THE NUMBER OF NON-ISOMORPHIC MODELS OF AN UNSTABLE FIRST-ORDER THEORY*

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ABSTRACT

It is proved that if T is an unstable (first-order) theory, $\lambda > |T| + \aleph_0$, then T has exactly 2^λ non-isomorphic models of cardinality λ. In fact we have stronger results: this is true for pseudo-elementary classes, and for almost every $\lambda \geqq |T| + \aleph_1$.

Suppose T is a (first-order) complete theory in the language L, $T_1 \supset T$, and $I(\lambda, T_1, T)$ is the maximal number of non-isomorphic models of T of cardinality λ, which are L-reducts of models of T_1. Our main theorem is

THEOREM 0.1. *If* $\lambda > |T_1| + \aleph_0$, *and* T *is unstable then* $I(\lambda, T_1, T) = 2^\lambda$.

In fact we can replace $\lambda > |T_1| + \aleph_0$ by $\lambda \geqq |T_1| + \aleph_1$, except in the case there is a family of 2^λ subsets of λ, each of cardinality λ, such that the intersection of any two sets from the family is finite. But this case is very rare (see Lemmas 3.2, 3.3).

CONJECTURE. If $\lambda \geqq |T_1| + \aleph_1$, and T is not superstable, then $2^\lambda = I(\lambda, T_1, T)$. Moreover, this is the best possible result.

There are many partial results toward this conjecture; some of them appeared in the notice [10], where a result weaker than Theorem 0.1 also appears. It holds for regular $\lambda > |T|$.

In the proof, we shall use Ehrenfeucht-Mostowski models (see [2], or the exposition Morley [6]). Ehrenfeucht in [1], using a property a little stronger than unstability, proves that if $\lambda = 2^\mu > |T_1|$, then $I(\lambda, T_1, T) \geqq 2$ (the property was the existence of an infinite set in a model M of T, and an antisymetric connected relation $\phi^M(x_1, \cdots, x_n)$ on it). Morley [5] improves his result, and improving

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this in Shelah [8] Theorem 2.13, it was proved that $I(\aleph_\alpha, T_1, T) \geqq |\alpha - \beta|$ if $|T_1| = \aleph_\beta$. Here we shall also use the theorem of Shelah [9], which says that if T is unstable, then there are $n < \omega$, and an infinite set of sequences of length n from a model M of T, and a relation $\phi^M(\bar{x}, \bar{y})$ which orders them (by [9], Theorem 4.7, this is weaker than Ehrenfeucht's demand). Other results on $I(\lambda, T_1, T)$ can be found in Keisler [4], Theorem 5.6, and Morley [7]. See [9, §OE, G] for a list of results. In Section 1 we define the Ehreufencht-Mostowski models, and show that we can, without loss of generality, assume additional assumptions on T_1 and T.

In Section 2 we prove a combinatorial theorem about ordered sets.

In Section 3 we prove that if $\lambda \geqq |T_1|$, $I(\lambda, T_1, T) < 2^\lambda$, then there is a family of 2^λ subsets of λ of cardinality λ, the intersection of any two of them is finite. We also show that this implies there is a regular $\aleph_\alpha \leqq \lambda$, such that $2^{|\alpha|} + 2^{\aleph_0} = 2^\lambda$. In Section 4 we show that $\lambda \geqq \aleph_\alpha > |T_1| + \aleph_0$, $\aleph_\alpha$ regular, implies $I(\lambda, T_1, T) \geqq 2^{|\alpha|} + 2^{\aleph_0}$, and so prove the main theorem.

NOTATIONS. Infinite cardinals will be denoted by $\lambda, \mu, \kappa, \chi$; ordinals by $i, j, k, l, \alpha, \beta, \gamma$; limit ordinal by δ; natural numbers by m, n, r. Ordinal is the set of the smaller ordinals, and a cardinal is an initial ordinal. M, N are models, a, b, c elements of models, x, y, z variables and ϕ, ψ formulas. $|A|$ is the cardinality of A, ω is the first infinite ordinal, and $\aleph_\alpha$ is the αth infinite cardinal. Sometimes we shall not differentiate between a predicate P and its interpretation P^M in the models (similarly for function symbols and terms).

1. For simplicity we assume that all the languages and theories will be infinite. A theory T in a (first-order) language L is a consistent set of sentences of L. Let T be a fixed complete theory in L, and T_1 a fixed theory in L_1, and $T \subset T_1$. Let $PC(T_1, T)$ be the class of L-reducts of models of T_1; and $I(\lambda, T_1, T)$ the maximal number of non-isomorphic models of $PC(T_1, T)$ in the cardinality λ. The theory T is unstable if for every λ there is a model M of T, and a set $A \subset |M|$ (= the set of elements of M) such that the elements of M realize over A more than λ complete types and $|A| \leqq \lambda$ (see Shelah [8]).

Our aim is

THEOREM 0.1. *If T is unstable, $\lambda > |T_1|$ $(= |T_1| + \aleph_0)$ then $I(\lambda, T_1, T) = 2^\lambda$. (In fact, we shall prove there for most λ it suffices to require $\lambda \geqq |T_1| + \aleph_1$.)*

Since for every $\lambda \geqq |T|$, clearly $I(\lambda, T_1, T) \leqq 2^\lambda$, *it suffices to prove* $I(\lambda, T_1, T) \geqq 2^\lambda$.

It is easy to see that $T_1 \subset T_2$ implies $I(\lambda, T_1, T) \geqq I(\lambda, T_{,2} T)$. As we want to to prove only $I(\lambda, T_1, T) \geqq 2^\lambda$, we can without loss of generality assume

ASSUMPTION 1. T_1 is complete.

By Shelah [9] Theorem 2.13:

LEMMA 1.1. *T is unstable iff T has a model M, and there is a natural number,* n, and sequences $\bar{a}^0, \bar{a}^1, \cdots, \bar{a}^m, \cdots \subset |M^n|$, *and a formula $\phi(\bar{x}, \bar{y})$ of L, such that for every $m, l < \omega$*

$$M \models \phi[\bar{a}^m, \bar{a}^l] \text{ iff } m < l.$$

In fact we can use here the conclusion of the lemma as a definition of unstability.

