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DETERMINING ABELIAN p-GROUPS FROM THEIR n-SOCLES

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Abstract: A necessary condition is given for a very wide separable p-group to be determined by its n-socle (the set of elements of order at most p^n). A group G is *very wide* if it has a direct summand of final rank $|G|$ which itself is a direct sum of cyclic groups. In the case, $n = 1$ (as Shelah has shown) this condition is sufficient but for $n > 1$ examples are given to show that this condition is not sufficient.

0. Introduction. Many important properties of a separable abelian p-group are determined by the socle of group viewed as a valuated vector space. To be precise let G be an abelian p-group (from now on when we refer to a group it will always be an abelian p-group and usually separable). Then $G[p^n] = \{g \in G\colon p^n g = 0\}$ is said to be the n-socle of G and $G[p]$ is called simply the socle. We view $G[p^n]$ as a valuated group by equipping it with the height function (i.e. $G[p^n]$ is viewed as having both a group structure and a function with domain $G[p^n]$ whose values are among the ordinals and ∞). In any group H the height of $h \in H$, written $\mathrm{ht}(h)$, is the least ordinal α so that $h \notin p^\alpha H$ if such an ordinal exists and ∞ otherwise. Here $p^0 H = H$, $p^{\alpha+1}H = p(p^\alpha H)$ and for limit ordinals δ, $p^\delta H = \cap_{\alpha<\delta} p^\alpha H$. A group is separable if the height of any element other than 0 is always finite. The length of a group is the supremum of the heights of its non-zero elements.

Whether or not a group G is a direct sum of cyclic groups (Σ-cyclic) is determined by $G[p]$ as a valuated vector space. More generally for any $n < \omega$, whether or not G is $p^{\omega+n}$-projective is determined by $G[p^{n+1}]$ as a valuated group. (See Nunke's characterization of $p^{\omega+n}$-projectives below.) In his solution to Crawley's problem for groups of cardinality $\aleph_1$, Megibben [6] was able to deal with valuated vector spaces rather than groups.

However the valuated vector space may not perfectly mirror the structure of the group. Consider the situation $H \subseteq G$ and $a \in G[p]$. Then looking at $H[p] \subseteq G[p]$ we can tell if the height of $a + H$ is $\geq \omega$ (in G/H). But it is not clear how to tell if the height is say ∞. The question we consider in this paper is

Which separable groups are determined by their n-socles?

For $n = 1$, there are two obvious classes of candidates; the Σ-cyclic groups and the torsion complete groups. Dugas and Vergohsen [2] showed assuming V = L that the only separable groups of cardinality $\aleph_1$ which are determined by their socles are the Σ-cyclic groups and the torsion complete groups. Shelah [12], independently, was able to prove this result for groups of arbitrary cardinality assuming GCH. That some hypothesis is needed can be seen from the independence results in [8]. There it is shown to be consistent that there is a group which is determined by its socle and which is neither Σ-cyclic nor torsion-complete.

One way to avoid the set theoretic difficulties is to restrict the class of groups. Shelah was able to prove the result with no set theoretic hypotheses for groups which are wide (as defined in his paper). A class of groups which is more easily understood are the very wide groups which are defined as follows.

Definition. A group G is *very wide* if $G = G_1 \oplus F$ where F is a Σ-cyclic group of final rank $|G|$.

In the current paper we will attempt to extend the results of [12] to determine which groups are determined by their n-socles. Let G be an abelian p-group. We say G is *determined by its n-socle* if for all H, $H[p^n] \cong G[p^n]$ as valuated groups implies $G \cong H$. The answer to our problem, suggested by analogy to the case $n = 1$, is that a very wide group G is determined by its n-socle if and only if G is $p^{\omega+n-1}$-projective. We will see that the methods of [12] generalize to arbitrary n. Indeed a secondary aim of this paper is to provide an account of the methods of [12] in the simple case where only very wide groups are considered. In the case $n = 1$, Shelah establishes that if a very wide separable group is determined by its socle then it is 0-pseudo-free (see the definitions below). Since the 0-pseudo-free groups coincide with the Σ-cyclic groups which are also the p^{ω}-projectives the theorem is proved. In the general case we will be able to show as well that if G is very wide, separable and determined by its n-socle then G is $(n-1)$-pseudo-free. As well the $p^{\omega+n-1}$-projectives are determined by their n-socles. However, as we will see in section 2, there are examples which show there are

$(n-1)$-pseudo-free groups which are not determined by their n-socles. Assuming $V = L$, we believe that there should be very wide separable groups which are determined by their n-socles which are not $p^{\omega+n-1}$-projective. However our attempt to construct such a group did not succeed. There does not seem to be a good characterization of the groups determined by their n-socles for $n > 1$.

Dugas and Vergohsen [2] also consider the problem of constructing two non-isomorphic groups with a predetermined n-socle (as before assuming $V = L$ and the groups have cardinality $\aleph_1$). Cutler [1] has also considered the problem of n-socles and he obtained results similar to those obtained by Dugas and Vergohsen.

1. Reconstructing Groups If a separable group A of cardinality $\aleph_1$ is not Σ-cyclic, then it is possible to understand why it is not Σ-cyclic. Namely we can write $A = \cup_{\alpha<\omega_1} A_\alpha$ where each A_α is countable, for each limit ordinal δ, $A_\delta = \cup_{\alpha<\delta} A_\alpha$, and $\{\delta\colon A_{\delta+1}/A_\delta$ has length $> \omega\}$ is stationary. The basic idea in [12] for the case where $n = 1$ is that the lengths of the quotients can be varied without changing the socle. In order to understand how separable groups of cardinality $> \aleph_1$ can fail to be Σ-cyclic, we need to use the notion of a λ-system. This notion is a generalization of being stationary and was introduced in [11] (see also [4]). We are aware that this definition is complicated at first reading. However we believe it is worth the effort to understand the definition.

