ISRAEL JOURNAL OF MATHEMATICS 89 (1995), 357–376

EVERY NULL-ADDITIVE SET IS MEAGER-ADDITIVE

BY

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ABSTRACT

It is proved that every null-additive subset of ${}^{\omega}2$ is meager-additive. Several characterizations of the null-additive subsets of ${}^{\omega}2$ are given, as well as a characterization of the meager additive subsets of ${}^{\omega}2$. Under CH, an uncountable null-additive subset of ${}^{\omega}2$ is constructed.

1. The basic definitions and the main theorem

1. *Definition:* (1) We define addition on ${}^{\omega}2$ as addition modulo 2 on each component, i.e., if $x, y, z \in {}^{\omega}2$ and $x + y = z$ then for every n we have $z(n) = x(n) + y(n) \pmod 2$. (2) For $A, B \subseteq {}^{\omega}2$ and $x \in {}^{\omega}2$ we set $x + A =^{\mathrm{df}} \{x + y\colon y \in A\}$, and we define $A + B$ similarly. (3) We denote the Lebesgue measure on ${}^{\omega}2$ by μ. We say that $X \subseteq {}^{\omega}2$ is **null-additive** if for every $A \subseteq {}^{\omega}2$ which is null, i.e. $\mu(A) = 0$, $X + A$ is null too. (4) We say that $X \subseteq {}^{\omega}2$ is **meager-additive** if for every $A \subseteq {}^{\omega}2$ which is meager also $X + A$ is meager.

2. THEOREM: *Every null-additive set is meager-additive.*

3. *Outline and discussion:* Theorem 2 answers a question of Pawlikowski. It will be proved in Section 2. In Section 3 we shall present direct characterizations

* Publ. No. 445. First version written in April 1991. Partially supported by the Basic Research Fund of the Israel Academy of Sciences and the Edmund Landau Center for Research in Mathematical Analysis, supported by the Minerva Foundation (Germany).
We thank Azriel Levy for rewriting the paper. The reader should thank him for including more proofs and making the paper self-contained.
Received February 5, 1992 and in revised form February 3, 1994

of the null-additive sets, and in Section 4 we shall do the same for the meager-additive sets.

It is obvious that every countable set is both null-additive and meager-additive. Are there uncountable null-additive sets, and even null-additive sets of cardinality $2^{\aleph_0}$? It will be shown in Section 5 that if the continuum hypothesis holds, then there is such a set. Haim Judah has shown that there is a model of ZFC in which all the null-additive sets are countable, but there are in it uncountable meager-additive sets. This is the model obtained by adding to L more than $\aleph_1$ Cohen reals. In this model the Borel conjecture holds, and therefore every null-additive set is strongly meager and hence countable. On the other hand, in this model the uncountable set of all constructible reals is meager-additive.

2. The proof of Theorem 2

4. *Notation:* (1) we shall use variables as follow: i, j, k, l, m, n for natural numbers, f, g, h for functions from ω to ω, $\eta, \zeta, \nu, \sigma, \tau$ for finite sequences of 0's and 1's, x, y, z for members of ${}^\omega 2$, A, B, X, Y for subsets of ${}^\omega 2$, and S, T for trees. (2) ${}^{\omega>}2 = \bigcup_{n<\omega} {}^n 2$. We shall denote subsets of ${}^{\omega>}2$ by U, V. For $\eta \in {}^{\omega>}2$, $U \subseteq {}^{\omega>}2$ and $x \in {}^\omega 2$ we shall write $\eta + x$ for $\eta + x \restriction \operatorname{length}(\eta)$, and $U + x$ for $\{\eta + x\colon \eta \in U\}$. (3) For $\eta, \nu \in {}^{\omega>}2$ we write $\eta \trianglelefteq \nu$ if ν is an extension of η. (4) A tree for us is a nonempty subset of ${}^{\omega>}2$ such that

(a) if $\eta \trianglelefteq \nu$ and $\nu \in T$ then also $\eta \in T$, and

(b) if $\eta \in T$ and $n > \operatorname{length}(\eta)$ then there is a ν of length n such that $\eta \trianglelefteq \nu$ and $\nu \in T$.

(5) For a tree T, $\operatorname{Lim}(T) = \{x \in {}^\omega 2\colon \text{for every } n < \omega \text{ we have } x \restriction n \in T\}$. (6) A tree T is said to be **nowhere dense** if for every $\eta \in T$ there is a $\tau \in {}^{\omega>}2$ such that $\eta \trianglelefteq \tau$ and $\tau \notin T$. A set $B \subseteq {}^\omega 2$ is nowhere dense if $B \subseteq \operatorname{Lim}(T)$ for some nowhere dense tree T. (7) For every $x, y \in {}^\omega 2$ we write $x \equiv y$ if $x(n) = y(n)$ for all but finitely many $n < \omega$. For $A \subseteq {}^\omega 2$, $A^{\text{fin}} =^{\text{df}} \{y \in {}^\omega 2\colon y \equiv x \text{ for some } x \in A\}$. (8) $U^{[\nu]} =^{\text{df}} \{\tau \in U\colon \tau \trianglelefteq \nu \text{ or } \nu \trianglelefteq \tau\}$ (read: U through ν). (9) $U^{\langle\nu\rangle} =^{\text{df}} \{\tau \in {}^{\omega>}2\colon \nu \frown \tau \in U\}$ (read: U above ν), and for $\eta \in {}^{\omega>}2$ we define $\eta^{\langle k\rangle} =^{\text{df}} \langle \eta(k+i)\colon i < \operatorname{length}(\eta) - k\rangle$. (10) For $\nu, \eta \in {}^{\omega>}2 \cup {}^\omega 2$ we write $\nu \sim_n \eta$ if $\operatorname{length}(\nu) = \operatorname{length}(\eta)$ and $\nu(i) = \eta(i)$ for every $n \leq i < \operatorname{length}(\nu)$. For $S \subseteq {}^{\omega>}2 \cup {}^\omega 2$ we define $S^{\sim n} = \{\nu\colon \nu \sim_n \eta \text{ for some } \eta \in S\}$.

5. *Outline of the proof:* Let $X \subseteq {}^{\omega}2$ be null-additive. It clearly suffices to prove that for every $A \subseteq {}^{\omega}2$ which is nowhere dense, $X+A$ is meager. Given a nowhere dense tree S we shall give a condition which is, as we prove in Lemma 6, sufficient for a tree T to be such that $T+S$ is nowhere dense. Then we shall split X into a union $X = \bigcup_{i=1}^{\infty} X_i$ such that, for each i, $X_i \subseteq \mathrm{Lim}\,(T_i)$ where T_i is a tree which satisfies that condition. Thus for a nowhere dense S, each set $X_i + \mathrm{Lim}\,(S) \subseteq \mathrm{Lim}\,(T_i+S)$ is nowhere dense, hence $X+\mathrm{Lim}\,(S) \subseteq \bigcup_{i=1}^{\infty} \mathrm{Lim}\,(T_i+S)$ is meager.

6. LEMMA: *Let T be a tree such that*

(a) *T is nowhere dense.*

(b) *$f = f_T$ is the function from ω to ω given by $f(n) = \min\{m :$ for every $\eta \in {}^{n}2$ there is a $\tau \in {}^{m}2$ such that $\eta \trianglelefteq \tau$ and $\tau \notin T\}$. Thus for every sequence η of length n there is a witness of length $\leq f(n)$ exemplifying that T is nowhere dense. Obviously, for every $n < \omega$, $f(n) > n$, and if $n < m$, then $f(n) \leq f(m)$.*

Let g be a function from ω to ω. We can find $\overline{n} = \langle n_i \colon i < \omega\rangle$ and $\overline{n}' = \langle n'_i \colon i < \omega\rangle$, increasing sequences of natural numbers such that

(c) *$f^{g(i)}(n_i) \leq n'_i < n_{i+1}$ for every $i < \omega$, where f^m denotes the m-th iteration of f.*

Then for every tree S which satisfies

(d) *S is of width $(\overline{n}', g)$, i.e., for every $i < \omega$ we have $|{}^{n'_i}2 \cap S| \leq g(i)$,*

$T+S$ is nowhere dense.

Proof: Let $\eta \in {}^{n_i}2$. We shall show the existence of an $\eta' \in {}^{n'_i}2$ such that $\eta \trianglelefteq \eta'$ and $\eta' \notin T+S$.