In fact we can assume that $n = 1$, because of the following construction. Let $F_0, \cdots, F_{n-1}$ be function symbols not in L_1, let M be a model of T_1. We define a model M^* as follows: its set of elements is $|M| \cup |M|^n$, and its relations will be the relations of M. The functions of M^* are the functions of M, and the functions $F_0^{M^*}, \cdots, F_{n-1}^{M^*}$ which are defined as follows:

$$\text{if } a \in |M|, \ F_k^{M^*}(a) = a$$

$$\text{if } a = \langle a_0, \cdots, a_{n-1} \rangle \in |M|^n, \ F_k^{M^*}(a) = a_k.$$

We define T_1^* as the (first-order) theory of M^*, and we define T^* similarly. It is easy to see that for every λ, $I(\lambda, T_1, T) = I(\lambda, T_1^*, T^*)$. It is also easy to see that T^* satisfies the conclusion of Lemma 1.1 for $n = 1$.

So we can assume

ASSUMPTION 2. In L there is a formula $x < y$ such that

A) T has a model M and elements $a_0, a_1, a_2, \cdots, a_m, \cdots$ of M such that for every $n, m < \omega$, $M \models a_n < a_m \Leftrightarrow n < m$

B) $(\forall xy)(x < y \rightarrow \neg\, y < x)$.

REMARKS. 1) If the formula $\phi(x,y)$ satisfies (A), then $\phi(x,y) \wedge \neg\, \phi(y,x)$ satisfies (A) and (B).

2) By the context we shall know what is the meaning of $<$.

A theory T^1 in a language L^1 contains its Skolem functions if for every formula $\phi(x, \bar{y})$ in L^1, there is a function symbol F in L^1 such that

$$(\forall \bar{y})[(\exists x)\phi(x, \bar{y}) \rightarrow \phi(F(\bar{y}), \bar{y})] \in T^1.$$

It is well known that for every T_1 in L_1 there is a theory T_2 in a language L_2, $|L_2| \leqq |L_1|$, $T_1 \subset T_2$, such that T_2 contains its Skolem functions. So we can assume, without loss of generality

ASSUMPTION 3. T_1 contains its Skolem functions and is complete.

By Assumptiona 2 and 3, T_1 has a model M_1 and elements $a_0, a_1, \cdots$ of M_1 such that $M \models a_n < a_m$ iff $n < m$; and that T_1 contains its Skolem functions. By the construction of Ehrenfeucht and Mostowski [2] it follows (See e.g. Morley [6]):

LEMMA 1.2. *For every ordered set I, T_1 has a model $M_1(I)$ and $a_s \in |M_1(I)|$ for every $s \in I$ such that*

1) *every element of $M_1(I)$ is of the form $\tau(a_{s_1}, \cdots, a_{s_n})$ where $s_1 < \cdots < s_n \in I$ and τ is a term in L_1.*

2) For every $s < t \in I$, $M_1(I) \models a_s < a_t$

3) For every formula $\phi(x_1, \cdots, x_n)$ of L_1,
and $s_1 < \cdots < s_n \in I$, $t_1 < \cdots < t_n \in I$

$$M_1(I) \models \phi[a_{s_1}, \cdots, a_{s_n}] \equiv \phi[a_{t_1}, \cdots, a_{t_n}]$$

4) For every element b of $M_1(I)$ there are $s_1 < \cdots < s_n \in I$ such that: if $t_1, t_2 \in I$, and $1 \leqq k \leqq n \Rightarrow t_1 < s_k \equiv t_2 < s_k$ and $1 \leqq k \leqq n \Rightarrow t_1 = s_k \equiv t_2 = s_k$ then $M_1(I) \models b < a_{t_1} \equiv b < a_{t_2}$, $M_1(I) \models a_{t_1} < b \equiv a_{t_2} < b$.

REMARK. Clearly (1) and (3) implies (4).

DEFINITION 1.1

A) $M(I)$ is the L-reduct of $M_1(I)$

B) $\langle a_s : s \in I \rangle$ will be called the skeleton of $M_1(I)$, and also of $M(I)$.

REMARK. Ehrenfeucht [1] uses a similar construction.

2. Let I, J denote non-void ordered sets (instead of ordered sets we shall say order). Their elements will be denoted by s, t and their orders by $<$. When there is a danger of confusion, the order of I is denoted by $<^I$. I^* is the converse of the order I.

Many times we shall assume implicitly that different ordered sets have no common elements. In fact, this requires many times the exchange of ordered sets by isomorphs of them, but we shall not mention it. $I + J$ is the sum of I and J, i.e. the elements of $I + J$ are the elements of I and J; and the order between elements of I remains, and similarly for J, and $s \in I$, $t \in J$ implies $s < t$. We define similarly $\sum_{k<l} I_k$ and $\sum_{s \in J} I_s$.

Ordinals and cardinals will also be considered sometimes as orders.

So $<$ is used as an order of ordered sets; as an order between ordinals and cardinals, and also as a predicate of L. Its meaning should be clear from the letters we use. Note that we shall write $\phi[\bar{a}]$ instead of $M \models \phi[\bar{a}]$, if it is clear what is M.

The central concepts of this section are

DEFINITION 2.1. Let M be a model with the relation $<$, which is antisymmetric. The sequence $\langle a_s : s \in I \rangle$ of elements of M is called a *nice sequence* (in M) if

A) $s < t \in I$ implies $a_s < a_t$

B) For every $b \in |M|$ there are $s_1 < \cdots < s_n \in I$ such that: for every $t_1, t_2 \in I$ *if* $1 \leqq k \leqq n \Rightarrow t_1 < s_k \equiv t_2 < s_k$ and $1 \leqq k \leqq n \Rightarrow t_1 = s_k \equiv t_2 = s_k$ *then* $b < a_{t_1} \equiv b < a_{t_2}$, $a_{t_1} < b \equiv a_{t_2} < b$.

Clearly, by (4) of Lemma 1.2

LEMMA 2.1. *The skeleton of $M(I)$ is a nice sequence, when $<$ is $<$ from Assumption* 2.

DEFINITION 2.2. The orders I, J will be called *contradictory* if there are no orders I_1, J_1 without last elements such that: there is a model M with a relation $<$ and in it nice sequences $\langle a_s : s \in I_1 + I^* \rangle$ and $\langle b_s : s \in J_1 + J^* \rangle$ such that:

1) for every $t \in J^*$ there is $s^0 \in I_1$, such that $s^1 > s^0$, $s^1 \in I_1$ implies there is $s > s^1$, $s \in I_1$, $a_s < b_t$.

2) for every $t \in I^*$ there is $s^0 \in J_1$, such that $s^1 > s^0$, $s^1 \in J_1$ implies there is $s > s^1$, $s \in J_1$, $b_s < a_t$.

