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CON(u > i)

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Summary. We prove here the consistency of $\mathfrak{u} > \mathfrak{i}$ where:

$$\mathfrak{u} = \mathrm{Min}\{|X|: X \subseteq \mathscr{P}(\omega) \text{ generates a non-principle ultrafilter}\},$$

$$\mathfrak{i} = \mathrm{Min}\{|\mathscr{A}|: \mathscr{A} \text{ is a maximal independent family of subsets of } \omega\}.$$

In this we continue Goldstern and Shelah [GlSh388] where Con($\mathfrak{r} > \mathfrak{u}$) was proved using a similar but different forcing. We were motivated by Vaughan [V] (which consists of a survey and a list of open problems). For more information on the subject see [V] and [GlSh388].

1 The single forcing

1.1 Definition. Let I be a proper ideal on ω containing the finite subsets. We define a forcing notion Q_I:

$$p \in Q_I \quad \text{iff} \quad p = (H, E, A) = (H^p, E^p, A^p) \quad \text{where}$$

(a) E is an equivalence relation on $\mathrm{Dom}\, E \subseteq \omega$,
(b) $\omega \setminus \mathrm{Dom}\, E \in I$,
(c) each E-equivalence class belongs to I,
(d) $A = \{x: x \in \mathrm{Dom}\, E,\ x = \mathrm{Min}(x/E)\}$,
(e) H is a function, $\mathrm{Dom}\, H = \omega$,
(f) for each $n \in \omega$, $H(n)$ is a function from ${}^A\{-1, 1\}$ to $\{-1, 1\}$ which depends on finitely many places only from $A \cap \{0, \ldots, n\}$, i.e. for some finite $w(n) \subseteq A \cap \{0, 1, \ldots, n\}$,

$$[\eta, \nu \in {}^A\{-1, 1\}\ \&\ \eta \restriction w(n) = \nu \restriction w(n) \Rightarrow H(n)[\eta] = H(n)[\nu]].$$

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For $i \in A$, we let x_i be the function that maps $\eta \in {}^{A}\{-1,1\}$ to $\eta(i)$. So $H(n)$ can be written as a Boolean combination of the functions x_i ($i \in A, i \leqq n$). We prefer to view $H(n)$ as a Boolean expression in the formal variables x_i (using operations max, min, $-$, and constants -1 and 1),

(g) if $n \in A$, $H(n)$ is x_n,

(h) if $n \in \mathrm{Dom}\, E \setminus A$, nEi and $i \in A$ then $H(n)$ is x_i or $-x_i$.

We define the partial order $\leqq$ (on Q_I) by $p \leqq q$ if:

(α) $\mathrm{Dom}\, E^p \supseteq \mathrm{Dom}\, E^q$, $\mathrm{Dom}\, E^q$ is a union of a family of E^p-equivalence classes,

(β) $E^p \restriction \mathrm{Dom}\, E^q$ refines E^q (hence $A^q \subseteq A^p$),

(γ) *if* $H^p(n) = x_i$, $n \in \mathrm{Dom}\, E^p$, then $H^q(n) = H^q(i)$; *if* $H^p(n) = -x_i$, $n \in \mathrm{Dom}\, E^p$, then $H^q(n) = -H^q(i)$,

(δ) if $n \in \omega \setminus \mathrm{Dom}\, E^p$, then*

$$H^q(n)[x_i\colon i \in A^q] = H^p(n)[\ldots, x_i, \ldots, H^q(j)[\ldots, x_\varepsilon, \ldots]_{\varepsilon \in A^q}, \ldots]_{\substack{i \in A^q \\ j \in A^p \setminus A^q}}.$$

1.1A Remark. The reader may worry about the absence of conditions for the case where $n \in \mathrm{Dom}\, E^p \setminus \mathrm{Dom}\, E^q$ [especially if $n = \min(\mathrm{Dom}\, E^p \setminus \mathrm{Dom}\, E^q)$]. The crucial difference between this forcing and the one in [GlSh388] is precisely that we don't impose any conditions other than (γ) in this case.

1.2 Claim. 1) *$Q_I = (Q_I, \leqq)$ is a partial order.*

2) *If $p \in Q_I$ and $E = E^p$ then $Q_I \restriction \{q\colon q \geqq p\}$ is isomorphic to $Q_{I/E}$ as follows: let $h\colon \mathrm{Dom}\, E \to \omega$ be $h(n) = |A^p \cap \mathrm{Min}(n/E)|$, $J = \{B \subseteq \omega\colon \{n\colon h(n) \in B\} \in I\}$, then $Q_I \restriction \{q\colon q \geqq p\}$ is isomorphic to Q_J.*

1.3 Definition. $\underset{\sim}{z}_{Q_I}$ is (the Q_I-name for) the set

$$\{n\colon \text{for some } p \in \underset{\sim}{G}_{Q_I},\ H^p(n) \text{ is constantly } 1\}.$$

1.4 Claim. *1) If $i < \omega$ and $A^p \cap (i+1) = \emptyset$ then $H^p(i)$ is constant.*

2) $p \Vdash$ "$\underset{\sim}{z}_{Q_I}(n) = \varepsilon$" ($\varepsilon = -1$ or $\varepsilon = 1$) iff $H^p(n)$ is constantly ε.

3) For each n the set $\{p \in Q_I\colon H^p(n)$ is constant$\}$ is a dense subset of Q_I.

4) If $p \in Q_I$, then

$$[\omega \setminus \{n\colon \textit{there are } p_{-1}, p_1 \geqq p \textit{ such that } p_\varepsilon \Vdash_{Q_I} \text{“}\underset{\sim}{z}_{Q_I}(n) = \varepsilon\text{” } \textit{for } \varepsilon = +1, -1\}] \in I.$$

Proof. E.g.

4) Let $p \in Q_I$, $n \in \mathrm{Dom}\, E$. We shall construct p_{-1}, p_1 as required. Let $\varepsilon \in \{-1, 1\}$, $i = \mathrm{Min}(n/E^p)$, $E^{p_\varepsilon} = E^p \restriction (\mathrm{Dom}\, E^p \setminus n/E)$, $A^{p_\varepsilon} = A^p \setminus \{i\}$. Lastly H^{p_ε} is defined as follows: $H^{p_\varepsilon}(j)$ is:

(a) constantly ε if $j \in i/E$, $H^p(j) = H^p(n)$,

(b) constantly $-\varepsilon$ if $j \in i/E$, $H^p(j) = -H^p(n)$,

(c) for $j \in \omega \setminus \mathrm{Dom}(E^p)$, $\eta \in A^{p_\varepsilon} \to \{-1, 1\}$ we let

$$(H^{p_\varepsilon}(j))(\eta) = (H^p(j))(\eta \cup \{\langle i, H^{p_\varepsilon}(i)\rangle\}),$$

(d) for $j \in \mathrm{Dom}(E^p) \setminus (n/E^p)$ we act as in (c), or less formally

$$H^{p_\varepsilon}(j) = H^p(j). \quad \Box_{1.4}$$

Remark. In similar cases later we shall be less formal.

