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PSEUDO PCF

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ABSTRACT

We continue our investigation on pcf with weak forms of the axiom of choice. Characteristically, we assume DC+$\mathscr{P}(Y)$ when looking at $\prod_{s\in Y}\delta_s$. We get more parallels of pcf theorems.

Anotated Contents

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0. Introduction

In the first section we deal with generalizing the pcf theory in the direction started in [Sh:938, §5], trying to understand the pseudo true cofinality of small products of regular cardinals. The difference with earlier works is that here we assume $\mathrm{AC}_{\mathscr{U}}$ for any set $\mathscr{U}$ of power $\leq|\mathscr{P}(\mathscr{P}(Y))|$ or, actually working harder, just $\leq|\mathscr{P}(Y)|$ when analyzing $\prod_{t\in Y}\alpha_t$, whereas in [Sh:497] we assumed $\mathrm{AC}_{\sup\{\alpha_t:t\in Y\}}$ and in [Sh:835] we have (in addition to $\mathrm{AC}_{\mathscr{P}(\mathscr{P}(Y))}$) assumptions like "$[\sup\{\alpha_t : t\in Y\}]^{\aleph_0}$ is well ordered". In [Sh:938, §1–§4] we assume only $\mathrm{AC}_{<\mu}+DC$ and consider $\aleph_1$-complete filters on μ, but in the characteristic case μ is a limit of measurable cardinals.

Note that generally in this work, though we try occasionally not to use DC, it will not be a real loss to assume it all the time. More specifically, we prove the existence of a minimal $\aleph_1$-complete filter D on Y such that $\lambda=$ ps-tcf$(\Pi\bar{\alpha},<_D)$

assuming $\mathrm{AC}_{\mathscr{P}(Y)}$ and (of course) DC and α_t of large enough cofinality. We then prove the existence of one generator, that is, of $X \subseteq Y$ such that $J^{\aleph_1\text{-comp}}_{\leq\lambda}[\bar{\alpha}] = J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}] + X$, see 1.5 and even (in 1.7) the parallel of the existence of a $<_{D_1}$-lub for an $<_D$-increasing sequence $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$, generalize the no-hole claim in 1.12, and give a bound on pp for non-fix points (in 1.10).

In §2 we further investigate true cofinality. In Claim 2.2, assuming AC_λ and D an $\aleph_1$-complete filter on Y, we start from ps-tcf$(\Pi\bar{\alpha}, <_D)$, dividing by $\mathrm{eq}(\bar{\alpha}) = \{(s,t) : \alpha_s = \alpha_t\}$. We also prove the composition Theorem 2.5: it tells us when ps-tcf$(\prod_i$ ps-tcf$(\prod_j \lambda_{i,j}, <_{D_i}), <_E)$ is equal to ps-tcf$(\prod_{(i,j)} \lambda_{i,j}, <_D)$.

We then prove the pcf closure conclusion, giving a sufficient condition for the operation ps-pcf$_{\aleph_1\text{-comp}}$ to be idempotent. Lastly, we revisit the generating sequence.

In §(3A) we measure $\prod_{t\in Y} g(t)$ modulo a filter D on Y for $g \in {}^Y(\mathrm{Ord}\backslash\{0\})$ in three ways and show they are almost equal in 3.2. The price is that we replace (true) cofinality by pseudo (true) cofinality, which is inevitable. We try to sort out the “almost equal” in 3.5–3.7.

In §(3B) we prove a relative of [Sh:513, §3], again dealing with depth (instead of rank as in [Sh:938]), adding some information even under ZFC. Assuming that the sequence $\langle D_n : n < \omega\rangle$ of filters has the independence property (IND) (see Definition 3.12), with D_n a filter on Y_n we can bound the depth of $({}^{(Y_n)}\zeta, <_{D_n})$ by ζ, for every ζ for many n's; see 3.13. Of course, we can generalize this to $\langle D_s : s \in S\rangle$. This is incomparable with the results of [Sh:938, §4]. See a continuation of [Sh:835] and [Sh:1005]

Note that the assumptions like IND$(\bar{D})$ are complementary to ones used in [Sh:835] to get considerable information. Our original hope was to arrive at a dichotomy. The first possibility will say that one of the versions of an axiom suggested in [Sh:835] holds, which means “for some suitable algebra” there is no independent ω-sequence; in this case [Sh:835] tells us much. The second possibility will be a case of IND, and then we try to show that there is a rank system in the sense of [Sh:938]. But presently for this we need too much choice. The dichotomy we succeed to prove is with small o-Depth in one side, the results of [Sh:835] on the other side. It would be better to have ps-o-Depth in the first side.

Question 0.1: [DC + $\mathrm{AC}_{\mathscr{P}(Y)}$]

Assume

(a) $\bar{\alpha} \in {}^Y \text{Ord}$,
(b) $\text{cf}(\alpha_t) \geq \text{hrtg}(\mathscr{P}(Y)$ for every $t \in Y$,
(c) $\lambda_t \in \text{pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ for $t \in Z$; in fact, $\lambda_t =$ ps-tcf$(\Pi\bar{\alpha}, <_{D_t}), D_t$ is an $\aleph_1$-complete filter on Y,
(d) $\lambda = \text{ps-tcf}_{\aleph_1-\text{comp}}(\langle \lambda_t : t \in Z \rangle)$,
(e) (a possible help) $X_t \in D_t, \langle X_t : t \in Y \rangle$ are pairwise disjoint.

(A) Now does $\lambda \in$ ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$? (See 2.5.)

(B) Can we say something about D_λ from [Sh:938, 5.9] improved in 1.2?

Question 0.2: How well can we generalize the RGCH, see [Sh:460] and [Sh:829]; the above may be relevant, see [Sh:938] and here in §(3C).

Recall

Notation 0.3: 1) For any set X let

$$\text{hrtg}(X) = \min\{\alpha : \alpha \text{ an ordinal such that there is no function from } X \text{ onto } \alpha\}.$$

2) $A \leq_{\text{qu}} B$ means that either $A = \emptyset$ or there is a function from A onto B.

Central in this work is

Definition 0.4: For a quasi order P we say P has **pseudo true cofinality** λ when λ is a regular cardinal and there is a sequence $\bar{\mathscr{F}}$ such that:

(a) $\mathscr{F} = \langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$,
(b) $\mathscr{F}_\alpha \subseteq P$,
(c) if $\alpha_1 < \alpha_2, p_1 \in \mathscr{F}_{\alpha_1}$ and $p_2 \in \mathscr{F}_{\alpha_2}$, then $p_1 \leq_P p_2$,
(d) if $q \in \mathscr{F}$, then for some $\alpha < \lambda$ and $p \in \mathscr{F}_\alpha$ we have $q <_P p_1$,
(e) $\lambda = \sup\{\alpha < \lambda : \mathscr{F}_\alpha \neq \emptyset\}$.

We may consider replacing AC_A by a more refined version, $\text{AC}_{A,B}$ defined below (e.g., in 1.1, 2.5), but we have not dealt with it systematically.

Definition 0.5: 1) $\text{AC}_{A,B}$ means: if $\langle X_a : a \in A \rangle$ is a sequence of non-empty sets then there is a sequence $\langle Y_a : a \in A \rangle$ such that $Y_a \subseteq X_a$ is not empty and $Y_a \leq_{\text{qu}} B$.

2) $\text{AC}_{A,<\kappa}, \text{AC}_{A,\leq B}$ are defined similarly but $|Y_a| < \kappa, |Y_a| \leq |B|$ respectively in the end.

Observation 0.6: 1) AC_A iff $\text{AC}_{A,1}$.

2) $\mathrm{AC}_{A,B}$ fails if $B = \emptyset$.
3) If $\mathrm{AC}_{A,B}$ and $|A_1| \le |A|$ and $B \le_{\mathrm{qu}} B_1$, then AC_{A_1,B_1}.

1. On pseudo true cofinality

We continue [Sh:938, §5].

Below we improve [Sh:938, 5.19] by omitting DC from the assumptions, but first we observe

Claim 1.1: *Assume* AC_Z.

(1) *We have* $\theta \ge \mathrm{hrtg}(Z)$ *when* $\bar\alpha = \langle \alpha_t : t \in Y\rangle$ *and* $\theta \in \text{ps-pcf}(\Pi\bar\alpha)$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(Z)$.

(2) *We have* $\mathrm{cf}(\mathrm{rk}_D(\bar\alpha)) \ge \mathrm{hrtg}(Z)$ *when* $\bar\alpha = \langle \alpha_t : t \in Y\rangle, t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(Z)$.

Proof. (1) If we have AC_α for every $\alpha < \mathrm{hrtg}(Z)$, then we can use [Sh:938, 5.7(4)] but we do not assume this. In general, let D be a filter on Y such that $\theta = \text{ps-tcf}(\Pi\bar\alpha, <_D)$, exists as we are assuming $\theta \in \text{ps-pcf}(\Pi\bar\alpha)$. Let $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \theta\rangle$ witness $\theta = \text{ps-tcf}(\Pi\bar\alpha, <_D)$, i.e., as in [Sh:938, 5.6(2)] or see 0.4 here; note $t \in Y \Rightarrow \alpha_t > 0$, as we are assuming $\mathscr{F}_\alpha \subseteq \Pi\bar\alpha$ is non-empty for some $\alpha < \theta$; also we can assume $\mathscr{F}_\alpha \ne \emptyset$ for every $\alpha < \theta$.

Toward a contradiction assume $\theta < \mathrm{hrtg}(Z)$. As $\theta < \mathrm{hrtg}(Z)$, there is a function h from Z onto θ, so the sequence $\langle \mathscr{F}_{h(z)} : z \in Z\rangle$ is well defined. As we are assuming AC_Z, there is a sequence $\langle f_z : z \in Z\rangle$ such that $f_z \in \mathscr{F}_{h(z)}$ for $z \in Z$. Now define $g \in {}^Y(\mathrm{Ord})$ by $g(s) = \bigcup\{f_z(s) : z \in Z\}$; clearly g exists and $g \le \bar\alpha$. But for each $s \in Y$, the set $\{f_z(s) : z \in Z\}$ is a subset of α_s of cardinality $\le \theta < \mathrm{hrtg}(Z)$, hence $<\ \mathrm{cf}(\alpha_s)$, hence $g(s) < \alpha_s$. Together $g \in \Pi\bar\alpha$ is a $<_D$-upper bound of $\bigcup\{\mathscr{F}_\varepsilon : \varepsilon < \theta\}$, a contradiction to the choice of $\bar{\mathscr{F}}$.

(2) Otherwise let $\theta = \mathrm{cf}(\mathrm{rk}_D(\bar\alpha))$ so $\theta < \mathrm{hrtg}(Z)$, let $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ be increasing with limit $\mathrm{rk}_D(\bar\alpha)$ and again let g be a function from Z onto θ. As AC_Z holds, we can find $\langle f_z : z \in Z\rangle$ such that for every $z \in Z$ we have $\mathrm{rk}_D(f_z) \ge \alpha_{h(z)}$ and $f_z <_D \bar\alpha$ and, without loss of generality, $f_z \in \Pi\bar\alpha$. Let $f \in \Pi\bar\alpha$ be defined by $f(t) = \sup\{f_{h(z)}(t) : z \in Z\}$ so $\mathrm{rk}_D(f) \ge \sup\{\alpha_z : z \in Z\} = \mathrm{rk}_D(\bar\alpha) > \mathrm{rk}_D(f)$, a contradiction. $\blacksquare_{1.1}$

Theorem 1.2 (The Canonical Filter Theorem): *Assume* $\mathrm{AC}_{\mathscr{P}(Y)}$.

Assume $\bar\alpha = \langle \alpha_t : t \in Y\rangle \in {}^Y\mathrm{Ord}$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(\mathscr{P}(Y))$ *and* $\partial \in \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar\alpha)$, *hence it is a regular cardinal. Then there is* $D = D^{\bar\alpha}_\partial$, *an*

$\aleph_1$-complete filter on Y such that $\partial = $ ps-tcf$(\Pi\bar{\alpha}/D)$ and $D \subseteq D'$ for any other such $D' \in \mathrm{Fil}^1_{\aleph_1}(D)$.

Remark 1.3: (1) By [Sh:938, 5.9] there are some such ∂ if DC holds.

(2) We work more to use just $\mathrm{AC}_{\mathscr{P}(Y)}$ and no more.

(3) If $\kappa > \aleph_0$ we can replace "$\aleph_1$-complete" by "κ-complete".

(4) If we waive "∂ regular" so just ∂, an ordinal, is a pseudo true cofinality of $(\Pi\bar{\alpha}, <_D)$ for $D \in \mathbb{D} \subseteq \mathrm{Fil}^1_{\aleph_1}(Y)$, exemplified by $\bar{\mathscr{F}}^D, \mathbb{D} \neq \emptyset$, the proof gives some $\partial', \mathrm{cf}(\partial') = \mathrm{cf}(\partial)$ and $\bar{\mathscr{F}}$ witnessing $(\Pi\bar{\alpha}, <_{D_*})$ has pseudo true cofinality ∂' where $D_* = \bigcap\{D : D \in \mathbb{D}\}$ for $\mathbb{D}$ as below.

Proof. Note that by 1.1

$\boxplus_1$ $\partial \geq \mathrm{hrtg}(\mathscr{P}(Y))$.

Let

$\boxplus_2$ (a) $\mathbb{D} = \{D : D$ is an $\aleph_1$-complete filter on Y such that $(\Pi\bar{\alpha}/D)$ has pseudo true cofinality $\partial\}$,

(b) $D_* = \bigcap\{D : D \in \mathbb{D}\}$.

Now obviously

$\boxplus_3$ (a) $\mathbb{D}$ is non-empty,

(b) D_* is an $\aleph_1$-complete filter on Y.

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : (Y \backslash A) \notin D\}$ and let $\mathscr{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$, equivalently $\mathscr{P}_* = \{A \subseteq Y : A \neq \emptyset \mod D$ for some $D \in \mathbb{D}\}$. As $\mathrm{AC}_{\mathscr{P}(Y)}$ holds also $AC_{\mathscr{P}_*}$ holds, so we can find $\langle D_A : A \in \mathscr{P}_*\rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathscr{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathscr{P}_*\}$; clearly

$\boxplus_4$ (a) $D_* = \bigcap\{D : D \in \mathbb{D}_*\}$,

(b) $\mathbb{D}_* \subseteq \mathbb{D}$ is non-empty.

As $\mathrm{AC}_{\mathscr{P}_*}$ holds, clearly

$(*)_1$ we can choose $\langle \bar{\mathscr{F}}^A : A \in \mathscr{P}_*\rangle$ such that $\bar{\mathscr{F}}^A$ exemplifies $D_A \in \mathbb{D}$ as in [Sh:938, 5.17,(1),(2)], so in particular $\bar{\mathscr{F}}^A$ is $\aleph_0$-continuous and, without loss of generality, $\mathscr{F}^A_\alpha \neq \emptyset, \mathscr{F}^A_\alpha \subseteq \Pi\bar{\alpha}$ for every $\alpha < \partial$.

For each $\beta < \partial$ let

$(*)_2$ $\mathbf{F}^1_\beta = \{\bar{f} = \langle f_A : A \in \mathscr{P}_*\rangle : \bar{f}$ satisfies $A \in \mathscr{P}_* \Rightarrow f_A \in \mathscr{F}^A_\beta\}$,

$(*)_3$ for $\bar{f} \in \mathbf{F}^1_\beta$ let $\sup\{f_A : A \in \mathscr{P}_*\}$ be the function $f \in {}^Y\mathrm{Ord}$ defined by $f(y) = \sup\{f_A(y) : A \in \mathscr{P}_*\}$,

$(*)_4$ $\mathscr{F}^1_\beta = \{\sup\{f_A : A \in \mathscr{P}_*\} : \bar{f} = \langle f_A : A \in \mathscr{P}_*\rangle$ belongs to $\mathbf{F}^1_\beta\}$.

Now

$(*)_5$ (a) $\langle \mathscr{F}^1_\beta : \beta < \partial\rangle$ is well defined, i.e., exists,

(b) $\mathscr{F}^1_\beta \subseteq \Pi\bar\alpha$.

[Why? Clause (a) holds by the definitions, clause (b) holds as $t \in Y \Rightarrow \ \mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(\mathscr{P}(Y))$.]

$(*)_6$ $\mathscr{F}^1_\beta \neq \emptyset$ for $\beta < \partial$.

[Why? As for $\beta<\lambda$, the sequence $\langle \bar{\mathscr{F}}^A_\beta : A \in \mathscr{P}_*\rangle$ is well defined (as $\langle \bar{\mathscr{F}}^A : A \in \mathscr{P}_*\rangle$ is) and $A \in \mathscr{P}_* \Rightarrow \mathscr{F}^A_\beta \neq \emptyset$, so we can use $\mathrm{AC}_{\mathscr{P}(Y)}$ to deduce $\mathscr{F}^1_\beta \neq \emptyset$.]

Define

$(*)_7$ (a) for $f \in \Pi\bar\alpha$ and $A \in \mathscr{P}_*$ let $\beta_A(f) = \min\{\beta < \partial : f < g \bmod D_A$ for every $g \in \mathscr{F}^A_\beta\}$,

(b) for $f \in \Pi\bar\alpha$ let $\beta(f) = \sup\{\beta_A(f) : A \in \mathscr{P}_*\}$.

Now

$(*)_8$ (a) for $A \in \mathscr{P}_*$ and $f \in \Pi\bar\alpha$, the ordinal $\beta_A(f) < \partial$ is well defined,

(b) for $f \in \Pi\bar\alpha$ the sequence $\langle \beta_A(f) : A \in \mathscr{P}_*\rangle$ is well defined.

[Why? Clause (a) holds because $\langle \mathscr{F}^A_\gamma : \gamma < \partial\rangle$ is cofinal in $(\Pi, \bar\alpha, <_{D_A})$, clause (b) holds by $(*)_7(a)$.]

$(*)_9$ (a) For $f \in \Pi\bar\alpha$ the ordinal $\beta(f)$ is well defined and $< \partial$,

(b) if $f \leq g$ are from $\Pi\bar\alpha$ then $\beta(f) \leq \beta(g)$.

[Why? For clause (a), first, $\beta(f)$ is well defined and $\leq \partial$ by $(*)_8$ and the definition of $\beta(f)$ in $(*)_7$(b). Second, recalling that ∂ is regular $\geq \mathrm{hrtg}(\mathscr{P}(Y)) \geq \mathrm{hrtg}(\mathscr{P}_*)$ clearly $\beta(f) < \partial$. Clause (b) is obvious.]

Now

$(*)_{10}$ (a) if $A \in \mathscr{P}_*, \gamma < \partial$ and $f \in \mathscr{F}^A_\gamma$ then $\beta_A(f) > \gamma$,

(b) if $\gamma < \partial$ and $f \in \mathscr{F}^1_\gamma$ then $\beta(f) > \gamma$.

[Why? Clause (a) holds because $\beta < \gamma \wedge g \in \mathscr{F}^A_\beta \Rightarrow g < f \bmod D_A$ and $\beta = \gamma \Rightarrow f \in \mathscr{F}^A_\gamma \wedge f \nleq f \bmod D_A$. Clause (b) holds because for some $\langle f_B : B \in \mathscr{P}_*\rangle \in \Pi\{\mathscr{F}^B_\gamma : B \in \mathscr{P}_*\}$ we have $f = \sup\{f_B : B \in \mathscr{P}_*\}$, hence $B \in \mathscr{P}_* \Rightarrow f_B \leq f$, hence in particular $f_A \leq f$; now recalling $\beta(f_A) > \gamma$ by clause (a) it follows that $\beta(f) > \gamma$.]

$(*)_{11}$ (a) For $\xi < \partial$ let $\gamma_\xi = \min\{\beta(f) : f \in \mathscr{F}^1_\xi\}$,

(b) for $\xi < \partial$ let $\mathscr{F}^2_\xi = \{f \in \mathscr{F}^1_\xi : \beta(f) = \gamma_\xi\}$.

$(*)_{12}$ (a) $\langle (\gamma_\xi, \mathscr{F}^2_\xi) : \xi < \partial\rangle$ is well defined, i.e., exists,

(b) if $\xi < \partial$ then $\xi < \gamma_\xi < \partial$.

[Why? γ_ξ is the minimum of a set of ordinals which is non-empty by $(*)_6$ and $\subseteq \partial$, by $(*)_9$(a), and all members are $> \gamma$ by $(*)_{10}$(b).]

$(*)_{13}$ For $\xi < \partial$ we have $\mathscr{F}^2_\xi \subseteq \Pi\bar{\alpha}$ and $\mathscr{F}^2_\xi \neq \emptyset$.

[Why? By $(*)_{11}$ as $\mathscr{F}^1_\xi \neq \emptyset$ and $\mathscr{F}^1_\xi \subseteq \Pi\bar{\alpha}$.]

$(*)_{14}$ We try to define $\beta_\varepsilon < \partial$ by induction on the ordinal $\varepsilon < \partial$.

$\varepsilon = 0$: $\beta_\varepsilon = 0$.

ε limit: $\beta_\varepsilon = \bigcup\{\beta_\zeta : \zeta < \varepsilon\}$.

$\varepsilon = \zeta + 1$: $\beta_\varepsilon = \gamma_{\beta_\zeta}$.

$(*)_{15}$ (a) If $\varepsilon < \partial$ then $\beta_\varepsilon < \partial$ is well defined $\geq \varepsilon$,

(b) if $\zeta < \varepsilon$ is well defined then $\beta_\zeta < \beta_\varepsilon$.

[Why? Clause (a) holds as ∂ is a regular cardinal, so the case ε limit is O.K.; the case $\varepsilon = \zeta + 1$ holds by $(*)_{12}$(b). As for clause (b) we prove this by induction on ε; for $\varepsilon = 0$ this is empty, for ε a limit ordinal use the induction hypothesis and the choice of β_ε in $(*)_{14}$, and for $\varepsilon = \xi + 1$, clearly by $(*)_{12}$(b) and the choice of γ_ε in $(*)_{14}$ we have $\beta_\xi < \beta_\varepsilon$ and use the induction hypothesis.]

$(*)_{16}$ If $f \in \Pi\bar{\alpha}$, then for some $g \in \bigcup\{\mathscr{F}^2_{\beta_\varepsilon} : \varepsilon < \partial\}$ we have $f < g \mod D_*$.

