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A THEOREM AND SOME CONSISTENCY RESULTS IN PARTITION CALCULUS

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0. Introduction

We prove the following results:

Theorem 1. *For all regular* $\kappa > \omega$, *if for all* $\lambda < \kappa$, $\lambda^{\aleph_0} < \kappa$, *then for all* $k < \omega$, $\kappa\omega \to (\kappa\omega, k)^2$.

Corollary 1. CH *implies that* $\omega_2\omega \to (\omega_2\omega, 3)^2$.

Theorem 2. *If* ZFC *is consistent, then so is* ZFC + GCH + $\omega_3\omega_1 \not\to (\omega_3\omega_1, 3)^2$.

Theorem 3. *If* ZFC *is consistent, then so is* ZFC + $\aleph_{c^+} \not\to (\aleph_{c^+}, \aleph_1)^2$.

Theorem 4. *If* ZFC + "*there exists a weakly compact cardinal*" *is consistent, then so is* ZFC + GCH + $(\forall k < \omega)\ \omega_3\omega_1 \to (\omega_3\omega_1, k)^2$.

Theorem 5. *If* ZFC *is consistent, then so is* ZFC + $\neg$CH + $\omega_2\omega \not\to (\omega_2\omega, 3)^2$.

A similar result, proved at the same time as the above, but which will appear in a sequel, [14], to this paper is:

Theorem 6. *If* ZFC + "*there exists* c^+ *measurable cardinals*" *is consistent, then so is* ZFC + $(\exists\lambda < \aleph_{c^+})\ 2^\lambda > \aleph_{c^+} + \aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$.

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Organization, notation, etc.

Theorems 1, 2, 3, are proved in Sections 1, 2, 3, respectively. In Section 4, we digress to give a general treatment of the technique of historicization, of which examples were given in Sections 2, 3. In Sections 5, 6, we prove Theorems 4, 5, respectively.

We assume the reader is thoroughly familiar with the basic facts about infinite cardinals, their cofinalities and their exponentiation, as well as with such standard notions and techniques of combinatorial set theory as filters and ideals, closed unbounded (club) and stationary sets, diagonal intersections, normality, Fodor's Lemma and Δ-systems, and is comfortable with one of the standard developments of forcing, viz. [6] or [7]. Our notation is intended to be a reasonably standard version of set-theoretic usage, or else to have a clear meaning, e.g. 'card(x)' (or 'card x') for the cardinality of x (we use 'power' and 'cardinality' and sometimes even 'size' interchangeably) and similarly for 'o.t.' and order type. The following litany should prevent us from being disappointed too often.

ZFC is the usual Zermelo–Fraenkel set theory with the Axiom of Choice. CH is the Continuum Hypothesis: $2^{\aleph_0} = \aleph_1$ (we also use $\mathfrak{c}$ to denote $2^{\aleph_0}$), and GCH is the Generalized Continuum Hypothesis: for all ordinals, α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. We use $x \setminus y$ to denote the set difference or the relative complement of y in x: $\{z \in x : z \notin y\}$. For cardinals κ, μ, $\mu^{<\kappa}$ denotes the cardinal sum of the μ^λ, for cardinals $\lambda < \kappa$. A normal sequence of elements of a complete, partially ordered set or class (usually the class of ordinals) is a monotone strictly increasing one with the property that for limit ordinals, $\delta > 0$, less than the length of the sequence, the δth term of the sequence is the least upper bound of the set of earlier terms.

When using partial orderings for forcing, we deviate from [6], [7], by using $p \leqslant q$ to mean that q gives more information. Were we worried about being consistent, we should, therefore, speak of generic *ideals* instead of generic filters and of *cofinal* sets instead of dense ones (but we don't, because, in this instance, we aren't!). A partial ordering, $\mathbb{P} = (P, \leqslant)$, has the κ-cc (we may, on occasion, say $\mathbb{P}$ is κ-cc) iff any antichain in $\mathbb{P}$ has power less than κ. As usual, we write ccc instead of $\aleph_1$-cc. If $\mathcal{F}$ is a filter in a Boolean algebra, $\mathcal{B}$, we use $\mathcal{F}^*$ for the ideal dual to $\mathcal{F}$, i.e., the set of complements of elements of $\mathcal{F}$. $\mathcal{F}^+$ denotes $\mathcal{B} \setminus \mathcal{F}^*$, and elements of $\mathcal{F}^+$ are called $\mathcal{F}$-positive. See [4] for background on the infinite games of perfect information, $G(\lambda, \mathcal{F}, \alpha)$ of Section 5. Generalizing from there, the second author coined the term 'κ-strategically-closed' to describe partial orderings $\mathbb{P}$ for which NONEMPTY has a winning strategy for the length κ-game where the players are required to generate an increasing sequence from $\mathbb{P}$, NONEMPTY playing at nonzero even stages, and losing if, at some stage she has no legal move. κ-strategic closure has many of the appealing properties of the more usual κ-closure, including not adding any new sequences of length $<\kappa$, when forcing, a form of the κ-Baire property.

For weakly compact cardinals, see [6]; the property we use in Section 5 can easily be obtained from the characterization of weakly compact cardinals as those uncountable cardinals, λ, for which the Compactness Theorem holds for $\mathcal{L}_{\lambda,\lambda}$. Here, $\mathcal{L}_{\lambda,\lambda}$ is the infinitary logic which extends first-order logic by allowing λ variables, and allowing as formation rules negation, conjunction and disjunction over sets of $<\lambda$ formulae and universal and existential quantification over blocks of fewer than λ variables, see [6, Section 32]. $\mathfrak{A} \prec_{\lambda,\lambda} \mathfrak{B}$ means that $\mathfrak{A}$ is an $\mathcal{L}_{\lambda,\lambda}$-elementary substructure of $\mathfrak{B}$, i.e., a substructure such that the truth of $\mathcal{L}_{\lambda,\lambda}$ statements with parameters from $\mathfrak{A}$ is preserved in passing from $\mathfrak{A}$ to $\mathfrak{B}$. The usual notion of Skolem function extends to this setting; a set of $\mathcal{L}_{\lambda,\lambda}$-Skolem functions for $\mathfrak{A}$ provides witnesses (in the form of elements of $[|\mathfrak{A}|]^{<\lambda}$, see below for this notation) for the truth in $\mathfrak{A}$ of existential $\mathcal{L}_{\lambda,\lambda}$ statements. The relevant fact is that closing a subset under a set of $\mathcal{L}_{\lambda,\lambda}$-Skolem functions yields an $\mathcal{L}_{\lambda,\lambda}$-elementary substructure of $\mathfrak{A}$. We use $\models$ for the satisfaction relation, i.e., $\mathfrak{A} \models \varphi$ indicates that the statement φ (of whatever language is appropriate) with parameters from $\mathfrak{A}$ is true in the model $\mathfrak{A}$.

We mention, but do not use, the following notions; with each, we supply a reference for it. L is Gödel's universe of constructible sets [6], [7]. $\square$ is a combinatorial principle formulated by Jensen [6], [7]. For morasses with or without built-in $\diamondsuit$, see [12], [13], [16], [17], [18]; for measurable cardinals and precipitous ideals, see [6].

For the reader who is not fluent in the partition calculus dialect of Hungarian, we define here the instances of the arrow (and related) notation which we shall need; a complete treatment of this material appears in [3], which contains a very complete bibliography of the earlier sources. For sets, X, and cardinals, κ, $[X]^\kappa$ is the set of subsets of X which have power κ; another notation for this is $\binom{X}{\kappa}$. Elements of $[X]^\kappa$ are sometimes called κ-sets, or κ-subsets of X. $[X]^{<\kappa}$ denotes the set of subsets of X of power less than κ; another notation for this is $\mathcal{P}_\kappa(X)$. $[X]^{\leq\kappa}$ is, of course $[X]^{<\kappa} \cup [X]^\kappa$. When ξ is an ordinal which is not a cardinal, we use $[X]^\xi$, $[X]^{<\xi}$, $[X]^{\leq\xi}$ in totally analogous fashion, but with 'power' replaced by 'order type'. This apparently leaves us without notation for, e.g., the set of all subsets of ω_1 which have *order type* ω, a proper subset of $[\omega_1]^{\aleph_0}$. While, for our present purposes, this is no loss, the potential ambiguity can be lessened by using $\aleph$'s as superscripts to emphasize the 'cardinal character' and ω's to emphasize the 'ordinal character', or resolved, by deciding that when both meanings are possible, the cardinality notion is intended, unless explicitly stated otherwise.

$\kappa \to (\lambda, \mu)^2$ is the assertion: whenever $F:[\kappa]^2 \to \{0, 1\}$, either there is $X \in [\kappa]^\lambda$ with $[X]^2 \subseteq F^{-1}[\{0\}]$, or there is $Y \in [\kappa]^\mu$ with $[Y]^2 \subseteq F^{-1}[\{1\}]$. X, (resp. Y) is called homogeneous for value 0 (resp. 1). We sometimes use graph-theoretic terminology in this context. We think of $[\kappa]^2$ as the complete (undirected) graph on κ vertices, or nodes; we think of 0 and 1 as 'colors' (in this paper, usually red and green respectively), and we think of F as an edge-coloring. In this context, a 3-set, e.g., is called a triangle; we could call homogeneous sets 'monochromatic', but we won't. On two occasions, we shall need to consider colorings by more than two colors. In this case, we display as many cardinals on the right of the $\rightarrow$ as there are colors; the conclusion is the disjunction, over the set of colors, of the assertions that there is a homogeneous set with power = the ith cardinal for the ith color. Clearly, this notation can be extended to ordinals in place of some (or all) of the cardinals with the obvious meaning. Statements written in this 'arrow notation' are called (ordinary) partition relations (with superscript 2), since the colorings, F, can be replaced in an obvious way by partitions. The negation of a partition relation is indicated by striking out the arrow thus: $\not\rightarrow$, and is called a negative (ordinary partition) relation (with superscript 2).

In this paper, other than those mentioned in the statements of the results, above, three partition relations will be especially important. Ramsey's theorem for superscript 2 and k colors, the great grandaddy of them all, can be stated: $\omega \rightarrow (\omega, \ldots, \omega)^2$ (k copies of ω). The Erdös–Rado theorem for superscript 2 and $\aleph_0$ reads: $\mathfrak{c}^+ \rightarrow (\mathfrak{c}^+, \aleph_1)^2$. Finally, the Erdös–Dushnik–Miller theorem for the uncountable cardinal κ reads: $\kappa \rightarrow (\kappa, \omega)^2$. We refer the reader to [3] for proofs of these theorems (and a host of others) and additional information on these notions. Combining Ramsey's theorem for k colors with the E–D–M theorem we get: $\kappa \rightarrow (\kappa, \omega, \ldots, \omega)^2$ by replacing our initial coloring (by $k+1$ colors) with a coloring by 2 'new' colors: the first color (of the original $k+1$ colors), and otherwise. A homogeneous ω-set for the second new color then has its edges colored by the remaining k original colors, and Ramsey's theorem yields the desired conclusion.

Discussion

In addition to Theorem 6, we plan to include two types of material in [14]. The first type is identical in status to Theorem 6: further results, or refinements or extensions of results or methods of this paper, *which we have already proved.* The second type deals with statements which are currently conjectures, but which, if settled, are to be dealt with in [14]. We indicate the second type by citing the statements involved by $[14]^{\mathrm{C}}$. While we shall not engage in assigning 'probabilities', we shall indicate a lower degree of confidence in a stated conjecture by adding additional superscript C's.

The proof of Theorem 1 will be discussed below under the heading of Historical Remarks. We shall limit ourselves, here, to noting that it is elementary! Theorems 2–6 are proved by forcing. The proofs actually prove more general statements which are discussed in the appropriate sections of the paper and are given in the general setting. In Theorems 2, 4 the questions are interesting primarily when $2^{\aleph_1} = \aleph_2$; for cardinals larger than $\aleph_1$, various patterns of cardinal exponentiation are possible for essentially trivial reasons, provided we start from a ground model where $2^{\aleph_1} = \aleph_2$. When an analogue of Theorem 2 is desired with $2^{\aleph_1} > \aleph_2$, Theorem 5 generalizes by increasing all (infinite) cardinals by one, provided we have CH in the ground model. We have proved an analogue of Theorem 4 with $2^{\aleph_1} > \aleph_2$; the method of proof actually generalizes and makes explicit ideas which are implicit in the proof we give for Theorem 4, thereby also improving this proof. This, as well as a similar situation for Theorem 6, will be discussed in more detail, below and will appear in [14].

In Theorem 3, we have considerable but not total freedom in producing a desired pattern of cardinal exponentiation consistent with the usual limitative results (one of which is that the negative relation implies that for some $\lambda < \aleph_{c^+}$, $2^\lambda > \aleph_{c^+}$). In particular, we obtain models of CH and others where CH fails, models where for all $\lambda < \aleph_{c^+}$, $\lambda^{\aleph_1} < \aleph_{c^+}$, and others where this fails, the two statements occurring independently. See also the portion of the Historical Remarks devoted to discussion of the relation $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$, and the discussion at the beginning of Section 3. It remains possible that refinements of our techniques will give the maximum possible freedom in this respect. This will appear in $[14]^{CC}$. A similar situation obtains as regards Theorem 6. Here, a refinement of our original proof *will* give the maximum possible freedom in arranging desired patterns of cardinal exponentiation. This will appear in [14]. We conjecture $[14]^{C}$ that $\omega_3\omega_1 \not\to (\omega_3\omega_1, 3)^2$ is a consequence of the existence of an $(\omega_2, 1)$-morass with built-in $\diamondsuit$ and, in particular that it (the negative relation) holds in L.

The partial orderings for Theorems 2, 3 involve the technique of *historicization* of a set of 'naive' conditions, which also appeared in [1], though not in such explicit fashion (we should note, in this connection that the function which this type of forcing was used to adjoin, there, has since been remarked by Velickovic to be a consequence of $\square$, using Todorcevic's 'walks' down the $\square$-sequence, [15]). This technique, introduced by the first author, seems destined to have wide applicability. The 'ideology' of this technique is that 'catastrophes' which will occur in 'naive' conditions will not occur in 'not so naive' conditions: ones which are generated from trivial conditions according to a prescribed recipe (these are called accessible conditions in Sections 2, 3). Increasing sequences of naive conditions, of successor length, arising from following the recipe are 'historical' conditions, as they contain the 'history' of the creation of the last term. The relevant items are 2.15 and 3.7. Other questions connected with Theorem 2 which we shall explore in $[14]^{C}$ include the question of generalizing Theorem 2 to other pairs of cardinals. The reader is referred to the discussion in 2.17.

On a formal level the (positive) relation of Theorem 4 can be 'motivated' by regarding it as a 'lifting' to $\omega_3\omega_1$ of a weak version of the Erdös–Dushnik–Miller Theorem for ω_1: $\omega_1 \to (\omega_1, k)^2$, for all $k < \omega$. In fact, our proof of Theorem 4 proceeds in this fashion, but a certain degree of 'indiscernibility', embodied by a filter existence property and provided by the ghost of the weakly compact cardinal, is required to define something akin to a 'derived coloring of $[\omega_1]^2$', whose homogeneous set, of either color, can be 'lifted'. A slightly different view of this is that given a coloring with no $\omega_3\omega_1$ red set, this can be 'canonized down' to a coloring of the set of 'columns' without a ω_1 red set, to which the weak E–D–M theorem can be applied, yielding a homogeneous green k-set for the derived coloring. The canonization is sufficiently strong to ensure that this gives a homogeneous green k-set for the initial coloring.

The proof of Theorem 4, below, proceeds in three stages. We first show that $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$ is a consequence of a stronger filter existence property (which is a stronger version of precipitousness of the dual ideal and which therefore implies that ω_3 is measurable in an inner model). We then show how the (strong) hypothesis can be cut down to a weaker one, which, we finally show, holds in the model where a weakly compact cardinal is made into ω_3, via the Lévy collapse, preserving ω_i, $i<3$.

In fact, the (weaker) filter existence property is not just an 'artifact' of weak compactness, but also a consequence. The improved version of this proof, alluded to above, proceeds by showing that the (weaker) filter existence property is preserved in certain kinds of generic extensions. In this setting, the third stage of the proof in this paper is best understood as proving that the Lévy collapse of a weakly compact cardinal to become $\aleph_3$ is the right kind of partial ordering. The extensions of Theorem 4, mentioned above, which will appear in [14], take this one step further by showing that the partial orderings for blowing up the power set of ω_1 to desired and possible values are also of the right type.

We intend to investigate the questions of lower bounds for the consistency strength of (GCH + $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$) and of the filter existence property [14]$^{\mathrm{C}}$. Regarding the latter, the obvious conjecture is that the filter existence property is equiconsistent to the existence of a weakly compact cardinal. Regarding the former, a possible approach is to modify the forcing used for Theorem 2 so that, in the envisioned morass application, built-in $\diamondsuit$ is not longer required, in which case by methods of [12], consistency strength of an inaccessible cardinal would follow.

Similarly, on a formal level, the positive relation of Theorem 6 can be viewed as a lifting to $\aleph_{\mathfrak{c}^+}$ of the instance of the Erdös–Rado Theorem: $\mathfrak{c}^+ \to (\mathfrak{c}^+, \aleph_1)^2$ (though the more accurate analogy would perhaps be the silly $\mathfrak{c}^+ \to (\mathfrak{c}^+, 1)^2$); however see also the discussion in the Historical Remarks, below, and sections 35 and 37 of [3] for additional (and more serious) motivation. Here too, this can also be looked at in terms of canonization.

Once again, our proof of Theorem 6 proceeds in this fashion, modulo some indiscernibility provided by the ghosts of the large cardinals and embodied in the existence of a system of filters; the existence of one such filter is the stronger filter existence property alluded to in the discussion of the proof of Theorem 4. Here, of course, the existence of the system of filters is known to be equiconsistent with the existence of $\mathfrak{c}^+$ measurable cardinals; we intend to investigate ([14]$^{\mathrm{CCC}}$, though we commit ourselves to no particular conjecture) the question of the consistency strength of $\aleph_{\mathfrak{c}^+} \to (\aleph_{\mathfrak{c}^+}, \aleph_1)^2$. Theorem 5 is the 'lightweight' of the paper: a truely obvious attempt, which succeeds.

Historical remarks

The partition relation $\omega_2\omega \to (\omega_2\omega, 3)^2$ is explicitly mentioned in [2] and was probably considered even earlier by Erdös and Hajnal; $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$ is an obvious generalization, one cardinal up. The history of $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$ is somewhat more recent. While the histories are somewhat different, one point of similarity is that while both (the first two relations on the one hand and the third on the other hand) had attracted considerable attention from combinatorial set theorists they had proved untractable; both remained completely open until Fall of 1985 when the results in this paper were obtained. At that time, they represented some of the simplest and most important instances of open problems from [2] (respectively [3]) which involved cardinals larger than ω_1.