Definition Suppose λ is a regular uncountable cardinal. A λ-*system* is a labeled subtree $\langle S, B_\eta, \lambda_\eta\colon \eta \in S\rangle$ of ${}^{<\omega}\lambda$ (i.e. S is a set of finite sequences of ordinals $< \lambda$ closed under subsequences) satisfying:

(1) $\lambda = \lambda_{\langle\rangle}$

(2) for all $\eta \in S$, λ_η is a regular cardinal

(3) $\eta \in S_f$(the terminal nodes of S) if and only if $\lambda_\eta = \omega$

(4) suppose η is not terminal then

(a) $E_\eta = \{i: \eta\hat{}\langle i\rangle \in S\}$ is stationary in λ_η

(b) for all $i \in E_\eta$, $\lambda_{\eta\hat{}\langle i\rangle} \leq |B_{\eta\hat{}\langle i\rangle}| < \lambda_\eta$

(c) if $i < j \in E_\eta$, then $B_{\eta\hat{}\langle i\rangle} \subsetneq B_{\eta\hat{}\langle j\rangle}$

(d) if $j \in E_\eta$ and j is a limit point of E_η, then $B_{\eta\hat{}\langle j\rangle} = \cup B_{\eta\hat{}\langle i\rangle}$ $(i < j, i \in E)$.

To simplify notation we let $\bar{B}_\eta = \cup_{\ell \leq \ell(\eta)} B_{\eta \restriction \ell}$ (where $\ell(\eta)$ denotes the length of η). Also if $\eta \in S$ and j is a limit point of E_η then we say $\eta\hat{}\langle j\rangle \in S_c$ and we can define $B_{\eta\hat{}\langle j\rangle} = \cup B_{\eta\hat{}\langle i\rangle}$ $(i < j, i \in E)$. Notice that any sequence $\eta_0, \eta_1, \ldots$ of elements of S such that η_{n+1} extends η_n must be finite, since the associated sequence of cardinals $\lambda_{\eta_0}, \lambda_{\eta_1}, \ldots$ is strictly decreasing.

If G is a group then a *λ-system of subgroups* for G is a sequence $\langle S, B_\eta, A_\nu, \lambda_\eta: \eta \in S, \nu \in S_f\rangle$ such that: $\langle S, B_\eta, \lambda_\eta: \eta \in S\rangle$ is a λ-system; if $i < j \in E_\eta$ and $\eta\hat{}\langle i\rangle = \nu \in S_f$ then $A_\nu \subsetneq B_{\eta\hat{}\langle j\rangle}$; for all $\nu \in S_f$, $|A_\nu| \leq \aleph_0$; for all η, A_η (if it exists) and B_η are subgroups of G; and for all $\eta\hat{}\langle i\rangle, \eta\hat{}\langle j\rangle \in S$, if $i < j$ and ν extends $\eta\hat{}\langle i\rangle$ then $B_\nu \subsetneq B_{\eta\hat{}\langle j\rangle}$. We call such a system a *λ-system of pure subgroups* if in addition all the subgroups are pure.

With these definitions in hand we can define one of the key concepts of this paper.

Definition A separable group G is *n-pseudo-free* if and only if there does not exist a λ-system $\langle S, B_\eta, A_\nu, \lambda_\eta: \eta \in S, \nu \in S_f\rangle$ of pure subgroups of G such that for all $\eta \in S_f$ the length of $A_\eta/\langle \bar{B}_\eta\rangle \geq \omega + n + 1$. (Notice the A_ν's play no real role in this definition, but it will be convenient to refer to them later on. It would be enough to have the length of $G/\langle \bar{B}_\eta\rangle \geq \omega + n + 1$, for all $\eta \in S_f$.)

By Shelah's singular compactness theorem [10] and the analysis in [11] a separable group G is Σ-cyclic if and only if G is 0-pseudo-free.

For convenience we will often assume that our groups are subgroups of some large torsion complete group $\bar{B}$. The effect of this is that if G is a pure subgroup of $\bar{B}$ then the height of an element $a \in G[p^n]$ in G is the same as its height calculated in $\bar{B}$. In particular if we can show that G, H are pure subgroups of $\bar{B}$ and $G[p^n] = H[p^n]$ then $G[p^n] \cong H[p^n]$ as valuated groups where the height functions are calculated in G and H respectively.

In order to show that only $(n-1)$-pseudo-free groups are determined by their n-socles we will prove results on realizing valuated groups as n-socles.

Theorem 1.1 *If G is very wide, pure subgroup of $\bar{B}$ then for all $n > 1$ there is a pure subgroup H of $\bar{B}$ so that H is n-pseudo-free and $H[p^n] = G[p^n]$.*

If fact we can do better. We can construct the group H so that it is $p^{\omega+n}$-projective. The next series of results prove this stronger theorem. Rather than defining what it means for a group to be $p^{\omega+n}$-projective we will give a characterization of these groups due to Nunke.

