatorname{rge} E| + |L| = |W_\Pi \setminus \operatorname{rge} E| + |E[L]| + |L \setminus \operatorname{dom} E| \\ &= |X| + |L \setminus \operatorname{dom} E| \leqslant |A| + |L \setminus \operatorname{dom} E| = |M_\Pi \setminus \operatorname{dom} E|, \end{aligned}$$

which implies that

$$|W_\Pi \setminus \operatorname{rge} E| \leqslant |M_\Pi \setminus \operatorname{dom} E| \leqslant \max(|M_\Pi \setminus \operatorname{dom} E|, \kappa),$$

and also that $|M_\Pi \setminus \operatorname{dom} E| \geqslant \kappa > 0$, so that E cannot be an espousal of Π, that is, Π is inespousable.

Now suppose that $\kappa > \aleph_0$ and κ is regular. Assume the inductive hypothesis that if $\mu \in \Upsilon$ and $\mu < \kappa$ and Σ is a μ-hindrance in a society then Σ is inespousable and $|W_\Sigma \setminus \operatorname{rge} E'| \leqslant \max(|M_\Sigma \setminus \operatorname{dom} E'|, \mu)$ for every partial espousal E' of Σ. Let $\bar{\Pi}$ be a hindering κ-tower such that $\bigcup \bar{\Pi} = \Pi$, let $l(\bar{\Pi}) = \bar{\Lambda}$ and let $\Lambda_\alpha = (M_\alpha, W_\alpha, K_\alpha)$ for each $\alpha < \kappa$. For each $\alpha \in \Phi(\bar{\Pi})$ let $\kappa_\alpha < \kappa$ be an element of Υ such that Λ_α is a κ_α-hindrance in Γ/Π_α.

Let E be a partial espousal of Π. For each $\alpha < \kappa$ let $E_\alpha = E \cap (M_\alpha \times W_\alpha)$, $A_\alpha = M_\alpha \setminus \operatorname{dom} E_\alpha$, $X_\alpha = W_\alpha \setminus \operatorname{rge} E_\alpha$. Then, by the inductive hypothesis,

$$|X_\alpha| \leqslant \max(|A_\alpha|, \kappa_\alpha) \qquad \alpha \in \Phi(\bar{\Pi}). \tag{9}$$

Let $\Xi = \{\alpha \in \Phi(\bar{\Pi}) : |X_\alpha| \leqslant |A_\alpha|\}$ and let $Q = \bigcup\{X_\alpha : \alpha \in \kappa \setminus \Xi\}$. If $\alpha \in \kappa \setminus \Xi$ then either $\alpha \in \kappa \setminus \Phi(\bar{\Pi})$, in which case Λ_α is maidenly and $|X_\alpha| = |W_\alpha| = 1$, or $\alpha \in \Phi(\bar{\Pi}) \setminus \Xi$, in which case $|X_\alpha| > |A_\alpha|$ by the definition of Ξ and therefore $|X_\alpha| \leqslant \kappa_\alpha$ by (9). Hence $|X_\alpha| < \kappa$ for every $\alpha \in \kappa \setminus \Xi$ and so $|Q| \leqslant \kappa$. Moreover, if $\alpha \in \Xi$ then

$|X_\alpha \setminus Q| = |X_\alpha| \leqslant |A_\alpha|$ by the definitions of Ξ and Q, and if $\alpha \in \kappa \setminus \Xi$ then $X_\alpha \subseteq Q$ and so $|X_\alpha \setminus Q| \leqslant |A_\alpha|$. Hence, by Lemma 8,

$$|W_\Pi \setminus \operatorname{rge} E| \leqslant |M_\Pi \setminus \operatorname{dom} E| + |Q| \leqslant |M_\Pi \setminus \operatorname{dom} E| + \kappa = \max(|M_\Pi \setminus \operatorname{dom} E|, \kappa).$$

