Classification over a Predicate II

§1 Introduction and preliminary facts.

Let T be a fixed complete first-order theory, and $P \in L(T)$ be a fixed monadic predicate.

Question: Describe the structure of $M \models T$ knowing $M \restriction P$.

When $(\forall x) \neg P(x) \in T$, this is a problem addressed [Sh 1], [Sh 4].

If $\forall x \; P(x) \in T$, there is an extremely strong structure theory. Gaifman dealt with the case "M has few ($\leq |P^M|$) automorphisms over P^M " and gets a representation theorem.

But for us the maximal structureness will be

"M is prime and even primary over P^M".

This is parallel to the case "T categorical in λ"; but this is stronger: remember that by Loewenheim Skolem Theorem T (if non- trivial) has models in all $\lambda \geq |T| + \aleph_0$. So the exact parallel will be "$\|M\|$, $M \restriction P$ determine M", or at least "dim (M,P), $M \restriction P$ determine M." If we are interested in the "categoricity theorem" (= uniqueness) we can restrict oneself to the case:

1.0 Hypothesis : $(\forall M \models T)(|P^M| = \|M\|)$ and even $(\exists \psi \in T)(\forall M \models \psi)$ $(|P^M| = \|M\|)$ (to avoid having to deal with the possibility that T is uncountable, and $(\forall M) \models T)[|P^M| = \|M\|]$ because of Chang's two cardinal theorem failing for all $\lambda \geq |T|$). The last condition is equivalent to: $[N \prec M \models T, \; P^M \subseteq N \Longrightarrow N = M]$.

We add * to the theorems assumings Hypothesis 1.0 (in our main conclusion here we shall do.)

This means that generally from P^M we cannot reconstruct M, not even its power.

We have start to deal with the problem in [Sh 2], but reading of it is not required (see there on other works on the subject of Gaifman Hodges and Pillay).

Section 3-4 are given almost as they were lectured in the seminar, hence are less formal but are more detailed and repetitious then usual. We do not try to save on set theoretic assumptions. In [Sh 1] the following classification is discussed.

unstable	*stable* – *superstable*	*superstable* – $\aleph_0$–*unstable* (*only* for *categories of models of countable theories*)	$\aleph_0$–*stable*

This corresponds to, roughly:

for every $p \in S(A)$:
stability $\Longrightarrow$ each $p \restriction \varphi$ is definable
superstability $\Longrightarrow p$ is almost definable over some finite $B \subseteq A$
$\aleph_0$-stability $\Longrightarrow p$ definable over some finite $B \subseteq A$.

We expect that the classification will be (this) $\times\ \omega$ with ω levels of complexity. Each time, for the unstable case, a non-structure theorem for $|T|^+$-saturated models, and for the unsuperstable T a non-structure theorem for $\aleph_\varepsilon$-saturated models. Only in the stable case we can continue to the next level. In fact it seemed that in order to get non-structure from unsuperstability we need first stability for all levels. We expect that the solution will be long, involving many branches. We concentrate on the stable/unstable dichotomy and quite saturated models. We shall use in "non-structure" proofs hypothesis like $G.C.H$, $V = L$ freely. If we do not do this we maybe forced to look at diagrams we get at approximation of less comfortable cofinalities;

if the properties are distinct the picture will be even more elaboborate. Let us explain more the expected classification.

n = - 1. Is every relation on P^M definable in M, also definable in $M \restriction P^M$?

1.1 Hypothesis: We assume, yes and even:

"every formula is equivalent (by T) to an atomic relation." (see [Sh 2])

n = 0. If M is saturated, $\|M\| = \lambda > |T|$, is M determined by $M \restriction P$? Its isomorphism type, yes but its isomorphism type over $M \restriction P$ not necessarily.

1.2 Hypothesis : For every $\bar{a} \in M \models T$ and φ, $p = tp_\varphi(\bar{a}, P^M)$ is definable (i.e. for some ψ_φ, and $\bar{c} \in P^M$; $\forall \bar{b} \in P^M[\varphi(\bar{x}, \bar{b}) \in p \iff \psi_\varphi(\bar{b}, c)]$. (see [Sh 2])

1.3 Theorem : If M is saturated, $\|M\| = \lambda > |T|$, then M is λ-prime over P^M among the λ saturated models, and is even λ-primary over it (i.e. $|M| = \{a_i : i < \alpha\}$, $tp(a_j, P^M \cup \{a_i, i < j\})$ is λ-isolated for λ regular; this proves uniqueness over P^M).

This is a weak structure theorem.

Proof: Note:

1.4 Fact: For every $\bar{c} \in M \models T$, $tp(\bar{c}, P^M)$ is $|T|^+$-isolated, in fact if $M \prec N$, then $tp(\bar{c}, P^M) \vdash tp(\bar{c}, P^N)$.

This follows from Hypothesis 1.2: for every φ there are $\psi_\varphi, \bar{c}_\varphi$ (ψ_φ does not depend on $\bar{c}$, only on $\ell(\bar{c}), \bar{c}_\varphi \subseteq P$) such that:

$$(\forall \bar{y} \subset P)[\varphi(\bar{c}, \bar{y}) \equiv \psi_\varphi(\bar{y}, \bar{c}_\varphi)].$$

So the formula $\Theta_\varphi(\bar{x}, \bar{c}_\varphi) = (\forall \bar{y} \subseteq P)[\varphi(\bar{x}, \bar{y}) \equiv \psi_\varphi(\bar{y}, \bar{c}_\varphi)]$ is satisfied by $\bar{c}$, its parameters are from P^M, so $\Theta_\varphi(\bar{x}, \bar{c}_\varphi) \in tp(\bar{c}, P^M)$ and easily $\Theta_\varphi(\bar{x}, \bar{c}_\varphi) \vdash tp_\varphi(\bar{c}, P^M)$. Hence,

$$\{\Theta_\varphi(x, \bar{c}_\nu) : \varphi \in L\} \begin{matrix} \subseteq tp(\bar{c}, P^M) \\ \vdash tp(\bar{c}, P^M) \end{matrix}$$

So $tp(\bar{c}, P^M)$ is $|T|^+$-isolated.

If $M \prec N$, then $N \models \Theta_\varphi(\bar{c}, \bar{c}_\varphi)$ hence

$$\{\Theta_\varphi(x,\bar{c}_\varphi):\varphi\in L\}\begin{matrix}\subseteq tp(\bar{c},N)\\ \vdash tp(\bar{c},N)\end{matrix}$$

but $\{\Theta_\varphi(x,\bar{c}_\varphi):\varphi\in L\}\subseteq tp(\bar{c},P^M)$.

Proof of the Theorem 1.3: Let $|M|=\{a_i:i<\lambda\}$. As $\lambda>|T|$, by the fact for $j<\lambda$ $tp(\langle a_i,i\le j\rangle,P^M)$ is isolated by a subset of power $\le|T|+|j|<\lambda$ (taking union on all finite subsequences). Hence $tp(a_j,P^M\cup\{a_i:i<j\})$ is λ-isolated. So M is λ-primary over P^M, etc. (see [Sh 1], Ch. IV).

1.5 Notation: Let $\mathfrak{C}$ be a very saturated model on T; we restrict ourselves to "small" elementary submodels of it. (see [Sh 1], Ch. I, §1).

1.6 Definition : $A\subseteq\mathfrak{C}$ is complete if $\mathfrak{C}\restriction(A\cap P)\prec\mathfrak{C}\restriction P^{\mathfrak{C}}$ and for every $\bar{a}\in A$ and φ there is $\bar{c}_{\varphi,\bar{a}}\subseteq A\cap P$ such that $\models\Theta_\varphi(\bar{a},\bar{c}_{\varphi,\bar{a}})$ (Θ_φ as previously). An equivalent formulation is: for every formula $\varphi(\bar{x},\bar{y})$ and $\bar{b}\in A$, if $\models(\exists\bar{x}\subseteq P)\,\varphi(\bar{x},\bar{b})$ then for some $\bar{a}\subseteq A\cap P$, $\models\varphi[\bar{a},\bar{b}]$.

Hence if $M\cap P^{\mathfrak{C}}\subseteq A\subseteq M$, then A is complete.

1.7 Remark: 1) If A,T are countable, this means (by the omitting type theorem):

$$\exists M(A\subseteq M\wedge M\cap P=A\cap P)$$

2) if $A\cap P$ is λ-saturated, $\lambda=|A|$ this means the same.

1.8 Definition : $S_*(A)=\{tp(\bar{a},A):A\cup\bar{a}$ complete, $\bar{a}\cap P=\phi\}$. Of course A complete, and let $S^m_*(A)=\{p\in S_*(A):p=tp(\bar{a},A),\ell(\bar{a})=m\}$.

1.9 Explanation: We are reconstructing M from P^M. It is reasonable to try to do this using intermediate A, $P^M\subseteq A\subseteq M$ but then the types in which we may be interested in realizing are only those from $S_*(A)$.

1.10 Explanation: From where comes the ω levels of the classification? We try to reconstruct M from P^M (e.g. in the case of categoricity). We let $\|M\|=\lambda_0$, let $M=\bigcup_{i<\lambda_0}M_i$, M_i increase continuously, $\|M_i\|=|T|+|i|$. This can be decomposed to λ_0 problems of:

"reconstruct M_{i+1} over $P^M\cup M_i$"

By Hypothesis 1.2 (see Fact) $tp_*(M_{i+1},P^{M_{i+1}} \cup M_i) \vdash tp_*(M_{i+1},P^M \cup M_i)$, so we have the reconstruction problem of M_{i+1} over $P^{M_{i+1}} \cup M_i$. We can decompose the diagram again, decreasing the power while increasing the diagram to 2^n sets. This is similar to [Sh 3] (and [Sh 4] XII §5, but there only the good cases occur). Note that if we allow $|P^M| < ||M||$ an extra complication arises.

If we have "good" behaviour for one power, every n, we can prove it for all larger powers. For each n we look at n-dimensional diagram $A = \bigcup_{w \subset n} A_w$ ($A_w \prec P^M$ if $0 \notin w$, $A_w \prec \mathfrak{C}$ if $0 \in w$), and ask about $|S_*^m(A)$. If we get stability (i.e. $|S_*^m(A)| \le |A|^{|T|}$), we can define $(n+1)$-diagrams [as we like to have that $tp_*(A_u,A_v)$ is determined by $tp_*(A_u,A_{v\cap u})$, and get some uniqueness we deal with them mainly when stability for n-diagrams was already proved, and 1.0 help simplifying]. If e.g. for every n the parallel of $\aleph_0$-stability holds, we would be able to prove "M is prime over P^M". From instability we will try to get non-structure theorems. We shall deal with ranks corresponding to stability (unstability .)

1.11 Definition : For every complete set A, for Δ_1,Δ_2 (sets of formulas $\varphi(\bar{x})$) we define $R = R_A^m(p,\Delta_1,\Delta_2,\lambda)$ (we sometimes omit A).
[the rank measure how close we are to:
p has a perfect set $\neq\phi$ of extensions in $S_*^m(A)$
Δ_1 is for "many extension"
Δ_2 is for "$A \cup \bar{x}$ is complete".]

We now define by induction when $R \ge \alpha$.

1) $R = -1 \iff p \vdash \bigvee_{\ell<\ell(\bar{x})} P(x_\ell)$

2) $R \ge 0 \iff R \ne -1 \iff p \quad \cup\{\neg P(x_\ell) : \ell < \ell(\bar{x})\}$ is finitely satisfiable.

3) $R \ge \delta$ (δ limit) $\iff R \ge i$ for every $i < \delta$.

4) $R \ge \alpha+1 \iff$ for every finite $q \subseteq p$ and cardinals μ,κ where [α odd $\Rightarrow \mu = 0$] , [α even $\Rightarrow \kappa = 0$] and $\mu + \kappa < \lambda$, and for every formula $\varphi_i(\bar{x},\bar{y}_i\bar{z}_i) \in \Delta_2$ and $\bar{b}_i \in A$ ($i < \kappa$) there are Δ_1–m–types $r_j (j \le \mu)$ pairwise

explicitly contradictories and $\bar{d}_i \in P^{\mathfrak{C}} \cap A(i \leq \kappa)$, such that:

$$R^m(\ q \cup r_j \cup \{(\forall \bar{z}_i \subset P)[\varphi_i(\bar{x},\bar{b}_i,\bar{z}_i) \equiv \psi_{\varphi_i}(\bar{z}_i,\bar{d}_i) : i < \kappa\}],\quad \Delta_1,\Delta_2) \geq \alpha$$

1.12 Remark: It does not matter whether we fix $\langle \psi_\varphi : \varphi \in L \rangle$ or just asked for "some suitable ψ_φ".

1.13 Definition : If K is a category of complete $A \subset \mathfrak{C}$ and some embeddings $f : A \to \mathfrak{C}$, then we can define R for K rather than for A, allowing in the definition to replace A by K-extension (i.e. the r_j but not $\bar{d}_i$ can be found there).

1.14 Claim: If $\mathfrak{C} \restriction A$ can be expanded to an $\aleph_0$-saturated model, and Δ_1,Δ_2 are finite, *then* $R^m_A(p,\Delta_1,\Delta_2,\aleph_0)$ is finite or ∞. (We make explicit the dependency on A).

Proof: By compactness. (similarly to [Sh 1], ch. II §2)

§2 Ranks and non-structure for n=1,2.

2.1 Remark: We concentrate on the case $\Delta_1,\Delta_2,\lambda$ finite, this lead to the "stable/unstable" dichotomy.
Of course the rank has obvious monotonicity and the finite character properties.

2.2 Claim: For every finite m $\Delta_1,\Delta_2,\lambda,n$ and $\varphi(\bar{x},\bar{y})$ there is a formula $\Theta(\bar{y})$ such that for any complete A and $\bar{a} \in A$

$$R^m_A(\varphi(\bar{x},\bar{a}),\Delta_1,\Delta_2,\lambda) \geq n \quad iff \quad \mathfrak{C} \restriction A \models \Theta[\bar{a}]$$

Proof: By induction on n.

2.3 Definition : 1) We say p is Δ_1-big (for A) *if* A is complete and $R^m_A(p,\Delta_1,\Delta_2,2) \geq \omega$ for ever finite Δ_2

2) A is unstable if for some finite Δ_1, $\{\bar{x} = x\}$ is Δ_1-big for A.

2.4 Lemma : Suppose A is complete and stable. Then $|S^m_*(A)| \leq |A|^{|T|}$.

Proof : For every $p \in S^m_*(A)$ we can find a complete $q_p \subseteq p$, of cardinality $\leq |T|$ such that for every finite Δ_1,Δ_2 : $R_A(p,\Delta_1,\Delta_2,2) = R^m_A(q_p,\Delta_1,\Delta_2,2)$. If

$|S^m_*(A)| > |A|^{|T|}$,

then for some finite Δ_1, $\{p \restriction \Delta_1 : p \in S^m_*(A)\}$ has power $> |A|^{|T|}$, there are $B \subseteq A, |B| \leq |T|$, q and $p, p_i \in S^m_*(A)$ for $i < (|A|^{|T|})^+$ such that $q_{p_i} = q_p \in S^m(B)$ hence $p_i \restriction B = p \restriction B$, and the $p_i \restriction \Delta_1$ are pairwise distinct. The rest is easy noting:

2.5 Fact: If A is complete, $p \in S_*(A)$, then $R_A(p, \Delta_1, \Delta_2, 2)$ is ∞ or is even.

2.6 Lemma : If $|A| = \lambda$, A complete, $\mathfrak{C} \restriction A$ saturated, A unstable, then $|S^m_*(A)| = 2^\lambda$.
In fact: there is a finite Δ_1 such that $|\{p \restriction \Delta_1 : p \in S^m_*(A)\}| = 2^\lambda$.

Proof: There is Δ_1 such that $R^m_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) > n$ for every finite Δ_2 and n. We define by induction on $\alpha < \lambda$ for every $\eta \in {}^\alpha 2$ an m-type p_η over A such that:

(1) $|p_\eta| < \aleph_0 + |\ell(\eta)|^+$

(2) for every finite Δ_2 $R^m_A(p_\eta, \Delta_1, \Delta_2, 2) \geq \omega$

(4) If $\alpha = \beta+1$, $\nu \in {}^\beta 2$ then for some $\varphi \in \Delta_1$, $\bar{c} \in A, \varphi(\bar{x}, \bar{c}) \in p_{\eta \frown \langle 0 \rangle}$ and $\neg\varphi(\bar{x}, \bar{c}) \in p_{\eta \frown \langle 1 \rangle}$.

(5) For every formula $\varphi(\bar{x}, \bar{a}, \bar{z})$, $\bar{a} \in A$, for some α, for every $\eta \in {}^\alpha 2$, for some $\bar{c} \in A \cap P$ $(\forall \bar{z} \subseteq P)\, (\varphi(\bar{x}, \bar{a}, \bar{z}) \equiv \psi_\varphi(\bar{z}, \bar{c})) \in p_\eta$.

For $\alpha = 0$, α limit no problem.