Theorem 1.2 [9] *A group H is $p^{\omega+n}$-projective if and only if there is a subgroup $A \subseteq H[p^n]$ so that H/A is Σ-cyclic.*

Proposition 1.3 *A separable $p^{\omega+n}$-projective group is n-pseudo-free.*

PROOF: Suppose $A \subseteq H[p^n]$ is such that H/A is Σ-cyclic. Choose a set X so that $\{x + A: x \in X\}$ is an independent set of generators for H/A. Suppose now that $\langle S, B_\eta, A_\nu, \lambda_\eta: \eta \in S, \nu \in S_f\rangle$ is a λ-system of pure subgroups of H which witnesses H is not n-pseudo-free. Let $C_0 = \bigcup_{\alpha \in E_{\langle\rangle}} B_{\langle\alpha\rangle}$. Since $E_{\langle\rangle}$ is stationary there is $\langle\alpha\rangle \in S$ and $X_0 \subseteq X$ so that $B_{\langle\alpha\rangle} + A = (\langle x: x \in X_0\rangle + A) \cap (C_0 + A)$. Proceeding by induction we can find $\eta\hat{}\langle i\rangle \in S_f$ so that if we let $C = \bigcup_{j \in E_\eta} B_{\eta\hat{}\langle j\rangle}$ there is $X_1 \subseteq X$ such that $\langle \bar{B}_{\eta\hat{}\langle i\rangle}\rangle + A = \langle \bar{B}_\eta \cup C \cup A\rangle \cap \langle X_1 \cup A\rangle$. Since the length of $\langle \bar{B}_\eta \cup C\rangle / \langle \bar{B}_{\eta\hat{}\langle i\rangle}\rangle$ is at least $\omega + n + 1$ and $A \subseteq H[p^n]$, $\langle \bar{B}_\eta \cup C\rangle + A / \langle \bar{B}_{\eta\hat{}\langle i\rangle}\rangle + A$ is not Σ-cyclic. But $(\langle \bar{B}_\eta \cup C \cup A\rangle + \langle X_1 \cup A\rangle) / \langle X_1 \cup A\rangle$ is separable since it is isomorphic to a subgroup of $(\langle X \cup A\rangle/A) / (\langle X_1 \cup A\rangle/A)$ which is Σ-cyclic. Further by the second isomorphism theorem the first group is isomorphic to $\langle \bar{B}_\eta \cup C \cup A\rangle / (\langle \bar{B}_\eta \cup C \cup A\rangle \cap \langle X_1 \cup A\rangle) = \langle \bar{B}_\eta \cup C\rangle + A / \langle \bar{B}_{\eta\hat{}\langle i\rangle}\rangle + A$. This is a contradiction. □

Proposition 1.4 *Suppose G is very wide pure subgroup of $\bar{B}$ then there is a $p^{\omega+m}$-projective group H which is a pure subgroup of $\bar{B}$ such that $H[p^m] = G[p^m]$.*

PROOF. Let G_1 and F be as in the definition of very wide. Choose $\{t_i^n: n < \omega$ and $i \in I_n\}$ a maximal pure independent subset of G_1, where the order of t_i^n is p^{n+1}. Next choose $\{x_\alpha: \alpha < \kappa\} \subseteq G_1[p^m]$ so that $\{p^{n-m+1}t_i^n: n < \omega$ and $i \in I_n\} \cup \{x_\alpha: \alpha < \kappa\}$ is a set of free generators for $G_1[p^m]$. (By convention we can assume that $p^{n-m+1}t_i^n$ denotes t_i^n when $m > n$.) Choose $a_{i\alpha}^n$ so that for each α, $x_\alpha = \sum_n \sum_i p^{n-m+1} a_{i\alpha}^n t_i^n$. For $k < \omega$, let $x_\alpha^k = \sum_{k<n} \sum_i p^{n-k} a_{i\alpha}^n t_i^n$. (The x_α^k are formed by removing some terms from the sum and then formally dividing by p.) If we let G_2 be the subgroup of $\bar{B}$ generated by the t_i^n's and the x_α^k's, then $G_2[p^m] = G_1[p^m]$ and G_2 is a pure subgroup of $\bar{B}$. So we can work with $G_2 \oplus F$ rather than $G_1 \oplus F$. Let F be generated independently by $\{s_j^n: n < \omega$ and $j \in J_n\}$, where the order of s_j^n is p^{n+1}. For simplicity we will assume each $J_n \supseteq \kappa$. For $k \geq m$, let $y_\alpha^k = x_\alpha^k + \sum_{n>k} p^{n+m-k+1} s_\alpha^n$. Finally let H be the group generated by the y_i^n's, s_j^n's and y_α^k's. Again it is not hard to see that $H[p^m] = G[p^m]$ and that H is a pure subgroup of $\bar{B}$. Finally $H / G_2[p^m]$ is Σ-cyclic. In fact it is independently generated by the images of the t_i^n's ($n > m$), y_α^k's and s_j^n's ($j \notin \kappa$ or $n < m$.) ◫

We have now completed the proof of Theorem 1.1.

Theorem 1.5 *If G is a pure subgroup of $\bar{B}$ and G is not $(m-1)$-pseudo-free, then there is H a pure subgroup of $\bar{B}$ such that $H[p^m] = G[p^m]$ and H is not m-pseudo-free.*

PROOF. Let $\langle S, B_\eta, A_\nu, \lambda_\eta: \eta \in S, \nu \in S_f\rangle$ witness that G is not $(m-1)$-pseudo-free. Let $C = \cup_{\alpha \in E_{\langle\rangle}} B_{\langle\alpha\rangle}$. Inductively choose $X = \{t_i^n: n < \omega, i \in$