As far as we have been able to ascertain, interest in $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$ developed while the final version of [3] was being written. What follows will be much clearer if it is borne in mind that during this period, the first author obtained, [9], a bound, in ZFC, on $2^{\aleph_0}$. Part of the proof involved proving that $(\aleph_\omega)^{\aleph_0} < \aleph_{c^+}$, from which it follows readily that for all $\lambda < \aleph_{c^+}$, $\lambda^{\aleph_0} < \aleph_{c^+}$. While this work is presented in Section 47 of [3], Sections 33, 37 were not modified to reflect this result. Thus the last inequality appears in these sections as an additional hypothesis. This was reasonable, before this theorem was known, since it was remarked that in the absence of this 'hypothesis', a Sierpinski partition (see Definition 19.5 and Section 37 of [3]) could be constructed to be a counterexample to the positive relation.

The question of $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$ appears as Problem 35.5 (under CH and the additional 'hypothesis' mentioned above), which the authors of [3] considered to be 'the most important' open ordinary partition relation for cardinals with superscript 2, and which they conjectured to hold; see p. 215 and ff. of [3] for additional discussion of the importance attributed to this problem. It is pointed out in Section 35, that, by Theorem 35.4 of [3], the positive relation holds unless $2^\lambda > \aleph_{c^+}$, for some $\lambda < \aleph_{c^+}$. The problem is discussed at greater length in Section 37 of [3], where it is shown that the negative relation cannot be established by a Sierpinski partition, again, under the additional 'hypothesis' on cardinal exponentiation. Thus, a proof of the positive relation, if forthcoming, could have been looked upon as a strengthening of the work of [9].

Concerning $\omega_2\omega \to (\omega_2\omega, 3)^2$ and $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$, to our knowledge, it was never envisioned that these two relations might go different ways. In this respect, it should be noted that Theorems 1 and 2 can be viewed as yet another illustration of how different ω and ω_1 are. The initial conjecture was that the (relevant instances of the) GCH should imply the positive relations. Perhaps this conviction that CH implies $\omega_2\omega \to (\omega_2\omega, 3)^2$ began to break down when, in the late 1960's, Hajnal [5] proved that under CH, $\omega_1\omega \not\to (\omega_1\omega, 3)^2$. Here, comparison of this result and Theorem 1 provides an instance of how different $\aleph_1$ and $\aleph_2$ are.

For whatever reason, a different conventional wisdom began to take hold, namely that, e.g., it should be possible to force over models of CH, using countably closed, $\aleph_2$-cc partial orderings, to obtain models where $\omega_2\omega \not\to (\omega_2\omega, 3)^2$. This was the state of affairs in 1981, when our work on this complex of problems began; the problem was brought to the attention of the second author as a candidate to be settled by the sort of forcing falling under the scope of our 'black-box' principles for morasses, [12], [13] or [16], [17], [18]. It was hoped, in this way, to show that $\omega_2\omega \not\to (\omega_2\omega, 3)^2$ follows from the existence of appropriate morasses, and in particular should hold in L. For reasons now explained by Theorem 1, all our efforts in this direction failed, several false claims of proofs notwithstanding. It briefly occurred to the second author, after the first false proof, that a direct construction from a suitable $(\omega_1, 1)$ morass might succeed, avoiding certain 'catastrophes' which provably occurred in a set of 'naive' conditions. While this idea was never pursued in this form, it is essentially the same idea which later succeeded *one cardinal up* to give Theorem 2, see above and below.

Our modus operandi was (with only slight exaggeration) to introduce more and more complicated side conditions on the partial ordering, the typical sequence of events being that by introducing side conditions 17–53, we were able to show that side conditions 1–16 were preserved under certain crucial operations, but that preserving side condition 36 seemed to require about 7 highly technical new side conditions and . . . and the whole process seemed to be diverging hopelessly, when in Fall of 1985, we decided to give it one last try, incorporating "several promising new ideas developed over the summer", and either settle the question or abandon it.

Immediately thereafter, the first author found the proof strategy for Theorem 1, which, in a very real sense, is the result of a dispassionate analysis of the obstacles we had encountered. It is not wildly inaccurate to paraphrase the basic idea (for $k = 3$) as: if there is no $\omega_2\omega$ red set and no green triangle, then certain patterns of greening which stop short of a green triangle are also ruled out lest they and the proscription of green triangles give rise to an $\omega_2\omega$ red set. It was precisely such patterns that the earliest side conditions had been introduced to forbid (these are one form of the 'catastrophes', alluded to above and below). But then, by analyzing and uniformizing their absence, canonizing, and thinning out the vertex set, a subcoloring is obtained where green edges are so sparse that there must be an $\omega_2\omega$ red set. It is striking that this purely combinatorial proof (which could have been found in the mid-1950's) gives the original conjecture after years during which the original conjecture was 'under a cloud'!

This breakthrough made, and the proof being highly specific to ω as the lower cardinal, hope remained for something like Theorem 2. What's more, we now knew that any side condition approach which seemed to generalize *down* to $\omega_2\omega$ was doomed to failure, so we began looking in more profitable directions, when the first author suggested (or rather resuggested, but the first time the approach seemed 'unduly complicated') the technique of historicization which had made its successful début in [1]. The rest is, if not history, at least Theorem 2 (and, in short order, Theorem 3, see below). It was noticed by the second author that the historical conditions are very similar to the initial segments of the 'sufficiently generic set' constructed to give our 'black-box' principles as consequences of morasses. This connection was made much firmer by a private communication from Baumgartner, according to whom the forcing of [1] in fact adds a simplified $(\omega_1, 1)$-morass!

Shortly thereafter, the first author realized the possibility of applying this kind of forcing to obtain Theorem 3. Theorems 4, 6 arose as natural attempts, using ideas of the first author which had appeared, *implicitly* in [10], [11]. In [10], among other things, it is shown (Theorem 3 of [10]) that if ZFC + "there exist infinitely many measurable cardinals" is consistent, then so is

$$\mathrm{ZFC} + (\aleph_2, \aleph_4, \dots, \aleph_{2n}, \dots) \to (\aleph_2, \aleph_4, \dots, \aleph_{2n}, \dots)_{\{((1)_m)_{\aleph_0} : m < \omega\}},$$

i.e., that if $(f_n : n < \omega)$ are such that $f_n : [\aleph_\omega]^{<\omega} \to \aleph_{2n+1}$, then for all n, there is $S_n \in [\aleph_{2n+2}]^{\aleph_{2n+2}}$ such that for all $0 < k < \omega$, all $n < \omega$, all $(n(l) : l < k)$ with $n \leq n(0) \leq n(1)$ and $(n(l) : 0 < l < k)$ strictly increasing, and $a_l, b_l \in S_{n(l)}$ $(l < k)$, $f_n\{a_0, \dots, a_{k-1}\} = f_n\{b_0, \dots, b_{k-1}\}$. Here not even the filter existence properties, let alone the question of their preservation, were formulated. Rather, it was shown directly that a similar combinatorial property holds for a sequence of ω measurable cardinals and that enough of this property is preserved when the nth measurable is Lévy collapsed to become $\aleph_{2n+2}$. After the fact, this proof can be 'split' in a fashion similar to the proof we shall give for Theorem 6 in [14].

1. Proof of Theorem 1

In this section, we prove Theorem 1. The simplest case, and chronologically the first one proved, is $\kappa = \omega_2$, $k = 3$ under CH (the general result was proved shortly thereafter). Of course if CH holds and $\kappa > \omega_1$ is regular and not the successor of an ω-cofinal cardinal, then this hypothesis holds so the paradigm is really a special case.

While the following is supposed to be directly readable, the reader in search of motivation is advised, on a first reading, to ignore the Propositions P_k of 1.2 and the induction on k and to extract a direct proof for $k = 2$ (which corresponds to getting a green triangle when there is no $\kappa\omega$ red set) from the ideas of 1.4–1.6 and the first part of 1.7. 1.1 performs some preliminary reductions. 1.2 introduces the Propositions P_k and points out some apparent strengthenings which are actually consequences. 1.3 gives a direct (and trivial) proof of P_1. 1.4–1.7 give the induction step for getting P_k from P_{k-1}. The basic approach is that if P_k fails, since there is to be no $\kappa\omega$ red set, certain configurations of greening cannot occur. By uniformizing their failure to occur, canonizing and thinning out, we eventually produce a subcoloring where green edges are so sparse as to violate P_{k-1}, which is a strong, uniform version of the statement that there is a homogeneous green k-set.

1.1. We regard $\kappa\omega$ as $\kappa\times\omega$, and for $\xi<\kappa\omega$, we define $\rho(\xi)$, $c(\xi)<\kappa$, ω, respectively, as the unique ρ, c such that $\xi=(\kappa c)+\rho$. We identify ξ with $(\rho(\xi), c(\xi))$. $A\subseteq\kappa\omega$ is called 'big' if o.t. $A=\kappa\omega$, i.e., under the above identification, there is $C\in[\omega]^{\aleph_0}$ and for each $n\in C$, $C_n\in[\kappa]^\kappa$, such that $\bigcup\{C_n\times\{n\}:n\in C\}\subseteq A$. If $X\subseteq\kappa\omega$, we let $\langle X\rangle^2=X^2\backslash$the diagonal, and we let

$$(X)^2=\{(\xi,\zeta)\in X^2:c(\xi)<c(\xi)\text{ and }\rho(\xi)\neq\rho(\zeta)\}.$$

Suppose $\kappa>\omega_1$ is regular and for all $\lambda<\kappa$, $\lambda^{\aleph_0}<\kappa$. We shall show that if $F:[\kappa\omega]^2\to\{\text{red, green}\}$ and there is no big $A\subseteq\kappa\omega$ which is homogeneous red for F, then, in a strong fashion (see the propositions, $P_k(F)$, introduced in 1.2, below), for all $k<\omega$, there is $A\in[\kappa\omega]^\kappa$ which is homogeneous green for F. In the remainder of this subsection, we treat F as fixed but arbitrary, while in the following subsections, F is sometimes a variable and sometimes fixed. Note that by Ramsey's theorem we may assume that

(*) for all $\alpha<\kappa$ and all $m<n<\omega$, $F\{(\alpha,m),(\alpha,n)\}=\text{red}$.

This is because if there were an α for which there were no infinite $C\subseteq\omega$ such that for $m<n$, $m,n\in C$, $F\{(\alpha,m),(\alpha,n)\}=\text{red}$, then there would be an infinite homogeneous green set for F, in the αth row. But then for all α there is $C_\alpha\in[\omega]^{\aleph_0}$ such that $\{\alpha\}\times C_\alpha$ is homogeneous red. But since $\kappa>\omega_1$, by our hypotheses, there is C, and $R\in[\kappa]^\kappa$ such that for all $\alpha\in R$, $C_\alpha=C$. Then, restricting to C as set of columns and R as set of rows, we have a subcoloring of F with (*), so it will suffice to obtain the desired conclusion for colorings with (*).

Similarly, we may assume that

(**) for all $n<\omega$ and all $\alpha<\beta<\kappa$, $F\{(\alpha,n),(\beta,n)\}=\text{red}$.

This is by (a weak version of) the Erdös–Dushnik–Miller theorem, since if there were an n for which there were no $R\in[\kappa]^\kappa$ such that for $\alpha<\beta$, $\alpha,\beta\in R$, $F\{(\alpha,n),(\beta,n)\}=\text{red}$, for all $k<\omega$, there would be a homogeneous green k-set in the nth column. Thus, for all n there is $R_n\in[\kappa]^\kappa$, as above. Restricting to $\bigcup\{R_n\times\{n\}:n<\omega\}$ (and renaming the αth member of R_n as α), we have a subcoloring of F which satisfies (**), so, again, it will suffice to obtain the desired conclusion for colorings satisfying (**).

Thus, we may regard F as defined on $\{\{\xi,\zeta\}\in[\kappa\omega]^2:\rho(\xi)\neq\rho(\zeta)$ and $c(\xi)\neq c(\zeta)\}$, and by taking the nodes in increasing c-order, as defined on $(\kappa\omega)^2$. We write $F(\alpha,m,\beta,n)$ for $F((\alpha,m),(\beta,n))$.

1.2. We introduce a family of propositions, $P_k(F)$, for $F:(\kappa\omega)^2\to\{\text{red, green}\}$. We shall let P_k denote the Proposition $(\forall F)\,P_k(F)$ where the quantifier is understood to range over colorings of $(\kappa\omega)^2$ by red and green. $P_k(F)$ is a strong uniform version of the assertion that if F has no big homogeneous red set, then there is a homogeneous green $(k+1)$-set for F. It will be easier to show by induction on k that P_k holds than if we tried to work directly with the weaker statements whose conclusions are that F has a homogeneous green $(k+1)$-set.

Definition. $P_k(F)$ states: if F has no big homogeneous red set, then for all $t<\omega$, for all $g:[\kappa\times\omega]^k\to[\omega]^{\le t}$, if g has the property that for all $a\in[\kappa\times\omega]^k$, $g(a)\cap\{c(x):x\in a\}=\emptyset$, *then* there are $\alpha_1<\cdots<\alpha_{k+1}<\kappa$ and $\{n_1,\ldots,n_{k+1}\}\in[\omega]^{k+1}$, with $n_{k+1}\notin g\{(\alpha_1,n_1),\ldots,(\alpha_k,n_k)\}$ such that

$$\{(\alpha_1,n_1),\ldots,(\alpha_{k+1},n_{k+1})\}\text{ is homogeneous green.}$$

Remark. For $a\in[\kappa\omega]^k$ and $n\notin\{c(x):x\in a\}$, let $\hat g(a,n)\stackrel{\text{def}}{=}\{\alpha<\kappa:\ \alpha>\max\{\rho(x):x\in a\}$ and $a\cup\{(\alpha,n)\}$ is homogeneous green$\}$. P_k implies that for all $F:(\kappa\omega)^2\to\{\text{red},\text{green}\}$ and all g, as in the statement of $P_k(F)$, there is no uniform bound $s<\omega$, such that for all $a\in[\kappa\omega]^k$ there are at most s many $n\notin(\{n_1,\ldots,n_k\}\cup g(a))$, with $\hat g(a,n)\neq\emptyset$. Suppose to the contrary that s is a uniform bound. For $a\in[\kappa\omega]^k$, let $g^*(a)=g(a)\cup\{n:\hat g(a,n)\neq\emptyset\}$. Then, g^*: $[\kappa\times\omega]^k\to[\omega]^{\le(s+t)}$, but for which the conclusion of $P_k(F)$ fails for g^*.

In fact, something apparently much stronger is true: there is no uniform bound, $s<\omega$, such that for all $a\in[\kappa\omega]^k$ there are at most s many $n\notin(\{c(x):x\in a\}\cup g(a))$, such that $\hat g(a,n)$ has power κ. This can be seen by thinning out a counterexample to get a counterexample to the previous statement, where we only require that $\hat g(a,n)\neq\emptyset$. We define a club $D\subseteq\kappa$ such that if $\delta\in D$, $a\in[\delta\times\omega]^k$, $n\notin(\{c(x):x\in a\}\cup g(a))$ and $\hat g(a,n)$ has power $<\kappa$, then $\hat g(a,n)\subseteq\delta$. We then restrict to $D\times\omega$ and call the resulting coloring F^*. By construction, F^* is a counterexample to the previous statement.

1.3. We shall show by induction on $k>0$ that P_k holds. For $k=1$, if $F:(\kappa\omega)^2\to\{\text{red},\text{green}\}$, with no big homogeneous red set, $t<\omega$ and $g:\kappa\times\omega\to[\omega]^{\le t}$, we easily canonize, i.e., we find, for $n<\omega$, $\bar g(n)\in[\omega]^{\le t}$ and $X_n\in[\kappa]^\kappa$ such that for all $\alpha\in X_n$, $g(\alpha,n)=\bar g(n)$. But then, we easily find $C\in[\omega]^{\aleph_0}$ such that for $n\in C$, $n\notin\bar g(n)$. Now we must be able to find (α_1,n_1), $(\alpha_2,n_2)\in\bigcup\{X_n\times\{n\}:n\in C\}$, $n_1\neq n_2$, such that $\{(\alpha_1,n_1),(\alpha_2,n_2)\}$ is green (lest $\bigcup\{X_n\times\{n\}:n\in C\}$ be a big homogeneous red set), proving P_1.

1.4. So, let $k>1$, and suppose that P_{k-1} has been proven. We prove P_k. Let $F:(\kappa\omega)^2\to\{\text{red},\text{green}\}$, with no big homogeneous red set, let $t<\omega$ and let $g:[\kappa\times\omega]^k\to[\omega]^{\le t}$. Suppose, towards a contradiction, that g is a counterexample to $P_k(F)$. We shall show that $P_{k-1}(F')$ fails for a coloring, F', isomorphic to a subcoloring of F. We first attempt to build, by an explicit recursion on $l<\omega$, for each $\xi<\kappa$, a system $((\alpha_l,n_l,a_{l,m}):l<m<\omega)$ such that, for $l<m<\omega$

and $x \in a_{l,m}$:

(!) (i) $a_{l,m} \in [\xi \times \omega]^{k-1}$,
(ii) $\max\{c(x): x \in a_{l,m}\} < n_l < n_m$,
(iii) $\xi \leqslant \alpha_l < \alpha_m$,
(iv) $a_{l,m} \cup \{(\alpha_l, n_l)\}$, $a_{l,m} \cup \{(\alpha_m, n_m)\}$ are both homogeneous green,
(v) $n_l \notin g(a_{l,m} \cup \{(\alpha_l, n_l)\})$.

We add a superscript ξ to the above notation when we are talking about the ξth system. By cardinality and pressing down arguments, in 1.5 below, we shall be able to conclude that there is a club $\Gamma \subseteq \kappa$ such that letting $\Gamma' = \{\delta \in \Gamma : \mathrm{cf}\,\delta > \omega\}$, for all $\delta \in \Gamma'$, there is no such system for δ. Thus, for $\delta \in \Gamma'$, the construction must break down. In 1.6, by canonization, we shall be able to uniformize the stage at which the construction broke down as well as the 'lower part' of the construction which we were able to carry out. We should note that the techniques and arguments of 1.4, 1.5 are the only place where the proof depends on the fact that the lower cardinal is ω.