How to satisfy (4)?:

As Δ_1 is finite we can code it by one formula (see [Sh 1] II 2.1); so let $\Delta_1 = \{\varphi(\bar{x}, \bar{y})\}$. What are the demands on $\bar{c}$? Write $\bar{z}$ for $\bar{c}$: $\{R^m_A(q \cup \{\varphi(\bar{x}, \bar{z})^{\mathbf{t}}\}, \Delta_1, \Delta_2, 2) \geq n$: for finite $q \subseteq p$ any $\mathbf{t}$ and any finite $\Delta_2, n\}$

(where $\mathbf{t}$ is false or truth, $\varphi^{truth} = \varphi$, $\varphi^{false} = \neg\varphi$)

By claim 2.2 each demand is first order in $\mathfrak{C} \restriction A$. As $\mathfrak{C} \restriction A$ is saturated, $|p_\nu| < \lambda = |A|$, it is enough to show any finitely many demands are satisfiable. By monotonicity in rank just *one* is enough; say $R^m_A(q \cup \{\varphi(\bar{x}, \bar{z})^{\mathbf{t}}, \} \Delta_1, \Delta_2, 2) \geq n$. But $R^n(q, \ldots.) \geq n+2$ and use this.

A similar proof works for (5).

■

2.7 Remark: Now there are theorems which give us for unstable A and $\mu \geq |T|$ an $A' \equiv A$, $|S_*^m(A')| > \mu \geq |A'|$.
But we shall be "easy" on the non-structure side, as this is not our main concern in these notes.

2.8 Question: Is some (= every) model stable?
Meanwhile we assume *no* and get some non-structure theorems, then we will assume *yes* and continue.

2.9 Note: We shall observe that: no $\Longrightarrow (\exists M \models T)(|M| > |P^M|)$

2.10 Theorem : Suppose that for some models $M \subseteq N$, cardinal μ, and finite Δ_1, $P^N \subseteq M$, $||M|| \leq \mu$, $|\{tp_{\Delta_1}(\bar{a}, M): \bar{a} \in N\}| \geq \mu^+$.
If $|T| < \lambda = \lambda^{<\lambda}, \diamondsuit_\lambda$, $2^\lambda < 2^{\lambda^+}$ and *then* there are 2^{λ^+} non-isomorphic models of T of power λ^+, with the same restriction to P.

Proof : Expand N to have enough set theory and get N^+, let $Q^{N^+} = M$. Let $N_{<>}$ be a saturated model of $Th(N^+)$ of power λ.
We define by induction on $\alpha < \lambda^+$ N_η, Γ_η (for $\eta \in {}^\alpha 2$) such that :

(1) N_η is saturated of power λ, elementarily equivalent to N^+, Γ_η a family of $\leq \lambda$ types omitted by N_η, moreover no one has a support over N_η in the sense of [Sh 5] (for carrying this we need $\diamondsuit_\lambda$).

(2) For $\beta < \ell(\eta)$; $N_{\eta \restriction \beta} \prec N_\eta$, $\Gamma_{\eta\restriction\beta} \subseteq \Gamma_\eta$, $P^{N_\eta} = P^{N_{<>}}$, and even $Q^{N_\eta} = Q^{N_{<>}}$.

(3) For $\alpha=\beta+1$, $\nu \in {}^\beta 2$, there is a $\Delta_1 - m$-type over $P^{N_{<>}}$ realized by $N_{\nu\frown<0>}$ and belonging to $\Gamma_{\nu\frown<1>}$.

For the continuation of the process in the limit we have to have more induction hypothesis as in the paper above; in the case $\alpha = \beta+1$, $\nu \in {}^\beta 2$ N_ν has a λ-saturated extension in which λ^+ $\Delta_m - m$-types complete over $Q^{N_{<>}}$ are realized. So there is one p_ν with no support $< \lambda$ over N_η. So let $\Gamma_{\nu\frown<1>} = \Gamma_\nu \cup \{p_\nu\}$, $N_{\nu\frown<0>}$ realizes p_ν, (we can get also the dual demand).

So, let for $\eta \in {}^{\lambda^+}2$: $N_\eta = \bigcup_\alpha N_{\eta\restriction\alpha}$. Over $Q^{N_{<>}}$ they are pairwise non-isomorphic; as $2^\lambda < 2^{\lambda^+}$, 2^{λ^+} of them are not isomorphic (even over ϕ) (easily, by [Sh 1, 1.2] and $N_\beta \restriction P^{N_\eta} = N_{<>}\restriction P^{N_{<>}}$ is the same.

Remark: We can eliminate the use of $\diamondsuit_\lambda$ by forgetting Γ_η by demanding that for $\alpha = \beta + 1$, $\nu \in {}^\beta 2$ there is a $\Delta_1{-}m{-}$type p over $P^{N_{<>}}$ which $N_{\eta\frown<0>}$ realize it whereas $N_{\eta\frown<1>}$ "says" it is omitted (and you can demand that you can interchange them.)

2.11 Remark: We can replace λ-saturated by λ-compact.

2.12 Theorem: Suppose that some model is unstable, but the hypothesis of the last theorem fails.
If $|T| \leq \lambda = \lambda^{<\lambda}, \diamondsuit_{\{\delta<\lambda^+ : cf(\delta)=\lambda\}}$, then the conclusion of the last theorem holds.

Remark: We can replace diamond by weak diamonds.

Proof : We define by induction on α for every $\eta \in {}^\alpha 2$ a model N_η such that:

(1) N_η is λ-saturated when $\ell(\eta)$ is a successor or $cf(\ell(\eta)) = \lambda$

(2) $|N_\eta| = \lambda(1+\ell(\eta))$

(3) $N_{\eta\restriction\beta} \prec N_\eta$, $P^{N_\eta} = P^{N_{<>}}$

Let $\left\langle\left\langle \eta_\delta, \nu_\delta, F_\delta \right\rangle : \delta < \lambda^+, cf\ \delta = \lambda, \lambda^\omega \text{ divides } \delta (\lambda^\omega \text{ is ordinal exponeniation})\right\rangle$ be a $\diamondsuit$-sequence i.e. $F_\delta:\delta \to \delta$, $\eta_\delta \neq \nu_\delta \in {}^\delta 2$ and for every $\eta \neq \nu \in {}^{\lambda^+}2$, and function $F:\lambda^+ \to \lambda^+$ for some (in fact a stationary set of) $\delta : \left\langle \eta_\delta, \nu_\delta, F_\delta \right\rangle = \left\langle n\restriction\delta, \nu\restriction\delta, F\restriction\delta \right\rangle$; so F maps δ to δ.

(4) For each δ, there is a type q over N_{η_δ} which is realized in $N_{\eta_\delta\frown<0>}$ and also in $N_{\eta_\delta\frown<1>}$ but $F_\delta(q)$ is not realized in any λ- saturated extension N^+ of $N_{\nu_\delta\frown<0>}$ or $N_{\nu_\delta\frown<1>}$ with $P^{N^+} = P^{N_{<>}}$.
If we succeed; there will be no problem.

For $\alpha = 0$, α **limit** : no problem.

$\alpha = \beta+1$ β **successor** : Over N_ν there is a Δ_1-big $p \in S_*^m(N_\nu)$. Let it be realized by $\bar{c}$, $N_\nu \cup \bar{c}$ is complete, hence (as $\mathfrak{C} \restriction P^{N_\nu}$ is λ-saturated of power λ) there are (for $e = 0,1$) λ-saturated $N_{\nu\char`^<e>}$ for power $\lambda$, such that $N_\nu \cup \bar{c} \subseteq N_{\nu\char`^<e>}$, $P^{N_{\nu\char`^<e>}} = P^{N_\nu}$.

$\alpha=\beta+1$, $cf\ \beta < \lambda$: $N_\eta = \bigcup_{\gamma<\beta} N_{\eta\restriction\gamma}$ is a complete set with P^{N_η} saturated (see below); hence we can find $N_{\eta\char`^<e>} \supseteq N_\eta$ saturated with the same P. *We use freely:*

2.13 Claim: If A is complete, $\mathfrak{C} \restriction (A \cap P)$ λ-saturated, $|A| = \lambda$[and $|T| < \lambda = \lambda^{<\lambda}$], then we can find $N, P^N \subseteq A \subseteq N$ [and N is λ- saturated]. (like the proof of the unstability Lemma 2.6, but simpler).

The next case is:

$\alpha=\beta+1$, $cf\ \beta = \lambda$ and w.l.o.g. $\langle N_\delta, \nu_\delta, F_\delta \rangle$ is defined. We define by induction on i a model N^i of power λ, $N^0 = N_{\nu_\delta}$, $N^j \prec N^i$ for $j < i$, $P^{N_i} = P^{N_0}$ and there is $\bar{c}_i \in N_{i+1}$ such that $tp_{\Delta_1}(\bar{c}_i, N^0)$ is not realized in N^i. We define as long as we can for $i < \lambda^+$.
If we can continue for $i < \lambda^+$ we get the hypothesis of the previous theorem. As for limits we have no problem, there is a last N^{i^*}, w.l.o.g. (by 2.13) it is λ-saturated. Let $N_{\nu_\delta\char`^<e>} = N^{i^*}$ for $e = 0,1$. Now $|S_*^m(N_{\eta_\delta})| > \lambda$, $|N^{i^*}| \le \lambda$, so for some $q_\delta \in S_*^m(N_{\eta_\delta})$, $F_\delta(q_\delta)$ is not realized in $N^{i^*}$. Choose $N_{\eta_\delta\char`^<e>}$ to realize q_δ (possible as $q_\delta \in S_*^m(N_{\eta_\delta})$ not just $\in S^m(N_{\eta_\delta})$). For $\rho \in 2^\beta \setminus \{\nu_\delta, \eta_\delta\}$ you have more freedom. (We could have made the situation symmetric).

* * *

So we have shown non-structure when some M is unstable. Let us relist our hypothesis:

T complete, P one place predicate

n = - 1 Hypothesis A=1.1: every formula is equivalent to a relation

n = 0 Hypothesis B=1.2: For every $\bar{a} \in \mathfrak{C}$, $tp(\bar{a}, P^{\mathfrak{C}})$ is definable

n = 1 Hypothesis C: For every M, $|S^m_*(M)| \leq ||M||^{|T|}$.

Note: For every M by B, $tp_*(M,P^M) \vdash tp_*(M,P^{\mathfrak{C}})$. The next stage is:

n = 2 Question D: Is every $M_0 \cup P^{M_1}$ stable, where $M_0 \prec M_1 \prec \mathfrak{C}$?

2.14 Theorem : Suppose the answer to question D is yes, $\lambda^{<\lambda} = \lambda, \lambda > |T| \geq \aleph_0$. If M is λ-saturated of power λ^+, then over P^M there is a λ-prime model
(So if $(\forall N \models T)((|N| = ||P||)$ then M is λ -prime over P^M)

2.15 Remark: Really: $\lambda^{<\lambda} \leq \lambda^+, \lambda > |T|$ is enough.

Proof: If $|P^M| \leq \lambda$ use the previous theorem 1.3.

Let $P^M = \bigcup_{i<\lambda^+} A_i$ increasing continuous, $\mathfrak{C} \restriction A_i \prec \mathfrak{C} \restriction P^M \prec \mathfrak{C} \restriction P$ and for $i = 0$, and i successor $\Longrightarrow \mathfrak{C} \restriction A_i$ is λ-saturated.

We define by inducton on i models M_i, increasing continuous, $M_i \cap P^{\mathfrak{C}} = A_i$, such that

(*) for every $\bar{c} \in M_{i+1}$ $tp(\bar{c}, M_i \cup A_{i+1})$ is λ-isolated

(**) M_0, M_{i+1} are λ-saturated, $||M_i|| = \lambda$.

(***) $tp(\bar{c}, A_0)$ is λ-isolated for $\bar{c} \in M_0$.

Why is this enough?
Let $M_0 = \{c_\alpha : \alpha < \lambda\}$ $M_{i+1} \setminus M_i = \{c_\alpha : \lambda(1+i) \leq \alpha < \lambda(1+i+1)\}$, maybe with repeatitions.

Now $tp_*(\{c_\beta : \beta \leq \alpha\}, A_0)$ is λ-isolated (as union of $< |\alpha|^+ + \aleph_0$ such types) but $tp_*(\{c_\beta, \beta \leq \alpha\}, A_0) \vdash tp_*(\{c_\beta, \beta \leq \alpha\}, P^M)$ so the latter is λ-isolated too; hence $tp_*(c_\alpha, P^M \cup \{c_\beta : \beta < \alpha\})$ is λ-isolated. Also for $(1+i)\lambda \leq \alpha < (1+i+1)\lambda$ $tp(c_\alpha, P^M \cup \{c_\beta, \beta' < \alpha\})$ is λ- isolated by:

2.16 Fact: If $A \cup \bar{a}$ is complete, then

$$tp(\bar{a}, A) \vdash tp(\bar{a}, A \cup P^{\mathfrak{C}})$$

Proof of Fact 2.16: For every $\bar{b} \in A$, $tp(\bar{a}\frown\bar{b}, A \cap P) \vdash tp(\bar{a}\frown\bar{b}, P^{\mathfrak{E}}$, hence $tp(\bar{a},(A \cap P^{\mathfrak{E}}) \cup \bar{b}) \vdash tp(\bar{a}, P^{\mathfrak{E}} \cup \bar{b})$, taking unions over all $\bar{b} \in A$ we get the fact.

∎

We know $tp_*(\{c_\beta:(1+i)\lambda \le \beta \le \alpha\}, M_i \cup A_{i+1})$ is λ-isolated and

$$tp_*(\{c_\beta : (1+i)\lambda \le \beta \le \alpha\}, M_i \cup A_{i+1}) \vdash tp_*(\{c_\beta:(1+i)\lambda < \beta \le \alpha\}, M_i \cup P^M)$$

by the Fact.

Hence the latter is λ-isolated, hence $tp(c_\alpha, \{c_\beta:(1+i)\lambda \le \beta < \alpha\} \cup M_i \cup P^M)$ is λ-isolated, but this is $tp(c_\alpha, \{c_\beta : \beta < \alpha\} \cup P^M)$. So $tp(c_\alpha, \{c_\beta : \beta < \alpha\} \cup P^M)$ is λ-isolated for every $\alpha < \lambda^+$, and this is enough.

We still have to define M_i

For $i = 0$, as A_0 is complete λ-saturated of power λ, there is M_0, $P^{M_0} = A_0$, M_0 λ- saturated and we know M_0 satisfies (***) necessarily.

Note:

2.17 Fact: If B is complete, $\lambda = \lambda^{<\lambda} > \|T\|$, $\mathfrak{E} \restriction (B \cap P^{\mathfrak{E}})$ is λ- saturated, $|B| = \lambda$, then there is a λ-saturated $N \supset B, N \cap P^{\mathfrak{E}} = B \cap P^{\mathfrak{E}}$.

For $i+1$: As $M_i \cup A_{i+1}$ is complete, and its intersection with $P^{\mathfrak{E}}$ is (A_{i+1}, which is) λ-saturated, clearly by 2.17 there is $N_i \supseteq M_i \cup A_{i+1}, N_i$ λ-saturated $P \cap N_i = A_{i+1}$. We define by induction on $\alpha < \lambda, c_\alpha \in N$ such that $tp(c_\alpha, M_i \cup A_{i+1} \cup \{c_\beta:\beta < \alpha\})$ is λ-isolated. By standard bookkeeping it is enough to prove that if $p(x_\alpha)$ is a type over $M_i \cup A_{i+1} \cup \{c_\beta : \beta < \alpha\}$ of power $< \lambda$ then it has a λ-isolated extension (over this set).

By the induction hypothesis there is a type

$$q(x_\beta : \beta < \alpha) \subseteq tp_*(\langle c_\beta, \beta < \alpha \rangle, M_i \cup A_{i+1})$$

of power $< \lambda$ such that $q(x_\beta : \beta < \alpha) \vdash tp_*(\langle c_\beta, \beta < \alpha \rangle, M_i \cup A_{i+1})$. Replace in $p(x_\alpha)$ the c_β's by x_β and get $p'(x_\beta : \beta \le \alpha)$. So $p' \cup q$ is finitely satisfiable (in N_i) and of power $< \lambda$ and is over $M_i \cup A_{i+1}$. Let $\{(\bar{y}_\gamma, \Delta_1^\gamma, \Delta_2^\gamma):\gamma < |\alpha| + |T|\}$ be the list of all triples $(\bar{y}, \Delta_1, \Delta_2)$; $\bar{y} \subseteq \{x_\beta : \beta \le \alpha\}$ is finite and $\Delta_1, \Delta_2 \subset L(T)$ are finite.