$I_n\}$ a maximal pure independent subset of C where the order of t_i^n is p^{n+1}. Further we can assume that for all $\eta \in S_f$ $X \cap \bigcup_{k \le \ell(\eta)} B_{\eta \upharpoonright k}$ is a maximal pure independent subset of $\sum_{k \le \ell(\eta)} B_{\eta \upharpoonright k}$. For each $\eta \in S_f$ choose $x_\eta \in A_\eta[p^m]$ so that, modulo $\sum_{k \le \ell(\eta)} B_{\eta \upharpoonright k}$, x_η has infinite height and the order of x_η is p^m. Note this means that x_η is in the closure of $X \cap \bigcup_{k \le \ell(\eta)} B_{\eta \upharpoonright k}$ and x_η witnesses that the length of $A_\eta + \sum_{k \le \ell(\eta)} B_{\eta \upharpoonright k} / \sum_{k \le \ell(\eta)} B_{\eta \upharpoonright k}$ is at least $\omega + m$. Further $\{p^n t_i^n : n < \omega \text{ and } i \in I_n\} \cup \{x_\eta : \eta \in S_f\}$ is pure independent in $C[p^m]$. So we can extend this set by adding on $\{y_\alpha : \alpha < \kappa\}$ to get an independent set of generators for $C[p^m]$. Choose integers $a_{i\eta}^n$ and $b_{i\alpha}^n$ so that $x_\eta = \sum_n \sum_i p^{n-m+1} a_{i\eta}^n t_i^n$ and $y_\alpha = \sum_n \sum_i p^{n-m+1} b_{i\alpha}^n t_i^n$. Now for all $k \ge m$ let $x_\eta^k = \sum_{n \ge k} \sum_i p^{n-k+1} a_{i\eta}^n t_i^n$ and $y_\alpha^k = \sum_{n \ge k} \sum_i p^{n-k+1} b_{i\alpha}^n t_i^n$. Now let H_1 be the subgroup of $\bar{B}$ generated by $X \cup \{x_\eta^k : k < \omega \text{ and } \eta \in S_f\} \cup \{y_\alpha^k : k < \omega \text{ and } \alpha < \kappa\}$. It is not hard to show that H_1 is a pure subgroup of $\bar{B}$ and that $H_1[p^m] = C[p^m]$.

In the construction of H_1, we have built a λ-system which witnesses that H_1 and any group which contains H_1 as a pure subgroup is not m-pseudo-free. (In fact it is not k-pseudo-free for any k.) The next lemma shows that there is a pure subgroup $H \supseteq H_1$ so that $H[p^m] = G[p^m]$. So modulo this lemma we are finished the proof. □

Lemma 1.6 *Suppose $A \subseteq G$ is a pure subgroup and G is a pure subgroup of $\bar{B}$ and H_1 is a pure subgroup of $\bar{B}$ such that $H_1[p^m] = A[p^m]$. Then there is $H \supseteq H_1$, a pure subgroup of $\bar{B}$ so that $H[p^m] = G[p^m]$.*

PROOF. Choose $X = \{t_i^n : n < \omega$ and $i \in I_n\}$ a maximal pure independent subset of A and $Y = \{s_j^n : n < \omega$ and $j \in J_n\}$ so that $X \cup Y$ is a maximal pure independent subset of G. Choose next $\{x_\alpha : \alpha < \kappa\}$ so that $\{x_\alpha : \alpha < \kappa\} \cup \{p^{n-m+1} s_j^n : n < \omega$ and $j \in J_n\}$ is an independent set of generators for $G[p^m]$ mod $A[p^m]$. Now for each $\alpha < \kappa$ choose integers $a_{i\alpha}^n$, $b_{j\alpha}^n$ so that $x_\alpha = \sum_n \sum_i p^{n-m+1} a_{i\alpha}^n t_i^n + \sum_n \sum_j p^{n-m+1} b_{j\alpha}^n s_i^n$. For $k \geq m$ let $x_\alpha^k = \sum_n \sum_i p^{n-k+1} a_{i\alpha}^n t_i^n + \sum_n \sum_j p^{n-k+1} b_{j\alpha}^n s_i^n$. Then we can let H be the subgroup of $\bar{B}$ generated by H_1 together with $\{s_j^n : n < \omega$ and $j \in J_n\} \cup \{x_\alpha^k : \alpha < \kappa$ and $m \leq k\}$. □

Putting Theorems 1.1 and 1.5 together we have proved our first main theorem.

Theorem 1.7 *If for some $1 \leq n < \omega$ a very wide separable p-group G is determined by its n-socle then G is $(n-1)$-pseudo-free.*

Corollary 1.8 [12] *A very wide separable group is determined by its socle if and only if it is Σ-cyclic.*

Theorem 1.7 appears (in different terminology) in [2] for the special case of groups of cardinality $\aleph_1$ and under the hypothesis that V = L, but without the restriction that the group be very wide. It is possible to prove a weak converse to Theorem 1.7, by adding the stronger assumption that the group is $p^{\omega+m}$-projective. However we will see in section 2, that the converse of 1.7 is not true and so (for separable groups) m-pseudo-free does not imply $p^{\omega+m}$-projective. We include a proof of the following theorem although it is essentially due to Fuchs (it also appears in [1]).

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Theorem 1.9 [5] *If G is $p^{\omega+m}$-projective, then G is determined by $G[p^{m+1}]$. In fact if $G[p^{m+1}] = H[p^{m+1}]$ and G, H are pure subgroups of some torsion complete group, then the isomorphism can be taken over $G[p]$.*

PROOF. Choose $P \subseteq G[p^m]$ so that G/P is Σ-cyclic. Next choose $\{x_\alpha: \alpha < \kappa\}$ so that $\{x_\alpha + P: \alpha < \kappa\}$ is a set of independent generators of $(G/P)[p]$ (as a valuated group) and for all n, p^n divides x_α if and only if p^n divides $x_\alpha + P$. Note that each $x_\alpha \in G[p^{m+1}]$. In G, choose y_α^n so that $p^n y_\alpha^n = x_\alpha$ and p^n is the maximal power of p which divides x_α. Similarly in H choose z_α^n so that $p^n z_\alpha^n = x_\alpha$. We attempt to define a map $\varphi\colon G \longrightarrow H$ by setting $\varphi \restriction P = 1_P$ and $\varphi(y_\alpha^n) = z_\alpha^n$. The only possible difficulty with such a definition is to show that the map is well defined. More exactly for integers a_α, b_α ($\alpha < \kappa$ all but finitely many 0) and elements $u, v \in P$, if $\sum_{\alpha<\kappa} a_\alpha y_\alpha^n + u = \sum_{\alpha<\kappa} b_\alpha y_\alpha^n + v$ then $\sum_{\alpha<\kappa} a_\alpha z_\alpha^n + u = \sum_{\alpha<\kappa} b_\alpha z_\alpha^n + v$. Since $\sum_{\alpha<\kappa} (a_\alpha - b_\alpha) y_\alpha^n + (u - v) = 0$, it also equals $0 \mod P$. Hence for all α, p^{n+1} divides $(a_\alpha - b_\alpha)$. Write $a_\alpha - b_\alpha$ as $p^n k_n$. So $\sum_{\alpha<\kappa} (a_\alpha - b_\alpha) y_\alpha^n + (u - v) = \sum_{\alpha<\kappa} k_\alpha p^n y_\alpha^n + (u - v) = \sum_{\alpha<\kappa} k_\alpha x_\alpha + (u - v) = \sum_{\alpha<\kappa} k_\alpha p^n z_\alpha^n + (u - v)$. Hence φ is well defined. Since $G[p] \subseteq \langle P \cup \{x_\alpha: \alpha < \kappa\}\rangle$, φ is the identity on $G[p]$. ◼

