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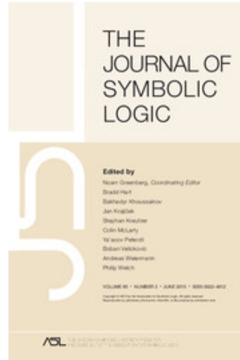

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UNIFORMIZATION, CHOICE FUNCTIONS AND WELL ORDERS IN THE CLASS OF TREES

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Abstract. The monadic second-order theory of trees allows quantification over elements and over arbitrary subsets. We classify the class of trees with respect to the question: does a tree T have a definable choice function (by a monadic formula with parameters)? A natural dichotomy arises where the trees that fall in the first class don't have a definable choice function and the trees in the second class have even a definable well ordering of their elements. This has a close connection to the uniformization problem.

§0. Introduction. The *uniformization problem* for a theory $\mathcal{T}$ in a language $\mathcal{L}$ can be formulated as follows: Suppose $\mathcal{T} \vdash (\forall \bar{Y})(\exists \bar{X})\psi(\bar{X}, \bar{Y})$ where ψ is an $\mathcal{L}$-formula and $\bar{X}$, $\bar{Y}$ are tuples of variables. Is there another $\mathcal{L}$-formula ψ^* such that

$$\mathcal{T} \vdash (\forall \bar{Y})(\forall \bar{X})[\psi^*(\bar{X}, \bar{Y}) \Longrightarrow \psi(\bar{X}, \bar{Y})] \quad \text{and} \quad \mathcal{T} \vdash (\forall \bar{Y})(\exists! \bar{X})\psi^*(\bar{X}, \bar{Y})?$$

Here $\exists!$ means "there is a unique".

The monadic second-order logic is the fragment of the full second-order logic that allows quantification over elements and over monadic (unary) predicates only. The monadic version of a first-order language $\mathcal{L}$ can be described as the augmentation of $\mathcal{L}$ by a list of quantifiable set variables and by new atomic formulas $t \in X$ where t is a first order term and X is a set variable. The monadic theory of a structure $\mathcal{M}$ is the theory of $\mathcal{M}$ in the extended language where the set variables range over all subsets of $|\mathcal{M}|$ and $\in$ is the membership relation.

Given a structure $\mathcal{M}$ we may ask the following question: is there a finite sequence $\bar{P}$ of subsets of $\mathcal{M}$ and a formula $\varphi(x, X, \bar{Z})$ in the monadic language of $\mathcal{M}$ such that

$$\mathcal{M} \models \varphi(a, A, \bar{P}) \Longrightarrow a \in A,$$
$$\mathcal{M} \models (\forall X)(\exists y)[X \neq \emptyset \Longrightarrow \varphi(y, X, \bar{P})]$$

and

$$\mathcal{M} \models \varphi(a, A, \bar{P}) \wedge \varphi(b, A, \bar{P}) \Longrightarrow a = b?$$

If the answer is positive we will say that $\mathcal{M}$ has a monadically definable choice function and that φ defines a choice function from non-empty subsets of $\mathcal{M}$. Note

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that if we let $\varphi(x, Y)$ be the formula that says "if Y is not empty then $x \in Y$" then a negative answer to the choice function problem for $\mathcal{M}$ implies a negative answer to the uniformization problem for the monadic theory of $\mathcal{M}$ (with φ being a counter-example).

The uniformization problem for the monadic theory of the tree $({}^{\omega>}2, \lhd)$ was first asked be Rabin ([6]). Here we continue the work by Gurevich and Shelah ([3]) who gave a negative answer by showing that $({}^{\omega>}2, \lhd)$ does not even have a monadically definable choice function. We ask what trees do have a monadically definable choice function.

Answering this question we split the class of trees into two natural subclasses, the class of wild trees and the class of tame trees and prove the following:

THEOREM. *Let T be a tree. If T is wild then there is no definable choice function on T (by a monadic formula with parameters). If T is tame then there is even a definable well ordering of the elements of T by a monadic formula (with parameters) $\varphi(x, y, \bar{P})$.*

Looking at the definitions and proofs we observe that a tree is tame [wild] if and only if its completion is tame [wild] and that the counter-examples for the choice function problem are either anti-chains or linearly ordered subsets of T. Hence we can prove:

CONCLUSION. *Let T be a tree and T' be its completion. Then the following are equivalent:*

(a) *T is tame.*
(b) *For some n, $\ell \in \mathbb{N}$, for every anti-chain/branch A of T there is a monadic formula $\varphi_A(x, X, \bar{P}_A)$ with quantifier depth $\leq n$ and $\leq \ell$ parameters from T, that defines a choice function from nonempty subsets of A.*
(c) *There is a monadic formula with parameters, $\psi(x, y, \bar{P})$ that defines a well ordering of the elements of T.*
(d) *There is a monadic formula, with parameters, $\psi'(x, y, \bar{P}')$ that defines a well ordering of the elements of T'.*

The 'positive' results on the existence of a definable well ordering (§§3 and 5) are elementary and do not require knowledge of monadic logic. The negative results (§§2, 3, and 4) are based on understanding of some composition theorems that hold for the monadic theory of trees. These facts are collected in §1.

More details and historical background can be found in [2] and [3].

§1. Composition theorems. In this section we will define partial theories and establish the technical tools that will be applied later. The composition theorems formalized here will enable us to compute partial theories of trees from partial theories of their parts. By using such theorems we will prove later that if for example a dense chain does not have definable choice function then a tree with a dense branch does not have a definable choice function as well.

DEFINITION 1.1. $(T, \lhd)$ is a *tree* if $\lhd$ is a partial order on T and for every $\eta \in T$, $\{\nu : \nu \lhd \eta\}$ is linearly ordered by $\lhd$. $\unlhd$ means $\lhd$ or $=$.

Note, a chain $(C, <)$ is a tree and so is a set without structure I.

DEFINITION 1.2. Let T be a tree.

(1) $S \subseteq T$ is a *convex subset* if $\eta, \nu \in S$ and $\eta \lhd \sigma \lhd \nu \in T$ implies $\sigma \in S$. When S is a convex subset of T we say that $(S, \lhd)$ is a *subtree* of $(T, \lhd)$. If T is a chain we use the term a *convex segment* or just a *segment*.
(2) $B \subseteq T$ is a *sub-branch* of T if B is convex and $\lhd$-linearly ordered.
(3) $B \subseteq T$ is a *branch* of T if B is a maximal sub-branch of T.
(4) $A \subseteq T$ is an *initial segment* of T if A is a sub-branch that is $\lhd$-downward closed. η is *above* [*strictly above*] an initial segment A if $\nu \in A \Longrightarrow \nu \unlhd \eta$ [$\nu \in A \Longrightarrow \nu \lhd \eta$]. In these cases we write $A \unlhd \eta$ [$A \lhd \eta$].
(5) For $\eta \in T$, $T_{\geq \eta}$ is the sub-tree $(\{\nu \in T : \eta \unlhd \nu\}, \lhd)$. $T_{>\eta}$ is the sub-tree $(T_{\geq \eta} \setminus \{\eta\}, \lhd)$. For $A \subseteq T$ an initial segment, $T_{\geq A}$ and $T_{>A}$ are defined naturally (and are equal if A does not have a $\lhd$-maximal element).
(6) For $\eta \in T$ we denote by $\mathrm{Suc}(\eta)$ or $\mathrm{Suc}_T(\eta)$ the set of $\lhd$-immediate successors of η (which may be empty).
(7) For $\eta, \nu \in T$ we denote the *common initial segment of η and ν in T* by $\eta \sqcap \nu$. This is defined to be the initial segment $\{\tau : \tau \unlhd \eta \ \& \ \tau \unlhd \nu\}$. However, when $\eta \sqcap \nu$ has a maximal element we may identify it with this element.
(8) If there is an $\eta \in T$ that satisfies $(\forall \nu \in T)[\eta \lhd \nu]$ we say that T has a root and denote η by $\mathrm{r}(T)$.
(9) $\eta, \nu \in T$ are *incomparable in* T and we write $\eta \perp \nu$, if neither $\eta \unlhd \nu$ nor $\nu \unlhd \eta$. $X \subseteq T$ is an *anti-chain of* T if X consists of pairwise incomparable elements of T.
(10) When $B \subseteq T$ is a sub-branch and $A \subseteq B$ is an initial segment we say that $\sigma \in T$ *cuts B at A* if for every $\eta \in A$ and $\nu \in B \setminus A$ we have $\eta \lhd \sigma \ \& \ \nu \perp \sigma$.
(11) A *gap* in T is a pair (A_1, A_2) where $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2$ is a sub-branch, A_1 is an initial segment, (so $\eta \in A_1, \nu \in A_2 \Longrightarrow \eta \lhd \nu$) without a $\lhd$-maximal element, A_2 without a $\lhd$-minimal element, and there is some $\sigma \in T$ that cuts $A_1 \cup A_2$ at A_1.
(12) *Filling a gap* (A_1, A_2) in T is adding a node τ to T such that $\eta \in A_1 \Longrightarrow \eta \lhd \tau$, $\nu \in A_2 \Longrightarrow \tau \lhd \nu$ and for every σ as above we have $\tau \lhd \sigma$.

DEFINITION 1.3. The *full binary tree* is the tree $({}^{\omega>}2, \lhd)$ where for sequences $\eta, \nu \in {}^{\omega>}2$, $\eta \lhd \nu$ means η is an initial segment of ν.

DEFINITION 1.4. The *monadic language of trees* $\mathscr{L}$ is the monadic version of the language of partial orders $\{\lhd\}$. Formally, we let $\mathscr{L} = (\mathrm{Sing}, \mathrm{Empty}, \lhd, \subseteq)$ where 'Sing' and 'Empty' are unary predicates, $<$ and $\lhd$ are binary relations. ($\mathscr{L}$ is a first order language).

Given a tree T we define the *monadic theory of T* as the first order theory of the model $\mathscr{M}_T := (\mathscr{P}(T); \mathrm{Sing}, \mathrm{Empty}, \lhd, \subseteq)$ where

$$\mathscr{M}_T \models \mathrm{Empty}(X) \iff X = \emptyset,$$
$$\mathscr{M}_T \models \mathrm{Sing}(X) \iff X = \{x\} \text{ for some } x \in T,$$
$$\mathscr{M}_T \models X \lhd Y \iff X = \{x\},\ Y = \{y\} \text{ and } T \models x \lhd y,$$

$\subseteq$ is interpreted in $\mathscr{M}_T$ as the usual inclusion relation.

We will not distinguish between T and $\mathscr{M}_T$ and write for example $T \models \mathrm{Sing}(X)$ and $T \models X \lhd Y$.

The definable relations $\trianglelefteq$ and $\in$ will be used freely thus we will write $T \models X \trianglelefteq Y$ and $T \models X \in Y$ (meaning $\mathscr{M}_T \models \mathrm{Sing}(X)\ \&\ X \subseteq Y$).

When T is a chain (linearly ordered set) we replace $\triangleleft$ and $\trianglelefteq$ by $<$ and $\leq$ respectively.

NOTE. Everything that is defined in 1.2 is definable by a monadic formula.

NOTATIONS. C, D and I denote chains. S, T and Γ denote trees.