We define by induction on γ a type r_γ in N_i over $M_i \cup A_{i+1}$ where r_γ is increasing of cardinality $< \lambda$, $r_{\gamma+1} = r_\gamma \cup r^\gamma(\bar{y}_\gamma)$, r^γ finite over $M_i \cup A_{i+1}$, the union consistent and $R(r^\gamma(\bar{y}_\gamma),\Delta_1,\Delta_2,2)$ is minimal where the rank is for $M_i \cup A_{i+1}$ (minimality: under the constraints required). As $M_i \cup A_{i+1}$ is stable and as A_{i+1} is λ-saturated, $N_i \cap P = A_{i+1}$, we can extend $r_{|\alpha|+|T|}$ to r' so that its domain is a set $C \subset A_{i+1} \cup M_i$ and $r' \restriction \bar{y} \in S_*^m(C)$ for any finite $\bar{y} \subseteq \{x_\beta : \beta \le \alpha\}$ of length m. Simply let $\langle c'_\beta : \beta \le \alpha \rangle$ be a sequence in N_i realizing $r_{|\alpha|+|T|}$; now choose $C_0 \subseteq A_{i+1}$ so that $\forall \bar{d} \subset \{c'_\beta : \beta \le \alpha\} \cup \mathrm{Dom}\ r_{|\alpha|+|T|}$, $tp(\bar{d},P^{N_{i+1}}) = tp(\bar{d},A_{i+1})$ is definable over C_0 and let $C = C_0 \cup \mathrm{Dom}\ r_{|\alpha|+|T|}$, $r' = tp(\langle c'_\beta : \beta \le \alpha \rangle, C)$.

By the definition of $R(\cdots)$, 2.5, and as for no Δ_1
$(\forall n)$ $(\forall$ finite $\Delta_2)$ $R_{A_{i+1} \cup M_i}(\bar{x} = x\ ,\Delta_1,\Delta_2,2) \ge n$, clearly r' has a unique complete extension over $A_{i+1} \cup M_i$ (using the construction of r^1).

So we have finished proving 2.14.

∎

2.18 Theorem : Suppose the answer (to Question D) is no, $\lambda = \lambda^{<\lambda} > |T|$. Let Q be the forcing of adding λ^+ Cohen subsets to λ. Then for some $A \prec P^{\mathfrak{C}}, |A| = \lambda^+$:

$\Vdash_Q$ "there are 2^{λ^+} λ-saturated models $M, P^M = A$, $\|M\| = \lambda^+$, pairwise non-isomorphic *over* A."

2.19 Remark: We can replace forcing by appropriate diamonds and get such models. Note that the answers to all our questions so far are absolute.

Proof : By assumption:

There is a triple: $P^{M^*} {\subset M^* \atop \subset P^{N^*}}$, $M^* \prec N^*$ whose union, $P^{N^*} \cup M^*$, is unstable. We can prove that there are many such triples. But for us it is enough to do the following. We define (in V) by induction on $i < \lambda^+, A_i$ such that A_i is strictly increasing, continuous, $|A_i| = \lambda$, $\mathfrak{C} \restriction A_i \prec \mathfrak{C} \restriction P$, A_0, A_{i+1} are λ-saturated and when $cf\ i = \lambda$ $(\mathbb{E} \restriction A_{i+1}, A_i) \equiv (\mathbb{E} \restriction P^{N^*}, P^{M^*})$ and when $cf\ i \in \{0,1,\lambda\}$ $(\mathbb{E} \restriction A_{i+1}, A_i)$ is λ-saturated.

For $i = 0$, i **limit** : no problem.

$1+i$, i *successor* or $cf\ i < \lambda$: easy.

$cf\ i = \lambda$: $\mathbb{E} \restriction A_i$ is λ a saturated of power λ by the induction-assumption. $Th(\mathbb{E} \restriction P^{N^*}, P^{M^*})$ has the λ- saturated model of power λ say (A, A^0), the A^0- part is saturated of power λ and has the theory of $\mathbb{E} \restriction P$, hence is isomorphic to A_i. We can identify them and choose A_{i+1} as A^1.

Now for any sequence $\langle r_i : i < \lambda^+, cf\ i = \lambda \rangle = \bar{r}$ of Cohen subsets of λ we describe how to build a λ-saturated model $M_{\bar{r}}$ of T with $P^{M_{\bar{r}}} = \bigcup A_i$.

Before this:

2.20 Fact: If M is a λ-saturated model of T, $\|M\| = \lambda$, $M \cap P^{\mathbb{E}} = A_i$, $cf\ i = \lambda$; *then* $M \cup A_{i+1}$ is a λ-saturated model of $Th(M^* \cup P^{N^*})$, and even $(M \cup A_{i+1}, A_i, M, A_{i+1})$ is a λ-saturated model of $Th(M^* \cup P^{N^*}, P^{M^*}, M^*, P^{N^*})$ (same argument as before plus use of 1.3).

We shall define $M_{\bar{r}} = \bigcup_{i<\lambda^+} M_{\bar{r},i}$, $M_{\bar{r},i}$ depends on $\bar{r} \restriction i$ only, $M_{\bar{r},i} \cap P^{\mathbb{E}} = A_i$, $M_{\bar{r},i+1}$ is λ-saturated. So in $S_*^m(M_{\bar{r},i} \cup A_{i+1})$ there is a perfect set homeomorphic to ${}^{\lambda}2$; we can (see 2.6) choose a tree $\{p_\eta : \eta \in {}^{\lambda>}2\}$ of types $p_\eta \in S_*^m(C^\eta_{\bar{r},i})$ $C^\eta_{\bar{r},i}$ increasing with η, $p_{\eta\frown<0>}$, $p_{\eta\frown<1>}$ explicitely contradictory, $C^\eta_{\bar{r},i} \subseteq M_{\bar{r},i} \cup A_{i+1}$ has power $\leq \ell(\eta) + \aleph_0$
and $(\forall c \in M_{\bar{r},i} \cup A_{i+1})$ $(\exists\ \alpha)$ $(\forall \eta \in 2^\alpha)$ $[c \in C^\eta_{\bar{r},i}]$.

Now r_i define a branch $\eta_i \in {}^{\lambda}2$ and we demand that M_{i+1} realizes $\bigcup_{i<\lambda} p_{\eta_i \restriction \xi}$. We can carry this as under our hypothesis since:

Fact A: $tp_*(M, P^M) \vdash tp_*(M, P^{\mathbb{E}})$

Fact B: If A is complete, $|A| \leq \lambda$, $A \cap P$ saturated of power λ then $(\exists N \supseteq A)[N$ is λ-saturated and $N \cap P = A \cap P]$.

Now, if we add to λ λ^+ Cohen subsets, there is no problem to define $\bar{r}_E$ (for

$E \subset \lambda^+, E \in V$ and $\langle\langle (S(E), g_E\rangle : E \subseteq \lambda^+, E \in V\rangle$ from V such that:

$\bar{r}_E \in V[r_j : j \in S(E) \subseteq \lambda^+]$,

$\bar{r}_E(i) = r_{g_E(i)}$, where $g_E : \lambda^+ \to S(E)$ is one to one and $g_E \in V$.

$E_1 \neq E_2 \Longrightarrow \{i < \lambda^+ : cf\ i = \lambda,\ r_{E_1}(i)$ does not appear as $\bar{r}_{E_2}(j)\}$ is stationary

[Easy, as there are $S_\xi \subseteq \{i < \lambda^+ : cf\ i = \lambda\}$ for $\xi < \lambda^+$ stationary pairwise disjoint]

Suppose $\underset{\sim}{f}$, a Q-name, is forced to be an isomorphism. As the forcing satisfies λ^+-cc there is a club $D \subseteq \lambda^+$, $D \in V$ such that :

f maps $M_{\bar{r}_{E_1},i}$ onto $M_{\bar{r}_{E_2},i}$ for $i \in D$ and $f \restriction M_{\bar{r}_{E_2},i}$ does not depend on $r_{E_1}(i)$ (in fact depend only on the generic sets $\{\bar{r}_E(j) : j < i\} \cup \{r : r$ does not appear in $\bar{r}_{E_1}, \bar{r}_{E_2}\}$). Choose $i \in D$, $cf\ i = \lambda$, $\bar{r}_{E_1}(i)$ does not appear in this $\bar{r}_{E_2}$. Let $V^+ = V[r_j : r_j \neq r_{E_1}(i)]$. Now $f \restriction M_{\bar{r}_E,i}$ is in the universe V^+, as well as the tree of types we have for $M_{\bar{r}_{E_1},i}$ after Fact 2.20. But in $M_{\bar{r}_{E_1}}(i+1)$ there is a type realized which $\notin V^+$, a contradiction.

■

§3 Introducing n-dimensional diagrams and on uniform local atomicity

3.1 Remark: In our non-structure theorems we prove something like: If ...., and λ is special e.g. $\lambda = \mu^+ = 2^\mu$, $\diamondsuit_\mu$ and $\diamondsuit_{\{\delta<\lambda : cf\ \delta = \mu\}}$ then over some $A \subset P^{\mathfrak{E}}$, $|A| = \lambda$, there are 2^λ models M with $P^M = A$ pairwise non-isomorphic over it. This excludes e.g. singular cardinals even if $V = L$. However in the cases we have dealt with we can really get 2^{λ^+} non-isomorphic models M_i, $P^{M_i} = A$ (non-isomorphic over it) with $|A| = \chi$ for any $\chi > \lambda$. Just iterate taking ultraproduct for D an ultrafilter over ω. So when our proof rests on omitting types of power μ, $\mu > \aleph_0$ this does not change much. For e.g. $\mu = \aleph_0$, we have to use indiscernibles instead; we shall return to this.

3.2. Let $\mathcal{P}(n) = \{w; w \subseteq \{0,1,\ldots,n-1\}\}$

$$\mathcal{P}^-(n) = \mathcal{P}(n) - \{n\} = \{w : w \subset \{0,1,\ldots,n-1\}\}$$

We shall deal with $I \subseteq \mathcal{P}(n)$ closed under subsets, mainly with $\mathcal{P}(n)$, $\mathcal{P}^-(n)$ and with (λ, I)-system $\langle A_s : s \in I \rangle$ $\lambda = \Sigma |A_s|$ such that

$$0 \notin s \Longrightarrow A_s \prec P^{\mathfrak{C}}$$

$$0 \in s \Longrightarrow A_s \prec \mathfrak{C}$$

$$A_s \cap P = A_{s \setminus \{0\}},\ A_s \cap A_t = A_{s \cap t}$$

and more.

We first deal with small n; for such systems we may ask about stability (of $\bigcup_{s \in I} A_s$), and existence (of M, $P^M \subseteq \bigcup_{s \in I} A_s \subseteq M$)

* * *

Note that:

for $\mathcal{P}^-(0)$ we get nothing

for $\mathcal{P}(0)$ we have just A_ϕ which is $\prec P^{\mathfrak{C}}$ (i.e. $A \subseteq P^{\mathfrak{C}}$ and $\mathfrak{C} \upharpoonright A_\phi \prec \mathfrak{C} \upharpoonright P^{\mathfrak{C}}$).

$\mathcal{P}^-(1) = \{\phi\}$

.sp $\mathcal{P}(1) = \{\phi, \{0\}\}$

So a $\mathcal{P}(1)$-system is $\langle A_{\{0\}}, A_\phi \rangle$

$A_{\{0\}}$ a model

A_ϕ its P-part

a $\mathcal{P}^-(1)$-system is just $A_\phi \prec P^{\mathfrak{C}}$ and the existence-problem is $\exists M (P^M = A_\phi)$. The stability just asks on $S_*(A)$ when $A \prec P^{\mathfrak{C}}$.

n=2: A $\mathcal{P}(2)$-system is $\langle A_\phi, A_{\{0\}}), A_{\{1\}}), A_{\{0,1\}} \rangle$.

For $\mathcal{P}^-(2)$ we have dealt with stability and existence. In this case automatically $tp_*(A_{\{0\}}, A_\phi) \vdash tp_*(A_{\{0\}}, A_{\{1\}})$.

n = 3: We have a cube, we add the demand

$$(A_{\{1\}},A_{\phi}) \prec (A_{\{1,2\}},A_{\{2\}}).$$

We shall assume that T absolutely has no two cardinal model (i.e. 1.0) (not always we shall use it).

3.3 Claim *: If $P^M \subseteq A \subseteq M$, A stable (and complete), then M is locally atomic over A[that is $\forall \bar b \in M, tp(\bar b, A)$ is locally isolated which means that for every $\varphi = \varphi(\bar x, \bar z)$, there is $\psi(\bar x, \bar a) \in tp(\bar b, A)$, $\psi(\bar x, \bar a) \vdash tp_{\varphi}(\bar b, A)$] and even uniformly so (i.e. ψ depends on φ only and not on $\bar b$, though $\bar a$ may still depend on $\bar b$).

Proof : First assume (M,A) is saturated of power λ. Then (see 3.4(2)) we can find $N, P^N \subseteq A \subseteq N$, $|N| = \{a_i : i < \lambda\}$ $tp(a_i, A \cup \{a_j, j < i\})$ is λ-isolated, hence we can embed N into M over A, by 1.6 the embedding is onto A, hence w.l.o.g. $N = M$. So for every $\bar b \in M, tp(\bar b, A)$ is λ-isolated. For some $q \subseteq tp(\bar b, A)$, $|q| < \lambda, q \vdash tp(\bar b, A)$. For every $\varphi = \varphi(\bar x, \bar y)$ let

$$\Gamma = q(\bar x_1) \cup q(\bar x_2) \cup \{\varphi(\bar x_2, \bar y), \neg\varphi(\bar x_1, \bar y), \bigwedge_{\ell < \ell(\bar y)} y_\ell \in A\}$$

(we have a predicate for A). Now Γ is not realized in M, because if $\bar x_1 \to \bar b_1$, $\bar x_2 \to \bar b_2$, $\bar y \to \bar d$ realized it then $\bar d \subseteq A$ and
$q_1 = q(\bar x) \cup \{\varphi(\bar x, \bar d)\}$ is consistent ($\bar b_1$ realized it)
$q_2 = q(\bar x) \cup \{\neg\varphi(x, \bar d)\}$ is consistent ($\bar b_2$ realized it)

contradicting "$q \vdash tp(\bar b, A)$."

So this holds if we replace q by some finite $q' \subseteq q$ hence by some formula $\psi_{\varphi,\bar b}(\bar x, \bar c_{\varphi}) \in tp(\bar b, A)$. So

$$\psi_{\varphi,\bar b}(\bar x, \bar c_{\varphi,\bar b}) \vdash tp_{\varphi}(\bar b, A), \quad \text{and} \quad \models \psi_{\varphi,\bar b}(\bar b, \bar c_{\varphi,\bar b})$$

Similarly we can deduce the uniformity from the $|T|^+$-saturativity.

3.3A Notation: 1) Let l.a. stand for locally atomic, u.l.a. stand for uniformly locally atomic.

2) Let $A \subseteq_t C$ means that if $\varphi(\bar z, \bar x) \in L$, $\bar c \in C$, $\bar a \in A$, $\mathfrak{C} \models \varphi[\bar c, \bar a]$ then there is $\bar c' \in A$ such that $\mathfrak{C} \models \varphi[\bar c', \bar a]$.

3.4 Claim: 1)* If A is complete, unstable and $|T|^+$-saturated, then over A there is an m-type p of power $\leq |T|$ with no $|T|^+$-isolated extension.

2) If A is complete stable, λ-saturated and $\lambda > |T|$, then

(a) for every m-type p over A of cardinality $< \lambda$ there is an m-type q over A, $p \subseteq q$, $|q - p| \leq |T|$, q has a unique extension in $S^m(A)$ and it is in $S^m_*(A)$.

(b) over A there is a primary model N, so necessarily $N \cap P^{\mathfrak{C}} = \cap P^{\mathfrak{C}}$.

3)* If A is complete, $A \cap P^{\mathfrak{C}}$ is λ-saturated and $P^M \subseteq A \subseteq M$ *then* M is λ-saturated.

Remark: We use "absolutely no two cardinal model" for 1) and 3)

3.5 Claim: Suppose A is complete, $A \subseteq_t B$ and $C \subseteq P^{\mathfrak{C}}$. then $A \subseteq_t B \cup C$.

Proof: Let $\bar{a} \in A$, $\bar{b} \in B$, $\bar{c} \in C$, and suppose $\models \varphi[\bar{c}, \bar{b}, \bar{a}]$.

Let $\psi(\bar{y}, \bar{x}) = (\exists z_0, z_1, \cdots)[\varphi(z_0, z_1, \ldots, \bar{y}, \bar{x}) \wedge \bigwedge_{\ell} P(z_0)]$, so clearly $\models \psi[\bar{b}, \bar{a}]$, hence for some $\bar{b}' \in A$ $\models \psi[\bar{b}', \bar{a}]$. As A is complete, and $\bar{a}, \bar{b}' \in A$ clearly for some $c_0', c_1', \ldots, \in A$, $\models \varphi[c_0', c_1', \ldots, \bar{b}', \bar{a}]$.

This proves $A \subseteq_t B \cup C$.

3.6 Claim: If $tp(\bar{b}, A)$ is locally isolated, $A \subseteq_t B$ *then* $tp(\bar{b}, A) \vdash tp(\bar{b}, B)$. If A' is l.a. [u.l.a.] over A, $A \subseteq_t B$ then A' is l.a. [u.l.a.] over $A \cup B$.

Proof: Easy.