Note that in the above theorem if we want φ to be the identity on $G[p^k]$ we will need G to be $p^{\omega+m}$-projective and that $G[p^{m+k}] = H[p^{m+k}]$. This restriction on how closely φ can resemble the identity is real.

Definition We say that G is *strongly determined* by $G[p^n]$ if for any H and isomorphism φ: $G[p^n] \longrightarrow H[p^n]$ (as valuated groups), there is an isomorphism φ: $G \longrightarrow H$ extending φ.

Theorem 1.10 *If G is a very wide separable group which is strongly determined by $G[p^k]$, then G is Σ-cyclic.*

PROOF. We will work inside a torsion complete group $\bar{B}$. Consider any very wide G, a pure subgroup of $\bar{B}$, which is not Σ-cyclic. Write $G = G_1 \oplus \oplus \langle s^n_\alpha$: $n < \omega$ and $\alpha < |G|\rangle$. (Here the order of s^n_α is p^{n+1}. We are not really guaranteed a large set of s^n_α for all α, only a large set for infinitely many n. The assumption that we we have s^n_α, for all n is only made to simplify the notation.)

Let A be a pure subgroup of G_1 such that A is not Σ-cyclic and $|A|$ is minimal. Let $\lambda = |A|$. We will now describe a particular filtration of A, relative to an ordered index set which is order isomorphic to λ. We could simplify the construction by the use of known facts about λ-systems. Although our proof will not use λ-systems, we will in part be repeating the construction of a particular λ-system from a non-Σ-cyclic groups. By the singular compactness theorem [10], λ is regular. Rather than indexing with ordinals $< \lambda$ we will index with finite sequences of ordinals which are to be lexicographically ordered. We will define the groups and the sequences of ordinals by induction on the length of the sequences. For $n = 1$, write $A = \cup A_\beta$ where (i) $A_0 = \{0\}$ and each A_β is pure, and (ii) if A/A_β is not λ-Σ-cyclic then $A_{\beta+1}/A_\beta$ is not Σ-cyclic and $A_{\beta+1}/A_\beta$ is of the minimum such cardinality, otherwise $|A_{\beta+1}/A_\beta| = \aleph_0$. Note that for all β, $|A_\beta| <$

$|A|$. As well, by the singular compactness theorem, for all β, $|A_{\beta+1}/A_\beta|$ is regular. Let $A_{\langle\beta\rangle} = A_\beta$. Let S_n denote the set of η for which A_η has been defined. Let η^{+n} be the successor (in the lexicographic order) of η inside S_n. Suppose now $\eta \in S_n$ is given. If $|A_{\eta^{+n}}/A_\eta| = \aleph_0$, then η has no extension in S_{n+1}. Otherwise, we can choose $A_{\eta\hat{}\langle\beta\rangle}$ $(\beta < |A_{\eta^{+n}}/A_\eta|)$ so that $A_{\eta\hat{}\langle 0\rangle} = A_\eta$ and the sequence $(A_{\eta\hat{}\langle\beta\rangle}/A_\eta \colon \beta < |A_{\eta^{+n}}/A_\eta|)$ satisfies properties (i) and (ii) with respect to $A_{\eta^{+n}}/A_\eta$. Notice that that $S = \cup\, S_n$ has no infinite branches, so it is well ordered by the lexicographic order (and order isomorphic to λ). Let η^+ denote the successor in S of η.

Inductively choose a basis (i.e. a maximal pure independent set) $Y = \{t_i^n \colon n < \omega \text{ and } i \in I_n\}$ where the order of t_i^n is p^{n+1} and for all η, $Y \cap A_\eta$ is a basis for A_η. Next choose inductively choose $X = \{x_\alpha \colon \alpha < \kappa\}$ so $X \cup \{p^{n+1-k}t_i^n \colon n < \omega$ and $i \in I_n\}$ is an independent set of generators for $A[p^k]$. (If $n < k$, then we declare $p^{n+1-k}t_i^n$ to be t_i^n.) Further $(X \cup \{p^{n+1-k}t_i^n \colon n < \omega$ and $i \in I_n\}) \cap A_\eta$ is an independent set of generators for $A_\eta[p^k]$. For convenience, we can assume that if A_{η^+}/A_η is not Σ-cyclic then there is $x_{\eta^+} \in A_{\eta^+} \setminus A_\eta$ so that $p^k x_{\eta^+}$ is in the closure of A_η and $x_{\eta^+} \in X$. For each α, write $x_\alpha = \sum_{n\geq k}\sum_{i\in I_n} p^{n+1-k}a_{i\alpha}^n t_i^n$. Now we will define the two groups. For all $\alpha < \kappa$ and $k \leq m$, define

$$y_\alpha^m = \sum_{n\geq m}\sum_{i\in I_n} p^{n+1-m}a_{i\alpha}^n t_i^n + \sum_{n>m} p^{n+k+1-m}s_\alpha^n$$

and define

$$z_\alpha^m = \sum_{n\geq m}\sum_{i\in I_n} p^{n+1-m}a_{i\alpha}^n t_i^n.$$

Note that $z_\alpha^k = x_\alpha = y_\alpha^k$. Let H_1 be the group generated by the t's, s's and

y's. Let H_2 be the group generated by the t's, s's and z's. It is easy to check that $H_1[p^k] = G[p^k] = H_2[p^k]$ and that all these groups are pure subgroups of $\bar{B}$.

