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SUPERSTABLE FIELDS AND GROUPS

G. CHERLIN*
Rutgers University, New Brunswick, NJ, U.S.A.

S. SHELAH**
The Hebrew University, Jerusalem, Israel

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We prove an indecomposability theorem for connected stable groups. Using this theorem we prove that all infinite superstable fields are algebraically closed, and we extend known results for ω-stable groups of Morley rank at most 3 to the corresponding class of superstable groups (Note: The logical notion of stability is unrelated to the notion of stability in finite group theory.)

1. Introduction

The main object of this paper is to give a proof of the following result:

Theorem 1. *Any infinite superstable field F is algebraically closed.*

(A field is called superstable if its first-order theory is superstable. The basic reference for the notions of stability theory and for all model-theoretic notions exploited in this paper is [12]. More accessible general introductions to the subject are in [11, 5].)

This extends the main result of [9], which treats the case of ω-stable fields.

Theorem 1 can be combined with results in [3, 6] to yield:

Corollary 2. *Any semisimple superstable ring R is the direct sum of a finite ring and finitely many full matrix rings:*

$$M_{n_i}(F_i)$$

over algebraically closed fields F_i. (Hence in fact R will be ω-stable of finite Morley rank.)

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The analysis of superstable division rings given in [3] can now be given significant notational simplification in view of Theorem 1.

The proof of Theorem 1 is in outline identical with the argument in [9]. The difference lies in a systematic use of connected groups (see Section 2) to replace two ad hoc algebraic constructions in [9]. That part of our proof which most closely parallels the arguments in [9] is given in our Section 3. The model-theoretic ingredients are supplied in Section 2, with the exception of the main technical result (the Indecomposability Theorem 34) which is discussed separately in Sections 4–6. The material in Section 4 completes the proof of Theorem 1; Sections 5 and 6 give variations on the same theme.

The remaining sections of the paper are devoted to the extension of the main results of [4] to the class of superstable groups of ∞-rank at most 3. (For the definition see Section 2.2.). This is in principle simply a matter of combining the Indecomposability Theorem with the various algebraic arguments of [4], but as it is necessary to rearrange all of the arguments involved, we have given the details at length. (From a psychological point of view [4] is a prerequisite for this material — and once our analysis arrives at a stage at which the remaining steps may be copied out of [4], we terminate the discussion.)

The expository article [5] can be viewed as a lengthy introduction to the present paper. Poizat has worked out a more systematic treatment of the model-theoretic aspects of stable groups connected with indecomposability theorems [15]. The conclusion appears to be that a more enthusiastic use of Shelah's "forking" makes life substantially simpler.

I. SUPERSTABLE FIELDS

2. Connected groups

We use the word "group" to mean what is usually called an *expansion* of a group, namely an algebraic system equipped with a binary operation $\cdot$ — *together* with possible additional operations and relations — such that the structure is a group with respect to the distinguished operation $\cdot$. The most important example of such a group is a *field* F, viewed as a group in two distinct ways, with the usual proviso that the underlying set of the multiplicative group F does not contain 0. We will see that this niggling over terminology has a nontrivial effect on the content of the following definition.

Definition 3. A group G is *connected* iff G has no proper definable subgroup of finite index.

Warning. When a field is viewed as a group as suggested above, the definable sets (that is the sets definable using the field operations) need not be definable from the single binary operation singled out for attention. Thus in Definition 3 the word "definable" means "definable in the *structure G*", without special reference to the group operation on G.

Unfortunately, the definition of connectivity also involves the word "subgroup", which of course refers directly to the specific group operation singled out for attention. Thus connectivity is a property of groups rather than structures (compare Theorems 6, 7 below).

The notion of connectivity has been studied in [2, 4, 13], and discussed at length in the expository [5]. To employ it one obviously needs existence theorems for connected groups.

Theorem 4. *If the group G is either ω-stable or else stable and* $\aleph_0$*-categorical, then G contains a unique connected group of finite index in G, denoted* G^0*, and* G^0 *is a normal subgroup of G.*

For a proof see [2]. As it happens, Theorem 4 is not applicable in the situations considered in the present paper. Indeed we have:

Example 5. The superstable group Z has no connected subgroup of finite index.

(The superstability follows most simply from Garavaglia's characterization of superstable modules in [7].)

Fortunately we will be able to prove:

Theorem 6. *If D is an infinite stable division ring, then the additive group of D is connected.*

Theorem 7. *If D is an infinite stable division ring, then the multiplicative group of D is connected.*

A proof of Theorem 6 is given in Section 2.3. We will devote Section 4 to the proof of Theorem 7. The application of these connectivity theorems is based on:

Theorem 8. (Surjectivity Theorem). *Let G be a connected superstable group and let*

$$h : G \to G$$

be a definable endomorphism of G whose kernel is finite. Then h is surjective.

The proof of this theorem is essentially model-theoretic. In conjunction with Theorems 6 and 7 is provides the algebraic information necessary to carry out Macintyre's argument (see Section 3).

2.2. *The surjectivity theorem*

We will base the proof of the Surjectivity Theorem 8 on the properties of the α-rank, which is defined below. Familiarity with the use of Morley rank in connection with ω-stable theories as presented in [10] will be found helpful, but is not essential.

Definition 9. Let T be a theory and let λ be a cardinal.

(1) A *rank function* for T is a function f which assigns ordinals to certain definable subsets of models of T, and which is monotone in the following sense: if $A \models T$, $S \subseteq S'$ are definable subsets of A, and $f(S')$ is defined, then $f(S)$ is also defined and $f(S) \leq f(S')$.

(2) A rank function f for T is *elementary* iff whenever A is a model of T, A' is an elementary extension of A, and S, S' are definable subsets of A, A' having the same defining formula (with parameters from A), then:

$$f(S) = f(S')$$

(and in particular $f(S)$ is defined iff $f(S')$ is defined).

(3) A rank function f for T satisfies the λ-*splitting condition* iff whenever S is a definable subset of a model A of T such that $f(S)$ is defined and $\mathcal{S} = \{S_\alpha\}$ is a class of at least λ mutually disjoint definable subsets of S, then:

$$f(S_\alpha) < f(S)$$

for some S_α.

(4) A rank function f for T is *total* iff f is defined for all definable subsets S of all models of T.

Fact 10. [12, Theorem II 3.14]. *If T is a superstable theory, then there is a total elementary rank function for T which satisfies the λ-splitting condition for some cardinal λ.*

Remark 11. Given a theory T and a cardinal λ, if one attempts to assign to each definable subset of a model of T the *least* ordinal compatible with the elementarity condition and the λ-splitting condition, then an inductive definition of a rank function inevitably emerges — in terms of an inductive definition of the sets of rank α (for each α). This rank function is optimal in two respects: it is defined on the largest possible domain, and takes on the least possible values there. Of course in general it need not be total.

This "minimal" rank function will be denoted λ-rank. (In the notation of [12] we have: λ-rank$(S) = R(\varphi, L, \lambda^+)$ where φ is a formula defining S.) Of course this function is defined relative to the given theory T.

Remark 12. The unspecified cardinal λ can easily be eliminated from the foregoing considerations. It can be shown that for all sufficiently large λ the ordinal λ-rank(S) is independent of λ (or undefined); see [12, Theorem 3.13].

Hence we may define:

$$\infty\text{-rank}(S) = \lim_{\lambda} \lambda\text{-rank}(S).$$

Then Fact 10 may be reformulated as follows:

> if T is a superstable theory, then ∞-rank is a total elementary rank function; and ∞-rank satisfies some λ-splitting condition.

The main connection with group theory lies in the following simple result:

Lemma 13. *If H is a definable subgroup of the superstable group G, then the following are equivalent:*
(1) ∞-rank$(H) < \infty$-rank(G);
(2) *The index of H in G is infinite.*

Proof. (2) → (1): Passing to a sufficiently saturated elementary extension of G, we may suppose that the index of H in G is arbitrarily large, and then apply the λ-splitting condition to the H-coset decomposition of G for some λ, noting that all cosets of H have the same ∞-rank as H itself.

(1) → (2): We need to see that if H is of finite index in G then ∞-rank$(H) = \infty$-rank(G). More generally, it is easy to see that if the definable set S is a finite union of definable sets S_i, then ∞-rank$(S) = \sup_i \infty$-rank(S_i). Cf. [12, Claim II 1.7].

Our proof of the Surjectivity Theorem will involve another property of ∞-rank:

Fact 14. *Let A be a superstable structure and let E be a definable equivalence relation on A having finite equivalence classes of bounded size. Let A/E denote the quotient structure, equipped with all relations and functions which are induced by definable relations and functions on A. Then:*

$$\infty\text{-rank}(A) = \infty\text{-rank}(A/E).$$

For the proof see [12, Claim V 7.2(6) and Theorem II 3.11].

Proof of the Surjectivity Theorem. Let h be a definable endomorphism of the superstable group G, and let H be the image of h. If the kernel of h is finite, then by Fact 14:

$$\infty\text{-rank}(H) = \infty\text{-rank}(G).$$

Then by Lemma 13 H is of finite index in G, and so by the connectivity of G we have $H = G$ as desired.

2.3. *Theorem 6*

As a rule the proof of a connectivity theorem depends on the use of certain chain conditions (see [5]). In the present case we will need the stable chain condition of Baldwin and Saxl.

Definition 15. Let G be a group, and let $\mathcal{J}$ be a collection of subgroups of G.

(1) The groups in $\mathcal{J}$ are said to be *uniformly definable* in G iff there is a single formula $\varphi(x, \bar{y})$ such that each group lying in $\mathcal{J}$ is definable by a formula of the form $\varphi(x, \bar{g})$ with $\bar{g}$ in G.

(2) G satisfies the CC-$\mathcal{J}$ (chain condition for $\mathcal{J}$) iff there is no infinite chain in $\mathcal{J}$, where a chain is a collection of groups linearly ordered by inclusion.

(3) G satisfies the *stable chain condition* iff G satisfies the CC-$\mathcal{J}$ for every family $\mathcal{J}$ of groups which can be obtained by closing a family $\mathcal{J}_0$ of uniformly definable groups under arbitrary intersections.

Fact 16. *If G is a stable group, then G satisfies the stable chain condition.*

The proof is implicit in [1, p. 274] and also in [5]. We will use this fact repeatedly in Section 7 and thereafter.

Example 17. Let $\mathcal{J}_0$ be the collection:

$$\{C(g) : g \in G\}$$

of all centralizers in G of single elements of G. Then the groups in $\mathcal{J}_0$ are uniformly definable. The closure of $\mathcal{J}_0$ under arbitrary intersection is the family of centralizers in G of arbitrary subsets of G.

The following, which is equivalent to Fact 16, is what one in fact actually proves:

Fact 18. *Let G be a stable group, $\mathcal{J}_0$ a collection of uniformly definable subgroups of G. Then:*

(1) *G satisfies the CC-$\mathcal{J}_0$.*

(2) *There is an integer n such that an arbitrary intersection of groups in $\mathcal{J}_0$ equals an intersection of at most n groups in $\mathcal{J}_0$.*

(Fact 18(2) shows that the closure of $\mathcal{J}_0$ under arbitrary intersections is again a family of uniformly definable subgroups of G, to which 18(1) applies.)

In the present connection we need only Fact 18(2).

Proof of Theorem 6. Let D be an infinite stable division ring. Let A be a definable subgroup of the additive group of D, of finite index in D. We will show that $A = D$.