§4 On $\mathcal{P}^-(3)$- systems and $\mathcal{P}^-(3)$ non- structure when there are unstable $\mathcal{P}^-(3)$- systems.

4.1 Definition : We define what is a $\mathcal{P}^-(3)$-system. It is $\mathbf{S} = \langle A_s : s \in \mathcal{P}^-(3) \rangle$ such that :

1) $A_\phi, A_{\{1\}}, A_{\{2\}}, A_{\{1,2\}} \prec \mathfrak{C} \restriction P$

2) The rest are $\prec \mathfrak{C}$.

3) $A_s \cap P^{\mathfrak{C}} = A_{s-\{0\}}$

4) $A_s \cap A_t = A_{s \cap t}$

5) $(A_{\{1,2\}}, A_{\{2\}}) \succ (A_{\{1\}}, A_{\phi})$

6) $A_{\{1,0\}}$ is uniformly locally atomic over $A_{\{1\}} \cup A_{\{0\}}$ and $A_{\{2,0\}}$ is uniformly locally atomic over $A_{\{2\}} \cup A_{\{0\}}$

Now 6) follows by previous hypothesis, for T absolutely with no two cardinal model, (see 3.3). We say **S** is stable if $\bigcup_s A_s$ is stable. **S** has the existence property if $(\exists M \supseteq \bigcup_s A_s) P^M \subseteq \bigcup_s A_s$.

4.2 Fact: Being a $\mathcal{P}^-(3)$-system depends on the first theory only [of $(\bigcup_{s \in \mathcal{P}^-(3)} A_s, \cdots A_s \cdots)_{s \in \mathcal{P}^-(3)}$] (because we have u.l.a. not just l.a.).

E.Question: Is there unstable $\mathcal{P}^-(3)$-system?

4.3 Theorem* : Suppose $\langle A_s^* : s \in \mathcal{P}^-(3) \rangle$ is unstable, $\lambda = \lambda^{<\lambda} > |T|$ and Q is the forcing of adding λ^{++}-Cohen subsets to λ(and $2^\lambda = \lambda^+$, $2^{\lambda^+} = \lambda^{++}$) and $\mu \geq \lambda^{++}$. Then in V^Q there are $2^{\lambda^{++}}$ non isomorphic models of T of power μ with the same P of power μ. [If e.g. $\mu^{<\lambda} = \mu$ then we can have λ-saturated models).

4.3A Remark: We do not try here to eliminate the set theory. We are more interested to show the dividing line is right.

4.4 Claim: Suppose for $\ell = 0,1$ $\langle A_s^\ell : s \in \mathcal{P}^-(3) \rangle$ is a $\mathcal{P}^-(3)$-system, $\langle A_s^\ell : s \in \mathcal{P}(\{1,2\}) \rangle$ is saturated of power $\lambda > |T|$, $\langle A_s^\ell : s \in \mathcal{P}(\{1,2\} \rangle$, ($\ell = 0,1$) are elementarily equivalent and A_s^ℓ is saturated of power λ when $0 \in s$. Then the two systems are isomorphic.

Proof : Obviously there is an isomorphism g from $\langle A_s^0 : s \in \mathcal{P}(\{1,2\}) \rangle$ onto $\langle A^{s^1} : s \in \mathcal{P}(\{1,2\}) \rangle$. Now we know (see 1.3) that: as $A_{\{0\}}^\ell$ is saturated of power λ, it is unique over $A_{\{1\}} \cap P^{\mathfrak{C}} = A_\phi^\ell$. So we can extend $g \restriction A_\phi^0$ to a isomorphism g_0 from $A_{\{0\}}^0$ onto $A_{\{0\}}^1$. Now (by 1.6, 2.16) we know that

$tp_*(A^\ell_{\{0\}}, A^\ell_\phi) \vdash tp_*(A^\ell_{\{0\}}, P^{\mathfrak{E}})$ hence $tp_*(A^\ell_{\{0\}}, A^\ell_\phi) \vdash tp(A^\ell_{\{0\}}, A^\ell_{\{1,2\}})$ hence $g^0 \stackrel{def}{=} g_0 \cup g$ is an elementary mapping. We know (by condition 6 of Definition 4.1) that $A^\ell_{\{2,0\}}$ is u.l.a over $A^\ell_{\{2\}} \cup A^\ell_{\{0\}}$, hence it is λ-atomic over it, so as it is λ-saturated it is unique over $A^\ell_{\{2\}} \cup A^\ell_{\{0\}}$. Hence $g^0 \restriction (A^0_{\{2\}} \cup A^0_{\{0\}})$ can be extended to an isomorphism g_1 from $A^0_{\{2,0\}}$ onto $A^1_{\{2,0\}}$. As we know $tp_*(A_{\{0,2\}}, A_{\{2\}}) \vdash tp_*(A_{\{0,2\}}, P^{\mathfrak{E}})$ also $tp_*(A_{\{0,2\}}, A_{\{2\}}) \vdash tp_*(A_{\{0,2\}}, A_{\{1,2\}})$ hence $tp_*(A_{\{0,2\}}, A_{\{2\}} \cup A_{\{0\}}) \vdash tp_*(A_{\{0,2\}}, A_{\{1,2\}} \cup A_{\{0\}})$ [note $A_{\{1,2\}} \cup A_{\{0\}} \subseteq A_{\{0,2\}}$] so necessarily $g^1 \stackrel{def}{=} g_1 \cup g^0$ is an elementary mapping. Now $A^\ell_{\{1,0\}}$ is also λ-prime over $A_{\{1\}} \cup A_{\{0\}}$ so again there is an isomorphism g_2 extending $g^1 \restriction (A^0_{\{1\}} \cup A^0_{\{0\}})$ to an isomorphism from $A^0_{\{1,0\}}$ onto $A^1_{\{1,0\}}$. So it suffices to prove that $g_2 \cup g^1$ is an elementary mapping. As $A^\ell_{\{1,0\}}$ is u.l.a. over $A^\ell_{\{1\}} \cup A^\ell_{\{0\}}$ it suffices to prove $tp_*(A^\ell_{\{0,1\}}, A^\ell_{\{1\}} \cup A^\ell_{\{0\}}) \vdash tp_*(A^\ell_{\{0,1\}}, A^\ell_{\{1,2\}} \cup A^\ell_{\{0,2\}})$,

for this, by 3.5 (and see 3.3A) it suffices to prove:

(*) $A^\ell_{\{1\}} \cup A^\ell_{\{0\}} \subseteq_t A^\ell_{\{1,2\}} \cup A^\ell_{\{0,2\}}$

Let $\bar{c}_s \in A^\ell_s$, $\bar{b}_s \in A^\ell_{s \cup \{2\}}$ for $s = \phi, \{0\}, \{1\}$ be such that $\models \varphi(\dots, \bar{c}_s, \dots, \bar{b}_s, \dots)_{s \in \mathcal{P}^-(2)}$. We shall show that there are $\bar{c}'_s \in A^\ell_s$, (for $s \in \mathcal{P}^-(2)$) such that $\models \varphi[\dots, \bar{c}'_s, \dots, \bar{b}_s, \dots]_{s \in \mathcal{P}^-(2)}$. As we have already proved that $tp_*(A^\ell_{\{0,2\}}, A^\ell_{\{0\}} \cup A^\ell_{\{2\}}) \vdash tp_*(A^\ell_{\{0,2\}}, A^\ell_{\{1,2\}} \cup A^\ell_{\{0\}})$, w.l.o.g. for some ψ_1, ψ_2:

a) $\mathfrak{E} \models \psi_1[\bar{c}_\phi, \bar{c}_{\{1\}}, \bar{b}_\phi, \bar{b}_{\{1\}}, \bar{b}_{\{0\}}]$

b) $\mathfrak{E} \models \forall \bar{y}_\phi, y_{\{1\}}, \bar{x}_\phi, x_{\{1\}}, \bar{x}_{\{0\}}([\psi_1(\bar{y}_\phi, \bar{x}_{\{1\}} \bar{x}_{\{1\}}, \bar{x}_\phi, \bar{x}_{\{0\}}) \rightarrow \psi_2(y_\phi, \bar{x}_\phi, \bar{x}_{\{0\}}])$

c) $\mathfrak{E} \models (\forall y_\phi, \bar{x}_\phi, \bar{x}_{\{0\}}) [\psi_2(\bar{y}_\phi, \bar{x}_\phi, \bar{x}_{\{0\}}) \rightarrow (\exists \bar{y}_{\{0\}}) \vartheta(\bar{y}_\phi, y_{\{0\}}, \bar{x}_\phi, \bar{x}_{\{0\}})]$

d) $\mathfrak{E} \models \forall \bar{y}_\phi, \bar{y}_{\{1\}}, \bar{y}_{\{0\}}, \bar{x}_\phi, \bar{x}_{\{1\}}, \bar{x}_{\{0\}})[\psi_1(y_\phi, \bar{y}_{\{1\}}, \bar{x}_\phi, x_{\{1\}}, \bar{x}_{\{0\}}) \wedge$

$$\vartheta(y_\phi, y_{\{0\}}, x_\phi, x_{\{0\}}) \rightarrow \varphi(\bar{y}_\phi, \bar{y}_{\{1\}}, \bar{y}_{\{0\}}, \bar{x}_\phi, \bar{x}_{\{1\}}, \bar{x}_{\{0\}})]$$

So in fact we have shown that w.l.o.g. $\bar{c}_{\{0\}}$ is empty [replace φ by ψ_1, (a) is the assumption; so suppose $\bar{c}'_\phi \in A^\ell_\phi$, $\bar{c}'_{\{1\}} \in A^\ell_{\{1\}}$ and $\models \psi_1[\bar{c}'_\phi, \bar{c}'_{\{1\}}, \bar{b}_\phi, \bar{b}_{\{1\}}, \bar{b}_{\{0\}}]$ hence by (b), $\models \psi_2[\bar{c}'_\phi, \bar{b}_\phi, b_{\{0\}}]$ and (c) $\models (\exists \bar{y}_{\{0\}}) \; \vartheta(\bar{c}'_\phi, \bar{y}_{\{0\}}, \bar{b}_\phi, \bar{b}_{\{0\}})$ and as $A^\ell_{\{0\}} \prec \mathfrak{E}$ for some $\bar{c}'_{\{0\}} \in A^\ell_{\{0\}}$, $\models \vartheta[\bar{c}'_\phi, \bar{c}'_{\{0\}}, \bar{b}_\phi, \bar{b}_{\{0\}})$, and by (d) we finish].

Then we can eliminate the use of $\bar{b}_{\{0\}}$ as $tp_\Delta(\bar{b}_{\{0\}}, P^{\mathfrak{C}})$ is isolated by some formula in $tp(\bar{b}_{\{0\}}, A_\phi)$ (for Δ a finite set of formulas). At last we know that $(A_{\{1\}}, A_\phi) \prec (A_{\{1,2\}}, A_{\{2\}})$.

In fact we have prove

4.5 Claim : If $\langle A_s : s \in \mathcal{P}^-(3) \rangle$ is $\mathcal{P}^-(3)$-system, and $\langle A_s : s \in \mathcal{P}(\{1,2\}) \rangle$ is λ-saturated then $\mathbf{S}$ is λ-saturated.

Proof of 4.3: The Hypothesis 4.3: There is a $\mathcal{P}^-(3)$-system $\langle A_s^* : s \in \mathcal{P}^-(3) \rangle$ such that $\bigcup_s A_s^*$ unstable.

Assumptions: $\lambda = \lambda^{<\lambda} > |T|$, $2^\lambda = \lambda^+$, $2^{\lambda^+} = \lambda^{++}$.

We first define $A_i (i < \lambda^{++})$ increasing continuous, $A_i \prec \mathfrak{C} \restriction P$, $|A_i| = \lambda^+$, A_i w.l.o.g. a set of ordinals $< \lambda^{++}$ $[cf(i) \in \{0, 1, \lambda^+\} \Longrightarrow A_i$ is saturated]. For each $j < \lambda^{++}, cf(j) = \lambda^+$, $i = j+1$ we define A_α^i, A_α^j for $\alpha < \lambda^+$ such that:

$A_\alpha^j \subseteq A_\alpha^i$, $|A_\alpha^i| = |A_\alpha^j| = \lambda$, $\bigcup_{\alpha<\lambda^+} A_\alpha^i = A_i$, $\bigcup_{\alpha<\lambda^+} A_\alpha^j = A_j$

(A_α^i, A_α^j) is an elementary chain (increasing-continuous) in α $(A_\alpha^i, A_\alpha^j) \equiv (A_{\{1\}}^*, A_\phi^*)$ and

$[cf\ (\alpha) = \lambda \Longrightarrow (A_{\alpha+1}^i, A_{\alpha+1}^j, A_\alpha^i, A_\alpha^j) \equiv (A_{\{1,2\}}^*, A_{\{2\}}^*, A_{\{1\}}^*, A_\phi^*)$, and is saturated.]

We do it by induction on i,

For i = 0 , or i limit: no problem.

i = j+1, cf j $:\neq \lambda^+$: no problem

i = j+1, cf j $= \lambda^+$: no real problem. First we define by induction on α, $(A_\alpha^i, A_\alpha^j) \equiv (A_{\{1\}}^*, A_\phi^*)$ a continuous increasing (in α) chain; $[cf\ \alpha \in \{0, 1, \lambda\} \Longrightarrow (A_\alpha^i, A_\alpha^j)$ is saturated], so that $\bigcup_{\alpha<\lambda^+} (A_\alpha^i, A_\alpha^j)$ will be saturated:

for $\alpha = 0$, or α *limit* or $\alpha = \beta + 1$, $cf\ (\beta) \neq \lambda$: no problem arise and take care of the saturation of the union

$\alpha = \beta + 1, cf\ \beta = \lambda$: Let $(A_{\{1,2\}}, A_{\{2\}}, A_{\{1\}}, A_\phi)$ be a saturated model of power λ of the theory of $(A_{\{1,2\}}^*, A_{\{2\}}^*, A_{\{1\}}^*, A_\phi^*)$. So $(A_{\{1\}}, A_\phi)$ and (A_β^i, A_β^j) are saturated models of the same power and theory; hence isomorphic, so w.l.o.g. equal and let

$$A^i_\alpha = A_{\{1,2\}} \qquad A^j_\alpha = A_{\{2\}}$$

Now $\bigcup_{\alpha<\lambda^+} A^j_\alpha$ is a saturated model of the theory of A^*_ϕ(= the theory of A^j_α) and has power λ^+, so it is isomorphic to A_i and w.l.o.g. they are equal. So we have defined A_i.

* * *

Now we define by induction on $i < \lambda^{++}$

$M_i \prec \mathfrak{C}$, such that:

a) $M_i \cap P^{\mathfrak{C}} = A_i$, M_i increasing continuous.

b) M_i is λ-constructible over A_i,

c) when $cf\ i \in \{0,1,\lambda^+\}$ M_i is λ-saturated and

d) if $j < i$ then M_i is λ-atomic over $M_j \cup A_i$;

We will define the M_i's in some forcing extension V^Q of V: but Q is λ-complete: so (when $cf\ i \in \{0,1,\lambda^+\}$) M_i is isomorphic over A_i to some $M_i' \in V$ [as over A_i there is in V a λ-prime model M_i' in fact a λ-primary one and this property is still true in V^Q. This property is also satisfied by M_i over A_i; so they are isomorphic: use the uniqueness of the λ-primary model (see [Sh 1], Ch. II, §5).

Specifically, Q will be "adding λ^{++}-Cohen subsets of λ, $\langle r^\alpha : \alpha < \lambda^+ \rangle$". For every sequence $\bar{r}, \bar{r} = \langle r_{i,\alpha} : i < \lambda^{++}, \alpha < \lambda^+ \rangle$ (where for some $h \in V$, $r_{i,\alpha} = r^{h(i,\alpha)}$, h one to one) we shall define a model $\overline{M}^{\bar{r}}$. For a while we suppress the superscript $\bar{r}$.

Case I: $i = 0$: by the proof of the existence of a λ-primary model over any λ-saturated $A \prec \mathfrak{C} \restriction P$, $|A| = \lambda^+$ (see 2.14).

Case II: i *limit:* The only problematic point is "M_i is λ-constructible over A_i, and M_i is λ-atomic over $M_j \cup A_i$ for $j < i$". Let $j < i$, every $\bar{c} \in M_i$

belongs

to M_ξ for some ξ , $j < \xi < i$, so by the induction hypothesis $tp(\bar c, A_\xi \cup M_j)$ is λ-isolated , but $M_j \cup A_\xi \cup \bar c$ is compete hence $tp(\bar c, A_\xi \cup M_j) \vdash tp(\bar c, A_i \cup M_j)$ so the latter is λ-isolated too. So M_i is λ-atomic over $M_j \cup A_i$.