Lower case and Greek letters (x, y, a, b, η, ν) are used to denote elements, upper case letters (X, Y, A, P, Q) denote subsets.

$\bar{a}$ and $\bar{P}$ denote finite sequences of elements and subsets, their lengths are $\lg(\bar{a})$ and $\lg(\bar{P})$. We will write $\bar{a} \in T$ and $\bar{P} \subseteq T$ instead of $\bar{a} \in {}^{\lg(\bar{a})}T$ and $\bar{P} \in {}^{\lg(\bar{P})}\mathscr{P}(T)$.

When $\bar{P}$ and $\bar{Q}$ are of the same length we will write $\bar{P} \cup \bar{Q}$ to denote $\langle P_0 \cup Q_0, \ldots, P_{\ell-1} \cup Q_{\ell-1}\rangle$. Similarly we write $\bigcup_{i\in I} \bar{P}^i$ (assuming $\lg(\bar{P}^i)$ is constant). $\bar{P} \cap S$ means $\langle P_0 \cap S, \ldots, P_{\ell-1} \cap S\rangle$.

$\bar{P}{}^\wedge\bar{Q}$ is the sequence $\langle P_0, \ldots, Q_0, \ldots\rangle$.

Next we define, following [7], the partial theories of a tree T. These are finite approximations of the monadic theory of T. $\mathrm{Th}^n(T; \bar{P})$ is essentially the monadic theory of $(T; \bar{P}, \triangleleft)$ restricted to sentences of quantifier depth n.

DEFINITION 1.5. For any tree T, $\bar{A} \subseteq T$, and a natural number n, define by induction

$$t = \mathrm{Th}^n(T; \bar{A}),$$

for $n = 0$:

$$t = \{\varphi(\bar{X}) : \varphi(\bar{X}) \in \mathscr{L},\ \varphi(\bar{X}) \text{ quantifier free},\ T \models \varphi(\bar{A})\},$$

for $n = m + 1$:

$$t = \{\mathrm{Th}^m(T; \bar{A}{}^\wedge B) : B \subseteq T\}.$$

$T_{n,\ell}$ is the set of all formally possible $\mathrm{Th}^n(T; \bar{P})$ where T is a tree and $\lg(\bar{P}) = \ell$.

NOTATION. When $x \in T$ we will usually write $\mathrm{Th}^n(T; x)$ instead of $\mathrm{Th}^n(T; \{x\})$.

FACT 1.6.

(A) For every formula $\psi(\bar{X}) \in \mathscr{L}$ there is an $n \in \mathbb{N}$ such that from $\mathrm{Th}^n(T; \bar{A})$ we can effectively decide whether $T \models \psi(\bar{A})$. We will call the minimal such n 'the depth of ψ' and write $\mathrm{dp}(\psi) = n$.

(B) If $m \geq n$ then $\mathrm{Th}^n(T; \bar{A})$ can be effectively computed from $\mathrm{Th}^m(T; \bar{A})$.

(C) Each $\mathrm{Th}^n(T; \bar{A})$ is hereditarily finite, and we can effectively compute the set $T_{n,\ell}$ from n and ℓ.

Next we recall the composition theorem for linear orders which states that the partial theory of a chain can be computed from the partial theories of its convex parts. This enables us to define the operation of addition of theories.

DEFINITION 1.7. If C, D are chains then $C + D$ is the chain that is obtained by adding a copy of D after C.

If $\langle C_i : i \in I\rangle$ is a sequence of chains then $\sum_{i\in I} C_i$ is the chain D that is the concatenation of the C_i's.

THEOREM 1.8 (composition theorem for linear orders).

(1) *If* $\lg(\bar{A}) = \lg(\bar{B}) = \lg(\bar{A}') = \lg(\bar{B}') = \ell$, *and*

$$\mathrm{Th}^m(C;\bar{A}) = \mathrm{Th}^m(C';\bar{A}') \quad and \quad \mathrm{Th}^m(D;\bar{B}) = \mathrm{Th}^m(D';\bar{B}')$$

then

$$\mathrm{Th}^m(C+D;\bar{A}\cup\bar{B}) = \mathrm{Th}^m(C'+D';\bar{A}'\cup\bar{B}').$$

(2) *If* $\mathrm{Th}^m(C_i;\bar{A}^i) = \mathrm{Th}^m(D_i;\bar{B}^i)$ *and* $\lg(\bar{A}^i) = \lg(\bar{B}^i) = l$ *for each* $i \in I$, *then*

$$\mathrm{Th}^m\Big(\sum_{i\in I} C_i;\bigcup_{i\in I}\bar{A}^i\Big) = \mathrm{Th}^m\Big(\sum_{i\in I} D_i;\bigcup_{i\in I}\bar{B}^i\Big).$$

PROOF. By [7, Theorem 2.4] (where a more general theorem is proved), or directly by induction on m. ⊣

NOTATION 1.9.

(1) When, for some $m,\ell\in\mathbb{N}$, $t_1, t_2, t_3 \in T_{m,\ell}$ then $t_1 + t_2 = t_3$ means: there are chains C and D such that

$$t_1 = \mathrm{Th}^m(C;A_0,\dots,A_{\ell-1}) \ \&\ t_2 = \mathrm{Th}^m(D;B_0,\dots,B_{\ell-1})$$
$$\&\ t_3 = \mathrm{Th}^m(C+D;\bar{A}\cup\bar{B}).$$

(By the composition theorem, the choice of C and D is immaterial.)

(2) $\sum_{i\in I}\mathrm{Th}^m(C_i;\bar{A}^i)$ is $\mathrm{Th}^m(\sum_{i\in I} C_i;\bigcup_{i\in I} A^i)$, (assuming $\lg(\bar{A}^i) = \lg(\bar{A}^j)$ for $i,j\in I$).

(3) If D is a sub-chain of C and $\bar{A}\subseteq C$ then $\mathrm{Th}^m(D;\bar{A})$ abbreviates $\mathrm{Th}^m(D;\bar{A}\cap D)$.

(4) For C a chain, $a<b\in C$ and $\bar{P}\subseteq C$ we denote by $\mathrm{Th}^n(C;\bar{P})\restriction_{[a,b)}$ the theory $\mathrm{Th}^n([a,b);\bar{P}\cap[a,b))$.

The class of trees has some weaker (but useful) composition theorems. First we define the composition of subtrees of the full binary tree following [3] and quote the respective composition theorem.

DEFINITION 1.10. Let $S\subseteq {}^{\omega>}2$ be a tree. A *grafting function* on S is a function g satisfying the following conditions:

(a) $\mathrm{dom}(g)\subseteq S\times\{0,1\}$,

(b) if $(x,0)\in\mathrm{dom}(g)$ then $x^\wedge\langle 0\rangle\notin S$ and if $(x,1)\in\mathrm{dom}(g)$ then $x^\wedge\langle 1\rangle\notin S$,

(c) every value $g(x,d)$ of g $(d\in\{0,1\})$ is a tree $\subseteq {}^{\omega>}2$.

The *composition* of a tree S and a grafting function g is the tree

$$S\cup\{x^\wedge\langle d\rangle^\wedge y : (x,d)\in\mathrm{dom}(g),\ y\in g(x,d)\}.$$

THEOREM 1.11 (composition theorem for binary trees). *Let* $S\subseteq {}^{\omega>}2$ *be a tree,* $N\subseteq {}^{\omega>}2$ *be the composition of* S *and a grafting function* g *and* $\bar{P}\subseteq N$ *with* $\lg(\bar{P})=\ell$.

Then, for every $n\in\mathbb{N}$ *there is* $m=m(n,\ell)\in\mathbb{N}$ *(effectively computable from* n *and* ℓ*) such that from* $\mathrm{Th}^m(S;\bar{P},\bar{L}^g(n,\bar{P}),\bar{R}^g(n,\bar{P}))$ *we can effectively compute*

$Th^m(N;\bar{P})$ where

$$L^g_t(n,\bar{P}) := \{\, x \in M : (x,0) \in \mathrm{dom}(g),\ \mathrm{Th}^n(g(x,0),\bar{P}) = t \,\},$$
$$R^g_t(n,\bar{P}) := \{\, x \in M : (x,1) \in \mathrm{dom}(g),\ \mathrm{Th}^n(g(x,0),\bar{P}) = t \,\},$$
$$\bar{L}^g(n,\bar{P}) := \langle\, L^g_t(n,\bar{P}) : t \in T_{n,\ell} \,\rangle$$

and

$$\bar{R}^g(n,\bar{P}) := \langle\, R^g_t(n,\bar{P}) : t \in T_{n,\ell} \rangle.$$

PROOF. This is Theorem 2 in §2.3 of [3]. (The language that is used there is different from our L but all the mentioned symbols are monadically inter-definable with some additional parameters with our $\lhd$.) ⊣

The next three theorems enable us to compute a partial theory $\mathrm{Th}^n(T;\bar{P})$ from partial theories of sub-structures of T. The proofs are by induction on n noting that $\mathrm{Th}^0(T;\bar{P})$ can express only statements as $P_i \subseteq P_j,\ P_i \lhd P_j,\ P_i = P_j$, $\mathrm{Empty}(P_i)$ and $\mathrm{Sing}(P_i)$ and that Th^{n+1} is a collection of n-theories. Everything is basically the same as in the previous case and we will not elaborate beyond that.

In the first case we are given a tree T a sequence $\bar{X} \subseteq T$ and an initial segment $A \subseteq T$. We would like to compute $\mathrm{Th}^n(T;\bar{X})$ from the theories of subtrees above A.

First, for x above A in T denote by $T_{A,x}$, the subtree

$$\{\, y \in T : (\exists z)[z \trianglelefteq x \ \&\ z \trianglelefteq y \ \&\ A \lhd z] \,\}.$$

Call x and y equivalent above A if x and y are above A and $T_{A,x} = T_{A,y}$ and let $\{\, T_i : i \in I_A \,\}$ list the equivalence classes above A (it's a collection of pairwise disjoint of sub-trees). Finally, let

$$T^*_A := T \setminus \bigcup_{i \in I_A} T_i = \{\, y \in T : \neg A \lhd y \,\}.$$

A typical case is when $\{\, v_i : i \in I \,\}$ is the set of immediate successors of some $\eta \in T$. In this case we are interested in the trees $\{\, T_{\geq v_i} : i \in I \,\}$ and $\{\, \tau : \tau \trianglelefteq \eta \vee \tau \perp \eta \,\}$.

THEOREM 1.12 (composition theorem for general successors). *Let T be a tree, let $\bar{X} \subseteq T$ with $\lg(\bar{X}) = \ell$ and let $A \subseteq T$ be an initial segment. Then, for every $n \in \mathbb{N}$, there is $m = m(n,\ell) \in \mathbb{N}$ (effectively computable from n and ℓ) such that from*

$$\mathrm{Th}^m(T^*_A;\bar{X}) \quad \textit{and} \quad \mathrm{Th}^m(I_A;\bar{P}^A(n,\bar{X}))$$

we can effectively compute $\mathrm{Th}^n(T;\bar{X})$ where

$$P^A_t(n,\bar{X}) := \{\, i \in I_A : \mathrm{Th}^n(T_i;\bar{X}) = t \,\}$$

and

$$\bar{P}^A(\bar{X}) := \langle\, P^A_t(n,\bar{X}) : t \in T_{n,\ell} \,\rangle.$$

Note, $\mathrm{Th}^m(I_A;\bar{P}^A(n,\bar{X}))$ is the m-theory of a set without structure—i.e., in the monadic language of equality.