For any nonzero element x of D let xA be the left scalar multiple of A by x. This is again an additive subgroup of D of finite index in D. A uniform definition of the left scalar multiples of A may be obtained from a definition of A. Hence by Fact 18(2) the intersection A_0 of all left scalar multiples of A can be reduced to a finite intersection, and hence the index of A_0 in D is finite.

On the other hand A_0 is closed under left multiplication by elements of D, i.e. A_0 is a left ideal of D. Since the index of A_0 in D is finite, $A_0 \neq (0)$, and hence $A_0 = D$, so $A = D$, as desired.

3. Theorem 1

We will use Theorems 6–8 of Section 2. (The proof of Theorem 7 is in Section 4.) The algebraic information needed is standard [16]:

Fact 19. *Let K be a Galois extension of prime degree q over the field F, and suppose $x^q - 1$ splits in F. If p is the characteristic of F, then K is as follows:*

(1) *If $p = q$, then K is generated over F by the solution of an equation $x^p - x = a$ for some $a \in F$.*

(2) *If $p \neq q$, then K is generated over F by the solution of an equation $x^q = a$ for some $a \in F$.*

K is said to be an *Artin–Schreier* extension of F in the first case, and a *Kummer* extension of F in the second case.

We will combine this with:

Lemma 20. *Let F be a superstable field. Then F is perfect and F has no Artin–Schreier or Kummer extensions.*

Proof. Let $h(x)$ be either of the following maps:

(1) $x \rightarrow x^p - x$ for x in F (if char. $F = p > 0$),

(2) $x \rightarrow x^n$ for $x \neq 0$ in F

where $n \geqslant 1$ is an arbitrary integer. Then h is a definable endomorphism of the additive group of F in the first case, and of the multiplicative group of F in the second case. In both cases the kernel of h is finite.

Since by Theorems 7, 8 both of these groups are connected, therefore in both cases Theorem 6 implies that h is surjective. This easily yields Lemma 20.

Proof of Theorem 1. Assume toward a contradiction that F is an infinite superstable field and that F is not algebraically closed. By Lemma 20 F is perfect, so it has a Galois extension of some finite degree n.

Consider all pairs of fields (F, K) satisfying:

(Gal) K is a Galois extension of finite degree over F and F is infinite and superstable.

Choose such a pair (F, K) in which the degree q of K over F is minimal (greater than one). A contradiction will be immediate from Fact 19, Lemma 20, and the following claim:

(Clm) q is prime and $x^q - 1$ splits in F.

Thus we need only to verify (Clm). First let r be a prime factor of q and let F_1 be the fixed field of an element of order r in $\mathrm{Gal}(K/F)$. F_1 is superstable: indeed F_1 is a finite-dimensional extension of F, hence is interpretable over F, and as such F_1 inherits the superstability of F (for more detail see [9]). Thus the pair (F_1, K) satisfies (Gal) above, and so the minimality of q yields $q = r$, q is prime.

Now let K_1 be the splitting extension of $x^q - 1$ over F. Then the degree of K_1 over F divides $q - 1$, so by the minimality of q we have $K_1 = F$ as claimed.

Thus the claim holds, yielding the desired contradiction.

II. INDECOMPOSABILITY THEOREMS

4. The indecomposability theorem for stable groups

In this section we will derive Theorem 7 from Theorem 6 and a general result concerning connected stable groups. Our basic tool will be the use of Δ-ranks for Δ finite as in [12, Chapter II], which we now review.

4.1. Δ-*rank*

Definition 21. Let T be a first-order theory and let Δ be a set of formulas $\varphi(x, \bar{y})$ in the language of T.

(1) For A a model of T let $\Delta(A)$ be the Boolean algebra generated by the subsets of A which can be defined by formulas of the form:

$$\varphi(x, \bar{a}) \quad (\varphi \in \Delta, \bar{a} \text{ in } A).$$

(2) If S is a definable subset of a model A of T and $\mathcal{S} = \{S_\alpha\}$ is an infinite family of subsets of S, then $\mathcal{S}$ is a Δ-*splitting* of S iff:

(i) The sets S_α are mutually disjoint subsets of S;

(ii) Each set S_α is the intersection of S with a set in $\Delta(A)$.

(3) A rank function f for T satisfies the Δ-*splitting condition* iff whenever S is a definable subset of a model of T for which $f(S)$ is defined and $S = \{S_\alpha\}$ is a Δ-splitting of S, then for some α:

$$f(S_\alpha) < f(S).$$

(4) The least elementary rank function which satisfies the Δ-splitting condition

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will be denoted:

Δ-rank.

(In the notation of [12] we have Δ-rank$(S) = R^1(S, \Delta, \aleph_0)$.)

(5) If A is a model of T, S is a definable subset of A, and Δ-rank(A) is defined we say that S is *Δ-small* iff

Δ-rank$(S) < \Delta$-rank(A).

We will be interested in the case in which Δ is a finite set of formulas in which case we are dealing with the so-called *local* rank functions.

Fact 22. *A theory T is stable iff for all finite sets of formulas Δ Δ-rank is total* [12, Section II.2].

Fact 23. *For S_1, S_2 definable subsets of a structure A and Δ a set of formulas:*

(sup) Δ-rank$(S_1 \cup S_2) = \sup(\Delta$-rank$(S_1), \Delta$-rank$(S_2))$

(one side is defined iff the other side is defined).

Corollary 24. *If A is a structure and Δ is a set of formulas such that Δ-rank(A) is defined, then the collection of Δ-small sets is an ideal of the Boolean algebra of all definable subsets of A.*

The notion of Δ-rank is supplemented by the notion of Δ-multiplicity (which Morley would have called "Δ-degree"). This is based on:

Fact 25. *Let Δ be a set of formulas and let A be a structure for which Δ-rank(A) is defined. Let I be the collection of Δ-small sets belonging to $\Delta(A)$. Then I is an ideal of $\Delta(A)$ and the quotient $\Delta(A)/I$ is a finite Boolean algebra.*

Definition 26. With the hypothesis and notation of Fact 25, the *Δ-multiplicity* of A is defined to be the number of atoms in $\Delta(A)/I$.

Since we will be making extensive use of Fact 25, we will rephrase it in a more explicit form.

Definition 27. Let A be a structure and let Δ be a set of formulas for which Δ-rank(A) is defined. For definable subsets X, Y of A define:

$X \equiv Y$ (or: $X \equiv_\Delta Y$ in A)

iff the symmetric difference of X and Y is Δ-small.

Then Fact 25 becomes:

Fact 28. *Let Δ be a set of formulas; let A be a structure for which* Δ-rank(A) *is defined, and let m be the Δ-multiplicity of A. Then there is a decomposition:*

(dec) $\quad A = A_1 \dot\cup \cdots \dot\cup A_m$

of A into m disjoint sets $A_1, \ldots, A_m$ satisfying:

(i) $A_i \in \Delta(A)$ *for* $i = 1, \ldots, m$,

(ii) Δ-rank$(A_i) = \Delta$-rank(A) *for* $i = 1, \ldots, m$.

The Δ-multiplicity m is the largest integer for which such a decomposition exists. Furthermore the decomposition (dec) is unique — up to the order of the pieces — modulo Δ-small sets; in other words if:

$$A = B_1 \dot\cup \cdots \dot\cup B_m$$

is a second such decomposition, then there is a (unique) permutation ρ of $1, \ldots, m$ such that

$$A_i \equiv B_{i\rho} \quad \text{for} \quad i = 1, \ldots, m.$$

Finally, for any S in $\Delta(A)$ there is a unique subset I of $\{1, \ldots, m\}$ for which

$$S \equiv \bigcup_I A_i.$$

Definition 29. With the above hypothesis and notation, a definable subset S of A is Δ-*indecomposable* iff

(1) Δ-rank$(S) = \Delta$-rank(A).

(2) S has Δ-multiplicity 1.

(For $S \in \Delta(A)$ this just means that S is an atom modulo the ideal of Δ-small sets.)

4.2. *Invariant sets*

We will be interested in studying the way in which a stable group G acts on the Boolean algebra of definable subsets of G under right or left translation by elements of G. Hence we introduce the following notions:

Definition 30. Let Δ be a set of formulas in a language L containing a binary operation $\cdot$, let T be a theory in this language, and let f be a rank function for T.

(1) Δ is T-*right invariant* iff for each formula $\varphi(x; \bar y)$ in Δ, for each model G of T, and for all $\bar a$, g in G, the formula:

$$\varphi(x \cdot g; \bar a)$$

is equivalent (in G) to an instance of a formula in Δ.

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(2) f is *right invariant* iff for any definable subset S of a model G of T for which $f(S)$ is defined and any $g \in G$:

$$f(S) = f(Sg).$$

Left invariance is defined similarly, and Δ (or f) is called invariant iff it is both left and right invariant.

Lemma 31. *If Δ is right invariant and T contains the theory of groups then* Δ-rank *is right invariant.*

Proof. The proof is entirely straightforward. The main point is that if Δ is right invariant, then $\Delta(G)$ is invariant under the action of G by right translation. It suffices to verify this assertion for a generator S of $\Delta(A)$ defined by a formula:

$$\varphi(x, \bar{a})$$

with $\varphi \in \Delta$ and $\bar{a}$ in G. But then for any element $g \in G$ the set Sg is defined by:

$$\varphi(x \cdot g^{-1}, \bar{a})$$

which by the right invariance of Δ is again defined by formula in Δ.

Now we will discuss the construction of invariant sets of formulas.

Definition 32. (1) If $\varphi(x;\bar{y})$ is a formula let $\hat{\varphi}(x;\bar{y}, z_1, z_2)$ be the formula $\varphi(z_1 \cdot x \cdot z_2; \bar{y})$.

(2) If Δ is a set of formulas let $\hat{\Delta} = \Delta \cup \{\hat{\varphi} : \varphi \in \Delta\}$.

Lemma 33. *For any set of formulas Δ and any theory T containing the theory of semigroups the set $\hat{\Delta}$ is T-invariant.*

Proof. Each formula $\varphi(x;\bar{y})$ of Δ is equivalent to the formula $\hat{\varphi}(x;\bar{y}, 1, 1)$, so it suffices to show that for each formula φ the set $\{\hat{\varphi}\}$ is invariant. Since T proves

$$\hat{\varphi}(u \cdot x \cdot v; \bar{y}, z_1, z_2) \equiv \hat{\varphi}(x; \bar{y}, z_1 \cdot u, v \cdot z_2),$$

this is clear.

4.2. *The indecomposability theorem*

The main result of this section will be:

Theorem 34. (The Indecomposability Theorem). *Let G be a stable group. Then the following are equivalent:*

(1) *G is connected.*

(2) *G is Δ-indecomposable for any finite invariant set of formulas Δ.*

(3) *For any finite set* Δ_0 *of formulas there is a finite invariant set of formulas* Δ *containing* Δ_0 *such that* G *is* Δ*-indecomposable.*

Using Theorem 34 it is possible to reduce Theorem 7 to Theorem 6. A fairly abstract version of this statement goes as follows:

Theorem 35. *Let* A *be a stable structure and let* X, Y *be definable subsets of* A. *Suppose that* A *is equipped with two binary operations* $+$ *and* $\cdot$ *such that:*

(i) $\langle A - X, + \rangle$ *and* $\langle A - Y, \cdot \rangle$ *are groups;*

(ii) *For every finite set of formulas* Δ_0 *there is a finite set* Δ *containing* Δ_0 *which is invariant relative to both* $+$ *and* $\cdot$ *(and* $\mathrm{Th}(A)$*), such that* X *and* Y *are* Δ*-small.*

Then $\langle A - X, + \rangle$ *is connected iff* $\langle A - Y, \cdot \rangle$ *is connected.*

(Slogan: connectivity is a property of the *structure* A rather than the *group* A; compare the comment after Definition 3.)