REMARK. By the definition of nice sequence, (1) implies that for every $t \in J^*$ there are $s^0 \in I^1$, $s^1 \in I^*$, such that $s^0 \leqq s \leqq s^1$ (in $I_1 + I^*$) implies $a_s < b_t$. Similarly for (2). We shall use this many times. Our aim in this section is to prove

THEOREM 2.2. *For every $\lambda > \aleph_0$, there are 2^λ orders of cardinality λ which are contradictory in pairs.*

We shall prove it by a series of lemmas.

DEFINITION 2.3. The cofinality of an order I, $cf(I)$ is the smallest cardinal λ, such that there is an increasing sequence $\langle s_k : k < \lambda \rangle$ of elements of I such that for every $s \in I$ there is $k < \lambda$ for which $s < s_k$.

If I has a last element, $cf(I)$ is not defined. The sequence $\langle s_k : k < \lambda \rangle$ is called a cofinal sequence in I. Clearly $cf(I)$ is a regular cardinal.

LEMMA 2.3. *If $cf(I) \neq cf(J)$ then I and J are contradictory orders.*

PROOF. Without loss of generality $cf(J) = \mu > \lambda = cf(I)$. Suppose I and J are not contradictory, and we shall get a contradiction. By Definition 2.2, there are orders I_1, J_1, a model M and nice sequences $\langle a_s: s \in I_1 + I^* \rangle$, $\langle b_t: t \in J_1 + J^* \rangle$ which satisfy the conditions mentioned in Definition 2.2. Let $\langle s_k: k < \lambda \rangle$ be a sequence cofinal in I, and $\langle t_k: k < \mu \rangle$ be a sequence cofinal in J. Let $k < \lambda$, then there is $t^k \in J_1$ such that $t \in J_1$, $t^k < t$ implies $b_t < a_{s_k}$ As $\langle b_t: t \in J_1 + J^* \rangle$ is a nice sequence, there is $l = l(k) < \mu$ such that (in $J_1 + J^*$) $t^k < t < t_l$ implies $b_t < a_{s_k}$. Hence $l(k) < i < \mu$ implies $b_{t_i} < a_{s_k}$.

As $\lambda < \mu$ and μ is a regular cardinal, there is $l^0 < \mu$ such that $l^0 \leqq i < \mu$, $k < \lambda$ implies $b_{t_i} < a_{s_k}$. As $\langle a_s: s \in I_1 + I^* \rangle$ is a nice sequence, $\langle s_k: k < \lambda \rangle$ is cofinal in I, and for every $k < \lambda$ $b_{t_{l^0}} < a_{s_k}$, there is $s^0 \in I_1$, such that $s > s^0$, $s \in I_1$ implies $b_{t_{l^0}} < a_s$. This contradicts Condition 1 in the Definition (2.2) of nice sequences. So we get a contradiction and so prove Lemma 2.3.

DEFINITION 2.4. (1) For any cardinal λ, D_λ will be the filter on λ whose basis is the set of closed and unbounded subsets of λ (closed by the order topology). (2) If $A, B \subset \lambda$, then $A \subset B (\operatorname{mod} D_\lambda)$ means $B \cup (\lambda - A) \in D_\lambda$. Similarly $A = B (\operatorname{mod} D_\lambda)$ means $A \subset B (\operatorname{mod} D_\lambda)$, $B \subset A (\operatorname{mod} D_\lambda)$.

REMARK. Solovay proved that for every regular λ, there are λ subsets of λ, disjoint by pairs, and $\neq 0 \ (\operatorname{mod} D_\lambda)$; (He improves previous results). For completeness we shall not use it, and prove a very weakened version of it—2.4.

REMARK. It is easy to see that D_λ is a filter, when $cf(\lambda) > \omega$, and $D_\lambda = \{A: A \subset \lambda$, and there is a closed set $B \subset A$, which is unbounded in $\lambda\}$. Moreover, if $\mu < cf(\lambda)$, then the intersection of μ sets from D_λ belongs to D_λ.

LEMMA 2.4. *There is a family of $\aleph_0$ subsets of $\aleph_1$, disjoint in pairs, such that every one of them is $\neq 0 \ (\operatorname{mod} D_{\aleph_1})$. In fact we can replace $\aleph_1$ by any cardinality of cofinality $> \aleph_0$.*

PROOF. Let us first prove

(*) if $A \subset \aleph_1$, $A \neq 0 \ (\operatorname{mod} D_{\aleph_1})$, then there is a set $B \subset A$, such that

$$B \neq 0 \ (\operatorname{mod} D_{\aleph_1}),\ A - B \neq 0 \ (\operatorname{mod} D_{\aleph_1})$$

Suppose A does not satisfy (*). Then it is easy to see that $\{A \cap C: C \in D_{\aleph_1}\}$ is an ultrafilter on A, which is closed under intersection of $\aleph_0$ sets. As $A \neq 0 \ (\operatorname{mod} D_{\aleph_1})$, $|A| = \aleph_1$ and so $\aleph_1$ is a measurable cardinal. A contradiction.

So (*) holds. Let us define by induction A_n for $n < \omega$ such that $\aleph_1 - \bigcup_{m<n} A_m \neq 0 \ (\operatorname{mod} D_{\aleph_1})$. For $n = 0$ the condition is satisfied. If A_m is defined for $m < n$, then by (*) there is $A_n \subset \aleph_1 - \bigcup_{m<n} A_m$ such that $A_n \neq 0 \pmod{D_{\aleph_1}}$, $(\aleph_1 - \bigcup_{m<n} A_m) - A_n = \aleph_1 - \bigcup_{m<n+1} A_m \neq 0 \pmod{D_{\aleph_1}}$. So the demand for $n+1$ is satisfied. Clearly $\{A_n : n < \omega\}$ is the required family.

If we replace $\aleph_1$ by λ, $cf(\lambda) > \aleph_0$, then there are two cases

A) λ is smaller than the first measurable cardinal. Then the proof is exactly as for $\aleph_1$.

B) λ is not smaller than the first measurable cardinal, and hence $> \aleph_\omega$. Then let $A_n = \{\delta : \delta < \lambda, \text{ and } cf(\delta) = \aleph_n\}$. It is easy to see that $\{A_n : n < \omega\}$ is the required family.

LEMMA 2.5. *If λ is a regular cardinal $> \aleph_0$, $I = \sum_{k<\lambda} I_k^*$, $J = \sum_{k<\lambda} J_k^*$, and*

$$\{k < \lambda : I_k, J_k \text{ are contradictory}\} \neq 0 \pmod{D_\lambda}$$

then I, J are contradictory.