[Why? Recall that $\beta_A(f)$ for $A \in \mathscr{P}_*$ and $\beta(f)$ are well defined ordinals $< \partial$ and $A \in \mathscr{P}_* \Rightarrow \beta_A(f) \leq \beta(f)$. Now let $\zeta < \partial$ be such that $\beta(f) < \beta_\zeta$, exists as we can prove by induction on ε (using $(*)_{15}$(b)) that $\beta_\varepsilon \geq \varepsilon$. As $\bar{\mathscr{F}}^A$ is $<_{D_A}$-increasing for $A \in \mathscr{P}_*$ clearly $A \in \mathscr{P}_* \wedge g \in \mathscr{F}^A_{\beta_\zeta} \Rightarrow f < g \mod D_A$. So by the definition of $\mathscr{F}^1_{\beta_\zeta}$ we have $A \in \mathscr{P}_* \wedge g \in \mathscr{F}^1_{\beta_\zeta} \Rightarrow f < g \mod D_A$, hence $g \in \mathscr{F}^1_{\beta_\zeta} \Rightarrow f < g \mod D_*$. As $\mathscr{F}^2_{\beta_\zeta} \subseteq \mathscr{F}^1_{\beta_\zeta}$ we are done.]

$(*)_{17}$ If $\zeta < \xi < \partial$ and $f \in \mathscr{F}^2_\zeta$ and $g \in \mathscr{F}_\xi$, then $f < g \mod D_*$.

[Why? As in the proof of $(*)_{16}$ but now $\beta(f) = \gamma_\zeta$.]

Together by $(*)_{13} + (*)_{16} + (*)_{17}$ the sequence $\langle \mathscr{F}^2_{\beta_\varepsilon} : \varepsilon < \partial \rangle$ is as required. $\blacksquare_{1.2}$

A central definition here is

Definition 1.4: (1) For $\bar{\alpha} \in {}^Y\mathrm{Ord}$ let

$$J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}] = \{X \subseteq Y : \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha} \restriction X) \subseteq \lambda\}.$$

So for $X \subseteq Y, X \notin J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$ iff there is an $\aleph_1$-complete filter D on Y such that $X \neq \emptyset \mod D$ and ps-tcf$(\Pi\bar{\alpha}, <_D)$ is well defined $\geq \lambda$ iff there is an $\aleph_1$-complete filter D on Y such that ps-tcf$(\Pi\bar{\alpha}, <_D)$ is well defined $\geq \lambda$ and $X \in D$.

(2) $J^{\aleph_1-\rm comp}_{\le\lambda}$ is $J^{\aleph_1-\rm comp}_{<\lambda^+}$ and we can use a set $\mathfrak{a}$ of ordinals instead of $\bar\alpha$.

Claim 1.5 (The Generator Existence Claim): *Let $\bar\alpha \in {}^Y(\mathrm{Ord}\backslash\{0\})$.*

(1) $J^{<\aleph_1-\rm comp}_{<\lambda}(\bar\alpha)$ *is an $\aleph_1$-complete ideal on Y for any cardinal λ except that it may be $\mathscr{P}(Y)$.*

(2) $[\mathrm{AC}_{\mathscr{P}(Y)}]$ *Assume $t\in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(\mathscr{P}(Y))$. If $\lambda \in$ ps-pcf$_{\aleph_1-\rm comp}(\bar\alpha)$ then for some $X \subseteq Y$ we have*

(A) $J^{\aleph_1-\rm comp}_{<\lambda^+}[\bar\alpha] = J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha] + X$,

(B) $\lambda =$ ps-tcf$(\Pi\bar\alpha, <_{J^{\aleph_1-\rm comp}_{=\lambda}[\bar\alpha]})$ *where* $J^{\aleph_1-\rm comp}_{=\lambda}[\bar\alpha] := J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha] + (Y\backslash X)$,

(C) $\lambda \notin$ ps-pcf$_{\aleph_1-\rm comp}(\bar\alpha \restriction (Y\backslash X))$.

Remark 1.6: Recall that if $\mathrm{AC}_{\mathscr{P}(Y)}$ then without loss of generality $\mathrm{AC}_{\aleph_0}$ holds. Why? Otherwise by $\mathrm{AC}_{\mathscr{P}(Y)}$ we have Y is well ordered and AC_Y, hence $|Y| = n$ for some $n < \omega$ and in this case our claims are obvious, e.g., 1.5(2), 1.7.

Proof. (1) Clearly $J^{<\aleph_1\text{-comp}}_{<\lambda}(\bar\alpha)$ is a $\subseteq$-downward closed subset of $\mathscr{P}(Y)$. If the desired conclusion fails, then we can find a sequence $\langle A_n : n < \omega\rangle$ of members of $J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha]$ such that their union $A := \bigcup\{A_n : n < \omega\}$ does not belong to it. As $A \notin J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha]$, by the definition there is an $\aleph_1$-complete filter D on Y such that $A \ne \emptyset \mod D$ and ps-tcf$(\Pi\bar\alpha, <_D)$ is well defined, so let it be $\mu = \mathrm{cf}(\mu) \ge \lambda$ and let $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ exemplify it.

As D is $\aleph_1$-complete and $A = \bigcup\{A_n : n < \omega\} \ne \emptyset \mod D$ necessarily for some $n, A_n \ne \emptyset \mod D$, but then D witnesses $A_n \notin J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha]$, a contradiction.

(2) Recall λ is a regular cardinal by [Sh:938, 5.8(0)] and $\lambda \ge \mathrm{hrtg}(\mathscr{P}(Y))$ by 1.1.

Let $D = D^{\bar\alpha}_\lambda$ be as in [Sh:938, 5.19] when DC holds, and as in 1.2 in general, i.e., $\Pi\bar\alpha/D$ has pseudo true cofinality λ and D contains any other such $\aleph_1$-complete filter on Y. Now if $X \in D^+$, then

$$\lambda = \text{ ps-tcf}_{\aleph_1-\rm comp}(\bar\alpha \restriction X, <_{(D+X)\cap\mathscr{P}(X)}),$$

hence $X \notin J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha]$, so

$(*)_1$ $X \in J^{\aleph_1-\rm comp}_{<\lambda}[\bar\alpha] \Rightarrow X = \emptyset \mod D$.

A major point is

$(*)_2$ some $X \in D$ belongs to $J^{\aleph_1-\rm comp}_{<\lambda^+}[\bar\alpha]$.

Why $(*)_2$? The proof will take awhile; assume that not, so we have $\mathrm{AC}_{\mathscr{P}(Y)}$, hence AC_D, so we can find $\langle(\bar{\mathscr{F}}^X, D_X, \lambda_X) : X \in D\rangle$ such that:

(a) λ_X is a regular cardinal $\geq \lambda^+$, i.e., $> \lambda$,
(b) D_X is an $\aleph_1$-complete filter on Y such that $X \in D_X$ and $\lambda_X =$ ps-tcf$(\Pi\bar{\alpha}, <_{D_X})$,
(c) $\bar{\mathscr{F}}^X = \langle \mathscr{F}^X_\alpha : \alpha < \lambda_X\rangle$ exemplifies that $\lambda_X =$ ps-tcf $(\Pi\bar{\alpha}, <_{D_X})$,
(d) moreover $\bar{\mathscr{F}}^X$ is as in [Sh:938, 5.17(2)], that is, it is $\aleph_0$-continuous and $\alpha < \lambda_X \Rightarrow \mathscr{F}^X_\alpha \neq \emptyset$.

Let

(e) $D^*_1 = \{A \subseteq Y$: for some $X_1 \in D$ we have $X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X\}$.

Clearly

(f) D^*_1 is an $\aleph_1$-complete filter on Y extending D.

[Why? First, clearly $D^*_1 \subseteq \mathscr{P}(Y)$ and $\emptyset \notin D^*_1$ as $X \in D \Rightarrow \emptyset \notin D_X$. Second, if $A \in D$ then $X \in D \wedge X \subseteq A \Rightarrow A \in D_X$ by clause (b), hence choosing $X_1 = A$ the demand for "$A \in D^*_1$" holds, so indeed $D \subseteq D^*_1$. Third, assume $\bar{A} = \langle A_n : n < \omega\rangle$ and "$A_n \in D^*_1$" for $n < \omega$; then for each A_n there is a witness $X_n \in D$, so by $\mathrm{AC}_{\aleph_0}$, recalling 1.6, there is an ω-sequence $\langle X_n : n < \omega\rangle$ with X_n witnessing $A_n \in D^*_1$. Then $X = \bigcap\{X_n : n < \omega\}$ belongs to D and witnesses that $A := \bigcap\{A_n : n < \omega\} \in D^*_1$ because every D_X is $\aleph_1$-complete. Fourth, if $A \subseteq B \subseteq Y$ and $A \in D^*_1$, then some X_1 witnesses $A \in D^*_1$, i.e., $X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X$; but then X_1 witnesses also $B \in D^*_1$.]

(g) Assume $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ is $<_D$-increasing in $\Pi\bar{\alpha}$, i.e., $\alpha < \lambda \Rightarrow \mathscr{F}_\alpha \subseteq \Pi\bar{\alpha}$ and $\alpha_1 < \alpha_2 \wedge f_1 \in \mathscr{F}_{\alpha_1} \wedge f_2 \in \mathscr{F}_{\alpha_2} \Rightarrow f_1 <_D f_2$ and $\mathscr{F}_\alpha \neq \emptyset$ for every or at least unboundedly many $\alpha < \lambda$; then $\bigcup_{\alpha<\lambda} \mathscr{F}_\alpha$ has a common $<_{D^*_1}$-upper bound.

[Why? For each $X \in D$ recall $(\Pi\bar{\alpha}, <_{D_X})$ has true cofinality λ_X which is regular $> \lambda$ hence by [Sh:938, 5.7(1A)] is pseudo λ^+-directed, hence there is a common $<_{D_X}$-upper bounded h_X of $\bigcup\{\mathscr{F}_\alpha : \alpha < \lambda\}$. As we have $\mathrm{AC}_{\mathscr{P}(Y)}$ we can find a sequence $\langle h_X : X \in D\rangle$ with each h_X as above. Define $h \in \Pi\bar{\alpha}$ by $h(t) = \sup\{h_X(t) : X \in D\}$; it belongs to $\Pi\bar{\alpha}$ as we are assuming $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(\mathscr{P}(Y)) \geq \mathrm{hrtg}(D)$. So $h \in \Pi\bar{\alpha}$ is a $<_{D_X}$-upper bound of $\bigcup\{\mathscr{F}_\alpha : \alpha < \lambda\}$ for every $X \in D$, hence by the choice of D^*_1 it is a $<_{D^*_1}$-upper bound of $\bigcup\{\mathscr{F}_\alpha : \alpha < \lambda\}$.]

But by the choice of D in the beginning of the proof we have $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$, so there is a sequence $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda\rangle$ witnessing it. By clause (f) we have $D \subseteq D_1^*$, so clearly $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda\rangle$ is also $<_{D_1^*}$-increasing, hence we can apply clause (g) to the sequence $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda\rangle$ and get a $<_{D_1^*}$-upper bound $f \in \Pi\bar{\alpha}$, a contradiction to the choice of $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda\rangle$ recalling 0.4(d) because $D \subseteq D_1^*$, contradiction. So $(*)_2$ really holds.

Choose X as in $(*)_2$; now

$(*)_3$ $D =$ dual$(J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}] + (Y\backslash X))$.

[Why? The inclusion $\supseteq$ holds by $(*)_1$ and $(*)_2$, i.e., the choice of X as a member of D. Now for every $Z \subseteq X$ which does not belong to $J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$, by the definition of $J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$ there is an $\aleph_1$-complete filter D_Z on Y to which Z belongs such that $\theta :=$ ps-cf$(\Pi\bar{\alpha}, <_D)$ is well defined and $\geq \lambda$. But $\theta \geq \lambda^+$ is impossible as we know that $Z \subseteq X \in J^{\aleph_1\text{-comp}}_{<\lambda^+}[\bar{\alpha}]$, so necessarily $\theta = \lambda$, hence by the choice of D by using 1.2 we have $D \subseteq D_Z$, hence $Z \neq \emptyset \bmod D$. Together we are done.]

$(*)_4$ $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_{J^{\aleph_1-\mathrm{comp}}_{=\lambda}})$; see clause (B) of the conclusion of 1.5(2).

[Why? By $(*)_3$, the choice of $J^{\aleph_1-\mathrm{comp}}_{=\lambda}[\bar{\alpha}]$, and as $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$ by the choice of D.]

$(*)_5$ $\lambda \notin$ ps-pcf$_{\aleph_1-\mathrm{comp}}(\bar{\alpha} \restriction (Y\backslash X))$.

[Why? Otherwise there is an $\aleph_1$-complete filter D' on Y such that $Y\backslash X \in D'$ and $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_{D'})$. But this contradicts the choice of D by using 1.2.]

So X is as required in the desired conclusion of 1.5(2): clause (B) by $(*)_4$, clause (C) by $(*)_5$ and clause (A) follows. Note that the notation $J^{\aleph_1-\mathrm{comp}}_{=\lambda}[\bar{\alpha}]$ is justified, as if X' satisfies the requirements on X then

$$X' = X \bmod J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]. \qquad \blacksquare_{1.5}$$

Conclusion 1.7: [$\mathrm{AC}_{\mathscr{P}(Y)}$] Assume $\bar{\alpha} \in {}^Y\mathrm{Ord}$ and each α_t a limit ordinal of cofinality $\geq$ hrtg$(\mathscr{P}(Y))$ and ps $-$ pcf$_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ is not empty.

(1) If $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq$ hrtg$(\mathrm{Fil}^1_{\aleph_1}(Y))$ then there is a function h such that:

- $\bullet_1$ the domain of h is $\mathscr{P}(Y)$,
- $\bullet_2$ Rang(h) includes ps $-$ pcf$_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ and is included in

  $$\text{ps} - \text{pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha}) \cup \{0\} \cup \{\mu : \mu = \sup(\mu \cap \text{ ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha}))\},$$

  also Rang(h) includes $\{\mathrm{cf}(\alpha_t) : t \in Y\}$, but see $\bullet_5$,
- $\bullet_3$ $A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B)$ and $h(A) = 0 \Leftrightarrow A = \emptyset$,

- $\bullet_4$ $h(A) = \min\{\lambda : A \in J^{\aleph_1-\text{comp}}_{\leq\lambda}[\bar{\alpha}]\}$,
- $\bullet_5$ if $h(A) = \lambda$ and $\text{cf}(\lambda) > \aleph_0$ then λ is regular and $\lambda \in \text{ps-tcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$, i.e., for some $\aleph_1$-complete filter D on Y we have $A \in D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \lambda$,
- $\bullet_6$ the set $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ has cardinality $< \text{hrtg}(\mathscr{P}(Y))$,
- $\bullet_7$ if $h(A) = \lambda$ and $\text{cf}(\lambda) = \aleph_0$ then we can find a sequence $\langle A_n : n < \omega\rangle$ such that $A = \bigcup\{A_n : n < \omega\}$ and $h(A_n) < \lambda$ for $n < \omega$,
- $\bullet_8$ $J^{\aleph_1-\text{comp}}_{<\lambda}[\bar{\alpha}] = \{A \subseteq Y : h(A) < \lambda\}$ when $\text{cf}(\lambda) > \aleph_0$,
- $\bullet_9$ if $\text{cf}(\text{otp}(\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha}))) > \aleph_0$ then $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ has a last member.

(2) Without the extra assumption of part (1), still there is h such that:

- $\bullet_1$ h is a function with domain $\mathscr{P}(Y)$,
- $\bullet_2$ the range of h is

$$\begin{aligned}\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha}) \cup \{0\} \cup \{\mu :\ & \mu = \sup(\mu \cap \text{ ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})) \text{ and } \text{cf}(\mu) = \aleph_0 \\ & \text{or just } \text{cf}(\mu) < \text{hrtg}(\text{ ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})) \text{ and} \\ & J^{\aleph_1-\text{comp}}_{<\mu}[\bar{\alpha}] \neq \cup\{J^{\aleph_1-\text{comp}}_{<\chi}[\bar{\alpha}] : \chi < \mu\}\},\end{aligned}$$

- $\bullet_3$ $A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B)$ and $h(A) = 0 \Leftrightarrow A = \emptyset$,
- $\bullet_4$ $h(A) = \min\{\lambda : A \in J^{\aleph_1-\text{comp}}_{\leq\lambda}[\bar{\alpha}]\}$,
- $\bullet_5$ if $h(A)=\lambda$ and $\text{cf}(\lambda) \geq \text{hrtg}(\text{ps-pcf}_{\aleph_1-\text{comp}}[\bar{\alpha}])$ then $\lambda \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$, i.e., there is an $\aleph_1$-complete filter D on Y such that $(\Pi\bar{\alpha}, <_D)$ has true cofinality λ,
- $\bullet_6$ as above,
- $\bullet_7$ as above,
- $\bullet_8$ as above.

(3) The set $\mathfrak{c} := \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$ satisfies $\mathfrak{c} \leq_{\text{qu}} \mathscr{P}(Y)$. If also AC_α holds for $\alpha < \text{hrtg}(\mathscr{P}(Y))$ or just $\text{AC}_{\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})}$ then we can find a sequence $\langle X_\lambda : \lambda \in \mathfrak{c}\rangle$ of subsets of Y such that for every cardinality μ, $J^{\aleph_1-\text{comp}}_{<\mu}[\bar{\alpha}]$ is the $\aleph_1$-complete ideal on Y generated by $\{X_\lambda : \lambda < \mu \text{ and } \lambda \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})\}$.

Proof. (1) Let $\Theta = \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$. We define the function h from $\mathscr{P}(Y)$ into Θ^+ which is defined as the closure of $\Theta \cup \{0\}$, i.e., $\Theta \cup \{\mu : \mu = \sup(\mu \cap \Theta)\}$, by $h(X) = \text{Min}\{\lambda \in \Theta^+ : X \in J^{\aleph_1-\text{comp}}_{\leq\lambda}[\bar{\alpha}]\}$. It is well defined as ps-pcf$_{\aleph_1-\text{comp}}(\bar{\alpha})$ is a set (see [Sh:938, 5.8(2)]), non-empty by an assumption and $J^{\aleph_1-\text{comp}}_{\leq\lambda}[\bar{\alpha}] = \mathscr{P}(Y)$ when $\lambda \geq \sup(\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha}))$. For this function, its range is included in Θ^+, but $\mathrm{otp}(\Theta^+) \le \mathrm{otp}(\Theta)+1$; also clearly $\bullet_1$ of the conclusion holds. Also, if $\lambda \in \Theta$ and X is as in 1.5(2) then $h(X) = \lambda$; so h is a function from $\mathscr{P}(Y)$ into Θ^+ and its range includes Θ, hence $|\Theta| < \mathrm{hrtg}(\mathscr{P}(Y))$ so $\bullet_2$'s first clause holds; the second clause of $\bullet_2$ holds as trivially $h(\emptyset) = 0$ and the definition of Θ^+, and the third clause by $t \in Y \Rightarrow h(\{t\}) = \mathrm{cf}(\alpha_t)$ holds. Now first by 1.1 we have $\theta \in \Theta \Rightarrow \theta \ge \mathrm{hrtg}(\mathscr{P}(Y))$, hence $\theta \in \Theta \Rightarrow \theta > \sup(\Theta \cap \theta)$, so the range of h is as required in $\bullet_2$.

Second, if $\lambda \in \Theta^+$ and $\mathrm{cf}(\lambda) = \aleph_0$ then clearly $\lambda \in \Theta^+ \backslash \Theta$, and we can find an increasing sequence $\langle \lambda_n : n < \omega \rangle$ of members of $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ with limit λ. For each n there is $X_n \in J^{\aleph_1-\mathrm{comp}}_{\le \lambda_n}[\bar{\alpha}] \backslash J^{\aleph_1\text{-comp}}_{<\lambda_n}[\bar{\alpha}]$ by 1.5(2), but $\mathrm{AC}_{\aleph_0}$ holds (see 1.6), hence such a sequence $\langle X_n : n < \omega \rangle$ exists. Easily $A := \cup\{X_n : n < \omega\} \in \mathscr{P}(Y)$ satisfies $h(A) = \lambda$, hence $\lambda \in \mathrm{Rang}(h)$. Third, if $\lambda = \sup(\text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha}))$ and $\mathrm{cf}(\lambda) > \aleph_0$, then $\bigcup_{\mu<\lambda} J_{<\mu}[\bar{\alpha}] \ne \mathscr{P}(Y)$ because Y does not belong to the union while $J_{<\lambda^+}(\bar{\alpha}) = \mathscr{P}(Y)$, so $h(Y) = \lambda$.

Fourth, assume $\lambda = h(A), \lambda \notin \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ and $\mathrm{cf}(\lambda) > \aleph_0$. We can find $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$, an increasing sequence with limit λ, but by the definition of h necessarily $\lambda \cap \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ is an unbounded subset of λ, so without loss of generality all are members of $\text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\Pi\bar{\alpha})$. Now $\langle J_i := J^{\aleph_1-\mathrm{comp}}_{<\lambda_i}[\bar{\alpha}] : i < \mathrm{cf}(\lambda) \rangle$ is a $\subseteq$-increasing sequence of $\aleph_1$-complete ideals on Y, no choice is needed, and by our present assumption $\aleph_0 < \mathrm{cf}(\lambda)$, hence the union $J = \bigcup\{J_i : i < \mathrm{cf}(\lambda)\}$ is an $\aleph_1$-complete ideal on Y and obviously $A \notin J$. So also $D_1 = \mathrm{dual}(J) + A$ is an $\aleph_1$-complete filter hence (recalling the extra assumption $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(\mathrm{Fil}^1_{\aleph_1}(Y))$), by [Sh:938, 5.9] for some $\aleph_1$-complete filter D_2 extending D_1 we have $\mu = \text{ps-tcf}(\Pi\alpha, <_{D_2})$ is well defined, so by 1.5(2) we have some $D_2 \cap J^{\aleph_1-\mathrm{comp}}_{\le\mu}[\bar{\alpha}] \ne \emptyset$; but $\emptyset = D_2 \cap J_i = D_2 \cap J^{\aleph_1-\mathrm{comp}}_{<\lambda_i}[\bar{\alpha}]$, hence $\mu \ge \lambda_i$. Hence $\mu \ge \lambda_i$ for every $i < \mathrm{cf}(\lambda)$; but λ is singular, so $\mu > \lambda$ and $\mu \in \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$. Hence $\chi := \min(\text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha}) \backslash \lambda)$ is well defined and $J^{\aleph_1-\mathrm{comp}}_{<\chi}[\bar{\alpha}] = J$ trivially $\chi \ge \lambda$, but as χ is regular while λ is singular clearly $\chi > \lambda$. But as $h(A) = \lambda < \chi$ we get that $A \in J^{\aleph_1-\mathrm{comp}}_{<\chi}[\bar{\alpha}]$, a contradiction to the definition of h.

So we have proved $\bullet_5$, the fifth clause of the conclusion. The other clauses follow from the properties of h.

(2) A similar proof.

(3) We define a function g with domain $\mathscr{P}(Y)$ by

$$g(A) = \min\{\lambda : A \in J_{<\lambda^+}[\bar{\alpha}]\}.$$

This function is well defined as if $\lambda = \mathrm{hrtg}(\Pi\bar{\alpha})$, then $A \subseteq Y \Rightarrow A \in J_{\leq\lambda}[\bar{\alpha}]$; and the cardinals are well ordered. Also $\mathfrak{c} \subseteq \mathrm{Rang}(h)$ because if $\lambda \in \mathfrak{c}$, then by 1.5(2), we are done recalling that we are assuming $\mathrm{AC}_{\mathscr{P}(Y)}$.