Suppose if possible that Π has an espousal F. If $\alpha \in \Phi(\bar{\Pi})$ then Λ_α is a κ_α-hindrance in Γ/Π_α and so is inespousable by the inductive hypothesis, and therefore $F[M_\alpha] \nsubseteq W_\alpha$. On the other hand, $F[M_\alpha] \subseteq \bigcup\{W_\theta : \theta \leqslant \alpha\}$ since $\Pi_{\alpha+1} \lhd \Gamma$. Therefore there exists $\phi(\alpha) < \alpha$ such that $F[M_\alpha] \cap W_{\phi(\alpha)} \neq \emptyset$. For each $\alpha \in \Phi(\bar{\Pi})$ select such a $\phi(\alpha)$, thus defining a regressive function $\phi : \Phi(\bar{\Pi}) \to \kappa$. By Lemma 1 there exist a κ-subset Θ of $\Phi(\bar{\Pi})$ and $\delta < \kappa$ such that $\phi[\Theta] = \{\delta\}$. Since $F[M_\alpha] \cap W_\delta = F[M_\alpha] \cap W_{\phi(\alpha)} \neq \emptyset$ for each $\alpha \in \Theta$, it follows that $\left|W_\delta \cap F\left[\bigcup_{\delta < \alpha < \kappa} M_\alpha\right]\right| \geqslant |\Theta| = \kappa$. Hence the set of ordinals θ for which $\left|W_\theta \cap F\left[\bigcup_{\theta < \alpha < \kappa} M_\alpha\right]\right| \geqslant \kappa$ is non-empty and so has a least element ε. Let $F_\varepsilon = F \cap (M_\varepsilon \times W_\varepsilon)$. Since $|W_\varepsilon| \geqslant \kappa$, Λ_ε cannot be maidenly and so must be a κ_ε-hindrance in Γ/Π_ε. Therefore, by the inductive hypothesis,

$$\max(|M_\varepsilon \setminus \operatorname{dom} F_\varepsilon|, \kappa_\varepsilon) \geqslant |W_\varepsilon \setminus \operatorname{rge} F_\varepsilon| \geqslant \left|W_\varepsilon \cap F\left[\bigcup_{\varepsilon < \alpha < \kappa} M_\alpha\right]\right| \geqslant \kappa$$

and consequently $|M_\varepsilon \setminus \operatorname{dom} F_\varepsilon| \geqslant \kappa$ (since $\kappa_\varepsilon < \kappa$), that is $|F[M_\varepsilon] \setminus W_\varepsilon| \geqslant \kappa$. Moreover $F[M_\varepsilon] \subseteq \bigcup_{\alpha \leqslant \varepsilon} W_\alpha$ since $\Pi_{\varepsilon+1} \lhd \Gamma$, and therefore (since κ is regular) $|F[M_\varepsilon] \cap W_\alpha| \geqslant \kappa$ for some $\alpha < \varepsilon$, which contradicts the definition of ε. This contradiction shows that Π is inespousable; Lemma 10 is proved.

We can now complete the proof of Theorem 3 as follows. By Theorem 2, every critical society is packed. From this it easily follows, using induction on κ, that every κ-obstruction in Γ is a κ-hindrance in Γ. From this, and Theorem 1, we see that if Γ is inespousable then there exists a κ-hindrance in Γ for some $\kappa \in \Upsilon$.

Conversely, suppose that there exists a κ-hindrance Π in Γ for some $\kappa \in \Upsilon$. Then $\Pi \lhd \Gamma$: this follows directly from the definition of κ-hindrance if $\kappa \leqslant \aleph_0$ and from the fact that $\Pi = \bigcup \bar{\Pi}$ for some tower $\bar{\Pi}$ in Γ if $\kappa > \aleph_0$. Hence any espousal E of Γ would induce an espousal $E \restriction M_\Pi$ of Π, contradicting Lemma 10. It follows that Γ is inespousable.

References

1. R. AHARONI, 'On the equivalence of two conditions for the existence of transversals', *J. Combin. Theory Ser. A*, 34 (1983), 202–214.
2. R. AHARONI, C. ST. J. A. NASH-WILLIAMS and S. SHELAH, 'A general criterion for the existence of transversals', *Proc. London Math. Soc.* (3), 47 (1983), 43–68.
3. T. JECH, *Set theory* (Academic Press, New York, 1978).
4. C. ST. J. A. NASH-WILLIAMS, 'Another criterion for marriage in denumerable societies', *Advances in graph theory: Proceedings of the Cambridge combinatorial conference* 1977 (ed. B. Bollobás, North-Holland, Amsterdam, 1978); *Ann. Discrete Math.*, 3 (1978), 165–179.
5. C. ST. J. A. NASH-WILLIAMS, 'Marriage in societies in which each woman knows countably many men, *Proceedings of the tenth southeastern conference on combinatorics, graph theory and computing,*

Vol. 1 (eds. F. Hoffman, D. McCarthy, R. C. Mullin and R. G. Stanton, Congressus Numerantium XXIII, Utilitas Mathematica Publishing Inc., 1979), pp. 103–115.

Department of Mathematics,
Technion—Israel Institute of
Technology,
Haifa 32000,
Israel.

Department of Mathematics,
University of Reading,
Whiteknights,
Reading,
Berkshire RG6 2AX.

Institute of Mathematics and Computer Science,
The Hebrew University of Jerusalem,
Givat Ram,
91904 Jerusalem,
Israel.