By (c) there is a sequence $m_0, \ldots, m_{g(i)}$ such that $m_0 = n_i$, $f(m_k) \leq m_{k+1}$ for $0 \leq k \leq g(i)$ and $m_{g(i)} = n'_i$. Let $\langle \tau_k \colon k < k_i\rangle$ enumerate the set ${}^{n'_i}2 \cap S$. By (d), $k_i \leq g(i)$. We define $\eta_k \in {}^{m_k}2$ for $0 \leq k \leq k_i$ by recursion as follows. Start with $\eta_0 = \eta$. Given $\eta_k \in {}^{m_k}2$, for $k < k_i$, we shall define $\eta_{k+1} \in {}^{m_{k+1}}2$ so that for no extension $\eta' \in {}^{n'_i}2$ of η_{k+1} shall we have $\eta' + \tau_k \in T$. We have $\eta_k + \tau_k \restriction m_k \in {}^{m_k}2$ and, by the definition of f and by the choice of the m_k's, $\eta_k + \tau_k \restriction m_k$ has an extension $\nu \in {}^{m_{k+1}}2$ such that $\nu \notin T$. If we take $\eta_{k+1} = \nu + \tau_k \restriction m_{k+1}$ then $\eta_k + \tau_k \restriction m_k \trianglelefteq \nu$ implies $\eta_k \trianglelefteq \eta_{k+1}$, $\eta_{k+1} \in {}^{m_{k+1}}2$ and $\eta_{k+1} + \tau_k \restriction m_{k+1} = \nu \notin T$, and therefore for every $\eta' \in {}^{n'_i}2$ such that $\eta_k \trianglelefteq \eta'$ we have $\eta' + \tau_k \notin T$. Let $\eta' = \eta_{k_i}$, and assume that $\eta' \in T+S$. Then, for some $k < k_i \leq g(i)$, we have $\eta' + \tau_k \in T$, contradicting our choice of $\eta_{k+1} = \eta' \restriction m_{k+1}$. Thus $\eta' \notin T+S$.

7. Lemma: *If S, T_i, $i \in \omega$ are trees and* $\mathrm{Lim}(S) \subseteq \bigcup_{i\in\omega} \mathrm{Lim}(T_i)$, *then for some* $\eta \in S$ *and* $j \in \omega$, $S^{[\eta]} \subseteq T_j$.

Proof: Suppose that this is not the case, i.e., for every $\eta \in S$ and $i < \omega$ there is a ζ such that $\zeta \in S^{[\eta]}$ and $\zeta \notin T_i$. Once there is such a ζ we can assume that $\eta \trianglelefteq \zeta$ and length$(\zeta) >$ length(η). We define now, by induction on i, η_i and k_i so that $k_i = \mathrm{length}(\eta_i)$, $k_0 = 0$, $\eta_0 = \langle\rangle$, $\eta_i \trianglelefteq \eta_{i+1}$, $k_i < k_{i+1}$, $\eta_{i+1} \in S$ and $\eta_{i+1} \notin T_i$. Let $y = \bigcup_{i\in\omega} \eta_i$. Since $\eta_i \in S$ for every $i \in \omega$, $y \in \mathrm{Lim}(S) \subseteq \bigcup_{i\omega} \mathrm{Lim}(T_i)$, hence for some $j \in \omega$, $y \in \mathrm{Lim}(T_j)$. However, $y \restriction k_{j+1} = \eta_{j+1} \notin T_j$, contradicting $y \in \mathrm{Lim}(T_j)$.

8. Lemma: *Let S and T be trees such that* $\mathrm{Lim}(S) \subseteq (\mathrm{Lim}(T))^{\mathrm{fin}}$. *Then there are* $k < \omega$, $\eta, \nu \in {}^k2$, $\eta \in S$ *such that* $S^{\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$.

Proof: For $n < \omega$, $\sigma_1, \sigma_2 \in {}^n2$ and $\sigma_2 \in T$ we define

$$T_{\sigma_1,\sigma_2} =^{\mathrm{df}} \{\tau : \tau \trianglelefteq \sigma_1\} \cup \{\sigma_1 \frown \tau : \sigma_2 \frown \tau \in T\}.$$

(This is the tree $T^{[\sigma_2]}$ with "σ_2 replaced by σ_1".) Clearly

$$(1) \qquad (\mathrm{Lim}(T))^{\mathrm{fin}} = \bigcup_{n<\omega,\, \sigma_1,\sigma_2\in {}^n2,\, \sigma_2\in T} \mathrm{Lim}(T_{\sigma_1,\sigma_2}).$$

Since there are only countably many T_{σ_1,σ_2}'s in (1), there are, by Lemma 7, a $\zeta \in S$ and $j < \omega$ such that $S^{[\zeta]} \subseteq T_{\sigma_1,\sigma_2}$. Clearly there is an η with $\zeta \trianglelefteq \eta$ and a ν with length$(\nu) =$ length(η) such that $S^{\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$. (If $\zeta \trianglelefteq \sigma_1$ then $\eta = \sigma_1$ and $\nu = \sigma_2$, else $\sigma_1 \trianglelefteq \zeta$ and then $\eta = \zeta$ and $\nu = \sigma_2 \frown \zeta \restriction [\mathrm{length}(\zeta), \mathrm{length}(\sigma_2))$.)

9. Lemma: *Let X be a null-additive set. Let T be a tree such that* $\mu(\mathrm{Lim}(T)) > 0$. *There is a tree T^* such that* $\mu(\mathrm{Lim}(T^*)) > 0$, *moreover for every $\eta \in T^*$ also* $\mu(\mathrm{Lim}(T^{*[\eta]})) > 0$, *and* $(({}^\omega 2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}}) + X) \cap \mathrm{Lim}(T^*) = \emptyset$, *and then*

$$X = \bigcup_{\eta\in T^*,\ \mathrm{length}(\zeta)=\mathrm{length}(\eta)} Y^X_{\eta,\zeta}$$

where $Y^X_{\eta,\zeta} = \{x \in X : \zeta \frown x^{\langle \mathrm{length}(\zeta)\rangle} + T^{*[\eta]} \subseteq T\}$.

Proof: Since $\mu(\mathrm{Lim}(T)) > 0$ then, as easily seen, $\mu((\mathrm{Lim}(T))^{\mathrm{fin}}) = 1$, hence $\mu({}^\omega 2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}}) = 0$. Since X is null-additive also $\mu(X + ({}^\omega 2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}})) = 0$. Hence there is a tree T^* such that

$$\mu(\mathrm{Lim}(T^*)) > 0 \quad \text{and} \quad (X + ({}^\omega 2 \setminus (\mathrm{Lim}(T))^{\mathrm{fin}})) \cap \mathrm{Lim}(T^*) = \emptyset.$$

Without loss of generality we can assume that T^* has been pruned so that, for $\eta \in T^*$, $\mu(\mathrm{Lim}\,(T^{*[\eta]})) > 0$.

Let $x \in X$; then

$$^{\omega}2 \setminus (x + (\mathrm{Lim}\,(T))^{\mathrm{fin}}) = x + ({}^{\omega}2 \setminus (\mathrm{Lim}\,(T))^{\mathrm{fin}}) \subseteq X + ({}^{\omega}2 \setminus (\mathrm{Lim}\,(T))^{\mathrm{fin}}).$$

Hence $({}^{\omega}2 \setminus (x+(\mathrm{Lim}\,(T))^{\mathrm{fin}})) \cap \mathrm{Lim}\,(T^*) \subseteq (X+({}^{\omega}2 \setminus (\mathrm{Lim}\,(T))^{\mathrm{fin}}) \cap \mathrm{Lim}\,(T^*) = \emptyset$, i.e., $\mathrm{Lim}\,(T^*) \subseteq x+(\mathrm{Lim}\,(T))^{\mathrm{fin}}$, and therefore $\mathrm{Lim}\,(x+(T^*)) = x+\mathrm{Lim}\,(T^*) \subseteq (\mathrm{Lim}\,(T))^{\mathrm{fin}}$. By Lemma 8 there are $\eta \in T^*$ and $\nu \in {}^{\mathrm{length}\,(\eta)}2$ such that $x^{\langle \mathrm{length}\,(\eta)\rangle} + T^{*\langle \eta \rangle} \subseteq T^{\langle \nu \rangle}$. Let $\zeta = \eta + \nu$; then $\zeta + \eta = \nu$ and therefore $\zeta \frown x^{\langle \mathrm{length}\,(\eta)\rangle} + T^{*[\eta]} \subseteq T^{[\nu]} \subseteq T$, hence $x \in Y^X_{\eta,\zeta}$.

10. LEMMA: *Let X be null-additive, and let $\overline{n} = \langle n_i \colon i < \omega \rangle$, $\overline{n}' = \langle n'_i \colon i < \omega \rangle$ be such that, for every $i < \omega$, $n_i < n'_i$ and $n'_i + i \cdot 2^{n'_i} \leq n_{i+1}$; then we can represent X as $\bigcup_{m<\omega} X_m$ such that, for each m, for some real $a_m \in (0,1)$ and S_m of width $(\overline{n}', g_{a_m})$ we have $X_m \subseteq \mathrm{Lim}\,(S_m)$, where for every real $a \in (0,1)$, g_a is the function on ω given by $g_a(0) = 1$, $g_a(i) = \max(1, \mathrm{int}(\log_2(a)/\log_2(1-2^{-i})))$ for $i > 0$, and for a real d, $\mathrm{int}(d)$ is the integral part of d.*

Proof: Since $n'_i + i \cdot 2^{n'_i} \leq n_{i+1}$ we can fix for each $0 < i < \omega$ a sequence $\langle u_{i,\tau} \colon \tau \in {}^{n'_i}2 \rangle$ of pairwise disjoint subsets of the interval $[n'_i, n_{i+1})$ having i members each. Let $B \subseteq {}^{\omega}2$ be given by

$$B = \{y \in {}^{\omega}2 \colon (\forall j > 0)(\exists k \in u_{j, y \restriction n'_j})\, y(k) = 1\}.$$

B is clearly a closed subset of ${}^{\omega}2$, hence for $T = \{y \restriction n \colon y \in B \wedge n \in \omega\}$ $B = \mathrm{Lim}\,(T)$.