Proof. We claim in fact that under the above hypotheses the following are equivalent:

(1) $\langle A - X, + \rangle$ is connected.

(2) $\langle A - Y, \cdot \rangle$ is connected.

(3) For every finite set Δ_0 of formulas there is a finite set Δ containing Δ_0, which is invariant relative to both $+$ and $\cdot$ (and $\mathrm{Th}(A)$), such that $A - X$ and $A - Y$ are Δ-indecomposable.

It suffices for example to prove that (1) is equivalent to (3). This follows directly from the corresponding equivalence in Theorem 1 (in the direction (1) $\to$ (3) we have the sets Δ given by (ii)).

The application to infinite stable division rings D is obtained by setting $A = D$, $X = \emptyset$, $Y = \{0\}$. Then once we verify hypothesis (ii) of Theorem 35 we will have as the conclusion: Theorem 6 is equivalent to Theorem 7 (see Section 2). Since Theorem 6 was proved in Section 2, Theorem 7 follows, and then the proof of Theorem 1 is complete.

It remains therefore to verify hypothesis (ii). Since X, Y are Δ-small for any Δ containing "$x = y$" it suffices to prove:

Lemma 36. *Let* T *be an extension of the theory of rings. Then any finite set* Δ *of formulas in the language of* T *is contained in a finite set of formulas which is* T*-invariant with respect to both* $+$ *and* $\cdot$.

Proof. Associate to any formula $\varphi(x, \bar{y})$ the formula

$$\hat{\varphi}(x; \bar{y}, z_1, z_2, z_3) = \varphi(z_1 x z_2 + z_3; \bar{y}).$$

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Then T proves:

$$\hat{\varphi}(uxv + w; \bar{y}, z_1, z_2, z_3) \equiv \hat{\varphi}(x; \bar{y}, z_1 u, vz_2, z_1 w z_2 + z_3)$$

and it follows that $\hat{\varphi}$ is invariant with respect to both $+$ and $\cdot$. The rest of the argument goes as in the proof of Lemma 33.

Thus Theorem 35 applies to infinite stable division rings, as claimed.

In connection with Theorem 35 it is natural to ask:

Question 37. If a stable structure A can be viewed as a group with respect to two operations, $+$ and $\cdot$, does connectivity of $\langle A, +\rangle$ imply connectivity of $\langle A, \cdot\rangle$?

It is not clear what use such a result would have, but on the other hand we will see in Section 5 that we get such a result easily if A is superstable via a simplified version of Theorem 34.

It remains to prove Theorem 34. We prove $(1) \rightarrow (2) \rightarrow (3) \rightarrow (1)$. The implication $(2) \rightarrow (3)$ follows from Lemma 33. The implication $(3) \rightarrow (1)$ is easy: suppose (3) holds and H is a subgroup of finite index in G defined by the formula $\varphi(x; \bar{a})$. Let Δ be a finite invariant set of formulas, containing φ, and such that G is Δ-indecomposable. Then since Δ-rank is invariant it follows that the cosets of H in G all have the same Δ-rank, and hence by Fact 23:

$$\Delta\text{-rank}(Hg) = \Delta\text{-rank}(G) \quad \text{for all} \quad g \in G.$$

Since G is Δ-indecomposable it follows that there is only one such coset, so $G = H$. This proves that G is connected, as desired.

It remains to be seen that $(1) \rightarrow (2)$.

4.4. *Theorem* 34: $(1) \rightarrow (2)$.

We consider a stable group G, which we will eventually take to be connected, and a finite invariant set Δ of formulas in the language of G. Let the Δ-multiplicity of G be m and fix a decomposition:

(dec) $\quad G = A_1 \dot{\cup} \cdots \dot{\cup} A_m$

of G into mutually disjoint indecomposable subsets of G lying in $\Delta(G)$.

For any element $g \in G$, since Δ is right invariant, right multiplication by g carries the decomposition (dec) to another decomposition:

$$G = A_1 g \dot{\cup} \cdots \dot{\cup} A_m g$$

of G into indecomposable subsets of G which lie in $\Delta(G)$. By Fact 28 there is a unique permutation $\rho = \rho_g$ characterized by:

$$A_i g \approx_\Delta A_{\rho i} \quad \text{for} \quad i = 1, \ldots, m.$$

Furthermore $\rho_{gh} = \rho_g\rho_h$ (this involves the right invariance of Δ-rank), or in other words the map:

$$\rho : g \to \rho_g$$

is a representation of G as a group of permutations of $1, \ldots, m$.

Let K be the kernel of ρ. Since the image of ρ is finite, K has finite index in G. We will prove:

Lemma 38. *If G is $\aleph_1$-saturated, then K is a definable subgroup of G.*

Assuming Lemma 38 we complete the proof of $(1) \to (2)$ (Theorem 34) as follows. With the above hypotheses and notation (notably: G, Δ, K) assume now that G is connected. We are to prove that $m = 1$. Since the notions involved are invariant under elementary extension, we may assume that G is $\aleph_1$-saturated.

Since K is a definable subgroup of finite index in G we have $K = G$. Making this more explicit, we have for every $g \in G$:

(fix) $\quad A_i g \equiv A_i, \quad i = 1, \ldots, m.$

Now consider the first-order theory consisting of the complete theory of G (with names for all elements of G) together with the following sentences involving an additional constant a:

$$\text{"}ag \in A_1\text{"} \quad \text{for each} \quad g \in G.$$

This theory is consistent, since (fix) implies that for any finite set $F \subseteq G$:

$$\Delta\text{-rank}\Big(\bigcap_{g \in F} A_1 g\Big) = \Delta\text{-rank}(A)$$

and hence:

$$\bigcap_{g \in F} A_1 g \neq \emptyset.$$

Let G' be a model of this theory. Then in G' we have:

(inc) $\quad aG \subseteq A'_1$

where A'_1 is the canonical extension of A_1 to G'.

It is easy to see that the inclusion (inc) implies $m = 1$. Indeed if $m > 1$ consider the inclusions:

(1) $aA_1 \subseteq A'_1 \cap aA'_1 = X$ (say).

(2) $aA_2 \subseteq A'_1 \cap aA'_2 = Y$ (say).

X, Y are disjoint subsets of A'_1 and X, Y are in $\Delta(G)$ because Δ is left invariant, so one of the two sets is Δ-small, since A'_1 is Δ-indecomposable. On the other hand neither aA_1 nor aA_2 is Δ-small, and so we appear to have the desired contradiction. There is, however, the technical point that e.g. aA_1 and X are defined in different groups. To conclude we therefore need the following:

Lemma 39. *Let Δ be a set of formulas, let A be a structure, and let A' be an elementary extension of A. Suppose that $S \in \Delta(A)$, X is definable in A', and $S \subseteq X$. If Δ-rank(X) is defined then*:

$$\Delta\text{-rank}(S) \leq \Delta\text{-rank}(X).$$

Proof. Straightforward by induction on Δ-rank. The point is that any Δ-splitting of S in A can be canonically extended to A' and will give a Δ-splitting of X if $S \in \Delta(A)$.

Thus to complete the proof of Theorem 4 we need only to prove the definability Lemma 38 above.

4.5. *A definability lemma*

We recast Lemma 38 in a more general form:

Lemma 40. *Let G be a group with a subgroup K of finite index. Suppose for some cardinal κ that K is the intersection of κ definable subsets of G and that G is κ^+-saturated. Then K is definable in G.*

Proof. Fix coset representatives $g_1, \ldots, g_k$ for K in G where k is the index of K in G. We may assume that $g_1 = 1$ and that $k > 1$. For $1 < i \leq k$ consider the following property of an unknown x:

(P$_i$) $\quad x \in K \cap Kg_i$.

In terms of the definable sets S_α $(\alpha < \kappa)$ whose intersection is K, we can construe (P$_i$) as a type in at most κ constants. Since this type is not realized in the κ^+-saturated group G, it is inconsistent. Thus if we make the harmless assumption that $\{S_\alpha\}$ is closed under finite intersection we may conclude that there is a set S_{α_i}, which by abuse of notation we will call S_i, satisfying:

$$S_i \cap S_i g_i = \emptyset \quad \text{for} \quad 1 < i \leq k.$$

Set $S = \bigcap_i S_i$. Then
(1) $K \subseteq S$;
(2) $S \cap \bigcup_{i>1} Kg_i \subseteq S \cap \bigcup_{i>1} Sg_i = \emptyset$.
Therefore $K = S$, so K is definable, as claimed.

Applying this with K, G chosen as in Section 4.4 shows that Lemma 38 follows if we can find a definition of K which can be put into the form of a countable conjunction of first-order conditions. For this it suffices to define K as the set of

$g \in G$ such that

(def) $\quad \Delta\text{-rank}(A_i g \cap A_i) = \Delta\text{-rank}(G) \quad \text{for} \quad i = 1, \ldots, m.$

(This works because $A_1, \ldots, A_m$ are Δ-indecomposable.)

To see that (def) has the right form we apply [12, Theorem II 2.2: (1)→(7)], which implies that the Δ-rank of G is a finite integer r, and [12, Lemma II 2.9(3)], which implies that the condition:

$$\Delta\text{-rank}(A_i g \cap A_i) = r$$

is equivalent to the consistency of the complete theory of G together with a certain countable first-order theory. Thus by the Compactness Theorem (def) can be put in the desired form.

This completes the proof of Lemma 38, and hence of Theorem 34.

5. More indecomposability theorems

5.1. *Results*

The main result of this section will be:

Theorem 41. *Let G be a superstable group. Then the following are equivalent:*

(1) *G is connected.*

(2) *G is indecomposable, i.e. given two disjoint definable subsets of G, at least one of them has smaller* ∞-rank *than G.*

If G is ω-stable another equivalent condition is:

(3) *G has Morley degree* 1.

(We will not discuss the ω-stable case, since the equivalence of (1) and (3) was already proved in [4] by a very similar argument.)

Condition (2) of Theorem 41 is somewhat unexpected, because in general a superstable structure does not even have finite multiplicity in the sense of ∞-rank (as an example take the additive group of the integers which has ∞-rank 1).

Theorem 41 follows from:

Theorem 42. *Let G be a superstable group and let S be a definable subset of G. Then the following are equivalent:*

(1) *There is a finite set of formulas Δ_0 such that for every finite invariant set Δ of formulas containing Δ_0,*

$$\Delta\text{-rank}(S) < \Delta\text{-rank}(G).$$

(2) $\infty\text{-rank}(S) < \infty\text{-rank}(G)$.

If G is ω-stable another equivalent condition is:

(3) $\text{rank}(S) < \text{rank}(G)$.

Here rank means Morley rank, i.e. Δ-rank where Δ is the set of all formulas. We omit the proof that (1) is equivalent to (3), even though it was not given in (4), because it is a trivial variant of the proof that (1) is equivalent to (2).