PROOF. Suppose I, J are not contradictory, and we shall get a contradiction.

So there are a model M, orders I^1, J^1 without last element and nice sequences $\langle a_s : s \in I^1 + I^* \rangle$, $\langle b_t : t \in J^1 + J^* \rangle$ satisfying the conditions mentioned in Definition 2.2.

As I_k, J_k are non-void, we can choose an element from each of them: $s_k \in I_k$, $t_k \in J_k$. Now we shall define the ordinal $k_l = k(l)$ for $l < \lambda$ such that:

I) $l < i < \lambda$ implies $k_l < k_i < \lambda$

II) for a limit ordinal δ, k_δ is the least upper bound of $\{k_l : l < \delta\}$

III) if $s \in I^*$, $s < s_{k(l+1)}$($<$ – in $I^1 + I^*$) then $a_s < b_{t_{k(l)}}$

IV) if $t \in J^*$, $t < t_{k(l+1)}$ then $b_t < a_{s_{k(l)}}$

Let us define k_l by induction:

A) $k_0 = 0$

B) Suppose k_l is defined, and we shall define k_{l+1}. As in the proof of Lemma 2.3, there is $s^0 \in I^*$ such that $s \in I^*$, $s < s^0$ implies $a_s < b_{t_{k(l)}}$. Similarly there is $t^0 \in J^*$ such that $t \in J^*$, $t < t^0$ implies $b_t < a_{s_{k(l)}}$. Let k_{l+1} be the first ordinal k such that $s_k < s^0$, $t_k < t^0$ (in $I^1 + I^*$, $J^1 + J^*$ respectively, of course).

C) If δ is a limit ordinal $< \lambda$, and k_l is defined for every $l < \delta$, then k_δ will be the least upper bound of $\{k_l : l < \delta\}$.

Clearly $\{k_l : l < \lambda\}$ is a closed unbounded subset of λ, and so is $\{k_\delta : \delta < \lambda, \delta$ is a limit ordinal$\}$. Hence $\{k_\delta : \delta$ is a limit ordinal$\} \in D_\lambda$. As $\{k < \lambda : I_k, J_k$ are contradictory$\} \neq 0 \bmod (D_\lambda)$, there is a $\delta < \lambda$ such that I_{k_δ}, J_{k_δ} are contradictory.

Let us define $\bar{I} = I_{k(\delta)}$, $\bar{J} = J_{k(\delta)}$, $\bar{I}_1 = (\sum_{k<k(\delta)} I_k^*)^*$, $\bar{J}_1 = (\sum_{k<k(\delta)} J_k^*)^*$.

Let N be a model, such that $|N| = |M|$, and $\langle a, b\rangle \in <^N$ iff $\langle b, a\rangle \in <^M$. Now the sequences $\langle a_s: s \in \bar{I}_1 + \bar{I}^*\rangle$, $\langle b_t: t \in \bar{J}_1 + \bar{J}^*\rangle$ in the model N, and the orders $\bar{I}$, $\bar{J}$, $\bar{I}_1$, $\bar{J}_1$ satisfy the condition mentioned in Definition 2.2; hence $\bar{I} = I_{k(\delta)}$, $\bar{J} = J_{k(\delta)}$ are not contradictory, in contradiction to the definition of δ. So I, J are contradictory, and we prove Lemma 2.5.

LEMMA 2.6. *There is a family K_1 of $2^{\aleph_0}$ orders of cardinality $\aleph_1$ which are contradictory in pairs.*

PROOF. By Lemma 2.4 there is a family $\{A_n: n < \omega\}$ of pairwise disjoint subsets of $\aleph_1$, each of which is $\neq 0 \pmod{D_{\aleph_1}}$.

For every subset B of ω and $k < \aleph_1$ let us define I^B_k as ω if $k \in A_n$, $n \in B$, and as $\aleph_1$ otherwise. Let $I^B = \Sigma_{k<\aleph_1}(I^B_k)^*$. We shall show $K_1 = \{I^B: B \subset \omega\}$ is the required family. Clearly $|K_1| = 2^{\aleph_0}$, and $|I^B| = \aleph_1$.

Now suppose $B, C \subset \omega$, $B \neq C$, and we shall show I^B, I^C are contradictory, K_1 is the required family. As $B \neq C$, there is $n < \omega$ such that $n \in B \Leftrightarrow n \notin C$. By Lemma 2.3, ω, $\aleph_1$ are contradictory, hence, for $k \in A_n$, I^B_k, I^C_k are contradictory. So

$$\{k < \aleph_1: I^B_k \; I^C_k \text{ are contradictory}\} \supset A_n \neq 0 \pmod{D_{\aleph_1}}.$$

Hence by Lemma 2.5, I^B, I^C are contradictory, and so K_1 satisfies our demands.

Now we shall prove the main theorem of this section.

THEOREM 2.2. *For every $\lambda > \aleph_0$ there is a family K_λ of 2^λ order of cardinality λ which are pairwise contradictory.*

PROOF. We shall define K_λ by induction on λ.

Case 1. $\lambda = \aleph_1$

Let $S = \bigcup\{2^\alpha: \alpha < \aleph_1\}$, i.e. S is the set of sequences of ones and zeros of length $< \aleph$. Clearly $|S| = 2^{\aleph_0}$. As by Lemma 2.6 there is a family K_1 of $2^{\aleph_0}$ pairwise contradictory orders of cardinality $\aleph_1$, we can name them such that $K_1 = \{I_\eta: \eta \in S\}$.

Now for every $\eta \in 2^{\aleph_1}$(i.e. η is a sequences of ones and zeros of length $\aleph_1$) we define

$$J_\eta = \sum_{k<\aleph_1} I^*_{\eta|k}$$

where $\eta | k$ is the sequence of the first k elements of η.

Clearly $|J_\eta| = \aleph_1$. Moreover if $\eta, \tau \in 2^{\aleph_1}$; $\eta \neq \tau$, then there is $k < \aleph_1$ such that $\eta | k \neq \tau | k$, hence

$$\{l < \aleph_1 : I_{\eta|l}, I_{\tau|l} \text{ are contradictory}\} \supset \{l_k < l < \aleph_1\} \in D_{\aleph_1}$$

Hence by Lemma 2.5 J_η, J_τ are contradictory. So clearly $K_{\aleph_1} = \{J_\eta : \eta \in 2^{\aleph_1}\}$ is the required family.