The properties of T in which we are interested are

(B0) $T \supseteq {}^{n_1}2$.

(B1) For each $\eta \in T \cap {}^{n'_i}2$ we have $\left|T^{[\eta]} \cap {}^{n_{i+1}}2\right| = 2^{(n_{i+1}-n'_i)}(1-2^{-i})$.

(B2) If $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^{n'_i}2$, $\nu^+_0, \ldots, \nu^+_{k-1} \in {}^{n_{i+1}}2$, $\eta + \nu_l \in T$, $\nu_l \trianglelefteq \nu^+_l$ for $l < k$ and $\nu_0, \ldots, \nu_{k-1}$ is with no repetitions, then

$$\left|\{\eta^+ \colon \eta \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\eta^+ + \nu^+_l \in T)\}\right| \leq 2^{n_{i+1}-n'_i}\left(1-2^{-i}\right)^k.$$

(B3) For every $\eta \in {}^{n'_i}2$ we have: $\eta \restriction n_i \in T$ implies $\eta \in T$.

These properties can be established by an obvious counting argument.

By (B0), (B1) and (B3) we have

$$\begin{aligned}\mu(\operatorname{Lim}(T)) &= \mu\left(\bigcap_{i=1}^{\infty}\{x \in {}^{\omega}2 \colon x \restriction n_i \in T\}\right)\\ &= \mu(\{x \in {}^{\omega}2 \colon x \restriction n_1 \in T\}) \cdot \prod_{i=1}^{\infty} \frac{\mu(\{x \in {}^{\omega}2 \colon x \restriction n_{i+1} \in T\})}{\mu(\{x \in {}^{\omega}2 \colon x \restriction n_i \in T\})}\\ &= 1 \cdot \prod_{i=1}^{\infty} \frac{|T \cap {}^{n_{i+1}}2|/2^{n_{i+1}}}{|T \cap {}^{n_{i+1}}2|/2^{n_i}} = \prod_{i=1}^{\infty}(1-2^{-i}) > 0.\end{aligned}$$

For the T which we constructed and the given X, let T^* and $Y_{\eta,\zeta} = Y^X_{\eta,\zeta}$ be as in Lemma 9. For $\rho \in {}^{\text{length}(\eta)}2$ let $Y_{\eta,\zeta,\rho} = \{y \in Y_{\eta,\zeta} \colon y \restriction \text{length}(\eta) = \rho\}$. Clearly

$$X = \bigcup_{\eta \in T^*,\ \text{length}(\eta)=\text{length}(\zeta)=\text{length}(\rho)} Y_{\eta,\zeta,\rho}. \tag{2}$$

Since there are only countably many $Y_{\eta,\zeta,\rho}$'s they can be taken to be the X_m's we are looking for, provided we show that every such $Y_{\eta,\zeta,\rho}$ is a subset of $\operatorname{Lim}(S)$ for some tree S of width $\langle \overline{n}', g_a\rangle$ for some real $0 < a < 1$. We shall see that this is indeed the case if we take $S = \{y \restriction m \colon y \in Y_{\eta,\zeta,\rho},\ m < \omega\}$ and $a = \mu(T^{*[\eta]})$; $a > 0$ by what we assumed about T^*. As, obviously, $Y_{\eta,\zeta,\rho} \subseteq \operatorname{Lim}(S)$, all we have to do is to show that S is of width $\langle \overline{n}', g_a\rangle$. We fix a $j \in \omega$.

We can choose a set $W \subseteq S \cap {}^{n_{j+1}}2$ such that the function mapping $\eta \in W$ to $\eta \restriction n'_j$ is one to one and onto $S \cap {}^{n'_j}2$.

We fix now η, ζ, ρ and denote $Y_{\eta,\zeta,\rho}$ by Y and the length of η, ζ, ρ by n. Let $z \in {}^{\omega}2$ be such that $z \restriction n = \zeta + \rho$ and $z(i) = 0$ for $i \geq n$. Then for every y such that $y \restriction n = \rho$ we have $y + z = \zeta \frown y^{\langle n\rangle}$. Therefore, by the definition of Y we have

$$\begin{aligned}Y &= \{y \in {}^{\omega}2 \colon y \restriction n = \rho,\ (\zeta \frown y^{\langle n\rangle}) + T^{*[\eta]} \subseteq T\}\\ &= \{y \in {}^{\omega}2 \colon y \restriction n = \rho,\ y + z + T^{*[\eta]} \subseteq T\};\end{aligned} \tag{3}$$

for every $y \in Y$ there is a unique $\tau \in W$ such that $\tau \restriction n'_j = y \restriction n'_j$ (τ may be $y \restriction n_{j+1}$). Clearly $|W| = |S \cap {}^{n'_j}2|$ and we denote $|W|$ by s, so it suffices to prove $s \leq g_a(j)$. If $n'_j \leq n$, then the only member of $S \cap {}^{n'_j}2$ is $\rho \restriction n'_j$ hence $s = 1$, so $s \leq g_a(j)$. We shall now deal with the case where $n'_j > n$. Let $\tau_0, \ldots, \tau_{s-1}$ be the members of W. For $m < s$, $\tau_m = y \restriction n_{j+1}$ for some $y \in Y$, hence, by (3),

$\tau_m + z + T^{*[\eta]} \subseteq T$ and therefore $(z + T^{*[\eta]}) \cap {}^{n_{j+1}}2 \subseteq \tau_m + T$. Since this holds for every $\tau \in W$ we have

$$z + T^{*[\eta]} \cap {}^{n_{j+1}}2 \subseteq \bigcap_{m<s} \tau_m + T. \tag{4}$$

Let us find out the size of $\bigcap_{m<s}(\tau_m + T)$. Let $\sigma \in {}^{n'_j}2$, and we shall ask how many members τ of $\bigcap_{m<s}(\tau_m + T)$ extend σ. Now $\tau \in \tau_m + T$ for each $m < s$ iff $\tau + \tau_m \in T$ for each $m < s$. If, for some $m < s$, $\sigma + \tau_m \restriction n'_j \notin T$, then also $\tau + \tau_m \notin T$, hence σ has no extension in $\bigcap_{m<s}(\tau_m + T)$. If, for every $m < s$, $\sigma + \tau_m \restriction n'_j \in T$, then by (B2) (where $\eta = \sigma$, $\nu_m = \tau_m \restriction n'_j$ and $\nu_m^+ = \tau_m$), since $\tau_m \restriction n'_j \neq \tau_l \restriction n'_j$ for $m \neq l$, the number of τ's such that $\sigma \trianglelefteq \tau \in {}^{n_{j+1}}2$ and $\tau + \tau_m \in T$ for every $m < s$ is $2^{n_{j+1}-n'_j}(1 - 2^{-j})^s$. Since there are $2^{n'_j}$ different σ's in ${}^{n'_j}2$ we have

$$\left|\bigcap_{m<s}(\tau_m + T)\right| \leq 2^{n_{j+1}} \cdot (1 - 2^{-j})^s. \tag{5}$$

On the other hand, since $\mu(T^{*[\eta]}) = a$, $T^{*[\eta]} \cap {}^{n_{j+1}}2$ has at least $a \cdot 2^{n_{j+1}}$ members, and so has $z + T^{*[\eta]} \cap {}^{n_{j+1}}2$. Comparing (4) with (5) we get $a \cdot 2^{n_{j+1}} \leq 2^{n_{j+1}}(1 - 2^{-j})^s$, i.e., $a \leq (1 - 2^{-j})^s$, $\log_2(a) \leq s \cdot \log_2(1 - 2^{-j})$, $s \leq \log_2(a)/\log_2(1 - 2^{-j})$.