In the second case we are given a tree T and a branch $B \subseteq T$. Now we would like to compute the theory of T from an enrichment of the theory of the branch B, that is a theory of a chain. This can be done by adding unary predicates that will tell us, for each node $\eta \in B$, the theory of the sub-tree consisting of the elements that cut B at η. However, we must take into account the possibility that B contains gaps. Thus, given a branch $B \subseteq T$ let $(B', \lhd)$ be the chain that is obtained by filling all the gaps in B. So B' is a subset of the completion of B as a linear ordering.

Now for $\eta \in B'$ let $T^{B'}_{\geq\eta} := T_{\geq\eta} \setminus B'$.

THEOREM 1.13 (composition theorem for branches). *Let T be a tree, $B \subseteq T$ a branch and $\bar{X} \subseteq T$ with $\lg(\bar{X}) = \ell$. Then, for every $n \in \mathbb{N}$ there is $m = m(n, \ell) \in \mathbb{N}$ (effectively computable from n and ℓ) such that from $\mathrm{Th}^m(B'; B, \bar{P}^{B'}(n, \bar{X}))$ we can effectively compute $\mathrm{Th}^n(T; \bar{X})$ where*

$$P^{B'}_t(n, \bar{X}) := \{\eta \in B' : \mathrm{Th}^m(T^{B'}_{\geq\eta}; \bar{X}) = t\}$$

and

$$\bar{P}^{B'}_t(n, \bar{X}) := \langle P^{B'}_t(n, \bar{X}) : t \in T_{n,\ell}\rangle.$$

Moreover, if $\bar{Y} \subseteq B$ then from $\mathrm{Th}^m(B'; B, \bar{P}^{B'}(n, \bar{X}), \bar{Y})$ we can effectively compute $\mathrm{Th}^n(T; \bar{X}{}^\wedge \bar{Y})$.

As we already know by [3], the binary tree does not have a definable choice function. We would like to reflect this fact in trees that embed it.

DEFINITION 1.14. Let T be a tree, by "$F: {}^{\omega>}2 \hookrightarrow T$ is an embedding" we mean F is 1-1 and for $\eta, \nu \in {}^{\omega>}2$, $\eta \lhd \nu \iff F(\eta) \lhd F(\nu)$, we also assume that T has a root and $F(\mathrm{r}({}^{\omega>}2)) = \mathrm{r}(T)$.

Now let $F: {}^{\omega>}2 \hookrightarrow T$ be an embedding and let $S \subseteq T$ be $F''({}^{\omega>}2)$. S is a tree (but not necessarily a sub-tree of T) that can be identified with ${}^{\omega>}2$.

For $x = F(\eta) \in S$ define x^0 $[x^1] \in S$ to be $F(\eta{}^\wedge\langle 0\rangle)$ $[F(\eta{}^\wedge\langle 1\rangle)]$.

For $Y \subseteq S$ an anti-chain (hence an anti-chain of T) let $\mathrm{Bush}(Y) := \{x \in T : (\exists y \in Y)[x \trianglelefteq y]\}$, (it's a subtree of T) and let $\mathrm{Bush}_S(Y) := \mathrm{Bush}(Y) \cap S$ (it's a subtree of S).

For every $y \in S$ denote $y^0 \sqcap y^1$ by y^i. It may be an element of T or an initial segment (see the convention in 1.2 (7)). Anyway, in the definitions below we think of the y^i's as elements. When y^i happens to be an initial segment, one should replace occurrences of "$x \trianglelefteq y^i$" by "$x \in y^i$".

For every $y \in S$ we define some subtrees of $T_{\geq y}$ (some of them may be trivial if for example $y = y^i$):

(0) $T_0(y) := T_{\geq y}$.

(1) $T_1(y) := \{x \in T : (\neg y^i \lhd x) \ \&\ (\exists z)[(z \trianglelefteq x) \ \&\ (y \lhd z \lhd y^i)]\}$.
[These are the elements that split from the segment (y, y^i).]

(2) $T_2(y) := \{x \in T : (y \trianglelefteq x) \ \&\ (\forall z)[(z \trianglelefteq y^i) \ \&\ (z \trianglelefteq x) \Longrightarrow (z \lhd y)]\}$.
[These are the elements that split from y but not from the segment (y, y^i).]

(3) $T_3(y) := \{x \in T : (\neg y^0 \lhd x) \ \&\ (\exists z)[(z \trianglelefteq x) \ \&\ (y^i \lhd z \lhd y^0)]\}$.
[These are the elements that split from the segment (y^i, y^0).]

(4) $T_4(y) := \{x \in T : (\neg y^1 \lhd x) \ \&\ (\exists z)[(z \trianglelefteq x) \ \&\ (y^i \lhd z \lhd y^1)]\}$.
[These are the elements that split from the segment (y^i, y^1).]

(5) $T_5(y) := \{x \in T : (y^i \trianglelefteq x) \,\&\, (\forall z)[(z \trianglelefteq x) \,\&\, (z \trianglelefteq y^0 \vee z \trianglelefteq y^1) \Longrightarrow (z \trianglelefteq y^i)]\}$.
[These are the elements that split from y^i but not from the segments (y^i, y^0) and (y^i, y^1).]
(6) $T_6(y) := T_{\geq y^0}$.
(7) $T_7(y) := T_{\geq y^1}$.

For $y \in S$, $\bar{X} \subseteq T$ with $\lg(\bar{X}) = \ell$, $\bar{t} = \langle t_0, t_1, \dots, t_7\rangle$ where $t_i \in T_{n,\ell}$, define $\bar{Q}_{\bar{t}}(n, \bar{X})$ by

$$y \in Q_{\bar{t}}(n, \bar{X}) \iff [\mathrm{Th}^n(T_0(y); \bar{X}) = t_0 \,\&\, \cdots \,\&\, \mathrm{Th}^n(T_7(y); \bar{X}) = t_7].$$

Let $Q_\emptyset := T \setminus S$.
Finally let $\bar{Q}(n, \bar{X})$ be $\langle Q_{\bar{t}}(n, \bar{X}) : \bar{t} \in {}^7(T_{n,\ell})\rangle {}^\wedge \langle Q_\emptyset\rangle$.

Note that every anti-chain $Y \subseteq S$ is definable from $\mathrm{Bush}_S(Y)$ and that S is definable from $\bar{Q}(n, \bar{X})$.

THEOREM 1.15 (composition theorem for embeddings). *Let T be a tree, $\bar{X} \subseteq T$ with $\lg(\bar{X}) = \ell$, $F\colon {}^{\omega>}2 \hookrightarrow T$ an embedding and let $S = F''({}^{\omega>}2)$. Then, for every $n \in \mathbb{N}$ there is $m = m(n, \ell) \in \mathbb{N}$ (effectively computable from n and ℓ) such that, following the above notations, for every anti-chain $Y \subseteq S$ and $y \in Y$, from* $\mathrm{Th}^m(\mathrm{Bush}_S(Y); y, \bar{Q}(n, \bar{X}))$ *we can effectively compute* $\mathrm{Th}^n(T; y, Y, \bar{X})$.

§2. Dense linear orders.

Every finite set A has a definable well ordering (by a formula with $\leq |A|$ parameters). This is not the case for infinite models.

CLAIM 2.1. *Let A be an infinite set without structure. Then there is no definable choice function on A. Moreover, if $|A| > 2^\ell$ then no formula with $\leq \ell$ parameters defines a choice function on A.*

PROOF. Let $\bar{P} = \langle P_0, \dots, P_{\ell-1}\rangle \subseteq A$ and suppose $\varphi(x, X, \bar{P})$ defines a choice function from subsets of A. Let $B = \{b_1, b_2\} \subseteq A$ be such that for every $i < \ell$,

$$b_1 \in P_i \iff b_2 \in P_i.$$

B exists if $|A| > 2^l$ and in particular if A is infinite. Clearly

$$A \models \varphi(b_1, B, \bar{P}) \iff A \models \varphi(b_2, B, \bar{P})$$

contradicting "φ chooses an element from B". ⊣

A chain C that embeds a dense linear order (hence the chain of rational numbers order $\mathbb{Q}$) does not have a definable choice function. The proof is by applying a Ramsey-like theorem for additive colorings from [7].

DEFINITION 2.2.
(a) A *coloring* of a chain C is a function $f\colon [C]^2 \to I$ where $[C]^2$ is the set of unordered pairs of distinct elements of C and I is a finite set (the set of colors).
(b) The coloring f is *additive* if for every $x_1 < y_1 < z_1$ and $x_2 < y_2 < z_2$ in C

$$[f(x_1, y_1) = f(x_2, y_2),\ f(y_1, z_1) = f(y_2, z_2)] \Longrightarrow f(x_1, z_1) = f(x_2, z_2).$$

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In this case a partial operation $+$ is well defined on I:

$$i_1 + i_2 = i_3 \iff (\exists x, y, z \in C)[x < y < z \ \& \ f(x,y) = i_1 \\ \& \ f(y,z) = i_2 \\ \& \ f(x,z) = i_3].$$

(Compare with 1.9 (1).)

(c) A sub-chain $D \subseteq C$ is *homogeneous* (for f) if there is an $i_0 \in I$ such that for every $x, y \in D$, $f(x,y) = i_0$.

THEOREM 2.3. *If f is an additive coloring of a dense chain C by a finite set I of colors, then there is an interval of C which has a dense homogeneous subset.*

PROOF. This is Theorem 1.3. in [7]. ⊣

CLAIM 2.4. *Let $(C, <)$ be a linear order that embeds a dense linear order. Then there is no definable choice function on C.*

PROOF. Let $\bar{P} \subseteq C$ with $\lg(\bar{P}) = \ell$ and suppose $\varphi(x, X, \bar{P})$ defines a choice function on C. Suppose $\mathrm{dp}(\varphi) = n$ (so from $\mathrm{Th}^n(C; x, X, \bar{P})$ we know if $\varphi(x, X, \bar{P})$ holds). Finally let $D \subseteq C$ be dense (in itself).