Suppose now that $\varphi\colon H_1 \longrightarrow H_2$ is an isomorphism which is the identity on $H_1[p^k]$. Since A is not Σ-cyclic, we can find an η so that A_{η^+}/A_η is not Σ-cyclic and $C \subseteq H_1$ such that: $C \cap A_{\eta^+} = A_\eta$; for all $t_i^n \in A_\eta$, $t_i^n \in \varphi(C)$; and for all $x_\alpha \in A_{\eta^+} \setminus A_\eta$ and $n < \omega$, $s_\alpha^n \notin C$. Choose α so that $x_{\eta^+} = x_\alpha$. Consider $\varphi(y_\alpha^{k+1})$. This element is $z_\alpha^{k+1} + u + v$, where $u \in \hat{A}[p]$ and v is a finite linear combination of elements of the form $p^n s_\beta^n$. (Here $\hat{A}[p]$ is the closure in $H[p]$ of $A[p]$.) Note that $\varphi(\langle C \cup A[p^k] \cup \{v\}\rangle) = \langle \varphi(C) \cup A[p^k] \cup \{v\}\rangle$. Now y_α^{k+1} has finite height modulo $\langle C \cup A[p^k] \cup \{v\}\rangle$ but $\varphi(y_\alpha^{k+1})$ has infinite height modulo $\langle \varphi(C) \cup A[p^k] \cup \{v\}\rangle$. □

2. Examples To complete the paper we give an example which shows that the (very wide) groups which are determined by their n-socles is a smaller class than the $(n-1)$-pseudo-free groups.

Theorem 2.1 *For all $n \geq 1$ there exists an n-pseudo-free group G which is not determined by $G[p^{n+1}]$.*

PROOF: For simplicity we will consider only the case where $n = 1$. The proof can be given using either $\diamondsuit$ or one of the Black Box principles. We will give the proof using $\diamondsuit$. The proof using the Black Box is a straightforward modification (see [4] for details on the Black Box and how to modify $\diamondsuit$ arguments). Of course the proof which uses $\diamondsuit$ is not a proof from the usual axioms of set theory, whereas the Black Box is provable in ZFC. As we will point out later the $\diamondsuit$ construction builds a group with additional properties. It is not provable in ZFC that a group with the additional properties exist. For the rest of the proof assume $\diamondsuit$.

The general strategy of the proof is to construct a group A so that $A[p^2]$ is rigid in the appropriate sense and then to use our control of $A[p^2]$ to complete the proof.

We will inductively define for $\alpha < \omega_1$, A_α and B_α so that $A_\alpha[p^2] = B_\alpha[p^2]$. Further if we let $A = \cup A_\alpha$ and $B = \cup B_\alpha$ then there is no embedding of A into $B \oplus \bigoplus_{n<\omega} \bigoplus_{\beta<\omega_1} \langle s^n_\beta \rangle = C$ where for all n and β the order of s^n_β is p^{n+1}. Before beginning the construction we will commit ourselves to having $\{x^n_i\colon n < \omega$ and $i < \alpha\}$ as a maximal pure independent subset of both A_α and B_α. Further we will construct the groups so that for all α, $A_{\alpha+1}$ and $B_{\alpha+1}$ are direct summands of A and B respectively. Because of the constraints on the construction, we know the torsion-completion of C (although we do not C itself). Let D denote the torsion-completion of C. The whole construction will take place inside D.

Using $\diamondsuit$, we have a sequence of maps $\{g_\alpha\colon \alpha < \omega_1\}$ where each $g_\alpha\colon \{x^n_i\colon n < \omega$ and $i < \alpha\} \longrightarrow D$.

We view any element of D as (uniquely) a countable sum of the x^n_α and the s^n_α. For $d \in D$, let supp d denote the set of x^n_α and s^n_α with a non-zero coefficient in the expression for d. Further we say that d has α as an *accumulation point* if and only if for all $n < \omega$ and $\beta < \alpha$ there is $n < m$ and $\beta < \gamma \leq \alpha$ so that either x^m_γ or s^m_γ is in supp d. We will construct A so that it satisfies the following condition.

(*) For all ordinals α there is an element $y \in A[p]$ so that for all elements $x \in A[p]$ there is some $k \in \mathbb{Z}$ so that $x + ky$ does not have α as an accumulation point. Further this element y is a member of $A_{\alpha+1}$.

The only interesting case of the construction occurs when δ is a limit ordinal and g_δ induces an embedding of A_δ into $B_\delta \oplus \underset{\alpha<\delta}{\oplus} \underset{n}{\oplus} \langle s_\alpha^n \rangle$. In the other cases, let $A_{\alpha+1} = A_\alpha \oplus \underset{n}{\oplus} \langle x_\alpha^n \rangle$ and $B_{\alpha+1} = B_\alpha \oplus \underset{n}{\oplus} \langle x_\alpha^n \rangle$ and at limit ordinals take unions. The main case breaks into three subcases.