So clearly $\mathfrak{c} \leq_{\mathrm{qu}} \mathscr{P}(Y)$, so as $\mathfrak{c}$ is a set of cardinals, clearly $\mathrm{otp}(\mathfrak{c}) < \mathrm{hrtg}(\mathscr{P}(Y))$, hence $|\mathfrak{c}| < \mathrm{hrtg}(\mathscr{P}(Y))$.

For the second sentence in 1.7(3), by the last sentence it suffices to assume $\mathrm{AC}_{\mathfrak{c}}$. For $\lambda \in \mathfrak{c}$ let $\mathscr{P}_\lambda = \{X \subseteq Y : X \text{ as in } 1.5(2)\}$, so $\mathscr{P}_\lambda \neq \emptyset$. By $\mathrm{AC}_{\mathfrak{c}}$ there is a sequence $\langle X_\lambda : \lambda \in \mathfrak{c}\rangle \in \prod_{\lambda\in\mathfrak{c}} \mathscr{P}_\lambda$. For $\lambda \in \mathfrak{c}$, let J^*_λ be the $\aleph_1$-complete ideal on Y generated by $\{X_\mu : \mu \in \mathfrak{c} \cap \lambda\}$, so by the definitions of $\mathscr{P}_\lambda$ we have $\mu < \lambda \wedge \mu \in \mathfrak{c} \Rightarrow X_\mu \in J_{\leq\mu}[\bar{\alpha}] \subseteq J_{<\lambda}[\bar{\alpha}]$, also $J_{<\lambda}[\bar{\alpha}]$ is $\aleph_1$-complete, hence $\lambda \in \mathfrak{c} \Rightarrow J^*_\lambda \subseteq J_{<\lambda}[\bar{\alpha}]$.

If for every λ equality holds, we are done, otherwise there is a minimal counterexample and use 1.5(2). $\blacksquare_{1.7}$

Definition 1.8: Assume $\mathrm{cf}(\mu) < \mathrm{hrtg}(Y)$ and μ is singular of an uncountable cofinality limit of regulars. We let

(a) $\mathrm{pp}^*_Y(\mu) = \sup\{\lambda : \text{for some } \bar{\alpha}, D \text{ we have}$

$$\begin{aligned} &(a) \quad \lambda = \ \text{ps-tcf}(\Pi\bar{\alpha}, <_D),\\ &(b) \quad D \text{ is an } \aleph_1 - \text{complete filter on } Y,\\ &(c) \quad \bar{\alpha} = \langle \alpha_t : t \in Y\rangle, \text{ each } \alpha_t \text{ regular},\\ &(d) \quad \mu = \ \lim_D \bar{\alpha}\}, \end{aligned}$$

(b) $\mathrm{pp}^+_Y(\mu) = \sup\{\lambda^+ : \lambda \text{ as above}\}$,

(c) similarly $\mathrm{pp}^*_{\kappa-\mathrm{comp},Y}(\mu), \mathrm{pp}^+_{\kappa-\mathrm{comp},Y}(\mu)$ restricting ourselves to κ-complete filters D; similarly for other properties,

(d) we can replace Y by an $\aleph_1$-complete filter D on Y, which means we fix D but not $\bar{\alpha}$ above.

Remark 1.9: (1) Of course, if we consider sets Y such that AC_Y may fail, it is natural to omit the regularity demands, so $\bar{\alpha}$ is just a sequence of ordinals.

(2) We may use $\bar{\alpha}$ a sequence of cardinals, not necessarily regular; see §3.

Conclusion 1.10 ($\mathrm{DC} + \mathrm{AC}_{\mathscr{P}(Y)}$): Assume $\theta = \mathrm{hrtg}(\mathscr{P}(Y)) < \mu, \mu$ is as in Definition 1.8, $\mu_0 < \mu$ and $\bar{\alpha} \in {}^Y(\mathrm{Reg} \cap \mu_0^+) \wedge \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha}) \neq \emptyset \Rightarrow \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha}) \subseteq \mu$. If $\sigma = |\mathrm{Reg} \cap \mu \backslash \mu_0| < \mu$ and $\kappa = |\mathrm{Reg} \cap \mathrm{pp}^+_Y(\mu) \backslash \mu_0|$ then $\kappa < \mathrm{hrtg}(\theta \times {}^Y\sigma)$.

Remark 1.11: In the ZFC parallel the assumption on $\mu_0 < \mu$ is not necessary.

Proof. Obvious by Definition [Sh:938, 5.6] noting Conclusion 1.7 above and 1.12 below. That is, letting $\Xi := \text{Reg} \cap \text{pp}^+_Y(\mu) \backslash \mu_0$ so $|\Xi| = \kappa$ and $\Lambda = \text{ Reg } \cap \mu \backslash \mu_0$, for every $\bar\alpha \in {}^Y\Lambda$ by Definition 1.8 the set $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$ is a subset of $\text{Reg} \cap \text{pp}^+_Y(\mu)\backslash\mu_0$, and by claim 1.7 it is a set of cardinality $< \text{hrtg}(\mathscr{P}(Y))$. By Definition 1.8 and Claim 1.12 below we have $\Xi = \bigcup\{\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bar\alpha) : \bar\alpha \in {}^Y\Lambda\}$. Clearly there is a function h with domain $\text{hrtg}(\mathscr{P}(Y)) \times {}^Y\sigma$ such that $\varepsilon < \text{hrtg}(\mathscr{P}(Y)) \wedge \bar\alpha \in {}^Y\sigma \Rightarrow (h(\varepsilon, \bar\alpha)$ is the ε-th member of $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$ if there is one, $\min(\Lambda)$ otherwise). So h is a function from $\text{hrtg}(\mathscr{P}(Y)) \times {}^Y\sigma$ onto a set including Ξ which has cardinality κ, so we are done. $\blacksquare_{1.10}$

Claim 1.12 (The No Hole Claim [DC]):

(1) *If* $\bar\alpha \in {}^Y\text{Ord}$ *and* $\lambda_2 \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$, *for transparency* $t \in Y \Rightarrow \alpha_t > 0$ *and* $\text{hrtg}(\mathscr{P}(Y)) \le \lambda_1 = \text{cf}(\lambda_1) < \lambda_2$, *then for some* $\bar\alpha' \in \Pi\bar\alpha$ *we have* $\lambda_1 = \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha')$.

(2) *In part (1), if in addition* AC_Y *then without loss of generality* $\bar\alpha' \in {}^Y\text{Reg}$.

(3) *If in addition* $\text{AC}_{\mathscr{P}(Y)} + \text{AC}_{<\kappa}$ *then even witnessed by the same filter (on* Y*).*

Proof. (1) Let D be an $\aleph_1$-complete filter on Y such that $\lambda_2 = \text{ps-tcf}(\Pi\bar\alpha, <_D)$; let $\langle \mathscr{F}_\alpha : \alpha < \lambda_2 \rangle$ exemplify this.

First assume $\text{hrtg}(\text{Fil}^1_{\aleph_1}(Y)) \le \lambda_1$; clearly $f \in \mathscr{F}_\alpha \Rightarrow \text{rk}_D(f) \ge \alpha$ for every $\alpha < \lambda_2$, hence in particular for $\alpha = \lambda_1$, hence there is $f \in {}^Y\text{Ord}$ such that $\text{rk}_D(f) = \lambda_1$ and now use [Sh:938, 5.9], but there we change the filter D, (extend it), so it is O.K. for part (1). In general, i.e., without the extra assumption $\text{hrtg}(\text{Fil}^1_{\aleph_2}(Y)) \le \lambda_1$, use 1.13(1),(2) below.

(2) Easy, too.

(3) Similarly using 1.13(3) below. $\blacksquare_{1.12}$

Claim 1.13: *Assume* $D \in \text{Fil}^1_\kappa(Y), \kappa > \aleph_0, \mathscr{F}_\alpha \subseteq {}^Y\text{Ord}$ *non-empty for* $\alpha < \delta$ *and* $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \delta \rangle$ *is* $<_D$*-increasing,* δ *a limit ordinal.*

(1) [DC] *There is* $f^* \in \Pi\bar\alpha$ *which satisfies* $f \in \bigcup\{\mathscr{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_D f^*$ *but there is no such* $f^{**} \in \Pi\bar\alpha$ *satisfying* $f^{**} <_D f$.

(2) $[\mathrm{AC}_{<\kappa}]$ *For* f^* *as above, let*

$$\begin{aligned} D_1 = D_{f^*,\bar{\mathscr{F}}} := \{Y \backslash A : & A = \emptyset \mod D \text{ or} \\ & A \in D^+ \text{ and there is } f^{**} \in {}^Y\mathrm{Ord} \text{ such that} \\ & f^{**} <_{D+A} f^* \text{ and } f \in \cup\{\mathscr{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_{D+A} f^{**}\}. \end{aligned}$$

Now D_1 *is a* κ*-complete filter and* $\emptyset \notin D_1, D_1$ *extends* D*, and if* $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ *then* $\langle \mathscr{F}_\alpha : \alpha < \delta\rangle$ *witnesses that* f^* *is a* $<_{D_1}$*-exact upper bound of* $\bar{\mathscr{F}}$*, hence* $(\prod_{y\in Y} f^*(y), <_{D_1})$ *has pseudo true cofinality* $\mathrm{cf}(\delta)$.

(3) $[\mathrm{DC} + \mathrm{AC}_{<\kappa} + \mathrm{AC}_{\mathscr{P}(Y)}]$ *If* $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ *then there is* $f' \in {}^Y\mathrm{Ord}$ *which is an* $<_D$*-exact upper bound of* $\bar{\mathscr{F}}$*, i.e.,*

$$f <_D f' \Rightarrow (\exists \alpha < \delta)(\exists g \in \mathscr{F}_\alpha)\ [f < g \mod D]$$

and

$$f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f <_D f'.$$

Proof. (1) If not, then by DC we can find $\bar{f} = \langle f_n : n < \omega\rangle$ such that:

(a) $f_n \in {}^Y\mathrm{Ord}$,
(b) $f_{n+1} < f_n \mod D$,
(c) if $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha$ and $n < \omega$ then $f < f_n \mod D$.

So $A_n = \{t \in Y : f_{n+1}(t) < f_n(t)\} \in D$, hence $\bigcap\{A_n : n < \omega\} \in D$, contradiction.

(2) First, clearly $D_1 \subseteq \mathscr{P}(Y)$ and by the assumption $\emptyset \notin D_1$. Second, if f^{**} witnesses $A \in D_1$ and $A \subseteq B \subseteq Y$ then f^{**} witnesses $B \in D_1$.

Third, we prove D_1 is closed under intersection of $< \kappa$ members, so assume $\zeta < \kappa$ and $\bar{A} = \langle A_\varepsilon : \varepsilon < \zeta\rangle$ is a sequence of members of D_1. Let $A := \bigcap\{A_\varepsilon : \varepsilon < \zeta\}$, $B_\varepsilon = Y \backslash A_\varepsilon$ for $\varepsilon < \zeta$ and $B'_\varepsilon = B_\varepsilon \backslash \bigcup\{B_\xi : \xi < \varepsilon\}$ and $B = \bigcup\{B_\varepsilon : \varepsilon < \zeta\}$. Clearly $B = Y\backslash A, A \subseteq Y$ and $\langle B'_\varepsilon : \varepsilon < \zeta\rangle$ is a sequence of pairwise disjoint subsets of Y with union B. But AC_ζ holds and $\varepsilon < \zeta \Rightarrow A_\varepsilon \in D_1$, hence we can find $\langle f^{**}_\varepsilon : \varepsilon < \zeta\rangle$ such that $f^{**}_\varepsilon \in {}^Y\mathrm{Ord}$ and if $A_\varepsilon \notin D$ then f^{**}_ε witnesses $A_\varepsilon \in D_1$. Let $f^{**} \in {}^Y\mathrm{Ord}$ be defined by $f^{**}(t) = f^{**}_\varepsilon(t)$ if $t \in B'_\varepsilon$ or $\varepsilon = 0 \wedge t \in Y\backslash B$; easily $B'_\varepsilon \in D^+ \wedge f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f < f^{**}_\varepsilon = f^{**} \mod (D + B'_\varepsilon)$ but $B = \bigcup\{B'_\varepsilon : \varepsilon < \zeta\}$ and D is κ-complete hence $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f < f^{**} \mod (D + B)$. So as $A = Y\backslash B$ clearly f^{**} witnesses $A = \bigcap_{\varepsilon<\zeta} A_\varepsilon \in D_1$ so D_1 is indeed κ-complete.

Lastly, assume $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ and we shall show that f^* is an exact upper bound of $\bar{\mathscr{F}}$ modulo D_1. So assume $f^{**} \in {}^Y\mathrm{Ord}$ and $f^{**} < f^* \mod D_1$ and we shall prove that there are $\alpha < \delta$ and $f \in \mathscr{F}_\alpha$ such that $f^{**} \leq f \mod D_1$.

Let $\mathscr{A} = \{A \in D_1^+$: there is $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha$ such that $f^{**} \leq f \bmod (D + A)\}$, yes, not D_1!

Case 1: For every $B \in D_1^+$ there is $A \in \mathscr{A}, A \subseteq B$.

For every $A \in \mathscr{A}$ let

$$\alpha_A = \min\{\beta : \text{ there is } f \in \mathscr{F}_\beta \text{ such that } f^{**} \leq f \bmod (D + A)\}.$$

So the sequence $\langle \alpha_A : A \in \mathscr{A}\rangle$ is well defined.

Let $\alpha(*) = \sup\{\alpha_A + 1 : A \in \mathscr{A}\}$; it is $< \delta$ as $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y)) \geq \mathrm{hrtg}(\mathscr{A})$.

Choose $f \in \mathscr{F}_{\alpha(*)}$ and let $B_f := \{t \in Y : f^{**}(t) > f(t)\}$. Now if $A \in \mathscr{A}$ (so $A \in D_2^+$) and $f' \in \mathscr{F}_{\alpha_A}$ witnesses this (i.e., $f^{**} \leq f' \mod (D + A)$), then $A \nsubseteq B_f$ as otherwise $f^{**} \leq f' < f < f^{**} \mod (D + A)$. So B_f contains no $A \in \mathscr{A}$, hence necessarily B_f is $= \emptyset \bmod D_1$ by the case assumption; this means that $f^{**} \leq f \mod D_1$. So recalling $f \in \mathscr{F}_{\alpha(*)} \subseteq \bigcup_{\alpha<\delta} \mathscr{F}_\alpha$, we have "$f$ is as required" thus finishing the proof of "f^* is an exact upper bound of $\bar{\mathscr{F}}$ mod D".

Case 2: $B \in D_1^+$ and there is no $A \in \mathscr{A}$ such that $A \subseteq B$.

For $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha$ let $B_f = \{t \in B : f(t) < f_{**}(t)\}$, and for $\alpha < \delta$ we define $\mathscr{B}_\alpha = \{B_f : f \in \mathscr{F}_\alpha\}$ and we define a partial function h from $\mathscr{P}(Y)$ into δ by $h(A) = \sup\{\alpha < \delta : A \in \mathscr{B}_\alpha\}$. As $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ necessarily $\alpha(*) = \sup \delta \cap \mathrm{Rang}(h)$ is $< \delta$. Choose $g \in \mathscr{F}_{\alpha(*)+1}$, hence

$$u := \{\alpha : \alpha \in [\alpha(*), \delta] \text{ and } B_g \in \mathscr{B}_\alpha\}$$

is an unbounded subset of δ.

Let $A = B \cap B_g$. Now if $A \in D^+$ then $\alpha \in u \Rightarrow \bigvee_{f\in\mathscr{F}_\alpha} f < f_{**} \bmod (D + A)$, but $\bar{\mathscr{F}}$ is $<_D$-increasing and $\delta = \sup(u)$ hence $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f < f_{**} \bmod (D + A)$, hence by the definition of D_1, f_{**} witnesses that $Y \backslash A \in D_1$, hence $A = \emptyset$ mod D_1. As $B \in D_1^+$ and $A = B \cap B_g$ it follows that $B \backslash B_g \in D_1^+$, and by the choice of $\mathscr{A}$ the set $B \backslash B_g$ belongs to $\mathscr{A}$. But $B \backslash B_g \subseteq B$ by its definition, so we get a contradiction to the case assumption.

(3) By [Sh:938, 5.12], without loss of generality, $\bar{\mathscr{F}}$ is $\aleph_0$-continuous. For every $A \in D^+$ the assumptions hold even if we replace D by $D + A$, and so there are D_1, f^* as in part (2); we are allowed to use part (1) as we have DC

and part (2) as we have $\mathrm{AC}_{<\kappa}$. As we are assuming $\mathrm{AC}_{\mathscr{P}(Y)}$ there is a sequence $\langle (D_A, f_A) : A \in D^+\rangle$ such that:

$(*)_1$ (a) D_A is a κ-complete filter extending $D + A$,
(b) $f_A \in {}^Y\mathrm{Ord}$ is a $<_{D_A}$-exact upper bound of $\bar{\mathscr{F}}$.

Recall $|A| \le_{\mathrm{qu}} |B|$ is defined as: A is empty or there is a function from B onto A. Of course, this implies $\mathrm{hrtg}(A) \le \mathrm{hrtg}(B)$.

Let $\bar{\mathscr{U}} = \langle \mathscr{U}_t : t \in Y\rangle$ be defined by

$$\mathscr{U}_t = \{f_A(t) : A \in D^+\} \cup \Big\{ \sup\{f(t) : f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha\}\Big\},$$

hence $t \in Y \Rightarrow 0 < |\mathscr{U}_t| \le_{\mathrm{qu}} \mathscr{P}(Y)$ even uniformly, so there is a sequence $\langle h_t : t \in Y\rangle$ such that h_t is a function from $\mathscr{P}(Y)$ onto $\mathscr{U}_t$, hence $|\prod_{t\in Y} \mathscr{U}_t| \le_{\mathrm{qu}} \mathscr{P}(Y) \times Y \le_{\mathrm{qu}} \mathscr{P}(Y \times Y)$; but $\mathrm{AC}_{\mathscr{P}(Y)}$ holds, hence Y can be well ordered. However, without loss of generality Y is infinite, hence $|Y \times Y| = Y$, so $|\prod_{t\in Y} \mathscr{U}_t| \le_{\mathrm{qu}} |\mathscr{P}(Y)|$.

Let

$$\mathscr{G} = \Big\{g : g \in \prod_{t\in Y} \mathscr{U}_t \text{ and not for every } f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha \text{ do we have } f < g \mod D\Big\},$$

so $|\mathscr{G}| \le |\prod_{t\in Y} \mathscr{U}_t| \le_{\mathrm{qu}} |\mathscr{P}(Y \times Y)| = |\mathscr{P}(Y)|$, hence $\mathrm{hrtg}(\mathscr{G}) \le \mathrm{hrtg}(\mathscr{P}(Y)) \le \mathrm{cf}(\delta)$.

Now for every $g \in \mathscr{G}$ the sequence $\langle\{\{t\in Y : g(t) \le f(t)\} : f \in \bigcup_{\beta<\alpha} \mathscr{F}_\beta\} : \alpha < \delta\rangle$ is a $\subseteq$-increasing sequence of subsets of $\mathscr{P}(Y)$, but $\mathrm{hrtg}(\mathscr{P}(Y)) \le \mathrm{cf}(\delta)$, hence the sequence is eventually constant and let $\alpha(g) < \delta$ be minimal such that

$$(*)_g \quad (\forall\beta)\Big[\alpha(g) \le \beta < \delta \Rightarrow \Big\{\{t \in Y : g(t) \le f(t)\} : f \in \bigcup_{\gamma<\beta} \mathscr{F}_\gamma\Big\} = \Big\{\{t \in Y : g(t) \le f(t)\} : f \in \bigcup_{\gamma<\alpha(g)} \mathscr{F}_\gamma\Big\}\Big].$$

But recalling $\mathrm{hrtg}(\mathscr{G}) \le \mathrm{cf}(\delta)$, the ordinal $\alpha(*) := \sup\{\alpha(g) : g \in \mathscr{G}\}$ is $< \delta$. Now choose $f^* \in \mathscr{F}_{\alpha(*)+1}$ and define $g^* \in \prod_{t\in Y} \mathscr{U}_t$ by $g^*(t) = \min(\mathscr{U}_t \backslash f^*(t))$, well defined as $\sup\{f(t) : t \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha\} \in \mathscr{U}_t$. It is easy to check that g^* is as required. $\blacksquare_{1.13}$

Observation 1.14: (1) Let D be a filter on Y.

If D is κ-complete for every κ then for every $f \in {}^Y\mathrm{Ord}$ and $A \in D^+$ there is $B \subseteq A$ from D^+ such that $f{\restriction}B$ is constant.

(2) If $\bar{\alpha} = \langle \alpha_s : s \in Y \rangle$ and $X_\varepsilon \subseteq Y$ for $\varepsilon < \zeta < \kappa$ and $X = \bigcup_{\varepsilon<\zeta} X_\varepsilon$ then $\mathrm{ps}-\mathrm{pcf}_{\kappa-\mathrm{comp}}(\bar{\alpha}{\restriction}X) = \bigcup_{\varepsilon<\zeta} \mathrm{ps}-\mathrm{pcf}_{\kappa-\mathrm{comp}}(\bar{\alpha}{\restriction}X_\varepsilon)$.

Remark 1.15: (1) Note that 1.14(1) is not empty; its assumptions hold when Y is an infinite set such that: for every $X \subseteq Y, |X| < \kappa \vee |Y \backslash X| < \kappa$ and $D = \{X \subseteq Y : |Y \backslash X| \nleq \kappa\}$.

Proof. Straightforward. $\blacksquare_{1.14}$

2. Composition and generating sequences for pseudo pcf

How much choice suffices to show $\lambda = \mathrm{ps}-\mathrm{tcf}(\prod_{(i,j)\in Y} \lambda_{i,j}/D)$ when λ_i is the pseudo true equality of $(\prod_{j\in Y_i} \lambda_{i,j}, <_{D_i})$ for $i \in Z$, where $Z = \{i : (i,j) \in Y\}$ and $Y_i = \{(i,j) : i \in Z, j \in Y_i\}$ and $\lambda =$ ps-tcf$(\prod_{i\in Z} \lambda_i, <_E)$? This is 2.5, the parallel of [Sh:g, Ch. II, 1.10, p. 12].