Clearly Theorem 42 can be applied to reduce the Indecomposability Theorem 41 to the previous Indecomposability Theorem 34. (The indecomposability condition of Theorem 34 now clearly implies the indecomposability condition of Theorem 41, and the latter easily implies connectedness.) It remains to prove Theorem 42.

5.2. *Large and small sets*

We will make use of the following purely group-theoretic notions (which are probably useless in unstable groups):

Definition 43. Let S be a subset of the group G.

(1) S is *large* iff there are finitely many elements $g_1, \ldots, g_k$ such that

$$G \subseteq \bigcup_i S g_i;$$

(2) S is *small* iff for every finite subset F of S there are arbitrarily many elements $g_1, g_2, \ldots, g_k$ such that:

(sml) $\quad g_i F \cap g_j S = \emptyset \quad \text{for} \quad i < j.$

Lemma 44. *If S is not small, then S is large.*

Proof. Suppose S is not small. Fix a finite subset $F = \{s_1, \ldots, s_n\}$ of S and a maximal integer k such that there are elements $g_1, \ldots, g_k$ satisfying (sml). Fix such elements $g_1, \ldots, g_k$.

For any $g \in G$, F together with $g_1, \ldots, g_k, g^{-1}$ does not satisfy (sml) whereas F together with $g_1, \ldots, g_k$ does satisfy (sml). Thus for any $g \in G$ there are s_i, g_j such that:

$$g_j s_i \in g^{-1} S, \quad \text{so} \quad g \in S s_i^{-1} g_j^{-1}.$$

In short:

$$G \subseteq \bigcup_{i,j} S s_i^{-1} g_j^{-1}$$

and we have proved that S is large, as claimed.

We are not claiming that a set cannot be both large and small. For stable groups this assertion is part of:

Theorem 45. *Let G be a stable group and let S be a definable subset of G. Then the following are equivalent:*

(1) *S is small.*

(2) *There is a finite set of formulas Δ_0 such that for every finite invariant set of formulas Δ containing Δ_0, S is Δ-small.*

(3) *S is not large.*

If G is super stable another equivalent condition is:

(4) ∞-rank$(S) < \infty$-rank(G).

Clearly Theorem 45 contains Theorem 42 (with the obvious extension for ω-stable groups). Since we have proved (3) → (1) (Lemma 44) it will suffice to prove (1) → (2) → (3) and (1) → (4) → (3). The implications (2) → (3), (4) → (3) are entirely straightforward since the rank functions involved in (2), (4) are invariant (right invariance would be adequate) and satisfy

(sup) $\quad f(S_1 \cup S_2) = \sup(f(S_1), f(S_2))$.

Hence it suffices to prove (1) → (2) and (1) → (4). The proofs, which are almost identical, make use of the machinery of [12, Chapter III], which we will now review.

5.3. *Forking*

Definition 46. Let S be a definable subset of a structure A. Let F be an infinite family of definable subsets of A.

(1) F is a family of *equiuniformly definable* subsets of A iff there is a single formula:

$$\varphi(x, \bar{y})$$

and an infinite indiscernible set I of sequences $\bar{a}$ from A such that the sets in F are exactly the sets defined by the formulas:

$$\varphi(x; \bar{a}) \quad (\bar{a} \in I).$$

(2) S *splits strongly within* A iff there are sets S_1, S_2 belonging to an infinite family of equiuniformly definable subsets of A such that:

S is contained in S_1 and is disjoint from S_2.

(3) S *splits strongly* iff A has an elementary extension A' within which the canonical extension S' of S to A' splits strongly.

(*Note*: the canonical extension S' of S is defined in A' by any formula which defines S in A. We will have occasion to make substantial use of this notion.)

(4) S *forks* iff for some elementary extension A' of A, S is a finite union of sets which split strongly (cf. [12, Theorem III 1.6]). This is called "Forking over the empty set" in [12].

We will need the following facts:

Fact 47. *If A is stable and S is a definable subset of A which forks then there is a finite set of formulas* Δ_0 *such that S is* Δ*-small for any finite set* Δ *containing* Δ_0. *Similarly,* ∞-rank$(S) < \infty$-rank(G) *if* ∞-rank(G) *exists* [12, Lemma III 1.2].

Fact 48. *If A is stable, A′ is an elementary extension of A, and S is a definable subset of A′ disjoint from A, then S forks* [12, Corollary III 4.10].

Fact 49. *If A is stable, S is a definable subset of A, and S′ is the canonical extension of S in an elementary extension of* A_1, *then S forks iff S′ forks* (*trivial*).

We return now to the proof of Theorem 45. Recall that it suffices to prove: (1) → (2) & (4).

Lemma 50. *If S is a small definable subset of the group G, then there is an elementary extension G′ of G which contains an infinite sequence of elements*

$$g_1, g_2, \ldots$$

such that

$$g_i S' \cap g_j S' \subseteq g_i(G' - G) \quad \text{for} \quad i < j.$$

Proof. Introduce constants $g_1, g_2, \ldots$ and consider the theory T consisting of the complete theory of G (with names for all elements) together with sentences saying:

$$\text{“}g_j s \notin g_i S\text{”} \quad \text{for} \quad i < j \quad \text{and} \quad s \in S.$$

By definition S is small iff T is consistent, so we may take a model G' of T. Then in G' we have:

$$g_i S \cap g_j S' = \emptyset \quad \text{for} \quad i < j$$

and hence:

$$g_i S' \cap g_j S' \subseteq g_i(S' - S) \subseteq g_i(G' - G) \quad \text{for} \quad i < j.$$

Proof of (1) → (2) & (4) (Theorem 45). We assume that S is a small definable subset of G and we adopt the notation of Lemma 50, assuming in addition (via Ramsey's Theorem and the Compactness Theorem) that $g_1, g_2, \ldots$ are indiscernible. We will prove

(i) $S' \cap g_1^{-1} g_2 S'$ forks;
(ii) $g_1 S' - g_2 S'$ forks.

This and Fact 47 will yield (2) and (4) because:

$$S' = (S' \cap g_1^{-1} g_2 S') \cup g_1^{-1}(g_1 S' - g_2 S')$$

and the rank functions involved in (2), (4) are invariant.

Now we have:

$$S' \cap g_1^{-1} g_2 S' \subsetneq G' - G$$

by Lemma 50, so Fact 48 proves (i). Finally, if $X = g_1S' - g_2S'$, then X is contained in g_1S' and is disjoint from g_2S' where g_1, g_2 belong to an infinite family of indiscernibles, so X splits strongly in G' and hence forks within G', proving (ii). This completes the argument.

5.4. *Question* 37

We can now supply a partial answer to Question 37 of Section 4.3.

Proposition 51. *Let A be a superstable structure, let X, Y be definable subsets of A such that:*

$$\infty\text{-rank}(X),\ \infty\text{-rank}(Y) < \infty\text{-rank}(A).$$

Suppose that A is equipped with two binary operations, + and ·, such that:

$$\langle A - X, + \rangle \quad \textit{and} \quad \langle A - Y, \cdot \rangle \textit{ are groups.}$$

Then $\langle A - X, + \rangle$ is connected iff $\langle A - Y, \cdot \rangle$ is connected.

This follows at once from Theorem 41, which implies that the connectivity of $\langle A - X, + \rangle$ or $\langle A - Y, \cdot \rangle$ is equivalent to the indecomposability of A. (Note that Proposition 51 is adequate for the proof of Theorem 7 in the superstable case.)

6. Variations

We will embark on the project of extending the results in [4] to a larger class of stable groups in Section 7. This involves a systematic use of "localization", i.e. getting along with a fixed finite set of first-order formulas in the course of a given argument, and an unsystematic use of detours around the spots where this is impossible.

In the present section we supply technical variants of the tools of Sections 2, 4 used in this subsequent analysis.

6.1. *$A - \Delta$-connected groups*

Definition 52. Let G be a group, Δ a finite set of formulas such that Δ-rank G is defined, and A a subgroup of G.

(1) $A - \Delta(G)$ is the subalgebra of $\Delta(G)$ consisting of sets which are closed under right multiplication by elements of A.

(2) G is $A - \Delta$-*connected* iff there is no definable subgroup H of finite index in

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G such that:

(i) for some $S \in \Delta(G)$ $H \approx_\Delta S$.

(ii) $A \subseteq H$.

(3) G is (right) $A - \Delta$-*indecomposable* iff there is no decomposition $G = S_1 \cup S_2$ of G such that:

(i) $S_i \in A - \Delta(G)$ for $i = 1, 2$

(ii) Δ-rank$(S_i) = \Delta$-rank(G) for $i = 1, 2$.

Two special cases are important: if $A = (1)$ we speak of Δ-connected and Δ-indecomposable groups, while if Δ is the set of all formulas we speak of A-connected and of A-indecomposable groups. (When $A = (1)$ and Δ contains all formulas then we are speaking of connected groups or of indecomposable groups, that is ω-stable groups of Morley degree 1.)

There will be an Indecomposability Theorem in the next subsection. It is convenient at this point to survey the methods for obtaining connected or indecomposable groups of various sorts, because the proof of our first result provides information needed for the proof of the Indecomposability Theorem.

Theorem 53. *Let G be a stable group and let Δ be a finite invariant set of formulas. Then G contains a unique maximal $\Delta(G)$-indecomposable subgroup H of finite index. H is normal in G.*

Proof. Let K be the kernel of the permutation representation of G induced by a decomposition:

(dec) $\quad G = G_1 \cup \cdots \cup G_m$

of G into Δ-indecomposable pieces, where m is the Δ-multiplicity of G. Since K is of finite index in G, we have:

$$\Delta(G)\text{-rank}(K) = \Delta\text{-rank}(G).$$

Now the argument in Section 4.4 yields an elementary extension G^1 of G and an element $g \in G_1^1$ such that:

(inc) $\quad gK \subseteq G_1^1$,

and then as in Section 4.4 it follows easily that K is $\Delta(G)$-indecomposable.

We will now show that any $\Delta(G)$-indecomposable subgroup of G is contained in K, which will complete the proof of the theorem. This proceeds in several steps.

Step 1: The action of G on $G_1, \ldots, G_m$ modulo $\approx_\Delta$ is transitive:

Fix $1 \leq i \leq m$. For $1 \leq j \leq m$ let G_{ij} be the set of g such that:

$$G_i g \equiv_\Delta G_j.$$

Then in an elementary extension G^1 of G there are elements $g_i \in G_i$ satisfying:

(inc-j) $\quad g_i G_{ij} \subseteq G_i^1$.

Then the sets G_{ij} are Δ-indecomposable (see the end of Section 4.4) and since G has Δ-multiplicity m the decomposition:

$$G = \bigcup_{i} G_{ij},$$

shows that $\Delta(G)$-rank$(G_{ij}) = \Delta$-rank(G) for each j.

Hence no G_{ij} is empty, and G acts transitively, as claimed.

Step 2: Define i by:

(i) $\quad \Delta(G)\text{-rank}(K \cap G_i) = \Delta\text{-rank}(G).$

(Since K is $\Delta(G)$-indecomposable this makes sense.) Let K_i be the isotropy group of G_i. *Then*: every $\Delta(G)$-indecomposable subgroup L of finite index in G is contained in K_i.