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* Here x_i is just -1 or 1 not the function x_i

1.5 Conclusion. $\Vdash_{Q_I}$"I does not generate a maximal ideal in V^{Q_I}".

1.6 Definition. 1) $p \leqq_n q$ iff $p \leqq q$ and $[k \in A^p \,\&\, |A^p \cap k| < n \Rightarrow k \in A^q]$.
2) If $u \subseteq A^p$, $h: u \to \{-1, 1\}$ then $q = p^{[h]}$ is defined as follows:

$$A^q = A^p \setminus u,$$

$$E^q = E^p \restriction \Big(\bigcup_{i \in A^p \setminus u} i/E^p \Big),$$

$H^q(n)$ is: $H^p(n)$ where we substitute $h(i)$ for x_i for $i \in u$, so in particular: *if* $n \in i/E^p$, $i \in u$, $H^p(n) = x_i$ *then* $H^q(n) = h(i)$ and *if* $n \in i/E^p$, $i \in u$, $H^p(n) = -x_i$ *then* $H^q(n) = -h(i)$.

1.7 Claim. *1) If $p \leqq q$, u a (finite) initial segment of A^p, $H^q(i)$ is constant for each $i \in u$ then for some unique $h: u \to \{1, -1\}$ we have $p \leqq p^{[h]} \leqq q$.*
2) If $p \in Q_I$, u is a finite initial segment of A^p then:
(i) for each $h \in {}^{u}\{-1, 1\}$ we have $p \leqq p^{[h]} \in Q_I$,
(ii) $\{p^{[h]}: h \in {}^{u}\{-1, 1\}\}$ is predense above p, and
(iii) for each such $h: u \to \{1, -1\}$ we have $H^{p^{[h]}}(i)$ is constant for each $i \in u$.
3) If $p \in Q_I$, u a finite initial segment of A^p, $|u| = n$, $p^{[h]} \leqq q \in Q_I$ then for some $r \in Q_I$, $p \leqq_n r \leqq q$, $r^{[h]} = q$.
4) $\leqq_n$ is a partial order on Q_I, $[p \leqq_{n+1} q \Rightarrow p \leqq_n q \Rightarrow p \leqq q]$.

1.8 Claim. *If $p \in Q_I$, $n < \omega$ are given, $\underset{\sim}{\tau}$ a Q_I-name of an ordinal, then there is $q \in Q_I$, $p \leqq_n q$ and (letting $u = \{i \in A^p: |A^p \cap i| < n\}$):*

$(*)_1$ *for every $h \in {}^{u}\{-1, 1\}$, $q^{[h]}$ forces a value to $\underset{\sim}{\tau}$,*

$(*)_2$ *for some set v of $\leqq 2^n$ ordinals, $q \Vdash$"$\tau \in v$".*

Proof. By 1.7(2)(ii), 1.7(3), and 1.7(4). □

1.9 Definition. Let I be an ideal on ω containing the finite subsets of ω.
1) E is an I-equivalence relation if:
(a) $\mathrm{Dom}\, E \subseteq \omega$,
(b) $\omega \setminus \mathrm{Dom}\, E \in I$,
(c) each E-equivalence class is in I.
2) $E_1 \leqq E_2$ *if* (both are I-equivalence relations and):
(i) $\mathrm{Dom}\, E_2 \subseteq \mathrm{Dom}\, E_1$,
(ii) $E_1 \restriction \mathrm{Dom}\, E_2$ refines E_2,
(iii) $\mathrm{Dom}\, E_2$ is the union of a family of E_1-equivalence classes.
3) $GM_I(E)$ is the following game. It lasts ω moves. In the nth move the first player chooses an I-equivalence relation E^1_n, $[n = 0 \Rightarrow E^1_0 = E]$, $[n > 0 \Rightarrow E^2_{n-1} \leqq E^1_n]$, and the second player chooses an I-equivalence relation E^2_n such that $E^1_n \leqq E^2_n$. In the end, the second player wins if

$$\bigcup \{\mathrm{Dom}\, E^2_n \setminus \mathrm{Dom}\, E^1_n : n > 0\} \in I \quad \text{(otherwise the first player wins)}.$$

1.10 Claim. *1) The game $GM_I(E)$ is not determined when I is a maximal ideal.*
2) $\mathscr{P}(\omega) \setminus I \models ccc$ is enough.

Proof. 1) As each player can imitate the other's strategy.
2) Easy, too, and will not be used in this paper.

1.11 Claim. *Suppose $p \in Q_I$, $\underset{\sim}{\tau}$ a Q_I-name of a function from ω to ordinals, $m<\omega$ and I a maximal (non-principal) ideal on ω (or just: the first player has no winning strategy in $GM_I(E^p)$). Then for some q, $p \leqq_m q \in Q_I$, and letting $A^q=\{i_\ell: \ell<\omega\}$ (in increasing order), $q^{[h]}$ forces a value to $\underset{\sim}{\tau} \upharpoonright (i_\ell+1)$ for any $h:\{i_0,\dots,i_\ell\} \to \{1,-1\}$ and any $\ell \geqq m$ (but $\ell<\omega$).*

Proof. For this we let $E=E^p \upharpoonright [\bigcup \{i/E^p: i \in A^p \text{ and } |i \cap A^p| \geqq m\}]$ and we shall define a strategy for the first player in $GM_I(E)$ during which the first player, on the side, chooses $p_0 \leqq p_1 \leqq \dots$.

Then as this is not a winning strategy, in some play in which the first player uses his strategy he loses and then $\langle p_\ell: \ell<\omega \rangle$ will have an upper bound as required.

In the nth move, the first player in addition to choosing E_n^1 chooses q_n, p_n, u_n such that:

(a) $p_0=q_0=p$,

(b) $p_n \leqq_{m+n} p_{n+1}$,

(c) u_0 is $\{i \in A^{p_0}: |i \cap A^{p_0}|<m\}$,

(d) $u_{n+1}=u_n \cup \{\operatorname{Min}(A^{q_{n+1}} \backslash u_n)\}$, so $|u_n|=m+n$,

(e) $E_n^1=E^{p_n} \upharpoonright \left(\operatorname{Dom} E^{p_n} \backslash \bigcup_{i \in u_n} i/E^{p_n}\right)$,

(f) q_{n+1} is as follows:

(f_1) $\operatorname{Dom} E^{q_{n+1}}=\operatorname{Dom} E^{p_n}$,

(f_2) $x E^{q_{n+1}} y$ *iff* (α) or (β) or (γ) holds where

(α) $x E_n^2 y$,

(β) $x,y \in (\operatorname{Dom} E^{p_n} \backslash \operatorname{Dom} E_n^2) \,\&\, x E^{p_n} y$ and for some $k \in u_n$ we have $x,y \in k/E^{p_n}$,

(γ) $x,y \in \bigcup \left\{k/E^{p_n}: k \in \operatorname{Dom} E^{p_n},\ k \notin \operatorname{Dom} E_n^2 \text{ and } k \notin \bigcup_{i \in u_n} i/E^{p_n}\right\}$,