11. *Proof of Theorem 2:* Let X be null-additive. As mentioned in subsection 5, it suffices to show that for every nowhere dense tree T, $X + \mathrm{Lim}\,(T)$ is meager. Let $f = f_T$ as in Lemma 6. Define by recursion $n_0 = 0$, $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1$ and $n_{i+1} = n'_i + i \cdot 2^{n'_i} + 1$. By Lemma 10, $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$, where for some $a_m \in (0,1)$ S_m is of width $\langle \overline{n}', g_{a_m} \rangle$, hence it suffices to show that if S is of width $\langle \overline{n}', g_a \rangle$ for some $a \in (0,1)$ then $\mathrm{Lim}\,(S) + \mathrm{Lim}\,(T) = \mathrm{Lim}\,(S + T)$ is meager. Let j be such that $\frac{1}{j+1} \leq a$ and let $\eta_1, \ldots, \eta_k$ be all the members of S of length n'_j. Then $S = \bigcup_{l=1}^k S^{[\eta_l]}$ and $\mathrm{Lim}\,(S) = \bigcup_{l=1}^k \mathrm{Lim}\,(S^{[\eta_l]})$. Therefore it suffices to prove that for $1 \leq l \leq k$, $\mathrm{Lim}\,(S_l) + \mathrm{Lim}\,(T)$ is meager and this follows once we show that $S_l + T$ is nowhere dense. To prove this we show that the requirements of Lemma 6 hold here for S_l, T. (a) and (b) hold by our choice of T and f. Let g be defined by $g(i) = 1$ for $i < j$ and $g(i) = g_a(i)$ for $i \geq j$. Now we shall see that (c) holds. For $i < j$ we have $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1 \geq f(n_i) + 1 = f^{g(i)}(n_i) + 1$, since $f(n) \geq n$ for every n, and for $i \geq j$ we have $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1 \geq f^{g_a(i)}(n_i) + 1 = f^{g(i)}(n_i) + 1$, since $a \geq \frac{1}{j+1} \geq \frac{1}{i+1}$ and the map $a \mapsto g_a(i)$ is a

decreasing function of a. Thus for every $i < \omega$, $f^{g(i)}(n_i) \le f^{g_{1/(i+1)}(i)}(n_i) \le n'_i$. (d) of Lemma 6 holds since, for $i < j$, $|^{n'_i}2 \cap S_l| = 1 = g(i)$ and, for $i \ge j$, $|^{n'_i}2 \cap S_l| \le |^{n'_i}2 \cap S| \le g_a(i) = g(i)$.

3. Characterization of the null-additive sets

12. Definition: By a **corset** we mean a non-decreasing function f from ω to $\omega \setminus \{0\}$ which converges to infinity (i.e., for every $n < \omega$, $f(m) > n$ for all sufficiently large m). For a corset f, we say that a tree T is **of width** f if for every $n < \omega$, $|T \cap {}^n2| \le f(n)$; and we say that T is **almost of width** f if $|T \cap {}^n2| \le f(n)$ for all sufficiently large n.

13. THEOREM: *For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:*

a. *X is null-additive.*
b. *For every corset f there is a tree S of width f such that $X \subseteq (\mathrm{Lim}\,(S))^{\mathrm{fin}}$.*
c. *For every corset f there are trees S_m, $m < \omega$, which are almost of width f such that $X \subseteq \bigcup_{m<\omega} (\mathrm{Lim}\,(S_m))^{\mathrm{fin}}$.*
d. *For every corset f there are trees S_m, $m < \omega$, of width f such that $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$.*

Proof: (b)→(c) is obvious.

(c)→(d): Let S be a tree almost of width f. Then for some k we have $|T \cap {}^n2| \le f(n)$ for all $n \ge k$. By (1) of Lemma 8,

$$(\mathrm{Lim}\,(S))^{\mathrm{fin}} = \bigcup_{\sigma_1,\sigma_2 \in {}^k2,\ \sigma_2 \in S} \mathrm{Lim}\,(S_{\sigma_1,\sigma_2}).$$

Each S_{σ_1,σ_2} is of width f since, for $n \le k$, we have $|S_{\sigma_1,\sigma_2} \cap {}^n2| = 1$ and for $n > k$ we have $|S_{\sigma_1,\sigma_2} \cap {}^n2| \le |S \cap {}^n2| \le f(n)$. Therefore, if $X \subseteq \bigcup_{m<\omega} (\mathrm{Lim}\,(S_n))^{\mathrm{fin}}$ as in (c) then each S_m can be replaced by countably many S_{σ_1,σ_2}'s and (d) holds.

(d)→(b): Let f be a corset. We can easily define by recursion a sequence $0 = n_0 < n_1 < \cdots$ of natural numbers and a corset f^* such that for all $j < \omega$ and $m \ge n_{j+1}$ we have $(j+1) \cdot 2^{n_j} \cdot f^*(m) \le f(m)$.

For a given corset f, if X satisfies (d) let S^*_m, $m < \omega$, be as in (d) for the corset f^*. We construct now a set $S \subseteq {}^{\omega>}2$ by defining $S \cap {}^m2$ by recursion on m. $S \cap {}^02 = \{\langle\rangle\}$. For $n_i < m \le n_{i+1}$ let

$$S \cap {}^m2 = \{\eta \in {}^m2\colon \eta \restriction n_i \in S \cap {}^{n_i}2 \text{ and } \eta \in S_j^{*\sim n_j} \text{ for some } j < i \vee j = 0\}.$$

S can be easily seen to be a tree, and clearly $(\mathrm{Lim}\,(S))^{\mathrm{fin}} \supseteq \bigcup_{n<\omega} \mathrm{Lim}\,(S^*_n) \supseteq X$. For $m \le n_1$ easily $|S \cap {}^m2| \le f(m)$, for $n_i \le m < n_{i+1}, i \ge 1$ we have $|S \cap {}^m2| \le \sum_{j\le i} |S_j^{*\sim n_j} \cap {}^m2| = \sum_{j\le i} 2^{n_j} |S^*_j \cap {}^m2| \le (i+1)\cdot 2^{n_j} \cdot f^*(m) \le f(m)$, thus S is of width f.

(d)$\to$(a): Assume now that (d) holds for X, and let $A \subseteq {}^\omega 2$, $\mu(A) = 0$; we shall prove that $\mu(X + A) = 0$. First we shall mention two lemmas from measure theory the proof of which is left to the reader.

LEMMA A: *For every tree T with $\mu(\mathrm{Lim}\,(T)) = a > 0$ and $\epsilon > 0$ there is an $N \in \omega$ such that for every $n \ge N$ there is a $t \subseteq {}^n2 \cap T$ such that $|t| \ge 2^n(a - \epsilon)$ and, for each $\eta \in t$, $\mu(\mathrm{Lim}\,(T^{[\eta]})) > 2^{-n}(1 - \epsilon)$.*

Using Lemma A one can prove

LEMMA B: *For every tree T with $\mu(\mathrm{Lim}\,(T)) > 0$, every $\epsilon > 0$ and every sequence $\langle \epsilon_i : 0 < i < \omega\rangle$ of positive reals there is a subtree T' of T and an increasing sequence $\langle n_i : i < \omega\rangle$ of natural numbers such that $n_0 = 0$, $\mu(\mathrm{Lim}\,(T')) > \mu(\mathrm{Lim}\,(T)) - \epsilon$ and*

(6) *for $i > 0$ and every $\eta \in {}^{n_i}2 \cap T'$,* $\mu(\mathrm{Lim}\,(T'^{[\eta]})) > 2^{-n_i}(1 - \epsilon_i)$.

By basic measure theory $\mu(A^{\mathrm{fin}}) = 0$, so there is a tree T such that $\mu(\mathrm{Lim}\,(T)) > 0$ and $\mathrm{Lim}\,(T) \cap A^{\mathrm{fin}} = \emptyset$ hence $(\mathrm{Lim}\,(T))^{\mathrm{fin}} \cap A = \emptyset$. Given $\epsilon < \mu(\mathrm{Lim}\,(T))$ and $\langle \epsilon_i : i < \omega\rangle$ as in Lemma B we obtain a subtree T' of T as in that lemma with $\mu(\mathrm{Lim}\,(T')) > 0$. The union of sufficiently many "finite translates" of T', i.e., trees T'_{σ_1,σ_2} as in (1) of Lemma 8, is a tree T'' satisfying (6) with $\mu(\mathrm{Lim}\,(T'')) \ge \frac{1}{2}$. $(\mathrm{Lim}\,(T''))^{\mathrm{fin}} = (\mathrm{Lim}\,(T'))^{\mathrm{fin}} \subseteq (\mathrm{Lim}\,(T))^{\mathrm{fin}}$ and hence $\mathrm{Lim}\,(T'') \cap A \subseteq (\mathrm{Lim}\,(T))^{\mathrm{fin}} \cap A = \emptyset$. We take now $\epsilon_i = 1/4(i+1)^3$ and take T to be T'' and we get $\mu(\mathrm{Lim}\,(T)) \ge \frac{1}{2}$ and

(7) for $i > 0$ and every $\eta \in {}^{n_i}2 \cap T$, $\mu(\mathrm{Lim}\,(T^{[\eta]})) > 2^{-n_i}\left(1 - \frac{1}{4(i+1)^3}\right)$.

Let f be the corset given by $f(n) = i + 1$ for $n_i \le n < n_{i+1}$. By (d) there are trees S_m of width f such that $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$. To show that $\mu(X + A) = 0$ it clearly suffices to show that, for every tree S of width f, $\mu(\mathrm{Lim}\,(S) + A) = 0$.