First consider $H = L \cap K$. Then H is a $\Delta(G)$-indecomposable subgroup of finite index in G. Hence there is a unique j such that:

(j) $\quad \Delta(G)\text{-rank}(H \cap G_j) = \Delta\text{-rank}(G).$

Comparison of (i) and (j) shows $i = j$. Hence i can also be characterized by:

(i^1) $\quad \Delta(G)\text{-rank}(L \cap G_i) = \Delta\text{-rank}(G).$

Then for $g \in L$ we define another j by:

(j^1) $\quad G_i g \equiv_\Delta G_j$

and conclude:

$$(L \cap G_i)g \equiv_\Delta L \cap G_j.$$

Thus

$$\Delta(G)\text{-rank}(L \cap G_j) = \Delta\text{-rank}(G),$$

so again $i = j$. Then (j^1) says:

$$g \in K_i$$

as claimed.

Step 3: $K_i = K$. (Combined with Step 2, this completes the argument.)

Let $1 \leq j \leq m$ be arbitrary. Let K_j be the isotropy group of G_j. It suffices to show that $K_i = K_j$.

Now K_j is $\Delta(G)$-indecomposable by the argument given in Step 1 (since $K_j = G_{jj}$ in that notation). Hence by Step 2

(∗) $\quad K_j \subseteq K_i.$

On the other hand K_j and K_i are conjugate as a consequence of Step 1; if:

$$G_j g \equiv_\Delta G_i$$

then:

(con) $\quad g^{-1}K_i g = K_i$.

Now from this we conclude easily that $K_i = K_j$ by a stability argument. Just let Δ^1 be any finite invariant set of formulas containing the definitions of K_i and K_j, and compute:

$$\Delta^1\text{-multiplicity}(K_j) = \Delta^1\text{-multiplicity}(K_i),$$

which together with (∗) yields $K_i = K_j$.

This completes the proof of the theorem.

Lemma 54. *Let G be a group with a subgroup A. Suppose only finitely many definable normal subgroups of G contain A. Then G contains a unique definable A-connected subgroup H of finite index, and H is normal in G.*

Proof. Let H be the intersection of all definable normal subgroups of finite index in G. Then H is a definable normal subgroup of finite index. Suppose H contains a definable subgroup K of finite index. Then by a standard argument K contains a smaller definable normal subgroup of G, also of finite index contradicting the choice of H.

The uniqueness assertion is straightforward.

Corollary 55. *Let G be a group with a subgroup A. Suppose either:*

(1) G − A consists of finitely many G-conjugacy classes, or

(2) G consists of finitely many double cosets modulo A. Then G contains a unique definable A-connected subgroup H of finite index, and H is normal in G.

6.2. *An indecomposability theorem*

Theorem 56. *Let G be a stable group, A a subgroup, and Δ a finite invariant set of formulas. Then the following are equivalent:*

(1) G is A − Δ-connected.

(2) G is A − Δ-indecomposable.

Corollary 57. *Let G be a stable group, A a subgroup. Then the following are equivalent:*

(1) G is A-connected.

(2) G is A − Δ-connected for all finite invariant Δ.

(3) G is A − Δ-indecomposable for all finite invariant Δ.

(4) For any finite set of formulas Δ_0 there is a finite invariant set Δ containing Δ_0 such that G is A − Δ-indecomposable.

It is clear that Theorem 56 proves Corollary 57.

Proof of Theorem 56. (2) → (1). Let H be a definable subgroup of finite index in G such that H contains A, and suppose that for some $S \in \Delta(G)$: $H \equiv_\Delta S$. We claim that if G is $A-\Delta$-indecomposable, then $H = G$.

Indeed, suppose $g \in G$ and $gH \neq H$. Then $gH \equiv_\Delta gS$. Set $X = S - gS$. The pair (X, gS) contradicts the $A-\Delta$-indecomposability of G, the main point being that

$$\Delta\text{-rank}(S - gS) = \Delta\text{-rank}(G)$$

since $S - gS \equiv_\Delta H - gH = H$.

(1) → (2). We modify the argument in Sections 4.4–4.5. Assume that G is $A-\Delta$-connected and fix a decomposition

(dec) $\quad G = G_1 \cup \cdots \cup G_m$

of G into Δ-indecomposable sets in $\Delta(G)$, where m is the Δ-multiplicity of G.

Now suppose $S \in A - \Delta(G)$. We claim that S or $G - S$ is Δ-small.

Fix $I \subseteq \{1, \ldots, m\}$ such that:

(S-dec) $\quad S \equiv_\Delta \bigcup_i G_i$.

Let SG denote the set of right translates Sg of S in G, and consider the quotient set:

$$X = SG/\equiv_\Delta$$

of SG modulo the equivalence relation $\equiv_\Delta$. If the index set I in (S-dec) has k elements, then X has at most $\binom{m}{k}$ elements, as one sees by letting G act on (S-dec) by right multiplication and recalling that G acts (modulo $\equiv_\Delta$) as a group of permutations of $G_1, \ldots, G_m$.

Thus G acts as a permutation group of the finite set X. Let K be the isotropy group of S in X, defined explicitly as the set of $g \in G$ for which:

(S-fix) $\quad Sg \equiv_\Delta S$.

Since the index of K in G is the order of the orbit of S in X, this index is finite. We can use the argument of Section 4.5 to show that K is definable in G if (as we may assume) G is $\aleph_1$-saturated. For this purpose it suffices to rephrase the condition (S-fix) above for membership in K as follows:

$$\Delta\text{-rank}(G_i g \cap S) = \Delta\text{-rank}(G) \quad \text{for} \quad i \in I.$$

Notice also that $A \subseteq K$, since $Sa = S$ for $a \in A$.

Now we will show that K differs from an element of $\Delta(G)$ by a Δ-small set. Let L be the kernel of the permutation representation of G acting on $G_1, \ldots, G_n$ modulo $\equiv_\Delta$. Then L is a subgroup of finite index in K, so it suffices to prove that L differs from an element of $\Delta(G)$ by a Δ-small set. In the proof of Theorem 53 we saw the following:

(1) There is a unique index i for which

$$\Delta\text{-rank}(L \cap G_i) = \Delta\text{-rank}(G).$$

(2) L equals the isotropy group of G_i.

From (1), (2) we may conclude:

(3) With i as in (1) and $g \in G - L$:

$$Lg \cap G_i \text{ is } \Delta\text{-small.}$$

It follows that $L \equiv_\Delta G_i$, since L is of finite index in G.

Now the rest is easy. K is a definable subgroup of finite index in G, containing A, and differing from an element of $\Delta(G)$ by a Δ-small set. Since G is $A - \Delta$-connected we conclude:

$$K = G.$$

Then as in the last part of Section 4.4 one finds an elementary extension G^1 of G in which there is an element s so that:

(inc) $\quad sG \subseteq S^1$.

In particular this yields:

(1) $sS \subseteq S^1 \cap sS^1 = X$ (say),

(2) $s(G - S) \subseteq S^1 \cap s(G^1 - S^1) = Y$ (say),

and one concludes as in Section 4.4 that either S or $G - S$ is Δ-small, as claimed.

This completes the proof of Theorem 56.

6.3. *A few lemmas*

We mention two useful properties of A-connected groups.

Lemma 58. *If K is a normal subgroup of the A-connected group G, then G/K is AK/K-connected.*

Lemma 59. *If N is a finite normal subgroup of the A-connected group G and N is contained in the centralizer of A, then N is contained in the center of G.*

(One looks at the kernel of the permutation representation given by the action of G on N via conjugation; cf. [5, § 3] for $A = (1)$.)

III. SUPERSTABLE GROUPS

7. Generalities

We now enter upon the extension of the results of [4] to broader classes of stable groups. Our basic idea is to replace ω-stability by superstability and Morley rank by ∞-rank in [4]; the main complication arising therefrom is the necessity for working with disconnected groups.

The main results of this part are as follows:

Theorem 62. (see Section 7.1). *A stable group of ∞-rank 1 is abelian-by-finite.*

Theorem 63. (see Section 8). *A stable group of ∞-rank 2 is solvable-by-finite.*

Theorem 64. (see Section 10). *A stable group of* ∞-rank 3 *which contains a definable subgroup of* ∞-rank 2 *is either solvable-by-finite or else contains a subgroup of finite index isomorphic to one of the groups*:

$$\mathrm{SL}(2, F) \quad \textit{or} \quad \mathrm{PSL}(2, F)$$

with F an algebraically closed field.

7.1. *Abelian subgroups*

The main result of this subsection will be:

Theorem 68. *Let G be an infinite $\aleph_0$-saturated group. If G is superstable, then G contains an infinite abelian subgroup.*

The next three lemmas can be replaced by trivial arguments under the hypotheses of Theorem 68, but they cast some light on the general case.

Lemma 69. *Let G be a stable group containing a normal subgroup N such that at least one G-conjugacy class S contained in N is infinite. Then N contains an infinite G-definable subgroup K which is normal in G.*

Proof. Let Δ be a finite invariant set of formulas containing the definition of S. For any integer k let

$$S^k = \{s_1 \cdot \ldots \cdot s_k : s_i \in S\}.$$

Let k be chosen so that:

$$\Delta\text{-rank}(S^k) = r$$

is as large as possible. For $g, h \in G$ define an equivalence:

$g \sim h$ iff $S^k g$ and $S^k h$ differ by a set of Δ-rank less than r (with a slight alteration of earlier notation we will write: $S^k g \equiv S^k h$).

Since $S^k g \subseteq S^{2k}$ for $g \in S^k$, it follows that S^k decomposes into finitely many equivalence classes (S^{2k} has Δ-rank r and finite multiplicity). Let X be one of these equivalence classes. It follows easily that X is definable, as in Section 4.5 (this depends on the fact that $X \subseteq S^k$, and that S^k meets only finitely many equivalence classes in G.) We may take X to have Δ-rank r. Fix $x \in X$.

Now let K be the equivalence class of 1 in G, i.e. the isotropy group of S^k modulo sets of lower rank. Notice that $Xx^{-1} \subseteq K$. We will now show that K is definable. The elementary extension argument at the end of Section 4.4 yields an elementary extension G^1 of G and an element $g \in S^k$ for which:

(inc) $\quad gK \subseteq (S^k)^1$.

From this one derives easily that any definable subset of K has Δ-rank at most r and Δ-multiplicity at most the Δ-multiplicity of S^k. Hence there is a maximal finite set

$$g_1, \ldots, g_k$$

of elements of K such that:

$$Xx^{-1}g_1, \ldots, Xx^{-1}h_k \text{ are mutually disjoint.}$$

Then for any $g \in X$ there is an i such that

$$Xx^{-1}g \cap Xx^{-1}g_i \neq \emptyset.$$

Thus:

$$K = \bigcup_i xX^{-1}Xx^{-1}g_i,$$

and it follows that K is definable, as claimed.

Clearly $K \subseteq N$, and since S^k is closed under conjugation K is normal in G. This completes the argument.

Lemma 70. *Let G be an infinite stable group with finite center in which the centralizer of an arbitrary element is finite of bounded order. Then G contains an infinite definable stable subgroup H such that all proper normal subgroups of H are contained in the center of G. In particular H is connected.*

Proof. Let Δ be a finite invariant set of formulas. We prove first:

(Δ) The collection of normal infinite subgroups of G which are in $\Delta(G)$ contains a unique minimal element.

Using $\Delta-2$-rank [12, Chapter II] it is easy to see that any normal infinite subgroup of G which is in $\Delta(G)$ contains a minimal such subgroup. Suppose now that H_1, H_2 are distinct minimal normal infinite $\Delta(G)$-subgroups, so that the intersection:

$$H = H_1 \cap H_2$$

must be finite. Then any element $h \in H_1$ centralizes an infinite subgroup of H_2, since commutation maps $H_1 \times H_2$ into H. This contradicts our assumptions, and establishes claim (A).