Now each M_j ($j < i$) is λ-constructible over A_j, hence over A_i. So (see [Sh 1 IV §3] $M_j = \bigcup_{\alpha<\lambda^+} M_{j,\alpha}$, $\|M_{j,\alpha}\| = \lambda$, $M_{j,\alpha}$ increasing continuous in α and M_j is λ-constructible hence λ-atomic over $A_j \cup M_{j,\alpha}$ and even over $A_i \cup M_{j,\alpha}$. Let $i = \bigcup_{\alpha<\lambda^+} W_\alpha$, $|W_\alpha| \le \lambda$, W_α increasing continuous, W_α with no last element. Let $N_\alpha = \bigcup_{j \in W_\alpha} M_{j,\alpha}$, so clearly $\|N_\alpha\| \le \lambda$. Let

$$C_0 = \{\alpha < \lambda^+ : \forall j < \xi \in W_\alpha,\ M_{\xi,\alpha} \cap M_j = M_{j,\alpha}\}$$

Clearly C_0 is a closed unbounded subset of λ^+.

Now for every $j < \xi < i$, M_ξ is λ-atomic over $M_j \cup A_\xi$ hence (as usual) over $M_j \cup A_i$, and for every $\bar c \in M_\xi$ there is $\alpha(\bar c, j) < \lambda^+$ such that $tp(\bar c, M_{j,\alpha(\bar c,j)} \cup (M_{\xi,\alpha(\bar c,j)} \cap A_\xi)) \vdash tp(\bar c, M_j \cup A_i)$ (are λ-isolated). Clearly $C_1 = \{\alpha \in C_0 : \forall \bar c (\forall j, \xi \in W_\alpha)[j < \xi \wedge \bar c \in M_{\xi,\alpha} \to \alpha(\bar c, j) < \alpha]\}$ is closed unbounded. It suffices to prove that for every $\alpha \in C_1$, $N_{\alpha+1}$ is λ-atomic over $N_\alpha \cup A_i$ (hence λ-constructible). (as we know N_0 is λ-atomic over N_i). First we prove that for every $j \in W_\alpha, M_j$ is λ-atomic over $N_\alpha \cup A_i$; let $\bar d \in M_j$ then as $\alpha \in C_1$, $tp(\bar d, M_{j,\alpha} \cup A_j)$ is λ-atomic hence $tp(\bar d, M_{j,\alpha} \cup A_i)$ is λ-atomic, so it suffices to prove $tp(\bar d, M_{j,\alpha} \cup A_i) \vdash tp(\bar d, \bar c \cup M_{j,\alpha} \cup A_i)$ for every $\bar c \in N_\alpha$. For any such $\bar c$, as W_α has no last element, for some ξ $\bar c \in M_{\xi,\alpha}$ $j < \xi \in W_\alpha$. Now $\alpha(\bar c, j) < \alpha$, hence $tp(\bar c, M_{j,\alpha(\bar c,j)} \cup (M_{\xi,\alpha(\bar c,j)} \cap A_\xi)) \vdash tp(\bar c, M_j \cup A_i)$ as $\bar d \in M_j$, this implies $tp(\bar c, M_{j,\alpha} \cup A_i) \vdash tp(\bar c, M_{j,\alpha} \cup A_i \cup \bar d)$ and by symmetry we get the conclusion. So we have proved that M_j is λ-atomic over $N_\alpha \cup A_i$, hence $\bigcup_{j \in W_\alpha} M_j$ is λ-atomic over $N_\alpha \cup A_i$, but $\bigcup_{j \in W_\alpha} M_j$ is $M_{\sup(W_\alpha)}$ and so we have proved it if $\sup(W_\alpha) = i$. Now if $\zeta \overset{def}{=} \sup W_\alpha < i$, then remember that we had proved that M_i is λ-atomic over $M_\zeta \cup A_i$; as we have just proved that M_ζ is λ-atomic over

$N_\alpha \cup A_i$, together we get that M_i is λ-atomic over $N_\alpha \cup A_i$.

Case III: $i = j + 1, cf\ j < \lambda^+$.

As M_j is λ-constructible over A_j, we can find $M_{j,\alpha}$, and $A_{i,\alpha}$ for $\alpha < \lambda^+$ such that, $M_j = \bigcup_{\alpha<\lambda^+} M_{j,\alpha}$ where $M_{j,\alpha}$ is increasing continuous (in α) $\|M_{j,\alpha}\| \le \lambda$, M_j λ-atomic over $M_{j,\alpha} \cup A_j$ (hence over $M_{j,\alpha} \cup A_i$), and $|A_{i,\alpha}| \le \lambda$, $A_i = \bigcup_{\alpha<\lambda^+} A_{i,\alpha}$. $A_{i,\alpha}$ increasing continuous in α, and $(A_{i,\alpha}, M_{j,\alpha}, A_{j,\alpha}) \prec (A_i, M_j, A_j)$ where $A_{j,\alpha} = A_{i,\alpha} \cap A_j = M_{j,\alpha} \cap A_j$, and when $cf\ \alpha \in \{0,1,\lambda\}$. $(A_{i,\alpha}, A_{j,\alpha})$ is λ-saturated, also when $cf\ \alpha \in \{0,1,\lambda\}$, $(A_{i,\alpha}, M_{j,\alpha}, A_{j,\alpha}) \prec_{L_{\lambda,\lambda}} (A_i, M_j, A_j)$.

We define by induction on α, $M_{i,\alpha}$ such that $A_{i,\alpha} \cup M_{j,\alpha} \subseteq M_{i,\alpha}$, $P^{M_{i,\alpha}} = A_{i,\alpha}$, $[cf\ \alpha \in \{0,1\lambda\} \rightarrow M_{i,\alpha}$ is λ-saturated$]$, $M_{i,\alpha}$ increasing continuous in α, and $M_{i,\alpha}$ is λ-atomic over $M_{i,\alpha} \cup M_{j,\alpha}$ and also over $A_{i,\alpha} \cup M_j$. For the last demand note that

(*) when $cf\ \alpha \in \{0,1,\lambda\}$, as $(A_{i,\alpha}, M_{j,\alpha}, A_{j,\alpha}) \prec_{L_{\lambda,\lambda}} (A_i, M_j, A_j)$ it suffices to prove that $M_{i,\alpha}$ is λ-atomic over $A_{i,\alpha} \cup M_{j,\alpha}$.

So for $\alpha = 0$ it is easy, by the last sentence, for α-limit there is no problem. For $\alpha = \beta + 1$, over $A_{i,\alpha} \cup M_{j,\alpha}$ there is a λ-atomic λ-saturated model $M_{i,\alpha}$, but why $M_{i,\beta} \subseteq M_{j,\alpha}$? As the previous is λ-atomic over $A_{i,\alpha} \cup M_{j,\alpha}$ ([prove it as you have proved (*) and for β limit we use $M_\beta = \bigcup_{\gamma<\beta} M_{\gamma+1}$) and as $\|M_{i,\beta}\| \le \lambda$, clearly $M_{i,\beta}$ is λ- constructible over $A_{i,\alpha} \cup M_{j,\alpha}$, and we can embed it into $M_{i,\alpha}$, over $A_{i,\alpha} \cup M_{j,\alpha}$ and so by renaming we can finish.

So $M_i \overset{\text{def}}{=} \bigcup_{\alpha<\lambda^+} M_{i,\alpha}$ is λ-atomic over $M_j \cup A_i$ (hence over $M_\xi \cup A_i$ for $\xi < i$ (see [Sh 1] ch. IV §3]) and is λ-saturated. We still have to show that it is λ-constructible over A_i. For this it suffices to prove $M_{i,\alpha+1}$ is λ-atomic over $M_{i,\alpha} \cup A_{j,\alpha+1}$ which we could have guaranteed this easily in the construction. More exactly, $M_{i,\alpha+1}$ is a λ-saturated model of cardinality λ extending $M_{i,\alpha} \cup A_{j,\alpha+1}$; Now if Γ is a set of $\le \lambda$ types over $M_{i,\alpha} \cup A_{j,\alpha+1}$ each with no support of power $<\lambda$ (i.e. no type q over $M_{i,\alpha} \cup A_{j,\alpha+1}$, (consistent), $|q| < \lambda$, $q \vdash p$ where p is the type from Γ), then there is a λ-saturated $M \supseteq A_{i,\alpha} \cup M_{j,\alpha+1}$, M omitting every $p \in \Gamma$. Now the other demands on $M_{i,\alpha+1}$ are of the form: omits some type; and to prove those types have no support $< \lambda$, it suffices to find a (λ-saturated M, $M \supseteq A_{i,\alpha} \cup M_{j,\alpha+1}$) omitting such a type for each $p \in \Gamma$ separately.

Case IV: $i = j+1, cf\ j = \lambda^+$.

We act exactly as in Case III, with one additional feature. When $\alpha = \beta + 1, cf\ \beta = \lambda$, we demand

(**) $$\left\langle M_{j,\alpha}, M_{i,\alpha}, M_{j,\beta}, A_{i,\alpha}, A_{j,\alpha}, A_{i,\beta}, A_{j,\beta} \right\rangle$$

$$= \left\langle A^*_{\{0,2\}}, A^*_{\{0,1\}}, A^*_{\{0\}}, A^*_{\{1,2\}}, A^*_{\{2\}}, A^*_{\{1\}}, A^*_{\phi} \right\rangle$$

[Remember $A_{j,\gamma}, A_{i,\gamma} (\gamma < \lambda^+)$ were defined in the first part of the proof, so that the relevant part of (**) holds. We then can define $M_{j,\gamma} (\gamma \geq 0)$, λ-saturated of power λ, $M_{j,\gamma} \cap P^{\mathfrak{C}} = A_{j,\gamma}$, and $M_{j,\gamma+1}$ is λ-atomic over $A_{j,\gamma+1} \cup M_{j;\gamma}$, by 2.14 w.l.o.g. $M_j = \bigcup_{\gamma} M_{j,\gamma}$. Now we defined by induction on γ, $M_{i,\gamma}$, λ-atomic over $M_{j,\gamma} \cup A_{i,\gamma}$. Clearly there is a λ-saturated model of cardinality λ elementarily equivalent to $\left\langle A^*_{\{0,2\}}, A^*_{\{0,1\}}, A^*_{\{0\}}, A^*_{\{1,2\}}, A^*_{\{2\}}, A^*_{\{1\}}, A_{\phi} \right\rangle$, and by 4.4 it is isomorphic to $\left\langle M_{j,\alpha}, M_{i,\alpha}, M_{j,\beta}, A_{i,\alpha}, A_{j,\alpha}, A_{i,\beta}, A_{j,\beta} \right\rangle$ so (**) holds].

So the left system is unstable so by 3.5 there is an m-type p over it of power $< \lambda$ with no λ-isolated extension over $M^j_\alpha \cup M^i_\beta \cup A^i_\alpha$, so in the construction we have a perfect (i.e. homeomorphic to M) set of possibilities an we use $r_{j,\beta}$ to decide (except here we do not use the Cohen sets, though once used we may continue to use it).

The non isomorphism is as in previous proofs.

Remark: We could simplify the proof of 4.3 by a more extensive use of 0.1.

§5 General system and relevant symmetry.

We change slightly the thing we analyze - we shall analyze "the possible existence of a λ-prime model over any $A \prec P^{\mathfrak{C}}$". Remember

Hypothesis: Every formula is equivalent to a relation.

In this section we shall deal with systems of the following kind:

5.1 Definition : A I-system is $\mathbf{S} = \langle A_s : s \in I \rangle$ $(I = I(\mathbf{S}))$ where

1) for some $n = n(I) = n(\mathbf{S})$, $\mathcal{P}(\{1, \ldots, n-1\}) \subseteq I \subseteq \mathcal{P}(n)$, I close under subsets

2) $A_s \cap A_t = A_{s \cap t}$

3) a) if $0 \notin s$, then $A_s \prec \mathfrak{C} \restriction P$, b) if $0 \in s$, $A_s \prec \mathfrak{C}$

4) $\langle A_s : s \in \mathcal{P}(\{1, \ldots, n-2\}) \rangle \prec \langle A_{s \cup \{n-1\}} : s \in \mathcal{P}(\{1, \ldots, n-2\}) \rangle$ are both systems (so the definition is by induction on n).

5) if $0 \in s$, A_s is u.l.a. over $\bigcup_{t \subset s} A_t$.

Remark: This is useful when no two cardinal models exist.

5.2 Definition : 1) A system $\mathbf{S}$ is stable if $\bigcup_{s \in I(\mathbf{S})} A_s^{\mathbf{S}}$ is

2) A system has the existence property if there is M, $P^M \subseteq \bigcup_s A_s^{\mathbf{S}} \subset M$.

3) The I-goodness holds if every I-system is stable.

4) $n^*(T)$ is sup $\{n+1 : \mathcal{P}^-(n)\text{-goodness holds}\}$ (so $n^*(T) \leq \omega$).

5) $n^{**}(T)$ is sup $\{n+1\text{: every } \mathcal{P}^-(n)\text{-system has the atomicity property}\}$.

where

6) $\langle A_s : s \in I \rangle$ has the atomicity property if for every $|T|^+$-saturated $\langle A_s^+ : s \in I \rangle \equiv \langle A_s : s \in I \rangle$, and m-type p over $\bigcup_{s \in I} A_s^+$ of cardinality

$\leq |T|$, has a $|T|^+$-isolated extension over $\bigcup_{s \in I} A_s$.

5.3 Lemma : 1) Being an I-system depends only on its first order theory.

2) Having the atomicity property (for an I-system) depends only on its first order theory.

3) If $\langle A_s : s \in I \rangle$ is a system, $n(T) > 0$, then so are $\langle A_s : s \in I \cap \mathcal{P}(n(I)-1) \rangle$ and $\langle A_{s \cup \{n(I)-1\}} : s \cup \{n(I)-1\} \in I,\ (n(I)-1) \not\in s \rangle$.

4) If $J \subseteq I$ satisfies (1) of 5.1 then $\langle A_s : s \in J \rangle$ is a J-system.

5)* If every model is stable (i.e., $|S_*^m(M)| \leq \|M\|^{|T|}$) then $n^*(T) = n^{**}(T)$, in fact stability and atomicity of $\mathcal{P}^-(n)$-systems are equivalent. (see 3.4(2)(a)). (Without 0.1 we get: stability implies atomicity.)

5.4 Lemma : For any system $\langle A_s : s \in I \rangle$, $(n = n(I))$:

a) if $0 \in s \in I$ then $tp_*(A_s, \bigcup_{t \subset s} A_t) \vdash tp_*(A_s, \bigcup \{A_t : t \in I, s \not\subseteq t\})$; moreover for every $\varphi(\bar{x}, \bar{y})$ and $\bar{c} \in A_s$ for some $\psi_\varphi(\bar{x}, \bar{b}_{\varphi,\bar{c}}) \in tp(\bar{c}, \bigcup_{t \subset s} A_t)$, $\psi_\varphi(\bar{x}, \bar{b}_{\varphi,\bar{c}}) \vdash tp_\varphi(\bar{c}, \bigcup\{A_t : t \in I, s \not\subseteq t\})$.

b) $\bigcup_{\substack{(n-1) \not\in s \\ s \in I}} A_s \subseteq_t \bigcup_{s \in I} A_s$, in fact: for $\bar{b}_t \in A_t (t \in I)$ such that $\models \varphi(\dots \bar{b}_t, \dots)$ we can find $\bar{b}_t^+ \in A_{s-\{n-1\}}$, such that $[(n-1) \not\in t \Longrightarrow \bar{b}_t^+ = \bar{b}_t]$, and $\models \varphi(\dots, b_t^+ \dots)$.

Proof : The proof is by simultaneous induction on $|I|$ (for all systems and both a) and b)). The proof is splitted to cases.

Proof of a):

Case 1: There is $t \in I$, $s \subset t$.

Then we can reduce the problem to one on $I^+ \subset I$ and use the induction hypothesis. So if not Case 1 $\{t : t \in I, s \not\subseteq t\} = I - \{s\}$.

Case 2 : not Case 1 and $(n-1) \not\in s$.

Let $\bar{c} \in A_s$, $\varphi(\bar{x},\bar{y})$ be a formula.

Let $J = \{t \in I:(n-1) \not\in t\}$, then by the induction hypothesis [as $|J| < |J \cup \{n-1\}| \leq |I|$, because $\{n-1\} \not\in J$, (and $\{n-1\} \in I$, as I is downward closed and $n = n(I)$). Note that $n-1 > 0$ as $n-1 \not\in s, 0 \in s$] for some $\psi_\varphi(\bar{x},\bar{b}_{\varphi,\bar{c}}) \in tp_*(A_s, \bigcup_{t \subset s} A_t)$, $\psi_\varphi(\bar{x},\bar{b}_{\varphi,\bar{c}}) \vdash tp_\varphi(\bar{c}, \bigcup_{\substack{t \neq s \\ t \in J}} A_t)$.