(f_3) $H^{q_{n+1}}(\ell)$ is: *first case* $\ell \in \omega \backslash \operatorname{Dom} E^{p_n}$ then

$$H^{q_{n+1}}(\ell)=H^{p_n}(\ell) \quad \text{or more exactly}$$

$$\begin{aligned} &H^{q_{n+1}}(\ell)[\dots,x_j,\dots]_{j \in A^{q_{n+1}}} \\ &\quad =H^{p_n}(\ell)[\dots,x_j,\dots,H^{q_{n+1}}(k)(\dots,x_\varepsilon,\dots)_{\varepsilon \in A^{q_{n+1}}},\dots]_{\substack{j \in A^{q_{n+1}} \\ k \in A^{p_n} \backslash A^{q_{n+1}}}} \end{aligned}$$

[no vicious circle as only $H^{q_{n+1}}(k)$ such that $k<\ell$ count];

second case $\ell \in \operatorname{Dom} E^{p_n} \backslash A^{q_{n+1}}$, $H^{p_n}(\ell)=x_i$ then

$$H^{q_{n+1}}(\ell)=H^{q_{n+1}}(i);$$

third case $\ell \in \operatorname{Dom} E^{p_n} \backslash A^{q_{n+1}}$, $H^{p_n}(\ell)=-x_i$ then

$$H^{q_{n+1}}(\ell)=-H^{q_{n+1}}(i);$$

fourth case $\ell \in A^{p_n} \backslash A^{q_{n+1}}$, then

$$H^{q_{n+1}}(\ell)=H^{p_n}(\operatorname{Min} \ell/E^{q_{n+1}}),$$

(g) $p_n \leqq_{m+n} q_{n+1} \leqq_{m+n+1} p_{n+1}$,

(h) if $h \in {}^{(u_{n+1})}\{-1,+1\}$ then $p_{n+1}^{[h]}$ forces a value to $\underset{\sim}{\tau} \upharpoonright ((\operatorname{Max} u_{n+1})+1)$.

(i) W.l.o.g. $\operatorname{Min} \operatorname{Dom} E_n^2 > \operatorname{Max} u_{n+1}$ so $\operatorname{Dom} E^{p_n} \backslash \operatorname{Dom} E_n^2 \subseteqq \bigcup \{k/E^{q_{n+1}}: k \in u_{n+1}\}$.

Now this strategy is well defined by Claim 1.8. In the nth move, if $n=0$ define p_0, q_0 by (a), u_0 by (c), and E_0^1 by (e). In the $(n+1)$-th move first define q_{n+1} by (f) [and check (g)], then use (d), to define u_{n+1} then choose p_{n+1} by (h) and 1.8, and lastly (e) to choose E_{n+1}^1 (the actual move). Now we can try to define a condition q as required in 1.11: $E^q=\lim_{n<\omega} E^{p_n}$ (i.e. $\operatorname{Dom} E^q=\bigcap_{n<\omega} \operatorname{Dom} E^{p_n}$, $x E^q y$ iff for every n

large enough, $xE^{p_n}y\big)$, $H^q(m)$ will be $H^{p_n}(m)$ for any large enough n (it is eventually constant) (formalistically its set of variables is decreasing, but the material one converges).

Now $\bigwedge_n p_n \leqq_{m+n} q$, *but* is $q \in Q_I$? Not necessarily; however, if

$$(\omega \setminus \operatorname{Dom} E^q) = \bigcap_{n<\omega} (\omega \setminus \operatorname{Dom} p^n) = \omega \setminus \bigcup \{i/E^{p_n}\colon i \in u_n,\ n<\omega\}$$

is in I, it does; and this occurs if the second player wins the play, which occurs for some such play (in which player I uses the strategy defined above) as by 1.10 player I has no winning strategy. $\square_{1.11}$

1.12 Conclusion. If I is a maximal ideal, then Q_I is ${}^{\omega}\omega$-bounding and even has the Sacks property. (See definitions in [Sh-b] or [Sh-f, VI, Sect. 2].)

1.13 Claim. *Assume I is a maximal ideal on ω (also $\mathscr{P}(\omega)/I \models ccc$ suffices). Then Q_I is proper (and even $(<\omega_1)$-proper and $(<\omega_1)$-strongly proper).*

Proof. Essentially combining the proofs of 1.10, 1.11; i.e. we simulate two plays, each finite initial segment is in the model, we take care of each Q_I-name of an ordinal from the model eventually, and take care that the second player wins at least one of them. $\square$

2 The maximal independent family

2.1 Definition. 1) For a family $\mathscr{B}$ of subsets of ω and partial function h from $\mathscr{B}$ to $\{1,-1\}$ let $\mathscr{B}^h = \bigcap \{A^{h(A)}\colon A \in \mathscr{B} \cap \operatorname{Dom} h\}$

$$\text{where} \quad A^1 = A, \quad A^{-1} = \omega \setminus A.$$

2) *$FF(\mathscr{B})$ is the family of finite partial functions from $\mathscr{B}$* to $\{1,-1\}$.

3) $\mathscr{A}$ denotes a family of subsets of ω which is independent

$$\text{(i.e. } h \in FF(\mathscr{A}) \Rightarrow \mathscr{A}^h \text{ infinite)}.$$

4) $AP = \{(\mathscr{A}, A)\colon A \subseteq \omega$ infinite, $\mathscr{A}$ a countable independent family of subsets of ω, moreover, $[h \in FF(\mathscr{A}) \Rightarrow |A \cap \mathscr{A}^h| = \aleph_0]\}$.

5) The order $\leqq$ on AP is

$$(\mathscr{A}_1, A_1) \leqq (\mathscr{A}_2, A_2) \quad \text{iff} \quad \mathscr{A}_1 \subseteqq \mathscr{A}_2 \,\&\, A_2 \subseteqq^* A_1$$
$$(A_1 \subseteqq^* A_2 \text{ means } A_1 \setminus A_2 \text{ is finite}).$$

6) For any $\mathscr{A}$,

$$\text{for} \quad A \subseteqq \omega \quad \text{let} \quad \mathscr{D}(A) = \{h \in FF(\mathscr{A})\colon A \cap \mathscr{A}^h \text{ is finite}\} \quad \text{and}$$

$$\mathrm{id}_{\mathscr{A}} = \{A \subseteqq \omega\colon \mathscr{D}(A) \text{ is dense in } FF(\mathscr{A})\} \text{equivalently:}$$

$$\mathrm{id}_{\mathscr{A}} = \{A \subseteqq \omega\colon \text{for every } h_0 \in FF(\mathscr{A}) \text{ for some } h_1,\ h_0 \subseteqq h_1 \in FF(\mathscr{A}) \text{ and } A \cap \mathscr{A}^{h_1} \text{ is finite}\}$$

[it is an ideal, increasing with $\mathscr{A}$ – why? If $A \in \mathrm{id}_{\mathscr{A}}$, $\mathscr{A}_1 \subseteqq \mathscr{A}_2$, $h_2 \in FF(\mathscr{A}_2)$ then $h_1 = h_2 \restriction \mathscr{A}_1 \in FF(\mathscr{A}_1)$ so there is $h' \in FF(\mathscr{A}_1)$ extending h_1, $A \cap \mathscr{A}^{h'} = \emptyset$, hence $A \cap \mathscr{A}^{h_2 \cup h'} = \emptyset$ as required] (if $\mathscr{A}$ is infinite we get the same ideal if we require "empty" in the definition of $\mathscr{D}(\mathscr{A})$ instead of "finite"). Note that for every dense $\mathscr{D}\subseteq FF(\mathscr{A})$, we have $\bigcap_{h\in\mathscr{D}}(\omega\backslash\mathscr{A}^h)$ belongs to $\mathrm{id}_{\mathscr{A}}$.