Call the group defined by (Δ) G_Δ, and set:

$$H = \bigcap_{\substack{\Delta \text{ finite} \\ \text{invariant}}} G_\Delta$$

(this is the intersection of a directed system). Taking G to be sufficiently saturated (and noting that the hypotheses are preserved by elementary extension), we may suppose that H is infinite. It is a normal subgroup of G and every noncentral conjugacy class in H is infinite, since centralizers of noncentral elements are finite. Thus Lemma 69 shows that H contains an infinite definable normal subgroup K of G. By construction:

$$K = H.$$

Thus H is definable in G, and is the smallest infinite definable normal subgroup of G. Now apply the same construction to obtain the smallest G-definable infinite subgroup N of H which is normal in H. It is clear that N is also normal in G, so $N = H$. Thus H has no proper definable infinite normal subgroup, and Lemma 69 shows easily that H has no infinite normal subgroup.

Finally, suppose F is a finite normal subgroup of H. Since H is clearly connected, F is contained in the center Z of H. But Z is a finite normal subgroup of G, and since noncentral elements have infinite conjugacy classes, Z is contained in the center of G. Thus F is central in G, and the proof of Lemma 70 is complete.

Lemma 71. *Let G be an infinite $\aleph_0$-saturated stable group containing no infinite abelian definable subgroup. Then there is an infinite stable $\aleph_0$-saturated simple group such that the centralizer of each element is finite of bounded order. Such a group is a torsion group of odd finite exponent.*

Proof. Applying the stable chain condition to infinite centralizers in G, we may assume that the hypotheses of Lemma 70 are satisfied, and take H as in the conclusion of Lemma 70. Then

(1) H is infinite, stable and $\aleph_0$-saturated;

(2) H has no infinite abelian subgroup;

(3) H has no noncentral proper normal subgroup;

(4) All centralizers of noncentral elements of H are finite of bounded order.

Let Z be the center of H. Then H/Z is an infinite stable $\aleph_0$-saturated simple group. Let a/Z be a nontrivial element of H/Z, and let C/Z be the centralizer of a/Z in H/Z. Since Z is finite, a commutes with a subgroup of finite index in C.

and it follows that C is finite, so that C/Z is finite. Thus H/Z has all the desired properties.

As to the final remark, such a group has odd exponent by [8, Theorem 2.1].

Remark 72. The existence of such a group is highly unlikely, but this question may involve combinatorial group theory essentially.

Proof of Theorem 68. By Lemma 71, if there is a counterexample G, then we may suppose G is infinite, superstable, and connected, and that the centralizer of every nontrivial element of G is finite of odd order. For $g \in G - 1$ the conjugacy class g^G of g in G may be identified with the coset space:

$$C(g) \setminus G,$$

and since the centralizer of each element is finite, it follows from Lemma 65 that the ∞-rank of g^G coincides with that of G.

Now if G is superstable, the Indecomposability Theorem 41 implies that $G - 1$ consists of a single conjugacy class, and the desired contradiction follows by an elementary group theoretic result given in [14]:

Fact 73. *Let G be a torsion group containing a single nontrivial conjugacy class. Then G is finite, of order at most 2.*

Corollary 74. *Let G be a stable group of* ∞-rank 1. *Then G is abelian-by-finite.*

Proof. By Theorem 68, G contains an infinite abelian definable subgroup A. If the index of A in G were not finite, it would follow easily that the ∞-rank of G would be at least 2 (as usual, consider cosets of A in G).

7.2. *Stable nilpotent groups*

Definition 75. The group G is *centralizer-connected* iff no concentral element has a centralizer of finite index in G (equivalently every conjugacy class with more than one element is infinite).

Lemma 76. *Any stable group has a centralizer-connected subgroup of finite index.*

Proof. Apply the stable chain condition to centralizers.

Lemma 77. *Let G be a centralizer-connected infinite nilpotent group. Then the center Z of G is infinite.*

Proof. If Z is finite, let a/Z be a nontrivial element of the center of G/Z. The conjugates of a all lie in the set aZ, which is finite, so a is central by Definition 75, which contradicts the choice of a.

Corollary 78. *Any infinite stable nilpotent group has a subgroup of finite index whose center is infinite.*

7.3. *Stable solvable groups*

Remark 79. Let G be a stable group. Let A be a maximal abelian subgroup or a maximal normal abelian subgroup. Then A is definable.

(In either case A is the center of its own centralizer, and the stable chain condition implies that this is a definable set.)

Lemma 80. *Let G be an infinite stable solvable centralizer-connected group. Then G contains an infinite normal abelian definable subgroup.*

Proof. Note that any finite normal subgroup F of G is central in G (since its centralizer in G has finite index).

Let Z be the center of G, which we may assume to be finite. Let B be the inverse image in G of a nontrivial normal abelian definable subgroup B/Z of G/Z. Then B is not central in G, so B is infinite. Furthermore B is nilpotent of class two. If B^0 is the intersection of all centralizers of finite index in B, then B^0 is normal in G, and by Lemma 77 its center is an infinite normal abelian subgroup of G.

Corollary 81. *If G is an infinite stable solvable group, then G contains an infinite abelian definable subgroup whose normalizer has finite index in G.*

8. Theorem 63

Recall Theorem 63: *A superstable group of* ∞-rank 2 *is solvable-by-finite.* The proof of this theorem will be divided into three subsections.

8.1. *Preliminary analysis*

We begin the analysis of a superstable group of ∞-rank 2. If G is not solvable-by-finite a contradiction will emerge. For the present we assume only:

(hyp 1) G is not abelian-by-finite.

Let A be an infinite abelian definable subgroup of G (Theorem 68). By (hyp 1) the index of A in G is infinite. It follows that A has ∞-rank 1.

Definition 82. Let G be a group with a subgroup A.

(1) The element $g \in G$ *quasinormalizes* A iff A and A^g are commensurable (i.e. $A \cap A^g$ is of finite index in both A and A^g). For G superstable an equivalent

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condition is

$$\infty\text{-rank}(A \cap A^g) = \infty\text{-rank}(A).$$

(2) The *quasinormalizer* of A is the group of all elements of G which quasinormalize A. It will be denoted $Q(A)$.

Lemma 83. *Let G be a stable group and let A be a definable subgroup of G. Then there is a definable subgroup A_0 of finite index in A such that:*

$$N(A_0) = Q(A).$$

(Note that if A has a connected subgroup A_0 of finite index, then this is obvious.)

Proof. Apply the stable chain condition to the family of groups of the form A^g where $g \in Q(A)$. Let A_0 be the intersection of all such groups. Since this can be reduced to a finite intersection, the index of A_0 in A is finite, and in particular:

$$Q(A_0) = Q(A).$$

By construction:

$$Q(A) \subseteq N(A_0).$$

hence:

$$Q(A_0) \subseteq N(A_0)$$

and the reverse inclusion is trivial. This completes the argument.

By a change in our notation we may assume the group A has been chosen in accordance with the above lemma:

(hyp 2) $Q(A) = N(A)$.

We now set $N = N(A)$.

The analysis now divides into an easy and a difficult case, according as the ∞-rank of N is 1 or 2.

Lemma 84. *If N has* ∞-rank 2, *then G is solvable-by-finite.*

Proof. It suffices to show that N is solvable-by-finite. Clearly N/A has ∞-rank 1 and hence is abelian-by-finite by Corollary 74. The result follows.

Accordingly we may now assume:

(hyp 3) [∞-rank$(N) = 1$.]

Lemma 85. *G contains an A-connected subgroup G_0 of finite index.*

Proof. For $g \in G - N$ the intersection $A \cap A^g$ is finite. It follows that for such g: AgA has ∞-rank 2. (To see this, consider the uniformly definable infinite sets

$$Agc$$

where c varies over cosets of $A \cap A^g$ in A.)

Since the double cosets AgA are uniformly definable, it follows that $G - N$ is a finite union of double cosets of A. On the other hand the index of A in N is finite, so N is also a finite union of double cosets (= simple cosets) of A.

Thus G breaks up into finitely many double cosets of A, and Corollary 55 applies.

Now by a change of notation we may assume:

(hyp 4) [G is A-connected.]

Notation 86. Z is the center of G.

From now on we assume:

(hyp 5) [G is not nilpotent-by-finite.]

Lemma 87. *Z is finite.*

This is proved like Lemma 84.

Now consider the group $H = G/Z$ and the subgroup $B = AZ/Z$. We claim that if the pair (G, A) is replaced by the pair (H, B) (so that N is replaced by $N(B)$) then the hypotheses (hyp 1–5) remain valid. This is clear for (hyp 5) (and similarly if G is not solvable-by-finite, then the same applies to H). For (hyp 4) see Lemma 58. For (hyp 2–3) it is sufficient to prove:

(1) $Q(AZ/Z) = Q(A)/Z$;

(2) $N(AZ/Z) = N(A)/Z$.

Now we clearly have:

$$[Q(A)/Z = N(A)/Z \subseteq N(AZ/Z) \subseteq Q(AZ/Z).]$$

Thus if suffices to prove:

$$[Q(AZ/Z) \subseteq Q(A)/Z]$$

and since Z is finite this is straightforward.

Lemma 88. *For $g \in H - N(B)$, $B \cap B^g = (1)$.*

Proof. Let $a \in A$ represent $b \in B \cap B^g$. Then:

$$a \in A^g Z.$$

It follows that A and A^g are contained in the centralizer $C(a)$ of a in G. Since $A \cap A^g$ is finite the set:

$$A \cdot A^g$$

has ∞-rank 2, and hence $C(a)$ has ∞-rank 2, and is therefore of finite index in G. Since G is A-connected we get

$$G = C(a),$$

so $a \in Z$, and $b = 1$ in H, as claimed.

Notation 89. We change our notation, writing G for H, A for B, N for $N(B)$, and Z for $Z(H)$. Then we have in addition to (hyp 1–5):

(hyp 6) For $g \in G - N$ $A \cap A^g = (1)$.

8.2. *The Bruhat Decompostion*

Now we can get detailed structural information concerning G. (Cf. [4, Lemma 4, §4.1].

Theorem 90. *If $w \in G - A$, then $G = A \cup AwA$. The element w may be chosen to be an involution (i.e. of order 2). Furthermore $A = N(A)$.*

Proof. We proceed in four steps.

Step 1: Fix g in $G - N$. Then $G = N \cup AgA$: As noted in the proof of Lemma 85, for g in $G - N$ the double coset AgA has ∞-rank 2. Now since G is A-connected, condition 4 of the relativized Indecomposability Theorem (Theorem 56) shows that there can be only one such double coset, as claimed.

Step 2: $G - N$ contains an involution. Fix $g \in G - N$. By Step 1 we can write:

$$g^{-1} = a_1 g a_2$$

with $a_1, a_2 \in A$. Let $w = ga_1$. Then $w^2 = a_1 a_2^{-1} \in A$. Setting $a = w^2$ we get:

$$a = a^w \in A \cap A^w = 1,$$

and thus w is an involution. Now fix such an involution.

Step 3: Let $K = N \cap A^w$. Then $N = A \rtimes K$ (semidirect product): Evidently $K \subseteq N$ normalizes A and $K \cap A = (1)$. It suffices therefore to show that $N = AK$.