Claim 2.1: *If $\boxplus$ below holds then for some partition (Y_1, Y_2) of Y and club E of λ we have*

$\oplus$ (a) *if $Y_1 \in D^+$ and $f, g \in \cup\{\mathscr{F}_\alpha : \alpha \geq \min(E)\}$ then $f = g \bmod (D+Y_1)$,*
(b) *if $Y_2 \in D^+$ then $\langle \mathscr{F}_\alpha : \alpha \in E\rangle$ is $<_{D+Y_2}$-increasing,*

where

$\boxplus$ (a) *λ is regular $\geq \mathrm{hrtg}(\mathscr{P}(Y)))$,*
(b) *$\mathscr{F}_\alpha \subseteq {}^Y\mathrm{Ord}$ for $\alpha < \lambda$ is non-empty,*
(c) *D is an $\aleph_1$-complete filter on Y,*
(d) *if $\alpha_1 < \alpha_2 < \lambda$ and $f_\ell \in \mathscr{F}_{\alpha_\ell}$ for $\ell = 1, 2$ then $f_1 \leq f_2 \bmod D$.*

Proof. For $Z \in D^+$ let

$(*)_1$ (a) $S_Z = \{(\alpha, \beta) : \alpha \leq \beta < \lambda$ and for some $f \in \mathscr{F}_\alpha$ and $g \in \mathscr{F}_\beta$ we have $f < g \bmod (D+Z)\}$,
(b) $S^+_Z = \{(\alpha, \beta) : \alpha \leq \beta < \lambda$ and for every $f \in \mathscr{F}_\alpha$ and $g \in \mathscr{F}_\beta$ we have $f < g \bmod (D+Z)\}$.

Note

$(*)_2$ (a) if $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ and $(\alpha_2, \alpha_3) \in S_Z$ then $(\alpha_1, \alpha_4) \in S_Z$,
(b) similarly for S^+_Z,
(c) if $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ and $(\alpha_1 \neq \alpha_2) \wedge (\alpha_3 \neq \alpha_4)$ and $(\alpha_2, \alpha_3) \in S_Z$ then $(\alpha_1, \alpha_4) \in S^+_Z$,
(d) $S_Z \subseteq S^+_Z$.

[Why? By the definitions.]

Let

$(*)_3$ $J:=\{Z \subseteq Y : Z \in \mathrm{dual}(D)$ or $Z \in D^+$ and $(\forall \alpha < \lambda)(\exists \beta)((\alpha,\beta) \in S^+_Z)\}$.

Next

$(*)_4$ (a) J is an $\aleph_1$-complete ideal on Y,

(b) if D is κ-complete then J is κ-complete[1],

(c) $J = \{Z \subseteq Y : Z \in \mathrm{dual}(D)$ or $Z \in D^+$ and $(\forall \alpha < \lambda)(\exists \beta)((\alpha,\beta) \in S_Z)\}$.

[Why? For clauses (a), (b) check, and for clause (c) recall $(*)_2$(c).]

Let

$(*)_5$ (a) for $Z \in J^+$ let

$$\alpha(Z) = \min\{\alpha < \lambda : \text{ for no } \beta \in (\alpha,\lambda) \text{ do we have } (\alpha,\beta) \in S_Z\},$$

(b) $\alpha(*) = \sup\{\alpha_Z : Z \in J^+\}$.

$(*)_6$ (a) For $Z \in J^+$ we have $\alpha(Z) < \lambda$,

(b) $\alpha(*) < \lambda$.

[Why? Clause (a) by the definition of the ideal J, and clause (b) as $\lambda = \mathrm{cf}(\lambda) \geq \mathrm{hrtg}(\mathscr{P}(Y))$.]

Let

$(*)_7$ (a) for $Z \in D^+$ let $f_Z : \lambda \to \lambda + 1$ be defined by $f_Z(\alpha) = \min\{\beta : (\alpha,\beta) \in S^+_Z \text{ or } \beta = \lambda\}$,

(b) $f_* : \lambda \to \lambda$ be defined by $f_*(\alpha) = \sup\{f_Z(\alpha) : Z \in D^+ \cap J\}$,

(c) $E = \{\delta : \delta \text{ a limit ordinal} < \lambda \text{ such that } \alpha < \delta \Rightarrow f_*(\alpha) < \delta\} \backslash \alpha(*)$.

Hence

$(*)_8$ (a) if $Z \in D^+ \cap J$ then f_Z is indeed a function from λ to λ,

(b) f_* is indeed a function from λ to λ,

(c) f_* is non-decreasing,

(d) E is a club of λ.

[Why? Clause (a) by the definition of J and of f_*, and clause (b) as $\lambda = \mathrm{cf}(\lambda) \geq \mathrm{hrtg}(\mathscr{P}(Y))$, and clause (c) by $(*)_2$, and clause (d) follows from (b)+(c).]

$(*)_9$ Let $\alpha_0 = \min(E), \alpha_1 = \min(E \backslash (\alpha_0 + 1))$, choose $f_0 \in \mathscr{F}_{\alpha_0}, f_1 \in \mathscr{F}_{\alpha_1}$ and let $Y_1 = \{y \in Y : f_0(y) = f_1(y)\}$ and $Y_2 = Y \backslash Y_1$,

$(*)_{10}$ (Y_1, Y_2, E) are as required.

[Why? Think.] $\blacksquare_{2.1}$

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[1] Not used. Note that AC_κ holds in the non-trivial case as $\mathrm{AC}_{\mathscr{P}(Y)}$ holds; see 1.14.

Claim 2.2: *We have* $\lambda = \mathrm{ps} - \mathrm{tcf}(\Pi\bar{\alpha}_1, <_{D_1}) = \mathrm{ps} - \mathrm{tcf}(\Pi\bar{\alpha}, <_D)$, *which means also that one of them is well defined iff the other is, when*

(a) $\bar{\alpha} \in {}^Y\mathrm{Ord}$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(Y)$,
(b) E *is the equivalence relation on* Y *such that* $sEt \Leftrightarrow \alpha_s = \alpha_t$,
(c) D *is a filter on* X,
(d) $Y_1 = Y/E$,
(e) $D_1 = \{Z \subseteq Y/E : \bigcup\{X : X \in Z\} \in D\}$, *so a filter on* Y_1,
(f) $\bar{\alpha}_1 = \langle \alpha_{1,y_1} : y_1 \in Y_1 \rangle$ *where* $y_1 = y/E \Rightarrow \alpha_{1,y_1} = \alpha_y$.

Proof. First, the "only if" direction holds by 2.3.

Second, for the "if direction", assume that $\mathrm{ps} - \mathrm{pcf}(\Pi\bar{\alpha}_1, <_{D_1})$ is well defined and call it λ_1. Let $\langle \mathscr{F}_{1,\alpha} : \alpha < \lambda \rangle$ witness this, for $f \in \mathscr{F}_{1,\alpha}$ let $f^{[0]} \in {}^Y\mathrm{Ord}$ be defined by $f^{[0]}(s) = f(s/E)$ and let $\mathscr{F}_\alpha = \{f^{[0]} : f \in \mathscr{F}_{1,\alpha}\}$. It is easy to check that $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ witnesses $\lambda_1 = \mathrm{ps} - \mathrm{tcf}(\Pi\bar{\alpha}, <_D)$ recalling $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(Y)$ by clause (d), so we have proved also the "if" implication. $\blacksquare_{2.2}$

By the following claims we do not really lose by using $\mathfrak{a} \subseteq \mathrm{Reg}$ instead $\bar{\alpha} \in {}^Y\mathrm{Ord}$ as by 2.4 below, without loss of generality, $\alpha_t = \mathrm{cf}(\alpha_t)$ (when AC_Y) and by 2.2.

Claim 2.3: *Assume* $\bar{\alpha} \in {}^Y\mathrm{Ord}$, $D \in \mathrm{Fil}(Y)$ *and* $\lambda = \text{ps-pcf}(\Pi\bar{\alpha}, <_D)$, *so* λ *is regular and* $y \in Y \Rightarrow \alpha_y < \lambda$.

If $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ *witnesses* $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ *and* $y \in Y \Rightarrow \mathrm{cf}(\alpha_y) \geq \mathrm{hrtg}(Y)$ *and* $\lambda \geq \mathrm{hrtg}(Y)$ *then for some e:*

(a) $e \in \mathrm{eq}(Y) = \{e : e$ *an equivalence relation on* $Y\}$,
(b) *the sequence* $\mathscr{F}_e = \langle \mathscr{F}_{e,\alpha} : \alpha < \lambda \rangle$ *witnesses* $\text{ps-tcf}(\langle \alpha_{y/e} : y \in Y/e \rangle, D/e \rangle)$ *where*
(c) $\alpha_{y/e} = \alpha_y, D/e = \{A/e : A \in D\}$ *where* $A/e = \{y/e : y \in A\}$ *and* $\mathscr{F}_{e,\alpha} = \{f^{[*]} : f \in \mathscr{F}_\alpha\}, f^{[*]} : Y/e \to \mathrm{Ord}$ *is defined by* $f^{[*]}(t/e) = \sup\{f(s) : s \in t/e\}$, *noting* $\mathrm{hrtg}(Y/e) \leq \mathrm{hrtg}(Y)$,
(d) $e = \{(s_1, s_2) : \alpha_{s_1} = \alpha_{s_2}\}$.

Proof. Let $e = \mathrm{eq}(\bar{\alpha}) = \{(y_1, y_2) : y_1 \in Y, y_2 \in Y$ and $\alpha_{y_1} = \alpha_{y_2}\}$. For each $f \in \Pi\bar{\alpha}$ let the function $f^{[*]} \in \Pi\bar{\alpha}$ be defined by $f^{[*]}(y) = \sup\{f(z) : z \in y/e\}$. Clearly $f^{[*]}$ is a function from $\prod_{y \in Y}(\alpha_y + 1)$ and it belongs to $\Pi\bar{\alpha}$ as $y \in Y \Rightarrow \mathrm{cf}(\alpha_y) \geq \mathrm{hrtg}(Y)$. Let $H : \lambda \to \lambda$ be $H(\alpha) = \min\{\beta < \lambda : \beta > \alpha$ and there are $f_1 \in \mathscr{F}_\alpha$ and $f_2 \in \mathscr{F}_\beta$ such that $f_1^{[*]} < f_2 \bmod D\}$, well defined as $\mathscr{F}$ is cofinal in $(\Pi\bar{\alpha}, <_D)$. We choose $\alpha_i < \lambda$ by induction on i by $\alpha_i = \bigcup\{H(\alpha_j)+1 : j < i\}$. So $\alpha_0 = 0$ and $\langle \alpha_i : i < \lambda\rangle$ is increasing continuous. Let

$$\mathscr{F}'_i = \{f^{[*]} : f \in \mathscr{F}_{\alpha_i} \text{ and there is } g \in \mathscr{F}_{H(\alpha_i)} = \mathscr{F}_{\alpha_{i+1}-1} \text{ such that } f^{[*]} < g \bmod D\}.$$

So

$(*)_1$ $\mathscr{F}'_i \subseteq \{f \in \Pi\bar{\alpha} : \mathrm{eq}(\bar{\alpha}) \text{ refine } \mathrm{eq}(f)\}$.

[Why? By the choice of $\mathscr{F}'_i$ and of e.]

$(*)_2$ $\mathscr{F}'_i$ is non-empty.

[Why? By the choice of $H(\alpha_i)$.]

$(*)_3$ If $i(1) < i(2) < \lambda$ and $h_\ell \in \mathscr{F}'_{i_\ell}$ for $\ell = 1, 2$, then $h_1 < h_2 \mod D$.

[Why? For $\ell = 1, 2$ let $g_\ell \in \mathscr{F}_{H(\alpha_{i(\ell)})}$ be such that $h_\ell = f^{[*]}_\ell < g_\ell \mod D$ exists by the definition of $\mathscr{F}'_{i(\ell)}$. But $H(\alpha_{i(1)}) < \alpha_{i(1)+1} \le \alpha_{i(2)}$, hence $g_1 \le f_2$ mod D so together $h_1 = f^{[*]}_1 < g_1 \le f_2 \le f^{[*]}_2 = h_2 \mod D$, hence we are done.]

$(*)_4$ $\bigcup_{i<\lambda} \mathscr{F}'_i$ is cofinal in $(\Pi\bar{\alpha}, <_D)$.

[Easy, too.]

Lastly, let $\mathscr{F}^+_i = \{f/e : e \in \mathscr{F}'_i\}$ where $f/e \in {}^{Y/e}\mathrm{Ord}$ is defined by $(f/e)(y/e) = f(y)$, clearly well defined. $\blacksquare_{2.3}$

Claim 2.4: *Assume* AC_Y *and* $\bar{\alpha}_\ell = \langle \alpha^\ell_y : y \in Y\rangle \in {}^Y\mathrm{Ord}$ *for* $\ell = 1, 2$. *If* $y \in Y \Rightarrow \mathrm{cf}(\alpha^1_y) = \mathrm{cf}(\alpha^2_y)$ *then* $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}_1, <_D)$ *iff* $\lambda = \mathrm{ps} - \mathrm{tcf}(\Pi\bar{\alpha}_2, <_D)$.

Proof. Straightforward. $\blacksquare_{2.4}$

Now we come to the heart of the matter

Theorem 2.5 (The Composition Theorem): [*Assume* AC_Z *and* $\kappa \ge \aleph_0$.]

We have $\lambda = \text{ps-tcf}(\prod_{(i,j)\in Y} \lambda_{i,j}, <_D)$ *and* D *is a* κ*-complete filter on* Y *when:*

(a) E *is a* κ*-complete filter on* Z,
(b) $\langle \lambda_i : i \in Z\rangle$ *is a sequence of regular cardinals,*
(c) $\lambda = \text{ps-tcf}(\prod_{i\in Z} \lambda_i, <_E)$,
(d) $\bar{Y} = \langle Y_i : i \in Z\rangle$,
(e) $\bar{D} = \langle D_i : i \in Z\rangle$,
(f) D_i *is a* κ*-complete filter on* Y_i,

(g) $\bar{\lambda} = \langle \lambda_{i,j} : i \in Z, j \in Y_i \rangle$ *is a sequence of regular cardinals (or just limit ordinals),*

(h) $\lambda_i = \text{ps-tcf}(\prod_{j \in Y_i} \lambda_{i,j}, <_{D_i})$,

(i) $Y = \{(i,j) : j \in Y_i$ *and* $i \in Z\}$,

(j) $D = \{A \subseteq Y$: *for some* $B \in E$ *we have* $i \in B \Rightarrow \{j : (i,j) \in A\} \in D_i\}$.

Proof. $(*)_0$ D is a κ-complete filter on Y.

[Why? Straightforward (and we do not need any choice).]

Let $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i, i \in Z \rangle$ be such that

$(*)_1$ (a) $\bar{\mathscr{F}}_i = \langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i \rangle$ witnesses $\lambda_i = \text{ps} - \text{tcf}(\prod_{j \in Y_i} \lambda_{i,j}, <_{D_i})$,

(b) $\mathscr{F}_{i,\alpha} \neq \emptyset$.

[Why? Exists by clause (h) of the assumption and AC_Z; for clause (b) recall [Sh:938, 5.6].]

By clause (c) of the assumption let $\bar{\mathscr{G}}$ be such that

$(*)_2$ (a) $\bar{\mathscr{G}} = \langle \mathscr{G}_\beta : \beta < \lambda \rangle$ witnesses $\lambda = \text{ps-tcf}(\prod_{i \in Z} \lambda_i, <_E)$,

(b) $\mathscr{G}_\beta \neq \emptyset$ for $\beta < \lambda$.

Now for $\beta < \lambda$ let

$$(*)_3 \quad \mathscr{F}_\beta := \{f : f \in \prod_{(i,j) \in Y} \lambda_{i,j} \text{ and for some } g \in \mathscr{G}_\beta \text{ and } \bar{h} = \langle h_i : i \in Z \rangle \in \prod_{i \in Z} \mathscr{F}_{i,g(i)} \text{ we have } (i,j) \in Y \Rightarrow f((i,j)) = h_i(j)\}.$$

$(*)_4$ The sequence $\langle \mathscr{F}_\beta : \beta < \lambda \rangle$ is well defined (so exists).

[Why? Obviously.]

$(*)_5$ If $\beta_1 < \beta_2, f_1 \in \mathscr{F}_{\beta_1}$ and $f_2 \in \mathscr{F}_{\beta_2}$ then $f_1 <_D f_2$.

[Why? Let $g_\ell \in \mathscr{G}_{\beta_\ell}$ and $\bar{h}_\ell = \langle h_i^\ell : i \in Z \rangle \in \prod_{i \in Z} \mathscr{F}_{i,g_\ell(i)}$ witnesses $f_\ell \in \mathscr{F}_{\beta_\ell}$ for $\ell = 1,2$. As $\beta_1 < \beta_2$ by $(*)_2$ we have $B := \{i \in Z : g_1(i) < g_2(i)\} \in E$. For each $i \in B$ we know that $g_1(i) < g_2(i) < \lambda_i$ and $h_i^1 \in \mathscr{F}_{i,g_i(i)}, h_i^2 \in \mathscr{F}_{i,g_2(i)}$; hence recalling the choice of $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i \rangle$, see $(*)_1$, we have $A_i \in D_i$, where for every $i \in Z$ we let $A_i := \{j \in Y_i : h_i^1(j) < h_i^2(j)\}$. As $\bar{h}_1, \bar{h}_2$ exists clearly $\langle A_i : i \in Z \rangle$ exist, hence $A = \{(i,j) : i \in B$ and $j \in A_i\}$ is a well-defined subset of Y and it belongs to D by the definition of D.

Lastly, $(i,j) \in A \Rightarrow f_1((i,j)) < f_2((i,j))$, shown above; so by the definition of D we are done.]

$(*)_6$ For every $\beta < \lambda$ the set $\mathscr{F}_\beta$ is non-empty.

[Why? Recall $\mathscr{G}_\beta \neq \emptyset$ by $(*)_2$(b) and let $g \in \mathscr{G}_\beta$. As $\langle \mathscr{F}_{i,g(i)} : i \in Z\rangle$ is a sequence of non-empty sets (recalling $(*)_2$(b)), and we are assuming AC_Z, there is a sequence $\langle h_i : i \in Z\rangle \in \prod_{i\in Z} \mathscr{F}_{i,g(i)}$. Let f be the function with domain Y defined by $f((i,j)) = h_i(j)$; so $g, \bar{h}$ witness $f \in \mathscr{F}_\beta$, so $\mathscr{F}_\beta \neq \emptyset$ as required.]

- $(*)_7$ If $f_* \in \prod_{(i,j)\in Y} \lambda_{i,j}$ then for some $\beta < \lambda$ and $f \in \mathscr{F}_\beta$ we have $f_* < f \bmod D$.

[Why? We define $\bar{f} = \langle f_i^* : i \in Z\rangle$ as follows: f_i^* is the function with domain Y_i such that

$$j \in Y_i \Rightarrow f_i^*(j) = f((i,j)).$$

Clearly $\bar{f}$ is well defined and for each i, $f_i^* \in \prod_{j\in Y_i} \lambda_{i,j}$, hence by $(*)_1$(a) for some $\alpha < \lambda_i$ and $h \in \mathscr{F}_{i,\alpha}$ we have $f_i^* < h \bmod D_i$, and let α_i be the first such α, so $\langle \alpha_i : i \in Z\rangle$ exists.

By the choice of $\langle \mathscr{G}_\beta : \beta < \lambda\rangle$ there are $\beta < \lambda$ and $g \in \mathscr{G}_\beta$ such that $\langle \alpha_i : i \in Z\rangle < g \bmod E$, hence $A := \{i \in Z : \alpha_i < g(i)\}$ belongs to E. So $\langle \mathscr{F}_{i,g(i)} : i \in Z\rangle$ is a sequence of non-empty sets, hence recalling AC_Z there is a sequence $\bar{h} = \langle h_i : i \in Z\rangle \in \prod_{i\in Z} \mathscr{F}_{i,g(i)}$. By the property of $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i\rangle$, we have $i \in A \Rightarrow f_i^* < h_i \bmod D_i$.

Lastly, let $f \in \prod_{(i,j)\in Y} \lambda_{i,j}$ be defined by $f((i,j)) = h_i(j)$. Easily $g, \bar{h}$ witness that $f \in \mathscr{F}_\beta$, and by the definition of D, recalling $A \in E$ and the choice of $\bar{h}$ we have $f_* < f \bmod D$, so we are done.]

Together we are done proving the theorem. $\blacksquare_{2.5}$

Conclusion 2.6 (The pcf closure conclusion.): Assume $\mathrm{AC}_{\mathscr{P}(\mathfrak{a})}$. We have $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{c})$ when:

(a) $\mathfrak{a}$ is a set of regular cardinals,

(b) $\mathrm{hrtg}(\mathscr{P}(\mathfrak{a})) \leq \min(\mathfrak{a})$,

(c) $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$.

Proof. Assume $\lambda \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{c})$, hence there is E, an $\aleph_1$-complete filter on $\mathfrak{c}$, such that $\lambda = \text{ps-tcf}(\Pi\mathfrak{c}, <_E)$. As we have $\mathrm{AC}_{\mathscr{P}(\mathfrak{a})}$ by 1.2 (as the D there is unique) there is a sequence $\langle D_\theta : \theta \in \mathfrak{c}\rangle, D_\theta$ an $\aleph_1$-complete filter on $\mathfrak{a}$ such that $\theta = \text{ps-tcf}(\Pi\mathfrak{a}, <_{D_\theta})$; also by 1.7 there is a function h from $\mathscr{P}(\mathfrak{a})$ onto $\mathfrak{c}$. Let $E_1 = \{S \subseteq \mathscr{P}(\mathfrak{a}) : \{\theta \in \mathfrak{c} : h^{-1}\{\theta\} \subseteq S\} \in E\}$. By Claim 2.2, the "if" direction with $\mathscr{P}(Y)$ here standing for Y there, we have $\lambda = \text{ps-tcf}(\Pi\{h(A) : A \in \mathscr{P}(\mathfrak{a})\}, <_{E_1})$ and E_1 is an $\aleph_1$-complete filter on $\mathscr{P}(\mathfrak{a})$.

Now we apply Theorem 2.5 with E_1, $\langle D_{h(\mathfrak{b})} : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\rangle$, λ, $\langle h(\mathfrak{b}) : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\rangle$, $\langle \theta : \theta \in \mathfrak{a}\rangle$ here standing for $E, \langle D_i : i \in Z\rangle, \lambda, \langle \lambda_i : i \in Z\rangle, \langle \lambda_{i,j} : j \in Y_i\rangle$ for every $j \in Z$ (constant here). We get a filter D_1 on $Y = \{(\mathfrak{b}, \theta) : \mathfrak{b} \in \mathscr{P}(\mathfrak{a}), \theta \in \mathfrak{a}\rangle$ such that $\lambda = \text{ps} - \text{tcf}(\Pi\{\theta : (\mathfrak{b}, \theta) \in Y\}, <_{D_1})$.

Now $|Y| = |\mathscr{P}(\mathfrak{a})|$ as $\mathfrak{a}$ can be well ordered (hence $\aleph_0 \le |\mathfrak{a}|$ or $\mathfrak{a}$ finite and all is trivial), so applying 2.2 again we get an $\aleph_1$-complete filter D on $\mathfrak{a}$ such that $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_D)$, so we are done. $\blacksquare_{2.6}$

Definition 2.7: Cornsider a set $\mathfrak{a}$ of regular cardinals.