We define

$$T^* = \{\eta \in {}^{\omega>}2 : \nu + \eta \in T \text{ for every } \nu \in S \text{ of the same length as } \eta\}.$$

We do not show that T^* is a tree but obviously, if $\zeta \trianglelefteq \eta \in T^*$, then $\zeta \in T^*$, thus $\mathrm{Lim}\,(T^*)$ is defined. If $\mu(\mathrm{Lim}\,(T^*)) > 0$ then, by a well-known property of the measure, $\mu(\mathrm{Lim}\,(T^*)^{\mathrm{fin}}) = 1$, hence in order to prove $\mu(\mathrm{Lim}\,(S) + A) = 0$ it suffices to prove $(\mathrm{Lim}\,(S) + A) \cap (\mathrm{Lim}\,(T^*))^{\mathrm{fin}} = \emptyset$. Assume $y \in (\mathrm{Lim}\,(S) + A) \cap (\mathrm{Lim}\,(T^*))^{\mathrm{fin}}$. Since $y \in (\mathrm{Lim}\,(T^*))^{\mathrm{fin}}$ there is a $y' \in {}^{\omega}2$ such that $y'(n) = y(n)$ for all sufficiently big n's and $y' \in \mathrm{Lim}\,(T^*)$. Since $y \in \mathrm{Lim}\,(S) + A$ there is an $x \in \mathrm{Lim}\,(S)$ such that $y + x \in A$, hence $y + x \notin (\mathrm{Lim}\,(T))^{\mathrm{fin}}$, hence $y' + x \notin \mathrm{Lim}\,(T)$. Therefore, for some n, $y' \restriction n + x \restriction n \notin T$, hence, by the definition of T^*, $y' \restriction n \notin T^*$ contradicting $y' \in \mathrm{Lim}\,(T^*)$.

We still have to prove that $\mu(\mathrm{Lim}\,(T^*)) > 0$. We shall prove, by induction on i, that

$$n_i \le n \le n_{i+1} \to |(T \setminus T^*) \cap {}^n2| \le 2^n \cdot \sum_{j<i} \frac{1}{4(j+1)^2}. \tag{8}$$

Once we establish (8) we notice that since

$$\mathrm{Lim}\,(T) \setminus \mathrm{Lim}\,(T^*) = \bigcup_{n<\omega} \mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 \colon x \restriction n \in T^*\},$$

and the set $\mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 \colon x \restriction n \in T^*\}$ is increasing with n hence

$$\begin{aligned}\mu(\mathrm{Lim}\,(T) \setminus \mathrm{Lim}\,(T^*)) &= \lim_{n\to\infty} \mu(\mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 : x \restriction n \in T^*\}) \\ &\le \lim_{n\to\infty} 2^{-n} |(T \setminus T^*) \cap {}^n2| \\ &\le \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{4(j+1)^2} \\ &= \sum_{j=0}^{\infty} \frac{1}{4(j+1)^2} = \frac{\pi^2}{24} < \frac{1}{2}\end{aligned}$$

and since $\mu(\mathrm{Lim}\,(T)) \ge \frac{1}{2}\ \mu(\mathrm{Lim}\,(T^*)) > 0$.

To prove (8), assume now $n_i \le n \le n_{i+1}$. By the definition of T^*

$$\begin{aligned}(T \setminus T^*) \cap {}^n2 =& \{\eta \in T \cap {}^n2 \colon (\exists \rho \in S \cap {}^n2)(\rho + \eta \notin T)\} \\ =& \{\eta \in T \cap {}^n2 \colon (\exists \rho \in S \cap {}^n2)(\eta \restriction n_i + \rho \restriction n_i \notin T)\} \\ & \cup \bigcup_{\rho \in S \cap {}^n2} \{\eta \in T \cap {}^n2 \colon \eta \restriction n_i + \rho \restriction n_i \in T \wedge \eta + \rho \notin T\} \\ \subseteq& \{\eta \in {}^n2 \colon \eta \restriction n_i \in T \setminus T^*\} \\ & \cup \bigcup_{\rho \in S \cap {}^n2} \{\eta \in {}^n2 \colon \eta + \rho \in \{\sigma \in {}^n2 \colon \sigma \restriction n_i \in T \wedge \sigma \notin T\}\}.\end{aligned}$$

Therefore

$$|(T \setminus T^*) \cap {}^n2| \le 2^{n-n_i}|(T \setminus T^*) \cap {}^{n_i}2| + |S \cap {}^n2||\{\sigma \in {}^n2\colon \sigma \restriction n_i \in T \wedge \sigma \notin T\}|.$$

For $i > 0$ we have, by the induction hypothesis,

$$|T \setminus T^* \cap {}^{n_i}2| \le 2^{n_i} \sum_{j<i} \frac{1}{4(j+1)^2}.$$

For $i = 0$ we have $(T \setminus T^*) \cap {}^{n_i}2 = \emptyset$ since $n_0 = 0$ and $\emptyset \in T^*$.

$$|S \cap {}^n2| \le f(n) = i \quad \text{and} \quad |\{\sigma \in {}^n2\colon \sigma \restriction n_i \in T \wedge \sigma \notin T\}| \le \frac{2^n}{4(i+1)^3},$$

by (7). Thus

$$|(T \setminus T^*) \cap {}^n2| \le 2^{n-n_i} \cdot 2^{n_i} \sum_{j<i} \frac{1}{4(j+1)^2} + (i+1) \cdot \frac{2^n}{4(i+1)^3} \le 2^n \sum_{j<i+1} \frac{1}{4(j+1)^2}$$

which is what we had to show.

(a)$\to$(c): Most of the proof follows that of Lemma 10. We need also the following Lemma 14, which will be proved later. Let f be a corset.

14. Lemma: *There is an infinite sequence $0 = n_0 < n_1 < n_2 < \cdots$ and a tree T such that for every $i \in \omega$ we have $f(n_{i+1}) > (i+1) \cdot 2^{i+1} + 1$ and*

(B1) *For each $\eta \in T \cap {}^{n_i}2$ we have $|T^{[\eta]} \cap {}^{n_{i+1}}2| = 2^{(n_{i+1}-n_i)} \cdot (1 - 2^{-(i+1)})$.*

(B2) *If $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^{n_i}2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n_{i+1}}2$, $\nu_j^+ \neq \nu_l^+$ for $j < l < k$, $\eta + \nu_l \in T$, $\nu_l \trianglelefteq \nu_l^+$ for $l < k$, then*

$$\left|\{\eta^+\colon \eta \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}\right| \le 2^{n_{i+1}-n_i}\left(1 - 2^{-(i+1)}\right)^{k-1}$$

Let $\langle n_i\colon i \in \omega\rangle$ and T be as in Lemma 14. As in the proof of Lemma 10 we get $\mu(\mathrm{Lim}\,(T)) > 0$. Let T^* and $Y_{\eta,\zeta} = Y^X_{\eta,\zeta}$ be as in Lemma 9 and let $Y_{\eta,\zeta,\rho}$, S and z be as in the proof of Lemma 10. All we have to do is to show that S is almost of width f. Let us fix η, ζ and ρ. We shall now see that

(9) If $\eta' \in T^{*[\eta]} \cap {}^{n_i}2$ then
$$|\{\eta^+\colon \eta' \trianglelefteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\}|/2^{(n_{i+1}-n_i)} \le (1 - 2^{-(i+1)})^{|S \cap {}^{n_i}2|-1}.$$

Let $\eta^+ \in T^{*[\eta]} \cap {}^{n_{i+1}}2$; then, by the definition of S (see (3)), if $\rho^+ \in S \cap {}^{n_{i+1}}2$ then $\rho^+ + \eta^+ + z \in T$. Thus

$$\{\eta^+ : \eta' \trianglelefteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\} \subseteq \{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2, (\forall \rho^+ \in S)\rho^+ + \eta^+ + z \in T\}.$$

Let us take in (B2) $\eta = \eta'$, $k = |S \cap {}^{n_i}2|$, $\{\tau_l : l < k\} = S \cap {}^{n_i}2$, $\{\tau_l^+ : l < k\} \subseteq S \cap {}^{n_{i+1}}2$, and, for $l < k$, $\tau_l^+ \restriction n_i = \tau_l$, $\nu_l = \tau_l + z$, $\nu_l^+ = \tau_l^+ + z$; hence $\nu_l = \nu_l^+ \restriction n_i$ for $l < k$. Since for $l < k$, $\nu_l^+ + z = \tau_l^+ \in S \cap {}^{n_{i+1}}2$ we have

$$\begin{aligned} &\{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2, (\forall \rho^+ \in S)(\rho^+ + \eta^+ + z \in T\} \\ &\subseteq \{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2, (\forall l < k)(\nu_l^+ + \eta^+ \in T)\}, \end{aligned}$$

therefore by (B2)

$$\begin{aligned} &|\{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2, (\forall \rho^+ \in S)(\rho^+ + \eta^+ + z \in T\}| \\ &\leq 2^{n_{i+1}-n_i}(1 - 2^{-(i+1)})^{|S \cap {}^{n_i}2|-1}, \end{aligned}$$

which establishes (9).