Case 2. $\lambda = \mu^+$

In this case the proof is, in fact identical to the proof of Case 1, with λ instead of $\aleph_1$, μ instead of $\aleph_0$, and K_μ instead of K_1.

Case 3. λ is a limit cardinal of cofinality $> \aleph_0$. Let μ be the cofinality of λ, and let $\lambda = \sum_{k<\mu}\lambda_k$, where for $k < l < \mu$, $\aleph_0 < \lambda_k < \lambda_l < \lambda$. Let $\mu_k = 2^{\lambda_k}$.

Clearly $\left|\prod_{l<k}\mu_k\right| = 2^{\lambda_k}$. By the induction hypothesis for every $k < \mu$ there is a family $K^k = \{I_\eta : \eta \in \prod_{l \leq k}\mu_l\}$ of 2^{λ_k} pairwise contradictory orders of cardinality λ_k. Now for every $\eta \in \prod_{l<\mu}\mu_l$ let $J_\eta = \sum_{l<\mu} I^*_{\eta|l}$ ($\eta \mid k$ is here the reduction of η, which can be looked at as a function with domain μ, to $k = \{l : l < k\}$). Clearly $\left|J_\eta\right| = \lambda$, and $\left|K_\lambda\right| = \left|\prod_{k<\mu}\mu_l\right| = \prod_{k<\mu}2^{\lambda_k} = 2^\lambda$ where $K_\lambda = \{J_\eta : \eta \in \prod_{l<\mu}\mu_l\}$. Clearly by Lemma 2.5 every two different orders from K_λ are contradictory, and so it is the required family.

Case 4. λ is a limit cardinal of cofinality $\aleph_0$. Let $\lambda = \sum_{n<\omega}\lambda_n$ where $n < m < \omega$ implies $\aleph_0 < \lambda_n < \lambda_m < \lambda$. Let $\mu_n = 2^{\lambda_n}$. Clearly $\left|\prod_{n \leq m}\mu_n\right| = \mu_m$. By the induction hypothesis for every $m < \omega$ there is a family $K^m = \{I_\eta : \eta \in \prod_{n<m}\mu_n\}$ of pairwise contradictory orders of cardinality λ_m. By Lemma 2.4 there is a family $\{A_n : n < \omega\}$ of pairwise disjoint subsets of $\aleph_1$, each of them $\neq 0 \pmod{D_{\aleph_1}}$. For simplicity assume $\bigcup_{n<\omega}A_n = \aleph_1$. Now for every $\eta \in \prod_{n<\omega}\mu_n$, and $k < \aleph_1$ let us define I^k_η: if $k \in A_n$, $I^k_\eta = I_{\eta|n}$.

Now for $\eta \in \prod_{n<\omega}\mu_n$ let

$$J_\eta = \sum_{k<\aleph_1} (I^k_\eta)^*.$$

Clearly $\left|J_\eta\right| = \lambda$, and $\eta \neq \tau$ implies J_η, J_τ are contradictory. Let $K_\lambda = \{J_\eta : \eta \in \prod_{n<\omega}\mu_n\}$, as $\left|K_\lambda\right| = \left|\prod_{n<\omega}\mu_n\right| = \prod_{n<\omega}2^{\lambda_n} = 2^{\sum_{n<\omega}\lambda_n} = 2^\lambda$, Case 4 is proved, and so Theorem 2.2.

3. Definition 3.1 1) A family K of sets is a *family of λ-disjoint sets* if $A, B \in K$ implies $\left|A \cap B\right| < \lambda$

2) A family K of subsets of A is (λ, μ)-proper if it is a family of μ-disjoint sets, and $B \in K$ implies $\left|B\right| \geq \lambda$. (We always assume $\lambda \geq \mu$.)

Theorem 3.1. *Suppose $\lambda \geq \left|T_1\right| + \aleph_1$. Then at least one of the following holds*:

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A) $I(\lambda, T_1, T) = 2^\lambda$

B) *There is a* $(\lambda, \aleph_0)$-*proper family of* 2^λ subsets of λ.

PROOF. By Theorem 2.2, there exists a family $K = \{I_k : k < 2^\lambda\}$ of pairwise contradictory orders of cardinality λ.

For every $k < 2^\lambda$, $l \leq \lambda$ let I_k^l be an order isomorphic to I_k, and s_k^l any element of I_k^l. Let us define for $k < 2^\lambda$, $J_k = \Sigma_{l \leq \lambda}(I_k^l)^*$, and $M_k = M(J_k)$. ($M(J_k)$ is defined by Definition 1.1.) and let $\langle a_s : s \in J_k \rangle$ be the skeleton of M_k. Clearly, for every $k < 2^\lambda$, the cardinality of M_k is λ.

Suppose A is not satisfied, and we shall prove that B holds. Let μ be any regular cardinal, $\lambda < \mu \leq 2^\lambda$. As A does not hold, $I(\lambda, T_1, T) < 2^\lambda$, and so there exists a model M such that $K = \{k < 2^\lambda : M_k \simeq M\}$ is of cardinality $\geq \mu$. Let $\langle b_s^k : s \in J_k \rangle$ be the image of the skeleton $\langle a_s : s \in J_k \rangle$ of M_k, by the isomorphism from M_k onto M. Clearly, by Lemma 2.1, $\langle b_s^k : s \in J_k \rangle$ is a nice sequence in M. For every $k \in K$, let $A_k = \{b_s^k : s = s_k^l, l < \lambda\} \subset |M|$. Clearly $|M|$ is a set of cardinality λ, and $\{A_k : k \in 2^\lambda\}$ is a family of $\geq \mu$ subsets of $|M|$, each of them of cardinality λ. ($|A_k| = \lambda$, because by the definition of a nice sequence $s < t \in J_k$ implies $b_s^k < b_t^k$ hence not $b_t^k < b_s^k$, and so $b_s^k \neq b_t^k$.)