Our ultimate goal is to obtain a subcoloring in which green edges are so sparse that, when this subcoloring is isomorphed onto a coloring, F', of $(\kappa\omega)^2$, $P_{k-1}(F')$ will fail, violating the induction hypothesis. This will be accomplished in 1.7. The following will useful:

Definition. If $\xi < \kappa$ and $y \in \kappa \times \omega$ with $\xi \leqslant \rho(y)$, let

$$G_\xi^{k-1}(y) = \{a \in [\xi \times \omega]^{k-1} : a \cup \{y\} \text{ is homogeneous green}\}.$$

For $\xi < \kappa$, we shall define the following by recursion: $l(\xi) \leqslant \omega$, for *finite* $l \leqslant l(\xi)$, a big $A_l^\xi \subseteq [\xi, \kappa) \times \omega$ and a countable subset, $b_l^\xi \subseteq [\xi \times \omega]^{k-1}$, and for $l < l(\xi)$, $x_l^\xi = (\alpha_l^\xi, n_l^\xi) \in A_l^\xi$. The construction will break down at stage l if A_j^ξ is defined for $j \leqslant l$, x_j^ξ is defined for $j < l$, but x_l^ξ is undefined. When this occurs, $l(\xi) = l$. If this never occurs, $l(\xi) = \omega$. We shall have the following additional properties of the construction: if $l + 1 < l(\xi)$, $A_{l+1}^\xi \subseteq (\alpha_l^\xi, \kappa) \times (n_l^\xi, \omega) \cap A_l^\xi$, and if $y \in A_l^\xi$, then, for all $j < l$, there is $a_j \in (G_\xi^{k-1}(y) \cap G_\xi^{k-1}(x_j^\xi) \cap b_{j+1}^\xi)$, such that if $l > 0$, $n_{l-1}^\xi \notin g(a_{l-1} \cup \{y\})$.

The construction is now clear: $A_0^\xi = [\xi, \kappa) \times \omega$, $b_0^\xi = \emptyset$; having defined A_j^ξ, b_j^ξ, x_j^ξ, for $j \leqslant l$, we know that there is a big set A^* of $y \in A_l^\xi$ such that there is $a \in (G_\xi^{k-1}(y) \cap G_\xi^{k-1}(x_l^\xi))$ with $n_l^\xi \notin g(a \cup \{y\})$. So, for $y \in A^*$, choose $u(y) \in G_\xi^{k-1}(y) \cap G_\xi^{k-1}(x_l^\xi)$ such that $n_l^\xi \notin g(u(y) \cup \{y\})$. For $n < \omega$ such that $\{y \in A^* : c(y) = n\}$ has power κ, clearly, there is $u_n \in G_\xi^{k-1}(y) \cap G_\xi^{k-1}(x_l^\xi)$ such that for κ-many $y \in A^*$ with $c(y) = n$, $u(y) = u_n$. We let $b_{l+1}^\xi =$ the set of u_n, for n as above. Then, $y \in A_{l+1}^\xi$ iff $y \in A^*$, $c(y) > n_l^\xi$, $\rho(y) > \alpha_l^\xi$ and there is $u \in (G_\xi^{k-1}(y) \cap b_{l+1}^\xi)$ with $n_l^\xi \notin g(u \cup \{y\})$. Finally, having defined A_j^ξ, for $j \leqslant l$, and

x_l^ξ, for $j<l$, we consider two cases. The first is that for some $x \in A_l^\xi$

(!!) there is a big set of $y \in A_l^\xi$ such that there is
$$a \in (G_\xi^{k-1}(y) \cap G_\xi^{k-1}(x)),\ c(x) \notin g(a \cup \{y\}).$$

Then, we take x_l^ξ to be the lexicographically least such x (first by $\rho(x)$, then by $c(x)$). The second case is when there is no such x, in which case x_l^ξ is undefined, $l(\xi)=l$ and the construction stops.

1.5. Lemma. *There is a club $\Gamma \subseteq \kappa$, such that letting $\Gamma' = \{\delta \in \Gamma : \mathrm{cf}\,\delta > \omega\}$, for all $\delta \in \Gamma'$, $l(\delta) < \omega$.*

Proof. If not, for a stationary $S \subseteq \{\delta < \kappa : \mathrm{cf}\,\delta > \omega\}$, $l(\delta)=\omega$. For $\delta \in S$, and $l<m<\omega$, choose $a_{l,m}^\delta \in G_\delta^{k-1}(x_l^\delta) \cap G_\delta^{k-1}(x_m^\delta)$ (it is immaterial for our present purposes whether or not $n_l^\delta \in g(a_{l,m}^\delta \cup \{x_m^\delta\})$. Let $\rho(\delta, l, m) = \max\{\rho(z) : z \in a_{l,m}^\delta\}$ and let $\rho(\delta) = \sup\{\rho(\delta, l, m) : l<m<\omega\}$. Thus, ρ presses down on S, so w.l.o.g. ρ is constant on S, say with value ρ_0. Further, $\{(a_{l,m}^\delta : l<m<\omega) : \delta \in S\}$ has power $\leqslant (\mathrm{card}(\rho_0))^{\aleph_0}$, which, by our hypotheses on cardinal exponentiation, is $<\kappa$. Thus, w.l.o.g., there is $(a_{l,m} : l<m<\omega)$ such that for all $\delta \in S$,
$$(a_{l,m}^\delta : l<m<\omega) = (a_{l,m} : l<m<\omega).$$
Finally, since $2^{\aleph_0} < \kappa$, w.l.o.g., there is $(n_l : l<\omega)$ such that for $\delta \in S$, $(n_l : l<\omega) = (n_l^\delta : l<\omega)$. But then $\{(\alpha_l^\delta, n_l) : l<\omega,\ \delta \in S\}$ would be a big homogeneous red set, contradiction.

1.6. By the lemma of 1.5, w.l.o.g. there is $l<\omega$ such that for all $\delta \in \Gamma'$, $l(\delta)=l$. This, however, means that for all $x \in A_l^\delta$,
$$\mathrm{CONT}(x) \stackrel{\mathrm{def}}{=} \{y \in A_l^\delta : (\exists a \in (G_\delta^{k-1}(y) \cap G_\delta^{k-1}(x)))\, c(x) \notin g(a \cup \{y\})\}$$
is small, i.e., for each $x \in A_l^\delta$ there is 'a lower left hand corner for x and δ', i.e., a $(\gamma_\delta(x), m_\delta(x))$, such that if $\gamma_\delta(x) < \beta$, $m_\delta(x) < m$ and $(\beta, m) \in A_l^\delta$ then for all $a \in G_\delta^{k-1}((\beta, m)) \cap G_\delta^{k-1}(x)$, $c(x) \in g(a \cup \{(\beta, m)\})$. Write $m_\delta(\alpha, n)$ for $m_\delta(x)$, when $x=(\alpha, n)$. $m_\delta(x)$ is the right hand boundary for common greening with x below δ.

Arguing as in the proof of the lemma of 1.5, we may suppose that, letting $\delta_0 = \min \Gamma'$, there are $a_{j,m} \in [\delta_0 \times \omega]^{k-1}$, for $j<m<l$, and b_j, countable subsets of $[\delta_0 \times \omega]^{k-1}$, for $j \leqslant l$, such that for all $\delta \in \Gamma'$, $a_{j,m} \in G_\delta^{k-1}(x_j^\delta) \cap G_\delta^{k-1}(x_m^\delta)$, $n_j \notin g(a \cup \{x_m^\delta\})$ and for $j \leqslant l$, $b_j^\delta = b_j$. But then, for all $j \leqslant l$, and all $\delta_1 < \delta_2$, both from Γ', $A_j^{\delta_2} = A_j^{\delta_1} \setminus [\delta_1, \delta_2) \times \omega$. For $\delta \in \Gamma'$, let $A^\delta = A_l^\delta$ and let $A = A^{\delta_0}$.

Lemma (Canonization of the right hand boundary for greening)**.** *W.l.o.g., we may assume that there is a big $A^* \subseteq A$ and an $\bar{m} : \omega \to \omega$ such that if $\delta \in \Gamma'$ and $(\alpha, n) \in A^\delta \cap A^*$, then $m_\delta(\alpha, n) = \bar{m}(n)$.*

Proof. For $\delta \in \Gamma'$, choose $((\xi_j^\delta, i_j^\delta):j<\omega) \in {}^\omega(A^\delta)$ such that $m_\delta(\xi_j^\delta, i_j^\delta)$, $i_j^\delta < i_{j+1}^\delta$. Thus, w.l.o.g. we may assume that there are $(i_j:j<\omega)$ and $(\bar{m}(i_j):j<\omega)$ such that for all $\delta \in \Gamma'$, $(i_j^\delta:j<\omega) = (i_j:j<\omega)$ and $(m_\delta(\xi_j^\delta, i_j^\delta):j<\omega) = (\bar{m}(i_j):j<\omega)$. Then, letting $A^* = \{(\xi_j^\delta, i_j^\delta):j<\omega,\ \delta \in \Gamma'\}$, and extending $\bar{m}$ in an arbitrary fashion to a function on ω, the proof is complete.

1.7. By thinning out A^* further, we may assume that

(#) for $(\alpha, n), (\beta, m) \in A^*$, if $\alpha < \beta$ and $n < m$ then for all $a \in [A^*]^{k-1} \cap G_\alpha^{k-1}(\alpha, n) \cap G_\alpha^{k-1}(\beta, m)$, $n \in g(a \cup \{(\beta, m)\})$.

We accomplish this by generating a club $D \subseteq \kappa$ such that for $\nu \in D$ and $\delta \in \Gamma' \cap \nu$, for all $j<\omega$, $\gamma(\xi_j^\delta, i_j^\delta) < \nu$. We then let $(\delta(\eta):\eta<\kappa)$ be the increasing enumeration of $\Gamma' \cap D$, and we thin out A^*, setting it $=$ $\{(\xi_n^{\delta(\omega\eta+n)}, i_n^{\delta(\omega\eta+n)}):\eta<\kappa,\ n<\omega\}$.

Now, we renumber the A^* which we have just constructed so as to identify it with $\kappa \times \omega$, in the following way: we identify $(\xi_n^{\delta(\omega\eta+n)}, i_n^{\delta(\omega\eta+n)})$ with (η, n). Let F' denote the resulting coloring of $(\kappa\omega)^2$. Note that the following property is inherited from A^* and (#):

(##) for all $a \in [\kappa\omega]^{k-1}$, there are at most $t+1$ n's such that $\hat{g}(a, n)$ has power κ (recall the Remark of 1.2 for the definition of $\hat{g}$).

This is because if there were $t+2$ such n's, letting n^* be the largest of them and letting $\gamma^* \in \hat{g}(a, n^*)$ be such that for all of the $t+1$ smaller n's, $\gamma^* \cap \hat{g}(a, n) \neq \emptyset$, there is one of the $t+1$ remaining smaller n's which $\notin g(a \cup \{(\gamma^*, n^*)\})$. Choose $\gamma \in \gamma^* \cap \hat{g}(a, n)$. Then, a, (γ, n), (γ^*, n^*) together contradict (#), remembering how the renumbering was done. Finally, however, recalling the last paragraph of the Remark of 1.2, (##) contradicts P_{k-1}.

2. Proof of Theorem 2

In this section, assuming GCH (actually just $2^{\aleph_1} = \aleph_2$), we define a partial ordering $\mathbb{P}$ which adds no new $\aleph_1$-sequences, preserves cofinalities, cardinal exponentiation (if we assume only that $2^{\aleph_1} = \aleph_2$, this (and the ground model value of $2^{\aleph_0}$) is all we preserve) and which adds a generic counterexample to $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$. $\mathbb{P}$ will be the historicization of a set of 'naive' conditions $\mathbb{P}_{\text{basic}}$. A general version of this technique of historicization will be presented in Section 4. See there and the Introduction for a discussion of how the historicization replaces side conditions.

The crucial result is 2.15, which says that the situation which the side conditions were supposed to prevent simply cannot arise in a 'naive' condition which is a component of a 'historical' condition. In 2.17 we discuss the possibility of generalizing this result to products of other pairs of cardinals, $\mu\kappa$, where $\kappa^+ < \mu$. These questions will be treated in a sequel to this paper, [14][C].

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2.1. Definition. If $\xi<\omega_3\omega_1$, define $\rho(\xi)<\omega_3$, $\gamma(\xi)<\omega_1$ such that $\xi=\omega_3\gamma(\xi)+\rho(\xi)$; $\gamma(\xi)$ is for the column of ξ, $\rho(\xi)$ is for the row of ξ. Sometimes we identify ξ with $(\rho(\xi), \gamma(\xi))$, thereby identifying $\omega_3\omega_1$ with $\omega_3\times\omega_1$.

2.2. Definition. If $a\subseteq\omega_3$, let $\tilde{a}=\{\xi:\rho(\xi)\in a\}$. Thus, $\tilde{a}$ is identified with $a\times\omega_1$. If $d\subseteq\omega_3\omega_1$, let $(d)^2=\{(x,y)\in d^2:\gamma(x)<\gamma(y)\}$.

2.3. Definition. $p\in P_{\text{basic}}$ iff $p=(a,b,c)$ where $a\in[\omega_3]^{\leq\aleph_1}$, $b:(\tilde{a})^2\to\omega_1$, $c:(\tilde{a})^2\to\{\text{gray, green, red}\}$ and

(1) $c(x,y)=\text{green}\Rightarrow\rho(y)<\rho(x)$,

(2) $c(x,y)=\text{red}\Leftrightarrow(\exists z)\,(c(x,z)=c(y,z)=\text{green})$,

(3) if $(x,y)\in(\tilde{a})^2$, then $c(x,z)=c(y,z)=\text{green}\Rightarrow\rho(z)<b(x,y)$.

We write $p=(a^p,b^p,c^p)$. If $p,q\in P_{\text{basic}}$, then we set

$$p\leqslant_{\text{basic}}q \quad\Leftrightarrow\quad a^p\subseteq a^q,\ b^p\subseteq b^q \text{ and } c^p\subseteq c^q.$$

$\mathbb{P}_{\text{basic}}=(P_{\text{basic}},\leqslant_{\text{basic}})$. Finally, if $p\in P_{\text{basic}}$, we let $p\in P_{\text{atomic}}\Leftrightarrow(\exists\alpha)\,a^p=\{\alpha\}$; note that by (1), (2), this means that range $c=\{\text{gray}\}$.

2.4. Definition. If x, y are sets of ordinals, we write $x\ll y$ to mean that all elements of x are less than the least element of y. If x, y are sets of ordinals, we say that (x,y) is a strong amalgamation pair iff there is d, a common initial segment of x and y, such that either $x\setminus d\ll y\setminus d$ (in this case we write $(x,y)\in$ u-AMP) or $y\setminus d\ll x\setminus d$ (in this case we write $(x,y)\in$ d-AMP). When this occurs, we write $e(x)$, $e(y)$ for $x\setminus d$, $y\setminus d$, respectively ($e(x)$ for the end of x).

2.5. Definition. If $a\subseteq\omega_3$ and $\pi:a\to\omega_3$ is order preserving, we define $\tilde{\pi}:\tilde{a}\to\omega_3\omega_1$ by: $\tilde{\pi}(x)=(\gamma(x),\pi(\rho(x)))$. We abuse notation by regarding $\tilde{\pi}$ as defined on $(\tilde{a})^2$; we write $\tilde{\pi}(x,y)$ for $(\tilde{\pi}(x),\tilde{\pi}(y))$.

If $p=(a,b,c)\in P_{\text{basic}}$, $a'\subseteq\omega_3$, o.t. $a'=$ o.t. a, let $\pi:a\to a'$ be the order isomorphism. We write $\pi(p)$ for the unique $q\in P_{\text{basic}}$ which is isomorphic to p via π, i.e., $a^q=a'$, $b^q(\tilde{\pi}(x,y))=b(x,y)$, $c^q(\tilde{\pi}(x,y))=c(x,y)$. We write $\pi:p\to q$. If, further, $(a,a')\in$ u-AMP (respectively d-AMP), we write $\pi:p\to_{\text{u}}q$ (respectively $\to_{\text{d}}$).

2.6. Lemma. *If $p\in P_{\text{basic}}$, $a'\subseteq\omega_3$, o.t. $a'=$ o.t. a, (a,a') is an amalgamation pair, $\pi:a\to a'$ is the order isomorphism, $q=\pi(p)$, then:*

(a) *there is $r\in P_{\text{basic}}$ with $p,q\leqslant_{\text{basic}}r$, $a^r=a^p\cup a^q$, and for $(x,y)\in((e(a))^{\sim})^2$, $b^r(x,\tilde{\pi}(y))=b^r(\tilde{\pi}(x),y)=b^p(x,y)$ and $c^r(x,\tilde{\pi}(y))$, $c^r(\tilde{\pi}(x),y)\neq\text{green}$;*

(b) *further, if $c^p(x,y)=\text{gray}$, there is $r\in P_{\text{basic}}$ with $p,q\leqslant_{\text{basic}}r$, $a^r=a^p\cup a^q$, for $(x',y')\in((e(a))^{\sim})^2\setminus\{(x,y)\}$, $c^r(x',\tilde{\pi}(y'))$, $c^r(\tilde{\pi}(x'),y')\neq\text{green}$, such that if $\pi:p\to_{\text{u}}q$, $c^r(\tilde{\pi}(x),y)=\text{green}$, and if $\pi:p\to_{\text{d}}q$, $c^r(x,\tilde{\pi}(y))=\text{green}$.*

Proof. (a) is clear: $c^r(x, \tilde{\pi}(y)) = c^r(\tilde{\pi}(x), y) = \text{red}$ iff

(1) there is $z \in \tilde{d}$ such that $c^p(x, z) = c^q(y, z) = \text{green}$.

For (b), if $(x', y') \in ((e(a))^{\sim})^2 \setminus \{(x, y)\}$, we say that (x', y') is *critical* iff

$$\begin{array}{lllll} & (2) & \pi : p \to_{\mathrm{u}} q, & x' = x, & c^p(y', y) = \text{green}, \\ \text{or} & (3) & \pi : p \to_{\mathrm{u}} q, & y' = x, & c^p(x', y) = \text{green}, \\ \text{or} & (4) & \pi : p \to_{\mathrm{d}} q, & x' = x, & c^p(y', y) = \text{green}, \\ \text{or} & (5) & \pi : p \to_{\mathrm{d}} q, & y' = x, & c^p(x', y) = \text{green}. \end{array}$$

If (x', y') is not critical, and (1) fails (for (x', y')), then $c^r(\tilde{\pi}(x'), y') = c^r(x', \tilde{\pi}(y')) = \text{gray}$. If (1) holds (for (x', y')), then $c^r(\tilde{\pi}(x'), y') = c^r(x', \tilde{\pi}(y')) = \text{red}$. So, suppose (x', y') is critical and (1) fails (for (x', y')). In cases (2) and (5), set $\text{red}(x', y') = (\tilde{\pi}(x'), y')$, $\text{gray}(x', y') = (x', \tilde{\pi}(y'))$. In cases (3) and (4), set $\text{gray}(x', y') = (\tilde{\pi}(x'), y')$, $\text{red}(x', y') = (x', \tilde{\pi}(y'))$.