(1) We say $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ is a **generating sequence for** $\mathfrak{a}$ when:

- (α) $\mathfrak{b}_\lambda \subseteq \mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$,
- (β) $J_{<\lambda^+}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}$ for $\lambda \in \mathfrak{c}$ hence $(\beta)' \cup \mathfrak{c} = \text{ps} - \text{tcf}_{\aleph_1-comp}(\mathfrak{a})$ and λ a card then $J_{<\lambda}[\mathfrak{a}]$ is the $\aleph_1$-complete ideal on $\mathfrak{a}$ generated by
$$\{\mathfrak{b}_\theta : \theta \in \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a}) \text{ and } \theta < \lambda\}.$$

(2) We say $\bar{\mathscr{F}}$ is a **witness for** $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \subseteq \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})\rangle$ when:

- (α) $\bar{\mathscr{F}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \mathfrak{c}\rangle$,
- (β) $\bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ witnesses $\text{ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$.

(3) Above, $\bar{\mathfrak{b}}$ is **closed** when $\mathfrak{b}_\lambda = \mathfrak{a}\cap \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}_\lambda)$.

(3A) Above, $\bar{\mathfrak{b}}$ is **smooth** when $\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda$.

(4) We say above $\bar{\mathfrak{b}}$ is **full** when $\mathfrak{c} = \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$.

Remark 2.8: (1) Note that 1.7 gives sufficient conditions for the existence of a full $\bar{\mathfrak{b}}$ as in 2.7(1).

(2) Of course, Definition 2.7 is interesting particularly when $\mathfrak{a} = \text{ps-pcf}_{\aleph_1-\text{com}}(\mathfrak{a})$.

THEOREM 2.9: *Assume* $\text{AC}_\mathfrak{c}$ *and* $\text{AC}_{\mathscr{P}(\mathfrak{a})}$. *Then* $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{c})$ *has a full closed generating sequence for* $\aleph_1$*-complete filters (see below) when:*

- (a) $\mathfrak{a}$ *is a set of regular cardinals,*
- (b) $\text{hrtg}(\mathscr{P}(\mathfrak{a})) \le \min(\mathfrak{a})$,
- (c) $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$.

Proof of 2.9.

$(*)_1$ $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{com}}(\mathfrak{c})$.

[Why? By 2.6 using $\text{AC}_{\mathscr{P}(\mathfrak{a})}$.]

$(*)_2$ There is a generating sequence $\langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ for $\mathfrak{a}$.

[Why? By 1.7(3) using also $\mathrm{AC}_{\mathfrak{c}}$.]

$(*)_3$ Let $\mathfrak{b}^*_\lambda = \text{ ps-pcf}_{\aleph_1-\mathrm{com}}(\mathfrak{b}_\lambda)$ for $\lambda \in \mathfrak{c}$.

$(*)_4$ (a) $\bar{\mathfrak{b}}^* = \langle \mathfrak{b}^*_\lambda : \lambda \in \mathfrak{c}\rangle$ is well defined,

(b) $\mathfrak{b}_\lambda \subseteq \mathfrak{b}^*_\lambda \subseteq \mathfrak{c}$,

(c) $\mathfrak{b}^*_\lambda = \text{ps-pcf}_{\aleph_1-\mathrm{com}}(\mathfrak{b}^*_\lambda)$,

(d) $\lambda = \max(\mathfrak{b}^*_\lambda)$,

(e) $\lambda \notin \mathrm{pcf}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$.

[Why? First, $\bar{\mathfrak{b}}^*$ is well defined as $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ is well defined. Second, $\mathfrak{b}_\lambda \subseteq \mathfrak{b}^*_\lambda$ by the choice of $\mathfrak{b}^*_\lambda$ and $\mathfrak{b}^*_\lambda \subseteq \mathfrak{c}$ as $\mathfrak{b}_\lambda \subseteq \mathfrak{a}$, hence

$$\mathfrak{b}^*_\lambda = \text{ ps-pcf}_{\aleph_1-\mathrm{com}}(\mathfrak{b}^*_\lambda) \subseteq \text{ ps-pcf}_{\aleph_1-\mathrm{com}}(\mathfrak{c}) = \mathfrak{c},$$

the last equality by 2.6, and third, $\mathfrak{b}^*_\lambda = \text{ ps-pcf}_{\aleph_1-\mathrm{com}}(\mathfrak{b}^*_\lambda)$ by Conclusion 2.6; it is easy to check that its assumption holds recalling $\mathfrak{b}_\lambda \subseteq \mathfrak{a}$. Fourth, $\lambda \in \mathfrak{b}^*_\lambda$ as $J_{=\lambda}[\mathfrak{a}]$ witnesses $\lambda \in \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{com}}(\mathfrak{b}_\lambda) = \mathfrak{b}^*_\lambda$ and $\max(\mathfrak{b}^*_\lambda) = \lambda$ by $(*)_2$ recalling Definition 2.7.

Lastly, note that

$$\mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{a}) = \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{b}_\lambda) \cup \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{a}\backslash\mathfrak{b}_\lambda)$$

by 1.14(2), hence $\mu \in \mathfrak{c}\backslash\mathfrak{b}^*_\lambda \Rightarrow \mu \in \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{a}\backslash\mathfrak{b}_\lambda)$; so if $\lambda \in \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$ by 2.6 it follows that $\lambda \in \mathrm{pcf}(\mathfrak{a}\backslash\mathfrak{b}^*_\lambda)$, which contradicts 1.7(3), 1.5(2), so $\lambda \notin \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$, that is, clause (e) holds.]

We can now choose $\bar{\bar{\mathscr{F}}}$ such that

$(*)_5$ (a) $\bar{\bar{\mathscr{F}}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \mathfrak{c}\rangle$,

(b) $\bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$,

(c) $\bar{\mathscr{F}}_\lambda$ witnesses $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$,

(d) if $\lambda \in \mathfrak{a}, \alpha < \lambda$ and $f \in \mathscr{F}_{\lambda,\alpha}$ then $f(\lambda) = \alpha$.

[Why? For each λ there is such $\bar{\bar{\mathscr{F}}}$ as $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$. But we are assuming $\mathrm{AC}_{\mathfrak{c}}$ and for clause (d) it is easy; in fact it is enough to use $\mathrm{AC}_{\mathscr{P}(\mathfrak{a})}$ and h as in 2.6, getting $\langle \bar{\bar{\mathscr{F}}}_{\mathfrak{b}} : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\rangle, \bar{\bar{\mathscr{F}}}_{\mathfrak{b}}$ witness $h(\mathfrak{b}) = \text{ ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}}[\mathfrak{a}])$ and putting $\langle \bar{\bar{\mathscr{F}}}_{\mathfrak{b}} : \mathfrak{b} \in h^{-1}\{\lambda\}\rangle$ together for each $\lambda \in \mathfrak{c}$.]

$(*)_6$ (a) For $\lambda \in \mathfrak{c}$ and $f \in \Pi\mathfrak{b}_\lambda$ let $f^{[\lambda]} \in \Pi\mathfrak{b}^*_\lambda$ be defined by: $f^{[\lambda]}(\theta) = \min\{\alpha < \lambda$: for every $g \in \mathscr{F}_{\theta,\alpha}$ we have $f{\restriction}\mathfrak{b}_\lambda \le (g{\restriction}\mathfrak{b}_\lambda) \bmod J_{=\theta}[\mathfrak{b}_\lambda]\}$,

(b) for $\lambda \in \mathfrak{c}$ and $\alpha < \lambda$ let $\mathscr{F}^*_{\lambda,\alpha} = \{(f{\restriction}\mathfrak{b}_\lambda)^{[\lambda]} : f \in \mathscr{F}_{\lambda,\alpha}\}$.

Now

$(*)_7$ (a) $f^{[\lambda]}{\restriction}\mathfrak{a} \ge f$ for $f \in \Pi\mathfrak{b}_\lambda, \lambda \in \mathfrak{c}$,

(b) $\langle \mathscr{F}^*_{\lambda,\alpha} : \lambda \in \mathfrak{c}, \alpha < \lambda \rangle$ is well defined (hence exists),

(c) $\mathscr{F}^*_{\lambda,\alpha} \subseteq \Pi \mathfrak{b}^*_\lambda$.

[Why? Obvious.]

$(*)_8$ Let J_λ be the $\aleph_1$-complete ideal on $\mathfrak{b}^*_\lambda$ generated by $\{\mathfrak{b}^*_\theta \cap \mathfrak{b}^*_\lambda : \theta \in \mathfrak{c} \cap \lambda\}$.

$(*)_9$ $J_\lambda \subseteq J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{b}^*_\lambda]$.

[Why? As for $\theta_0, \dots, \theta_n, \dots \in \mathfrak{c} \cap \lambda$ by 1.14(2) we have

$$\begin{aligned} \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bigcup\{\mathfrak{b}^*_{\theta_n} : n < \omega\}) &= \bigcup\{\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}^*_{\theta_n}) : n < \omega\} \\ &= \bigcup\{\mathfrak{b}^*_{\theta_n} : n < \omega\} \in J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{c}]. \ ] \end{aligned}$$

$\odot_1$ If $\lambda \in \mathfrak{c}$ and $\alpha_1 < \alpha_2 < \lambda$ and $f_\ell \in \mathscr{F}_{\lambda,\alpha_\ell}$ for $\ell = 1, 2$ then $f_1^{[\lambda]} \leq f_2^{[\lambda]} \bmod J_\lambda$.

[Why? Let $\mathfrak{a}_* = \{\theta \in \mathfrak{b}_\lambda : f_1(\theta) \geq f_2(\theta)\}$, hence by the assumption on $\langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda \rangle$ we have $\mathfrak{a}_* \in J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{a}]$, hence we can find a sequence $\langle \theta_n : n < \mathbf{n} \leq \omega \rangle$ such that $\theta_n \in \mathfrak{c} \cap \lambda$ and $\mathfrak{a}_* \subseteq \mathfrak{b}_* := \bigcup\{\mathfrak{b}_{\theta_n} : n < \mathbf{n}\}$, hence $\mathfrak{c}_* :=$ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{a}_*) \subseteq \bigcup\{\mathfrak{b}^*_{\theta_n} : n < \mathbf{n}\} \in J_\lambda$. So it suffices to prove $f_1^{[\lambda]} \restriction (\mathfrak{b}^*_\lambda \backslash \mathfrak{c}_*) \leq f_2^{[\lambda]} \restriction (\mathfrak{b}^*_\lambda \backslash \mathfrak{c}_*)$, so let $\theta \in \mathfrak{b}^*_\lambda \backslash \bigcup_n \mathfrak{b}^*_{\theta_n}$. By $(*)_4$(d) we have $\theta \leq \lambda$, let $\alpha := f_2^{[\lambda]}(\theta)$, so by the definition of $f_2^{[\lambda]}(\theta)$ we have

$$(\forall g \in \mathscr{F}_{\theta,\alpha})((f_2 \restriction \mathfrak{b}_\lambda) \leq (g \restriction \mathfrak{b}_\lambda) \bmod J_{=\theta}[\mathfrak{b}_\lambda]).$$

But $\mathfrak{a}_* \subseteq \bigcup_n \mathfrak{b}_{\theta_n}$ and $n < \omega \Rightarrow \theta \notin \mathfrak{b}^*_{\theta_n} = \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}_{\theta_n})$, hence by 1.14(2) we have $\theta \notin \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bigcup_n \mathfrak{b}_{\theta_n})$, hence $\bigcup_n \mathfrak{b}_{\theta_n} \in \mathbf{J}^{\aleph_1-\text{comp}}_\theta[\mathfrak{b}_\lambda]$, hence $\mathfrak{a}_* \in J^{\aleph_1-\text{comp}}_{=\theta}[\mathfrak{b}_\lambda]$. So (first, inequality by the previous sentence and the choice of $\mathfrak{a}_*$, second, by the earlier sentence)

$$(f_1 \restriction \mathfrak{b}_\lambda) \leq (f_2 \restriction \mathfrak{b}_\lambda) \leq (g \restriction \mathfrak{b}_\lambda) \quad \bmod J^{\aleph_1-\text{comp}}_{=\theta}[\mathfrak{b}_\lambda],$$

hence by the definition of $f_1^{[\lambda]}$, $f_1^{[\lambda]}(\theta) \leq \alpha = f_2^{[\lambda]}(\theta)$. So we are done.]

$\odot_2$ If $\lambda \in \mathfrak{c}$ and $g \in \Pi \mathfrak{b}^*_\lambda$ then for some $\alpha < \lambda$ and $f \in \mathscr{F}_{\lambda,\alpha}$ we have $g < f \bmod J_\lambda$.

[Why? We choose $\langle h_\theta : \theta \in \mathfrak{b}^*_\lambda \rangle$ such that $h_\theta \in \mathscr{F}_{\theta,g(\theta)}$ for each $\theta \in \mathfrak{b}^*_\lambda$. Let $h_1 \in \Pi \mathfrak{b}^*_\lambda$ be defined by $h_1(\kappa) = \sup\{h_\theta^{[\lambda]}(\kappa) : \kappa \in \mathfrak{b}_\theta \text{ and } \theta \in \mathfrak{b}^*_\lambda\}$, hence there are $\alpha < \lambda$ and $h_2 \in \mathscr{F}_{\lambda,\alpha}$ such that $h_1 \leq h_2 \bmod J_{=\lambda}[\mathfrak{a}]$. Now $f := h_2^{[\lambda]} \in \Pi \mathfrak{b}^*_\lambda$ is as required, in particular $f \in \mathscr{F}^*_{\lambda,\alpha}$.]

$\odot_3$ The sequence $\langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda \rangle$ witness $\lambda = \text{ps}-\text{tcf}(\Pi \mathfrak{b}^*_\lambda, <_{J_\lambda})$.

[Why? In $(*)_7$(b), (c) $+ \odot_1 + \odot_2$.]

$\odot_4$ If $\lambda \in \mathfrak{c}$ then $J_{<\lambda} = J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{b}^*_\lambda]$.

[Why? By $(*)_4, (*)_8, (*)_9$ and $\odot_3$.]

So

$\odot_5$ $\bar{\mathfrak{b}}^* = \langle \mathfrak{b}^*_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\mathfrak{c}$.

[Why? By $\odot_4$, $(*)_7$ recalling that $\lambda \notin \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$ by $(*)_4$(e).] $\blacksquare_{2.9}$

Remark 2.10: Clearly $\bar{\mathfrak{b}}^*$ is full and closed, but what about smoooth? Is this necessary for generalizing [Sh:460]?

Discussion 2.11: Naturally the definition now of $\bar{\mathscr{F}}$ as in 2.7(2) for $\Pi\mathfrak{a}$ is more involved, where $\bar{\mathscr{F}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \text{ps-pcf}_{\kappa-\text{com}}(\mathfrak{a})\rangle, \bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ exemplifies ps-tcf$(\Pi\mathfrak{a}, J_{=\lambda}(\mathfrak{a}))$.

Claim 2.12: $[\text{DC} + \text{AC}_{<\kappa}]$ *Assume*

(a) $\mathfrak{a}$ *a set of regular cardinals,*
(b) κ *is regular* $> \aleph_0$,
(c) $\mathfrak{c} = \text{ps-pcf}_{\kappa-\text{comp}}(\mathfrak{a})$,
(d) $\min(\mathfrak{a})$ *is* $\geq \text{hrtg}(\mathscr{P}(\mathfrak{c}))$ *or at least* $\geq \text{hrtg}(\mathfrak{c})$,
(e) $\bar{\mathscr{F}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \mathfrak{c}\rangle$, $\bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ *witness*[2] $\lambda = \text{ps-tcf}(\Pi\gamma, <^{J^{\kappa-\text{comp}}}_{=\lambda}[\mathfrak{a}])$.

Then

$\boxplus$ *for every* $f \in \Pi\mathfrak{a}$ *for some* $g \in \Pi\mathfrak{c}$, *if* $g \leq g_1 \in \Pi\mathfrak{c}$ *and* $\bar{h} \in \Pi\{\mathscr{F}_{\lambda,g_1(\lambda)} : \lambda \in \mathfrak{c}\}$ *then* $(\exists\mathfrak{d} \in [\mathfrak{c}]^{<\kappa})(f < \sup\{h_\lambda : \lambda \in \mathfrak{d}\})$.

Proof. Let $f \in \Pi\mathfrak{a}$. For each $\lambda \in$ ps-pcf$_{\kappa-\text{com}}(\mathfrak{a})$ let

$$\alpha_{f,\lambda} = \min\{\alpha < \lambda : f < g \mod J_{=\lambda}[\mathfrak{a}] \text{ for every } g \in \mathscr{F}_{\lambda,\alpha}\},$$

so clearly each α_f is well defined hence $\bar{\alpha} = \langle \alpha_{f,\lambda} : \lambda \in \text{ps-pcf}_{\kappa-\text{com}}(\mathfrak{a})\rangle$ exists. So $g = \langle \alpha_{f,\lambda} : \lambda \in \mathfrak{c}\rangle \in \Pi\mathfrak{c}$ is well defined. Assume $g_1 \in \Pi\mathfrak{c}$ and $g \leq g_1$. Let $\langle h_\lambda : \lambda \in \mathfrak{c}\rangle$ be any sequence from $\prod_{\lambda\in\mathfrak{c}} \mathscr{F}_{\lambda,g_1(\lambda)}$; at least one exists when $\text{AC}_\mathfrak{c}$ holds. Let $\mathfrak{a}_{f,\lambda} = \{\theta \in \mathfrak{a} : f(\theta) < h_\lambda(\theta)\}$ so $\langle \mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{c}\rangle$ exists, and we claim that for some $\mathfrak{d} \in [\mathfrak{c}]^{<\kappa}$ we have $\mathfrak{a} = \bigcup\{\mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{d}\}$. Otherwise let J be the κ-complete ideal on $\mathfrak{a}$ generated by $\{\mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{c}\}$; it is a κ-complete ideal. So by [Sh:938, 5.9], applicable by our assumptions, there is a κ-complete ideal J_1 on $\mathfrak{a}$ extending J such that $\lambda_* =$ ps-tcf$(\Pi\mathfrak{a}, <_{J_1})$ is well defined. So $\lambda_* \in \mathfrak{c}$ and $\mathfrak{a}_{f,\lambda_*} \in J_1$, easy contradiction. $\blacksquare_{2.12}$

[2] So we are assuming it is well defined, now if $\text{AC}_{\mathscr{P}(Y)}$ such $\bar{\mathscr{F}}$ exists.

CLAIM 2.13: $[\mathrm{AC}_{\aleph_0}]$ *We can uniformly define a* $\aleph_0$*-continuous witness for* $\lambda = \mathrm{ps}-\mathrm{pcf}_{\kappa-\mathrm{comp}}(\bar{\Pi}\bar{\alpha}, <_D)$ *where:*

(a) $\bar{\alpha} \in {}^Y\mathrm{Ord}$,

(b) *each* α_t *is a limit ordinal with* $\mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(S)$,

(c) λ *is regular* $\geq \mathrm{hrtg}(S)$,

(d) $\bar{\bar{\mathscr{F}}}=\langle \bar{\mathscr{F}}_a : a \in S\rangle$ *satisfies: each* $\bar{\mathscr{F}}_a$ *is a witness for* $\lambda=\mathrm{pcf}_{\kappa-\mathrm{comp}}(\Pi\bar{\alpha},<_D)$,

(e) *if* $a \in S$ *then* $\bar{\mathscr{F}}_a$ *is* $\aleph_0$*-continuous and* $f_1, f_2 \in \mathscr{F}_{a,\alpha} \Rightarrow f_1 = f_2 \mod D$.

Proof. $(*)_0$ $\mathrm{hrtg}(S \times S)$ is $\leq \lambda$ and $\leq \mathrm{cf}(\alpha_t)$ for $t \in Y$.

[Why? As $\lambda, \mathrm{cf}(\alpha_t)$ are regular cardinals.]

For $a, b \in S$ let

$(*)_1$ (a) $E_{a,b} = \{\delta < \lambda :$if $\alpha < \delta$ then for some $\beta \in (\alpha, \delta)$ and $f_1 \in \mathscr{F}_{a,\alpha}, f_2 \in \mathscr{F}_{b,\beta}$ we have $f_1 < f_2 \mod D\}$,

(b) define $g_{a,b} : \lambda \to \lambda$ by

$$g_{a,b}(\alpha) = \min\{\beta < \lambda :\text{there are } f_1 \in \mathscr{F}_{a,\alpha} \text{ and } f_2 \in \mathscr{F}_{b,\beta} \text{ such that } f_1 < f_2 \mod D\},$$

$(*)_2$ $g_{a,b}$ is well defined.

[Why? As $\bar{\mathscr{F}}_b$ is cofinal in $(\Pi\bar{\alpha}, <_D)$.]

$(*)_3$ $g_{a,b}$ is non-decreasing.

[Why? As $\bar{\mathscr{F}}_a$ is $<_D$-increasing.]

Hence

$(*)_4$ $E_{a,b} = \{\delta < \lambda : \delta$ a limit ordinal and $(\forall \alpha < \delta)(g_{a,b}(\alpha) < \delta)\}$.

Also

$(*)_5$ $E_{a,b}$ is a club of λ.

[Why? By its definition, $E_{a,b}$ is a closed subset of λ and it is unbounded as $\mathrm{cf}(\lambda) = \lambda > \aleph_0$, because for every $\alpha < \lambda$ letting $\alpha_0 = \alpha, \alpha_{n+1} = g_{a,b}(\alpha_n)+1 < \lambda$ clearly $\beta := \bigcup\{\alpha_n : n < \omega\}$ is $< \lambda$ and $\gamma < \delta \Rightarrow (\exists n)(\gamma < \alpha_n) \Rightarrow (\exists n)(g_{a,b}(\gamma) < \alpha_{n+1}))$.]

$(*)_6$ Let $g : \lambda \to \lambda$ be $g(\alpha) = \sup\{g_{a,b}(\alpha) : a, b \in S\}$.

$(*)_7$ g is a (well-defined) non-decreasing function from λ to λ.

[Why? "Non-decreasing trivial", and it is "into λ" as $\mathrm{hrtg}(S \times S) \leq \lambda$, recalling $(*)_0$.]

$(*)_8$ $E = \bigcap\{E_{a,b} : a, b \in S\} = \{\delta < \lambda : (\forall \alpha < \delta)(g(\alpha) < \delta)\}$ is a club of λ.

[Why? Like $(*)_7$.]

$(*)_9$ Let $E_1 = \{\delta \in E : \mathrm{cf}(\delta) = \aleph_0\}$ so $E_1 \subseteq \lambda = \sup(\lambda), \mathrm{otp}(E_1) = \lambda$.