Subcase 1: For all $n < \omega$, $\alpha < \delta$ and $k \in \mathbb{Z}$ there is $m \geq n$ and $\alpha < \beta < \delta$ such that $g_\delta(p^m x_\beta^m) \neq kp^m x_\beta^m$. Choose an increasing sequence $\{\beta(n)\colon n < \omega\}$ with limit δ and an increasing sequence $\{m(n)\colon n < \omega\}$ of natural numbers so that for all $k \in \mathbb{Z}$ $\{n\colon g_\delta(p^{m(n)} x_{\beta(n)}^{m(n)}) \neq kp^{m(n)} x_{\beta(n)}^{m(n)}\}$ is infinite. As well we can assume that $\beta(n+1)$ is greater than any α such that x_α^k or s_α^k is in supp $g_\delta(x_{\beta(n)}^{m(n)})$ for some k. As a consequence of this assumption, if $x_{\beta(n)}^{m(n)}$ is in supp $g_\delta(p^k x_{\beta(k)}^{m(k)})$ then $k = n$. (The assumption bars $x_{\beta(n)}^{m(n)}$ from being in supp $g_\delta(p^k x_{\beta(k)}^{m(k)})$, for $n > k$. For $n < k$, $x_{\beta(n)}^{m(n)}$ cannot be in supp $g_\delta(p^k x_{\beta(k)}^{m(k)})$ since g_δ does not decrease heights.) By taking a subsequence, we can assume that for all n the height of $g_\delta(p^{m(n)} x_{\beta(n)}^{m(n)})$ is less than the height of $g_\delta(p^{m(n+1)} x_{\beta(n+1)}^{m(n+1)})$. So there is α such that for all k, $\sum_{k<n} g_\delta(p^{m(n)} x_{\beta(n)}^{m(n)})$ has α as an accumulation point. Let α be the least such ordinal. If $\alpha < \delta$, we may need to thin the sequence again. By taking a subsequence if necessary we can assume that $\sum_n g_\delta(p^{m(n)} x_{\beta(n)}^{m(n)})$ is barred by hypothesis (*) from being an element of $C[p]$ (i.e. there is no $y \in A_{\alpha+1}[p]$

and k so that $\sum_n g_\delta(p^{m(n)}x^{m(n)}_{\beta(n)}) + ky$ does not have α as an accumulation point). Let $y^0_\delta = \Sigma\, p^{m(n)}x^{m(n)}_{\beta(n)}$ and for $k > 0$ let

$$y^k_\delta = \sum_{n\geq k} p^{m(n)-k}x^{m(n)}_{\beta(n)} + \sum_{n\geq k} p^{n+1-k}x^n_\delta.$$

Now let $A_{\delta+1} = \langle A_\delta \cup \{y^k_\delta\colon k < \omega\} \cup \{x^n_\delta\colon n < \omega\}\rangle$ and let $B_{\delta+1} = \langle B_\delta \cup \{y^k_\delta\colon k < \omega\} \cup \{x^n_\delta\colon n < \omega\}\rangle$.

It should be clear that $A_{\delta+1}$ and $B_{\delta+1}$ satisfy the inductive hypotheses including (*). Let us now see that, if A is constructed according to our promises, then g_δ cannot be extended to an embedding g of A into C. Since $g(y^0_\delta) = \sum_n g_\delta(p^{m(n)}x^{m(n)}_{\beta(n)})$, it is enough to see that $g(y^0_\delta) \notin C[p]$. Let α be the accumulation point of $g(y^0_\delta)$ considered in the construction. If $\alpha < \delta$, we have already guaranteed that $g(y^0_\delta) \notin C[p]$. If $\alpha = \delta$, then for some k not congruent to 0 modulo p, $g(y^0_\delta) + ky^0_\delta$ does not have δ as an accumulation point. So $g(y^0_\delta) = \sum_n -k\, p^{m(n)}x^{m(n)}_{\beta(n)} + a$, for some a in $B_\delta \oplus \bigoplus_{\alpha<\delta}\bigoplus_n \langle s^n_\alpha\rangle$. Hence for all but finitely many n, $g(p^{m(n)}x^{m(n)}_{\beta(n)}) = -kp^{m(n)}x^{m(n)}_{\beta(n)}$, a contradiction. In the following cases, similar arguments are used but we will not give as many details.

Subcase 2. There is an integer $k_0 \neq 0$, a natural number n_0 and an ordinal $\alpha_0 < \delta$ so that for $\alpha < \beta < \delta$ and $n < m$ $g_\delta(p^m x^m_\beta) = k_0 p^m x^m_\beta$ (i.e. subcase 1 does not hold) but for all $n < \omega$, $k \in \mathbb{Z}$ and $\alpha < \delta$ there is $m > n$ and $\alpha < \beta < \delta$ so that $g_\delta(p^{m-1}x^m_\beta) \neq kp^{m-1}x^m_\beta$. Notice in this case for $m > n_0$ and $\beta > \alpha_0$ that $g_\delta(p^{m-1}x^m_\beta) - k_0 p^{m-1}x^m_\beta = z^m_\beta$ has order p. Choose increasing sequences $\{\beta(n)\colon n < \omega\}$ of ordinals with limit δ and $\{m(n)\colon n < \omega\}$ of natural numbers such that for all n, $\alpha_0 < \beta(n)$, $n_0 <$

$m(n)$ and for all k, $\{n: g_\delta(p^{m(n)-1}x^{m(n)}_{\beta(n)}) - kp^{m(n)-1}x^{m(n)}_{\beta(n)} \neq 0\}$ is infinite. As well we can assume that $\beta(n+1)$ is greater than any α such that x^k_α or s^k_α is in supp $g_\delta(x^{m(n)}_{\beta(n)})$ for some k, and that $m(n) + 1 < m(n+1)$. By thinning we can assume that for all n, the height of $z^{m(n)}_{\beta(n)}$ is less than the height of $z^{m(n+1)}_{\beta(n+1)}$. So there is a such that for all k, $\sum_{k<n} z^{m(n)}_{\beta(n)}$ has α as an accumulation point. Let α be the least such ordinal. As before, if $\alpha < \delta$, we can assume that $\sum_n z^{m(n)}_{\beta(n)}$ is barred from being an element of $A[p]$. Let $y^0_\delta = \sum p^{m(n)}x^{m(n)}_{\beta(n)}$ and for $k > 0$ let

$$y^k_\delta = \sum_{n\geq k} p^{m(n)-k}x^{m(n)}_{\beta(n)} + \sum_{n\geq k} p^{n+1-k}x^n_\delta.$$

Let $A_{\delta+1} = \langle A_\delta \cup \{y^k_\delta: k < \omega\} \cup \{x^n_\delta: n < \omega\}\rangle$ and let $B_{\delta+1} = \langle B_\delta \cup \{y^k_\delta: k < \omega\} \cup \{x^n_\delta: n < \omega\}\rangle$.