In all cases, set $c^r(\text{red}(x', y')) = \text{red}$, $c^r(\text{gray}(x', y')) = \text{gray}$, $b^r(\text{gray}(x', y')) = b^p(x', y')$, $b^r(\text{red}(x', y')) = \max(b^p(x', y'), \gamma(y) + 1)$. Finally, if (x', y') is not critical, set

$$b^r(\tilde{\pi}(x'), y') = b^r(x', \tilde{\pi}(y')) = b^p(x', y').$$

2.7. Remarks. In (a), r is called the no-green amalgamation of p, q; if $\pi : p \to_{\mathrm{u}} q$, we say r is the no-green, up amalgamation of p, q; if $\pi : p \to_{\mathrm{d}} q$, we say r is the no-green, down amalgamation of p, q; we write $r = \mathrm{Am}(p, a', \mathrm{u}, \emptyset)$, $r = \mathrm{Am}(p, a', \mathrm{d}, \emptyset)$, respectively. In (b), r is called the $\{(x, y)\}$ amalgamation of p, q; 'up' and 'down' are added as before and as appropriate; we write $r = \mathrm{Am}(p, a', \mathrm{u}, \{(x, y)\})$, $r = \mathrm{Am}(p, a', \mathrm{d}, \{(x, y)\})$, as appropriate. Also, if $\pi : p \to_{\mathrm{u}} q$, we write $\text{green}(x, y) = \text{green}(p, q) = (\tilde{\pi}(x), y)$; if $\pi : p \to_{\mathrm{d}} q$, we write $\text{green}(x, y) = \text{green}(p, q) = (x, \tilde{\pi}(y))$.

2.8. Lemma. *$\mathbb{P}_{\text{basic}}$ is ω_2-closed. In fact, the triple of unions of the coordinates of the terms of an increasing sequence of length $<\omega_2$ is the least upper bound of the sequence.*

Proof. Clear.

2.9. Definition. $p \in P \Leftrightarrow p = (p_\alpha : \alpha \leqslant \theta(p))$, where $\theta = \theta(p) < \omega_2$, $\alpha < \beta \leqslant \theta \Rightarrow p_\alpha \leqslant_{\text{basic}} p_\beta$, $p_0 \in P_{\text{atomic}}$ and:

(1) if $0 < \lambda = \bigcup \lambda \leqslant \theta$, then letting $a_\alpha = a^{p_\alpha}$, etc., $p_\lambda = (\bigcup \{a_\alpha : \alpha \leqslant \theta\}, \bigcup \{b_\alpha : \alpha \leqslant \theta\}, \bigcup \{c_\alpha : \alpha \leqslant \theta\})$;

(2) if $\alpha < \theta$, then there is (unique) $(a'_\alpha, \varepsilon_\alpha, o_\alpha)$ such that (a_α, a'_α) is an amalgamation pair and $p_{\alpha+1} = \mathrm{Am}(p_\alpha, a'_\alpha, \varepsilon_\alpha, o_\alpha)$. For $\alpha < \theta$, if $o_\alpha \neq \emptyset$, we let x_α, y_α be such that $o_\alpha = \{(x_\alpha, y_\alpha)\}$ and we let $\pi_\alpha : a_\alpha \to a'_\alpha$ be the order isomorphism.

2.10. Definition. If $p \in P$, $\alpha < \theta$, we define $p^{\langle\{\alpha\}\rangle} \in P$ as follows. We write q for $p^{\langle\{\alpha\}\rangle}$, and we set $q = (q_\beta : \beta \leqslant \theta)$, where, for $\alpha < \beta \leqslant \theta$, $q_\beta = p_\beta$, and for $\beta \leqslant \alpha$, $q_\beta = (\pi_\alpha \restriction a_\beta)(p_\beta)$.

Note that for $\beta < \alpha$, we have $\pi'_\beta : \pi''_\alpha a_\beta \to \pi''_\alpha a'_\beta$; in fact, $\pi'_\beta = \{(\pi_\alpha(x), \pi_\alpha(y)) : (x, y) \in \pi_\beta\}$, i.e., $\pi'_\beta = \pi_\alpha \circ \pi_\beta \circ (\pi_\alpha)^{-1}$, so, letting $\pi_\alpha(\pi_\beta) = \pi'_\beta$, we have $\pi_\alpha \circ \pi_\beta = \pi_\alpha(\pi_\beta) \circ \pi_\alpha$. Of course, the analogue of π'_β for $\beta = \alpha$ is just $(\pi_\alpha)^{-1}$.

If $h \in [\theta]^{<\omega}$ and $h \neq \emptyset$, let $(\alpha_i : i < \mathrm{card}(h))$ be the decreasing enumeration of h. Define p^i, for $i \leqslant \mathrm{card}(h)$, by recursion: $p^0 = p$; $p^{i+1} = (p^i)^{\langle\{\alpha_i\}\rangle}$. Let $p^{\langle h\rangle} = p^{\mathrm{card}(h)}$.

Remarks. For definiteness, we have defined $p^{\langle h\rangle}$ using the decreasing enumeration of h. It will be useful, however, to note that the order in which we took the α_i was, in fact, immaterial. So, let $p \in P$, $\beta < \alpha < \theta$. We show that $(p^{\langle\{\alpha\}\rangle})^{\langle\{\beta\}\rangle} = (p^{\langle\{\beta\}\rangle})^{\langle\{\alpha\}\rangle}$. For $q \in P$, with $\theta(q) > \alpha$, let $q^* = (q^*_\xi : \xi \leqslant \theta(q)) = q^{\langle\{\alpha\}\rangle}$ and let $q^\# = (q^\#_\xi : \xi < \theta(q)) = q^{\langle\{\beta\}\rangle}$. The key observation was made above, where we observed that $\pi_\alpha \circ \pi_\beta$ (which is the function associated with the order 'first β then α') $= \pi_\alpha(\pi_\beta) \circ \pi_\alpha$ (which is the function associated with the order 'first α then β'). This is what permits us to show that for $\xi \leqslant \beta$, $(p^\#_\xi)^* = (p^*_\xi)^\#$. The case of $\beta < \xi \leqslant \alpha$ is easier, and the case of $\alpha < \xi \leqslant \theta$ is trivial. If $h = \emptyset$, let $p^{\langle h\rangle} = p$.

2.11. Definition. If $p, q \in P$, let $q \leqslant p \Leftrightarrow \theta(q) \leqslant \theta(p)$ and for some finite (possibly empty) $h \subseteq \theta(p)$, $q \subseteq p^{\langle h\rangle}$. We let $q \leqslant_{\mathrm{pure}} p \Leftrightarrow$ we can take $h = \emptyset$, i.e., $q \subseteq p$. We let $\mathbb{P} = (P, \leqslant)$.

Remarks. It is not difficult to verify that $\leqslant$ is transitive, using the Remarks of 2.10. Further, and this will be crucial in the proof of $\aleph_2$-strategic-closure, if $q \leqslant p$, we can find r with $p \leqslant r$ and $q \leqslant_{\mathrm{pure}} r$, simply by choosing r with $p^{\langle h\rangle} \subseteq r$, where $q \subseteq p^{\langle h\rangle}$. Finally, note that $\mathbb{P}$ is really a partial pre-order, but we happily refuse to give in to the formalism of equivalence classes.

2.12. Definition. If $p \in P$, let $a = a(p) = a_\theta$, let $\lg p = \mathrm{o.t.}\ a$. We also let $b(p) = b_\theta$, $c(p) = c_\theta$. Let $s^p = (\xi^\theta_i : i < \lg p) = (\xi_i : i < \lg p)$ be the increasing enumeration of a. Note that there is a unique $q \in P$ with $a(q) = \lg p$, $\theta(q) = \theta(p)$ and whose coordinate basic conditions are coordinatewise isomorphic to the coordinates of p. We obtain this q by setting $q_\alpha = (s^p)^{-1}(p_\alpha)$ for $\alpha \leqslant \theta$. We denote this q by τ^p.

2.13. We now state (the proof is trivial: this is the payoff for the definition of $\leqslant$) the amalgamation property for $\mathbb{P}$.

Lemma. *If $p, q \in P$, $\tau^p = \tau^q$ and $(a(p), a(q))$ is an amalgamation pair, let $\theta = \theta(p) = \theta(q)$, let $\varepsilon = \mathrm{up}$ if $e(a(p)) \ll e(a(q))$; otherwise, let $\varepsilon = \mathrm{down}$. Let $o \in \{\emptyset\} \cup \{\{(x, y)\} : (x, y) \in (e(a(p)))^2$ and $c^{p_\theta}(x, y) = \mathrm{gray}\}$. Let $r = \mathrm{Am}(p_\theta, a(q), \varepsilon, o)$. Then $p, q \leqslant p^\wedge r, q^\wedge r$. In particular, p, q are compatible.*

Proof. From the construction of τ^p in 2.12 it is clear that $\theta(\tau^p) = \theta(p)$ and that $s^p(\tau^p_{\theta(p)}) = p_{\theta(p)}$. Therefore, if $\tau^p = \tau^q$, $\theta(p) = \theta(q)$, so letting $\theta =$ the common value, $s^q \circ (s^p)^{-1}(p_\theta) = q_\theta$, so the conclusion is by 2.6.

Corollary. $(2^{\aleph_1} = \aleph_2)$ $\mathbb{P}$ *has the* $\aleph_3$*-c.c.*

Proof. The usual argument using the amalgamation property given by the lemma, the Δ-system lemma (for $[\omega_3]^{\leqslant\aleph_1}$) and the fact that $\mathrm{card}(\{\tau^p : p \in P\}) = \aleph_2$; the latter two use that $2^{\aleph_1} = \aleph_2$.

2.14. Lemma. $\mathbb{P}$ *is* ω_2*-strategically-closed.*

Proof. NONEMPTY's strategy, at even successor stages, $\lambda + 2n + 2$, is to play a pure extension of $p^{\lambda+2n}$, which also extends $p^{\lambda+2n+1}$ (as remarked in 2.11 this is always possible). This enables her, at non-zero limit stages, λ, to take $p^\lambda = \bigcup\{p^\alpha : \alpha < \lambda,\ \alpha \text{ is even}\} \cup \{(\theta, (a, b, c))\}$, where, for $\alpha < \lambda$,

$$p^\alpha = ((a^\alpha_\xi, b^\alpha_\xi, c^\alpha_\xi) : \xi < \theta_\alpha), \qquad \theta = \bigcup\{\theta_\alpha : \alpha < \lambda\},$$
$$a = \bigcup\{a^\alpha_{\theta_\alpha} : \alpha < \lambda\}, \qquad b = \bigcup\{b^\alpha_{\theta_\alpha} : \alpha < \lambda\}, \qquad c = \bigcup\{c^\alpha_{\theta_\alpha} : \alpha < \lambda\}.$$

A somewhat more delicate argument shows that $\mathbb{P}$ is actually ω_1-closed (i.e., countably closed), but $\mathbb{P}$ is not ω_2-closed.

2.15. $\mathbb{P}$'s generic partition is $\mathring{C} =$ (the canonical term for)

$$\bigcup\{c^p : (\exists p \in \mathring{G})\ (\exists \alpha \leqslant \theta(p))\ p = p_\alpha\}.$$

We now show that the generic partition doesn't have a homogeneous red set which meets $\aleph_1$ columns.

Lemma. *There is no* $p \in P_{\text{basic}}$ *which is* p_α *for some* p *in* P (such p will be called *accessible*), *whose* c *has such a homogeneous red set* (there will be inaccessible p whose c do!).

Before proving the lemma, note that it suffices to do so, since then, suppose, towards a contradiction, that some p^0 forces that there is a homogeneous red set $\mathring{X}$ meeting $\aleph_1$ columns, say p^0 forces that $\mathring{x}_\xi \in \mathring{X}$ $(\xi < \omega_1)$ and $(\mathring{\gamma}_\xi : \xi < \omega_1) \in {}^{\omega_1}\omega_1$ is strictly increasing and for $\xi < \omega_1$, $\gamma(\mathring{x}_\xi) = \mathring{\gamma}_\xi$. Using the winning strategy, we generate sequences p^ξ, q^ξ, x_ξ, γ_ξ, such that $p^\xi \leqslant_{\text{pure}} p^\zeta$ for $\xi < \zeta < \omega_1$, $p^\xi \leqslant q^\xi \leqslant p^{\xi+1}$, $\gamma(x_\xi) = \gamma_\xi$, $(\gamma_\xi : \xi < \omega_1)$ is strictly increasing and for $\xi < \omega_1$, q^ξ forces that $\mathring{x}_\xi = x_\xi$ and $\mathring{\gamma}_\xi = \gamma_\xi$. But then, by $\aleph_2$-strategic-closure, there is p extending all the p^ξ, and in the accessible condition, $(a(p), b(p), c(p))$, $\{x_\xi : \xi < \omega_1\}$ is homogeneous red meeting $\aleph_1$ columns, contradicting the lemma. So, we turn to the:

Proof of lemma. Towards a contradiction, suppose there is such a p, say $p = p_\alpha$, where $\boldsymbol{p} \in P$. Clearly, we may assume that $\alpha = \theta = \theta(\boldsymbol{p})$, and suppose that we have chosen $\boldsymbol{p}$ to have θ minimal such that p_θ is a counterexample. Then θ must be a limit ordinal of cofinality ω_1. We adopt the notation of 2.9. Let $p = (a, b, c)$, let $R \subseteq \tilde{a}$ be homogeneous red such that R meets $\aleph_1$ columns, i.e., $\Gamma = \Gamma(R) = \{\gamma < \omega_1 : (\exists x \in R)\, \gamma(x) = \gamma\}$ has power $\aleph_1$. We shall obtain a contradiction by obtaining such a homogeneous red R^* in p_{β^*}, where $\beta^* < \theta$.

Without loss of generality assume that R meets each column in at most a singleton. Let $(u_i : i < \omega_1)$ be the unique enumeration of R such that, letting $\gamma_i = \gamma(u_i)$, $(\gamma_i : i < \omega_1)$ is monotone increasing. Let $\beta_i = \beta(u_i) =$ the least $\beta < \theta$ such that $u_i \in a_\beta$. Note that the leastness of θ guarantees that whenever $X \in [\omega_1]^{\aleph_1}$, $\{\beta_i : i \in X\}$ is cofinal in θ. Thus, by further thinning, we can assume that $(\beta_i : i < \omega_1)$ is strictly increasing, so by omitting at most one element from R, we may assume that for all $u \in R$, $\beta(u) > 0$, in which case, $\beta(u)$ is a successor ordinal.

2.15.1. We now develop the histories of the $x \in \tilde{a}$. The histories will be finite; the analysis is like a direct limit analysis. So, suppose $\boldsymbol{q} \in P$. If $\theta(\boldsymbol{q}) = 0$ and $z \in (a(\boldsymbol{q}))^{\sim}$, let $\text{hist}(\boldsymbol{q}, z) = \{(0, z)\}$. If $\theta(\boldsymbol{q}) = \lambda = \bigcup \lambda > 0$, and $z \in (a(\boldsymbol{q}))^{\sim}$, let $\beta < \lambda$ be least such that $z \in (a^{q_\beta})^{\sim}$ and let $\text{hist}(\boldsymbol{q}, z) = \text{hist}(\boldsymbol{q} \mid \beta + 1, z)$. Finally, if $\theta(\boldsymbol{q}) = \beta + 1$, and $z \in (a(\boldsymbol{q}))^{\sim}$, we consider two cases: $z \in (a^{q_\beta})^{\sim}$, or not. In the first case, $\text{hist}(\boldsymbol{q}, z) = \text{hist}(\boldsymbol{q} \mid \beta + 1, z)$, while in the second case, we must have $z \in (e(a')^{q_\beta})^{\sim}$, i.e., $z = \pi_\beta(z')$, where $z' \in (e(a^{q_\beta}))^{\sim}$. In this case, we let $\text{hist}(\boldsymbol{q}, z) = \text{hist}(\boldsymbol{q} \mid \beta + 1, z') \cup \{(\beta + 1, z)\}$. We let $\text{hist}_1(\boldsymbol{q}, z) = \{\beta : (\exists x)(\beta, x) \in \text{hist}(\boldsymbol{q}, z)\}$, and similarly for hist_2.

2.15.2. For $u \in R$, let $\text{hist}(u)$, $\text{hist}_1(u)$, $\text{hist}_2(u) = \text{hist}(\boldsymbol{p}, u)$, $\text{hist}_1(\boldsymbol{p}, u)$, $\text{hist}_2(\boldsymbol{p}, u)$, respectively. Note that $\beta(u) = \max \text{hist}_1(u)$. W.l.o.g. for some $n > 1$, $\text{card}\, \text{hist}(u_i) = n$, for all i. Let $\beta_{i,j}$ be the decreasing enumeration of $\text{hist}_1(u_i)$; thus, $\beta_{i,0} = \beta_i$, $\beta_{i,n-1} = 0$. Let $u_{i,j}$ be such that $(\beta_{i,j}, u_{i,j}) \in \text{hist}(u_i)$; thus $u_{i,0} = u_i$. Let $0 < m < n$ be least such that $\{b_{i,m} : i < \omega_1\}$ is bounded in θ. This is well defined since $\{\beta_{i,0} : i < \omega_1\}$ is unbounded in θ, while $\{\beta_{i,n-1} : i < \omega_1\}$ is $\{0\}$. Let $\beta^* = \sup\{\beta_{i,m} : i < \omega_1\}$, so $\beta^* < \theta$. By thinning, we may assume that if $k_1 < k_2$ then:

(!) $\beta_{k_2,m} < \beta_{k_1,m-1}$ and $\beta_{k_1} < \beta_{k_2,m-1}$,

(!!) $\beta_{k_1,m} \leqslant \beta_{k_2,m}$,

(!!!) for all $j < m$, letting $\beta(j) = \beta_{k_1,j}$ and $\bar{\beta}(j) = \beta(j) - 1$, $\gamma(u_{k_2}) > \gamma(y_{\bar{\beta}(j)})$, whenever $y_{\bar{\beta}(j)}$ is defined (refer back to 2.9 for this notation).

2.15.3. We now define a partition $F : (< \mid \omega_1) \to \{\text{purple}\} \cup m \times (m + 1)$: $F(i_1, i_2) = \text{purple}$ iff for no $(j_1, j_2) \in m \times m + 1$, do we have:

(#) $(u_{i_1,j_1}, u_{i_2,j_2}) = \text{red}(u', v')$, where (u', v') is critical at stage $\beta - 1$ and $(u_{i_1,j_1}, u_{i_2,j_2})$ was created at stage β (refer back to 2.6 for this notion; recall also the division into cases (1) and otherwise of the proof of (b) of the lemma there, since the argument of 2.15.4 below will amount to showing that in circumstances of vital interest to us it is only (1) which can arise).