7) In 6) let $\mathrm{fil}_{\mathscr{A}}$ be the dual filter.

2.2 Claim. *1) If $(\mathscr{A}_n, A_n)\leqq(\mathscr{A}_{n+1}, A_{n+1})$ for $n<\omega$, in AP, then for some A, $\left(\bigcup_m \mathscr{A}_m, A\right)\in AP$ and*

$$(\forall n)\left[(\mathscr{A}_n, A_n)\leqq\left(\bigcup_m \mathscr{A}_m, A\right)\right].$$

2) If $(\mathscr{A}, A)\in AP$ then for some $B\subseteq A$, $B\notin\mathscr{A}$ and

$$(\mathscr{A}, A)\leqq(\mathscr{A}\cup\{B\}, A)\in AP.$$

3) If $(\mathscr{A}, A)\in AP$, E an equivalence relation on ω, each equivalence class finite, then for some B:

$$(\mathscr{A}, A)\leqq(\mathscr{A}, B)\in AP,$$

$$E\restriction B \text{ is equality}.$$

4) If $(\mathscr{A}, A)\in AP$, E an equivalence relation on ω, $h_0\in FF(\mathscr{A})$ then for some h_1, B we have:

(a) $h_0\subseteq h_1\in FF(\mathscr{A})$;
(b) $(\mathscr{A}, A)\leqq(\mathscr{A}, B)\in AP$;
(c) $E\restriction(\mathscr{A}^{h_1}\cap B)$ is equality or has one equivalence class.

Proof. E.g.

1) Let $FF(\mathscr{A}_n)=\{h_{n,\ell}: \ell<\omega\}$; now choose by induction on n $\langle k_{n,m,\ell}: m\leqq n, \ell\leqq n\rangle$ such that: $k_{n,m,\ell}\in A_n\cap\mathscr{A}_n^{h_{m,\ell}}$ [possible as $h_{m,\ell}\in FF(\mathscr{A}_n)$ as $\mathscr{A}_m\subseteq\mathscr{A}_n$ (when $m\leqq n$)]. Lastly let $A=\{k_{n,m,\ell}: n<\omega, m\leqq n, \ell\leqq n\}$.

2) Let $FF(\mathscr{A})=\{h_n: n<\omega\}$, and choose by induction on n, $k_n^1\in A\cap\mathscr{A}^{h_n}\backslash\{k_\ell^2: \ell<n\}$ and $k_n^2\in A\cap\mathscr{A}^{h_n}\backslash\{k_\ell^1: \ell\leqq n\}$. Then let $B=\{k_n^1: n<\omega\}$.

3) Let $FF(\mathscr{A})=\{h_n: n<\omega\}$, choose by induction on $n<\omega$, $k_n\in A\cap\mathscr{A}^{h_n}\backslash\bigcup\{k_\ell/E: \ell<n\}$. Let $B=\{k_n: n<\omega\}$.

(Note that $\bigcup\{k_\ell/E: \ell<n\}$ is finite as each E-equivalence class is finite.)

4) Let $\{h^n: n<\omega\}=\{h\in FF(\mathscr{A}): h_0\subseteq h\}$. Now we try to choose by induction on n, $k_n\in\mathscr{A}^{h^n}\backslash\bigcup\{k_\ell/E: \ell<\omega\}$. If we succeed let $h_1=h_0$ and $B=(A\backslash\mathscr{A}^{h_0})\cup\{k_n: n<\omega\}$, clearly it is as required. So assume that for some n, we have chosen $k_0, \dots, k_{n-1}$ but we cannot choose k_n. Now try to choose by induction on $\ell\leqq n$, $h^{n,\ell}\in FF(\mathscr{A})$ increasing with ℓ, such that: $h^{n,0}=h^n$, and $\mathscr{A}^{h^{n,\ell+1}}\cap(k_\ell/E)$ is finite. If we succeed, $\mathscr{A}^{h^{n,n}}\cap\bigcup_{\ell<n}(k_\ell/E)$ is finite (as a finite union of finite sets), while $\mathscr{A}^{h^{n,n}}\backslash\bigcup_{\ell<n}(k_\ell/E)$ is empty by the choice of n. So necessarily for some $\ell<n$, $h^{n,\ell}$ is defined while we cannot define $h^{n,\ell+1}$. Let $h_1=h^{n,\ell}$, $B=(\mathscr{A}^{h_1}\cap(k_\ell/E)\cup(A\backslash\mathscr{A}^{h_1})$; clearly they are as required. □

2.3 Claim (CH). *There is $\langle(\mathscr{A}_i, A_i): i<\omega_1\rangle$, such that $\left(\text{let } \mathscr{A}_*=\bigcup_{i<\omega_1}\mathscr{A}_i\right)$:*

(a) $(\mathscr{A}_i, A_i)\in AP$,
(b) $i<j<\omega_1 \Rightarrow (\mathscr{A}_i, A_i)\leqq(\mathscr{A}_j, A_j)$,
(c) $\mathscr{A}_{i+1}\backslash\mathscr{A}_i\neq\emptyset$,
(d) for each i for some $A\in\mathscr{A}_{i+2}\backslash\mathscr{A}_i$, $A\subseteq A_i$,
(e) for any $A\subseteq\omega$ and $h_0\in FF(\mathscr{A}_*)$ there is h_1 such that:

$$h_0\subseteq h_1\in FF(\mathscr{A}_*),$$

$$\mathscr{A}_*^{h_0}\subseteq A \quad or \quad \mathscr{A}_*^{h_1}\cap A=\emptyset,$$

(f) for any equivalence relation E on ω and $h_0 \in FF(\mathscr{A}_*)$ there is h_1 such that:

$$h_0 \subseteq h_1 \in FF(\mathscr{A}_*),$$

$$E \restriction \mathscr{A}^{h_1} \text{ is equality or has one equivalence class},$$

(g) if E is an equivalence relation on ω, each equivalence class finite, then for some i, $E \restriction A_i$ is the equality,

(h) $\mathrm{id}_{\mathscr{A}_*}$ *is the ideal generated by $\{\omega \backslash A_i: i < \omega_1\}$; moreover, for every $A \in \mathrm{id}_{\mathscr{A}_*}$ for unboundedly many $i < \omega_1$, $A \cap A_i = \emptyset$,*

(i) for $n \neq m$ for uncountably many i, $n \in A_i \,\&\, m \notin A_i$,

(j) $\mathscr{A}_*$ is a maximal independent family.