By 2.3 there is an $A \subseteq D$, dense inside an interval of D, hence in itself, homogeneous with respect to the coloring $f(a,b) = \mathrm{Th}^{n+5}(C; \bar{P})\restriction_{[a,b)}$, (see Notation 1.9 (4)) that is, for some $t^* \in T_{n+5,\ell}$:

$$[a,b,c,d \in A \ \& \ a < b \ \& \ c < d] \Longrightarrow [\mathrm{Th}^{n+5}(C; \bar{P})\restriction_{[a,b)} = \mathrm{Th}^{n+5}(C; \bar{P})\restriction_{[c,d)} = t^*].$$

Let $\mathbb{Z}$ be the set of integers and choose $X \subseteq A$ of order type $\mathbb{Z}$, denote $X := \{x_n : n \in \mathbb{Z}\}$. Suppose our choice function picks x_m from X, i.e.

$$(*) \qquad C \models \varphi(x_m, X, \bar{P}) \ \& \ \bigwedge_{k \neq m} C \models \neg\varphi(x_k, X, \bar{P}).$$

Let $C_0 = \{c \in C : x_i \in X \Longrightarrow c < x_i\}$ and $C_1 = \{c \in C : x_i \in X \Longrightarrow x_i < c\}$. Let $t_0 = \mathrm{Th}^n(C; \bar{P})\restriction_{C_0}$ and $t_1 = \mathrm{Th}^n(C; \bar{P})\restriction_{C_1}$. So

$$\mathrm{Th}^n(C; \bar{P}) = t_0 + \sum_{i \in \mathbb{Z}} \mathrm{Th}^n(C; \bar{P})\restriction_{[x_i, x_{i+1})} + t_1.$$

Now denote:

$$t_0' := \mathrm{Th}^n(C; x_m, X, \bar{P})\restriction_{C_0} = \mathrm{Th}^n(C; \emptyset, \emptyset, \bar{P})\restriction_{C_0},$$
$$t_1' := \mathrm{Th}^n(C; x_m, X, \bar{P})\restriction_{C_1} = \mathrm{Th}^n(C; \emptyset, \emptyset, \bar{P})\restriction_{C_1},$$
$$t' := \mathrm{Th}^n(C; x_l, X, \bar{P})\restriction_{[x_k, x_{k+1})}, \quad \text{when } k \neq l \text{ this is } \mathrm{Th}^n(C; \emptyset, x_k, \bar{P})\restriction_{[x_k, x_{k+1})},$$
$$t^{(k)} := \mathrm{Th}^n(C; x_k, X, \bar{P})\restriction_{[x_k, x_{k+1})} = \mathrm{Th}^n(C; x_k, x_k, \bar{P})\restriction_{[x_k, x_{k+1})}.$$

Now $\emptyset$ is definable and x_k is the first element in the segment $[x_k, x_{k+1})$ hence also definable. So, as we started with $n + 5$ (which is an overkill),

- t_0' and t_1' do not depend on m,
- t_0 determines t_0' and t_1 determines t_1',
- t^* determines t' and $t^{(k)}$.

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We also have, for every $k \in \mathbb{Z}$:

$$(**) \qquad \mathrm{Th}^n(C; x_k, X, \bar{P}) = t'_0 + \sum_{\substack{j \in \mathbb{Z} \\ j<k}} t' + t^{(k)} + \sum_{\substack{j \in \mathbb{Z} \\ j>l}} t' + t'_1.$$

It follows (by homogeneity and the above remarks) that for every $k \in \mathbb{Z}$:

$$(\alpha) \qquad t^{(k)} = t^{(m)},$$

$$(\beta) \qquad \sum_{\substack{j \in \mathbb{Z} \\ j<k}} t' = \sum_{\substack{j \in \mathbb{Z} \\ j<m}} t',$$

$$(\gamma) \qquad \sum_{\substack{j \in \mathbb{Z} \\ j>k}} t' = \sum_{\substack{j \in \mathbb{Z} \\ j>m}} t'.$$

So by $(**)$, for every $k \in \mathbb{Z}$

$$\mathrm{Th}^n(C; x_k, X, \bar{P}) = \mathrm{Th}^n(C; x_m, X, \bar{P}).$$

Hence

$$C \models \varphi(x_k, X, \bar{P}) \iff C \models \varphi(x_m, X, \bar{P})$$

Contradicting $(*)$ = "φ chooses x_m from X". ⊣

§3. **Scattered orders.** A chain is scattered if it does not embed a dense chain. We will define Hdeg, the Hausdorff degree of scattered chains, and show that a scattered chain $(C, <)$ has a definable well ordering if and only if $\mathrm{Hdeg}(C) < \omega$ and that $\mathrm{Hdeg}(C) \geq \omega \Rightarrow$ there is no definable choice function on C.

Definition 3.1. We define by recursion the Hausdorff degree of a scattered chain $(C, <)$:

- $\mathrm{Hdeg}(C) = 0$ if and only if C is finite
- $\mathrm{Hdeg}(C) = \alpha$ if and only if $\bigwedge_{\beta<\alpha} \mathrm{Hdeg}(C) \neq \beta$ and $C = \sum_{i \in I} C_i$ where I is well ordered or inversely well ordered and for every $i \in I$,

$$\bigvee_{\beta<\alpha} \mathrm{Hdeg}(C_i) = \beta.$$

Claim 3.2.

(1) *C is a scattered chain if and only if* $\mathrm{Hdeg}(C)$ *is well defined (i.e., there is one and only one ordinal such that* $\mathrm{Hdeg}(C) = \alpha$).

(2) *Let C be a scattered chain with* $\mathrm{Hdeg}(C) = \alpha$, *$C'$ the completion of C and $D \subseteq C'$. Then C' and D are scattered and* $\mathrm{Hdeg}(D) \leq \mathrm{Hdeg}(C') = \alpha$.

Proof.

(1) By [4].

(2) By induction on α. ⊣

Claim 3.3. *For every $n \in \mathbb{N}$ there is a formula $\varphi_n(x, y, \bar{Z})$ with* $\lg(\bar{Z}) \leq n-1$ *such that if C is a scattered chain with* $\mathrm{Hdeg}(C) \leq n$, *then there are $\bar{P} \subseteq C$ with such that $\varphi_n(x, y, \bar{P})$ defines a well ordering of C.*

PROOF. We will show, by induction on $n \in \mathbb{N}$ the existence of a formula $\psi_n(x, y, \bar{Z})$ with $\bar{Z} = \langle Z_1, \dots, Z_{n-1} \rangle$ such that for every scattered chain C with $\mathrm{Hdeg}(C) = n$ there are $\bar{P} \subseteq C$ such that $\psi_n(x, y, \bar{P})$ well orders C. $\varphi(x, y, \bar{Z})$ that should apply to chains C with $\mathrm{Hdeg}(C) \leq n$ will be of the form

$$(Z_{n-1} \neq \emptyset \rightarrow \psi_n) \ \& \ \big((Z_{n-1} = \emptyset \wedge Z_{n-2} \neq \emptyset) \rightarrow \psi_{n-1}\big) \dots .$$

For $n = 0$ define $\psi_n(x, y) := x < y$. For $n = 1$, if $\mathrm{Hdeg}(C) = 1$ then either C is well ordered or inversely well ordered. The monadic sentence

$$\theta := (\forall X)\big[X \neq \emptyset \rightarrow (\exists x \in X)[(\forall y \in X)(x \leq y)]\big]$$

distinguishes between these cases. Let then

$$\psi_1(x, y) := \big(\theta \rightarrow (x < y)\big) \ \& \ \big(\neg\theta \rightarrow (x > y)\big).$$

To finish suppose $\mathrm{Hdeg}(C) = n + 1$, so $C = \sum_{i \in I} C_i$ where I is well ordered or inversely well ordered and each $\mathrm{Hdeg}(C_i)$ is n. By the induction hypothesis there is a sequence $\langle \bar{P}^i : i \in I \rangle$ with $\bar{P}^i \subseteq C_i$ where $\bar{P}^i = \langle P^i_1, \dots, P^i_{n-1} \rangle$ such that $\psi_n(x, y, \bar{P}^i)$ well orders each C_i. For $0 < k < n$ let $P_k := \bigcup_{i \in I} P^i_k$ (disjoint union).

Let $P_n := \bigcup\{ C_i : i \text{ an even ordinal} \}$. Using P_n define an equivalence relation $\sim$ on C by $x \sim y$ if and only if $\bigwedge_i (x \in C_i \iff y \in C_i)$. The definition is by the formula

$$\begin{aligned} e(x, y, P_n) := {} & [x \in P_n \iff y \in P_n] \\ & \& \ (\forall z)[(x < y < z \vee y < z < x) \Longrightarrow (x \in P_n \iff z \in P_n)]. \end{aligned}$$

Similarly we can define the $\sim$-equivalence classes $[x]$. Now there is a formula $\theta'(P_n)$ such that $C \models \theta'(P_n)$ if and only if I is well ordered:

$$\begin{aligned} \theta'(P_n) := (\forall X)\big[&[X \neq \emptyset \wedge (\forall x, y \in X)\neg e(x, y, P_n)] \\ &\rightarrow [(\exists x \in X)[(\forall y \in X)(x \leq y)]]\big]. \end{aligned}$$

$\psi_{n+1}(x, y, \bar{Z})$ is defined by:

$$\begin{aligned} \big[\theta'(Z_n) \wedge [x \not\sim y] \rightarrow x < y\big] \ & \& \ \big[\neg\theta'(Z_n) \wedge [x \not\sim y] \rightarrow x > y\big] \\ & \& \ \big[[x \sim y] \rightarrow \psi_n(x, y, \bar{Z} \cap [x])\big]. \end{aligned}$$

⊣

Next we prove that scattered chains of infinite Hdeg don't have a definable choice function (hence a well ordering). It suffices to look only at special chains: ${}^{n\geq}\omega$ with the 'alternating' lexicographic order.

DEFINITION 3.4. We define for every $n < \omega$ a model $\mathcal{M}^n$ in the language consisting of a binary relation $<^n$:

(a) The universe of $\mathcal{M}^n$, which will be denoted by M^n, is the tree ${}^{n\geq}\omega$.

(b) Let, for every $\eta \in {}^{n\geq}\omega$, $<_\eta$ be a linear ordering of $\mathrm{Suc}(\eta) := \{ \eta^\frown\langle k \rangle : k < \omega \}$ such that if $\mathrm{lev}(\eta)$ is even then $k < l \Longrightarrow \eta^\frown\langle k \rangle <_\eta \eta^\frown\langle l \rangle$, and if $\mathrm{lev}(\eta)$ is odd then $k < l \Longrightarrow \eta^\frown\langle l \rangle <_\eta \eta^\frown\langle k \rangle$. (So $<_\eta$ orders $\mathrm{Suc}(\eta)$ with order type ω if η is in an even level and with order type ω^* if η is in an odd level.)

(c) $<^n$ is the lexicographic order induced by the orders $<_\eta$ of immediate successors.

$(M^n, <^n)$ is hence a chain. Note, the 'usual' partial order $\lhd$ on ${}^{n\geq}\omega$ (being an initial segment), is not definable in $\mathscr{M}^n$.

DEFINITION 3.5. We define by induction the scattered chains C_n and C_n^*:

$$C_1 := \omega, \qquad C_1^* := \omega^*,$$

$$C_2 := \sum_{i\in\omega} \omega^*, \quad C_2^* := \sum_{i\in\omega^*} \omega,$$

and in general:

$$C_n := \sum_{i\in\omega} C_n^*, \quad C_n^* := \sum_{i\in\omega^*} C_n.$$

DEFINITION 3.6. $f : \mathscr{M}^n \hookrightarrow C$ is an embedding of $\mathscr{M}^n$ in a scattered chain $(C, <)$ if f is 1-1 and $\sigma <^n \tau \Longrightarrow f(\sigma) < f(\tau)$.

FACT 3.7. Let C be a scattered chain with $\mathrm{Hdeg}(C) \geq n + 1$. Then there is an embedding $f : \mathscr{M}^n \hookrightarrow C$.

PROOF. Clearly the following hold:

(α) For a scattered chain C, $\mathrm{Hdeg}(C) = n \Longrightarrow [C_n \subseteq C \text{ or } C_n^* \subseteq C]$.

(β) $\mathscr{M}^n \subseteq \mathscr{M}^{n+1}$.