Note that $\neg(\#)$ includes the possibility that $c(u_{i_1,j_1}, u_{i_2,j_2}) \neq \text{red}$. If (#) is true for some (j_1, j_2), we let $F(i_1, i_2) =$ the lexicographically least such (j_1, j_2).

Let us assume that R has been thinned to also guarantee that for $i_1 < i_2 < i_3 < \omega_1$:

(!!!!) for all $(j_1, j_2) \in m \times (m+1)$, $b(u_{i_1,j_1}, u_{i_2,j_2}) < \gamma(u_{i_3})$.

We now argue that there is no triple, $i_1 < i_2 < i_3$, such that for some j_1, j_2, j_3, $F(i_1, i_3) = (j_1, j_3)$ and $F(i_2, i_3) = (j_2, j_3)$. Suppose, towards a contradiction that this occurs. There are two cases, according to whether $j_3 < m$. If so, then, letting $\beta = \beta_{i_3,j_3}$, we must have $\gamma(y_{\beta-1}) < b(u_{i_1,j_1}, u_{i_2,j_2})$, but by (!!!!), we would have $b(u_{i_1,j_1}, u_{i_2,j_2}) < \gamma(u_{i_3})$ which is absurd. If $j_3 = m$, it is an immediate consequence of (!!!), for $k_1 = i_1$, $k_2 = i_3$, that there is no $i_1 < i_3$ and j_1 such that $F(i_1, i_3) = (j_1, m)$, since, if so, in this case, the edge $\{u_{i_1,j_1}, u_{i_3,m}\}$ is created at stage $\beta \overset{\text{def}}{=} \beta_{i_1,j_1}$. Then, letting $\beta = \bar{\beta} + 1$, under the hypotheses on i_1, i_3, j_1, by (!!!) we must have $\gamma(u_{i_3,m}) = \gamma(u_{i_3}) < \gamma(y_{\bar{\beta}}) < \gamma(u_{i_3})$!

But, $\omega_1 \to (\omega_1, \omega, \ldots, \omega)^2$ ($m^2 + m\ \omega$'s), so there is an uncountable homogeneous purple subset of ω_1, which may as well be ω_1.

2.15.4. Note that this means that for all $i_1 < i_2 < \omega_1$ and all $(j_1, j_2) \in m \times (m+1)$, $c(u_{i_1,j_1}, u_{i_2,j_2}) = \text{red}$. This is easily seen for all $j_2 \leq m$, with $j_1 = 0$, using that $F(i_1, i_2) = \text{purple}$ (and of course that $c(u_{i_1,0}, u_{i_2,0}) = c(u_{i_1}, u_{i_2}) = \text{red}$). But then, letting $(j_1 + 1, j_2)$ be lexicographically least such that $c(u_{i_1,j_1+1}, u_{i_2,j_2}) \neq \text{red}$, if $j_2 = 0$, we get a contradiction to the fact that $F(i_1, i_2) \neq (j_1, j_2)$, while if $j_2 > 0$, we get a contradiction to the fact that $F(i_1, i_2) \neq (j_1 + 1, j_2 - 1)$. In particular, this means that for all $i_1 < i_2 < \omega_1$, $c(u_{i_1,m}, u_{i_2,m}) = \text{red}$. But this contradicts the leastness of θ, since then we have $\{u_{i,m} : i < \omega_1\}$, a homogeneous red set in p_{β^*}. This completes the proof of the lemma.

2.16. Now let G be generic for $\mathbb{P}$. Let $C^\# = \bigcup \{c^p : (\exists p \in G)(\exists \alpha \leq \theta(p))\, p = p_\alpha\}$. We define a coloring, C, by green and red which we obtain from $C^\#$. First note that, while $A \overset{\text{def}}{=} \bigcup \{a(p) : p \in G\}$ may not be ω_3, nevertheless, $\operatorname{card} A = \aleph_3$, which suffices for our purposes, since we can identify $[\tilde{A}]^2$ with $[\omega_3\omega_1]^2$ via the increasing enumeration of A. To see that $\operatorname{card} A = \aleph_3$, it suffices, since $\mathbb{P}$ preserves cofinalities, to show that A is cofinal in ω_3, i.e., that for all $\xi < \omega_3$, $\{p \in P : \xi < \sup(a(p))\}$ is dense. This is an easy consequence of the amalgamation property.

So, we define C as follows. If $\{x, y\} \in [\tilde{A}]^2$ and $\gamma(x) = \gamma(y)$, let $C\{x, y\} = \text{red}$. If $(x, y) \in (\tilde{A})^2$, define $C\{x, y\}$ as follows: if $C^\#(x, y) = \text{gray}$, set $C(x, y) = \text{red}$, otherwise let $C\{x, y\} = C^\#(x, y)$. We now prove:

Lemma. *C has no homogeneous red set of order type $\omega_3\omega_1$.*

Proof. Suppose, towards a contradiction, that X is such a set; let $\mathring{X}$ be a $\mathbb{P}$-name for X and suppose that $\boldsymbol{q} \in P$ and:

$$\boldsymbol{q} \Vdash \text{"}\mathring{X} \text{ is homogeneous red for } \mathring{C} \text{ and o.t. } \mathring{X} = \omega_3\omega_1\text{"}.$$

Working in the ground model V, we easily obtain the following: $((\boldsymbol{p}^\iota, R_\iota) : \iota < \omega_3)$, such that $\boldsymbol{q} \leqslant \boldsymbol{p}^\iota$, $R_\iota \in [(a(\boldsymbol{p}^\iota)^\frown]^{\aleph_1}$, R_ι meets each column in at most one point, $\{\rho(x) : x \in R_\iota\}$ is cofinal in $a(\boldsymbol{p}^\iota)$ and $\boldsymbol{p}^\iota \Vdash$ "$R_\iota \subseteq \mathring{X}$". Let $\theta_\iota = \theta(\boldsymbol{p}^\iota)$ and let $(a_\iota, b_\iota, c_\iota) = p^\iota = p^\iota_{\theta_\iota}$. Note that for $(x, y) \in (R_\iota)^2$, $c_\iota(x, y) \neq$ green. By 2.15, there is such (x, y) such that $c_\iota(x, y) =$ gray. We can find stationary $S \subseteq \omega_3$ such that for $\iota < j$, $\iota, j \in S$, $\tau^{\boldsymbol{p}^\iota} = \tau^{\boldsymbol{p}^j}$ and $(a_\iota, a_j) \in$ u-AMP. Note that $\lg p_\iota = \lg p_j$ and that if $\pi : a_\iota \to a_j$ is the order isomorphism, $\pi : p_\iota \to p_j$. Let $(x, y) \in (R_\iota)^2$ such that $c_\iota(x, y) =$ gray. But then let $p^* = \mathrm{Am}(p_\iota, a_j, \mathrm{u}, \{(x, y)\})$ and let $\boldsymbol{p}^* = \boldsymbol{p}^{\iota\wedge} p^*$, so $\boldsymbol{p}^\iota, \boldsymbol{p}^j \leqslant \boldsymbol{p}^*$. But then $\boldsymbol{p}^* \Vdash$ "$R_\iota \cup R_j \subseteq \mathring{X}$" and of course $\boldsymbol{q} \leqslant \boldsymbol{p}^*$, so $\boldsymbol{p}^* \Vdash$ "$\mathring{X}$ is homogeneous red for $\mathring{C}$", but this is contradicted by the fact that $c^{p^*}(\bar{\pi}(x), y) =$ green! This completes the proof of the lemma and of Theorem 2 for $\omega_3\omega_1$.

2.17. Remarks. As mentioned in the introduction, 2.1–2.16 generalize easily from $\omega_3\omega_1$ to (the ordinal product) $\kappa^{++}\kappa$ for any regular uncountable κ. In fact κ^{++} can be replaced by many regular non-weakly-compact cardinals $\mu > \kappa^+$, though some details of the arguments of 2.15, 2.16 must be modified. The case of $\mu = \tau^+$ for regular $\tau > \kappa^+$ is most straightforward: we use conditions of size $<\tau$. If τ is a large cardinal, the preservation of the large cardinal properties of τ is not problematical, unless these properties are already rather strong, in which case use of appropriate filters to stabilize the histories of conditions becomes a possibility. This, as well as the cases when μ is Mahlo and we wish to preserve the Mahlo property and the case when μ is the successor of a singular cardinal will be treated in the sequel, [14]$^{\mathrm{C}}$. The case of inaccessible but non-Mahlo μ appears to be the first truly difficult case. When we want to preserve large cardinal properties of κ, problems do not arise until the desired property is μ-supercompact or more; in this case arguments along the lines of Laver's [8] may work. These questions will also be investigated in [14]$^{\mathrm{C}}$.

3. Proof of Theorem 3

In this section we define a partial ordering $\mathbb{P}$ which, under appropriate hypotheses introduced below, preserves cofinalities and the ground model values of $2^{\aleph_0}$, $2^{\aleph_1}$ and which adds a generic counterexample to $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$. Once again, $\mathbb{P}$ will be the historicization of a $\mathbb{P}_{\mathrm{basic}}$. The treatment parallels Section 2. 3.7 corresponds to 2.15 and, once again is the most substantial item.

There are various possibilities for the cardinal exponentiation in the extension; the desired pattern will determine our choice of the size of the conditions. This is most clearly understood in a somewhat more general setting. We have the following cardinal parameters: θ, λ, κ, μ, obeying the following requirements: $\theta>\omega$, θ is regular, $\kappa=(2^{<\theta})^+$, λ is singular with $\mathrm{cf}(\lambda)=\kappa$ (and we pick an increasing κ-sequence λ_i, $i<\kappa$, of cardinals cofinal in λ), $\theta\leqslant\mu$, $2^\mu\leqslant\kappa^+$. The conditions will have size μ. Thus in our intended setting, $\theta=\aleph_1$, $\kappa=\mathfrak{c}^+$, $\lambda=\aleph_\kappa$, $\lambda_i=\aleph_i$, and $\aleph_1\leqslant\mu$, $2^\mu\leqslant\mathfrak{c}^{++}$. In the simplest setting, we will want the μ^{++}-cc, which will require that $2^\mu=\mu^+$ ($\mathbb{P}$ will always have the $(2^\mu)^+$-cc). If $\mu<2^{<\theta}$, this imposes a perhaps unwanted restriction on $2^{<\theta}$ (i.e. on $2^{\aleph_0}$, in our intended interpretation). Another natural case is $\mu=\kappa$, but this imposes a perhaps unwanted restriction on 2^κ (in our intended interpretation, $2^{(\mathfrak{c}^+)}$). If we don't require $2^\mu=\mu^+$, then, in the extension, all cardinals $\leqslant\mu^+$ will be preserved, the ground model's $(2^\mu)^+$ will become μ^{++} and all larger cardinals will be preserved, so $2^\mu=\mu^+$ will hold in the extension. We will, however, always want to preserve the regularity of κ, whether by pseudo-closure properties or by chain-conditions.

3.1. Let $\theta>\omega$ be regular, let $\kappa=(2^{<\theta})^+$, let $(\lambda_i : i<\kappa)$ be a normal sequence of cardinals with $\lambda_0=0$, and let $\lambda=\bigcup\{\lambda_i : i<\kappa\}$. For $i<\kappa$, let $I_i=$ the ordinal interval $[\lambda_i, \lambda_{i+1})$. If $\alpha<\lambda$, let $i(\alpha)$ be the unique $i<\kappa$ with $\alpha\in I_i$. For $a\subseteq\lambda$, let $I(a)=\{i(\alpha):\alpha\in a\}$. Let μ be a cardinal with $\theta\leqslant\mu\leqslant\kappa$ and $2^\mu=\mu^+$ (we have discussed, above, the situations that arise if we no longer require $2^\mu=\mu^+$). The existence of such a μ constitutes our cardinal exponentiation hypothesis.

Definition. $p\in P_{\text{basic}}$ iff $p=(a^p, c^p)=(a, c)$, where $a\in[\lambda]^{\leqslant\mu}$, $c:[a]^2\to\{\text{green}, \text{red}\}$, and for $i<\kappa$, if $\{\alpha, \beta\}\in[a\cap I_i]^2$, then $c\{\alpha, \beta\}=\text{red}$. If $p, q\in P_{\text{basic}}$, $p\leqslant_{\text{basic}} q$ iff $a^p\subseteq a^q$ and $c^p\subseteq c^q$. $\mathbb{P}_{\text{basic}}=(P_{\text{basic}}, \leqslant_{\text{basic}})$.

3.2. Proposition. *$\mathbb{P}_{\text{basic}}$ is μ^+-closed with unions of the coordinates as upper bounds.*

Proof. Clear.

3.3. Suppose $a\in[\lambda]^{\leqslant\mu}$. If $\pi:a\to a'$ is one-to-one, where $a'\subseteq\lambda$, π is *tame* if whenever $\alpha, \beta\in a$ and $i(\alpha)=i(\beta)$, then $i(\pi(\alpha))=i(\pi(\beta))$. When π is tame, we define $\bar\pi:I(a)\to I(a')$, by $\bar\pi(i(\alpha))=i(\pi(\alpha))$. When π is a bijection and tame, and $\bar\pi$ is one-to-one, we define $q=(a', c')$, the unique condition isomorphic to p via π by: $c'\{\pi(\alpha), \pi(\beta)\}=c\{\alpha, \beta\}$, and we write $\pi:p\to q$ or $q=\pi(p)$.

Lemma. *Suppose $\pi:p\to q$ and both of the following conditions hold:*

(a) $\pi\,|(a^p\cap a^q)=\mathrm{id}|\,(a^p\cap a^q)$,

(b) $\bar\pi\,|(I(a^p)\cap I(a^q))=\mathrm{id}|\,(I(a^p)\cap I(a^q))$.

Suppose further that $\alpha\in a^p\setminus(a^p\cap a^q)$, $\beta\in a^q\setminus(a^p\cap a^q)$ and that $i(\alpha)\neq i(\beta)$. Let $c:[a^p\cup a^q]^2\to\{\text{green}, \text{red}\}$, $c^p\cup c^q\subseteq c$ and if $\gamma\in a^p\setminus(a^p\cap a^q)$, $\delta\in a^q\setminus(a^p\cap a^q)$ and $\{\gamma, \delta\}\neq\{\alpha, \beta\}$, then $c\{\gamma, \delta\}=\text{red}$. Then $p, q\leqslant_{\text{basic}}((a^p\cup a^q), c)$.

Proof. Clear, from our above hypotheses on π, $\bar{\pi}$ and since we have required that $i(\alpha) \neq i(\beta)$. When $c\{\alpha, \beta\} = \text{red}$, we have the all red amalgamation. When $c\{\alpha, \beta\} = \text{green}$, we have the $\{\alpha, \beta\}$-green amalgamation. The all-red amalgamation and the different $\langle \alpha, \beta \rangle$-green amalgamations constitute the admissible amalgamations.

3.4. Definition. $(a, c) \in P_0$ iff *for all* $i < \kappa$, $\text{card}(a \cap I_i) \leq 1$ and range $c = \{\text{red}\}$. $\boldsymbol{p} \in P$ iff $\boldsymbol{p} = (p_\xi : \xi \leq \eta(\boldsymbol{p}))$, where $\eta = \eta(\boldsymbol{p}) < \mu^+$, for $\xi \leq \eta$, $p_\xi = (a_\xi, c_\xi) \in P_{\text{basic}}$, $p_0 \in P_0$, for nonzero limit $\delta \leq \eta$, $a_\delta = \bigcup \{a_\xi : \xi < \delta\}$, $c_\delta = \bigcup \{c_\xi : \xi < \delta\}$, and, finally, for $\xi < \eta$, there are (unique) q_ξ and π_ξ satisfying the hypotheses of the lemma of 3.3, such that $p_{\xi+1}$ is an admissible amalgamation of p_ξ and q_ξ. If $p \in P_{\text{basic}}$, p is *accessible* iff for some $\boldsymbol{p} \in P$ and some $\xi \leq \eta(\boldsymbol{p})$, $p = p_\xi$.

3.5. If $\boldsymbol{p} \in P$ and (adopting the notation of 3.4) $\xi < \eta$, we define $\boldsymbol{p}^{\langle\{\xi\}\rangle}$ by writing $\boldsymbol{r}$ for $\boldsymbol{p}^{\langle\{\xi\}\rangle}$, setting $\eta(\boldsymbol{r}) = \eta$, setting $r_\zeta = p_\zeta$, for $\xi < \zeta \leq \eta$, and setting $r_\zeta = (\pi \mid a_\zeta)(p_\zeta)$ for $\zeta \leq \xi$ (thus, in particular, $r_\xi = q_\xi$). If $h \in [\eta]^{<\omega}$ and $h \neq \emptyset$, let $(\xi_i : i < \text{card}(h))$ be the decreasing enumeration of h, and define $\boldsymbol{p}^i$ for $i \leq \text{card}(h)$ by recursion: $\boldsymbol{p}^0 = \boldsymbol{p}$, $\boldsymbol{p}^{i+1} = (\boldsymbol{p}^i)^{\langle\{\xi_i\}\rangle}$. Let $\boldsymbol{p}^{\langle h \rangle} = \boldsymbol{p}^{\text{card}(h)}$. If $h = \emptyset$, let $\boldsymbol{p}^{\langle h \rangle} = \boldsymbol{p}$. The remarks of 2.10 carry over to this setting.

3.6. Definition. If $\boldsymbol{p}, \boldsymbol{q} \in P$, let $\boldsymbol{q} \leq \boldsymbol{p} \Leftrightarrow \eta(\boldsymbol{q}) \leq \eta(\boldsymbol{p})$ and for some finite (possibly empty) $h \subseteq \eta(p)$, $\boldsymbol{q} \subseteq \boldsymbol{p}^{\langle h \rangle}$. We let $\boldsymbol{q} \leq_{\text{pure}} \boldsymbol{p}$ iff we can take $h = \emptyset$, i.e., $\boldsymbol{q} \subseteq \boldsymbol{p}$. We let $\mathbb{P} = (P, \leq)$. The remarks of 2.11 carry over to this setting.

3.7. We now have the analogue of 2.15 (although the proof is somewhat simpler here), showing that historicization replaces side conditions in eliminating unwanted configurations from basic conditions. This time, the unwanted configuration is a large *green* set.