Proof. Straightforward. □

3 The iteration

3.1 Theorem. ($2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2, \diamondsuit_{\{\delta < \aleph_2: cf\delta = \aleph_1\}}$). *There is a forcing notion P such that:*

(i) P is proper of cardinality $\aleph_2$ satisfying $\aleph_2$ – c.c.

(ii) Forcing with P preserves cardinalities and cofinalities, $V^P \models 2^{\aleph_0} = \aleph_2$.

(iii) In V^P, $\mathfrak{u} = \aleph_2 > \aleph_1 = \mathfrak{i}$.

Remark. We prove more on V^P.

Proof. Let $(\mathscr{A}_i, A_i)$ $(i < \omega_i)$, $\mathscr{A}_*$ be as in 2.3. We define a CS iteration $\bar{Q} = \langle P_\alpha, Q_\beta: \alpha \leqq \omega_2, \beta < \omega_2 \rangle$, each Q_β of the form Q_{I_β}, I_β a P_β-name of a maximal ideal on ω (containing all finite subsets of ω) such that:

(*) *if* in $V^{P_{\omega_2}}$, I is a P_{ω_2}-name of a maximal non-principal ideal on ω *then* for some α, $I_\alpha \subseteq I''$.

This is possible as $\diamondsuit_{\{\delta < \aleph_2: cf(\delta) = \aleph_1\}}$ holds. Let $P = P_{\omega_2}$.

Now each Q_α is proper (1.13) of cardinality $\aleph_1$, for $\alpha < \aleph_2 \Vdash_{P_\alpha}$ "CH", P_α has a dense subset of power $\aleph_1$ (proved by induction on $\alpha < \omega_2$) hence ([Sh-b] or [Sh-f, III]) $P = P_{\omega_2}$ satisfies (i). Now (ii) follows. ($2^{\aleph_0} > \aleph_1$, as each Q_α adds a new real.)

Now $\mathfrak{u} > \aleph_1$ by (*) above and 1.5, hence (as $2^{\aleph_0} = \aleph_2$) $\mathfrak{u} = \aleph_2$. We are left with proving $\mathfrak{i} = \aleph_1$; of course, it suffices to prove that $\mathscr{A}_*$ is a maximal independent subfamily of $\mathscr{P}(\omega)$.

Now we shall prove for $\alpha \leqq \omega_2$ the following four statements; clearly $\otimes^4_\alpha$ (for $\alpha = \omega_2$) gives the maximality of $\mathscr{A}$ and thus finishes the proof of 3.1:

> $\otimes^1_\alpha$ in V^{P_α}, for every sequence $\langle \tau_n: n < \omega \rangle \in V^P$ of ordinals, and $f: \omega \to \omega$ diverging to infinity, f from V, there is $\langle w_n: n < \omega \rangle \in V$ such that: $\bigwedge_n \tau_n \in w_n$ and $|w_n| \leqq 1 + f(n)$.
>
> (This is "P_α has the Sacks property" which each Q_α satisfies by 1.11, and P_α satisfies by the preservation theorem [Sh-f, VI, Sect. 2] (or [Sh-b, V4.3] – where we use also ω-properness there, but it holds here; or see [Sh326, Appendix 2.4]).)
>
> $\otimes^2_\alpha$ in V^{P_α}, for every dense open $A \subseteq {}^{\omega >}\omega$, there is B such that: $B \in V$, $B \subseteq A$, B dense open subset of ${}^{\omega >}\omega$ (we can replace ${}^{\omega >}\omega$ by e.g. ${}^{\omega >}2$ or ${}^{\omega >}\{-1, 1\}$).

To show that each individual Q_α has this property, let $p \in Q_I$, $p \Vdash$ "$\underset{\sim}{A} \subseteq {}^{\omega>}\omega$ is dense open". We follow the proof of 1.11, but in point (h) we require now that for some fixed enumeration $\langle w_n : n<\omega\rangle$ of the basic open neighborhoods of ${}^{\omega>}\omega$:

$$p_{n+1} \Vdash \text{"} v_n \subseteq \underset{\sim}{A} \text{" for some basic open } v_n \subseteq w_n \text{ (not a name!)}.$$

This can be achieved as follows. Given q_{n+1}, fix an enumeration $\langle h_i : i<2^{n+m+1}\rangle$ of all $h \in {}^{(w_{n+1})}\{-1,1\}$. Define conditions $p_{n+1,i}$ such that $q_{n+1} \leqq_{n+m+1} p_{n+1,i} \leqq_{n+m+1} p_{n+1,i+1}$ and $p^{[h_i]}_{n+1,i} \Vdash$ "$v^{i+1}_n \subseteq \underset{\sim}{A}$", where the v^i_n's are basic open neighborhoods such that $v^{i+1}_n \subseteq v^i_n$ and $v^0_n = w_0$. This is possible since $\underset{\sim}{A}$ is dense open. Then put $p_{n+1} = p_{n+1,2^{n+m+1}-1}$, and $v_n = v_n^{2^{n+m+1}}$.

The property just shown for every individual Q_α is preserved under CS iterations by [Sh-f, XVIII, 3.7] or [Sh-f, VI, 2.x].

> $\otimes^3_\alpha$ every member of $(\mathrm{id}_{\mathscr{A}_*})^{V^{P_\alpha}}$ is included in a member of $(\mathrm{id}_{\mathscr{A}_*})^V$.
>
> (Why? It follows by $\otimes^2_\alpha$ and the definition of $\mathrm{id}_{\mathscr{A}}$ – note that if $A \in \mathrm{id}_{\mathscr{A}_*}$ in V^{V_α} then for some $i<\omega_1$, $A \in \mathrm{id}_{\mathscr{A}_i}$ [see 2.1(6)], now letting $\mathscr{A}_i = \{B_n : n<\omega\}$, if $A \in \mathrm{id}_{\mathscr{A}_i}$ then $\{h$: for some n, $h:\{B_0,\dots,B_{n-1}\}\to\{-1,1\}$ and we have $A \cap \mathscr{A}^h_i = \emptyset\}$ is open and dense, hence it includes some dense open [in $FF(\mathscr{A}_i)$] set $Y \subseteq \{h$: for some n, $h:\{B_0,\dots,B_{n-1}\}\to\{-1,1\}\}$ from V, let $A^Y = \bigcap_{h\in Y}(\omega\backslash\mathscr{A}^h)$, so $A^Y \in V$, $A^Y \in \mathrm{id}_{\mathscr{A}_i} \subseteq \mathrm{id}_{\mathscr{A}_*}$ and $A \subseteq A^Y$.)
>
> $\otimes^4_\alpha$ in V^{P_α}, for each $h^* \in FF(\mathscr{A}_*)$ for every $A \subseteq \mathscr{A}^{h^*}_*$, either A includes a member of $(\mathrm{fil}_{\mathscr{A}_*})^V + \mathscr{A}^{h^*}_*$ (see definition below) *or* A is disjoint to some $\mathscr{A}^h_*$, $h^* \subseteq h \in FF(\mathscr{A}_*)$, where $(\mathrm{fil}_{\mathscr{A}_*})^V + \mathscr{A}^{h^*}_* = \{X \subseteq \omega$: there is $A \in (\mathrm{fil}_{\mathscr{A}_*})^V$ such that $A \cap \mathscr{A}^{h^*}_* \subseteq X\}$.