(γ) There is an embedding $g : \mathscr{M}^n \hookrightarrow C_n$.

Now assume $\mathrm{Hdeg}(C) = n + 1$ and use (α). In the case $C_{n+1} \subseteq C$ we have by (γ) an embedding $g : \mathscr{M}^{n+1} \hookrightarrow C$ and by (β) an embedding $f : \mathscr{M}^n \hookrightarrow C$. In the case $C_{n+1}^* \subseteq C$ we have, by the definition of C_{n+1}^*, $C_n \subseteq C_{n+1}^*$ and by (γ) an embedding $f : \mathscr{M}^n \hookrightarrow C$. ⊣

CONCLUSION 3.8. *Let C be a scattered chain with* $\mathrm{Hdeg}(C) \geq \omega$. *Then, for every $n < \omega$ there is an embedding of $\mathscr{M}^n$ into C.*

LEMMA 3.9. *Let C be scattered. Suppose $F : [C]^2 \to \{j_1, \dots, j_{n-1}\}$ is an additive coloring. Then, if* $\mathrm{Hdeg}(C) \geq n + 1$, *there is a subset $X \subseteq C$ of order type $\mathbb{Z}$, homogeneous with respect to F.*

PROOF. Without loss of generality C is $(\mathscr{M}_n, <^n)$: As $\mathrm{Hdeg}(C) \geq n + 1$ there is an embedding $f : \mathscr{M}^n \hookrightarrow C$. Now $F \circ f : [M_n]^2 \to \{j_1, \dots, j_{n-1}\}$ is an additive coloring and if $Y \subseteq M_n$ is homogeneous of order type $\mathbb{Z}$ (with respect to $F \circ f$) then so is $X = f''(Y)$ (with respect to F).

NOTATION. We will write $(T, <)$ instead of $({}^{n\geq}\omega, <^n)$. $T_{\geq\eta}$ and $T_{>\eta}$ are as usual.

The plan is the following: We will thin out T to get a subtree $A^* \subseteq T$ of height n such that for $\eta \in A^*$ $|\mathrm{Suc}_{A^*}(\eta)| = \aleph_0$. A^* will satisfy:

$$(*) \quad \left[\bigwedge_{i<4}(\sigma_i \in A^*) \;\&\; \bigwedge_{i<4}(\mathrm{lev}(\sigma_i) = n) \;\&\; \big(\mathrm{lev}(\sigma_0 \sqcap \sigma_1) = \mathrm{lev}(\sigma_2 \sqcap \sigma_3)\big)\right] \Longrightarrow [F(\sigma_0, \sigma_1) = F(\sigma_2, \sigma_3)].$$

(Here $\sigma \sqcap \tau$ is always an element and not an initial segment.)

Assuming such an A^* can be obtained we define, for $0 < k \leq n$, t_k to be the color $F(\sigma, \tau)$ for $\sigma, \tau \in A^*$, with level n and with $\text{lev}(\sigma \sqcap \tau) = n - k$. As we have only $n - 1$ colors there are $0 < l < r \leq n$ such that $t_\ell = t_r$. Using the fact that F is additive we can prove that $t_\ell = t_{\ell+1}$ as well:

Let $\sigma < \tau$ be in A^* such that $\text{lev}(\sigma) = \text{lev}(\tau) = n$ and $\text{lev}(\sigma \sqcap \tau) = n - r$, (so $F(\sigma, \tau) = t_r$). Then find $\rho \in A^*$ with $\sigma < \rho < \tau$, $\text{lev}(\rho) = n$, $\text{lev}(\sigma \sqcap \rho) = n - (l+1)$ and $\text{lev}(\rho \sqcap \tau) = n - r$. What we get is the following equation:

$$t_r = F(\sigma, \tau) = F(\sigma, \rho) + F(\rho, \tau) = t_{\ell+1} + t_r$$

but $t_r = t_\ell$ hence

$$(\dagger) \qquad t_\ell = t_{\ell+1} + t_\ell.$$

Imitate this computation: let $\sigma < \tau$ be in A^* be such that $\text{lev}(\sigma) = \text{lev}(\tau) = n$ and this time $\text{lev}(\sigma \sqcap \tau) = n - (\ell+1)$, (so $F(\sigma, \tau) = t_{\ell+1}$) and find $\rho \in A^*$ with $\sigma < \rho < \tau$, $\text{lev}(\rho) = n$, $\text{lev}(\sigma \sqcap \rho) = n - (\ell + 1)$ and $\text{lev}(\rho \sqcap \tau) = n - \ell$. What we get is the following equation:

$$t_{\ell+1} = F(\sigma, \tau) = F(\sigma, \rho) + F(\rho, \tau) = t_{\ell+1} + t_\ell$$

hence

$$(\ddagger) \qquad t_{\ell+1} = t_{\ell+1} + t_\ell.$$

Combining $(\dagger)$ and $(\ddagger)$ we get $t_{\ell+1} = t_\ell$.

Finding $0 < k < n$ with $t_k = t_{k+1}$ pick $\eta \in A^*$ with $\text{lev}(\eta) = n - (k + 1)$. Let

$$\text{Suc}_{A^*}(\eta) = \{ \eta{}^\wedge\langle \ell_i \rangle : i < \omega \}$$

$(\langle \ell_i : i < \omega \rangle$ strictly increasing) and denote $\nu_i = \eta{}^\wedge\langle \ell_i \rangle$.

Assuming $n - (k + 1)$ is even we get $\ell_i < \ell_j \Longrightarrow A^* \models \nu_i < \nu_j$. For each ν_i with $i > 0$ choose $\sigma_i \in A^*$ with $\text{lev}(\sigma_i) = n$ such that ν_i is an initial segment of σ_i. By the definition of the linear order in T hence in A^*, $0 < i < j < \omega \Longrightarrow \sigma_i < \sigma_j$. Moreover, as $i \neq j \Longrightarrow \sigma_i \sqcap \sigma_j = \eta$, we get for every i and j

$$F(\sigma_i, \sigma_j) = t_{k+1} = t_k.$$

Hence $\langle \sigma_i : 0 < i < \omega \rangle$ is a homogeneous sequence of order type ω. Returning to ν_0 we have $\text{lev}(\nu_0) = n - k < n$ let

$$\text{Suc}_{A^*}(\nu_0) = \{ \nu_0{}^\wedge\langle m_i \rangle : i < \omega \}$$

$(\langle m_i : i < \omega \rangle$ strictly increasing) and denote $\rho_i = \nu_0{}^\wedge\langle m_i \rangle$. As now $n - k$ is odd we get $m_i < m_j \Longrightarrow A^* \models \rho_i > \rho_j$. For each ρ_i choose $\tau_i \in A^*$ with $\text{lev}(\tau_i) = n$ such that ρ_i is an initial segment of τ_i. Now we have $i < j < \omega \Longrightarrow \tau_i > \tau_j$ and as $i \neq j \Longrightarrow \tau_i \sqcap \tau_j = \nu_0$, we get for every i and j

$$F(\sigma_i, \sigma_j) = t_k.$$

Hence $\langle \tau_i : i < \omega \rangle$ is a homogeneous sequence of order type ω^*. Clearly for every $i < \omega$ and $0 < j < \omega$ we have $A^* \models \tau_i < \sigma_j$ and $\tau_i \sqcap \sigma_j = \eta$ (hence $F(\tau_i, \sigma_j) = t_{k+1} = t_k$). Therefore

$$X := \{ \tau_i : i < \omega \} \cup \{ \sigma_j : 0 < j < \omega \}$$

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is the required homogeneous subset of order type $\mathbb{Z}$.

When $n-(k+1)$ is odd the τ_i's that extend ν_0 (which is the maximal element in $\mathrm{Suc}_{A^*}(\eta)$) are of order type ω and the $<$-smaller σ_i's are of order type ω^* so X is again as required.

We are left now with the task of defining the subtree $A^* \subseteq T$ that will satisfy $(*)$. This will be done by induction going down with levels. Arriving to a node η we will have defined for each $\nu = \eta^\wedge\langle i\rangle \in \mathrm{Suc}_T(\eta)$ a sub-tree $A_{\geq\nu} \subseteq T_{\geq\nu}$, in the next step we will choose an infinite $B_\eta \subseteq \omega$. $A_{\geq\eta}$ will be

$$\{\eta\} \cup \bigcup\{ A_{\geq\eta^\wedge\langle i\rangle} : \min(B_\eta) < i \in B_\eta \}.$$

A^* is $A_{\geq\langle\rangle}$.

Denote for $0 \leq \ell < n$

$$\bigoplus_\ell := \left[\bigwedge_{i<4}(\mathrm{lev}(\sigma_i) = n) \ \& \ (\mathrm{lev}(\sigma_0 \sqcap \sigma_1) = \mathrm{lev}(\sigma_2 \sqcap \sigma_3) = \ell)\right]$$
$$\Longrightarrow [F(\sigma_0,\sigma_1) = F(\sigma_2,\sigma_3)].$$

(So $(*)$ means $\bigoplus_{n-1}$ & $\bigoplus_{n-2}$ & $\cdots$ & $\bigoplus_0$).

Assume without loss of generality that n is odd.

STEP 1. Given $\eta \in T$ with $\mathrm{lev}(\eta) = n-1$ pick an infinite set $B_\eta^{n-1} \subseteq \omega$ such that for some color j_η^{n-1}

$$k < \ell \in B_\eta^{n-1} \Longrightarrow F(\eta^\wedge\langle k\rangle, \eta^\wedge\langle \ell\rangle) = j_\eta^{n-1}.$$

Let $o_\eta = \min(B_\eta^{n-1})$ and let $A_{\geq\eta} \subseteq T_{\geq\eta}$ be

$$\{\eta\} \cup \{\eta^\wedge\langle k\rangle : o_\eta < k \in B_\eta^{n-1}\}.$$

$A_{\geq\eta}$ clearly satisfies $\bigoplus_{n-1}$.

STEP 2. Given $\eta \in T$ with $\mathrm{lev}(\eta) = n-2$ we have defined j_ν^{n-1}, B_ν, o_ν and $A_{\geq\nu}$ for every $\nu \in \mathrm{Suc}_T(\eta)$. Pick an infinite $B_\eta^1 \subseteq \omega$ such that $k, \ell \in B_\eta^1 \Longrightarrow j_{\eta^\wedge\langle k\rangle}^{n-1} = j_{\eta^\wedge\langle \ell\rangle}^{n-1}$. Call the common color j_η^{n-1}. Clearly

$$A_{\geq\eta}^1 := \{\eta\} \cup \bigcup\{ A_{\geq\eta^\wedge\langle i\rangle} : i \in B_\eta^1 \}$$

satisfies $\bigoplus_{n-1}$.