Lemma. *If* $p = (a, c) \in P_{\text{basic}}$ *is accessible, then there is no* $\Gamma \in [a]^\theta$, *homogeneous green for c.*

Proof. Towards a contradiction, suppose that there is a counterexample, $p = (a, c)$, and Γ is a witness. Since any initial segment (with length a successor ordinal) of a $\boldsymbol{p} \in P$ is in turn in P, we may assume that $p = p_\eta$, for $\boldsymbol{p} = (p_\xi : \xi \leq \eta) = ((a_\xi, c_\xi) : \xi \leq \eta) \in P$ and further that whenever $\boldsymbol{r} = (r_\xi : \xi \leq \eta') \in P$ and $\xi < \eta$, then r_ξ is not a counterexample to the lemma, i.e., that η is the least ξ such that some counterexample to the lemma occurs as r_ξ, for some $\boldsymbol{r} = (r_\zeta : \zeta \leq \eta') \in P$. So, η is a limit ordinal with $\text{cf}(\eta) = \theta$.

For $\alpha \in \Gamma$, let $\xi(\alpha)$ be the least ξ such that $\alpha \in a_\xi$. By the leastness of η, whenever $\Gamma' \in [\Gamma]^\theta$, $\{\xi(\alpha) : \alpha \in \Gamma'\}$ is cofinal in η, so we can find $\Gamma' \in [\Gamma]^\theta$ such that on Γ', $\xi(\alpha)$ is a monotone increasing function of α. Note that, by the definition of P_{basic}, for all $i < \kappa$, $\text{card}(\Gamma \cap I_i) \leq 1$. As in 2.15.1, we develop the complete histories of points in a; as there, we denote the history of α (in p) by hist(α) and we use $\mathrm{hist}_1(\alpha)$, $\mathrm{hist}_2(\alpha)$ to denote the projections of hist(α) on the first and second coordinates, respectively. Thus, $\xi(\alpha) = \max \mathrm{hist}_1(\alpha)$, and if $\zeta \in \mathrm{hist}_1(\alpha)$, then either $\zeta = 0$ or ζ is a successor ordinal, in which case $\bar{\zeta}$ denotes its predecessor.

Let $(\alpha_i : i < \theta)$ be an enumeration of Γ and let $(\nu_i : i < \theta)$ be the increasing enumeration of a club of η of order type θ. We easily find a club, C, of θ such that for $i \in C$ and $j < i$, $\xi(\alpha_j) < \nu_i$, so assume, w.l.o.g., that this is true for all $i < \theta$. Similarly, there is a stationary subset, S, of θ and an $n < \omega$ such that for all $i \in S$, card(hist(α_i)) $= n$, so assume, w.l.o.g., that this holds for all $i < \theta$. Further, on a stationary set, S, of limit ordinals below θ, the function $h(i) \overset{\text{def}}{=} \max(\mathrm{hist}_1(\alpha_i \cap \nu_i))$ is constant, say with value σ. W.l.o.g. there is $m < n$, such that for all $i < \theta$, $\mathrm{card}(\mathrm{hist}_1(\alpha_i) \setminus \sigma + 1) = m$. Let $\mathrm{hist}(\alpha_i) = \{(\xi_{i,j}, \alpha_{i,j}) : j < n\}$, where $(\xi_{i,j} : j < n)$ is decreasing; thus $\alpha_{i,0} = \alpha_i$, $\xi_{i,j} = \xi(\alpha_i)$, and $\xi_{i,m} = \sigma$. Note, further, that for all $i_1 < i_2$, all $(j_1, j_2) \in n \times n$, $i(\alpha_{i_1,j_1}) \neq i(\alpha_{i_2,j_2})$. This is by the definition of admissible amalgamation and by the definition of P_{basic}, since the $\bar{\pi}$ used in admissible amalgamations are required to be 1–1 functions.

Now, define $F : (< \mid \theta) \to \{\text{purple}\} \cup (m+1) \times (m+1)$ by $F(i_1, i_2) = \text{purple}$ iff there is no $(j_1, j_2) \in (m+1) \times (m+1)$ such that $c\{\alpha_{i_1,j_1}, \alpha_{i_2,j_2}\} = \text{green}$ and, at the stage this edge was created, the $\{\alpha_{i_1,j_1}, \alpha_{i_2,j_2}\}$-green amalgamation was used; otherwise, $F(i_1, i_2) =$ the lexicographically least such (j_1, j_2). We claim that there is no (j, j') which has an F-homogeneous set of size 3. Towards a contradiction, suppose X is F-homogeneous for (j, j'), $\mathrm{card}\, X > 2$. First, note that if $j' < m$, then for all $i_1 < i_2$, both from X, $\xi_{i_1,j}, \xi_{i_1,j'} < \xi_{i_2,j'}$. Thus, if further, $i_3 \in X \setminus i_2 + 1$, then $\{\alpha_{i_1,j}, \alpha_{i_3,j'}\}$, $\{\alpha_{i_2,j}, \alpha_{i_3,j'}\}$ were both created at stage $\xi_{i_3,j'}$, but then two different amalgamations would have been used at this stage, contradiction! So, we may suppose that $j' = m$. Next, note that $j < m$, since otherwise for $i_1 < i_2$, both in X, $\xi_{i_1,j} = \xi_{i_1,m} = \sigma = \xi_{i_2,m} = \xi_{i_2,j'}$, i.e., $\alpha_{i_1,j}$, $\alpha_{i_2,j'}$ were created at the same stage and can not have been colored green *at this stage because of the* $\{\alpha_{i_1,j}, \alpha_{i_2,j'}\}$-*green-amalgamation*! There remains the case where $j < m$, but here, if $i_1 < i_2 < i_3$, all from X, then $\{\alpha_{i_1,j}, \alpha_{i_3,j'}\}$, $\{\alpha_{i_1,j}, \alpha_{i_2,j'}\}$ were both created at stage $\xi_{i_1,j}$, but then two different amalgamations would have been used at this stage, contradiction!

Then, since $\theta \to (\theta, \omega, \ldots, \omega)$ ($(m+1)^2$ ω's), there is a homogeneous purple set of power θ, which we may as well suppose is all of θ. Once again, as in 2.15.4, we show (easily, for $j = 0$ and arbitrary j', and then by induction on $j \leq m$ for arbitrary j', using homogeneous purple) that for all $i_1 < i_2$, and all $(j, j') \in (m+1) \times (m+1)$, $c\{\alpha_{i_1,j}, \alpha_{i_2,j'}\} = \text{green}$. Once again, this means that $\{\alpha_{i,m} : i < \theta\}$ is a size θ (recall that for $i_1 < i_2 < \theta$, $i(\alpha_{i_1,m}) \neq i(\alpha_{i_2,m})$) homogeneous green set in p_σ, contradicting the leastness of η. This completes the proof of the lemma.

Here, as in 2.15, the lemma suffices to show that there is no homogeneous

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green set of power θ for the generic coloring; the argument is exactly like that preceding the proof of the lemma of 2.15.

3.8. Lemma. *$\mathbb{P}$ is μ^+ strategically closed and has the μ^{++}-c.c.*

Proof. As in Section 2, NONEMPTY's strategy, at stage $\delta+2n+2$, is to play a *pure* extension of $\boldsymbol{p}^{\delta+2n}$ which also extends $\boldsymbol{p}^{\delta+2n+1}$, which permits her, at limit stages, $\delta>0$, to play

$$\boldsymbol{p}^{\delta}=\bigcup\{\boldsymbol{p}^{\alpha}:\alpha<\delta,\ \alpha \text{ is even}\}\cup\{(\eta,(a,c))\},$$

where, for $\alpha<\delta$,

$$\boldsymbol{p}^{\alpha}=((a^{\alpha}_{\xi},c^{\alpha}_{\xi}):\xi<\eta_{\alpha}),\qquad \eta=\bigcup\{\eta_{\alpha}:\alpha<\delta\},$$
$$a=\bigcup\{a^{\alpha}_{\eta_{\alpha}}:\alpha<\delta\},\qquad c=\bigcup\{c^{\alpha}_{\eta_{\alpha}}:\alpha<\delta\}.$$

A somewhat more delicate argument shows that $\mathbb{P}$ is actually ω_1-closed (i.e., countably closed), but $\mathbb{P}$ is not ω_2-closed.

For the μ^{++}-c.c., we must repeat the 'blueprint analysis' of conditions as in 2.12. If $\boldsymbol{p}\in P$, let $a=a(\boldsymbol{p})=a_{\eta}$, let $I=I(\boldsymbol{p})=\{i(\alpha):\alpha\in a\}$, let $\lg\boldsymbol{p}=\text{o.t. } a$. Let $s^{\boldsymbol{p}}=(\alpha^{\eta}_{\xi}:\xi<\lg\boldsymbol{p})=(\alpha_{\xi}:\xi<\lg\boldsymbol{p})$ be the increasing enumeration of a and let $\bar{s}^{\boldsymbol{p}}=(\iota^{\eta}_{\xi}:\xi<\text{Ilg}\,\boldsymbol{p})=(\iota_{\xi}:\xi<\text{Ilg}\,\boldsymbol{p})$ be the increasing enumeration of I. Let $c=c(\boldsymbol{p})=c_{\eta}$. The construction here will be somewhat more complicated than in 2.12, due essentially to the following difficulty: for certain $\xi<\text{Ilg}\,\boldsymbol{p}$, possibly $\text{card}(a\cap I_{i_{\xi}})>\text{card}\, I_{\xi}$. However, if we proceed formally, without demanding that the $\tau^{\boldsymbol{p}}$ we produce actually be in $\mathbb{P}$, we can skirt this difficulty. So we define $a^*\in[\text{Ilg}\,\boldsymbol{p}\times\mu^+]^{\leqslant\mu}$ and order preserving $\sigma:a^*\to a$, by:

$$a^*\cap(\{\xi\}\times\mu^+)=(\{\xi\}\times\text{o.t.}(a\cap I_{i_{\xi}}));$$

$\sigma(\xi,\zeta)=$ the ζth element of $(a\cap I_{i_{\xi}})$. Then, for $\nu\leqslant\eta$, we define a^*_{ν} to be $\sigma^{-1}[a_{\nu}]$ (so $a^*_{\eta}=a^*$) and we take c^*_{ν} to be the coloring of a^*_{ν} induced by σ and c_{ν}. Finally, $\tau^{\boldsymbol{p}}=((a^*_{\nu},c^*_{\nu}):\nu\leqslant\eta)$. Clearly, if $\eta(\boldsymbol{p})=\eta(\boldsymbol{q})$ and $(a(\boldsymbol{q}),c(\boldsymbol{q}))=\pi(a(\boldsymbol{p}),c(\boldsymbol{p}))$ for a (unique) π satisfying the hypotheses of 3.3, including the hypotheses (a), (b) of the lemma, then $\tau^{\boldsymbol{q}}=\tau^{\boldsymbol{p}}$. Further, $\tau^{\boldsymbol{p}}$ reconstructs $\boldsymbol{p}$, given $s^{\boldsymbol{p}}$, simply by putting the lexicographic order on a^* and mapping the ξth element of a^* to α_{ξ} (in fact, this is σ).

We now state (the proof is trivial: this is the payoff for the definition of $\leqslant$) the amalgamation property for $\mathbb{P}$:

If $\boldsymbol{p},\boldsymbol{q}\in P$, $\tau^{\boldsymbol{p}}=\tau^{\boldsymbol{q}}$, $\text{o.t.}(a(\boldsymbol{p}))=\text{o.t.}(a(\boldsymbol{q}))$ and, letting $\pi:a(\boldsymbol{p})\to a(\boldsymbol{q})$ be the order isomorphism, π is tame and satisfies the other hypotheses of 3.3 (including (a) and (b) of the lemma) then, letting r be any admissible amalgamation of $p_{\eta(\boldsymbol{p})}$ and $q_{\eta(\boldsymbol{p})}$ (note that since $\tau^{\boldsymbol{p}}=\tau^{\boldsymbol{q}}$, $\eta(\boldsymbol{p})=\eta(\boldsymbol{q})\stackrel{\text{def}}{=}\eta$ and $\pi:p_{\eta}\to q_{\eta}$), $\boldsymbol{p},\boldsymbol{q}\leqslant\boldsymbol{p}^\wedge r$, $\boldsymbol{q}^\wedge r$. In particular, $\boldsymbol{p}$, $\boldsymbol{q}$ are compatible.

Corollary. $(2^{\mu}=\mu^+)$ *$\mathbb{P}$ has the μ^{++}-c.c.*

Proof. We have two main cases: $\kappa \in \{\mu, \mu^+\}$, and otherwise. In either case, since $2^\mu = \mu^+$, we know that $\mathrm{card}(\{\tau^p : p \in P\}) = \mu^+$. In the first case, since $\mu^+ = 2^\mu$, we know that $\mathrm{card}(\{I(a) : (a, c) \in P_{\text{basic}}\}) = \mu^+$. Thus, in the first case, given μ^{++} conditions, p^ξ, we may assume that they all have the same τ and the same $I(a_\eta)$. In the second case, we cannot stabilize the $I(a_\eta)$; instead we must apply the Δ-system lemma for $[\kappa]^\mu$ (which is available, since $\mu^{++} \leqslant \kappa$ in this case) to obtain a size μ^{++} Δ-system among the $I(a_\eta)$. This achieved, in either case, we finish by applying the Δ-system lemma for $[\lambda]^\mu$ to obtain a size μ^{++} Δ-system among the a_η. But then, by the amalgamation property for $\mathbb{P}$, any two of the remaining conditions are compatible.

3.9. Lemma. $\Vdash_{\mathbb{P}}$ *"there is no homogeneous red set of power λ".*

Proof. We show, in fact, that it is forced that there is no homogeneous red set R and no $i_1 < i_2 < \kappa$ for which $\mu^{++} \leqslant \mathrm{card}(R \cap I_{i_j})$, $j = 1, 2$. Suppose, towards a contradiction, that p forces that $(\mathring{R}, i_1, i_2)$ has this property. Then we easily obtain, for $j = 1, 2$ and $\xi < \mu^{++}$, $p \leqslant p^\xi \in P$ and $\alpha_{j,\xi} \in (a^\xi_{\eta_\xi} \cap I_{i_j}) \setminus \bigcup \{a^\zeta_{\eta_\zeta} : \zeta < \xi\}$ such that p^ξ forces $\alpha_{j,\xi} \in \mathring{R}$, $j = 1, 2$. By 3.8, w.l.o.g. the p^ξ satisfy the hypotheses of the amalgamation property for $\mathbb{P}$. But then, e.g., p^0, p^1 can be amalgamated by the $\{\alpha_{0,0}, \alpha_{1,1}\}$-green-amalgamation, contradiction!

4. Historicization, the general case

In this section, we generalize from the two examples in Sections 2, 3, to present the technique of historicization of a set of 'naive' conditions, satisfying certain properties. We omit proofs, referring the reader to the relevant items in Sections 2, 3.

We have cardinal parameters, μ, λ, which have essentially the same meaning as in Section 3: we assume that $\omega \leqslant \mu < \lambda$. We also assume that we are given a partial ordering $\mathbb{P}_{\text{basic}}$ satisfying the following properties:

(1) $\mathbb{P}_{\text{basic}}$ is μ-closed with least upper bounds.

(2) D_α is a predense subset of P_{basic}, for $\alpha < \lambda$; we let $\bar{D}_\alpha$ = the dense open set generated by D_α.

(3) For each $p \in P_{\text{basic}}$, the set $\mathrm{rlm}(p) \overset{\text{def}}{=} \{\alpha : p \in \bar{D}_\alpha\}$ has power $\leqslant \mu$ (this notation and approach are taken from Velleman, [16]; our approach of terms and indiscernible sequences, [12], would be an (equivalent) alternative, and is implicit in 2.12, and 3.8); further for each α there is $p \in \bar{D}_\alpha$ with $\mathrm{rlm}(p) = \{\alpha\}$.

(4) We have an equivalence relation, $\sim$, on P_{basic}, with $\leqslant 2^\mu$ classes, such that if $p \sim q$, then o.t. $\mathrm{rlm}(p)$ = o.t. $\mathrm{rlm}(q)$.

(5) There is a family, $\mathfrak{AM}$, of μ^+ symmetric functions Am_ν, $\nu < \mu^+$, such that each $\mathrm{dom}\,\mathrm{Am}_\nu$ is a symmetric subset of $P_{\text{basic}} \times P_{\text{basic}}$, such that if $(p, q) \in \mathrm{dom}\,\mathrm{Am}_\nu$, then $p \sim q$ and, letting $\pi : \mathrm{rlm}(p) \to \mathrm{rlm}(q)$ be the order isomorphism,

$\pi \mid (\mathrm{rlm}(p) \cap \mathrm{rlm}(q))$ is the identity and $p, q \leqslant \mathrm{Am}_\nu(p, q)$; if $(p, p) \in \mathrm{dom}\, Am_\nu$ then $\mathrm{Am}_\nu(p, p) = p$.

(6) If $p \in P_{\text{basic}}$ then, for sufficiently large ν, whenever $(a_\alpha : \alpha < \mu^{++})$ is a Δ-system from $[\lambda]^{\text{o.t. rlm}(p)}$, with $a_0 = \mathrm{rlm}(p)$, there is $0 < \alpha < \mu^{++}$ and q, with $\mathrm{rlm}(q) = a_\alpha$ and $(p, q) \in \mathrm{dom}\, \mathrm{Am}_\nu$ (in particular, this holds if for all $\alpha < \mu^{++}$, $a_\alpha = a_0$, i.e., for sufficiently large ν, $(p, p) \in \mathrm{dom}\, \mathrm{Am}_\nu$).

(7) If $(p, q), (p, q') \in (\mathrm{dom}\, \mathrm{Am}_\nu \cap \mathrm{dom}\, \mathrm{Am}_{\nu'})$ and $\mathrm{Am}_\nu(p, q) = \mathrm{Am}_{\nu'}(p, q')$, then $q = q'$ and $\nu = \nu'$.

(8) If $(p, q) \in \mathrm{dom}\, \mathrm{Am}_\nu$, $g : \mathrm{dom}\, g \to \lambda$ is order preserving with $\mathrm{rlm}(\mathrm{Am}_\nu(p, q)) \subseteq \mathrm{dom}\, g$, then there is unique $(p', q') \in \mathrm{dom}\, \mathrm{Am}_\nu$ with $p \sim p'$, $q \sim q'$, $\mathrm{rlm}(p') = g''\, \mathrm{rlm}(p)$, $\mathrm{rlm}(q') = g''\, \mathrm{rlm}(q)$; further, $\mathrm{rlm}(\mathrm{Am}_\nu(p', q')) = g''\, \mathrm{rlm}(\mathrm{Am}_\nu(p, q))$.