$(*)_{10}$ For $\delta \in E$ of cofinality $\aleph_0$ let

$$\mathscr{F}_\delta = \{\sup\{f_n : n < \omega\} : \text{for some } a \in S \text{ and } \bar\alpha = \langle \alpha_n : n < \omega\rangle \text{ increasing of cofinality } \aleph_0 \text{ we have } \langle f_n : n < \omega\rangle \in \prod_n \mathscr{F}_{a,\alpha_n}\rangle\}.$$

$(*)_{11}$ $\langle \mathscr{F}_\delta : \delta \in E_1\rangle$ is $<_D$-increasing cofinal in $(\Pi\bar\alpha, <_D)$, in particular $\mathscr{F}_\delta \neq \emptyset$.

[Why? $\mathscr{F}_\delta \neq \emptyset$ as $\delta \in E, \mathrm{cf}(\delta) = \aleph_0$ and $\mathrm{AC}_{\aleph_0}$.]

We can correct $\langle \mathscr{F}_\delta : \delta \in E_1\rangle$ to be $\aleph_0$-continuous easily (and as in [Sh:938, §5]). $\blacksquare_{2.13}$

Question 2.14: If 2.9 suffices to assume $\mathrm{AC}_{\mathscr{P}(\mathfrak{a})}$ (and omit $\mathrm{AC}_{\mathfrak{a}}$), we can conclude that $\mathfrak{c} = \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c})$ has a full closed generating sequence.

We may try to repeat the proof of 2.9, only in the proof of $(*)_5$ we use Claim 2.15 below.

CLAIM 2.15: *In 2.9 we can add "$\bar{\mathfrak{b}}$ is weakly smooth", which means $\theta \in \mathfrak{b}_\lambda \Rightarrow \theta \notin \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c}\backslash\mathfrak{b}_\lambda)$.*

Proof. Let $\langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ be a full closed generating sequence.

We choose $\mathfrak{b}^1_\lambda$ by induction on $\lambda \in \mathfrak{c}$ such that

$(*)_1$ (a) $J_{\le\lambda}[\mathfrak{a}] = J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^1_\lambda$,
(b) $\mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{b}^1_\lambda) = \mathfrak{b}^1_\lambda$,
(c) $\max(\mathfrak{b}^1_\lambda) = \lambda$,
(d) if $\theta \in \mathfrak{b}^1_\lambda$ then $\mathfrak{b}^1_\lambda \supseteq \mathfrak{b}^1_\theta \mod J = \theta[\mathfrak{a}]$, i.e., $\mathfrak{b}^1_\theta\backslash\mathfrak{b}_\lambda \in J^{\aleph_1-\mathrm{comp}}_{<\theta}[\mathfrak{a}]$.

Arriving at λ let $\mathfrak{d}_\lambda = \{\theta \in \mathfrak{b}_\lambda : \mathfrak{b}^1_\theta\backslash\mathfrak{b}^1_\lambda \notin J^{\aleph_1-\mathrm{comp}}_{<\theta}[\mathfrak{a}]\}, \mathfrak{d}^1_\lambda = \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{d}_\lambda)$.

Now

$(*)_2$ $\mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{d}_\lambda) \subseteq \mathfrak{b}_\lambda \cap \lambda$.

[Why? $\subseteq \mathfrak{b}_\lambda$ is obvious, if $\not\subseteq \lambda$ recall $\mathfrak{d}^1_\lambda = \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{d}_\lambda)$; now $\mathfrak{d}_\lambda \subseteq \mathfrak{b}_\lambda$ hence $\mathfrak{d}^1_\lambda \subseteq \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{b}_\lambda) \subseteq \lambda^+$. So the only problematic case is $\lambda \in \mathfrak{d}^1_\lambda = \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{d}_\lambda)$. But then, as $\mathfrak{d}_\lambda \subseteq \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c}\backslash\mathfrak{b}_\lambda)$ by the composition theorem we have $\lambda \in \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c}\backslash\mathfrak{b}_\lambda)$, contradicting an assumption on $\bar{\mathfrak{b}}$.]

$(*)_3$ There is a countable $\mathfrak{e}_\lambda \subseteq \mathfrak{d}^1_\lambda$ such that $\mathfrak{d}^1_\lambda \subseteq \bigcup\{\mathfrak{b}^1_\sigma : \sigma \in \mathfrak{e}\}$.

[Why? Should be clear.]

Lastly, let $\mathfrak{b}^1_\lambda = \bigcup\{\mathfrak{b}^1_\theta : \theta \in \mathfrak{e}_\lambda\} \cup \mathfrak{b}_\lambda$ and check. $\blacksquare_{2.15}$

3. Measuring reduced products

3(A). On ps-$\mathbf{T}_D(g)$. Now we consider some ways to measure the size of ${}^\kappa\mu/D$ and show that they essentially are equal; see Discussion 3.9 below.

Definition 3.1: Let $\bar\alpha = \langle \alpha_t : t \in Y\rangle \in {}^Y\mathrm{Ord}$ be such that $t \in Y \Rightarrow \alpha_t > 0$.

(1) For D a filter on Y let

$$\text{ps-}\mathbf{T}_D(\bar\alpha) = \sup\{\mathrm{hrtg}(\mathbf{F}) : \mathbf{F} \text{ is a family of non-empty subsets of } \Pi\bar\alpha \text{ such that for every } \mathscr{F}_1 \neq \mathscr{F}_2 \text{ from } \mathbf{F} \text{ we have } f_1 \in \mathscr{F}_1 \wedge f_2 \in \mathscr{F}_2 \Rightarrow f_1 \neq_D f_2\},$$

recalling $f_1 \neq_D f_2$ means $\{t \in Y : f_1(t) \neq f_2(t)\} \in D$.

(2) Let

$$\text{ps-}\mathbf{T}_{\kappa-\mathrm{comp}}(\bar\alpha) = \sup\{\mathrm{hrtg}(\mathbf{F}) : \text{for some } \kappa\text{-complete filter } D \text{ on } Y, \mathbf{F} \text{ is as above for } D\}.$$

(3) If we allow $\alpha_t = 0$ just replace $\Pi\bar\alpha$ by

$$\Pi^*\bar\alpha := \{f : f \in \prod_t(\alpha_t + 1) \text{ and } \{t : f(t) = \alpha_t\} = \emptyset \mod D\}.$$

Theorem 3.2: $[\mathrm{DC} + \mathrm{AC}_{\mathscr{P}(Y)}]$ *Assume that D is a κ-complete filter on Y and $\kappa > \aleph_0$ and $g \in {}^Y(\mathrm{Ord}\setminus\{0\})$; if g is constantly α we may write α. The following cardinals are equal or at least $\lambda_1, \lambda_2, \lambda_3$ are $\mathrm{Fil}^1_\kappa(D)$-almost equal which means: for $\ell_1, \ell_2 \in \{1,2,3\}$ we have $\lambda_{\ell_1} \leq^{\mathrm{sal}}_{\mathrm{Fil}^1_\kappa(D)} \lambda_{\ell_2}$, which means if $\alpha < \lambda_{\ell_1}$ then α is included in the union of S sets each of order type $< \lambda_{\ell_2}$:*

(a) $\lambda_1 = \sup\{|\mathrm{rk}_{D_1}(g)|^+ : D_1 \in \mathrm{Fil}^1_\kappa(D)\}$,

(b) $\lambda_2 = \sup\{\lambda^+ :$ *there are* $D_1 \in \mathrm{Fil}^1_\kappa(D)$ *and a* $<_{D_1}$ *-increasing sequence* $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ *such that* $\mathscr{F}_\alpha \subseteq \prod_{t\in Y} g(t)$ *is non-empty*$\}$,

(c) $\lambda_3 = \sup\{\mathrm{ps} - \mathbf{T}_{D_1}(g) : D_1 \in \mathrm{Fil}^1_\kappa(D)\}$.

Remark 3.3: (1) Recall that for D a κ-complete filter on Y we let

$$\mathrm{Fil}^1_\kappa(D) = \{E : E \text{ is a } \kappa\text{-complete filter on } Y \text{ extending } D\}.$$

(2) The conclusion gives slightly less than equality of $\lambda_1, \lambda_1, \lambda_3$.
(3) See 3.10(6) below; by it $\lambda_2 = \text{ps-Depth}^+({}^\kappa\mu, <_D)$, recalling 3.10(5).
(4) We may replace κ-complete by $(\leq Z)$-complete if $\aleph_0 \leq |Z|$.
(5) Compare with Definition 3.10.
(6) Note that those cardinals are $\leq \mathrm{hrtg}(\Pi^* g)$.

Proof. Stage A: $\lambda_1 \leq^{\mathrm{sal}}_{\mathrm{Fil}^1_\kappa(D)} \lambda_2, \lambda_3$.

Why? Let $\chi < \lambda_1$, so by clause (a) there is $D_1 \in \mathrm{Fil}^1_\kappa(D)$ such that $\mathrm{rk}_{D_1}(g) \geq \chi$. Let

$$X_{D_2} = \Big\{\alpha < \chi : \text{some } f \in \prod_{t\in Y} g(t) \text{ satisfies}^3\ D_2 = \mathrm{dual}(J[f, D_1]) \text{ and } \alpha = \mathrm{rk}_{D_1}(f)\Big\},$$

for any $D_2 \in \mathrm{Fil}^1_\kappa(D_1)$. By [Sh:938, 1.11(5)] we have $\chi = \bigcup\{X_{D_2} : D_2 \in \mathrm{Fil}^1_\kappa(D_1)\}$.

Now

$\odot$ $D_2 \in \mathrm{Fil}^1_\kappa(D_1) \Rightarrow |\mathrm{otp}(X_{D_2})| < \lambda_2, \lambda_3$; this is enough.

Why does this hold? Letting

$$\mathscr{F}_{D_2,i} = \{f \in {}^Y\mu : \mathrm{rk}_{D_1}(f) = i \text{ and } J[f, D_1] = \mathrm{dual}(D_2)\},$$

by [Sh:938, 1.11(2)] we have: $i < j \wedge i \in X_{D_2} \wedge j \in X_{D_2} \wedge f \in \mathscr{F}_{D_2,i} \wedge g \in \mathscr{F}_{D_2,j} \Rightarrow f < g \mod D_2$, so by the definitions of λ_2, λ_3 we have $\mathrm{otp}(X_{D_2}) < \lambda_2, \lambda_3$.

Stage B: $\lambda_2 \leq^{\mathrm{sal}}_{\mathrm{Fil}^1_\kappa(D)} \lambda_1, \lambda_3$, moreover $\lambda_2 \leq \lambda_1, \lambda_3$.

Why? Let $\chi < \lambda_2$ and let D_1 and $\langle \mathscr{F}_\alpha : \alpha < \chi\rangle$ exemplify $\chi < \lambda_2$. Let $\gamma_\alpha = \min\{\mathrm{rk}_{D_1}(f) : f \in \mathscr{F}_\alpha\}$ so easily $\alpha < \beta < \chi \Rightarrow \gamma_\alpha < \gamma_\beta$, hence $\mathrm{rk}_D(g) \geq \chi$. So $\chi < \lambda_1$ by the definition of λ_1, and as for $\chi < \lambda_3$ this holds by Definition 3.1(2) as $\alpha < \beta \wedge f \in \mathscr{F}_\alpha \wedge g \in \mathscr{F}_\beta \Rightarrow f < g \bmod D_1 \Rightarrow f \neq g \bmod D_1$ as $\chi^+ = \mathrm{hrtg}(\chi) \leq \lambda_3$.

Stage C: $\lambda_3 \leq^{\mathrm{sal}}_{\mathrm{Fil}^1_\kappa(D)} \lambda_1, \lambda_2$.

Why? Let $\chi < \lambda_3$. Let $\langle \mathscr{F}_\alpha : \alpha < \chi\rangle$ exemplify $\chi < \lambda_3$. For each $\alpha < \chi$ let $\mathbf{D}_\alpha = \{\mathrm{dual}(J[f, D]) : f \in \mathscr{F}_\alpha\}$, so a non-empty subset of $\mathrm{Fil}^1_\kappa(Y)$. Now for every $D_1 \in \mathbf{D}_* := \bigcup\{\mathbf{D}_\alpha : \alpha < \lambda\}$ let $X_{D_1} = \{\alpha < \chi : D_1 \in \mathbf{D}_\alpha\}$ and for $\alpha \in X_{D_1}$ let $\zeta_{D_1,\alpha} = \min\{\mathrm{rk}_D(f) : f \in \mathscr{F}_\alpha \text{ and } D_1 = \mathrm{dual}(J[f, D])\}$ and let

[3] Recall $\mathrm{dual}(J[f, D_1]) = \{X \subseteq Y : X \in D_1 \text{ or } \mathrm{rk}_{D_1+(X\setminus Y)}(f) > \mathrm{rk}_{D_1}(f)\}$.

$\mathscr{F}_{D_1,\alpha} = \{f \in \mathscr{F}_\alpha : D_1 = J[f,D]$ and $\mathrm{rk}_{D_1}(f) = \zeta_{D_1,\alpha}\}$ so a non-empty subset of $\mathscr{F}_\alpha$ and clearly $\langle(\zeta_{D_1,\alpha}, \mathscr{F}_{D_1,\alpha}) : \alpha \in X_{D_1}\rangle$ exists.

Now

(a) $\alpha \mapsto \zeta_{D_1,\alpha}$ is a one-to-one function with domain X_{D_1} for $D_1 \in \mathbf{D}_*$,
(b) $\chi = \bigcup\{X_{D_1} : D_1 \in \mathbf{D}_*\}$ noting $\mathbf{D}_* \subseteq \mathrm{Fil}^1_\kappa(D)$,
(c) for $D \in \mathbf{D}_*$, if $\alpha < \beta$ are from X_{D_1} and $\zeta_{D_1,\alpha} < \zeta_{D_1,\beta}, f \in \mathscr{F}_{D_1,\alpha}, g \in \mathscr{F}_{D_2,\beta}$ then $f < g$ mod D_1.

[Why? For clause (a), if $\alpha \neq \beta \in X_{\zeta_1}, f \in \mathscr{F}_{D_1,\alpha}, g \in \mathscr{F}_{D_1,\beta}$ then $f \neq g$ mod D, hence by [Sh:938, 1.11] we have $\zeta_{D_1,\alpha} \neq \zeta_{D_1,\beta}$. For clause (b), it follows by the choices of $\mathbf{D}_\alpha, X_{D_1}$. Lastly, clause (c) follows by [Sh:938, 1.11(2)].]

Hence (by clause (c))

(d) $\mathrm{otp}(X_{D_1})$ is $< \lambda_2$ and is $\leq\ \mathrm{rk}_{D_1}(g)$ for $D_1 \in \bigcup\{\mathbf{D}_\alpha : \alpha < \chi\} \subseteq \mathrm{Fil}^1_\kappa(D)$.

Together clause (d) shows that $D \in \mathbf{D}_* \Rightarrow |X_D| < \lambda_1, \lambda_2$ so, by clause (b), $\lambda_3 \leq^{\mathrm{sal}}_{\mathrm{Fil}^1_*(D)} \lambda_1, \lambda_2$, hence we are done. $\blacksquare_{3.2}$

Observation 3.4: If D is a filter on Y and $\bar{\alpha} \in {}^Y(\mathrm{Ord}\backslash\{0\})$ then

$$\begin{aligned}\mathrm{ps} - \mathbf{T}_D(\bar{\alpha}) = \sup\{\lambda^+ :\ & \text{there is a sequence } \langle\mathscr{F}_\alpha : \alpha < \lambda\rangle \\ & \text{such that } \mathscr{F}_\alpha \subseteq \Pi\bar{\alpha}, \mathscr{F}_\alpha \neq \emptyset \\ & \text{and } \alpha \neq \beta \wedge f_1 \in \mathscr{F}_\alpha \wedge f_2 \in \mathscr{F}_\beta \Rightarrow f_1 \neq_D f_2\}.\end{aligned}$$

Proof. Clearly the new definition gives a cardinal $\leq$ ps $- \mathbf{T}_D(\bar{\alpha})$. For the other inequality assume $\lambda <$ ps $- \mathbf{T}_D(\bar{\alpha})$, so there is $\mathbf{F}$ as there such that $\lambda < \mathrm{hrtg}(\mathbf{F})$. As $\lambda < \mathrm{hrtg}(\mathbf{F})$ there is a function h from $\mathbf{F}$ onto λ. For $\alpha < \lambda$ define $\mathscr{F}'_\alpha = \bigcup\{\mathscr{F} : \mathscr{F} \in \mathbf{F}$ and $h(\mathscr{F}) = \alpha\}$. So $\langle\mathscr{F}'_\alpha : \alpha < \lambda\rangle$ exists and is as required. $\blacksquare_{3.4}$

Concerning Theorem 3.2 we may wonder "when does λ_1, λ_2 being S-almost equal imply they are equal?". We consider a variant this time for sets (or powers, not just cardinals).

Definition 3.5: (1) We say **the power of $\mathscr{U}_1$ is S-almost smaller than the power of** $\mathscr{U}_2$, or write $|\mathscr{U}_1| \leq |\mathscr{U}_2|$ mod S or $|\mathscr{U}_1| \leq^{\mathrm{alm}}_S |\mathscr{U}_2|$ when: we can find a sequence $\langle u_{1,s} : s \in S\rangle$ such that $\mathscr{U}_1 = \bigcup\{u_{1,s} : s \in S\}$ and $s \in S \Rightarrow |\mathscr{U}_{1,s}| \leq |\mathscr{U}_2|$.

(2) We say the power $|\mathscr{U}_1|, |\mathscr{U}_2|$ are **S-almost equal** (or $|\mathscr{U}_1| = |\mathscr{U}_2|$ mod S or $|\mathscr{U}_1| =^{\mathrm{alm}}_S |\mathscr{U}_2|$) when $|\mathscr{U}_1| \leq^{\mathrm{alm}}_S |\mathscr{U}_2| \leq^{\mathrm{alm}}_S |\mathscr{U}_2|$.

(3) Let $|\mathscr{U}_1| \le^{\rm alm}_{<S} |\mathscr{U}_2|$ be defined naturally.

(4) In particular, this applies to cardinals.

(5) Let $|\mathscr{U}_1| <^{\rm alm}_S |\mathscr{U}_2|$ mean there is a sequence $\langle u_{1,s} : s \in S\rangle$ with union $\mathscr{U}_1$ such that $s \in S \Rightarrow |\mathscr{U}_s| < |\mathscr{U}_2|$.

(6) Let $|\mathscr{U}_1| \le^{\rm sal}_S |\mathscr{U}_2|$ mean that if $|\mathscr{U}| < |\mathscr{U}_1|$ then $|\mathscr{U}| <^{\rm alm}_S |\mathscr{U}_2|$.

Observation 3.6: (1) If $|\mathscr{U}_1| \le |\mathscr{U}_2|$ and $S \ne \emptyset$ then $|\mathscr{U}_1| \le^{\rm alm}_S |\mathscr{U}_2|$.

(2) If $\lambda_1 \le \lambda_2$ and $S \ne \emptyset$ then $\lambda_1 \le^{\rm sal}_S \lambda_2$.

(3) If $\lambda_2 = \lambda_1^+$ and $\mathrm{cf}(\lambda_2) < \mathrm{hrtg}(S)$ then the power of λ_2 is S-almost smaller than S.

Proof. Immediate. $\blacksquare_{3.6}$

Observation 3.7: (1) The cardinals λ_1, λ_2 are equal when $\lambda_1 =^{\rm alm}_S \lambda_2$ and $\mathrm{cf}(\lambda_1)$, $\mathrm{cf}(\lambda_2) \ge \mathrm{hrtg}(\mathscr{P}(S))$.

(2) The cardinals λ_1, λ_2 are equal when $\lambda_1 =^{\rm alm}_S \lambda_2$ and λ_1, λ_2 are limit cardinals $> \mathrm{hrtg}(\mathscr{P}(S))$.

(3) If $\lambda_1 \le^{\rm alm}_S \lambda_2$ and $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ then $\lambda_1 \le^{\rm alm}_{<\partial} \lambda_2$.

(4) If $\lambda_1 \le^{\rm alm}_{<\theta} \lambda_2$ and $\mathrm{cf}(\lambda_1) \ge \theta$ then $\lambda_1 \le \lambda_2$.

(5) If $\lambda_1 \le^{\rm alm}_{<\theta} \lambda_2$ and $\theta \le \lambda_2^+$ then $\lambda_1 \le \lambda_2^+$.

Proof. (1) Otherwise, let $\partial = \mathrm{hrtg}(\mathscr{P}(S))$, without loss of generality $\lambda_2 < \lambda_1$, and by part (3) we have $\lambda_1 \le^{\rm alm}_{<\partial} \lambda_2$, and by part (4) we have $\lambda_1 \le \lambda_2$, a contradiction.

(2) Otherwise, letting $\partial = \mathrm{hrtg}(\mathscr{P}(S))$, without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \le^{\rm alm}_{<\partial} \lambda_2$, but $\partial < \lambda_2$ is assumed and $\lambda_2^+ < \lambda_1$, as λ_2 is a limit cardinal, so together we get a contradiction to part (5).

(3) If $\langle u_s : S \in S\rangle$ witnesses $\lambda_1 \le^{\rm alm}_S \lambda_2$, let

$$w = \{\alpha < \lambda_1 : \text{ for no } \beta < \alpha \text{ do we have } (\forall s \in S)(\alpha \in u_s \equiv \beta \in u_s)\}$$

so clearly $|w| < \mathrm{hrtg}(\mathscr{P}(S)) = \theta$, and for $\alpha \in w$ let

$$v_\alpha = \{\beta < \lambda_1 : (\forall s \in S)(\alpha \in u_s \equiv \beta \in u_s)\}$$

so $\langle v_\alpha : \alpha \in w\rangle$ witnesses $\lambda_1 \le^{\rm alm}_w \lambda_2$, hence $\lambda_1 \le^{\rm alm}_{<\theta} \lambda_2$.

(4), (5) Let $\sigma < \theta$ be such that $\lambda_1 \le^{\rm alm}_\sigma \lambda_2$ and let $\langle u_\varepsilon : \varepsilon < \sigma\rangle$ witness $\lambda_1 \le^{\rm alm}_\sigma \lambda_2$, that is $|u_\varepsilon| \le \lambda_2$ for $\varepsilon < \sigma$ and $\bigcup\{u_\varepsilon : \varepsilon < \sigma\} = \lambda_1$.

For part (4), if $\lambda_2 < \lambda_1$, then we have $\varepsilon < \sigma \Rightarrow |u_\varepsilon| < \lambda_1$, but $\mathrm{cf}(\lambda_1) > \sigma$ hence $|\{\bigcup\{u_\varepsilon : \varepsilon < \sigma\}| < \lambda_1$, a contradiction.