For any n in N since $nw \notin N$ we may write:

$$nw = a_1 w a_2$$

with $a_1, a_2 \in A$. Then $a_1^{-1} n = a_2^w \in A^w \cap N = K$. Thus $n \in a_1 K$ and $N = AK$.

Step 4. $K = (1)$ (and hence $N = A$): Consider $a \in K^w$. Then $a \in A$ and $a^w \in N$. We claim that $a^g \in N$ for all $g \in G$. This is clear if $g \in N$ while if $g = a_1 w a_2$ with

$a_1, a_2 \in A$, then:

$$a^g = (a^w)^{a_2}$$

is also in N. Thus a^G is contained in N, as claimed.

Now set $B = \langle a^g : g \in G \rangle$. Then $B \triangleleft G$ and B is contained in N. If $a \neq 1$ we will now obtain a contradiction.

By (hyp 6) since $a \in A$ therefore a is noncentral, and hence $C(a)$ has infinite index. This implies that B is infinite. Since $[N : A \cap B]$ is finite, therefore $[B : A \cap B] < \infty$. Conjugating by w,

$$[B : A^w \cap B] < \infty, \quad \text{so} \quad [B : A \cap A^w \cap B] < \infty,$$

contradicting (hyp 6).

Thus $a = 1$, $K = (1)$, and $N = A$, completing the proof.

The double coset decomposition described in Theorem 90 is called the *Bruhat decomposition* of G. The motivation for this is described in [4, Section 4.4].

Lemma 91. *If $g \in G$ and the index of the centralizer $C(g)$ of g in G is finite, then $g = 1$. (In particular $Z = (1)$).*

Proof. Let F be the subgroup of G consisting of elements whose centralizer in G has finite index. Let H be the centralizer of F. Then H is of finite index in G (Lemma 76), hence is not nilpotent-by finite. It follows as in the proof of Lemma 87 that $F \cap H$ is finite, and hence F is a finite normal subgroup of G.

We will now show that any finite normal subgroup N_0 of G is trivial. Since $A^g \subseteq A \cdot N_0$ for $g \in N_0$, it follows that:

$$N_0 \subseteq Q(A) = A$$

by (hyp 2) and Theorem 66. Then since A is abelian N_0 centralizes A, so $N_0 \subseteq Z$ by Lemma 59. However,

$$Z \cap A = (1)$$

by (hyp 6). This proves that N_0—and in particular F—is trivial.

8.3. *Conjugacy classes*

In addition to the double coset decomposition of Section 8.2 we will have to acquire information concerning conjugacy classes in G.

Lemma 92. *With the hypotheses and notations of Section 8.2, G contains a definable subgroup K of finite index such that no conjugacy class of K has* ∞-rank 2.

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Proof. Set:

$$X = \bigcup_{g \in G} A^g.$$

Then clearly the ∞-rank of X is 2 (using (hyp 6)). Furthermore, for $a \in A$ the centralizer $C(a)$ of a in G contains A, hence has ∞-rank at least one, and it follows that the conjugacy class of a has ∞-rank at most one.

The problem then is to study the conjugacy classes of elements outside x. Since conjugacy classes are uniformly definable, there can be at most finitely many of ∞-rank 2 in G, say $C_1, \ldots, C_k$.

Suppose we are able to find definable subgroups K_i of finite index in G such that:

$$K_i \cap C_i = \emptyset \quad (i = 1, \ldots, k).$$

Then we may set $K = \bigcap K_i$ and we will be done.

It therefore suffices to consider a single conjugacy class C of ∞-rank 2, and to find a definable subgroup of finite index in G which is disjoint from C.

Now given any definable set $S \subseteq G$, we will have the equivalent:

$$\Delta\text{-rank}(S) = \infty\text{-rank}(S)$$

for all sufficiently large finite sets Δ of formulas. (Cf. [12, Chapter II Example 1.10 ($\lambda = \infty$) and Theorem 3.13 ($\lambda = \aleph_0$)].) Fix a finite invariant set Δ of formulas satisfying:

(1) Δ-rank$(C) = 2$;

(2) The formulas "$x = y$", $x \in y^{G}$", "$x \in A^y$", "$x \in X$" belong to Δ (any additional parameters occurring in the last two formulas should be replaced by free variables).

By Theorem 53 we may fix a Δ-indecomposable normal definable subgroup K of finite index in G. We claim:

$$K \cap C = \emptyset.$$

If on the contrary the intersection is nonempty, then:

$$C \leqslant K$$

since K is normal in G. Since K is Δ-indecomposable and the sets X and C are in $\Delta(G)$, therefore:

$$\Delta\text{-rank}(K \cap X) < 2.$$

This, however, yields an immediate contradiction, since the family:

$$\{(K - (1)) \cap A^g : g \in G\}$$

gives a Δ-splitting of K-(1) into infinite pieces (as $K \cap A^g$ is of finite index in A^g). This contradiction completes the argument.

We will have further use for this particular subgroup K in the next lemma. In particular note that by Lemma 91,

$$A = C(a)$$

for any nontrivial $a \in A$, so we may assume that the formulas defining A and X in Δ are formalizations of:

$$\text{“}x \in C(y)\text{”}$$

$$\text{“}x \in \bigcup_{G} C(y^g)\text{”}.$$

Assume now:

(hyp 7) [G is not solvable-by-finite.]

Lemma 93. *The group K constructed in the proof of Lemma 92 is contained in X.*

Proof. Suppose $b \in K - X$ and let $Y = \bigcup_{g \in G} C(b^g)$. We claim

(1) $X \cap Y = (1)$;

(2) Δ-rank$(K \cap X) = \Delta$-rank$(K \cap Y) = 2$ for large Δ. This will contradict the Δ-indecomposability of K.

As far as (1) is concerned, if the intersection of X and Y is nontrivial, then we can assume there is a nontrivial element a in:

$$A \cap C(b).$$

Then we get $a = a^b \in A \cap A^b$, so by (hyp 6):

$$b \in N(A) = A \quad \text{(Lemma 88)},$$

contradicting $b \in K - X$.

As for (2), we showed above:

$$\Delta\text{-rank}(K \cap X) = 2.$$

We consider $K \cap Y$.

Now $C(b)$ is infinite since ∞-rank$(b^G) < 2$. Hence $C(b^g) \cap K$ is infinite for all $g \in G$. If Δ-rank$(C(b)) = 2$, then (2) is trivial, so assume:

$$\Delta\text{-rank } C(b) = 1.$$

Then with the help of Lemma 83 and the hypothesis (hyp 7) it is easy to see that the index of the quasinormalizer

$$Q(C(b))$$

in G is infinite. Letting g run over coset representatives in G modulo $Q(C(b))$, we claim:

$$\{(K \cap C(b^g)) - (1)\}$$

is a Δ-splitting of $K \cap Y$ into infinite pieces. This will complete the proof of (2).
All that needs to be proved then is that the intersections

$$C(b) \cap C(b^g)$$

are trivial if $g \in G - Q(C(b))$. This is an easy variation of the proof of Lemma 88.

Thus the proof of Lemma 70 is complete.

Corollary 94. *A contains an involution.*

Proof. It is clear that the group G we considered originally (before the changes in notation) contained an involution. Hence the same argument proves that K contains an involution, and then Lemma 93 implies that A contains an involution.

Proof of Theorem 63. We derive a contradiction from the above analysis of a counterexample.

Let $i, j \in K$ be involutions in distinct conjugates of A. Let $a = ij$. By Lemma 93 we may assume $a \in A$. Note $a \neq 1$.

By a trivial computation:

$$a^i = a^{-1} \in A \cap A^i.$$

Hence by (hyp 6) and Lemma 88 $i \in N(A) = A$. Similarly $j \in A$ contradicting the choice of i and j.

9. Solvable groups of ∞-rank

We need a more precise analysis of groups of ∞-rank 2 for use in the analysis of groups of ∞-rank 3. We will study the solvable nonnilpotent groups of ∞-rank 2. The main example of such a group is the semidirect product:

$$F_+ \times F^*$$

of the additive and multiplicative groups of an algebraically closed field F, where F^* acts on F_+ by multiplication. The general case will turn out to be not too far from this example.

Our main result will be:

Theorem 95. *Let G be a superstable group of ∞-rank 2 which is not nilpotent-by-finite. Then G contains a subgroup H of finite index such that:*

(1) *the center Z of H is finite;*

(2) *the quotient H/Z is isomorphic to a semidirect product:*

$$F_+ \times F^*$$

of the additive and multiplicative groups of an algebraically closed field F, F acting on F, by multiplication.*

Lemma 96. *With the hypotheses of Theorem 95, G contains a connected abelian subgroup of* ∞-rank 1 *whose normalizer is of finite index in G*

Proof. We may assume that G is solvable and centralizer connected (Definition 75). Let A be an infinite normal abelian definable subgroup of G (Lemma 80). Then the index of A in G is infinite, so ∞-rank$(A) = 1$. The center Z of G is finite since G is not nilpotent-by-finite.

Fix $a \in A - Z$. The conjugacy class of a in G is infinite, since otherwise the centralizer of a would disconnect G. Since A has ∞-rank 1, there can be only finitely many such conjugacy classes. If follows that A contains only finitely many normal subgroups of G.

Now any definable subgroup B of G which is a subgroup of A of finite index in A must contain a normal definable subgroup of G which is again of finite index in A (apply the stable chain condition to the conjugates of B). It follows that if A^0 is the smallest definable subgroup of A which is normal in G and of finite index in A, then A^0 is connected. This proves the lemma.

Lemma 97. *Let G be a superstable group of* ∞-rank 2 *which is not nilpotent-by-finite. Then G contains a connected subgroup of finite index.*

Proof. We may take G to be solvable, centralizer-connected. Then the conjugacy class of any noncentral element is infinite. Since G is not nilpotent-by-finite, the center Z_0 of G is finite.

By Lemma 80, we can fix an infinite normal abelian subgroup A of G, and by the proof of Lemma 96 A will contain a connected subgroup U of finite index. U is again a normal abelian subgroup of G. Let $Z = U \cap Z_0$, and fix $u \in U - Z$. Let C be the centralizer of u in G.

Now C contains U and ∞-rank$(U) = 1$, so U has finite index in C. Let this index be called k. Our main claim is:

(ind) for any definable subgroup H of finite index in G, the index of H in G is bounded by k.

This will yield the conclusion of the lemma at once, so it suffices to verify (ind).

Fix H a definable subgroup of finite index in G, and consider the conjugacy class u^H of u in H. Applying Corollary 57 and noting that u^H is an infinite subset of $U - Z$, we conclude easily that $U - Z$ reduces to the single conjugacy class u^H. In particular u^H is invariant under conjugation by G, so for $g \in G$ we can solve the equation:

$$u^h = u^g$$

with $h \in H$, so $gh^{-1} \in C$, $g \in CH$, and thus $G \subseteq CH$.

On the other hand $U \subseteq C \cap H$ (since U is abelian and connected), so the index of H in G is at most k.

Lemma 98. *Let G be a connected nonnilpotent centerless group with* ∞-rank $(U) = 1$. *Then for some algebraically closed field F, G is isomorphic to the semidirect product:*

$$F_+ \times F^*$$

of the additive and multiplicative groups of F, where F^* acts on F_+ by multiplication.

Proof. By Lemma 96 we may fix a connected abelian normal subgroup U of G having ∞-rank 1. (This will turn out to be a copy of F_+.) Fix $u \in U-(1)$. As in the proof of Lemma 97 it follows that $U-(1) = u^G$.