Note: that by $\oplus^3_\alpha$, $\oplus^4_\alpha$ is equivalent to

> $\otimes^5_\alpha$ in V^{P_α} for each $h^* \in FF(\mathscr{A}_*)$ for every $A \subseteq \mathscr{A}^{h^*}_*$ for some h, $h^* \subseteq h \in FF(\mathscr{A}_*)$ and $A \cap A^h_* = \emptyset$ or $\mathscr{A}^h_* \subseteq A$.
>
> [Why? Clearly $\otimes^4_\alpha \Rightarrow \otimes^5_\alpha$. So assume $\otimes^5_\alpha$ and we shall prove $\otimes^4_\alpha$, so let $h^* \in FF(\mathscr{A}_*)$. If for some $h, h^* \subseteq h \in FF(\mathscr{A}_*)$ we have $A \cap \mathscr{A}^h_* = \emptyset$ then the second possibility in the conclusion of $\otimes^4_\alpha$ holds. If there is no such h, then (by $\otimes^5_\alpha$ applied to h) for every h, $h_* \subseteq h \in FF(\mathscr{A}_*)$, there is h' such that: $h \subseteq h' \in FF(\mathscr{A}_*)$ and $\mathscr{A}^{h'}_* \subseteq A$. So $\mathscr{A}^{h^*}\backslash A$ belongs to $(\mathrm{id}_{\mathscr{A}_*})^{V^{P_\alpha}}$ hence by $\otimes^3_\alpha$ we know $\mathscr{A}^{h^*}\backslash A$ is a subset of some $A' \in (\mathrm{id}_{\mathscr{A}_*})^V$ which is as required in the first possibility of the conclusion of $\otimes^4_\alpha$.]

We prove $\otimes^4_\alpha$ by induction on α. For notational simplicity let $h^* = \emptyset$.

First case: $\alpha = 0$ – by 2.3, part (e).

Second case: $\alpha = \beta+1$. We work in V^{P_β}.

So let $p \in Q_{I_\beta}$, $\underset{\sim}{A} \subseteq \omega$ a Q_{I_β}-name of a subset of ω, p forces $\underset{\sim}{A}$ is a counterexample. By 1.2(2) without loss of generality p is trivial; i.e. E^p is equality on ω (replacing I_β by some I_β/E) and by 1.11 without loss of generality from $r_{Q_{I_\beta}} \restriction n$ we can compute $\underset{\sim}{A} \cap (n+1)$.

If for some $\varrho \in {}^n\{1,-1\}$, $n<\omega$, $Y_\varrho =: \{m : p^{[\varrho]} \nVdash_{Q_{I_\beta}}$ "$m \notin \underset{\sim}{A}$"$\}$ is not in $\mathrm{fil}_{\mathscr{A}_*}$ (we can use $p^{[\varrho]}$ as p is trivial); then note: $Y_\varrho \in V^{P_\beta}$ and V^{P_β} satisfies the induction hypothesis so apply it to Y_ϱ, but the first possibility in $\oplus^4_\beta$ fails. Hence there is $h \in FF(\mathscr{A}_*)$ for which $\mathscr{A}^h_* \cap Y_\varrho = \emptyset$, so $p^{[\varrho]} \Vdash$ "$\mathscr{A}^h_* \cap \underset{\sim}{A} = \emptyset$" as required. So assume that there is no such ϱ. Remember ${}^{\omega>}\{1,-1\} = \bigcup_n {}^n\{-1,1\}$.

Now for each $\varrho \in {}^{\omega>}\{1,-1\}$ and $m \in Y_\varrho$ there is $q = q_{\varrho,m}$ where $p^{[\varrho]} \leqq q \in Q_{I_\beta}$ such that $q \Vdash$ "$m \in \underset{\sim}{A}$"; by an assumption in the beginning of the second case, there is $v_{\varrho,m}$, $\varrho \trianglelefteq v_{\varrho,m} \in {}^{\omega>}\{1,-1\}$ ($\trianglelefteq$ means being an initial segment) such that $p^{[v_{\varrho,m}]} \Vdash_{Q_{I_\beta}}$ "$m \in \underset{\sim}{A}$". Let $n: \omega \to \omega$ be defined by (note: $\ell g v_{\varrho,m} \geqq \ell g \varrho$)

$$h(n) = \mathrm{Max}[\{n+1\} \cup \{\ell g(v_{\varrho,m}) : \varrho \in {}^{n\geqq}\{1,-1\},\ m \leqq n,\ m \in Y_\varrho\}].$$

So by $\otimes^3_\beta$ and 2.3(h), for each $\varrho \in {}^{\omega>}\{1,-1\}$ there is $i(\varrho) \in \omega_1$ such that $A_{i(\varrho)} \subsetneqq Y_\varrho$. So for some $i(*) < \omega_1$ for every $i \geqq i(*)$, $\bigwedge_{\varrho \in {}^{\omega>}\{1,-1\}} A_i \subseteq^* Y_\varrho$. Let $f: \omega \to \omega$ be such that: $\bigwedge_n h(n) \leqq f(n)$ and for $\varrho \in {}^n\{1,-1\}$, $n < \omega$, $A_{i(*)} \setminus Y_\varrho \subsetneqq f(n)$; there is such $f \in V^{P_\beta}$, hence such $f \in V$ (by $\otimes^1_\beta$). Choose by induction on $\ell < \omega$, $n_\ell \in A_{i(*)} \cup \{0\}$ as follows: $n_0 = 0$, $n_{\ell+1}$ is the first $n \in A_{i(*)}$ such that $n > n_\ell$ and $\bigwedge_{m \leqq n_\ell} f(m) < n$ (possible as $A_{i(*)}$ is infinite). Define an equivalence relation E^0 on ω: mE^0k iff $\bigvee_\ell (m,k \in [n_{3\ell}, n_{3\ell+3}))$. This is an equivalence relation on ω with each class finite, and $E^0 \in V$ as $f \in V$. So by 2.3(g) there is i_1, $i(*) < i_1 < \omega_1$ such that $|A_{i_1} \cap [n_{3\ell}, n_{3\ell+3})| \leqq 1$ for every ℓ. Define an equivalence relation E^1 on A_{i_1}: mE^1k iff $m,k \in A_{i_1}$ and $[m = k \vee k < m \leqq f(k) \vee m < k \leqq f(m)]$. E^1 is an equivalence relation by the defining property of i_1. Easily $E^1 \in V$, each E^1-equivalence class has at most two members. Define an equivalence relation E^2 on ω: mE^2k iff $m = k$ or mE^1k. So again applying 2.3(g) for some i_2 with $i_1 < i_2 < \omega$, we have: each E^2-equivalence class contains at most one member of A_{i_2}. By 2.3(h), without loss of generality $A_{i_2} \subsetneqq (A_{i(*)} \cap A_{i_1}) \setminus [0, f(0)]$. As we could rename $i(*)$ as i_2, without loss of generality:

$$n \in A_{i(*)} \cup \{0\} \Rightarrow f(n) < \mathrm{Min}[A_{i(*)} \setminus (n+1)].$$