For part (5), for $\varepsilon < \sigma$, let $u'_\varepsilon = u_\varepsilon \setminus \bigcup\{u_\zeta : \zeta < \varepsilon\}$ and so $\mathrm{otp}(u'_\varepsilon) \le \mathrm{otp}(u_\varepsilon) < |u_\varepsilon|^+ \le \lambda_2^+$, so easily $|\lambda_1| = |\bigcup\{u_\varepsilon : \varepsilon < \sigma\}| = |\bigcup\{u'_\varepsilon : \varepsilon < \sigma\}| \le \sigma \cdot \lambda_2^+ \le \lambda_2^+ \cdot \lambda_2^+ = \lambda_2^+$. $\blacksquare_{3.7}$

Similarly

Observation 3.8: (1) If $\lambda_1 <^{\mathrm{alm}}_S \lambda_2$ and $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ then $\lambda_1 <^{\mathrm{alm}}_{<\partial} \lambda_2$.

(2) If $\lambda_1 <^{\mathrm{alm}}_{<\theta} \lambda_2$ and $\mathrm{cf}(\lambda_1) \ge \theta$ then $\lambda_1 < \lambda_2$.

(3) If $\lambda_1 <^{\mathrm{alm}}_{<\theta} \lambda_2$ and $\theta \le \lambda_2^+$ then $\lambda_1 \le \lambda_2$.

(4) If $\lambda_1 \le^{\mathrm{sal}}_S \lambda_2$ and $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ then $\lambda_1 <^{\mathrm{sal}}_{<\partial} \lambda_2$.

(5) If $\lambda_1 \le^{\mathrm{sal}}_{<\theta} \lambda_2$ and $\partial \le \lambda_2^+, \theta < \lambda_2$ and $\mathrm{cf}(\lambda_2) \ge \theta$ then $\lambda_1 \le \lambda_2$.

(6) If $\lambda_1 \le^{\mathrm{sal}}_{<\theta} \lambda_2$ and $\theta \le \lambda_2^+$ then $\lambda_1 \le \lambda_2^+$.

Discussion 3.9: (1) We like to measure $({}^Y\mu)/D$ in some ways and show their equivalence, as was done in ZFC. Natural candidates are:

(A) $\mathrm{pp}_D(\mu)$: say of length of increasing sequence $\bar{P}$ (not $\bar{p}$!, i.e., sets) ordered by $<_D$,

(B) $\mathrm{pp}^+_Y(\mu) = \sup\{\mathrm{pp}^+_D(\mu) : D$ an $\aleph_1$-complete filter on $Y\}$,

(C) As in 3.1.

(2) We may measure ${}^Y\mu$ by considering all ∂-complete filters.

(3) We may be more lenient in defining "same cardinality". For instance:

(A) we define when sets have similar powers, say by divisions to $\mathscr{P}(\mathscr{P}(Y))$ sets, we measure $({}^Y\mu)/\approx_{\mathscr{P}(\mathscr{P}(Y))}$, where $\approx_B$ is the following equivalence relation on sets:

$X \approx_B Y$ when we can find sequences $\langle X_b : b \in B\rangle, \langle Y_b : b \in B\rangle$ such that:

(a) $X = \bigcup\{X_b : b \in B\}$,

(b) $Y = \bigcup\{Y_b : b \in B\}$,

(c) $|X_b| = |Y_b|$;

(B) we may demand more: the $\langle X_b : b \in B\rangle$ are pairwise disjoint and the $\langle Y_b : b \in B\rangle$ are pairwise disjoint;

(C) we may demand less: e.g.

(c)′ $|X_b| \le_* |Y_b| \le_* |X_b|$,

and/or

(c)$_*$ $(\forall b \in B)(\exists c \in B)(|X_b| \le |Y_c|)$ and $(\forall b \in B)(\exists c \in B)(|Y_b| \le |X_c|)$.

Note that some of the main results of [Sh:835] can be expressed this way.

(D) rk-sup$_{Y,\partial}(\mu)$ = rk-sup $\{\mathrm{rk}_D(\mu) : D$ is ∂-complete filters on $Y\}$;
(E) for each non-empty $X \subseteq {}^Y\mu$ let

$$\begin{aligned}\mathrm{sp}^1_\alpha(X) =&\{(D,J) : D \text{ an } \aleph_1\text{-complete filter on } Y,\\ &\qquad J = J[f,D], \alpha = \mathrm{rk}_D(f) \text{ and } f \in X\},\\ \mathrm{sp}_1(X) =&\bigcup\{\mathrm{sp}^1_\alpha(X) : \alpha\};\end{aligned}$$

(F) *question*: If $\{\mathrm{sp}(X_s) : s \in S\}$ is constant, can we bound J?
(G) X, Y are called **connected** when $\mathrm{sp}(X_1), \mathrm{sp}(X_2))$ are non-disjoint or equal.

(4) We hope to prove, at least sometimes, $\gamma := \Upsilon({}^Y\mu) \leq \mathrm{pp}_\kappa(\mu)$, that is, we like to imitate [Sh:835] without the choice axioms on ${}^\omega\mu$. So there is $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ witnessing $\gamma < \Upsilon({}^Y\mu)$. We define

$$u = u_{\bar f} = \{\alpha\colon \text{ there is no } \bar\beta \in {}^\omega\alpha \text{ such that } (\forall it \in Y)(f_\alpha(t) \in \{f_{\beta_n}(t) : n < \omega\})\}.$$

You may say that $u_{\bar f}$ is the set of $\alpha < \delta$ such that f_α is "really novel".

By DC this is O.K., i.e.,

$\boxplus_1$ for every $\alpha < \delta$ there is $\bar\beta \in {}^\omega(u_{\bar f} \cap \alpha)$ such that $(\forall t \in Y)(f_\alpha(t)) = \{f_{\beta_n}(t) : n < \omega\}$.

Next, for $\alpha \in u_{\bar f}$ we can define $D_{\bar f,\alpha}$, the $\aleph_1$-complete filter on Y generated by $\{\{t \in Y : f_\beta(t) = f_\alpha(t)\} : \beta < \alpha\}$. So clearly $\alpha \neq \beta \in u_{\bar f} \wedge D_{\bar f,\alpha} = D_{\bar f,\beta} \Rightarrow f_\alpha \neq_D f_\beta$. Now for each pair $\bar D = (D_1, D_2) \in \mathrm{Fil}^4_Y$ (i.e., for the $\aleph_1$-complete case) let $\Lambda_{\bar f,\bar D} = \{\alpha \in u_{\bar f} : D_{\bar f,\alpha} = D_1 \text{ and } J[f_\alpha, D_1]\} = \mathrm{dual}(D_2)$. So γ is the union of $\leq \mathscr{P}(\mathscr{P}(Y))$-sets (as $|Y| = |Y| \times |Y|$, well ordered).

So

$(*)_1$ $\gamma \leq \mathrm{hrtg}(({}^Y\omega \times {}^\omega(\mu))$,
$(*)_2$ u is the union of $\mathscr{P}(\mathscr{P}(\kappa))$-sets each of cardinality $< \mathrm{pp}^+_{Y,\aleph_1}(\mu)$.

(I) What about $\mathrm{hrtg}({}^\kappa\mu) <$ ps-pp$_{Y,\aleph_1}(\mu)$?

We are given $\langle \mathscr{F}_\alpha : \alpha < \kappa\rangle \neq F_\alpha \neq \emptyset, \mathscr{F}_\alpha \subseteq \mu, \alpha \neq \beta \Rightarrow \mathscr{F}_\alpha \cap \mathscr{F}_\beta = \emptyset$.

Easier: looking modulo a fix filter D,

$(*)_2$ for $D \in \mathrm{Fil}_{Y,\aleph_1}$, let $\mathscr{F}_{\alpha,D} = \{f \in \mathscr{F}_\alpha : \neg(\exists g \in \mathscr{F}_\alpha)(g <_D f)\}$.

Maybe we have somewhere a bound on the size of $\mathscr{F}_{\alpha,D}$.

3(B). Depth of reduced power of ordinals. Our intention has been to generalize a relative of [Sh:460], but actually we are closed to [Sh:513, §3]. So as there we use IND but unlike [Sh:938] rather than with rank we deal with depth.

Definition 3.10: (1) Let $\mathrm{suc}_X(\alpha)$ be the first ordinal β such that we cannot find a sequence $\langle \mathscr{U}_x : x \in X\rangle$ of subsets of β, each of order type $< \alpha$ such that $\beta = \bigcup\{\mathscr{U}_x : x \in X\}$.

(2) We define $\mathrm{suc}^{[\varepsilon]}_X(\alpha)$ by induction on ε naturally: if $\varepsilon = 0$ it is α, if $\varepsilon = \zeta+1$ it is $\mathrm{suc}_X(\mathrm{suc}^{[\zeta]}_X(\alpha))$, and if ε is a limit ordinal then it is $\bigcup\{\mathrm{suc}^{[\zeta]}_X(\alpha) : \zeta < \varepsilon\}$.

(3) For a quasi-order P let the **pseudo ordinal depth of** P, denoted by ps-o-Depth(P), be

$$\sup\{\gamma\colon \text{there is a } <_P\text{-increasing sequence } \langle X_\alpha\colon \alpha<\gamma\rangle \text{ of non-empty subsets of } P\}.$$

(4) o-Depth(P) is defined similarly, demanding $|X_\alpha| = 1$ for $\alpha < \gamma$.

(5) Omitting the "ordinal" means γ is replaced by $|\gamma|$; similarly in the other variants including omitting the letter o in ps-o-Depth.

(6) Let

$$\text{ps-o-Depth}^+(P) = \sup\{\gamma + 1 : \text{there is an increasing sequence } \langle X_\alpha : \alpha < \gamma\rangle \text{ of non-empty subsets of } P\}.$$

Similarly for the other variants, e.g. without o we use $|\gamma|^+$ instead of $\gamma + 1$ in the supremum.

(7) For D a filter on Y and $\bar\alpha \in {}^Y(\mathrm{Ord} \setminus \{0\})$ let

$$\text{ps-o-Depth}^+_D(\bar\alpha) = \text{ps-o-Depth}^+(\Pi\bar\alpha, <_D).$$

Similarly for the other variants and we may allow $\alpha_t = 0$ as in 3.1(3).

(8) Let ps-o-depth$^+_D(\bar\alpha)$ be the cardinality of ps-o-Depth$^+_D(\bar\alpha)$.

Remark 3.11: Note that 1.13 can be phrased using this definition.

Definition 3.12: (0) We say $\mathbf{x}$ is a **filter** ω**-sequence** when

$$\mathbf{x} = \langle (Y_n, D_n) : n < \omega\rangle = \langle Y_{\mathbf{x},n}, D_{\mathbf{x},n} : n < \omega\rangle$$

is such that D_n is a filter on Y_n for each $n < \omega$; we may omit Y_n as it is $\bigcup\{Y : Y \in D\}$ and may write D if $\bigwedge_n D_n = D$.

(1) Let $\mathrm{IND}(\mathbf{x}), \mathbf{x}$ has the **independence property**, mean that for every sequence $\bar{F} = \langle F_{m,n} : m < n < \omega \rangle$ from $\mathrm{alg}(\mathbf{x})$ (see below), there is $\bar{t} \in \prod_{n<\omega} Y_n$ such that $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{t}{\restriction}(m,n])$. Let $\mathrm{NIND}(\mathbf{x})$ be the negation.

(2) Let $\mathrm{alg}(\mathbf{x})$ be the set of sequences $\langle F_{n,m} : m < n < \omega \rangle$ such that $F_{m,n} : \prod_{\ell=m+1}^{n} Y_\ell \to \ \mathrm{dual}(D_n)$.

(3) We say $\mathbf{x}$ is **κ-complete** when each $D_{\mathbf{x},n}$ is a κ-complete filter.

Theorem 3.13: *Assume* $\mathrm{IND}(\mathbf{x})$, *where* $\mathbf{x} = \langle (Y_n, D_n) : n < \omega \rangle$ *is as in Definition 3.12,* D_n *is* κ_n*-complete,* $\kappa_n \geq \aleph_1$.

(1) $[\mathrm{DC} + \mathrm{AC}_{Y_n}$ *for* $n < \omega]$ *For every ordinal* ζ, *for infinitely many* n, $\text{ps-o-Depth}({}^{(Y_n)}\zeta, <_{D_n}) \leq \zeta$.

(2) [DC] *For every ordinal* ζ, *for infinitely many* n, $\text{o-Depth}({}^{(Y_n)}\zeta, <_{D_n}) \leq \zeta$; *equivalently there is no* $<_{D_n}$*-increasing sequence of length* $\zeta + 1$.

Remark 3.14: (0) Note that the present results are incomparable with [Sh:938, §4]—the loss is using depth instead of rank and possibly using "pseudo".

(1) [Assume $\mathrm{AC}_{\aleph_0}$] If, for every n, we have $\mathrm{rk}_{D_n}(\zeta) > \ \mathrm{suc}_{\mathrm{Fil}^1_\kappa(D_n)}(\zeta)$ then for some $D^1_n \in \ \mathrm{Fil}^1_{\aleph_1}(Y_n)$ for $n < \omega$ we have $\mathrm{NIND}(\langle Y_n, D^1_n) : n < \omega \rangle$. (Why? By [Sh:938, 5.9]). But we do not know much on the D^1_n's.

(2) This theorem applies to, e.g., $\zeta = \aleph_\omega, Y_n = \aleph_n, D_n = \ \mathrm{dual}(J^{\mathrm{bd}}_{\aleph_n})$. So even in ZFC, it tells us things not covered by [Sh:513, §3]. Note that Depth and pcf are closely connected but only for sequences of length $\geq \mathrm{hrtg}(\mathscr{P}(Y))$, and see 3.19 below.

(3) If we assume $\mathrm{IND}(\langle Y_{\eta(n)}, D_{\eta(n)} : n < \omega \rangle)$ for every increasing $\eta \in {}^\omega\omega$, which is quite reasonable, then in Theorem 3.13 we can strengthen the conclusion, replacing "for infinitely many n" by "for every $n < \omega$ large enough".

(4) Note that 3.13(2) is complimentary to [Sh:835].

Observation 3.15: (1) If $\mathbf{x}$ is a filter ω-sequence, $\mathbf{x}$ is $\aleph_1$-complete and $n_* < \omega$ and $\mathrm{IND}(\mathbf{x}{\restriction}[n_*, \omega)$ then $\mathrm{IND}(\mathbf{x})$.

(2) If $\mathbf{x}$ is a filter ω-sequence and $\mathrm{IND}(\mathbf{x})$ and $\eta \in {}^\omega\omega$ is increasing and $\mathbf{y} = \langle Y_{\mathbf{x},\eta(n)}, D_{\mathbf{x},\eta(n)} : n < \omega \rangle$ then $\mathbf{y}$ is a filter ω-sequence and $\mathrm{IND}(\mathbf{y})$.

Proof. (1) Let

$$\bar{F} = \langle F_{n,m} : n < m < \omega \rangle \in \mathrm{alg}(\mathbf{x}),$$

so $\langle F_{n,m} : n \in [n_*, \omega)$ and $m \in (n, \omega) \rangle$ belongs to $\mathrm{alg}(\mathbf{x}{\restriction}[n_*, \omega)$, hence by the assumption "$\mathrm{IND}(\mathbf{x}{\restriction}[n_*, \omega))$" there is $\bar{t} = \langle t_n : n \in [n_*, \omega) \rangle \in \prod_{n \geq n_*} Y_n$ such that $t_n \notin F_{n,m}(\bar{t}{\restriction}(n,m))$ when $n_* \le n < \omega$. Now by downward induction on $n < n_*$ we choose $t_n \in Y_n$ such that $t_n \notin F_{n,m}(\langle \bar{t}{\restriction}[n+1,m])$ for $m \in [n+1,\omega)$. This is possible as the countable union of members of $\mathrm{dual}(D_n)$ is not equal to Y_n. We can carry the induction and $\langle t_n : n < \omega\rangle$ is as required to verify $\mathrm{IND}(\mathbf{x})$.

(2) Let $\bar{F} = \langle F_{i,j} : i < j < \omega\rangle \in \mathrm{alg}(\mathbf{y})$. For $m < n$ we define $F'_{m,n}$ as the following function from $\prod_{k=m-1}^{n} Y_{\mathbf{x},k}$ into $\mathrm{dual}(D_{\mathbf{x},m})$ by

- if $i < j, m = \eta(i), n = \eta(j)$ and $\bar{s} = \langle s_k : k \in (m,n]\rangle \in \prod_{k=m+1}^{n} Y_{\mathbf{x},k}$ then $F'_{m,n}(\bar{s}) = F_{i,j}(\langle s_{\eta(i+k)} : k \in [1, j-i)]\rangle)$,
- if there are no such i, j then $F_{m,n}$ is constantly $\emptyset$.

As $\mathrm{IND}(\mathbf{x})$ holds there is $\bar{t} \in \prod_n Y_{\mathbf{x},k}$ such that $m < n \Rightarrow t_m \notin F_{m,n}(\bar{t}{\restriction}(m,n))$. Now $\bar{t}' = \langle t_{\eta(k)} : k < \omega\rangle \in \prod_n Y_{\mathbf{x},\eta(n)} = \prod_n Y_{\mathbf{y},n}$ is necessarily as required. $\blacksquare_{3.15}$

Proof of Theorem 3.13. We concentrate on proving part (1); part (2) is easier, (i.e., below each $\mathscr{F}_{n,\varepsilon}$ is a singleton, hence so is $\mathscr{G}^1_{m,n,\varepsilon}$, so there is no need to use AC_{Y_n}).

Assume this fails. So for some $n_* < \omega$ for every $n \in [n_*,\omega)$ there is a counterexample. As $\mathrm{AC}_{\aleph_0}$ holds we can find a sequence $\langle \bar{\mathscr{F}}_n : n \in [n_*,\omega)\rangle$ such that:

$\odot$ for $n \in [n_*,\omega)$
  (a) $\bar{\mathscr{F}}_n = \langle \mathscr{F}_{n,\varepsilon} : \varepsilon \le \zeta\rangle$,
  (b) $\mathscr{F}_{n,\varepsilon} \subseteq {}^{Y_n}\zeta$ is non-empty,
  (c) $\bar{\mathscr{F}}_n$ is a $<_{D_n}$-increasing sequence of sets, i.e., $\varepsilon_1 < \varepsilon_2 \le \zeta \wedge f_1 \in \mathscr{F}_{n,\varepsilon_1} \wedge f_2 \in \mathscr{F}_{n,\varepsilon_2} \Rightarrow f_1 <_{D_n} f_2$.

Now by $\mathrm{AC}_{\aleph_0}$ we can choose $\langle f_n : n \in [n_*,\omega)\rangle$ such that $f_n \in \mathscr{F}_{n,\zeta}$ for $n \in [n_*,\omega)$.

$(*)$ Without loss of generality $n_* = 0$.

[Why? As $\mathbf{x}{\restriction}[n_*,\omega)$ satisfies the assumptions on $\mathbf{x}$ by 3.15(2).]

Now

$\boxplus_1$ for $m \le n < \omega$ let $Y^0_{m,n} = \prod_{\ell=m}^{n-1} Y_\ell$ and for $m,n < \omega$ let $Y^1_{m,n} := \bigcup\{Y^0_{k,n} : k \in [m,n]\}$, so $Y^0_{m,n} = \emptyset = Y^1_{m,n}$ if $m > n$ and $Y^0_{m,n} = \{\langle\rangle\} = Y^1_{m,n}$ if $m = n$; so if $\eta \in Y^0_{m+1,n}$ and $s \in Y_m, t \in Y_{n+1}$ we define $\langle s\rangle{}^\frown\eta \in Y^0_{m,n}$ and $\eta{}^\frown\langle t\rangle \in Y_{m+1,n+1}$ naturally;

$\boxplus_2$ for $m \le n$ let $\mathscr{G}^1_{m,n}$ be the set of functions g such that:
  (a) g is a function from $Y^1_{m,n}$ into $\zeta+1$,
  (b) $\langle\rangle \ne \eta \in Y^1_{m,n} \Rightarrow g(\eta) < \zeta$,

(c) if $k \in [m, n)$ and $\eta \in Y^0_{k+1,n}$ then the sequence $\langle g(\langle s\rangle \hat{}\ \eta) : s \in Y_k\rangle$ belongs to $\mathscr{F}_{k,g(\eta)}$;

$\boxplus_3$ $\mathscr{G}^1_{m,n,\varepsilon} := \{g \in \mathscr{G}^1_{m,n} : g(\langle\rangle) = \varepsilon\}$ for $\varepsilon \leq \zeta$ and $m \leq n < \omega$.

Now the sets $\mathscr{G}^1_{m,n}$ are non-trivial, i.e.

$\boxplus_4$ if $m \leq n$ and $\varepsilon \leq \zeta$ then $\mathscr{G}^1_{m,n,\varepsilon} \neq \emptyset$.

[Why? We prove it by induction on n: First, if $n = m$ this is trivial because the unique function g with domain $\{\langle\rangle\}$ and value ε belongs to $\mathscr{G}^1_{m,n,\varepsilon}$. Next, if $m < n$ we choose $f \in \mathscr{F}_{n-1,\varepsilon}$, hence the sequence $\langle \mathscr{G}^1_{m,n-1,f(s)} : s \in Y_{n-1}\rangle$ is well defined and by the induction hypothesis each set in the sequence is non-empty. As $\mathrm{AC}_{Y_{n-1}}$ holds there is a sequence $\langle g_s : s \in Y_{n-1}\rangle$ such that $s \in Y_{n-1} \Rightarrow g_s \in \mathscr{G}^1_{m,n-1,f(s)}$. Now define g as the function with domain $Y^1_{m,n}$:

$$\begin{aligned} g(\langle\rangle) =& \varepsilon, \\ g(\nu \hat{}\ \langle s\rangle) =& g_s(\nu) \quad \text{for } \nu \in Y^1_{m,n-1} \text{ and } s \in Y_n. \end{aligned}$$

It is easy to check that $g \in \mathscr{G}^1_{m,n,\varepsilon}$, indeed so $\boxplus_4$ holds.]

$\boxplus_5$ If $g, h \in \mathscr{G}^1_{m,n}$ and $g(\langle\rangle) < h(\langle\rangle)$ then there is an (m,n)-witness Z for (h, g) which means (just being an (m,n)-witness means we omit clause (d)):

(a) $Z \subseteq Y^1_{m,n}$ is closed under initial segments, i.e., if $\eta \in Y^0_{k,n} \cap Z$ and $m \leq k < \ell \leq n$ then $\eta{\restriction}[\ell, n) \in Y^0_{\ell,n} \cap Z$,

(b) $\langle\rangle \in Z$,

(c) if $\eta \in Z \cap Y^0_{k+1,n}, m \leq k < n$ then $\{s \in Y_k : \langle s\rangle \hat{}\ \eta \in Z\} \in D_k$,

(d) if $\eta \in Z$ then $g(\eta) < h(\eta)$.

[Why? By induction on n, similarly to the proof of $\boxplus_4$.]