Now fix $b \in G - C(U)$ and set $T = C(b)$. Form the set of commutators:

$$X = \{[b, u]: u \in U\}.$$

Clearly $T \cap U$ is finite and hence X is infinite. Furthermore since G/U is abelian, $X \subseteq U$. But U is connected of ∞-rank 1, and it follows that $U - X$ is finite.

We claim now that $UT = G$. It suffices to prove that UT has finite index in G. Fix $g \in G - UT$ and consider $[b, g]$. If $[b, g] \in X$ it follows easily that $g \in UT$. Hence $[b, g] \in U - X$. However, $U - X$ is finite, and if $[b, g_1] = [b, g_2]$ then $g_1 \in Tg_2$, so it follows that $G\text{-}UT_0$ contains only finitely many right T-cosets. Hence a fortiori UT is of finite index in G, and we conclude that $G = UT$, as claimed.

Since $G = UT$ and $U \cap T$ is finite, it follows that T is infinite and ∞-rank$(T) = 1$. Let T_0 be an infinite abelian definable subgroup of T. Then $G = UT_0$ (since G is connected) and $U \cap T_0 = (1)$, since G is centerless. Make a small change of notation, writing T for T_0. So far we have obtained a semidirect product decomposition

$$G = U \dot{\times} T.$$

For $t \in T$ define:

$$\hat{t} = tut^{-1}.$$

We claim the map:

$$\hat{}: T \to U-(1)$$

is a 1–1 onto map. It is onto since $U-(1) = u^G = u^{UT} = u^T$, and 1–1 since from $\hat{s} = \hat{t}$ we conclude easily that the centralizer of st^{-1} contains

$$T \cup \{u\},$$

and this is a set of generators for G, so $st^{-1} \in Z(G) = (1)$.

Now we can convert T into the multiplicative group of a field. Adjoin to T a formal symbol 0, and extend the multiplication on T to $T\cup\{0\}$ by the rule:

$$x\cdot 0=0\cdot x=0.$$

Define also $\hat{0}=1$ (the identity element of U). Let $F=T\cup\{0\}$, and define addition on F by:

$$(x+y)^\wedge=\hat{x}+\hat{y}$$

(on the right + denotes the group operation restricted to U).

It is easy to verify that F is a field, cf. [4, Theorem 1 of Section 4.2]. Furthermore F is superstable, hence algebraically closed. Thus the proof of the lemma is complete.

Lemma 99. *Let G be a connected nonnilpotent group of ∞-rank 2. Then for some algebraically closed field f, G is isomorphic to a semidirect product:*

$$F_+\times T$$

where T is a connected abelian divisible subgroup of G containing the center Z of G and such that

$$T/Z\simeq F^\cdot$$

via an isomorphism which transforms the action of T/Z on F_+ via conjugation into the action of $F^\cdot$ by multiplication.

Proof. By an argument that we have used repeatedly, the center Z of G is finite and G/Z is centerless (since the center of G/Z is finite and pulls back to a finite normal subgroup of G, which is necessarily central). The previous lemma yields a factorization:

$$G/Z\simeq F_+\times F^\cdot$$

for some algebraically closed field F. Let U_1, T_1 be the inverse images of F_+, $F^\cdot$ in G.

Both U_1 and T_1 contain abelian subgroups U, T of finite index, and we may take U to be normal in G. Then by the proof of Lemma 96 we may even take U to be connected.

Now UT has ∞-rank 2, so $UT=G$. Next we will show:

(int) $\quad U\cap Z=(1)$;

then since clearly $U\cap T\subseteq Z$ it follows that:

(spl) $\quad U\cap T=(1), \qquad G=U\times T.$

Our claim (int) is proved as follows. Fix $u\in U-Z$ and define:

$$\hat{t}=tut^{-1}$$

for $t \in T$. Since U is connected Corollary 57 shows that $\hat{T}$ is cofinite in U ($\hat{T} = u^G$, so $\hat{T}$ is infinite). If $U \cap Z$ is nontrivial it follows that for some $t_1 \neq t_2$ in T we get an equation:

$$\hat{t}_1 = z\hat{t}_2 \quad \text{with} \quad z \in U \cap Z, \quad Z \neq 1.$$

Then modulo Z we have $\hat{t}_1 = \hat{t}_2$, so in F^* (viewed as a sub-group of G/Z) we get $t_1/Z = t_2/Z$. However, this yields:

$$\hat{t}_1 = \hat{t}_2, \quad \text{so} \quad z = 1,$$

a contradiction.

Thus (int) is proved, and (spl) follows. In particular it now follows that T is connected. Now T is of ∞-rank 1 and connected, so it follows easily that T is either of prime exponent or divisible. Since $T/Z \cap T$ is an algebraically closed field, we must have T divisible.

Finally we show that $Z \subseteq T$. If $z = ut \in Z$, where $u \in U$, $t \in T$, then

$$a = a^z = a^t \quad \text{for} \quad a \in U$$

and hence t centralizes both U and T. Then $t \in Z$ and $u \in Z \cap U = (1)$, so that $z = t \in T$, as claimed.

Proof of Theorem 95. Combine Lemmas 97 and 99.

At this point we can get extra information simply by repeating arguments in [4]. The following result occurs in [4, §4.2 as Theorems 3 and 4].

Theorem 100. *Let G, Z, T, F, be as in the statement of Lemma 99 and write U for F_+ viewed as a normal subgroup of G. Then:*

(1) *If H is a subgroup of G such that the structure*

$$G_H = \langle G : H \text{ distinguished} \rangle$$

has ∞-rank 2 and so that U, T are connected in G_H, then H is definable in G. If H is infinite and unequal to G, then H has one of the following two forms:

(i) *$U \times L$ with $L \subseteq T$ finite;*

(ii) *T^u with $u \in U$.*

(2) *Let α be an automorphism of G such that the structure*

$$G_\alpha = \langle G; \alpha \rangle$$

has ∞-rank 2 and so that U, T are connected in G_α. Suppose that for some $n > 0$

$$\alpha^n = 1$$

as an automorphism of G. Then α is an inner automorphism.

10. Groups of ∞-rank 3.

Definition 101. A superstable group G of ∞-rank 3 is *good* if it contains a definable subgroup of ∞-rank 2, and is *bad* otherwise.

This section is devoted to a proof of Theorem 64, which reads as follows: *A good group of* ∞-rank 3 *is either solvable-by-finite or contains a subgroup of finite index isomorphic to one of the groups*:

$$\mathrm{SL}(2, F) \quad \textit{or} \quad \mathrm{PSL}(2, F)$$

with F an algebraically closed field.

To avoid unnecessary repetition of arguments given in detail in [4] we will restrict ourselves to the proof of the following, which is all that is needed to carry out the arguments in [4] using ∞-rank rather than Morley rank.

Lemma 102. *Let G be a stable group of* ∞-rank 3 *and let B be a definable subgroup of* ∞-rank 2. *Assume that G is not solvable-by-finite. Then*:

(1) *B contains a connected nonnilpotent definable subgroup of finite index*;

(2) *G contains a connected subgroup of finite index.*

(One also needs all the information in Section 9, which is why we went through it in detail.)

We break the proof of this lemma up into several pieces

Lemma 103. *Let G be a superstable group of* ∞-rank 3, *and suppose that G is not solvable-by-finite. Let B be a definable subgroup of G having* ∞-rank 2. *Then B is not nilpotent.*

Proof. Since we may replace B by any definable subgroup of finite index in B, we may take B to be solvable and centralizer-connected (Theorem 63, Lemma 76), and so that $Q(B) = N(B)$ (Lemma 83). Furthermore we may take G to be centralizer-connected.

If B is normal in G, then easily G is solvable-by-finite. Therefore we may fix $x \in G$ so that the group:

$$A_0 = B \cap B^x$$

has infinite index in B, and hence ∞-rank(A_0) is at most 1. In fact A_0 has ∞-rank exactly 1, since otherwise A_0 would be finite and it would follow easily that the set:

$$B \cdot B^x$$

has ∞-rank 4.

Hence A_0 contains an infinite abelian definable subgroup A. Let C be the centralizer in G of A. If $C = G$, then A is normal in G and ∞-rank(G/A) is at most 2. It then follows that ∞-rank(C) is at most 2, contradicting our assumptions.

Thus $C \neq G$ and since G is centralizer-connected it follows that C has ∞-rank at most 2. This implies easily that either $C \cap B$ or $C \cap B^x$ has ∞-rank 1. Without loss of generality, $C \cap B$ has ∞-rank 1.

Now the center Z of B is infinite by Lemma 77. Then $A \cdot Z \subseteq C \cap B$, and it follows that $A \cap Z$ is infinite. Repeat the foregoing analysis with $A \cap Z$ in place of A; then $B \subseteq C$, and we conclude $C \cap B^x$ has ∞-rank 1, so

$$A \cap Z(B) \cap Z(B^x)$$

is infinite. But the centralizer of this last group contains both B and B^x, contradicting the foregoing analysis. Thus we have arrived at a contradiction.

The first part of Lemma 102 is now easily obtained.

Proof of Lemma 102(1). By Lemma 103, B is not nilpotent-by-finite. Then the analysis of such groups in Section 9 yields the result.

Now using Theorem 100 and more or less direct calculations it is possible to prove:

Lemma 104. *Let G be a superstable group of* ∞-rank 3. *Assume that G is not solvable-by-finite and that B is a connected definable subgroup,* ∞-rank(B) = 2. *Let U, T be as in Lemma 99 (we know that B is not nilpotent by Lemma 103). Then G equals the set:*

$$U \cdot N(T) \cdot U.$$

(The details will be found in the statement and proof of [4, Section 5.1, Lemma 3].)

Lemma 105. *With the notations and hypotheses of Lemma 104, G is a finite union of double cosets of B.*

Proof. Since B is clearly of finite index in $N(B)$, there are only finitely many double cosets (= simple cosets) of the form:

$$BxB = Bx \quad (x \in N(B)).$$

On the other hand by Lemma 104 any double coset of B may be written

$$BxB \quad (x \in N(T)),$$

so it suffices to consider the double cosets corresponding to elements $x \in N(T) - N(B)$. It will suffice to show that such double cosets have ∞-rank 3.

Fix $x \in N(T) - N(B)$. Notice then that:

$$T = B \cap B^x.$$

(Clearly T is contained in the intersection, and Theorem 100(1) yields the reverse inclusion.)
Now to complete the proof it will suffice to show that:

$$BxU$$

has ∞-rank 3, and for this it suffices to show that

$$bxu = x \quad \text{implies} \quad u = 1 \quad \text{for} \quad b \in B,\ u \in U.$$

Assume therefore that $bxu = x$. Then:

$$T = T^x = T^{bxu},$$

so

$$T^{bx} = T^{u^{-1}} \subseteq B^x \cap B = T.$$

Thus $T = T^u$ and a trivial computation shows $u \in T \cap U = (1)$. This completes the argument.

Proof of Lemma 102(2). Let B be a connected definable subgroup of ∞-rank 2. Then G breaks up into finitely many double cosets of B, hence contains a B-connected subgroup H of finite index by Corollary 55. Since B is connected it follows that H is connected, and the Lemma is proved.

Thus we have obtained the starting point for a proof of Theorem 64, and the rest of the proof goes as in [4, Section 5.1].

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