Taking care of $\bigoplus_{n-2}$ let $k, \ell \in B_\eta^1$, $\sigma_k := \eta^\wedge\langle k\rangle$ and $\sigma_\ell := \eta^\wedge\langle \ell\rangle$. Let $r_0 < r_1 < r_2$ be in $B_{\sigma_k} \setminus \{o_{\sigma_k}\}$ and $s_0 < s_1$ be in $B_{\sigma_\ell} \setminus \{o_{\sigma_\ell}\}$. Define $\tau = \sigma_k{}^\wedge\langle o_{\sigma_k}\rangle$, $\tau_0 = \sigma_k{}^\wedge\langle r_0\rangle$, $\tau_1 = \sigma_k{}^\wedge\langle r_1\rangle$, $\tau_2 = \sigma_k{}^\wedge\langle r_2\rangle$, $\rho = \sigma_\ell{}^\wedge\langle o_{\sigma_\ell}\rangle$, $\rho_0 = \sigma_\ell{}^\wedge\langle s_0\rangle$ and $\rho_1 = \sigma_\ell{}^\wedge\langle s_1\rangle$. As we assume n is odd we get $\tau < \tau_0 < \tau_1 < \tau_2 < \rho < \rho_0 < \rho_1$. Now:

(α) $F(\tau,\rho) = F(\tau_1,\rho)$,
[as $F(\tau,\rho) = F(\tau,\tau_2) + F(\tau_2,\rho) = F(\tau_1,\tau_2) + F(\tau_2,\rho) = F(\tau_1,\rho)$].

(β) $F(\tau,\rho) = F(\tau_2,\rho)$,
[as $F(\tau,\rho) = F(\tau,\tau_3) + F(\tau_3,\rho) = F(\tau_2,\tau_3) + F(\tau_3,\rho) = F(\tau_3,\rho)$].

(γ) $F(\tau_1,\rho) = F(\tau_2,\rho)$,
[by (α) and (β)].

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(δ) $F(\tau, \rho_1) = F(\tau, \rho_2)$
[as $F(\tau, \rho_1) = F(\tau, \rho) + F(\rho, \rho_1) = F(\tau, \rho) + F(\rho, \rho_2) = F(\tau, \rho_2)$].
(ε) $F(\tau_1, \rho_1) = F(\tau_1, \rho_2) = F(\tau_2, \rho_1) = F(\tau_2, \rho_2)$ and this is equal to

$$F(\tau, \rho) + j^{n-1}_{\sigma_\ell}.$$

(Note: when n is even we can apply similar considerations by reversing the order.)

By our previous choices $j^{n-1}_{\sigma_\ell}$ is j^{n-1}_η. We conclude that if $\nu_1, \nu_2 \in A^1_{\geq\eta}$ with $\operatorname{lev}(\nu_1) = \operatorname{lev}(\nu_2) = n$, $\nu_1 \sqcap \nu_2 = \eta$ and say $\eta \lhd \sigma_1 \lhd \nu_1$, $\eta \lhd \sigma_2 \lhd \nu_2$ then $F(\nu_1, \nu_2)$ is a function of $\sigma_1{}^\wedge\langle o_{\sigma_1}\rangle$ and $\sigma_2{}^\wedge\langle o_{\sigma_2}\rangle$, i.e., of σ_1 and σ_2. Denote $F(\nu_1, \nu_2) = g(\sigma_1, \sigma_2)$. Now choose an infinite $B_\eta \subseteq B^1_\eta$ such that $k < \ell \in B_\eta \Longrightarrow g(\eta{}^\wedge\langle k\rangle, \eta{}^\wedge\langle \ell\rangle)$ is constant. Let $o_\eta := \min(B_\eta)$ and let

$$A_{\geq\eta} := \{\eta\} \cup \bigcup\{ A_{\geq\eta{}^\wedge\langle k\rangle} : o_\eta < k \in B_\eta \}.$$

$A_{\geq\eta}$ satisfies $\bigoplus_{n-1}$ and $\bigoplus_{n-2}$ (and j^{n-1}_η is implicitly defined).

STEP m. Given $\eta \in T$ with $\operatorname{lev}(\eta) = n - m$ we have defined

$$\bar{\jmath}_\nu := \langle j^{n-1}_\nu, \dots, j^{n-m+1}_\nu \rangle,$$

B_ν, o_ν and $A_{\geq\nu}$ for every $\nu \in \operatorname{Suc}_T(\eta)$. Pick an infinite $B^1_\eta \subseteq \omega$ such that $k, \ell \in B^1\eta \Longrightarrow \bar{\jmath}_{\eta{}^\wedge\langle k\rangle} = \bar{\jmath}_{\eta{}^\wedge\langle \ell\rangle}$. Call the common sequence $\bar{\jmath}_\eta = \langle j^{n-1}_\eta, \dots, j^{n-m+1}_\eta \rangle$. Clearly

$$A^1_{\geq\eta} := \{\eta\} \cup \bigcup\{ A_{\geq\eta{}^\wedge\langle i\rangle} : i \in B^1_\eta \}$$

satisfies $\bigoplus_{n-1}, \dots, \bigoplus_{n-m+1}$.

Using the canonical branches

$$\eta \lhd \eta{}^\wedge\langle k\rangle = \sigma_k \lhd \tau_{n-m+2} \lhd \cdots \lhd \tau_0$$

and

$$\eta \lhd \eta{}^\wedge\langle \ell\rangle = \sigma_\ell \lhd \rho_{n-m+2} \lhd \cdots \lhd \rho_0$$

where

$$\tau_{n-m+i} = \tau_{n-m+i+1}{}^\wedge\langle o_{\tau_{n-m+i+1}}\rangle \quad \text{and} \quad \rho_{n-m+i} = \rho_{n-m+i+1}{}^\wedge\langle o_{\rho_{n-m+i+1}}\rangle$$

we can verify, as in Step 2, that when $k, \ell \in B^1_\eta$, $\sigma_k = \eta{}^\wedge\langle k\rangle$, $\sigma_\ell = \eta{}^\wedge\langle \ell\rangle$, $\sigma_k \lhd \tau$, $\sigma_\ell \lhd \rho$ (so $\tau \sqcap \rho = \eta$) and $\operatorname{lev}(\tau) = \operatorname{lev}(\rho) = n$, $F(\tau, \rho)$ depends only on σ_k and σ_ℓ. Denote such values by $g(\sigma_k, \sigma_\ell)$ and pick an infinite $B_\eta \subseteq B^1_\eta \subseteq \omega$ such that $k, \ell \in B_\eta \Longrightarrow g(\eta{}^\wedge\langle k\rangle, \eta{}^\wedge\langle \ell\rangle)$ is constant. Let $o_\eta = \min(B_\eta)$.

$$A_{\geq\eta} := \{\eta\} \cup \bigcup\{ A_{\geq\eta{}^\wedge\langle i\rangle} : o_\eta < i \in B_\eta \}$$

satisfies $\bigoplus_{n-m}$ and the previous $\bigoplus$'s.

$A^* := A_{\geq\langle\rangle}$ satisfies $\bigoplus_{n-1}, \dots, \bigoplus_0$ hence $(*)$. $\dashv$

CONCLUSION 3.10. *For every $m, \ell \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that if C is a scattered chain and* $\operatorname{Hdeg}(C) \geq n + 1$ *then C does not have a definable choice function by a formula with quantifier depth $\leq m$ and with $\leq \ell$ parameters.*

PROOF. Let n be $|T_{m+5,\ell}|$. Suppose $\varphi(x, X, \bar{P})$ defines a choice function on a scattered chain C with $\mathrm{Hdeg}(C) \geq n+1$. The additive coloring $F : [C]^2 \to T_{m,\ell}$ that is defined by $F(a,b) = \mathrm{Th}^{m+5}(C; \bar{P})\restriction_{[a,b)}$ has, by the previous lemma, a homogeneous subset $A \subseteq C$ of order type $\mathbb{Z}$. As in the proof of Claim 2.4, φ can not choose an element from A. ⊣

§4. Wild trees.

Large sets without structure, dense chains, scattered chains with large Hausdorff degree and the binary tree are prototypes of structures without a monadically definable choice function.

Respectively, wild trees are trees that have a large amount of splitting (4.2 (1)(i)) or have 'wild' branches (4.2 (1)(ii)(iii)), or embed the binary tree (4.2 (1)(iv)). Thus, using the composition theorems, there are no definable choice functions in the class of wild trees (4.7).

DEFINITION 4.1. Let $(T, \lhd)$ be a tree and $A \subseteq T$ an initial segment.

(1) The binary relation $\sim^0_A$ on $T \setminus A$ is defined by

$$x \sim^0_A y \iff (\forall t \in A)[t \lhd x \equiv t \lhd y].$$

(It is an equivalence relation that says "x and y cut A at the same place".)

(2) The binary relation $\sim^1_A$ on $T \setminus A$ is defined by

$$x \sim^1_A y \iff [x \sim^0_A y] \ \& \ (\exists z \in T \setminus A)[z \trianglelefteq x \ \& \ z \trianglelefteq y \ \& \ z \sim^0_A x].$$

(It's an equivalence relation that refines $\sim^0_A$ by dividing each $\sim^0_A$-equivalence class into disjoint subtrees.)

DEFINITION 4.2.

(1) A tree T is called *wild* if either
  (i) $\sup\{\, |T_{>A}/\sim^1_A| : A \subseteq T \text{ an initial segment}\,\} \geq \aleph_0$, or
  (ii) There is a branch $B \subseteq T$ and an embedding $f : \mathbb{Q} \to B$, or
  (iii) All the branches of T are scattered but $\sup\{\,\mathrm{Hdeg}(B) : B \text{ a branch of } T\,\} \geq \omega$, or
  (iv) There is an embedding $f : {}^{\omega>}2 \hookrightarrow T$.

(2) A tree T is *tame for* (n^*, k^*) if the value in (i) is $\leq n^*$, the value in (iii) is $\leq k^*$ and (ii) and (iv) do not hold.

(3) A tree T is *tame* if T is tame for (n^*, k^*) for some $n^*, k^* \in \mathbb{N}$.

CLAIM 4.3. *If T is a wild tree and* (1)(i) *of* 4.2 *holds then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on T.*

PROOF. We will use the composition theorem for general successors 1.12: Suppose $\varphi(x, X, \bar{Q})$ defines a choice function on T, $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{Q}) = \ell$. Given an initial segment $A \subseteq T$ let $T \setminus A/\sim^1_A = \{\, T_i : i \in I_A \,\}$ and by our assumption, there is an initial segment $A \subseteq T$ such that $|I_A| > |T_{n,\ell+1}|$. For every $i \in I_A$ pick $x_i \in T_i$. Now there are $\alpha, \beta \in I_A$ such that $\mathrm{Th}^n(T_\alpha; x_\alpha, \bar{Q}) = \mathrm{Th}^n(T_\beta; x_\beta, \bar{Q})$.

Denote, for $t \in T_{n,\ell+2}$, $P^A_t(\alpha) = \{\, i \in I_A : \mathrm{Th}^n(T_i; x_\alpha, \{x_\alpha, x_\beta\}, \bar{Q}) = t \,\}$ and let $\bar{P}^A(\alpha) = \langle\, P^A_t(\alpha) : t \in T_{n,\ell+2} \,\rangle$. By 1.12 there is some $m \in \mathbb{N}$ such that from

$$\mathrm{Th}^m(T^*_A; x_\alpha, \{x_\alpha, x_\beta\}, \bar{Q}) \quad \text{and} \quad \mathrm{Th}^m(I_A; \bar{P}^A(\alpha))$$

we compute $\mathrm{Th}^n(T; x_\alpha, \{x_\alpha, x_\beta\}, \bar{Q})$. Similarly, replacing x_α by x_β, from

$$\mathrm{Th}^m(T_A^*; x_\beta, \{x_\alpha, x_\beta\}, \bar{Q}) \quad \text{and} \quad \mathrm{Th}^m\big(I_A; \bar{P}^A(\beta)\big)$$

we compute $\mathrm{Th}^n(T; x_\beta, \{x_\alpha, x_\beta\}, \bar{Q})$. ($T_A^*$ is $T \setminus \bigcup_{i \in I_A} T_i$). Now

(i) $\mathrm{Th}^m(T_A^*; x_\alpha, \{x_\alpha, x_\beta\}, \bar{Q}) = \mathrm{Th}^m(T_A^*; x_\beta, \{x_\alpha, x_\beta\}, \bar{Q}) = \mathrm{Th}^m(T_A^*; \emptyset, \emptyset, \bar{Q})$.