We shall prove now that $k \neq l \Rightarrow |A_k \cap A_l| < \aleph_0$ in order to prove Theorem 3.1. So suppose $|A_k \cap A_l| \geq \aleph_0$. Then there is a sequence $\langle c_n : n < \omega \rangle$ of different elements of $A_k \cap A_l$. Let $c_n = b_{s(n)}^k$, $s(n) = s_k^{i(n)}$; $c_n = b_{t(n)}^l$, $t(n) = s_k^{j(n)}$. Without lose of generality, $i(n), j(n)$ are increasing functions (otherwise we replace $\langle c_n : n < \omega \rangle$ by a suitable subsequence). Let $\delta_1 = \sup\{i(n) : n < \omega\}$, $\delta_2 = \sup\{j(n) : n < \omega\}$. It is easy to see $\langle c_n : n < \omega \rangle ^\frown \langle b_s^k : s \in (I_k^{\delta_1}) \rangle^*$ and

$$\langle c_n : n < \omega \rangle ^\frown \langle b_s^l : s \in (I_l^{\delta_2})^* \rangle$$

are nice sequences (in M). Looking at the conditions of Definition 2.2, it is clear that this implies $I_k^{\delta_1}, I_l^{\delta_2}$ are not contradictory. As $I_k^{\delta_1}$ is isomorphic to I_k, and $I_l^{\delta_2}$ is isomorphic to I_l, it follows I_k, I_l are not contradictory, a contradiction.

So if $I(\lambda, T_1, T) < 2^\lambda$, then for every regular $\mu \leq 2^\lambda$ there is a $(\lambda, \aleph_0)$-proper family of subsets of $|M|$ of cardinality $\geq \mu$, where $\| M \| = \lambda$. From this B can be proved easily.

LEMMA 3.2. *If there is a* $(\lambda, \aleph_0)$-*proper family of* 2^λ *subsets of* λ, $\lambda > \aleph_0$, *then there is a regular* $\aleph_\alpha \leq \lambda$ *such that* $2^\lambda = 2^{|\alpha|} + 2^{\aleph_0}$.

For proving this we shall first prove

LEMMA 3.3. *Let* K *be a* (λ, μ) *proper family of subsets of* κ, *and* μ *be regular*

A) *If* $\kappa^\mu = \kappa$, *then* $|K| \leqq \kappa$. *Moreover, always* $|K| \leqq \kappa^\mu$.

B) *If* $\kappa^\mu > \kappa$, *but* $\chi < \kappa$ *implies* $\chi^\mu < \kappa$, *and* λ *is greater than the cofinality of* κ *then* $|K| \leqq \kappa$

C) $\kappa = \aleph_{\alpha+\gamma}$, $\chi < \aleph_\alpha$ *implies* $\chi^\mu < \aleph_\alpha$, *and for every* $\beta \leqq \gamma$, $\aleph_{\alpha+\beta} \geqq \mu_0 + |\beta|^+$; *and* $\lambda \geqq \mu_0 + |\gamma|^+$, *then* $|K| \leqq \kappa$, *where* $\mu_0 = \mu + [cf(\alpha)]^+$.

PROOF OF LEMMA 3.3. Let $K = \{A_k : k < |K|\}$.

A) Let for every k, B_k be a subset of A_k of cardinality μ. (As $|A_k| \geqq \lambda \geqq \mu$, there is such a subset.) Now if $k \neq l$, then $B_k \cap B_l \subset A_k \cap A_l$, hence $|B_k \cap B_l| < \mu$, and, as $|B_k| = |B_l| = \mu$, this implies $B_k \neq B_l$. So $|K| = |\{A_k : k < |K|\}| = |\{B_k : k < |K|\}| \leqq |\{B : B \subset K, |B| = \mu\}| = \kappa^\mu$. So clearly A follows.

B) Let $\chi < \lambda$ be the cofinality of κ, and let $\kappa = \sum_{i<\chi} \kappa_i$. For every k clearly $\lambda \leqq |A_k| = |\bigcup_{i<\chi}(A_k \cap \kappa_i)| \leqq \sum_{i<\chi}|A_k \cap \kappa_i|$. As clearly $\chi \leqq \mu \leqq \lambda$, $\chi < \lambda$, there is $i = i(k)$ such that $|A_k \cap \kappa_i| \geqq \chi^+ + \mu$, and let B_k be a subset of $A_k \cap \kappa_{i(k)}$ of cardinality μ. Then clearly $k \neq l$ implies $B_k \neq B_l$, and $|K| = |\{A_k : k < |K|\}| = |\{B_k : k < |K|\}| \leqq |\bigcup_{i<\chi}\{B : B \subset \kappa_i, |B| = \mu\}| = \sum_{i<\chi} \kappa_i^\mu = \kappa$.
So B follows immediately.

C) We shall prove it by induction on γ.

Case 1. $\gamma = 0$. This is Part B) of our lemma (or part A)

Case 2. $\gamma = i + 1$.

As $\aleph_{\alpha+i} \geqq \mu_0 + |i|^+$, and $\aleph_{\alpha+i+1}$ is a regular cardinal, for every k there is $j = j(k) < \aleph_{\alpha+i+1}$ such that $|A_k \cap j(k)| \geqq \mu_0 + |i|^+$. If $|K| > \aleph_{\alpha+i+1}$, then there is j such that $|\{k < |K| : j(k) = j\}| > \aleph_{\alpha+i+1} > \aleph_{\alpha+i}$. So $\{A_k \cap j : k < |K|, j(k) = j\}$ is clearly a $(\mu_0 + |i|^+, \mu)$-proper family of subsets of j, which is a set of cardinality $\aleph_{\alpha+i}$. By the induction hypothesis we get a contradiction. Hence $|K| \leqq \aleph_{\alpha+i+1} = \kappa$.

Case 3. γ is a limit ordinal.

Let $k < |K|$. If for every $i < \gamma$, $|A_k \cap \aleph_{\alpha+i}| < \mu_0 + |\gamma|^+$, then as μ is a regular cardinal $|A_k \cap \aleph_{\alpha+\gamma}| < \mu_0 + |\gamma|^+$. So there is $i = i(k) < \gamma$ such that $|A_k \cap \aleph_{\alpha+i}| \geqq \mu_0 + |\gamma|^+$. If $|K| > \aleph_{\alpha+\gamma}$ then there is i^0 such that $|\{k : i(k) = i^0\}| > \aleph_{\alpha+\gamma}$, and so $\{A_k \cap \aleph_{\alpha+i^0} : k < |K|\}$ is a $(\mu_0 + |\gamma|^+, \mu)$-proper family of subsets of $\aleph_{\alpha+i^0}$, of cardinality $> \aleph_{\alpha+\gamma} > \aleph_{\alpha+i^0}$. This contradicts the induction hypothesis. Hence $|K| \leqq \aleph_{\alpha+\gamma} = \kappa$.

PROOF OF LEMMA 3.2. Clearly by Lemma 3.3A, $2^\lambda = \lambda^{\aleph_0}$. Let $\aleph_\beta$ be the first cardinal such that $\aleph_\beta^{\aleph_0} = \lambda^{\aleph_0}$. Clearly $\aleph_\beta = \aleph_0$, or $\chi < \aleph_\beta$ implies $\chi^{\aleph_0} < \aleph_\beta$.