So for no $\bar{d} \in \bigcup_{\substack{t \neq s \\ t \in J}} A_t$. $\models (\exists \bar{x})[\psi_\varphi(\bar{x},\bar{b}_{\varphi,\bar{c}}) \wedge \varphi(\bar{x},\bar{d})] \wedge (\exists \bar{x})[\psi_\varphi(\bar{x},\bar{b}_{\varphi,\bar{c}}) \wedge \neg \varphi(\bar{x},\bar{d})]$. Applying the induction hypothesis to $I-\{s\}$ (for (b)) we see that $\bigcup\{A_t : t \in J-\{s\}\} \subseteq_t \bigcup\{A_t : t \in I-\{s\}\}$. So also in $\bigcup\{A_t : t \in I-\{s\}\}$ we cannot find $\bar{d}$ as above. So $\psi_\varphi(\bar{x},\bar{b}_{\bar{\varphi},\bar{c}}) \vdash tp_\varphi(\bar{c}, \bigcup\{A_t : t \in I-\{s\}\})$, as required.

Case 3: Not 1 nor 2 and there is v, a maximal member of I, $0 \in v, v \neq s, (n-1) \not\in v$. So v,s are $\subset$- incomparable.

By using the induction hypothesis for $I-\{v\},s$ and case 2 for I,v we see that

$$tp_*(A_s, \bigcup_{t \subset s} A_t) \vdash tp_*(A_s, \bigcup\{A_t : t \in I,\ t \neq v,s\})$$
$$tp_*(A_v, \bigcup_{t \subset v} A_t) \vdash tp_*(A_s, \bigcup\{A_t : t \in I,\ t \neq v\})$$

Together we get the first close of (a). As for the second: we can treat our system as an $|I|$-sorted model, find a $|T|^+$-saturated elementary extension, so also there we get the first close of (a). By saturativity we get the ψ_φ and note that its property is preserve by elementary equivalence.

Case 4: For some $t \in I$, $0 \not\in t$ and $t \cup \{0\} \not\in I$.

Let $J_0 = \{v \in I : 0 \not\in v\}$, $J_1 = \{v \in I : 0 \not\in v,\ v \cup \{0\} \in I\}$, $J_2 = \{v \in I : 0 \in v\}$. We shall prove that $tp(\bigcup_{u \in J_2} A_u, \bigcup_{u \in J_1} A_u) \vdash tp(\bigcup_{u \in J_2} A_u, \bigcup_{u \in J_0} A_u)$ (by [Sh 1, ch. IV §2 §3], this suffices for the first phrase of (a),) then proceed as in Case 3.

For each $v \in J_2$, as $tp_*(A_v, \bigcup_{u \subset v} A_u)$ is u.l.a. by the induction hypothesis $\bigcup_{u \subset v} A_u \subseteq_t \bigcup \{A_u : v \not\subseteq u, \ u \in J_2 \cup J_1\}$, hence by 3.5 $\bigcup_{u \subset v} A_u \subseteq_t \bigcup \{A_u : v \not\subseteq u, u \in I\}$, together we get $tp_*(A_v, \bigcup_{u \subset v} A_u) \vdash tp(A_v, \bigcup \{A_u : v \not\subseteq u, u \in I\})$. By [Sh 1, IV 3.3] this gives $tp(\bigcup_{v \in J_2} A_v, \bigcup_{u \in J_1} A_u) \vdash tp(\bigcup_{v \in J_2} A_v, \bigcup_{u \in J_0} A_u)$.

Case 5: not cases 1,2,3,4.

So $(n-1) \in s$ [as not Case 2] and $(\forall t \in I)(t \cup \{n-1\} \in I)$, [if t is a counterexample, as I is downward closed w.l.o.g. t is maximal in I; as $t \cup \{n-1\} \notin I$ clearly $t \not\subseteq s$, $n-1 \notin t$, by "not case 4" $t \cup \{0\} \in I$ hence by t's maximality $0 \in t$, and we get Case 3, contradiction]. So $I = J \cup \{t \cup \{n-1\} : t \in J\}$ where $J = \{t \in I : (n-1) \notin I\}$. We apply the induction hypothesis to $\langle A_{s \cup \{n-1\}} : s \in J \rangle$, $s - \{n-1\}$ (remember 5.3(3)) so

$$tp_*(A_s, \bigcup \{A_t : t \subset s, (n-1) \in t\}) \vdash tp_*(A_s, \bigcup \{A_t : t \neq s, s - \{n-1\}\})$$

hence the first close of (a) follows (by Ax VII of [Sh 1, ch IV §1]) and we prove (a) as in the Case 3.

Proof of (b) of 5.4: Now we prove (b) of 5.4. Let $J = \{t \in I : (n-1) \notin t\}$.

First replace our system by a $|T|^+$-saturated one. Then by increasing the $\bar{b}_s$ to sequences of length $< |T|^+$ we can assume for each $s \in I$: if $0 \in s$ then $tp(\bar{b}_s, \bigcup_{t \subset s} \bar{b}_t) \vdash tp(\bar{b}_s, \bigcup \{A_t : t \in I, s \not\subseteq t\})$. Now we define the $\bar{b}_s^+$. If $(n-1) \notin s$ let $\bar{b}_s^+ = \bar{b}_s$. Next choose $\langle \bar{b}_s^+ : s \in I, 0 \notin s, (n-1) \in s \rangle$, so that $\bar{b}_s^+ \in A_{s-\{n-1\}}$ and in the model $\langle A_{s \cup \{n-1\}} : s \in \mathcal{P}(\{1, \ldots, n-2\}) \rangle$ it realizes over $\bigcup \{\bar{b}_s : s \in \mathcal{P}(\{1, \ldots, n-2\}\}$ the same type as $\langle \bar{b}_s : s \in I, 0 \notin I, (n-1) \in s \rangle$ (possible by (4) of Definition 5.1). For the others, define by induction on $|s|$, $\bar{b}_s^+$ such that $tp(\cdots {}^\frown \bar{b}_t {}^\frown \cdots)_{t \subseteq s} = tp(\cdots {}^\frown \bar{b}_t^+ {}^\frown \cdots)_{t \subseteq s}$, and simultaneously prove that the mapping $\bar{b}_s \to \bar{b}_s^+$ defined so far is elementary (for $\mathfrak{C}$).

5.5 Conclusion: 1)* Suppose $\lambda = \lambda^{<\lambda} > |T|$, and $\langle A_s^\ell : s \in I \rangle$ is a system, $\langle A_s^\ell : 0 \notin s \in I \rangle$ is λ-saturated, each A_s^ℓ is λ-saturate and of power λ and

$\langle A_s^0 : 0 \notin s \in I \rangle \equiv \langle A_s^1 : 0 \notin s \in I \rangle$. Then the two systems are isomorphic.

2) If in (1) we do not assume 1.0, we need $(A_s^\ell, c)_{c \in \bigcup_{t \subset s} A_t^\ell}$ is λ-saturated when $0 \in s \in I$.

5.6 Conclusion: 1) If $\langle A_s : s \in I \rangle$ is an I-system, $\mathcal{P}(\{1, \ldots, n-1) \subseteq J \subseteq I$ then $\bigcup_{s \in I} A_s$ is u.l.a. over $\bigcup_{s \in J} A_s$.

2) If $\langle A_s : s \in I \rangle$ is an I-system, then for $s \in I$, $A_s \cap P^{\mathfrak{C}} = A_{s-\{0\}}$.

§6 A proof of the existence property.

6.0 Hypothesis: $n^{**}(T) = \omega$.

6.1 Theorem * : Suppose T is countable and $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ is a system satisfying:

(*) $0 \in s \in \mathcal{P}^-(n) \Longrightarrow A_s$ is $\mathbf{F}^\ell_{\aleph_0}$- constructible over $\bigcup_{t \subset s} A_t$.

Then there is a model M $\mathbf{F}^\ell_{\aleph_0}$-constructible over $\bigcup_s A_s$, u.l.a. over it, and $P^M \subseteq \bigcup_s A_s$. So the existence property holds for such systems.

Proof : The proof is broken to some claims.

6.2 Claim: If $A \subseteq_t C$, B is $\mathbf{F}^\ell_{\aleph_0}$-constructible over A, then B is $\mathbf{F}^\ell_{\aleph_0}$-constructible over C (by the same sequence), $tp_*(B,A) \vdash tp_*(B,C)$, and $A \cup B \subseteq_t C$.

Proof : See [Sh 1 , Ch. XII].

6.3 Claim *: If M is $\mathbf{F}^\ell_{\aleph_0}$-constructible over $\bigcup A_s \subseteq M$, $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ is a system *then* M is u.l.a. over $\bigcup_s A_s$.

Proof : W.l.o.g. (by easy set theory) for some $\lambda > |\bigcup_s A_s| + |T|$,

$\lambda = \lambda^{<\lambda}$ so let $\langle A'_s : s \in \mathcal{P}^-(n)\rangle$ be a saturated elementary extension of $\langle A_s : s \in \mathcal{P}^-(n)\rangle$. By 6.2 $tp_*(M, \bigcup_s A'_s)$ is λ-isolated so there is a λ-primary N, $M \cup \bigcup_s A'_s \subseteq N$. Hence N is u.l.a. over $\bigcup A_s$ see 3.3 and we finish by the next Fact (6.4).

6.4 Fact: If $A \prec C$, and B is u.l.a. over C, $tp_*(B,A) \vdash tp_*(B,C)$ then B is u.l.a. over A (witnessed in the same way).

Remark: Note that we assume $A \prec C$, i.e. $\mathfrak{C} \restriction A \prec C \restriction \mathfrak{C}$, not just $A \subseteq_t C$.

Proof : Let $\bar{b} \in B$, $\varphi \in L$, then for some $\psi_\varphi(\bar{x}, \bar{c}_{\varphi,\bar{b}}) \in tp(\bar{b}, C)$ $\psi_\varphi(\bar{x}, \bar{c}_{\varphi,\bar{b}}) \vdash tp_\varphi(\bar{b}, C)$. As $tp_*(B,A) \vdash tp_*(B,C)$ there is $\vartheta(\bar{x}, \bar{a}) \in tp_*(\bar{b}, A)$, $\vartheta(\bar{x}, \bar{a}) \vdash \psi_\varphi(\bar{x}, \bar{c}_{\varphi,\bar{b}})$. So

$$\models (\forall \bar{x})[\vartheta(\bar{x}, \bar{c}_{\varphi,\bar{b}}) \rightarrow \psi_\varphi(\bar{x}, \bar{c}_{\varphi,\bar{b}})] \wedge$$
$$(\forall \bar{y} \in C)[(\forall \bar{x})(\psi_\varphi(\bar{x}, c_{\varphi,b}) \rightarrow \varphi(\bar{x}, \bar{y})) \vee (\forall \bar{x})(\psi_\varphi(\bar{x} c_{\varphi,\bar{b}}) \rightarrow \neg \varphi(\bar{x}, \bar{y}))]$$

so there is $\bar{c}'_{\varphi,b} \in A$ with those properties.

6.5 Claim: If $\langle A_s : s \in \mathcal{P}^-(n)\rangle$ is a system, satisfying (*) (from 6.1) $\lambda = \sum_s |A_s| > |T|$ then we can define $\langle A^\alpha_s : s \in \mathcal{P}^-(n)\rangle$ $(\alpha < \lambda)$ such that

(1) $\sum_s |A^\alpha_s| = |\alpha| + |T|$

(2) $\langle A^\alpha_s : s \in \mathcal{P}^-(n)\rangle \prec \langle A_s : s \in \mathcal{P}^-(n)\rangle$

(3) $\langle A^\alpha_s : s \in \mathcal{P}^-(n)\rangle$ is increasing continuous in α.

(4) $\langle A^\alpha_s : s \in \mathcal{P}^-(n)\rangle$ is a system, as well as $\langle A^\alpha_s : s \in \mathcal{P}^-(n+1), s \neq \{0, \dots, n-1\}\rangle$ (where $A^\alpha_{s \cup \{n\}} = A^{\alpha+1}_s$ for $s \in \mathcal{P}^-(n)$), satisfying (*) in both cases.

Proof : Easy. [Sh 1, ch. IV, §3]

Proof of 6.1: We prove it by induction on $\lambda = \sum_s |A_s|$ (for all n simultaneously).

Easily of the three properties demanded of M in 6.1 the first implies the second (by 6.3) and the third (apply u.l.a. for the formula $x=y$). Remember T is countable.

Case 1: $\lambda = \aleph_0$.

So $A \overset{def}{=} \bigcup_s A_s$ is countable. By Hypothesis 6.0, easily for every $\varphi(\bar{x},\bar{a})$ $\bar{a} \in A$, $\models \exists \bar{x}\, \varphi(\bar{x},\bar{a})$, and $\varphi_1(\bar{x},\bar{y})$ there is $\vartheta(\bar{x},\bar{a}_1)$, $\bar{a}_1 \in A$, and $\models (\exists \bar{x})[\varphi(\bar{x},\bar{a}) \wedge \vartheta(\bar{x},\bar{a}_1)]$ and $\vartheta(\bar{x},\bar{a}_1) \vdash tp_{\varphi_1}(\bar{c},A)$ for some $\bar{c}$. [otherwise replace $\langle A_s : s \in \mathcal{P}^-(n)\rangle$ by an elementarily equivalent $|T|^+$-saturated system and get contradiction to 5.2(6)]. So we can define by induction on n, $\varphi_n(x_0, \ldots, x_n,\bar{a}_n)$, $\bar{a}_n \in A$ such that $\models (\exists x_0, \ldots, x_n)\varphi_n(x_0, \ldots, x_n,\bar{a}_n)$, $\models (\forall x_0, \ldots, x_{n+1})$ $(\varphi_{n+1}(x_0, \ldots, x_{n+1}, \bar{a}_{n+1}) \rightarrow \varphi_n(x_0, \ldots, x_n, \bar{a}_n))$ and for every $\psi = \psi(x_0, \ldots, x_n; \bar{y})$ for some $k > n$ and $c_0, \ldots, c_n$ $(\exists x_{n+1} \cdots x_k)$ $\varphi_n(x_0, \ldots, x_k,\bar{a}_k) \vdash tp(\langle c_0, \ldots, c_n\rangle A)$ $\{\varphi_n(x_0, \ldots, x_n,\bar{a}_n) : n < \omega\}$ is complete over A (in $\{x_n : n < \omega\}$) and is the complete diagram over A of a model as required (remember Ax VII (of [Sh 1 , Ch. IV. §1]. holds for $\mathbf{F}^{\ell}_{\aleph_0}$.)

Case 2: $\lambda > |T|$.

Define $A_s^\alpha(\alpha < \lambda)$ by 6.5. We now define by induction on α, a model M_α, so that M_α is $\mathbf{F}^{\ell}_{\aleph_0}$-constructible over $\bigcup_s A_s^\alpha$, $\bigcup_s A_s^\alpha \subseteq M_\alpha$, also if α is limit $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, and $\alpha = \beta + 1$ M_α is $\mathbf{F}^{\ell}_{\aleph_0}$-constructible over $\bigcup_s A_s^\alpha \cup M_\beta$. We should prove for each α, that $\langle A_s^\alpha : \alpha \in \mathcal{P}^-(n+1)\rangle$ is a system where $A_n^\alpha = M_\alpha$, this follows by 6.5(4) and noting M_n is u.l.a. over $\bigcup\{A_s^\alpha : s \in \mathcal{P}^-(n)\}$ by 6.3.

6.6 Theorem * : Suppose T is countable. If M,N are $\aleph_1$-saturated, with $M \restriction P = N \restriction P$ then M,N are isomorphic over P^M. (by 3.4(3) the $\aleph_1$-saturation of $M \restriction P$ implies that of M).

Proof : Over P^M there is a $\mathbf{F}^{\ell}_{\aleph_0}$ primary model M^+, so M^+ is $\mathbf{F}^{t}_{\aleph_1}$-primary and $\mathbf{F}^{t}_{\aleph_1}$-prime. So it can be elementarily embedded into M over P^M hence its image is equal to M. Similarly for N.

This theorem is made more interesting by the following (not using 6.0 anymore):

6.7 Fact: Assume for every M, P^M is stable.

If there are $M_0 \prec M, P^M \subseteq M_0 \neq M_1$ then for every $\lambda \geq |T| + |\delta|$ we can find M and $a_i \in M (i < \delta)$ such that :

(a) $i \neq j \Longrightarrow a_i \neq a_j$, $\|M\| = \lambda = |P^M|$,

(b) for $i < j$ $\quad tp(a_j, P^M \cup \{a_\alpha : \alpha < i\}) = tp(a_i, P^M \cup \{a_\alpha : \alpha < i\})$.

(c) moreover for every $\bar{b} \in M$ there is $i(\bar{b}) < \delta$ such that: if $i(b) \leq i < j$, $tp(a_j, P^M \cup \bar{b} \cup \{a_\alpha : \alpha < i\}) = tp(a_i, P^M \cup \bar{b} \cup \{a_\alpha : \alpha < i\})$

Proof : W.l.o.g. M_0 is λ^+-saturated. Let $a \in M_1 - M_0$ and define by induction on $i < \delta$, $N_i \prec M_0$, $\|N_i\| = \lambda$ and a_i such that $\bigcup_{j<i} N_j \cup \{a_j : j < i\} \subseteq N_i$ and $a_i \in M_0$ realizes $tp(a, N_i)$. By claim 2.16 (or 1.4) $\bigcup_{i<\delta} N_i$,$\{a_i : i < \delta\}$ are as required.