Let $\langle k(n) : n < \omega \rangle$ list $A_{i(*)} \cup \{0\}$, and for $\varrho \in {}^{k(n)}\{1,-1\}$ let v_ϱ be such that $\varrho \triangleleft v_\varrho \in {}^{k(n+2)}\{1,-1\}$, $p^{[v_\varrho]} \Vdash$ "$k(n+1) \in \underset{\sim}{A}$". It is easy to check v_ϱ exists: $k(n+1) \in A_\varrho$ as $A_{i(*)} \setminus Y_\varrho \subsetneqq f(k(n)) < k(n+1)$ and $k(n+1) \in A_{i(*)}$, and $\ell g(v_{\varrho,k(n+1)}) \leqq h(k(n+1)) \leqq f(k(n+1)) < k(n+2)$, so any $v, v_{\varrho,k(m+1)} \trianglelefteq v \in {}^{k(n+2)}\{1,-1\}$ will be as required.

Now if $B \subsetneqq \omega$, satisfies $[\ell, m \in B \;\&\; \ell \neq m \Rightarrow |\ell - m| > 2]$ and, $[\ell \in B \Rightarrow \ell > 2]$, then we can define p_B which is potentially an element of Q_{I_β} (and $> p$), as follows:

(a) $\mathrm{Dom}(E^{p_B}) = \omega \setminus \bigcup \{[k(n-1), k(n+1)) : n \in B\}$,

(b) E^{p_B} is the identity,

(c) $H^{p_B}(i) = x_i$ for $i \in \mathrm{Dom}\, E^{p_B}$,

(d) if $\ell \in \omega \setminus \mathrm{Dom}(E^{p_B})$, so for some $n \in B$, $k(n-1) \leqq \ell < k(n+1)$ and we want to define

$$H(\ell)(\varrho) = v_{\varrho \restriction k(n-1)}(\ell)$$

but some $\varrho(m)$, $m < \ell$ should be computed by $H(\ell)$, so we define $H(\ell)$ by induction on ℓ, naturally. Let us do it more formally: Suppose $k(n-1) \leqq \ell < k(n+1)$, and $H(m)$ has been defined for $m < \ell$. To define $H^{p_B}[x_i : i \in A^{p_B}]$ (the x_i again represent just minus one's and one's), find

$$\varrho = \langle \ldots, x_i, \ldots, H^{p_B}(j)[\ldots, x_\varepsilon, \ldots], \ldots \rangle_{\substack{i \in k(n-1) \cap \mathrm{Dom}\, E^p \\ j \in k(n-1) \setminus \mathrm{Dom}\, E^p}},$$

and let

$$H^{p_B}[x_i\colon i\in A^{p_B}]=v_{\varrho\restriction k(n-1)}(\ell).$$

Easily:

(*) if $\bigcup_{n\in B}[k(n-1),k(n+1))\in \underset{\sim}{I}_\beta$ then $p\leqq p_B\in Q_{\underset{\sim}{I}_\beta}$ and

$$p_B\Vdash_{Q_{\underset{\sim}{I}_\beta}}\text{“}\{k(n)\colon n\in B\}\subseteq \underset{\sim}{A}\text{”}.$$

So it suffices to find $B\subseteq\omega$ such that: $\bigcup[k(n-1),k(n+1))\in I_\beta$ and $\{k(n)\colon n\in B\}\in \mathrm{fil}_{\mathscr{A}_*}$ or just for some $h\in FF(\mathscr{A}_*)$, $\{k(n)\colon n\in B_0\}\in \mathrm{fil}_{\mathscr{A}_*}+\mathscr{A}_*^h$ (remember $\oplus_\alpha^5$).

As in the paragraph above for some $B_0\subseteq\omega$, $\{k(n)\colon n\in B\}\in \mathrm{fil}_{\mathscr{A}_*}$, and $[m,n\in B_0\ \&\ m\neq n\Rightarrow |m-n|>2]$. We can find contradictory $h_1,h_2\in FF(\mathscr{A}_*)$, so $\mathscr{A}_*^{h_1}\cap\mathscr{A}_*^{h_2}=\emptyset$ so without loss of generality $\mathscr{A}_*^{h_1}\in I_\beta$, so $B=\{k(n)\colon n\in\mathscr{A}_*^{h_1}$ and $n\in B_0\}$ is as required. [Note that actually $\ell g(v_{\varrho,m})=\max\{m,\ell g\varrho\}$ is O.K.)

Third case: α limit: By 3.2 below applied with α, $\bar{Q}\restriction\alpha$, $(\mathrm{fil}_{\mathscr{A}})^V$, $\{\omega\backslash\mathscr{A}_*^h\colon h\in FF(\mathscr{A}_*)\}$ here standing for δ, $\bar{Q}$, D, F there. $\square_{3.1}$

3.2 Lemma. *Suppose*

(a) D is a family of non-empty subsets of ω, containing the co-bounded subsets, closed under (finite) intersection and for every countable $\mathscr{B}\subseteq D$ for some $A\in D$ we have $\bigwedge_{B\in\mathscr{B}} A\subseteq^* B$; we denote by $[D]$ the filter D generates,

(b) F is a family of subsets of ω, $X\in F\Rightarrow X\notin[D]$,

*(c) D is Ramsey; i.e. if $\langle A_n\colon n<\omega\rangle$ is a partition of ω, $\omega\backslash A_n\in D$ then we can find $k_n\in A_n$, $\{k_n\colon n<\omega\}\in D$,****

(d) if $X\subseteq\omega$, $X\notin[D]$ then for some $A\in F$, $X\subseteq^* A$,

(e) if $X\subseteq\omega$ and $X\cap A=\emptyset$ for some $A\in D$ then $X\subseteq B$ for some $B\in D$.

If $\bar{Q}=\langle P_\alpha,Q_\beta\colon \alpha\leqq\delta,\ \beta<\delta\rangle$ is a CS proper iteration of ${}^\omega\omega$-bounding proper forcing notions, such that for $\alpha<\delta$, $\Vdash_{P_\alpha}$“if $X\subseteq\omega$, $X\notin[D]^{V^{P_\alpha}}$ then for some $A\in F$, $X\subseteq^* A$” [i.e. (d) holds in V^{P_α}] then this holds for $\alpha=\delta$.

Proof. Also here we could have used the general preservation theorems of [Sh-f, XVIII, Sect. 2] (see 3.11 there).