(9) tells us how T^* grows from the level n_i to the level n_{i+1} and therefore $|T^* \cap {}^{n_i}2| \cdot 2^{-n_i} \leq \prod_{j<i}(1 - 2^{-(j+1)})^{|S \cap {}^{n_j}2|-1}$. Let $c_0 = \mu(\operatorname{Lim} T^*)$. We know that $c_0 > 0$ and we can assume $c_0 < 1$. Then

$$-\infty < \log c_0 \leq \log(|T^* \cap {}^{n_i}2| \cdot 2^{-n_i}) \leq \sum_{j<i}\left(\log(1 - 2^{-(j+1)}) \cdot (|S \cap {}^{n_j}2| - 1)\right).$$

Since $\log(1 - x) \leq -\frac{1}{2}x$ we get

$$\sum_{j<i} 2^{-(j+2)} \cdot (|S \cap {}^{n_{i+1}}2| - 1) \leq \log \frac{1}{c_0}.$$

We shall denote $4 \log \frac{1}{c_0}$ by c, so $\sum_{j<i} 2^{-j} \cdot (|S \cap {}^{n_j}2| - 1) \leq c$, and for every j, $2^{-j}(|S \cap {}^{n_j}2| - 1) \leq c$, hence $|S \cap {}^{n_j}2| \leq c \cdot 2^j + 1$. For $j > c$ we have, by our choice of the n_i's, $f(n_j) > j \cdot 2^j + 1 > c \cdot 2^j + 1 \geq |S \cap {}^{n_j}2|$, hence S is almost of width f.

Lemma 14 follows immediately from the following Lemma.

15. Lemma: *For every $n \in \omega$ and $0 < p < 1$ there is an $N > n$ such that, for every $n' \geq N$ and $t \subseteq {}^n2$, there is a $t' \subseteq {}^{n'}2$ which satisfies the following (i)–(iii).*

(i) *For each $\zeta \in t'$, $\zeta \restriction n \in t$.*

(ii) *For each* $\eta \in t$, $|t'^{[\eta]}| \geq 2^{n'-n} \cdot p$.

(iii) *If* $0 < k \leq 2^n$, $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^n2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n'}2$, $\nu_j^+ \neq \nu_l^+$ *for* $j < l < k$, $\eta + \nu_l \in t$, $\nu_l = \nu_l^+ \restriction n$ *for* $l < k$, *then*

$$|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2, (\forall l < k)\, \eta^+ + \nu_l^+ \in t'\}| \leq 2^{n'-n} p^{k-1}.$$

Proof: We shall prove the lemma by the probabilistic method. Let $n' > n$ and let $A = \{\eta^+ \in {}^{n'}2 : \eta^+ \restriction n \in t\}$. We construct a subset A^* of A as follows. We take a coin which yields heads with probability p. For each $\eta^+ \in A$ we toss this coin and we put η^+ in A^* iff the coin shows heads. We shall see that if we take $t' = A^*$ then, for sufficiently large n', the probability that (ii) holds has a positive lower bound which does not depend on n' while the probability that (iii) holds is arbitrarily close to 1. Hence there is an N and a t' as claimed by the lemma. We prove first two lemmas.

LEMMA 16: *For* $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ *as in Lemma 15 there are reals* $c_1, c_2 > 0$ *which depend only on* p, n *and* k *such that*

$$\Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2, \bigwedge_{l<k} \eta^+ + \nu_l^+ \in A*\}| \geq p^{k-1} 2^{n'-n}\Big) < c_1 e^{-c_2 \cdot 2^{n'}}.$$

Proof: We denote $2^{n'-n}$ with m. We set $({}^{n'}2)^{[\eta]} = \{\eta_j^+ : j < m\}$. Let G be the graph on m given by

$$iGj \text{ iff } \{\eta_i^+ + \nu_l^+ : l < k\} \cap \{\eta_j^+ + \nu_l^+ : l < k\} \neq \emptyset.$$

Obviously, each $i < m$ has at most k^2 neighbors in G hence, by a well known theorem, m can be decomposed into $k^2 + 1$ pairwise disjoint sets $B_0, \ldots, B_{k^2}$ such that, for every $i \leq k^2$, if $j, l \in B_i$ and $j \neq l$ then jGl does not hold. Let $d < \frac{1}{2}\min\{p^{l-1} - p^l : l \leq 2^n\} = \frac{1}{2} p^{2^n-1}(1-p) > 0$.

$$\begin{aligned} &\Pr\Big(|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \geq m \cdot p^{k-1}\Big) \\ (10) \qquad &\leq \Pr\Big(|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)\Big) \quad \text{since } p^k + d < p^{k-1}. \end{aligned}$$

Assume that

$$\begin{aligned} &\text{for every } i \leq k^2 \text{ such that } |B_i| \geq \frac{dm}{2k^2+2} \\ (11) \qquad &\text{we have } |\{j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \leq |B_i|\big(p^k + \frac{d}{2}\big); \end{aligned}$$

then

$$\{j < m\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\} \subseteq \bigcup_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} \{j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\} \quad \bigcup_{i \le k^2,\, |B_i| < \frac{dm}{2k^2+2}} B_i$$

hence

$$\begin{aligned} &|j < m\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \\ &\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} |j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| + \sum_{i \le k^2,\, |B_i| < \frac{dm}{2k^2+2}} |B_i| \\ &\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} |B_i|(p^k + \frac{d}{2}) + \sum_{i \le k^2,\, |B_i| < \frac{dm}{2k^2+2}} |B_i|, \qquad \text{by (11)} \\ &\le m(p^k + \frac{d}{2}) + (k^2+1)\frac{dm}{2k^2+2} = m(p^k + d). \end{aligned}$$

Therefore the event $|\{j < m\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)$ is incompatible with (11), so we continue the inequality (10) by

$$\begin{aligned} &\le \Pr\Big(\bigvee_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} (|\{j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2}))\Big) \\ &\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} \Pr\Big(|\{j \in B_i\colon \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\Big). \end{aligned} \tag{12}$$

For a fixed $j < m$ the events $\eta_j^+ + \nu_l^+ \in A^*$ for different l's are independent, hence $\Pr\left(\bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\right) = p^k$. For a fixed i the events $\bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*$ for different j's in B_i are independent since, by the definition of the B_i's, if $j_1, j_2 \in B_i$ and $j_1 \neq j_2$ then $\eta_{j_1}^+ + \nu_{l_1}^+ \neq \eta_{j_2}^+ + \nu_{l_2}^+$. We have here $|B_i|$ independent events, each with probability p^k. By a formula of probability theory (see, e.g., the formula $\Pr[X > a] < e^{-2a^2/n}$ in Spencer [2], p. 29)

$$\Pr\Big(\{j \in B_i\colon \bigwedge_{k<l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|p^k + \epsilon\Big) < e^{-\frac{2\epsilon^2}{|B_i|}}$$

and, taking $\epsilon = \frac{1}{2}|B_i|d$, we get

$$\Pr\Big(\{j \in B_i\colon \bigwedge_{k<l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\Big) < e^{-\frac{d^2|B_i|}{2}}.$$

Continuing (12) we get

$$\leq \sum_{i\leq k^2,\,|B_i|\geq \frac{dm}{2k^2+2}} e^{-\frac{d^2|B_i|}{2}} \leq \sum_{i\leq k^2,\,|B_i|\geq \frac{dm}{2k^2+2}} e^{-\frac{d^2}{2}\frac{dm}{2k^2+2}} \leq (k^2+1)e^{-\frac{d^3 2^{n'-n}}{4k^2+4}}.$$

Combining this with the inequalities (10) and (12) we get

$$\begin{aligned}&\Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ \bigwedge_{l<k} \eta^+ + \nu_l^+ \in A^*\}| \geq p^{k-1}2^{n'-n}\Big)\\ &< (k^2+1)e^{-\frac{d^3 2^{n'-n}}{4k^2+4}} = (k^2+1)e^{-\frac{d^3 2^{-n} 2^{n'}}{4k^2+4}}.\end{aligned}$$

Since $d = \frac{1}{2}p^{2^n-1}(1-p)$ this proves Lemma 16.