6.8 Lemma : 1) Under the assumption of 6.7, if the conclusion of 6.1 holds then when $|T| < \lambda < \mu$ there is a model M^*, $\|M^*\| = |P^{M^*}| = \mu$, so that there are $a_i (i < \delta)$ as there (for M^*) when $\delta = \lambda$ but not when $\lambda < cf\ \delta \leq \mu$.

2) If 1.0 fails, λ regular $2^\lambda = \lambda^+$, then we can find M, $a_i (i < \lambda)$ as in 6.7, P^M is saturated, $\|M\| = \|P^M\| = \lambda^+$.

Proof : 1) Let M ,$a_i (i < \lambda)$ be as there, choose $A, P^M \in A \prec \mathfrak{C} \restriction P$, $|A| = \lambda$ and let M^* be $\mathbf{F}^{\ell}_{\aleph_0}$-constructible over $M \cup A$. By the P^M's stability, $a_i (i < \lambda)$ has the property in M^* too. Suppose $\langle c_i : i < \delta \rangle$ has the property (in M, i.e. a),b),c) of 6.7) too and $\lambda < cf\ \delta$. By [Sh 1 Ch IV §3] we can find $N \prec M^*$, $M \subseteq N$, $\|N\| = \lambda$, N closed enough (under history of the construction and the function $\bar{b} \to c_{i(\bar{b})}, c_i \to c_{i+1}$), so that M^* is $\mathbf{F}^{\ell}_{\aleph_0}$-constructible over $N \cup A$, and $\langle a_i : a_i \in N \rangle$ has the property in N and $cf(\sup\{i : a_i \in N\}) > |T|$. Then

$tp(a_{\sup\{i:a_i \in N\}}, N \cup A)$ is not $\aleph_1$- isolated, contradiction.

2) Left to the reader.

§7 Manipulations with systems for an arbitrary theory.

7.0 Discussion: We are dealing with several kinds of I-systems, so we shall use the name "$I-x-$system", x a latin letter to differentiate. For Definition 5.1 we use $x=a$ and say it is for $\mathfrak{C}$ or for T.

7.1 Definition: We call $\mathbf{S} = \langle A_s : s \in I \rangle$ an $I-b-$system for T if:

1) for some $n = n(I) = n(S)$, $I \subseteq \mathcal{P}(n)$, $I \not\subseteq \mathcal{P}(n-1)$, I closed under subsets.

2) $\langle A_s : s \cup \{n-1\} \in I, n-1 \notin s \rangle \prec \langle A_{s \cup \{n-1\}} : s \cup \{n-1\} \in I, n-1 \notin s \rangle$

3) each A_s is a model of T.

4) $\langle A_s : s \in J \rangle$ is a I-system when $J = \{s \in I : (n-1) \notin s\}$.

5) If $n-1 \in t \in I$ then $A_t \cap (\bigcup \{A_s : n-1 \notin s \in I\}) \subseteq \bigcup \{A_s : n-1 \notin s \in I, s \cup \{n-1\} \in I\}$.

7.2 Fact: 1) For $\mathbf{S}$ to be an $I-b-$system for T depends on its first order theory only.

2) If $\langle A_s : s \in I \rangle$ is an $I-b-$system then $A_s \cap A_t = A_{s \cap t}$ for any $s, t \in I$.

Proof: 1) Check.

2) Prove it by induction on n. If $n-1 \notin s \cup t$-trivial using condition 4). If $n-1 \in s \cap t$, by condition (4) and the induction hypothesis $A_{s-\{n-1\}} \cap A_{t-\{n-1\}} = A_{s \cap t-\{n-1\}}$ and use condition (2). If $n-1 \in s, n-1 \notin t$, then $s \cap t = (s-\{n-1\}) \cap t$, and again $A_{s-\{n-1\}} \cap A_t = A_{s \cap t}$ by condition (4), and by (2) $A_{s-\{n-1\}} = A_s \cap (\bigcup \{A_v : v \cup \{n-1\} \in I, n-1 \neq I\}$, and we finish by (5).

7.3 Fact: $\langle A_s : s \in \mathcal{P}(n) \rangle$ is a $\mathcal{P}(n)-b$-system for $\mathfrak{E} \restriction P$ *iff* $\langle A_{s-1} : s \in \mathcal{P}(n+1), 0 \notin s \rangle$ is a $\mathcal{P}(\{1,\ldots,n\})-a$-system (for an integer k let $s-k = \{i-k : i \in k\}$ $s+k$ is defined similarly.)

7.4 Fact: If $n > 0$, $\langle A_s : s \in \mathcal{P}(n) \rangle$ is a $\mathcal{P}(n)-b$-system for T and for $s \in \mathcal{P}(n-1)$ $B_s = (A_{(s+1)\cup\{0\}}, A_{s+1})$ then $\langle B_s : s \in \mathcal{P}(n-1) \rangle$ is a $\mathcal{P}(n-1)-b$-system for $T_1 = Th(A_{\{0\}}, A_\phi)$

Proof: We prove it by induction on n

n=1: so $\mathcal{P}(n-1) = \mathcal{P}(0) = \{\phi\}$, so $\langle B_s : s \in \mathcal{P}(n-1) \rangle$ consistent of one model, of T_1 of course:

n+1:

Condition: 1) is trivial.

Condition: 2) We should prove

$$\langle B_s : s \in \mathcal{P}(n-1) \rangle \prec \langle B_{s\cup\{n-1\}} : s \in \mathcal{P}(n-1) \rangle$$

(looking what I is).

This is equivalent to

$$\langle \langle A_{(s+1)\cup\{0\}}, A_{s+1} \rangle : s \in \mathcal{P}(n-1) \rangle \prec \langle \langle A_{(s+1)\cup\{0,n\}}, A_{(s+1)\cup\{n\}} \rangle : s \in \mathcal{P}(n-1) \rangle$$

which is equivalent to

$$\langle A_s : s \in \mathcal{P}(n) \rangle \prec \langle A_{s\cup\{n\}} : s \in \mathcal{P}(n) \rangle$$

which holds as $\langle A_s : s \in \mathcal{P}(n+1) \rangle$ is a $\mathcal{P}(n+1)-b$-system.

Condition: 3) we know $\langle A_s : s \in \mathcal{P}(n) \rangle$ is a $\mathcal{P}(n)-b$-system hence by the induction hypothesis for $s \in \mathcal{P}(n-1)$, $(A_{(s+1)\cup\{0\}}, A_{(s+1)}) \equiv (A_{\{0\}}, A_\phi)$. As we have proved condition 2), for $s \in \mathcal{P}(n-1)$ $B_s \prec B_{s\cup\{n-1\}}$ i.e. $(A_{(s+1)\cup\{0\}}, A_{s+1}) \equiv (A_{(s+1)\cup\{0,n-1\}}, A_{(s+1)\cup\{n-1\}})$, so the condition holds for $s \cup \{n-1\}$ when $s \in \mathcal{P}(n-1)$.

So it holds for every $s \in \mathcal{P}(n)$, as required.

Condition: 4) Easy.

Condition 5): Obvious, (by 5) for $\langle A_s : s \in \mathcal{P}(n+1)\rangle$).

7.5 Lemma: 1) Suppose $\langle M_s : s \in \mathcal{P}(n)\rangle$ is a $\mathcal{P}(n)-b$-system for T, $\lambda = \sum_s \|M_s\|$, $\lambda > |T|$. Then we can find $\langle M_{s,\alpha} : s \in \mathcal{P}(n)\rangle$ $(\alpha < \lambda)$ such that:

(i) $\langle M_{s,\alpha} : s \in \mathcal{P}(n)\rangle \prec \langle M_s : s \in \mathcal{P}(n)\rangle$

(ii) $\|M_{s,\alpha}\| = |T| + |\alpha|$.

(iii) Let for $\alpha < \lambda^+, s \in \mathcal{P}(n)$,

$$M_s^\alpha = M_s^\alpha, M^\alpha_{s \cup \{n\}} = M_s^{\alpha+1}$$

Then $\langle M_s^\alpha : s \in \mathcal{P}(n+1)\rangle$ is a $\mathcal{P}(n+1)-b$-system for T.

2) If $\langle M_s : s \in \mathcal{P}(n)\rangle$ is κ-saturated $(\forall \alpha < \lambda)[|\alpha|^{<\kappa} < \lambda]$, $2^{|T|} < \lambda$ then we can demand $\langle M_{s,\alpha} : s \in \mathcal{P}(n)\rangle$ is κ-saturated when $cf\ \alpha \in \{0,1\}$ or $cf\ \alpha \geq \kappa$, but then $\|M_{s,\alpha}\| \leq (|T| + |\alpha|)^{<\kappa}$ (if we ask just for κ-compact then $\|M_{s,\alpha}\| \leq (|T| + |\alpha|)^{<\kappa}$.

(2a) We can even demand this for each $I \subseteq \mathcal{P}(n)$ separately.

3) If $\lambda = \kappa^+$, $\kappa = \kappa^{<\kappa} > |T|$, and $\langle M_s : s \in \mathcal{P}(n)\rangle$ is saturated, then we can also demand $\langle M_s^\alpha : s \in \mathcal{P}(n+1)\rangle$ is saturated when $cf\ \alpha \notin [\aleph_0, \kappa)$. (but $\|M_{s,\alpha}\| = \kappa$) We can, except for some unbounded non stationary subset determine its theory as that of $\langle N_s : s \in \mathcal{P}(n+1)\rangle$ a $\mathcal{P}(n+1)-b$-system, provided that $\langle N_s : s \in \mathcal{P}(n)\rangle \equiv \langle M_s : s \in \mathcal{P}(n)\rangle$.

Proof: 1) Easy, 2) Easy, 3) See proofs in §4.

7.6 Lemma: Suppose $\lambda = \lambda^{<\lambda}$, and $2^{\lambda^{+\ell}} = \lambda^{+\ell+1}$ for $\ell < n$. Suppose $\langle A_s^* : s \in \mathcal{P}(n)\rangle$ is a $\mathcal{P}(n)-b$-system for T. Let $J = J_{\lambda,n} \stackrel{def}{=} \{\eta : \eta$ a sequence of ordinals of length $\leq n, \eta(\ell) < \lambda^{+(n-\ell)}\}$.

Then we can define models $M_{\eta,t}$ $(\eta \in J,\ t \in \mathcal{P}(\ell(\eta)))$ of T such that:

(i) $M_{\eta,t}$ is a model of T of power $\lambda^{+(n-\ell(\eta))}$, it is saturated provided that $(\forall \ell < \ell(\eta))[cf\ \eta(\ell) \in \{0, 1, \lambda^{+(n-\ell)}\} \vee \ell \in t]$.

(ii) if $\eta \in J, t \in \mathcal{P}(\ell(\eta))$ and $\ell(\eta) < n$, then

a) $M_{\eta,t} = \bigcup \{M_{\eta\frown<i>,t} : i < \lambda^{(n-\ell(\eta))}\}$

b) if $\eta\frown<\delta> \in J$ δ limit then $M_{\eta\frown<\delta>,t} = \bigcup_{i<\delta} M_{\eta\frown<i>,t}$

c) $M_{\eta\frown<i>,t\cup\ell(n)} = M_{\eta\frown<i+1>}$.

(iii) for each $\eta \in J, \mathbf{S}^\eta \overset{def}{=} \langle M_{\eta,t} : t \in \mathcal{P}(\ell(\eta))\rangle$ is a $\mathcal{P}(\ell(\eta))-b$ -system.

(iv) if $(\forall \ell < \ell(\eta))[cf\ \eta(\ell) \in \{0,1,\lambda^{+(n-\ell)}\}]$ then $\mathbf{S}_\eta$ is saturated and its theory is that of $\langle A_s^* : s \in \mathcal{P}(\ell(\eta))\rangle$.

Proof: We prove it by induction on n for all possible T, $\langle A_s^* : s \in \mathcal{P}(n)\rangle$. For each $n > 0$ we define by induction on $\alpha < \lambda^{+n}$ the models $M_{<\alpha>,\phi}$ and $M_{\eta,t}(\eta(0) < \alpha,)$ $\eta \in J, t \in \mathcal{P}(\ell(\eta))$, such that when $cf\ \alpha \in \{0,1,\lambda^{+(n-1)}\}$, $M_{<\alpha>,\phi}$ is a saturated, of cardinality $\lambda^{+(n-1)}$ $M_{<\alpha>} \models T$, $M_{<\alpha>,\{1\}}$ is saturated, $\langle M_{<\alpha>} : \alpha < \lambda^{+n}\rangle$ an elementary chain, for α limit, $M_{<\alpha>} = \bigcup_{\beta<\alpha} M_{<\beta>}$, for $\alpha = \beta + 1$ $M_{<\alpha>,\phi} = M_{<\beta>,\{1\}}$.
For α limit or zero - no problem. For $\alpha = \beta + 1$, $cf\ \beta \notin \{0,1,\lambda^{+(n-1)}\}$, we let $M_{<\alpha>,\{1\}}$ be a saturated elementary extension of $M_{<\beta>}$ of power $\lambda^{+(n-1)}$ and then use 7.5 (2a). For $\alpha = \beta + 1, \beta=0$ for $M_{<\alpha>}$ there is no problem and then use 7.5. For $\alpha = \beta+1$, $cf\ \beta \in \{1,\lambda^{+(n-1)}\}$, $M_{<\beta>,\phi}$ is saturated. We use the induction hypothesis for $n-1$, and T_1 from 7.4 (starting there with $\langle A_s^* : s \in \mathcal{P}(n)\rangle$). Getting $(M^1_{\eta,t}, M^0_{\eta,t})$ $\eta \in J_{\lambda,n-1}$. So $M^0_{<\alpha>,\phi}$ is a model of T of power $\lambda^{+(n-1)}$, saturated hence $\cong M_{<\beta>,\phi}$ so w.l.o.g. it is $M_{<\beta>}$, let $M_{<\alpha>,\phi} = M^1_{<>} = M_{<\beta>,\{0\}}$, $M_{<\alpha>\frown\eta(t-\{0\})-1}$ is $M^0_{\eta,t}$ if $0 \notin t$, and is $M^1_{\eta,(t-\{0\})-1}$ if $0 \in t$.

7.7. Lemma. Suppose $\lambda^{<\lambda} > |T|$, $2^{\lambda^{+\ell}} = \lambda^{+\ell+1}$ for $\ell < n$ and $\langle A_s^* : s \in \mathcal{P}(n)\rangle$ a $\mathcal{P}(n)-b$ -system for T. Let

$W(\lambda) = \{\delta < \lambda^+ : cf\ \delta = \lambda\}$ $\qquad W^+(\lambda) = \{\delta+1 : \delta \in W(\lambda)\}$, and $W^*(\lambda) = W(\lambda) \cup W^+(\lambda)$.

$J'_{\lambda,n} = \{\eta : \eta$ a sequence of ordinals of length $\leq n$, $\eta(\ell) < \lambda^{+(n-\ell)}$,

$$[\ell+1 < \ell(\eta) \Longrightarrow \eta(\ell) \in W^*(\lambda^{+(n-\ell-1)})]\}$$

$J^a_{\lambda,n} = \{\eta \in J'_{\lambda,n}$: for every $\ell+1 < \ell(\eta)$, $\eta(\ell) \in W(\lambda^{+(n-\ell)})\}$.

$J''_{\lambda,n} = \{\eta \in J'_{\lambda,n} : cf\ (\eta(\ell(\eta)-1)) \in W^*(\lambda^{+(\ell(\eta)-1)})\}$.

Then we can define models $M_\eta (\eta \in J'_{\lambda,n})$ of T such that:

(i) M_η is a model of T of power $\lambda^{+(n-\ell(\eta))}$

(ii) if $\eta \in J''_{\lambda,n}$ then M_η is saturated.

(iii) if $\eta \in J'_{\lambda,n}$ is not maximal then
$i < j \Longrightarrow M_{\eta\frown\langle i\rangle} \prec M_{\eta\frown\langle j\rangle}$; $M_\eta = \bigcup\limits_i M_{\eta\frown\langle i\rangle}$; for δ limit $M_{\eta\frown\langle\delta\rangle} = \bigcup\limits_{i<\delta} M_{\eta\frown\langle i\rangle}$.

(iv) For each $\eta \in J^a_{\lambda,n}$ we define a $\mathcal{P}(\ell(\eta))-b$-system $\mathbf{S}^\eta$: $\mathbf{S}^\eta = \left\langle M_t^{\mathbf{S}^\eta} : t \in \mathcal{P}(\ell(\eta))\right\rangle$, $M_t^{\mathbf{S}^\eta} = M_{\nu(\eta,t)}$ where $\ell(\nu(\eta,t)) = \ell(\eta)$ and

$$\nu(\eta,t)(\ell) = \begin{cases} \eta(\ell) & \text{if } \ell < \ell(\eta) \quad \ell \notin t \\ \eta(\ell)+1 & \text{if } \ell < \ell(\eta) \quad \ell \in t \end{cases}$$

We shall want:

(iv) If $\eta \in J''_{\lambda,n}$, $\mathbf{S}^\eta$ is saturated and $\equiv \left\langle A_s^* : s \in \mathcal{P}(\ell(\eta))\right\rangle$.