$\boxplus_6$ (a) We can find $\bar{g} = \langle g_n : n < \omega\rangle$ such that $g_n \in \mathscr{G}^1_{0,n,\zeta}$ for $n < \omega$;

(b) for $\bar{g}$ as above for $n < \omega, s \in Y_n$ let $g_{n+1,s} \in \mathscr{G}^1_{0,n}$ be defined by $g_{n+1,s}(\nu) = g_{n+1}(\nu \hat{}\ \langle s\rangle)$ for $\nu \in Y_{0,n}$.

[Why? Clause (a) by $\boxplus_4$ as $\mathrm{AC}_{\aleph_0}$ holds, clause (b) is obvious by the definitions in $\boxplus_2 + \boxplus_3$.]

We fix $\bar{g}$ as in $\boxplus_6$(a) for the rest of the proof.

$\boxplus_7$ There is $\langle\langle Z_{n,s} : s \in Y_n\rangle : n < \omega\rangle$ such that $Z_{n,s}$ witness $(g_n, g_{n+1,s})$ for $n < \omega, s \in Y_n$.

[Why? For a given $n < \omega, s \in Y_n$ we know that $g_{n+1}(\langle s\rangle) < \zeta = g_n(\langle\rangle)$, hence $Z_{n,s}$ as required exists by $\boxplus_5$. By AC_{Y_n} for each n a sequence $\langle Z_{n,s} : s \in Y_n\rangle$ as required exists, and by $\mathrm{AC}_{\aleph_0}$ we are done.]

$\boxplus_8$ $Z_n := \{\langle\rangle\} \cup \{\nu\hat{}\langle s\rangle : s \in Y_{n-1}, \nu \in Z_{n-1,s}\}$ is a $(0, n)$-witness.

[Why? By our definitions.]

$\boxplus_9$ There is $\bar{F}$ such that:

(a) $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle$,

(b) $F_{m,n} : Y^1_{m+1,n+1} \to \mathrm{dual}(D_m)$,

(c) $F_{m,n}(\nu)$ is $\{s \in Y_m : \nu\hat{}\langle s\rangle \notin Z_{n-1}\}$ when $\nu \in Z_n$ and is $\emptyset$ otherwise.

[Why? As clauses (a), (b), (c) define $\bar{F}$.]

$\boxplus_{10}$ $\bar{F}$ witnesses $\mathrm{IND}(\langle(Y_n, D_n) : n < \omega\rangle)$ fails.

Why? Clearly $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle$ has the right form.

So toward a contradiction assume $\bar{t} = \langle t_n : n < \omega\rangle \in \prod_{n<\omega} Y_n$ is such that

$(*)_1$ $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{t}\restriction(m, n])$.

Now

$(*)_2$ $\bar{t}\restriction[m, n) \in Z_n$ for $m \le n < \omega$.

[Why? For each n, we prove this by downward induction on m. If $m = n$ then $\bar{t}\restriction[m, n) = \langle\rangle$, but $\langle\rangle \in Z_n$ by its definition. If $m < n$ and $\bar{t}\restriction[m+1, n) \in Z_n$ then $t_m \notin F_{m,n-1}(\bar{t}\restriction(m, n])$ by $(*)_1$, so $\bar{t}\restriction[m, n) = \langle t_m\rangle\hat{}(\bar{t}\restriction[m + 1, n)) \in Z_n$ holds by clause $\boxplus_9$(c).]

$(*)_3$ $g_{n+1}(\bar{t}\restriction[m, n]) < g_n(\bar{t}\restriction[m, n))$.

[Why? Note that Z_{n,t_n} is a witness for (g_n, g_{n+1,t_n}) by $\boxplus_7$. So by $\boxplus_5$ (see clause (d) there) we have $\eta \in Z_{n,t_n} \Rightarrow g_{n+1,t_n}(\eta) < g_n(\eta)$. But $m < n \Rightarrow \bar{t}\restriction[m, n] \in Z_{n+1} \Rightarrow \bar{t}\restriction[m, n) \in Z_{n,t_n}$, the first implication by $(*)_2$, the second implication by the definition of Z_{n+1} in $\boxplus_8$. Hence by $\boxplus_6$(b) and the last sentence, and by the sentence before last $g_{n+1}(\bar{t}\restriction[m, n]) = g_{n+1,t_n}(\bar{t}\restriction[m, n)) < g_n(\bar{t}\restriction[m, n))$ as required. So $(*)_3$ holds indeed.]

So for each $m < \omega$ the sequence $\langle g_n(\bar{t}\restriction[m, n) : n \in [m, \omega)\rangle$ is a decreasing sequence of ordinals, a contradiction. Hence there is no $\bar{t}$ as above, so indeed $\boxplus_{10}$ holds. But $\boxplus_{10}$ contradicts an assumption, so we are done. $\blacksquare_{3.13}$

Remark 3.16: (1) Note that in the proof of 3.13 there was no use of completeness demands; it is still natural to assume $\aleph_1$-completeness because: if D'_n is the $\aleph_1$-completion of D_n then $\mathrm{IND}(\langle D'_n : n < \omega\rangle)$ is equivalent to $\mathrm{IND}(D_n : n < \omega)$.

(2) Recall that by [Sh:513, 2.7], iff $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then for every $\lambda > \aleph_\omega$ for infinitely many $n < \omega$ we have $(\forall \mu < \lambda)(\mathrm{cf}(\mu) = \aleph_n \Rightarrow \mathrm{pp}(\mu) \leq \lambda)$.

(3) Concerning 3.17 below recall that:

(A) if Y_n is a regular cardinal, D_n witnesses Y_n is a measurable cardinal, then clause (a) of 3.17 holds, but [Sh:938, §4] says more:

(B) if $\mu = \mu^{<\mu}$ and $\mathbb{P}_\mu$ is the Levy collapse, of a measurable λ cardinal to be μ^+ with D a normal ultrafilter on λ, then $\Vdash_{\mathbb{P}_\mu}$ "the filter which D generates is as required in (b) with μ in the role of Z_n", by Jech–Magidor–Mitchell–Prikry [JMMP80].

So we can force that $n < \omega \Rightarrow Y_n = \aleph_{2n}$.

(4) So

(a) if $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1}$ and $\aleph_\omega$ divides $\delta, \mathrm{cf}(\delta) < \aleph_\omega$ and $\delta < \aleph_\delta$, then $\mathrm{pp}(\aleph_\delta) < \aleph_{|\delta|^+}$,

(b) we can replace $\aleph_\omega$ by any singular $\mu < \aleph_\mu$,

(c) if, e.g., $\delta_n < \lambda_n = \aleph_{\delta_n}, \delta_n < \delta_{n+1}$ and $\mathrm{cf}(\delta_n) < \lambda_0$ for $n < w$, then, except at most one n, $\mathrm{pp}(\aleph_{\lambda_n}) \geq \aleph_{\lambda_n^+}$.

(5) We had thought that maybe: (in ZFC) if μ is singular and $\mathrm{pp}(\mu) > \aleph_{\mu^+}$ then some case of IND follows, because by [Sh:513, 2.7] this holds if $\mu < \aleph_{\mu^+}$ (even getting $\mathrm{IND}(\langle \mathrm{dual}(J^{\mathrm{bd}}_{\lambda_n}) : n < \omega\rangle)$. Moreover for some increasing sequence $\langle \lambda_n : n < \omega\rangle$ of regular cardinals $< \mu$ (with limit μ if $\mathrm{cf}(\mu) = \aleph_0$ and $\subseteq \{\lambda^+ : \lambda \in E\}$ for any club E of μ if $\mathrm{cf}(\mu) > \aleph_0$) provided that $\mu = \aleph_\delta \wedge |\delta|^{\aleph_0} < \mu$; see [Sh:513, 2.8]).

Claim 3.17: [DC] *For* $\mathbf{x} = \langle Y_n, D_n : n < \omega\rangle$ *with each* D_n *being an* $\aleph_1$*-complete filter on* Y_n*, each of the following is a sufficient condition for* $\mathrm{IND}(\mathbf{x})$*, letting* $Y(< n) := \prod_{m<n} Y_m$ *and for* $m < n$*, let*

$$Z_{m,n} = \left\{ t : t \text{ is a function from } \prod_{\ell=m+1}^{n-1} Y_\ell \text{ into } Y_m \right\}$$

and let $Z_n = \prod_{m<n} Z_{m,n}$*:*

(a) D_n *is a* $(\leq Z_n)$*-complete ultrafilter,*

(b) • D_n *is a* $(\leq Z_n)$*-complete filter,*

- *for each n in the following game $\partial_{\mathbf{x},n}$ the non-empty player has a winning strategy. A play lasts ω-moves. In the k-th move the empty player chooses $A_k \in D_n$ and $\langle X^k_t : t \in Z_n\rangle$, a partition of A_k and the non-empty player chooses $t_k \in Z_n$. In the end the non-empty player wins the play if $\bigcap_{k<\omega} X^k_{t_k}$ is non-empty,*

(c) *like clause* (b) *but in the second part the non-empty player instead of t_k chooses $S_k \subseteq Z_n$ satisfying $|S_k| \le_X |S|$ and every $D_{\mathbf{x},n}$ is $(\le S)$-complete, S is infinite,*

(d) *if $m < n < \omega$ then D_m is $(\le \prod^n_{k=m+1} Y_k)$-complete*[4].

Proof. Straightforward; for instance:

Clause (b):

Let $\langle\, \mathbf{st}_n : n < \omega\rangle$ be such that $\mathbf{st}_n$ is a winning strategy of the non-empty player in the game $\partial_{\mathbf{x},n}$.

Let $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle \in \mathrm{alg}(\mathbf{x})$ and we should find a member of $\prod_n Y_n$ as required in Definition 3.12(2). We now, by induction on $i < \omega$, choose the following objects satisfying the following condition:

$(*)_i$ (a) for $k < m$ and $j < i$, $G_{j,k,m}$ is a function from $\prod^m_{\ell=k+1} Y_\ell$ into Y_k;

(b) (α) for $m < \omega$, $\langle(\bar{X}_{j,m}, \mathbf{t}_{j,m}) : j < i\rangle$ is an initial segment of a play of the game $\partial_{\mathbf{x},m}$ in which the non-empty player uses the strategy $\mathbf{st}_m$;

(β) we have $\bar{X}_{j,m} = \langle X_{j,m,\mathbf{t}} : \mathbf{t} \in Z_m\rangle$ and $\mathbf{t}_{j,m} = \langle t_{j,k,m} : k < m\rangle$ and $t_{j,k,m} \in Z_k$ and we have $X_{j,m,\mathbf{t}} = \bigcap_{k<m} X_{j,k,m,t_k}$ when $\mathbf{t} = \langle t_k : k < m\rangle, \bigwedge_k t_k \in Z_{k,m}$;

(c) (α) $Y_{j,m}$ is Y_m if $j = 0$;

(β) $Y_{j,m}$ is $\cap\{X_{\iota,m,k,t_{j,k,m}} : \iota < j\} \subseteq Y_m$ if $j \in (0, i)$;

(d) (α) If $j = 0 < i$ then $G_{j,k,m}$ is $F_{k,m}$;

(β) if $j \in (0, i)$ then $G_{j,k,m}$ is defined by: for $\langle y_{k+1}, \ldots, y_m\rangle \in \prod^m_{\ell=k+1} Y_\ell$ we have

$$G_{j,k,m}(\langle y_{k+1}, \ldots, y_m\rangle) = G_{j-1,k,m+1}(\langle y_{k+1}, \ldots, y_{m+1}\rangle)$$

for any $y_{m+1} \in Y_{j,m+1}$ (so the value does not depend on y_{m+1}!);

[4] So the Y_k's are not well ordered! If $\alpha < \mathrm{hrtg}(Y_n) \Rightarrow D_n$ is $|\alpha|^+$-complete, then $\alpha^{Y_n}/D_n \cong \alpha$. If α is a counterexample, D projects onto a uniform $\aleph_1$-complete filter on some $\mu \le \alpha$.

(e) for $k < m$ and $t \in Z_{k,m}$ let $X_{j,k,m,t}$ be

$$\{y \in Y_m : \text{if } \langle y_{k+1}, \dots, y_{m-1}\rangle \in \prod_{\ell=k+1}^{m-1} Y_\ell \text{ then } G_{j,k,m}(y_{k+1}, \dots, y_{m-1}, y) = (y_{k+1}, \dots, y_{m-1})\}.$$

Clearly $(*)_0$ holds emptily.

For $i \geq 1$, let $j = i - 1$; clearly $\langle Y_{j,m} : m < \omega\rangle$ is well defined by (c), hence we can define $\langle X_{j,k,m,t} : t \in Z_{k,m}\rangle$ by clause (e) and let

$$X_{j,m,\mathbf{t}} = \bigcap\{X_{j,k,m,t_k} : k < m\}$$

when $\mathbf{t} = \langle t_k : k < m\rangle$.

So $\bar{X}_{j,m} = \langle X_{j,m,\mathbf{t}} : \mathbf{t} \in Z_m\rangle$ is a legal j-move of the empty player in the game $\eth_{\mathbf{x},m}$, so we can use $\mathbf{st}_m$ to define $\mathbf{t}_{j,m} = \langle t_{j,k,m} : k < m\rangle$ as the j-th move of the non-empty player.

Lastly, the function $G_{j,k,m}$ is well defined. Having carried the induction, for each m clearly $\langle(\bar{X}_{j,m}, \mathbf{t}_{j,m}) : j < \omega\rangle$ is a play of the game $\eth_{\mathbf{x},n}$ in which the non-empty player uses the strategy $\mathbf{st}_n$, hence win in the play, so $\bigcap\{X_{j,m,\mathbf{t}_{j,m}} : j < \omega\}$ is non-empty, so by $\mathrm{AC}_{\aleph_0}$ we can choose $\bar{y} = \langle y_m : m < \omega\rangle$ such that $y_m \in \bigcap\{X_{j,m,\mathbf{t}_{j,m}} : j < \omega\}$.

It is easy to see that $\bar{y}$ is as required in Definition 3.12(2). $\blacksquare_{3.17}$

Conclusion 3.18: [DC] Assume $\langle \kappa_n : n\rangle$ is increasing and κ_n is measurable as witnessed by the ultrafilter D_n or just D_n is a uniform[5] $\Upsilon(\mathscr{P}(\kappa_{n-1}))$-complete ultrafilter on κ_n.

Then for every ordinal ζ, for every large enough n we have $\text{o-Depth}^+_{D_n}(\zeta) \leq \zeta$.

Proof. By 3.17 we know that $\mathrm{IND}(\langle D_n : n < \omega\rangle)$ and by 3.13(2) we get the desired conclusion. $\blacksquare_{3.18}$

Claim 3.19: *(*ZFC *for simplicity).*

If (A), *then* (B), *where*

(A) (a) $\lambda_n = \mathrm{cf}(\lambda_n)$ *and* $(\lambda_n)^{<\lambda_n} < \lambda_{n+1}$ *and* $\mu = \Sigma\{\lambda_n : n < \omega\}$ *and* $\lambda = \mu^+$,

(b) $\mathbb{P}_n$ *is the natural* λ_n*-complete* λ_n^+*-c.c. forcing adding* $\langle \underset{\sim}{f}_{n,\alpha} : \alpha < \lambda\rangle$ *of members of* ${}^{\lambda_n}(\lambda_n)$, $<_{J^{\mathrm{bd}}_{\lambda_n}}$*-increasing,*

(c) $\mathbb{P}$ *is the product* $\prod_n \mathbb{P}_n$ *with full support;*

[5] Recall $\Upsilon(A) = \min\{\theta$: there is no one-to-one function from θ into $A\}$.

(B) *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\text{NIND}(\langle \text{dual}(J^{\text{bd}}_{\lambda_n}) : n < \omega \rangle)$ *and a cardinal* θ *is not collapsed if* $\theta \notin (\mu^+, \mu^{\aleph_0}]$.

Proof. So $p \in \mathbb{P}_n$ iff p is a function from some $u \in [\lambda^+]^{<\lambda_n}$ into $\cup \{^{\zeta}(\lambda_n) : \zeta < \lambda_n\}$, ordered by $\mathbb{P}_n \models$ "$p \leq q$" iff $\alpha \in \text{Dom}(q) \Rightarrow \alpha \in \text{Dom}(q) \wedge p(\alpha) \subseteq q(\alpha)$. Now use 3.13. $\blacksquare_{3.19}$

3(C). Bounds on the depth. We continue 3.2. We try to get a bound for singulars of uncountable cofinality, say for the depth, recalling that depth, rank and ps-T_D are closely related.

Hypothesis 3.20: D is an $\aleph_1$-complete filter on a set Y.

Remark 3.21: Some results do not need the $\aleph_1$-completeness.

Claim 3.22: *Assume* $\bar{\alpha} \in {}^Y\text{Ord}$.

(1) [DC] *(No-hole Depth) If* $\zeta + 1 \leq$ ps-o-Depth$^+_D(\bar{\alpha})$ *then, for some* $\bar{\beta} \in {}^Y\text{Ord}$, *we have* $\bar{\beta} \leq \bar{\alpha} \bmod D$ *and* $\zeta + 1 =$ ps-o-Depth$^+(\bar{\beta})$.
(2) *In Definition 3.1 we may allow* $\mathscr{F}_\varepsilon \subseteq {}^Y\text{Ord}$ *such that* $g \in \mathscr{F}_\varepsilon \Rightarrow g < f \bmod D$.
(3) *If* $\bar{\beta} \in {}^Y\text{Ord}$ *and* $\bar{\alpha} = \bar{\beta} \bmod D$ *then* ps-o-Depth$^+(\bar{\alpha}) =$ ps-o-Depth$^+(\bar{\beta})$.
(4) *If* $\{y \in Y : \alpha_y = 0\} \in D^+$ *then* ps-o-Depth$^+(\bar{\alpha}) = 1$.
(5) *Similarly for the other versions of depth from Definition 3.10.*

Proof. (1) By DC, without loss of generality, there is no $\bar{\beta} <_D \bar{\alpha}$ such that $\zeta + 1 \leq$ ps-o-Depth$^+(\bar{\beta})$. Without loss of generality, $\bar{\alpha}$ itself fails the desired conclusion hence $\zeta + 1 <$ ps-o-Depth$^+(\beta)$. By parts (3), (4), without loss of generality, $s \in Y \Rightarrow \alpha_s > 0$. As $\zeta + 1 <$ ps-o-Depth$^+(\bar{\alpha})$ there is a $<_D$-increasing sequence $\langle \mathscr{F}_\varepsilon : \varepsilon < \zeta + 1 \rangle$ with $\mathscr{F}_\varepsilon$ a non-empty subset of $\Pi\bar{\alpha}$. Now any $\bar{\beta} \in \mathscr{F}_\zeta, \zeta + 1 \leq$ ps-o-Depth$^+(\bar{\beta})$ as witnessed by $\langle \mathscr{F}_\varepsilon : \varepsilon < \zeta \rangle$, recalling part (2), contradicting the extra assumption on $\bar{\alpha}$ (being $<_D$-minimal such that...).

(2) Let $\mathscr{F}'_\varepsilon = \{f^{[\bar{\alpha}]} : f \in \mathscr{F}_\varepsilon\}$, where $f^{[\bar{\alpha}]}(s)$ is $f(s)$ if $f(s) < \alpha_s$ and is zero otherwise.

(3), (4) Obvious.

(5) Similarly. $\blacksquare_{3.22}$

Claim 3.23: [DC + AC$_Y$] *If* $\bar{\alpha}, \bar{\beta} \in {}^Y\text{Ord}$ *and* D *is a filter on* Y *and* $s \in Y \Rightarrow |\alpha_s| = |\beta_s|$ *then* ps-$\mathbf{T}_D(\bar{\alpha}) = \text{ps} - \mathbf{T}_D(\bar{\beta})$.

Proof. Straightforward. $\blacksquare_{3.23}$

Assuming full choice the following is a relative of the Galvin–Hajnal theorem.

THEOREM 3.24: [DC + AC$_Y$] *Assume* $\alpha(1) < \alpha(2) < \lambda^+$, $\text{ps-o-Depth}^+(\lambda) \leq \lambda^{+\alpha(1)}$ *and* $\xi = \text{hrtg}({}^Y\alpha(2)/D)$. *Then* $\text{ps-o-Depth}^+_D(\lambda^{+\alpha(2)}) < \lambda^{+(\alpha(1)+\xi)}$.

Proof. Let $\Lambda = \{\mu : \lambda^{+\alpha(1)} < \mu \leq \lambda^{\alpha(1)+\xi}\}$, and for every $\mu \in \Lambda$

- $(*)_1$ let $\mathscr{F}_\mu = \mathscr{F}(\mu) = \{f : f \in {}^Y\{\lambda^{+\alpha} : \alpha < \alpha(2)\}$ and $\mu = \text{ps-Depth}^+_D(f)\}$,
- $(*)_2$ obviously $\langle \mathscr{F}_\mu : \mu \in \Lambda\rangle$ is a sequence of pairwise disjoint subsets of ${}^Y\alpha(2)$ closed under equality modulo D.

By the no-hole-depth Claim 3.22(1) above we have

- $(*)_3$ if $\mu_1 < \mu_2$ are from Λ and $f_2 \in \mathscr{F}_{\mu_2}$ then for some $f_1 \in \mathscr{F}_{\mu_1}$ we have $f_1 < f_2 \mod D$,
- $(*)_4$ $\xi > \sup\{\zeta + 1 : \mathscr{F}(\lambda^{+(\alpha+\zeta)}) \neq \emptyset\}$ implies the conclusion.

Lastly, as $\xi = \text{hrtg}({}^Y\alpha(2)/D)$ we are done. $\blacksquare_{3.24}$

Remark 3.25: (0) Compare this with Conclusion 1.10.

(1) We may like to lower ξ to $\text{ps-Depth}^+_D(\alpha(2))$; toward this let

- $(*)_1$ $\mathscr{F}'_\mu = \{f \in \mathscr{F}_\mu :$ there is no $g \in \mathscr{F}_\mu$ such that $g < f \mod D\}$.

By DC

- $(*)_2$ if $f \in \mathscr{F}_\mu$ then there is $g \in \mathscr{F}'_\mu$ such that $g \leq_D f \mod D$.

(2) Still the sequence of those $\mathscr{F}'_\mu$ is not $<_D$-increasing.

Instead of counting cardinals we can count regular cardinals.

THEOREM 3.26: [DC+AC$_Y$] *The number of regular cardinals in the interval* $(\lambda^{+\alpha(1)}, \text{ps-depth}^+_D(\lambda^{+\alpha(2)})$ *is at most* $\text{hrtg}({}^Y\alpha(2)/D)$ *when:*

(a) $\alpha(1) < \alpha(2) < \lambda^+$,
(b) $\kappa > \aleph_0$,
(c) *D is a κ-complete filter on Y*,
(d) $\lambda^{+\alpha(1)} = \text{ps-Depth}_D(\lambda)$.

Proof. Straightforward, using the No-Hole Claim 1.12. $\blacksquare_{3.26}$

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