17. LEMMA: *There are c_3, c_4 which depend only on p and n such that*

$$\begin{aligned}&\Pr\Big(\bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+}\\ &\qquad |\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l<k)\, \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n}p^{k-1}\Big)\\ &\leq c_3(2^{n'})^{2^n} e^{-c_4 2^{n'}}\end{aligned} \tag{13}$$

where $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ are as in (iii) *of Lemma 15.*

Proof: By our requirements on $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ there are at most 2^n possible k's and η's and $(2^{n'})^{2^n}$ sequences $\langle \nu_0^+, \ldots, \nu_{k-1}^+\rangle$, while $\nu_0, \ldots, \nu_{k-1}$ are determined by $\nu_0^+, \ldots, \nu_{k-1}^+$ and n. Therefore we get, by Lemma 16,

$$\begin{aligned}&\Pr\Big(\bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+}\\ &\qquad \big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l<k)\, \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n}p^{k-1})\big)\\ &\leq \sum_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+}\\ &\qquad \Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l<k)\, \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n}p^{k-1}\Big)\\ &\leq 2^n \cdot 2^n \cdot (2^{n'})^{(2^n)} \cdot c_1 e^{-c_2 2^{n'}}.\end{aligned}$$

Proof of Lemma 15 (continued): For each $\eta^+ \in {}^{n'}2$ such that $\eta \trianglelefteq \eta^+$, $\eta^+ \in A^*$ if the coin shows heads and different tosses are independent, $|A^{*[\eta]}|$ is a binomial random variable with expectation $2^{n'-n}p$. By the central limit theorem of probability theory (see, e.g., Feller [1, Ch. 7]) the limit, as $n' \to \infty$, of $\Pr\left(|A^{*[\eta]}| \geq 2^{n'-n}p\right)$ is $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2}$, hence there is an N such that, for every $n' \geq N$, $\Pr\left(|A^{*[\eta]}| \geq 2^{n'-n}p\right) \geq \frac{1}{3}$. For different $\eta \in t$ the random variables $|A^{*[\eta]}|$ are idependent, hence

$$\Pr\left(\bigwedge_{\eta \in t} (|A^{*[\eta]}| \geq 2^{n'-n}p)\right) \geq \frac{1}{3^{|t|}} \geq \frac{1}{3^{2^n}}. \tag{14}$$

The right-hand side of (13) clearly vanishes as $n \to \infty$. Let us take N to be such that, for $n' \geq N$, the right-hand side of (13) is $< 3^{-2^n}$. Therefore we have, by (13) and (14),

$$\begin{aligned}\Pr\Big(&\bigvee_{k,\eta,\nu_0,\dots,\nu_{k-1},\nu_0^+,\dots,\nu_{k-1}^+} \\ &(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l < k)\, \eta^+ + \nu_l^+ \in A^*\}| < 2^{n'-n}p^{k-1}) \\ &\wedge \bigwedge_{\eta \in t} (|A^{*[\eta]}| \geq 2^{n'-n}p)\Big) > 0.\end{aligned} \tag{15}$$

By (15) there is a t' as required by the lemma.

4. Characterization of the meager-additive sets

18. THEOREM: *For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:*
   (a) *X is meager additive.*
   (b) *For every sequence $n_0 < n_1 < n_2 < \cdots$ of natural numbers there is a sequence $i_0 < i_1 < \cdots$ of natural numbers and a $y \in {}^\omega 2$ such that, for every $x \in X$ and for every sufficiently big $k < \omega$, there is an $l \in [i_k, i_{k+1})$ such that $x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$.*

Proof: Throughout this proof, if $x \in {}^\omega 2 \cup {}^{\omega >}2$, $k, l \in \omega$ and $k < l$, then $x \restriction [k, l)$ will denote the sequence $\xi \in {}^{l-k}2$ such that $\xi(i) = x(k+i)$ for all $i < l - k$.

(b)$\to$(a): In order to prove (a) it clearly suffices to show that $X + \mathrm{Lim}(T)$ is meager for every nowhere dense tree T.

For a nowhere dense tree T, let $\langle n_i : i < \omega \rangle$ be an ascending sequence of natural numbers such that $n_0 = 0$ and, for every $i \in \omega$, there is a sequence $\nu_i \in {}^{n_{i+1}-n_i}2$ such that, for every $\tau \in {}^{n_i}2$, $\tau \frown \nu_i \notin T$. Let $\langle i_j : j < \omega \rangle$ and y be as in (b); then, by (b), $X = \bigcup_{k \in \omega} X_k$ where

$$X_k = \{x \in X : (\forall m \geq k)(\exists l \in [i_m, i_{m+1}))\, x \restriction [n_l, n_{l+1}) = y \restriction (n_l, n_{l+1})\}.$$

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It clearly suffices to prove that $X_k + \mathrm{Lim}\,(T)$ is nowhere dense.

Let $\tau \in {}^{n_{i_m}}2$ for some $m \geq k$; we shall show that τ has an extension which is not in $X_k + \mathrm{Lim}\,(T)$. Let $\nu = \nu_{i_m} \frown \nu_{i_m+1} \frown \cdots \frown \nu_{i_{m+1}-1}$ and let $\rho = y \restriction [n_{i_m}, n_{i_{m+1}}) + \nu$. We show that no extension z of $\tau \frown \rho$ is in $X_k + \mathrm{Lim}\,(T)$. Suppose $\tau \frown \rho \trianglelefteq z \in X_k + \mathrm{Lim}\,(T)$; then $z = x + w$, $x \in X_k$, $w \in \mathrm{Lim}\,(T)$. Therefore $\tau = \tau_1 + \tau_2$ and $\rho = \rho_1 + \rho_2$ such that $\tau_1 \frown \rho_1 \trianglelefteq x$ and $\tau_2 \frown \rho_2 \trianglelefteq w$, hence $\tau_2 \frown \rho_2 \in T$. Let $\xi \in {}^{n_{i_m}}2$ be such that $\xi(j) = 0$ for every $j < n_{i_m}$, and let $\rho' = \xi \frown \rho$, $\rho'_1 = \xi \frown \rho_1$, $\rho'_2 = \xi \frown \rho_2$. Clearly $\rho' = \rho'_1 + \rho'_2$. Since $x \in X_k$ there is, by (b), an $l \in [i_m, i_{m+1})$ such that $x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$. Since $\tau_1 \frown \rho_1 \trianglelefteq x$ we have $\rho'_1 \restriction [n_{i_m}, n_{i_m+1}) = x \restriction [n_{i_m}, n_{i_m+1})$ and hence $\rho_1 \restriction [n_l, n_{l+1}) = x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$. Therefore, by the definition of ρ and ν,

$$\begin{aligned} y \restriction [n_l, n_{l+1}) + \rho'_2 \restriction [n_l, n_{l+1}) &= \rho'_1 \restriction [n_l, n_{l+1}) + \rho'_2 \restriction [n_l, n_{l+1}) = \rho' \restriction [n_l, n_{l+1}) \\ &= y \restriction [n_l, n_{l+1}) + \nu_l, \end{aligned}$$

hence $\rho'_2 \restriction [n_l, n_{l+1}) = \nu_l$. By the definition of ν_l, $\tau_2 \frown \rho_2 \notin T$, contradicting $\tau_2 \frown \rho_2 \in T$.

(a)$\to$(b): Let X be meager-additive. Let $\langle n_i \colon i < \omega \rangle$ be an ascending sequence of natural numbers. Let $B = \{x \in {}^{\omega}2 \colon \forall j (\exists k \in [n_j, n_{j+1}))\, x(k) \neq 0\}$ and $T = \{x \restriction n \colon x \in B, n \in \omega\}$. Clearly $B = \mathrm{Lim}\,(T)$ is nowhere dense, so $X + \mathrm{Lim}\,(T)$ is meager, hence there are nowhere dense trees S_n, $n \in \omega$ such that, for every n, $S_n \subseteq S_{n+1}$ and $X + \mathrm{Lim}\,(T) \subseteq \bigcup_{n \in \omega} S_n$. We define now $\langle i_l \colon l < \omega \rangle$, an ascending sequence of natural numbers, and $\langle \nu_l \colon l < \omega \rangle$, a sequence in ${}^{\omega >}2$, by recursion as follows. Let $i_0 = 0$. Given i_l let ν_l and i_{l+1} be such that $\nu_l \in {}^{n_{i_{l+1}} - n_{i_l}}2$ and, for every $\rho \in {}^{n_{i_l}}2$, $\rho \frown \nu_l \notin S_l$; there are such ν_l and i_{l+1} since S_l is nowhere dense. Let $y \in {}^{\omega}2$ be given by $y \restriction [n_{i_l}, n_{i_{l+1}}) = \nu_l$ for every $l < \omega$. We shall now prove that $\langle i_l \colon l < \omega \rangle$ and y are as required by (b).

Let $x \in X$, so $\mathrm{Lim}\,(x+T) = x + \mathrm{Lim}\,(T) \subseteq X + \mathrm{Lim}\,(T) \subseteq \bigcup_{n \in \omega} S_n$. Therefore, by Lemma 7 (where we take $x + T$ for S) there is an $\eta \in T$ and $n \in \omega$ such that $x + T^{[\eta]} \subseteq S_n$. Let k be such that $k \geq n$ and $i_k \geq \mathrm{length}\,(\eta)$. By $x + T^{[\eta]} \subseteq S_n$ we have $x \restriction n_{i_{k+1}} + (T^{[\eta]} \cap {}^{n_{i_{k+1}}}2) \subseteq S_n \subseteq S_k$. Thus for every $\rho \in T^{[\eta]} \cap {}^{n_{i_{k+1}}}2$, $x \restriction n_{i_{k+1}} + \rho \in S_k$, hence, by the definition of ν_k and y,

$$x \restriction [n_{i_k}, n_{i_{k+1}}) + \rho \restriction [n_{i_k}, n_{i_{k+1}}) \neq \nu_k = y \restriction [n_{i_k}, n_{i_{k+1}})$$

and therefore $x \restriction [n_{i_k}, n_{i_{k+1}}) - y \restriction [n_{i_k}, n_{i_{k+1}}) \neq \rho \restriction [n_{i_k}, n_{i_{k+1}})$, i.e.,

$$x \restriction [n_{i_k}, n_{i_{k+1}}) - y \restriction [n_{i_k}, n_{i_{k+1}}) \notin \{\rho \restriction [n_{i_k}, n_{i_{k+1}}) \colon \rho \in T^{[\eta]}\}.$$

Since $i_k > \text{length}(\eta)$ this can happen, by the definition of T, only if for some $i_k \leq j < i_{k+1}$, $x \restriction [n_j, n_{j+1}) - y \restriction [n_j, n_{j+1})$ is identically zero, and this is what we had to prove.