We then define the set of historical conditions based on the preceding data by: $\boldsymbol{p} \in P$ iff $\boldsymbol{p} = (p_\xi : \xi \leqslant \eta)$, where:

(a) for some α, $\mathrm{rlm}(p_0) = \{\alpha\}$,

(b) $\xi < \zeta \leqslant \eta \Rightarrow p_\xi \leqslant_{\text{basic}} p_\zeta$,

(c) for limit $\delta \in (0, \eta]$, p_δ is the least upper bound of $\{p_\xi : \xi < \delta\}$,

(d) if $\xi < \eta$, for some $\nu < \mu^+$ and some q (which will be unique by (7) above), $p_{\xi+1} = \mathrm{Am}_\nu(p, q)$.

As in 2.9, 3.4, we define q_ξ, π_ξ to be the unique pair witnessing the conclusion of (d). Then, as in 2.10, 3.5, we define the $\boldsymbol{p}^{\langle h \rangle}$ and, by (8), above, the remark of 2.10 carries over. We define $\boldsymbol{q} \leqslant \boldsymbol{p}$, as before, iff $\eta(\boldsymbol{q}) \leqslant \eta(\boldsymbol{p})$ and there is $h \in [\eta(\boldsymbol{p})]^{<\omega}$ such that $\boldsymbol{q} \subseteq \boldsymbol{p}^{\langle h \rangle}$, the definition of $\leqslant_{\text{pure}}$ being, as before, that $\boldsymbol{q} \subseteq \boldsymbol{p}$. The remark of 2.11 carries over. The 'blueprint analysis' is essentially as before: $\tau^{\boldsymbol{p}}$ is the sequence of length $\eta(\boldsymbol{p})$ whose ξth term is the triple $(t, \nu, (g, h))$, where t is the $\sim$-equivalence class of p_ξ, ν is such that $p_{\xi+1} = \mathrm{Am}_\nu(p_\xi, q_\xi)$, and $g : \text{o.t.}\, \mathrm{rlm}(p_\xi) \to \text{o.t.}\, \mathrm{rlm}(p_{\xi+1})$, $h : \mathrm{rlm}(q_\xi) \to \mathrm{rlm}(p_{\xi+1})$ are order preserving such that the ith member of $\mathrm{rlm}(p_\xi)$ (resp. $\mathrm{rlm}(q_\xi)$) is the $g(i)$th (resp. the $h(i)$th) member of $\mathrm{rlm}(p_{\xi+1})$. Then, $\tau^{\boldsymbol{p}}$ reconstructs $\boldsymbol{p}$ from the increasing enumeration of $\mathrm{rlm}(p_{\eta(\boldsymbol{p})})$. The amalgamation property is either trivial or false: given $\boldsymbol{p}$, $\boldsymbol{q}$ with $\tau^{\boldsymbol{p}} = \tau^{\boldsymbol{q}}$, then either there is $\nu < \mu^+$ with $(p_\eta, q_\eta) \in \mathrm{dom}\, \mathrm{Am}_\nu$ (where η is the common value of $\eta(\boldsymbol{p})$, $\eta(\boldsymbol{q})$), in which case $\boldsymbol{p}, \boldsymbol{q} \leqslant \boldsymbol{p}^\wedge \mathrm{Am}_\nu(p_\eta, q_\eta)$, or there may be no such ν. However, (6) guarantees that, if $2^\mu = \mu^+$, then the amalgamation will exist often enough that we recover the μ^{++}-c.c. The proof of μ^+-strategic closure is as before.

5. Proof of Theorem 4

We shall accomplish the stated objective in three stages. We first show that if there is a normal filter on κ^{++} which has a property formally stronger than precipitousness (of its dual ideal), but see [4], then $\kappa^{++}\kappa \to (\kappa^{++}\kappa, k)^2$, for all $k < \omega$. This is done in 5.1–5.4. We then analyze this proof and note that the hypothesis can be scaled down to a considerably weaker filter existence property on κ^{++}. This is done in 5.5. Finally, we show, in 5.6, that if a weakly compact cardinal, λ, is turned into τ^+ for some regular τ, $\kappa \leqslant \tau$, via the Lévy collapse $\mathrm{Coll}(\tau, \lambda)$, then in the extension, (λ, κ) has the weak filter existence property. As usual, in the interest of clarity, the result is stated for $\kappa^{++}\kappa$ but the arguments are presented in a much more general setting and prove correspondingly general statements.

5.1. Let $\omega < \kappa$, $\kappa^{++} \leqslant \lambda$, λ regular, and let $\alpha \leqslant \kappa^+$ (in the simplest setting, $\lambda = \kappa^{++}$). Let $\mathcal{F}$ be a filter on λ. Let $A_0 = \lambda$. $G(\lambda, \mathcal{F}, \alpha)$ is the game: EMPTY and NONEMPTY pick $\mathcal{F}$-positive sets A_ξ, $0 < \xi < \delta \leqslant \alpha$, generating a $\subseteq$-decreasing sequence, where EMPTY plays at odd stages, NONEMPTY at non-zero even stages, and NONEMPTY loses if for some non-zero even $\delta < \alpha$, NONEMPTY has no legal move (in which case δ must be a positive limit ordinal). $\mathrm{Pr}(\lambda, \mathcal{F}, \alpha)$ is the statement: $\mathcal{F}$ is a normal filter on λ with the property that NONEMPTY has a winning strategy in $G(\lambda, \mathcal{F}, \alpha)$.

5.2. We assume that $\kappa > \omega$, $\kappa^{++} \leqslant \lambda$ is regular, $\mathcal{F}$ is a normal filter on λ and that $\mathrm{Pr}(\lambda, \mathcal{F}, \kappa+1)$. We adopt the notation of 2.1, 2.2, but now for the ordinal $\lambda\kappa$, and now, for $X \subseteq \lambda\kappa$, we let $(X)^2$ denote $\{(x, y) : \gamma(x) \neq \gamma(y)$ and $\rho(x) < \rho(y)\}$ and for any set Y, we let $\langle Y\rangle^2 = Y^2 \setminus$the diagonal. Our choice of $A_{i,0}$, $i < \kappa$, below, as pairwise disjoint allows us to ignore edges (x, y) where $\rho(x) = \rho(y)$. The case where $\gamma(x) = \gamma(y)$ is dealt with below. Suppose that $f : [\lambda\kappa]^2 \to \{\text{red}, \text{green}\}$. Our goal, realized in 5.3, 5.4, is to show that either f has a homogeneous red set of type $\lambda\kappa$ or f has a homogeneous green set of type k. Let $(A_{i,0} : i < \kappa)$ be a partition of λ into $\mathcal{F}$-positive sets.

Using $\mathrm{Pr}(\lambda, \mathcal{F}, \kappa)$, we shall define $\subseteq$-decreasing sequences $(A_{i,\xi} : \xi < \kappa)$ of $\mathcal{F}$-positive sets, for all $i < \kappa$, by recursion on ξ, i.e., we shall define, at stage ξ, all the $A_{i,\xi}$, $i < \kappa$, having already defined all the $A_{i,\zeta}$, $\zeta < \xi$, $i < \kappa$. We shall have that $A_i \overset{\text{def}}{=} \bigcap \{A_{i,\xi} : \xi < \kappa\}$ will be $\mathcal{F}$-positive and that the A_i will have strong pseudo-homogeneity properties. These will allow us to define a (non-classical logic's version of a) derived coloring, $f' : \langle\kappa\rangle^2 \to \{\text{red}, \text{green}\}$, to which we shall, in a suitable sense, apply a weak Erdös–Dushnik–Miller theorem for κ: $\kappa \to (\kappa, k)^2$. The properties of the A_i will allow us to 'lift' the conclusion of this theorem to $\lambda\kappa$. Actually, there are two (very similar) 'applications of Erdös–Dushnik–Miller plus lifting': one to the 'coloring' of $[\kappa]^2$ obtained from f' by looking at increasing edges in $\langle\kappa\rangle^2$, the other to the 'coloring' obtained by looking at decreasing edges.

We need a preliminary application (actually κ-many simultaneous applications) of the weak E–D–M theorem, this time for λ and k, to handle the edges $((\alpha, i), (\beta, i))$, where $\alpha < \beta < \lambda$, i.e., to be able to regard f as defined on $(\lambda\kappa)^2$. So, by the E–D–M theorem, either there is a homogeneous green set of type k, or for all $\gamma < \kappa$, there is $X_\gamma \in [\lambda]^\lambda$ such that for $\alpha < \beta \in X_\gamma$, $f\{(\alpha, \gamma), (\beta, \gamma)\} =$

red. In the latter case, we may clearly suppose that each $X_\gamma = \lambda$, so vertical edges are red. This lets us regard f as defined on $(\lambda\kappa)^2$, which we do from now on.

$(A_{i,\xi}: \xi < \kappa)$ will be obtained as a run of the game where NONEMPTY plays by the winning strategy, starting from $A_{i,0}$; thus EMPTY plays the $A_{i,2\alpha+1}$, NONEMPTY the $A_{i,2\alpha}$.

To set up the plays, we let $((i_\alpha, j_\alpha): \alpha < \kappa)$ enumerate $\langle \kappa \rangle^2$. At stage $2\alpha + 1$, if $i \notin \{i_\alpha, j_\alpha\}$, EMPTY repeats the previous move, i.e., $A_{i,2\alpha+1} = A_{i,2\alpha}$. Otherwise, EMPTY proceeds as follows: he defines $A^l_{i_\alpha,2\alpha+1}$, $A^l_{j_\alpha,2\alpha+1}$, by induction on $l \leqslant 2$, starting from $A^0_{i_\alpha,2\alpha+1} = A_{i_\alpha,2\alpha}$, $A^0_{j_\alpha,2\alpha+1} = A_{j_\alpha,2\alpha}$. We identify green with 1, red with 2. So, if $l < 2$, having defined $A^l_{i_\alpha,2\alpha+1}$, $A^l_{j_\alpha,2\alpha+1}$, *if*: there are $\mathscr{F}$-positive $A'_{i_\alpha}, A'_{j_\alpha} \subseteq A^l_{i_\alpha,2\alpha+1}$, $A^l_{j_\alpha,2\alpha+1}$, respectively, with $f((\gamma, i_\alpha), (\delta, j_\alpha)) \neq l + 1$, for all $\gamma \in A'_{i_\alpha}$, and all $\delta \in A'_{i_\alpha} \setminus \gamma + 1$, *then* $(A^{l+1}_{i_\alpha,2\alpha+1}, A^{l+1}_{j_\alpha,2\alpha+1})$ is chosen to be such a pair. Otherwise $(A^{l+1}_{i_\alpha,2\alpha+1}, A^{l+1}_{j_\alpha,2\alpha+1}) = (A^l_{i_\alpha,2\alpha+1}, A^l_{j_\alpha,2\alpha+1})$. Finally, $A_{i_\alpha,2\alpha+1} = A^2_{i_\alpha,2\alpha+1}$, $A_{j_\alpha,2\alpha+1} = A^2_{j_\alpha,2\alpha+1}$. We then let $A_i = \bigcap\{A_{i,\xi}: \xi < \kappa\}$, for $i < \kappa$.

Now, let $l = 1, 2$ (recall our identification of colors with these integers), let $(i, j) \in \langle \kappa \rangle^2$. We claim that

(*) If there are $\mathscr{F}$-positive $A'_i, A'_j \subseteq A_i$, A_j, respectively, such that for all $\rho \in A'_i$ and all $\sigma \in A'_j \setminus \rho + 1$, $f((\rho, i), (\sigma, j)) \neq l$, *then* for all $\rho \in A_i$ and all $\sigma \in A_j \setminus \rho + 1$, $f((\rho, i), (\sigma, j)) \neq l$.

This is clear by construction. When the conclusion of (*) holds, we write $f'(i, j) \neq l$. When $f'(i, j) \neq$ one color, we write $f'(i, j) =$ the other color. Of course $f'(i, j) = l \Rightarrow \neg(f'(i, j) \neq l)$, but not conversely, and similarly for the contrapositives.

5.3. Proposition. *W.l.o.g., for* $i < j < \kappa$, $f'(i, j) \neq$ green.

Proof. It will suffice to show that either there is a homogeneous green set of type k or there is $I \in [\kappa]^\kappa$ such that for $i, j \in I$, if $i < j$ then $f'(i, j) \neq$ green (since then we can simply replace κ by I as the column set). So suppose that there is no such I. Then, by the weak E–D–M Theorem, there are $i_1 < \cdots < i_k < \kappa$ with $1 \leqslant m < n \leqslant k \Rightarrow \neg(f'(i_m, i_n) \neq \text{green})$. We need a:

Lemma. *If* $\neg(f'(i, j) \neq l)$, $D_i, D_j \subseteq A_i, A_j$, *respectively are* $\mathscr{F}$*-positive, then there is* $X \in \mathscr{F}$ *such that for all* $\alpha \in D_i \cap X$, $D'_j(\alpha)$ *is* $\mathscr{F}$*-positive, where*

$$D'_j(\alpha) = \{\beta \in D_j \setminus \alpha : f((\alpha, i), (\beta, j)) = l\}.$$

Proof of lemma. Suppose not. Let $S = \{\alpha \in D_i : D'_j(\alpha) \in \mathscr{F}^*\}$, so S is $\mathscr{F}$-positive. For $\alpha \in S$, let $X_\alpha \in \mathscr{F}$ be such that for all $\beta \in (D_j \cap X_\alpha) \setminus \alpha$, $f((\alpha, i), (\beta, j)) \neq l$. Let $X = \triangle_{\alpha \in S} X_\alpha$, so $X \in \mathscr{F}$.

Let $A'_i = S$, $A'_j = D_j \cap X$. Thus, A'_i, A'_j are $\mathscr{F}$-positive. To obtain the sought after contradiction, it suffices to show that if $\alpha \in S$, $\beta \in A'_j \setminus \alpha$, then $f((\alpha, i), (\beta, j)) \neq l$. But, since $A_i \cap A_j = \emptyset$ and $\alpha \leqslant \beta$, $\alpha < \beta$, so $\beta \in X_\alpha$, so $f((\alpha, i), (\beta, j)) \neq l$, and the lemma is proved.

Returning to the proof of the Proposition, we define by recursion on $1 \leqslant m \leqslant k$ and $m < n \leqslant k$, the following: $B_m \subseteq A_{i_m}$, and when $m < k$: $B_{m,n}$, $C_{m,n} \subseteq A_{i_n}$, A_{i_m}, respectively, $\alpha_m \in B_m \cap \bigcap \{C_{m,n} : m < n \leqslant k\}$ (when $m = k$, we get $\alpha_k \in B_k$), and we set $B'_{m,n} = \{\beta \in B_{m,n} \setminus \alpha_m : f((\alpha_m, i_m), (\beta, i_n)) = \text{green}\}$. We shall have B_m, $B_{m,n}$, $B'_{m,n}$ $\mathcal{F}$-positive, $C_{m,n} \in \mathcal{F}$.

Set $B_1 = A_{i_1}$, $B_{1,n} = A_{i_n}$, $1 < n \leqslant k$. If $m < k$, having defined B_m and the $B_{m,n}$, we let $C_{m,n}$ be as guaranteed by the lemma for $i = i_m$, $j = i_n$, $D_i = B_m$, $D_j = B_{m,n}$, we choose $\alpha_m \in B_m \cap \bigcap \{C_{m,n} : m < n \leqslant k\}$, and we let $B_{m+1} = B'_{m,m+1}$. If $m + 1 < k$, for $m + 1 < n \leqslant k$, we let $B_{m+1,n} = B'_{m,n}$. Finally, if $m = k$ we choose $\alpha_k \in B_k$.

We claim that $\{x_m : 1 \leqslant m \leqslant k\}$ is homogeneous green, where $x_m \stackrel{\text{def}}{=} (\alpha_m, i_m)$, which yields the desired conclusion. Of course $f(x_m, x_{m+1}) = \text{green}$, $m < k$, since by construction $\alpha_{m+1} \in B_{m+1} = B'_{m,m+1}$. On the other hand, if $1 \leqslant m' < m$, then, since $B_{m+1} = B'_{m,m+1} \subseteq B'_{m',m+1}$, $f(x_{m'}, x_{m+1}) = \text{green}$.

5.4. We now conclude the proof that $\Pr(\lambda, \mathcal{F}, \kappa)$ implies that $\lambda\kappa \to (\lambda\kappa, k)^2$, by showing that our arbitrary coloring f is not a counterexample. We consider two cases: first, that there is $T \in [\kappa]^\kappa$ such that for $i, j \in T$ and $i < j$, $f'(j, i) \neq \text{green}$. In this case, we shall show, using 5.3, that $\bigcup \{A_i \times \{i\} : i \in T\}$ is homogeneous red. In the second case, when there is no such T, as in 5.3, we shall obtain $i_1 < \cdots < i_k < \kappa$, such that for $1 \leqslant m < n \leqslant k$, $\neg(f'(i_n, i_m) \neq \text{green})$, whence, by imitating the proof in 5.3, we get a homogeneous green set of type k, but working from right to left, this time. Since this last argument is exactly as before, we consider only the first case. The key (and simple) observation is, once again, that for $i < j < \kappa$, $A_i \cap A_j = \emptyset$, so if $i, j \in T$, $\alpha \in A_i$, $\beta \in A_j$, then either $\alpha < \beta$, in which case $f((\alpha, i), (\beta, j)) = \text{red}$, by 5.3, or $\beta < \alpha$, in which case $f((\alpha, i), (\beta, j)) = \text{red}$ by hypothesis on T.

Remark. In fact, 5.1–5.4 actually show that if $\operatorname{card} \delta = \kappa$ and $\delta \to (\delta', k)^2$, $\Pr(\lambda, \mathcal{F}, \kappa)$ implies that $\lambda\delta \to (\lambda\delta', k)^2$, simply by obtaining A_i, as above, but for $i < \delta$.

5.5. We now consider a filter existence property which is weaker than $\Pr(\lambda, \mathcal{F}, \kappa)$ but which, we shall see, is still sufficient to lift the E–D–M theorem to $\lambda\kappa$.

Definition. $\mathrm{HF}(\lambda)$ is the closure of λ under the formation of pairs. If $\mathfrak{P} \subseteq \mathcal{P}(\mathrm{HF}(\lambda))$, $G^{\mathfrak{P}}(\lambda, \mathcal{F}, \alpha)$ is the variant of $G(\lambda, \mathcal{F}, \alpha)$ where both players are require to play moves from $\mathcal{P}(\lambda) \cap \mathfrak{P}$.