Now, we will explain why g_δ cannot be extended to an embedding g of A into C where $g(p^n x^n_\delta) = k_0 p^n x^n_\delta$ for all but finitely many n. Suppose not, consider $g(y^1_\delta) - k_0 y^1_\delta = \sum_n z^{m(n)}_{\beta(n)} + \sum_{n<m} g(x^n_\delta) - k_0 x^n_\delta$ for some m. By subtracting $\sum_{n<m} g(x^n_\delta) - k_0 x^n_\delta$, we have $\sum_n z^{m(n)}_{\beta(n)}$ is an element of C. Let α be the accumulation point of $\sum_n z^{m(n)}_{\beta(n)}$ used in the construction. If $\alpha < \delta$, this sum is not in $C[p]$ by the construction. Suppose $\alpha = \delta$. Arguing as case 1, we can show that there is k so that for all but finitely many n, $g(p^{m(n)-1}x^{m(n)}_{\beta(n)}) = kp^{m(n)-1}x^{m(n)}_{\beta(n)}$. Again we have a contradiction.

Subcase 3. There is $k \in \mathbb{Z}$, $n_0 < \omega$ and $\alpha_0 < \delta$ so that for all $n < m$ and $\alpha < \beta < \delta$ $g_\delta(p^{m-1}x^m_\beta) = kp^{m-1}x^m_\beta$. Choose an increasing sequences $\{m(n): n < \omega\}$ of natural numbers and $\{\beta(n): n < \omega\}$ of ordinals with limit δ such that for all n, $m(n) > n_0$ and $\beta(n) > \alpha_0$. Let $y^0_\delta = \sum_n p^{m(n)}x^{m(n)}_{\beta(n)}$ and for

$k > 0$ let $y^k_\delta = \sum_{n \geq k} p^{m(n)-k} x^{m(n)}_{\beta(n)} + \sum_{n \geq k} p^{n+1-k} x^n_\delta$. Now let $B_{\delta+1} = \langle B_\delta \cup \{y^k_\delta : k < \omega\} \cup \{x^n_\delta : n < \omega\}\rangle$. We now consider two strategies for extending A_δ. For $k = 0, 1$, let $z^k_\delta = y^k_\delta$. For $k > 1$, let $z^k_\delta = y^k_\delta + \sum_{n > k} p^{n+2-k} x^n_\delta$. Let $A^1 = \langle A_\delta \cup \{y^k_\delta : k < \omega\} \cup \{x^n_\delta : n < \omega\}\rangle$ and $A^2 = \langle A_\delta \cup \{z^k_\delta : k < \omega\} \cup \{x^n_\delta : n < \omega\}\rangle$. We claim that g_δ does not extend to both g^1, g^2 from A^1 and A^2 respectively to C such that for all but finitely many n, $g^i(p^{n-1}x^n_\delta) = kp^{n-1}x^n_\delta$. Otherwise $g^1(y^2_\delta) - g^2(y^2_\delta) \in C[p]$ and has $\delta+1$ as an accumulation point. There are no elements of $C[p]$ with this property. Choose $A_{\delta+1}$ to be either A^1 or A^2 so that g_δ does not have an extension as above.

Assume now that $g: A \longrightarrow C$ is an embedding. Let $E = \{\delta : g \restriction A_\delta = g_\delta\}$. Suppose first there is no α_1 and n_1 and k_1 such that for all $\beta > \alpha_1$ and $n > n_1$, $g(p^n x^n_\beta) = k_1 p^n x^n_\beta$. In this case there is $\delta \in E$, so that δ falls under subcase 1. But in the construction of $A_{\delta+1}$, g_δ was prevented from extending to an embedding of A into C. Hence α_1, n_1, k_1 exist as above. Next suppose there is no α_2 and n_2 and k_2 such that for all $\beta > \alpha_2$ and $n > n_2$, $g(p^{n-1}x^n_\beta) = k_2 p^{n-1} x^n_\beta$. In this case there is $\delta \in E$, so that $\delta > \alpha_1$ and δ falls under subcase 2. Since $\delta > \alpha_1$, g_δ was prevented from extending to an embedding of A into C. So α_2, n_2, k_2 exist as above. If $\delta > \alpha_1, \alpha_2$ and $\delta \in E$, then δ falls under subcase 3. So g_δ was prevented from extending to an embedding of A into C. In any case the existence of g leads to a contradiction. $\square$

The group constructed in the preceding example under the assumption of $\diamondsuit$ has the additional property that it is $\aleph_1$-separable. There may not be examples with this additional property. In [7], it is shown that it is consistent that an $\aleph_1$-separable group of cardinality $\aleph_1$ is $p^{\omega+n}$-projective if and only if it is n-pseudo-free. Our example is, of course, not $p^{\omega+1}$-projective. Assuming $\diamondsuit$, an example is given in [7] (using the techniques of [3]) of a n-pseudo-free $\aleph_1$-separable groups of cardinality $\aleph_1$ which are not $p^{\omega+n}$-projective.

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