In the first case, clearly $\aleph_\alpha = \aleph_1$ satisfies our demand: it is regular and $2^\lambda = \lambda^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph_1^{\aleph_0}$. So assume $\aleph_\alpha > \aleph_0$. By Lemma 3.3B, C for any γ_0 if for every $\gamma \leqq \gamma_0$, $\aleph_{\beta+\gamma} > |\gamma|$, then $\aleph_{\beta+\gamma} < \lambda$. Hence $\aleph_\alpha = \aleph_{\beta+\aleph_\beta+1}$ is a regular cardinal $< \lambda$, and $2^\lambda \geqq 2^\alpha + 2^{\aleph_0} \geqq 2^{\aleph_\beta} \geqq \aleph_\beta^{\aleph_0} = 2^\lambda$. So $\aleph_\alpha$ is the required cardinal.

From Theorem 3.1 and Lemma 3.2, it clearly suffices, in order to prove the main theorem, to prove that if $\aleph_\alpha$ is regular, $\lambda \geqq \aleph_\alpha > |T_1|$, then $I(\lambda, T_1, T) \geqq 2^{\aleph_0} + 2^{|\alpha|}$. This will be done in the following section.

4. The aim of this section is to prove

THEOREM 4.1. *If $\aleph_\alpha$ is a regular cardinal, $\lambda \geqq \aleph_\alpha > |T_1|$, then $I(\lambda, T_1, T) \geqq 2^{|\alpha|} + 2^{\aleph_0}$.*

As was said in the end of the last section, by Theorem 3.1 and Lemma 3.2, this will end the proof of the main theorem.

PROOF. For every $k < \lambda$ let $I^{0,k}$ be order isomorphic to the order of the rationals. Let $I^\lambda = \sum_{k<\lambda} I^{0,k}$, $I^0 = I^{0,1}$, $J^0 = (\sum_{k<\aleph_1} I^{0,k})^*$. For every $A \subset \aleph_\alpha$ and $k < \aleph_\alpha$, let I_k^A be an order isomorphic to J^0 if $k \in A$, and to I^0 if $k \notin A$. Let $I^A = (\sum_{k<\aleph_\alpha} I_k^A) + I^\lambda$.

Later we shall prove

LEMMA 4.2. *If $A, B \subset \aleph_\alpha$, $A \neq B (\mathrm{mod}\, D_{\aleph_\alpha})$ then $M(I^A)$, $M(I^B)$ are not isomorphic.*

Let us show that Theorem 4.1 follows from this lemma.

It is easy to see that if $\{A^k: k < \mu\}$ is a family of subsets of $\aleph_\alpha$, and any two of them are not equal $(\mathrm{mod}\, D_{\aleph_\alpha})$, then $\{M(I^{A^k}): k < \mu\}$ is a family of non-isomorphic models of $PC(T_1, T)$. It is easily seen that the cardinality of each of them is λ, hence $I(\lambda, T_1, T) \geqq \mu$.

It is also easily seen that if $\{A_k: k < \kappa\}$ is a family of pairwise disjoint subsets of $\aleph_\alpha$, each of them $\neq 0$ (mod $D_{\aleph_\alpha}$), then $\{\bigcup_{k \in B} A_k: B \subset \kappa\}$ is a family of 2^κ subsets of $\aleph_\alpha$, any two of them are $\neq$(mod $D_{\aleph_\alpha}$). By the previous paragraph this implies $I(\lambda, T_1, T) \geqq 2^\kappa$. By Lemma 2.4 it follows $I(\lambda, T_1, T) \geqq 2^{\aleph_0}$.

Now for every regular cardinal $\aleph_\beta < \aleph_\alpha$, let $A_\beta = \{\delta < \aleph_\alpha: \delta$ is a limit ordinal of cofinality $\aleph_\beta\}$. Clearly $A_\beta \neq 0$ (mod $D_{\aleph_\alpha}$), and $\beta \neq \gamma$ implies $A_\beta \cap A_\gamma = 0$. (If $\alpha < \omega$ it follows from 2.4.) Hence $I(\lambda, T_1, T) \geqq 2^{|\alpha|}$. So we prove the theorem, and it remains to prove Lemma 4.2.

PROOF OF LEMMA 4.2. Suppose $A, B \subset \aleph_\alpha$, and $M(I^A) \simeq M(I^B)$. We should prove $A = B (\mathrm{mod}\, D_{\aleph_\alpha})$. By the symmetry of our assumptions, it suffices to prove $A \subset B$ (mod $D_{\aleph_\alpha}$).

Let F be an isomorphism from $M(I^A)$ onto $M(I^B)$, and let $\langle a(s): s \in I^A \rangle$ be the skeleton of $M(I^A)$, and $\langle b(t): t \in I^B \rangle$ be the skeleton of $M(I^B)$. For every $k < \aleph_\alpha$, let us choose $s_k \in I^A_k$. It is known (Lemma 1.2) that, for any k, there is a term τ of L_1 and $t_1 < \cdots < t_{n(k)} \in I^B$ such that

$$F[a(s_k)] = \tau^{M(I^B)}[b(t_1), \cdots, b(t_{n(k)})].$$

(In the future we shall write only τ.) For every s_k, we choose fixed such τ and $t_1, \cdots$. As $\aleph_\alpha$ is a regular cardinal $> |T_1|$, there is a set C such that it satisfies

Condition 1. $C \subset \aleph_\alpha$ $|C| = \aleph_\alpha$, and for every $k \in C$, there are $t^1_k < \cdots < t^n_k \in I^B$ such that $F[a(s_k)] = \tau[b(t^1_k), \cdots, b(t^n_k)]$.