(ii) $$\bar{P}^A(\alpha) = \bar{P}^A(\beta)$$

as $\mathrm{Th}^n(T_\alpha; x_\alpha, \bar{Q}) = \mathrm{Th}^n(T_\beta; x_\beta, \bar{Q})$ and as for $i \in I_A \setminus \{\alpha, \beta\}$

$$\mathrm{Th}^n(T_i; x_i, \{x_\alpha, x_\beta\}, \bar{Q}) = \mathrm{Th}^n(T_i; x_i, \{x_\alpha, x_\beta\}, \bar{Q}) = \mathrm{Th}^n(T_i; \emptyset, \emptyset, \bar{Q}).$$

Therefore

(iii) $$\mathrm{Th}^m\big(I_A; \bar{P}^A(\alpha)\big) = \mathrm{Th}^m\big(I_A; \bar{P}^A(\beta)\big).$$

It follows that

$$\mathrm{Th}^n(T; x_\alpha, \{x_\alpha, x_\beta\}, \bar{Q}) = \mathrm{Th}^n(T; x_\beta, \{x_\alpha, x_\beta\}, \bar{Q})$$

hence

$$T \models \varphi(x_\alpha, \{x_\alpha, x_\beta\}, \bar{Q}) \iff T \models \varphi(x_\beta, \{x_\alpha, x_\beta\}, \bar{Q}).$$

So φ cannot choose an element from $\{x_\alpha, x_\beta\}$, a contradiction. $\dashv$

Claim 4.4. *If T is a wild tree and* (1)(ii) *of* 4.2 *holds then no monadic formula $\varphi(x, X, \bar{Q})$ defines a choice function on T.*

Proof. Let $B \subseteq T$ be a branch that embeds $\mathbb{Q}$. We will apply the composition 1.13 and reflect a choice function on T to a choice function on B, contradicting Claim 2.4.

So assume that $\varphi(x, X, \bar{Q})$ defines a choice function on T where $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{Q}) = \ell$. By 1.13 there is an $m \in \mathbb{N}$, a chain $(B', \lhd')$ with $(B, \lhd) \subseteq (B', \lhd)$ and a sequence of parameters $\bar{P} \subseteq B'$ such that from $\mathrm{Th}^m(B'; B, \bar{P})$ we can compute $\mathrm{Th}^n(T; \bar{Q})$. Define, for $\eta \lhd \nu \in B$,

$$f(\eta, \nu) = \mathrm{Th}^{m+5}(B'; B, \bar{P}) \restriction_{[\eta, \nu)}.$$

f is an additive coloring hence by 2.3 there is $Y = \{\eta_i\}_{i \in \mathbb{Z}}$, of order type $\mathbb{Z}$, homogeneous with respect to f. As in the proof of 2.4 we have:

$$i, j \in \mathbb{Z} \Longrightarrow \mathrm{Th}^m(B'; \eta_i, Y, \bar{P}) = \mathrm{Th}^m(B'; \eta_j, Y, \bar{P})$$

and (by the 'moreover' clause in 1.13) this implies

$$i, j \in \mathbb{Z} \Longrightarrow \mathrm{Th}^n(T; \eta_i, Y, \bar{Q}) = \mathrm{Th}^n(T; \eta_j, Y, \bar{Q}).$$

Hence

$$i, j \in \mathbb{Z} \Longrightarrow [T \models \varphi(\eta_i, Y, \bar{Q}) \iff T \models \varphi(\eta_j, Y, \bar{Q})]$$

and this contradicts "φ chooses an element from Y". $\dashv$

Claim 4.5. *If T is a wild tree and* (1)(iii) *of* 4.1 *holds then no monadic formula $\varphi(x, X, \bar{Q})$ defines a choice function on T.*

PROOF. Combine the two previous proofs: Suppose $\varphi(x, X, \bar{Q})$ defines a choice function on T with $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{Q}) = \ell$.

By (1)(iii) for every $k \in \mathbb{N}$ there is a branch $B \subseteq T$ with $\mathrm{Hdeg}(B) > k$. By conclusion 3.10 a formula with depth n and ℓ parameters cannot define a choice function from subsets of branches with large enough Hausdorff degree. By the composition Theorem 1.13, the extra structure in T makes no difference. ⊣

CLAIM 4.6. *Let T be a tree and $F: {}^{\omega>}2 \hookrightarrow T$ be a tree embedding. Then no monadic formula $\varphi(x, X, \bar{P})$ defines a choice function on T.*

PROOF. First, we may assume, without loss of generality, that T has a root (adding a root will not effect the existence of a choice function) and that $F(\mathrm{r}({}^{\omega>}2)) = \mathrm{r}(T)$. The proof in §5 of [3] shows the following:

(∗) for every $\bar{Q} \subseteq {}^{\omega>}2$ and $m \in \mathbb{N}$ there is an infinite anti-chain $Y \subseteq {}^{\omega>}2$ such that for every $y \in Y$ there is $y^* \neq y$ in Y with $\mathrm{Th}^m(\mathrm{Bush}_{\omega>2}(Y); y, \bar{Q}) = \mathrm{Th}^m(\mathrm{Bush}_{\omega>2}(Y); y^*, \bar{Q})$.

($\mathrm{Bush}_{\omega>2}(Y) := \{x \in {}^{\omega>}2 : (\exists y \in Y)[x \trianglelefteq y]\}$.)

Assume $\varphi(x, X, \bar{P})$ defines a choice function on T with $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{P}) = \ell$. Denote $F''({}^{\omega>}2) = S \subseteq T$. Let $\bar{Q} = \bar{Q}(n, \bar{P}) \subseteq S$ be a sequence of parameters as in the composition theorem 1.13 and let $m = m(n, \ell)$ be as there. As $S \cong {}^{\omega>}2$ it follows by (∗) that there is an infinite anti-chain $Y \subseteq S$ such that

(∗∗) for each $y \in Y$ there is $y^* \neq y$ in Y with $\mathrm{Th}^m(\mathrm{Bush}_S(Y); y, \bar{Q}) = \mathrm{Th}^m(\mathrm{Bush}_S(Y); y^*, \bar{Q})$.

Now assume $T \models \varphi(y, Y, \bar{P})$. By 1.13 $\mathrm{Th}^m(\mathrm{Bush}_S(Y); y, \bar{Q})$ determines $\mathrm{Th}^n(T; y, Y, \bar{P})$ hence by (∗∗) there is $y^* \neq y$ in Y with

$$\mathrm{Th}^n(T; y, Y, \bar{P}) = \mathrm{Th}^n(T; y^*, Y, \bar{P})$$

therefore

$$T \models \varphi(y, Y, \bar{P}) \iff T \models \varphi(y^*, Y, \bar{P}).$$

So φ fails to choose an element from Y. ⊣

We conclude

THEOREM 4.7. *If T is a wild tree, then T does not have a monadically definable choice function. Moreover, every candidate fails to choose from either linearly ordered subsets* (4.4, 4.5) *or anti-chains* (4.3, 4.6).

§5. Tame trees. Not only that tame trees have definable choice functions, they even have definable well orderings of their elements.

DEFINITION 5.1. Let T be a tree. For $\eta \in T$ we define by recursion a rank function $\mathrm{rk}(\eta)$ by:

$$\mathrm{rk}(\eta) \geq \alpha + 1 \iff \text{there are } \nu_1, \nu_2 \in T \text{ with } \eta \trianglelefteq \nu_1 \text{ and } \eta \trianglelefteq \nu_2 \text{ such that } \nu_1 \perp \nu_2,\ \mathrm{rk}(\nu_1) \geq \alpha \text{ and } \mathrm{rk}(\nu_2) \geq \alpha.$$

If $\mathrm{rk}(\eta)$ is not defined we stipulate $\mathrm{rk}(\eta) = \infty$.

FACT 5.2.

(1) $\eta \triangleleft \nu \in T \Longrightarrow \mathrm{rk}(\nu) \le \mathrm{rk}(\eta)$ where $\le$ has the obvious meaning.

(2) ${}^{\omega>}2$ is not embeddable in a tree $T \iff$ for every $\eta \in T$, $\mathrm{rk}(\eta) \ne \infty$.

(3) If $\eta \triangleleft \nu_1$, $\eta \triangleleft \nu_2$ and $\nu_1 \perp \nu_2$ then $\mathrm{rk}(\nu_1) < \mathrm{rk}(\eta)$ or $\mathrm{rk}(\nu_2) < \mathrm{rk}(\eta)$.

PROOF. Straightforward. ⊣

LEMMA 5.3. *Let T be a tame tree. Then there are $\bar{Q} \subseteq T$ and a monadic formula $\varphi(x, y, \bar{Q})$ that defines a well ordering of T.*

PROOF. Suppose that T is (n^*, k^*)-tame (n^* bounds splittings and k^* bounds Hausdorff degrees of branches). We will partition T into a disjoint union of sub-branches, indexed by the nodes of a well founded tree Γ and reduce the problem of a well ordering of T to a problem of a well ordering of Γ. The tameness will enable us to define Γ in T and to well order the set of immediate successors of each node of Γ. The well ordering of T will be induced by the lexicographic order of Γ.

STEP 1 (Defining Γ). Let $\lambda = |T|^+$. Define by induction on α a set $\Gamma_\alpha \subseteq {}^\alpha\lambda$ (this is a our set of indices), for every $\eta \in \Gamma_\alpha$ define a tree $T_\eta \subseteq T$ and a branch $A_\eta \subseteq T_\eta$.

$\alpha = 0$: Γ_0 is $\{\langle\rangle\}$, $T_{\langle\rangle}$ is T and $A_{\langle\rangle}$ is any branch (i.e., a maximal linearly ordered subset) of T.

$\alpha = 1$: Look at $(T \setminus A_{\langle\rangle})/{\sim^1_{A_{\langle\rangle}}}$ (see Definition 4.1), it's a disjoint union of trees and name it $\langle T_{\langle i\rangle} : i < i^* \rangle$, let $\Gamma_1 := \{ \langle i\rangle : i < i^* \}$ and for every $\langle i\rangle \in \Gamma_1$ let $A_{\langle i\rangle}$ be a branch of $T_{\langle i\rangle}$.

$\alpha = \beta + 1$: For $\eta \in \Gamma_\beta$ denote $(T_\eta \setminus A_\eta)/{\sim^1_{A_\eta}}$ by $\{ T_{\eta^\wedge\langle i\rangle} : i < i_\eta \}$, let $\Gamma_\alpha = \{ \eta^\wedge\langle i\rangle : \eta \in \Gamma_\beta,\ i < i_\eta \}$ and choose $A_{\eta^\wedge\langle i\rangle}$ to be a branch of $T_{\eta^\wedge\langle i\rangle}$.