$Q(\lambda, \alpha)$ states: ($\lambda^{<\lambda} = \lambda$ and) whenever $\mathfrak{P} \in [\mathcal{P}(\mathrm{HF}(\lambda))]^{\leqslant\lambda}$ and F_i is a θ_i-ary function from $\mathcal{P}(\mathrm{HF}(\lambda))$ to $\mathcal{P}(\mathrm{HF}(\lambda))$, where $\theta_i < \lambda$, for $i < \lambda$, then there is $\mathfrak{P}^* \in [\mathcal{P}(\lambda)]^\lambda$, which includes $\mathfrak{P}$, is closed under the F_i and there is $\mathcal{F}$, a $(\mathfrak{P}^* \cap {}^\lambda\lambda)$-normal filter on $\mathfrak{P}^* \cap \mathcal{P}(\lambda)$ such that NONEMPTY has a winning strategy in $G^{\mathfrak{P}^*}(\lambda, \mathcal{F}, \alpha + 1)$.

Lemma. *$Q(\lambda, \kappa) \Rightarrow (\forall k < \omega)\, \lambda\kappa \to (\lambda\kappa, k)^2$; further the Remark of* 5.4 *carries over to this context.*

Proof. Let μ be a regular cardinal, sufficiently large so that the E–D–M theorem for κ and k and for λ and k holds in $(H(\mu), \in)$ and, in any case, much larger than 2^λ, let $f:[\lambda\kappa]^2 \to \{\text{red}, \text{green}\}$, let $\mathfrak{P} = \mathrm{HF}(\lambda) \cup \{\mathrm{HF}(\lambda)\}$. Let the F_i be a collection of 'modified' $L_{\lambda,\lambda}$-Skolem functions for $\mathcal{M} \overset{\text{def}}{=} (H(\mu), \in, \mathcal{P}(\mathrm{HF}(\lambda)), f)$ (modified, by restricting to sequences from $\mathcal{P}(\mathrm{HF}(\lambda))$ of the right lengths, and giving value $\emptyset$ to sequences where the real Skolem functions' values are undefined or are not subsets of $\mathrm{HF}(\lambda)$). Let $\mathfrak{P}^*$, $\mathcal{F}$ be as guaranteed by $Q(\lambda, \kappa)$ for $\mathfrak{P}$ and the F_i.

Let N be the closure of $\mathfrak{P}$ under the unmodified Skolem functions from which we obtained the F_i, so $(N, \in, \mathcal{P}(\mathrm{HF}(\lambda)), f) \prec_{L_{\lambda,\lambda}} \mathcal{M}$, and in particular, N is closed for sequences of length $<\lambda$. By coding, this means that such sequences of elements of $\mathfrak{P}^*$ can be regarded as lying in $\mathfrak{P}^*$; we shall abuse notation by doing so, thereby ignoring the coding. In order to carry out 5.1–5.4, we appeal to the winning strategy for $G^{\mathfrak{P}^*}(\lambda, \mathcal{F}, \kappa)$ to obtain a sequence $(A_i : i < \kappa)$ satisfying (*) of 5.2. Once this is accomplished, the remainder of the argument of 5.3–5.4 can be carried out in N, since we have built enough set theory, including the needed instances of the E–D–M theorem, into $\mathcal{M}$ and therefore into N.

The A_i are obtained essentially as before, in 5.2, by playing the same kind of runs of the game, *outside N, but using only N-sets* (i.e., $\mathfrak{P}^*$-sets). Thus, EMPTY's plays are decided upon by looking for the $\mathcal{F}$-positive $A^l_{i_\alpha}$, $A^l_{j_\alpha} \subseteq A^l_{i_\alpha,2\alpha+1}$, $A^l_{j_\alpha,2\alpha+1}$ *inside* $\mathfrak{P}^*$, and NONEMPY plays by the winning strategy for $G^{\mathfrak{P}^*}(\lambda, \mathcal{F}, \kappa)$ which is guaranteed by $Q(\lambda, \kappa)$. Now the runs of the game have been constructed outside N, but the individual moves are N-sets, so since N is closed for κ-sequences, in fact these runs lie inside N! Thus the A_i can be taken to lie in $\mathfrak{P}^*$. This completes the proof.

5.6. Lemma. *Suppose that in the ground model, V, λ is weakly compact, that $\omega < \kappa < \tau < \lambda$, κ is a cardinal, τ is regular and that V' is $V[G]$, where G is $\mathbb{P}$-generic$/V$ and $\mathbb{P}$ is the Lévy-collapse of λ to become τ^+. Then, in V', $Q(\lambda, \kappa)$.*

Proof. Clearly, in V', $\lambda^{<\lambda} = \lambda$. So, let $\mathring{\mathfrak{P}} = (\mathring{X}_i : i < \lambda)$, $\mathring{F} = (\mathring{F}_i : i < \lambda)$ be given. We seek $\mathring{\mathfrak{P}}^*$ as required. Let $(N, \in) \prec_{L_{\lambda,\lambda}} (H((2^{2^\lambda})^+, \in)$, $\operatorname{card} N = \lambda$, with $\mathring{\mathfrak{P}}, \mathring{F} \in N$. Now, in V, since λ is weakly compact, we can find a κ-complete ultrafilter, E, on $\mathcal{P}(\lambda) \cap N$ which is N-normal, i.e., if $f \in N$, f is regressive on an E-set, then f is constant on an E-set. One way of obtaining such an E is to let N' be such that $N \prec_{L_{\lambda,\lambda}} N'$, N' has a new λth ordinal, say λ^*, and take $E = \{A \in N : N' \models \text{“}\lambda^* \in A\text{”}\}$. Let $\boldsymbol{p} = (p^{\boldsymbol{p}}_\alpha : \alpha \in S^{\boldsymbol{p}}) = (p_\alpha : \alpha \in S) \in I$ iff $S \in E$ and $\boldsymbol{p}$ is a Δ-system of $\mathbb{P}$-conditions. For $\boldsymbol{p} \in I$, let $\mathring{S}(\boldsymbol{p}) = \{\alpha \in S^{\boldsymbol{p}} : p^{\boldsymbol{p}}_\alpha \in \mathring{G}^{\mathbb{P}}\}$; thus $S(\boldsymbol{p})$ is the term: “α if $p^{\boldsymbol{p}}_\alpha$”. We now prove three Facts. The forewarned reader (and the reader is now forewarned) will find it easy, while verifying the proof of Fact a and the first two paragraphs of Fact b, to verify that they can be carried out inside

N, since $(N, \in) \prec_{L_{\lambda,\lambda}} (H(\mu), \in)$. Let $\mathring{\mathscr{F}}$ be the (canonical term for the) filter generated in V' by E.

Fact a. *For $p \in I$, it is forced by the empty condition that "either $\mathring{S}(p) = \emptyset$ or $\mathring{S}(p)$ is $\mathring{\mathscr{F}}$-positive".*

Proof of Fact a. Let $p \in I$. Then, there is $y \in [\tau]^{<\tau}$ and for $\alpha \in S^p$, there is $x_\alpha \in [\lambda]^{<\tau}$ such that p^p_α is a function from $x_\alpha \times y$ such that for all $\xi \in x_\alpha$ and all $\zeta \in y$, $p^p_\alpha(\xi, \zeta) \in \xi$. Further, the $(x_\alpha : \alpha \in S^p)$ form a Δ-system, say with heart h, and for all $\{\alpha, \beta\} \in [S^p]^2$, $p^p_\alpha \mid (h \times y) = p^p_\beta \mid (h \times y)$. Now if $q \in P$ and q is incompatible with $p^p_\alpha \mid (h \times y)$ for some (all) $\alpha \in S^p$, then q forces that $\mathring{S}(p) = \emptyset$, while if q is compatible with $p^p_\alpha \mid (h \times y)$, then there is $\alpha_0 < \lambda$ such that for $\alpha \in S^p \setminus \alpha_0$, q does not force that $\alpha \notin \mathring{S}(p)$. Fact a is then clear.

Fact b. *It is forced by the empty condition that "if $\gamma < \tau$, $p^\beta \in I$ and $\mathring{S}(p^\beta)$ is $\mathring{\mathscr{F}}$-positive for $\beta < \gamma$, and for $\eta < \beta < \gamma$, $\mathring{S}(p^\beta) \subseteq \mathring{S}(p^\eta)$, then for some $q \in I$, $\mathring{S}(q)$ is $\mathring{\mathscr{F}}$-positive and for all $\beta < \gamma$, $\mathring{S}(q) \subseteq \mathring{S}(p^\beta)$".*

Proof of Fact b. Let $r \in P$ force the hypothesis of the statement in quotes; without loss of generality, r decides the value of γ and of each p^β. Let h^β be the heart of the βth Δ-system and let q^β be $p^{p^\beta}_\alpha \mid h^\beta$ for some (all) $\alpha \in S^{p^\beta}$. Then, by (the proof of) Fact a, for all $\beta < \gamma$, r forces that $q^\beta \in \mathring{G}$, so $\bigcup \{q^\beta : \beta < \gamma\} \subseteq r$.

Further, for $\eta < \beta < \gamma$, r forces that $\mathring{S}(p^\beta) \subseteq \mathring{S}(p^\eta)$. Again by the proof of Fact a, since for each $\beta < \gamma$, r forces that $\mathring{S}(p^\beta)$ is $\mathring{\mathscr{F}}$-positive, for each β there is $\alpha_0(\beta)$ such that for $\alpha \in S^{p^\beta} \setminus \alpha_0(\beta)$, r does not force that $\alpha \notin \mathring{S}(p^\beta)$. Taking the sup of the $\alpha_0(\beta)$, we get $\alpha^* < \lambda$ such that for all $\eta < \beta < \gamma$ and all $\alpha \in S^{p^\beta}$, $\alpha \in S^{p^\eta}$ (lest r force that $\alpha \notin \mathring{S}(p^\eta)$!), i.e., $(S^{p^\beta} \setminus \alpha^* : \beta < \gamma)$ is $\subseteq$-decreasing. Similarly, if $\eta < \beta < \gamma$, $\alpha \in S^{p^\beta} \setminus \alpha^*$, then, since r forces that $\mathring{S}(p^\beta) \subseteq \mathring{S}(p^\eta)$, $r \cup p^{p^\beta}_\alpha$ forces that $p^{p^\eta}_\alpha \in \mathring{G}$, i.e., $r \cup p^{p^\eta}_\alpha \subseteq r \cup p^{p^\beta}_\alpha$.

Note that for $\beta < \gamma$, $S^{p^\beta} \setminus \alpha^* \in E$, and that the sequence $(S^{p^\beta} \setminus \alpha^* : \beta < \gamma) \in N$; this is because N is closed for sequences of length $<\lambda$, which, in turn, is because $(N, \in) \prec_{L_{\lambda,\lambda}} (H(\mu), \in)$. Thus, since E is λ-complete, $S \in E$, where $S = \bigcap \{S^{p^\beta} \setminus \alpha^* : \beta < \gamma\}$. For $\alpha \in S$, let $q_\alpha = \bigcup \{p^{p^\beta}_\alpha : \beta < \gamma\}$. It is easy to verify that $q \stackrel{\text{def}}{=} (q_\alpha : \alpha \in S) \in I$, and that r is compatible with the common restriction of the q_α's to the heart of the Δ-system, $\{\operatorname{dom} q_\alpha : \alpha \in S\}$. Thus, by the proof of Fact a, again, r forces that q is as required by the conclusion of the statement in quotes. Note that if each p^β, $\beta < \gamma$ was taken to lie in N, then so would q, again since then we would have the sequence $(p^\beta : \beta < \gamma) \in N$. This completes the proof of Fact b.

Fact c. *For $\mathring{S} \in N$, it is forced by the empty condition that "if $\mathring{S}$ is positive for $\mathring{\mathscr{F}}$, then there is $p \in I \cap N$ such that $\mathring{S}(p)$ is $\mathring{\mathscr{F}}$-positive and $\mathring{S}(p) \subseteq \mathring{S}$".*

Proof of Fact c. Let $\mathring{S} \in N$ and let $r \in P$ force the hypothesis of the statement in quotes. Let $\alpha \in S^*$ iff r does not force that $\alpha \notin \mathring{S}$. Thus $S^* \in N$, and clearly we must have $S^* \in E$. For $\alpha \in S^*$, let r_α force $\alpha \in \mathring{S}$, where r_α is chosen to extend r.

Note that such a sequence $(r_\alpha : \alpha \in S^*)$ can be constructed inside N. Since E is N-normal, we can find $S^{**} \subseteq S^*$, $S^{**} \in E$, such that $p \overset{\text{def}}{=} (r_\alpha : \alpha \in S^{**}) \in I$. Then clearly r forces that p is as required (again using the proof of Fact a to see that r forces that $\mathring{S}$ is $\mathring{\mathscr{F}}$-positive), completing the proof of Fact c.

Finally, we let $\mathring{\mathfrak{P}}^* =$ (the canonical term for) $\mathscr{P}(\mathrm{HF}(\lambda)) \cap N[\mathring{G}]$ and we let $\mathring{\mathscr{F}}$ be as above. Clearly it is forced by the empty condition that "$\mathring{\mathfrak{P}} \subseteq \mathring{\mathfrak{P}}^*$ and that $\mathring{\mathfrak{P}}^*$ is closed under $\mathring{F}$", since $\mathring{\mathfrak{P}}, \mathring{F} \in N$. NONEMPTY's strategy in $G^{\mathring{\mathfrak{P}}^*}(\lambda, \mathscr{F}, \kappa+1)$ is, at even successor stages, to play an $\mathring{S}(p)$, where $p \in I \cap N$, as guaranteed for $\mathring{S} =$ EMPTY's previous play by Fact c, and, at positive limit ordinals, to play an $\mathring{S}(q)$, as guaranteed by Fact b. This completes the proof of the lemma, and with it, the proof of Theorem 4.

6. Proof of Theorem 5

In this section we introduce a ccc partial ordering of finite conditions which adds a counterexample to $\omega_2\omega \to (\omega_2\omega, 3)^2$ (and $\aleph_2$ reals). For uncountable regular κ, if $\lambda^+ < \kappa \Rightarrow 2^\lambda < \kappa$, the construction generalizes to $\kappa^{++}\kappa$, using conditions of size $<\kappa$. We adopt the conventions of Section 1 concerning notation (and abuse thereof) about $\omega_2\omega$.

6.1. Definition. $f \in P$ iff $f : \operatorname{dom} f \to \{\text{red}, \text{green}\}$, where $\operatorname{dom} f \in [[\omega_2\omega]^2]^{<\omega}$, f has no green triangle, and if $f\{x, y\} = \text{green}$, then

(*) $\{x, y\}$ is a descending edge, i.e., either $(\rho(x) < \rho(y)$ and $c(x) > c(y))$ or $(\rho(y) < \rho(x)$ and $c(y) > c(x))$.

If $f, g \in P$, then $f \leqslant g$ iff $f \subseteq g$. $\mathbb{P} = (P, \leqslant)$.

6.2. Lemma. *$\mathbb{P}$ has the ccc.*

Proof. If $(f_\alpha : \alpha < \omega_1) \in {}^{\omega_1}P$, there is an uncountable $A \subseteq \omega_1$, and a g such that $\{\operatorname{dom} f_\alpha : \alpha \in A\}$ is a Δ-system, say with root r, and for all $\alpha \in A$, $f_\alpha \restriction r = g$. But then, if $\alpha < \beta$, both from A, f_α, f_β can be amalgamated by making all new mixed edges red.

6.3. Clearly it is forced by the empty condition that the generic coloring has no green triangle. To complete the proof, we need a

Lemma. *It is forced by the empty condition that the generic coloring has no homogeneous red $\omega_2\omega$ set.*

Proof. Suppose, towards a contradiction, that f_0 forces that $\mathring{A}$ is a homogeneous red set of order type $\omega_2\omega$. If $\delta < \omega_2$ and $\operatorname{cf}\delta = \aleph_1$, choose, for $0 < n < \omega$: $p_n^\delta \in P$,

$\alpha_n^\delta \in [\delta, \omega_2)$, $k_n^\delta \in (n, \omega)$, such that, letting $p_0^\delta = f_0$, $(p_n^\delta : n < \omega)$ is increasing and for all $n < \omega$, p_{n+1}^δ forces that $(\alpha_n^\delta, k_n^\delta) \in \mathring{A}$. For such δ, $h(\delta) < \delta$, where $h(\delta) \stackrel{\text{def}}{=} \inf\{\beta \leqslant \delta : (\forall n) \operatorname{dom} p_n^\delta \cap \delta \times \omega \subseteq \beta \times \omega\}$, so w.l.o.g. assume that h is constant, say with value β_0. Now, thin out by a club C, so that for all $\delta \in C$, if $\delta' < \delta$, then for all n, $\operatorname{dom} p_n^{\delta'} \subseteq \delta \times \omega$.

For each δ, choose $0 < m(\delta) < \omega$, such that $\operatorname{dom} p_1^\delta \subseteq \omega_2 \times m(\delta)$; w.l.o.g. $m(\delta) = m$ for all δ. W.l.o.g., we may assume that we have g, a finite partial function from $\beta_0 \times \omega$ to {red, green}, such that for all δ, $p_{m+1}^\delta \restriction (\beta_0 \times \omega) = g$. But, then for $\delta' < \delta$, $p_1^\delta \restriction (\beta_0 \times \omega) \subseteq g$ (since $p_1^\delta \leqslant p_{m+1}^\delta$) and therefore, since $\operatorname{dom} p_{m+1}^{\delta'} \subseteq \delta$ and $(\operatorname{dom} p_1^\delta \setminus \beta_0 \times \omega) \subseteq [\delta, \omega_2) \times \omega$, p_0^δ and $p_{m+1}^{\delta'}$ are compatible (as functions). Moreover, they can be amalgamated in the following way: all new mixed edges are red *except* $\{(\alpha_0^\delta, k_0^\delta), (\alpha_m^{\delta'}, k_m^{\delta'})\}$. We must check that there is no green triangle. This, however, is clear, since any such triangle must have a vertex, z, in $\beta_0 \times \omega$, one green edge from $(\alpha_0^\delta, k_0^\delta)$ to z and another green edge from $(\alpha_m^\delta, k_m^\delta)$ to z; then, by the definition of m, $c(z) < m$, since $\operatorname{dom} p_1^\delta \subseteq \omega_2 \times m$, but by $(*)$ of the definition of P, $k_m^{\delta'} < m$, contradicting that $k_n^\delta > n$. Thus, this amalgamation, call it q, is a condition extending both p_1^δ and $p_{m+1}^{\delta'}$, and therefore forcing that $(\alpha_0^\delta, k_0^\delta)$, $(\alpha_m^{\delta'}, k_m^{\delta'})$ are both in the homogeneous red set $\mathring{A}$, while coloring $\{(\alpha_0^\delta, k_0^\delta), (\alpha_m^{\delta'}, k_m^{\delta'})\}$ green, contradiction!

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