Proof: Like 7.6, only simpler.

§8 The structure theory we can still get when $k < n^{**}(T)$

8.1 Claim: If $A \subseteq^t_\lambda C$, and B is $\mathbf{F}^t_\lambda$-constructible over A, then B is $\mathbf{F}^t_\lambda$-constructible over C (by the same construction) and $tp_*(B,A) \vdash tp(B,C)$.

Proof: See [Sh 1, Ch. XI].

Remark: 1) $A \subseteq^t_\lambda C$ if every m-type of power $< \lambda$ over A realized in C is realized in A.

2) The same holds for $\subseteq^s_\lambda$, but we ignore this distinction (important for $\lambda = |T|$).

3) Remember M is λ-compact if every m-type over M of power $<\lambda$, finitely satisfiable in M is realized in M.

8.2 Claim*: If M is $\mathbf{F}^t_\lambda$-constructible over $\bigcup_s A_s \subseteq M$, $\langle A_s : s \in \mathcal{P}^-(n)\rangle$ a λ-compact $\mathcal{P}^-(n)-a$ –system and $n < n^{**}(T)$ *then* M is u.l.a. over $\bigcup A_s$.

Proof: W.l.o.g. for some $\mu > \Sigma |A_s| + |T|$, $\mu = \mu^{<\mu}$ and let $\langle A_s' : s \in \mathcal{P}^-(n)\rangle$ be a μ-compact elementary extension of $\langle A_s : s \in \mathcal{P}^-(n)\rangle$ which has power μ. As $\langle A_s : s \in \mathcal{P}^-(n)\rangle$ is λ-compact clearly $\bigcup A_s \subseteq^t_\lambda \bigcup_s A_s'$ (in case of saturation instead compactness - even $\subseteq^s_\lambda$) so by 8.1 M is $\mathbf{F}^t_\lambda$-constructible over $\bigcup_s A_s'$, so $tp(M, \bigcup_s A_s) \vdash tp(M, \bigcup_s A_s')$ hence there is a μ-primary model N over $\bigcup_s A_s'$, $M \subseteq N$. We know (see 3.3) N is u.l.a. over $\bigcup_s A_s'$. So for every $\bar c \in M$ and φ there is a $\psi = \psi(x, \bar b) \in tp(\bar c, \bigcup A_s')$. $\psi \vdash tp_\varphi(\bar c, \bigcup_s A_s')$. But we know $tp(\bar c, \bigcup_s A_s) \vdash tp(\bar c, \bigcup_s A_s')$ hence for some $\vartheta \in tp(\bar c, \bigcup_s A_s)$ $\vartheta \vdash \psi$. So $\vartheta \in tp(\bar c, \bigcup A_s), \vartheta \vdash tp_\varphi(\bar c, \bigcup_s A_s^{(0)})$. We get M is l.a. over $\bigcup A_s$. But we want u.l.a.

This follows from 6.4.

8.3 Claim*: Let $\mathbf{S} = \langle A_s : s \in I\rangle$ be an $I-a$ –system and $\lambda > |T|$. $\mathbf{S}$ is λ-saturated *iff* $\langle A_s : s \in I \cap \mathcal{P}(\{1, \ldots, n(I)-1\})\rangle$ is λ-saturated and each $M_s (s \in I,\ 0 \in s)$ is λ-saturated.

Proof: $\Longrightarrow$ trivial.

$\Longleftarrow$: We prove it by induction on $|I|$. Let $p = p\ (x_0, \ldots, x_{m-1})$ be an m-type over $\mathbf{S}$. $|\mathrm{Dom}\ p| < \lambda$ and p is finitely satisfiable in $\mathbf{S}$. If $I = \mathcal{P}(\{1, \ldots, n(I)-1\})$ this is trivial. Otherwise choose $t \in I$, $0 \in t$, t maximal, and let $J = I - \{t\}$. W.l.o.g.

$$\{x_0 \in A_t - \bigcup_{s \subset t} A_s, \ldots, x_{k-1} \in A_t - \bigcup_{s \subset t} A_s,$$
$$x_k \notin A_t - \bigcup_{s \subset t} A_s, \ldots, x_{m-1} \in A_t - \bigcup_{s \subset t} A_s\} \subseteq p.$$

As A_t is u.l.a. over $\bigcup_{s \subset t} A_s$ (and 5.1) *there is* $\psi(\bar x, \bar y) \in L(T)$ such that for every

$a \in A_t - \bigcup_{s \subset t} A_s$, for some $\bar{b} \in \bigcup_{s \subset t} A_s$, $\models \psi[a,\bar{b}]$ and $\psi(x,\bar{b}) \vdash \{x \neq e : e \in \bigcup\{A_s : s \in J\}$ (this in $\mathfrak{C}$, so w.l.o.g. ψ is atomic, we shall not mention such things). So $p \cup \{\psi(x_0,\bar{y}_0)$, $\bar{y}_0 \subseteq \bigcup_{s \in J} A_s$, $(\forall z \in \bigcup_{s \in J} A_s)(\neg\psi(z,\bar{y}_0)\}$ is finitely satisfiable in **S**. So w.l.o.g. for $\ell < k$ $(\exists\bar{y})[\psi(x_\ell,\bar{y}) \wedge \bar{y} \subseteq \bigcup_{s \in J} A_s \wedge (\forall z \in \bigcup_{s \in J} A_s) \neg\psi(z,\bar{y})] \in p$. Now let $\bar{x}^0 = \langle x_0, \ldots, x_{k-1} \rangle$ $\bar{x}^1 = \langle x_k, \ldots, x_{m-1} \rangle$, and

$$p^1 = p \cup \{(\forall \bar{z} \subseteq \bigcup_{s \in I} A_s)[\varphi(\bar{x}^0,\bar{z}) \equiv \psi_\varphi(\bar{z},\bar{y}_\varphi)] \wedge \bar{y}_\varphi \subseteq \bigcup_{s \subset t} A_s : \varphi \in L\}.$$

Let p^2 be the closure of p^1 under conjunctions. Let $p^3 = \{(\exists\bar{x}^0)\vartheta : \vartheta \in p^2\}$. By the induction hypothesis p^3 is realized say by $\bar{x}^1 \to \bar{a}^1, \bar{y}_\varphi \to \bar{b}_\varphi$ for $\varphi \in L$ (you may argue that p^3 has $|T|$ variable not some $m' < \omega$, but λ-compactness implies this). Now we can find $\bar{a}^0$ realizing $\{\psi_\varphi(\bar{x}^0,\bar{b}_\varphi) : \varphi \in L\}$. Still we do not know that $\bar{a}^\wedge\bar{a}^1$ realizes p - it may contain formulas which are not atomic. But our conclusion follows from:

8.4 Claim: Let $\langle A_s : s \in I \rangle$ be an I–a–system, $0 \in t \in I$, t maximal. Let $\langle \psi_\varphi : \varphi \in L \rangle$ wittness the u.l.a. of A_t over $\bigcup_{s \subset t} A_s$ $\bar{d}^1, \bar{d}^2 \in \bigcup_{s \subset t} A_s$, $\bar{c}^1 \bar{c}^2 \in A_s - \bigcup_{s \subset t} A_s$, $\models \psi_\varphi(\bar{c}^\ell, b^\ell_\varphi)$, $\bar{b}^\ell_\varphi \in \bigcup_{s \subset A} A_s$, $(\cdots, \bar{b}^2_\varphi \cdots, \bar{d}^2) = tp(\cdots, b^1_\varphi \cdots, \bar{d}^1)$ *then* in $\langle A_s : s \in I \rangle$ the sequences $\bar{c}^1 \frown \bar{d}^1$, $\bar{c}^2 \frown \bar{d}^2$ realizes the same type.

Proof: Again as in the previous claim; then some automorphism of $\langle A_s' : \in J \rangle$ take $\bar{d}^1$ to $\bar{d}^2$ and b^1_φ to b^2_φ. Then there is an automorphism of N embedding it taking $\bar{c}^1$ to $\bar{c}^2$.

8.5 Claim: Suppose $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ is an I–a–system, λ-compact, and $\mu = \Sigma |A_s| > |T|$.

Then we can find $\langle A_s^\alpha : s \in \mathcal{P}^-(n) \rangle$ $(\alpha < \mu)$ such that

(1) $|A_s^\alpha| < \mu$, if $\mu = \chi^+$, $|A_s^\alpha| = \chi$,

(2) $\langle A_s^\alpha : s \in \mathcal{P}^-(n) \rangle \prec \langle A_s : s \in \mathcal{P}^-(n) \rangle$.

(3) If $J \subseteq \mathcal{P}^-(n)$, $\langle A_s : s \in J \rangle$ is λ-compact $(\forall \alpha < \mu)$ $[|\alpha|^{<\lambda} < \mu]$ then $\langle A_s^{\alpha+1} : s \in J \rangle$ is λ-compact (also for $\alpha = -1$).

(4) If A_s is λ-constructible over $\bigcup_{t \subset s} A_t$ then there is an $\mathbf{F}_\lambda^t$-construction $\langle a_i^s, B_i^s : i < \mu \rangle$ of A_s over $\bigcup_{t \subset s} A_t$ such that for each α for some $j(\alpha)$,

$$A_s^\alpha - \bigcup_{t \subset s} A_t^\alpha = \{a_i^s : i < j(\alpha)\}\ ,\ (\forall i < j(\alpha) B_i^s \subseteq \bigcup_{t \subset s} A_t^\alpha$$

Proof: Easy (for (4) see [Sh 1 Ch IV §3]

8.6 Claim *: A complete set A is stable iff it has the atomicity property provided.

Proof: W.l.o.g. A is saturated of power μ, $\mu = \mu^{<\mu} > |T|$. Now easily stability implies atomicity. So assume atomicity for A, so there is M, λ-primary over A . Let $(M', A') \equiv (M, A)$ be saturated of power μ, so w.l.o.g. $A = A'$ and $M \prec M'$. By the hypothesis 1.0 $M = M'$. Hence M' is atomic over A, so by the saturation M' is u.l.a. over A. Also for every $p \in S_*^m(A)$ there is a λ-saturated $M'' \supseteq A \supseteq P^{M''}$ realizing p, but as again w.l.o.g. $M = M''$, p is λ-isolated, hence $\mathbf{F}_{\aleph_0}^a$-isolated. From here atomicity is easy.

8.7 Lemma: Suppose $|T|\ \mu < \lambda = \lambda^{<\lambda}$, $2^{\lambda^{+\ell}} = \lambda^{+\ell+1}$ for $\ell < k+n$.

1)* In the definition of an $I-a$-system we can omit "A_s is u.l.a. over $\bigcup_{t \subset s} A_t$ for $s \in I, 0 \in s$" when $|s| < n^{**}(T)$.

2) $n^*(T) = n^{**}(T)$.

3) If for $\ell = 1,2$ $\langle A_s^\ell : s \in I \rangle$ is a $\mathcal{P}^-(n)-a$-system, $\langle A_{s+1}^\ell : s+1 \in I \rangle$ is saturated of power μ with first order theory not depending on ℓ, $n(I) < n^{**}(T)$ *then* $\langle A_s^\ell : s \in I \rangle \cong \langle A_s^2 : s \in I \rangle$.

4)* If $k + n < n^{**}(T)$, $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ and $\mathcal{P}^-(n)-a$-system, $|A_s| \leq \lambda^{+k}$, $A_{\{1,\ldots,n-1\}}$ is λ- saturated *then* over $\bigcup_s A_s$ there is a λ-primary model $M, P^M = A_{\{1,\ldots,n-1\}}$.

Proof: 1) By 3.3,

2) See 8.6.

3) We prove by induction on n (similar proof occurs previoulsy) we start with an isomorphism from $\langle A^1_{s+1} : s+1 \in I \rangle$ onto $\langle A^2_{s+1} : s+1 \in I \rangle$ and extend it step by step. For this we have to prove $A_t (0 \in t \in I)$ is μ-primary over $\bigcup_{s \subset t} A_s$, for this it suffices to prove it is u.l.a. over $\bigcup_{s \subset t} A_s$, which follows by 3.3 if we have proved 2).

4) We prove it by induction on k. For $k = 0$, $\bigcup_s A_s$ is stable, so there is a λ-primary model over it but

8.8 Claim: If A is complete, $A \cap P^{\mathfrak{C}}$ is λ- compact, $p \in S^m(A)$ is λ-isolated *then* $p \in S^m_*(A)$.

For $k+1$: Use 8.5 to get $A^\alpha_s (\alpha < \lambda^{+(k+1)}$. Now we define by induction on $\alpha, A^\alpha_{\mathcal{P}(n)}$ so that

(i) $A^0_{\mathcal{P}(n)}$ is λ-primary over $\bigcup \{A^0_s : s \in \mathcal{P}^-(n)\}$.

(ii) $A^{\alpha+1}_{\mathcal{P}(n)}$ is λ-primary over $\bigcup \{A^{\alpha+1}_s : s \in \mathcal{P}^-(n)\} \cup A^\alpha_{\mathcal{P}(n)}$.

(iii) $A^\delta_{\mathcal{P}(n)} = \bigcup_{\alpha<\delta} A^\alpha_{\mathcal{P}(n)}$.

(iv) $A^\alpha_{\mathcal{P}(n)}$ is u.l.a. over $\bigcup \{A^\alpha_s : s \in \mathcal{P}^- n)\}$ and is a model,

(v) $A^\alpha_{\mathcal{P}(n)} \cap P^{\mathfrak{C}} = A^\alpha_{\mathcal{P}(n-1)+1}$ (and $tp_*(A^\alpha_{\mathcal{P}(n)}, \bigcup \{A_s : s \in \mathcal{P}^-(n)\} \vdash tp_*(A^\alpha_{\mathcal{P}(n)}, \bigcup \{A_s : s \in \mathcal{P}^-(n)\})$. The induction step (for α) is by the induction hypothesis for k (as $|A^{\alpha+1}_s| \leq \lambda^{+k}$) and 7.7 for α successor, and remember 7.5(3).

§9 Non structure when $n^{**}(T) < \omega$ and there is no two cardinal model

9.0 Hypothesis : $P^N \subseteq M \prec N \Longrightarrow M = N$; every formula is equivalent to a relation (for T).

9.1 Main Theorem : Suppose $\lambda = \lambda^{<\lambda}$, $2^{\lambda^{+\ell}} = \lambda^{+\ell+1}$ for $\ell < n \overset{def}{=} n^{**}(T)$, Q

is the forcing of adding λ^{+n} Cohen subset to V say $\langle r_\eta : \eta \in J'_{\lambda,n} \rangle$. (see 7.7).

Then in V^Q there are $2^{(\lambda^{+n})}$ model M_i, $\|M_i\| = |P^{M_i}| = \lambda^{+n}$ which pairwise are not isomorphic over P^M; really we can make $\|M_i\| = |P^{M_i}| = \mu$, for any $\mu \geq \lambda^{+n}$.

Proof : Let $\langle A_s : s \in \mathcal{P}^-(n) \rangle$ be an $I-a$-system which is unstable. Working in V let $A_\eta(\eta \in J'_{\lambda,n})$ be as in 7.7 [A_η standing for M_η $\langle A^*_{s+1} : s \in \mathcal{P}^-(n-1) \rangle$ for $\langle A^*_s : s \in \mathcal{P}^-(n-1) \rangle$) and Th ($\mathfrak{C} \restriction P$) for T]. Define a well ordering $<^*$ on $J'_{\lambda,n}$: $\eta \leq^* \nu$ iff $\eta = \nu \restriction \ell(\eta)$ or $(\exists \ell)[\eta \restriction \ell = \nu \restriction \ell \wedge \eta(\ell) < \nu(\ell)]$. For $A \subseteq J'_{\lambda,n}, A \in V$, we now define for each n by induction on $<^*$ a model N^A_η such that

(i) $N^A_\eta \cap P^{\mathfrak{C}} = A_\eta$, $N^A_\eta \prec \mathfrak{C}$.

(ii) if $\eta \in J'_{\lambda,n}$ is not maximal then $[i < j \Longrightarrow N^A_{\eta \frown \langle i \rangle} \prec N^A_{\eta \frown \langle j \rangle}]$ for δ limit $N^A_{\eta \frown \langle \delta \rangle} = \bigcup_{i<\delta} N^A_{\eta \frown \langle i \rangle}$ and $N^A_\eta = \bigcup_i N^A_{\eta \frown \langle i \rangle}$:

(iii) if $s \subseteq t \subseteq \mathcal{P}(\ell(\eta))$ then $N_{\nu(\eta,s)} \subseteq N^A_{\nu(\eta,t)}$.

(iv) The construction of $\langle N_\eta : \eta <^* \nu \rangle$ is done in $V[\langle r_\eta : \eta <^* \nu, \eta \in A \rangle]$.

(where by renaming assume Q odd the sets $\langle r_\eta : \eta \in J'_{\lambda,n} \rangle$, r_η a function from λ to $\{0,1\}$.

There are no particular problems (especially if you have read §4).

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