Let $p\in P_\delta$, $p\Vdash$“$\underset{\sim}{X}\subseteq\omega$”, it suffices to find q, $p\leqq q\in P_\delta$ and either $A\in F$ such that $q\Vdash_{P_\delta}$“$\underset{\sim}{X}\subseteq^* A$” or $A\in D$, $q\Vdash_{P_\delta}$“$A\subseteq^*\underset{\sim}{X}$”. As each P_α $(\alpha\leqq\delta)$ is ${}^\omega\omega$-bounding (by the preservation theorem [Sh-f, VI, Sect. 2], proof of $\otimes_1$ in 3.1), $[D]$ is a Ramsey filter in V^{P_α} for $\alpha\leqq\delta$.

For sufficiently large χ, let $N\prec(H(\chi),\varepsilon,<_\chi^*)$ be countable such that p, $\underset{\sim}{X}$, F, D, $\bar{Q}$ belong to N. We can assume that *for no* $\alpha\in\delta\cap N$ and p' satisfying $p\leqq p'\in N\cap P_\delta$ and $q\in P_\alpha$ such that $p'\restriction\alpha\leqq q$, q is (N,P_α)-generic and $G_\alpha\subseteq P_\alpha$ generic over V such that $q\in G_\alpha$ *do we* have in $V[G_\alpha]$

$$\{n\colon p'\Vdash_{P_\delta/G_\alpha}\text{“}n\notin\underset{\sim}{X}\text{”}\}\notin[D]^{V[G_\alpha]}$$

(as in $V[G_\alpha]$, (d) still holds).

** Equivalently in the following game player I has no winning strategy: I chooses $A_n\in[D]$, II chooses $k_n\in A_n$; player II wins the play if $\{k_n\colon n<\omega\}\in[D]$. The filter $\mathrm{fil}_{\mathscr{A}_*}$ has a base linearly ordered by $\subseteq^*$ into order type ω_1^*, and is therefore a p-filter. It is also a q-filter by 2.3(g). It is well known that a filter is Ramsey iff it is simultaneously a p-filter and a q-filter

Without loss of generality $\delta=\omega$, $\underset{\sim}{X}\cap\{n\}$ is a P_n-name above p (more exactly, above $p\restriction n$) (as in [Sh-b] or [Sh-f, III]). We can find $\langle p^0_\ell: \ell<\omega\rangle\in N$, $\langle k_\ell: \ell<\omega\rangle\in N$, $p^0_\ell \Vdash_{P_\omega}$ "$k_\ell\in\underset{\sim}{X}$", $p^0_\ell\leqq p^0_{\ell+1}\in P_\omega$, $\{k_\ell: \ell<\omega\}\in D$ (use the game).

Let $A^*\in D$ be such that $(\forall A\in D\cap N)[A^*\subseteq^* A]$ and $A^*\subseteq\{k_\ell: \ell<\omega\}$.

We define by induction on n, p_n, q_n such that:

(a) $q_n\in P_n$, $q_{n+1}\restriction n=q_n$, q_n is (N_n, P_n)-generic,

(b) p_n is a P_n-name of a member of $P_\omega\cap N$,

(c) $p_n\leqq q_n$,

(d) $p_n\leqq p_{n+1}$,

(e) if $q_n\in G_n\subseteq P_n$, G_n generic over V, *then* in $V[G_n]$ we can find $\langle p^n_\ell: \ell<\omega\rangle\in N[G_n]\cap(P_\delta/G_n)$, $p_n[G_n]\leqq p^n_\ell\leqq p^n_{\ell+1}$, $\{p^n_\ell: \ell<\omega\}\subseteq P_\omega/G_n$, and for some $B_n\in D\cap N$, $p^n_\ell\Vdash_{P_\omega/G_n}$ "$B_n\cap\ell\subseteq\underset{\sim}{X}\cap\ell$", and $A^*\subseteq B_n$.

If we succeed, $\bigcup_n q_{n+1}\restriction\{n\}\in P_\delta$ force $\underset{\sim}{X}\supseteq A^*$. (Why? By our assumption, $p_{n+1}[G_{n+1}]$ decides the truth value of "$n\in X$". If $n\in A^*$, then the existence of p^{n+1}_{n+1} and (N, P_{n+1})-genericity of q_{n+1} assure us that no $q'\geqq q$ can force n not to be in X. The p^n_ℓ for $n\neq\ell$ are needed only to keep the inductive argument going.) For $n=0$ – we have taken care of it choosing p^0_ℓ, A^*. So let us do the induction step and work in $V[G_n]$ ($q_n\in G_n\subseteq P_n$, G_n generic over V).

So $\langle p^n_\ell: \ell<\omega\rangle\in N[G_n]$ is defined. Working in $V[G_n]^{Q_n}$ we can find, for each ℓ, $\langle p^{n,\ell}_m: m<\omega\rangle$, $p^n_\ell\leqq p^{n,\ell}_m\leqq p^{n,\ell}_{m+1}$ in $P_\omega/\underset{\sim}{G}_{n+1}$, $p^{n,\ell}_m\Vdash$ "$\underset{\sim}{X}\cap m\supseteq Y^n_\ell\cap m$, $Y^n_\ell\in D$" (use D is Ramsey); so there are Q_n-names for them, $\underset{\sim}{Y}^n_\ell$, $\langle p^{n,\ell}_m: m<\omega\rangle$. Clearly without loss of generality those Q_n-names belong to $N[G_n]$. Hence, for $\ell<\omega$ there is $p'_{n,\ell}\in Q_n\cap N[G_n]$, $p^n_\ell\leqq p'_{n,\ell}\in p_\omega/G_n$, $p'_{n,\ell}$ forces $\underset{\sim}{Y}^n_\ell=Y^n_\ell$ (so is as above), so without loss of generality $\langle p'_{n,\ell}, Y^n_\ell: \ell<\omega\rangle\in N[G_n]$, and there is $Y\in D\cap N$, $\bigwedge_\ell Y\subseteq^* Y^n_\ell$. Necessarily, $A^*\subseteq^* Y$. Note: $Y^n_\ell\cap\ell=B\cap\ell\supseteq A^*\cap\ell$, also the function $h:\omega\to\omega$, defined by $h(\ell)=\mathrm{Min}\{n: n>\ell$ and $\sup(Y\setminus Y^n_\ell)<n\}$ belongs to $N[G_n]$. As D is Ramsey, for some $\{k_i: i<\omega\}\in D\cap N[G_n]$, $\bigwedge_i h(k_i)<k_{i+1}$, so for some i^*, $[i^*\leqq k\in A^*\Rightarrow (k,h(k))\cap A^*=\emptyset]$ (we use the forcing being ${}^\omega\omega$-bounding to get D in N rather than in $N[G_n]$). So for some ℓ, $A^*\subseteq Y^n_\ell$ and we can continue. Choose $q_{n+1}\in p_{n+1}$, $q_{n+1}\restriction n=q_n$, $q_{n+1}(n)\in Q_n$ is $(N[G_n], Q_n[G_n])$-generic and above $p'_{n,\ell}$ and is as required. $\square_{3.2}$

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