α limit: Let $\Gamma_\alpha = \{ \eta \in {}^\alpha\lambda : \bigwedge_{\beta<\alpha} \eta\restriction_\beta \in \Gamma_\beta,\ \bigwedge_{\beta<\alpha} T_{\eta\restriction_\beta} \ne \emptyset \}$, let for $\eta \in \Gamma_\alpha$ $T_\eta = \bigcap_{\beta<\alpha} T_{\eta\restriction_\beta}$ and A_η be a branch of T_η. (T_η may be empty.)

Now, at some stage $\alpha \le |T|^+$ we have $\Gamma_\alpha = \emptyset$ and let $\Gamma = \bigcup_{\beta<\alpha} \Gamma_\beta$. Clearly $\{ A_\eta : \eta \in \Gamma \}$ is a partition of T into disjoint sub-branches.

NOTATION. Having two trees T and Γ, to avoid confusion, we use x, y, s, t for nodes of T and η, ν, σ for nodes of Γ.

STEP 2 (Γ is well founded). By tameness of T, for every $x \in T$, $\mathrm{rk}(x)$ is defined (i.e., $< \infty$). We would like to show that Γ contains no infinite branch. For that, we have to restrict the choice of the branches $A_\eta \subseteq T_\eta$.

For $\eta^\wedge\langle i\rangle \in \Gamma$ define $\gamma_{\eta,i}$ as $\max\{ \mathrm{rk}(t) : t \in T_{\eta^\wedge\langle i\rangle} \}$. The maximum is obtained by Fact 5.3 and by the definition of $\sim^1$ (from which it follows that for every τ_1, τ_2 in $T_{\eta^\wedge\langle i\rangle}$ there is $\sigma \in T_{\eta^\wedge\langle i\rangle}$ such that $\sigma \trianglelefteq \tau_1$ and $\sigma \trianglelefteq \tau_2$).

PROVISO. For every $\eta \in \Gamma$ and $i < i_\eta$, the sub-branch $A_{\eta^\wedge\langle i\rangle}$ contains every $s \in T_{\eta^\wedge\langle i\rangle}$ with $\mathrm{rk}(s) = \gamma_{\eta,i}$.

Now, choose the A_η's by abiding the proviso there is no infinite branch in Γ. Otherwise, suppose $\{\eta_n\}_{n<\omega}$ is $\triangleleft$ increasing in Γ and choose $s_n \in A_{\eta_n}$, with $\mathrm{rk}(s_n) = \gamma_{\nu_n,i}$ (where $\eta_n = \nu_n{}^\wedge\langle i\rangle$). It follows

$$\mathrm{rk}(s_0) > \mathrm{rk}(s_1) > \mathrm{rk}(s_2) > \cdots$$

hence $\langle \mathrm{rk}(s_n) : n < \omega \rangle$ is an infinite, strictly decreasing sequence of ordinals, a contradiction.

STEP 3 (Definability of Γ). We will show that "x and y belong to the same A_η" is expressible by a monadic formula (with parameters). For that choose for each $\eta \in \Gamma$ a representative $s_\eta \in A_\eta$ and let $Q := \{ s_\eta : \eta \in \Gamma \}$. Let $h\colon T \to \{d_0, \dots, d_{n^*}\}$ be a coloring that satisfies

(i) $h\restriction_{A_{\langle\rangle}} = d_0$,

(ii) for $\eta{}^\wedge\langle i\rangle \in \Gamma$, $h\restriction_{A_{\eta^\wedge\langle i\rangle}}$ is constant.

(iii) for $i < j < i_\eta$, if $s_{\eta^\wedge\langle i\rangle} \sim^0_{A_\eta} s_{\eta^\wedge\langle j\rangle}$ then $h\restriction_{A_{\eta^\wedge\langle i\rangle}} \neq h\restriction_{A_{\eta^\wedge\langle j\rangle}}$.

There is no difficulty to define h (clause (iii) is taken care of by (n^*, k^*)-tameness).

Define a sequence $\langle D_0, \dots, D_{n^*}\rangle$ of subsets of T by $x \in D_i$ if and only if $h(x) = d_i$. Now "$\bigvee_\eta[x, y \in A_\eta]$" is defined by

$$\theta(x, y, \bar{D}) := [(x \trianglelefteq y) \vee (y \trianglelefteq x)] \ \& \ \left[\bigvee_i (x \in D_i \equiv y \in D_i)\right]$$

$$\& \ (\forall z)\left[[(x \trianglelefteq z \trianglelefteq y) \vee (y \trianglelefteq z \trianglelefteq x)] \to \left[\bigvee_i (x \in D_i \equiv z \in D_i)\right]\right].$$

Let, for $x \in T$, $A_{\eta(x)} = A_x$ be the sub-branch to which x belongs. A_x is definable from $\{x\}$ and $\bar{D}$ and in particular each A_η is definable from $\{s_\eta\} = Q \cap A_\eta$ and $\bar{D}$ so there is a monadic formula $\chi(s_\eta, X, \bar{D})$ saying "$X = A_\eta$". We would like now to interpret the partial order of Γ in T.

By the construction, $\Gamma \models \eta \triangleleft \nu$ if and only if every element of A_ν cuts A_η, i.e., is above an initial segment and is incomparable with a final segment of A_η. Let the partial order $<$ on sub-branches be defined by $X < Y$ if and only if for some $\eta, \nu \in \Gamma$, $X = A_\eta$ and $Y = A_\nu$ and $\Gamma \models (\eta \triangleleft \nu)$. Now "$X < Y$" is definable by

$$\phi(X, Y, Q, \bar{D}) := (\exists s_\eta, s_\nu \in Q)[\chi(s_\eta, X, \bar{D}) \wedge \chi(s_\nu, Y, \bar{D})]$$

$$\& \ (\exists v, w \in X)\big[(v \triangleleft (Q \cap Y)) \wedge (w \perp (Q \cap Y))\big].$$

Caution! if T has a root this is not true for $A_{\langle\rangle}$ and a $\leq n^*$ $A_{\langle i\rangle}$'s. To fix that we may have to add $\leq n^*$ parameters (for $A_{\langle i_1\rangle}, \dots, A_{\langle i_{n^*}\rangle}$) but there is no problem with that.

So ϕ and θ interpret $(\Gamma, \triangleleft)$ in T.

STEP 4 (Well ordering of immediate successors in Γ). As each A_η has Hausdorff degree $\leq k^*$, we can choose a sequence $\bar{P}^\eta = \langle P^\eta_0, \dots, P^\eta_{k^*-1}\rangle \subseteq A_\eta$ and use it to define a well ordering of A_η by a monadic formula $\varphi_{k^*}(x, y, \bar{P})$ as in Claim 3.3. Let $\bar{P} \subseteq T$ be $\bigcup_{\eta\in\Gamma} \bar{P}^\eta$ (the union is disjoint in each coordinate) and let

$$\varphi(x, y, \bar{P}) := \big(\theta(x, y, \bar{D}) \ \& \ \varphi_{k^*}(x, y, \bar{P} \cap A_x)\big).$$

This defines a partial order on T such that the restriction to each sub-branch A_η is a well order.

Now as "$\nu \in \mathrm{Suc}_\Gamma(\eta)$" is definable (as a relation between s_ν and s_η), so is the set $A^+_\eta := \{ s_{\eta^\wedge\langle i\rangle} : i < i_\eta \}$ (from s_η, Q and $\bar{D}$). The order on A_η induces an order on $\{s_{\eta^\wedge\langle i\rangle}/\sim^0_{A_\eta}\}$ that is embeddable in the completion of A_η and therefore has

Hausdorff degree $\leq k^*$ as well. (To compare $s_{\eta\,\wedge\langle i\rangle}$ and $s_{\eta\,\wedge\langle j\rangle}$ compare the initial segment of A_η below $s_{\eta\,\wedge\langle i\rangle}$ and the initial segment of A_η below $s_{\eta\,\wedge\langle j\rangle}$). Thus, using a sequence of parameters $\langle Q_1^\eta, \dots, Q_{k^*}^\eta\rangle$, we can define a well order on $\{s_{\eta\,\wedge\langle i\rangle}/\sim^0_{A_\eta}\}$. To compare $s_{\eta\,\wedge\langle i\rangle}$'s that are $\sim^0_{A_\eta}$-equivalent but not $\sim^1_{A_\eta}$-equivalent (each such collection has $\leq n^*$ elements), fix once and for all an ordering between the colors $\{d_0, \dots, d_{n^*}\}$.

As before, the sequence $\bar{Q} := \bigcup_{\eta\in\Gamma} \bar{Q}_\eta$ enables us to define a partial order on Q such that its restriction to each A_η^+ is a well order. This defines a well order on sets of immediate successors in Γ.

STEP 5 (Well ordering T). Using the parameters Q, $\bar{D}$, $\bar{P}$, and $\bar{Q}$ define a well order on the elements of T by $x < y$ if and only if one of the following:

(i) x and y belong to the same A_η and $x < y$ according to the well order on A_η,

(ii) $x \in A_\eta$, $y \in A_\nu$ and $\Gamma \models (\eta \lhd \nu)$,

(iii) $x \in A_\eta$, $y \in A_\nu$, $\sigma = \eta \sqcap \nu$, $\sigma^\wedge\langle i\rangle \lhd \eta$, $\sigma^\wedge\langle j\rangle \lhd \nu$ and $s_{\sigma\,\wedge\langle i\rangle} < s_{\sigma\,\wedge\langle j\rangle}$ according to the well order on A_σ^+. ($\sigma = \eta \sqcap \nu$ is easily definable as a relation between s_σ, s_η and s_ν.)

We have defined a linear order $<$ on the elements of T in which each A_η is a convex subset and well ordered. Moreover on Γ (that is on the set of representatives Q), $<$ is the lexicographical ordering where each set of immediate successors is well ordered. As Γ is well founded we have defined a well order of T. ⊣

We conclude:

THEOREM 5.4. *Let T be a tree. If T is wild then there is no definable choice function on T (by a monadic formula with parameters). If T is tame then there is even a definable well ordering of the elements of T by a monadic formula (with parameters) $\varphi(x, y, \bar{P})$.*

A tree is tame [wild] if and only if its completion (the tree obtained by completing each branch) is tame [wild]. By this and Theorem 4.7 we have:

CONCLUSION. *Let T be a tree and T' be its completion. Then the following are equivalent:*

(a) *T is tame.*

(b) *For some $n, \ell \in \mathbb{N}$, for every anti-chain/branch A of T there is a monadic formula $\varphi_A(x, X, \bar{P}_A)$ with $\mathrm{dp}(\varphi) \leq n$, $\bar{P} \subseteq T$ and $\lg(\bar{P}) \leq \ell$, that defines a choice function from nonempty subsets of A.*

(c) *There is a monadic formula, with parameters, $\psi(x, y, \bar{P})$ that defines a well ordering of the elements of T.*

(d) *There is a monadic formula, with parameters, $\psi'(x, y, \bar{P}')$ that defines a well ordering of the elements of T'.* ⊣

REMARK. In a forthcoming paper ([5]) we solve the full uniformization problem for the monadic theory of trees.

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