For simplicity we shall assume $\aleph_1 < \aleph_\alpha$. Clearly there is no decreasing sequence of length $\aleph_2$ in I^B, and also not in I^A. As $\aleph_2 \leqq \aleph_\alpha$, every sequence of length $\aleph_\alpha$ whose elements are from I^B (or I^A), has a subsequence of length $\aleph_\alpha$, which is (strictly) increasing, or is constant. Also every sequence from I^B of length $\aleph_\alpha$ has a subsequence of length $\aleph_\alpha$, such that either all its elements are from I^λ or from $\sum_{k<\aleph_\alpha} I^A_k$. Applying this to $\langle t^1_k : k \in C \rangle$, we find that there is $C^1 \subset C$, $|C^1| = \aleph_\alpha$ such that

1) either $k, l \in C^1$, $k < l \Rightarrow t^1_k < t^1_l$, or $k, l \in C^1 \Rightarrow t^1_k = t^1_l$

2) either for every $k \in C^1, t^1_k \in I^\lambda$, or for every $k \in C^1$ $t^1_k \in \sum_{l<\aleph_\alpha} I^A_l$

Applying this to C^1 and $\langle t^2_k : k \in C^1 \rangle$, we can find $C^2 \subset C^1$ which satisfies (1) and (2) also for the t^2_k's. We can continue n times, and so prove there is a set $\subset \aleph_\alpha$, which we shall call again for simplicity C, such that

Condition 2. $C \subset \aleph_\alpha$, $|C| = \aleph_\alpha$, and for every $1 \leqq m \leqq n$ exactly one of the following conditions is satisfied:

A) for every $k \in C$, $t^m_k \in I^\lambda$; and $l < k \in C \Rightarrow t^m_l < t^m_k$; and $l < k \in C, t^m_l \in I^{0,i}$ $\Rightarrow t^m_k \notin I^{0,i}$

B) there is t^m such that $k \in C \Rightarrow t^m_k = t^m$

C) for every $k \in C$, $t^m_k \in \sum_{l<\aleph_\alpha} I^B_l$; and for every $l < k \in C$, $t^m_l < t^m_k$; and $l < k \in C$, $t^m_i \in I^A_l \Rightarrow t^m_k \notin I^A_i$.

Similarly, by proving the existence of suitable subsets of C, we can assume that C satisfies also the following conditions:

Condition 3. if m and r satisfy $1 \leqq m < r \leqq n$ then exactly one of the following possibilities is satisfied:

A) $k < l \in C$ implies $t^m_k < t^r_k < t^m_l < t^r_l$

B) for every $k, l \in C, t^m_k < t^r_l$

REMARK. Remember that for every k, $t_k^m < t_k^r$; and for every $k < l \in C$, $t_k^m \leqq t_l^m$, $t_k^r \leqq t_l^r$,

Condition 4. if m satisfies (C) from Condition 2, $l < k \in C$, $t_l^m \in I_i^B$, $i < \aleph_\alpha$, then $i < k$ (as ordinals) and t_k^m does not belong to $\sum_{j<i} I_j^B$.

It is obvious that C^*, the set of accumulation points of C, is a closed subset of $\aleph_\alpha$ of cardinality $\aleph_\alpha$. Hence $C^* \in D_{\aleph_\alpha}$. As we want to prove $A \subset B \pmod{D_{\aleph_\alpha}}$, it suffices to prove $A \cap C^* \subset B \cap C^*$. So we assume $\delta \in C^*$, $\delta \in A$, but $\delta \notin B$. We shall get a contradiction and so prove Lemma 4.2.

By Condition 4 it is easy to see that if t_δ^B is any element we chose from I_δ^B, and m satisfies C from Condition 2, *then* $k \in C$, $k < \delta \Rightarrow t_k^m < t_\delta^B$; and $t' \notin I_\delta^B$, $t' < t_\delta^m$ implies there is $k \in C$, $k < \delta$ such that $t' < t_k^m$. As $\delta \in A$, there is a decreasing sequence $\langle s^i : i < \aleph_1 \rangle$, $s^i \in I_\delta^A$, such that there is no element of I_δ^A which is smaller than every s^i. Let $F[a(s^i)] = \tau^i[b(t^{i,1}), \cdots, b(t^{i,n(i)})]$. As in I_δ^B there is no first element, for every $i < \aleph_1$ there is $w(i) \in I_\delta^B$ such that

$$t^{i,1}, \cdots, t^{i,n(i)} \notin \{t : t \in I_\delta^B,\ t < w(i)\}$$

As $\delta \notin B$, $\left| I_\delta^B \right| = \aleph_0$, and so there is $w \in I_\delta^B$ such that $\left| \{ i < \aleph_1 : w(i) = w \} \right| = \aleph_1$. So without lose of generality we can assume that for every $i < \aleph_1$, $w(i) = w$.

For every m satisfying (A) from Condition 2 let δ^m be the least upper bound of $\{ j < \lambda :$ there is $k < \delta$, $t_k^m \in I^{0,j} \subset I^\lambda \subset I^B \}$. As in the last paragraph, we can find $w_m \in I^{0,\delta m}$ for each $1 \leqq m \leqq n$ such that we can assume that for every $i < \aleph_1$

$$t^{i,1}, \cdots, t^{i,n(i)} \notin \{ t : t \in I^{0,\delta^m}, t < w_m \}$$

Now it can be easily seen that there are $t^1, \cdots, t^n \in I^B$ such that:

1) if m satisfies (B) from Condition 2 then t^m has already been defined

2) if m satisfies (C) from Condition 2 then $t^m < w$, $t^m \in I_\delta^A$

3) if $m < r$, m, n satisfy (C) from Condition 2 then $t^m < t^r$

4) if m satisfies (A) from Condition 2 then $t^m < w_m$, $t^m \in I^{0,\delta^m}$

5) if $m < r$; m, r satisfy (A) from Condition 2 then $t^m < t^r$.

Let $a = \tau[b(t^1), \cdots, a(t^m)]$.. By the definition of $M(I^B)$ (see Lemma 1.2, 3) it follows easily that: for every $k \in C$, $k < \delta$ $F[a(s_k)] < a$; and for every $i < \aleph_1$, $a < F[a(s^i)]$. It should be noted also that there is no $s \in I^A$ such that: $i < \aleph_1 \Rightarrow s < s^i$; $k \in C, k < \delta \Rightarrow s_k < s$. It is easy to show that this implies $\langle F[a(s)] : s \in I^A \rangle$ is not a nice sequence. But $\langle a(s) : s \in I^A \rangle$ is the skeleton of $M(I^A)$, hence by Lemma 2.1 it is a nice sequence (in $M(I^A)$). As F is an isomorphism from $M(I^A)$ onto $M(I^B)$, this implies that $\langle F[a(s)] : s \in I^B \rangle$ is also a nice sequence. A contradiction.

So $\delta \in A$, $\delta \in C^*$ implies $\delta \in B$, hence $A \cap C^* \subset B \cap C^*$, hence $A \subset B(\mathrm{mod}\, D_{\aleph_\alpha})$. So $A = B(\mathrm{mod}\ D_{\aleph_\alpha})$, and we prove Lemma 4.2. As has been said, this implies Theorem 4.1, and so the main theorem.

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