5. An uncountable null-additive set

19. THEOREM: *If the continuum hypothesis holds, then there is an uncountable null-additive set.*

Proof: Let, by CH, $\langle f_\alpha \colon \alpha < \omega_1 \rangle$ be a sequence containing all corsets and let $\langle T_\alpha \colon \alpha < \omega_1 \rangle$ be a sequence containing all perfect trees. Let E be the set of all limit ordinals $\delta < \omega_1$ such that, for every $\alpha, \beta < \delta$ and $n < \omega$, there is a $\gamma < \delta$ such that

$$T_\gamma \subseteq T_\alpha, \quad T_\gamma \cap {}^n 2 = T_\alpha \cap {}^n 2$$

$$\text{and, for all } m, \quad |T_\gamma \cap {}^m 2| \leq \max(|T_\alpha \cap {}^m 2|, f_\beta(m)).$$

Clearly E is closed. For every $\alpha, \beta < \omega_1$ there is a perfect tree T such that $T \subseteq T_\alpha$, $T \cap {}^n 2 = T_\alpha \cap {}^n 2$ and, for all $m < \omega$, we have $|T \cap {}^m 2| \leq \max(|T_\alpha \cap {}^m 2|, f_\beta(m))$. This tree T is T_γ for some $\gamma < \omega_1$. By a simple closure argument this implies that E is unbounded.

We need now the following lemma which will be proved later.

20. LEMMA: *There is an increasing and continuous sequence $\langle \delta_\zeta \colon \zeta < \omega_1 \rangle$ of ordinals in E such that for every $\zeta < \omega_1$, $k < \omega$ and $\alpha < \delta_\zeta$ there is an ordinal γ which is good for (ζ, α, k), where by γ* **is good for** *(ζ, α, k) we mean that*

(16)
(i) $\gamma < \delta_{\zeta+1}$,
(ii) $T_\gamma \subseteq T_\alpha$, $T_\gamma \cap {}^k 2 = T_\alpha \cap {}^k 2$,
(iii) *for all $\xi \leq \zeta$ such that $\delta_\xi > \alpha$ and for every $\epsilon < \delta_\zeta$, there is a $\beta < \delta_\xi$ such that $T_\gamma \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ.*

For $\xi < \omega_1$ let γ_ξ be the γ which is good for $(\xi, 0, 0)$. We choose $\eta_\xi \in \text{Lim}(T_{\gamma\xi}) \setminus \{\eta_\beta \colon \beta < \xi\}$, and let $X = \{\eta_\xi \colon \xi < \omega_1\}$. X is clearly uncountable. We shall prove that X is null-additive by proving that X satisfies condition (c) of Theorem 13. For a given corset f, $f = f_\epsilon$ for some $\epsilon < \omega_1$. Let $\xi < \omega_1$ be such that $\delta_\xi > \epsilon$. Let $Z = \{\beta < \delta_{\xi+1}\colon T_\beta$ is almost of width $f_\epsilon\}$. We shall see that $X \subseteq \{\eta_\zeta\colon \zeta \leq \xi\} \cup \bigcup_{\beta\in Z} \mathrm{Lim}\,(T_\beta)$. Since Z and ξ are countable, condition (c) of Theorem 13 holds.

Let $\zeta > \xi$; it suffices to prove that $\eta_\zeta \in \mathrm{Lim}\,(T_\beta)$ for some $\beta \in Z$. $\epsilon < \delta_\xi$ and, since γ_ζ is good for $\alpha = k = 0$, there is a $\beta < \delta_\xi$ such that $T_{\gamma_\zeta} \subseteq T_\beta$, and T_β is of width f_ϵ. Thus $\beta \in Z$ and $\eta_\zeta \in \mathrm{Lim}\,(T_{\gamma\zeta}) \subseteq \mathrm{Lim}\,(T_\beta)$.

Proof of Lemma 20: We define $\langle \delta_\zeta\colon \zeta < \omega_1\rangle$ as follows. δ_0 is the least member of E. For a limit ordinal ζ, we set $\delta_\zeta = \bigcup_{\xi<\zeta} \delta_\xi$. Since $\delta_\xi \in E$ for $\xi < \zeta$, also $\delta_\zeta \in E$. We shall now define $\delta_{\zeta+1}$. We shall assume, as an induction hypothesis, that for each $\xi < \zeta$ the lemma holds. For each $\alpha < \delta_\zeta$ and $k < \omega$ we shall find a $\gamma(\alpha, k)$ which is good for (ζ, α, k) and we shall choose $\delta_{\zeta+1}$ to be the least member of E greater than all these $\gamma(\alpha, k)$'s.

First we shall show that what the lemma claims holds for the case where ζ is a successor or 0. Whenever we shall write $\zeta - 1$ we shall assume that ζ is a successor. Let $\alpha < \delta_\zeta$ and $k < \omega$ be given, and let $\{\epsilon_n\colon n < \omega\} = \{\epsilon\colon \epsilon < \delta_\zeta\}$. We define sequences $\langle \alpha_n\colon n < \omega\rangle$ and $\langle k_n\colon n < \omega\rangle$ so that

(a) $k_0 = k$. If $\zeta = 0$ or $\alpha < \delta_{\zeta-1}$ then $\alpha_0 = \alpha$. If $\alpha \geq \delta_{\zeta-1}$ then α_0 is an ordinal which is good for $(\zeta - 1, \alpha, k)$. In any case $\alpha_0 < \delta_\zeta$, $T_{\alpha_0} \subseteq T_\alpha$ and $T_{\alpha_0} \cap {}^k2 = T_\alpha \cap {}^k2$.

(b) $\alpha_{n+1} < \delta_\zeta$.

(c) $T_{\alpha_{n+1}} \subseteq T_{\alpha_n}$.

(d) $T_{\alpha_{n+1}} \cap {}^{k_n}2 = T_{\alpha_n} \cap {}^{k_n}2$.

(e) $T_{\alpha_{n+1}}$ is almost of width f_{ϵ_n}.

(f) $k_{n+1} > k_n$ and every $\eta \in T_{\alpha_{n+1}} \cap {}^{k_n}2$ has at least two extensions in $T_{\alpha_{n+1}} \cap {}^{k_{n+1}}2$.

There are indeed such sequences $\langle \alpha_n\colon n < \omega\rangle$ and $\langle k_n\colon n < \omega\rangle$. (a) determines k_0 and α_0; if $\alpha < \delta_{\zeta-1}$, then there is an α_0 as in (a) by the induction hypothesis. δ_ζ is in E and let us take $\alpha_n, \epsilon_n, k_n, \alpha_{n+1}$ for α, β, n, γ in the definition of E, then $\delta_\zeta \in E$ says that there is an α_{n+1} which satisfies (b)–(e). Since $T_{\alpha_{n+1}}$ is perfect there is a k_{n+1} as in (f).

Let $T = \bigcap_{n\in\omega} T_{\alpha_n}$. By (c), (d), (f) T is a perfect tree, hence it is T_γ for some $\gamma < \omega_1$. Since T, and therefore also γ, depend on α and k, we denote γ by $\gamma(\alpha, k)$. As is easily seen $T_{\gamma(\alpha,k)} \subseteq T_\alpha$, $T_{\gamma(\alpha,k)} \cap {}^k2 = T_\alpha \cap {}^k2$, and, for every $\epsilon < \delta_\zeta$, $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{l+1}} \subseteq T_\alpha$, where l is such that $\epsilon = \epsilon_l$. This means that (iii) of (16) holds for $\xi = \zeta$. We shall have to show that (iii) holds for $\xi < \zeta$ and to deal with the case where ζ is a limit ordinal.

If ζ is a limit ordinal let $\langle \zeta_n : n < \omega \rangle$ be an increasing sequence such that $\delta_{\zeta_0} > \alpha$ and $\bigcup_{n<\omega} \zeta_n = \zeta$. We construct the sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$ as in the case where ζ is a successor, except that (a), (b), (e) are replaced by

(a′) $k_0 = k$, $\alpha_0 = \alpha$.

(b′) $\alpha_n < \delta_{\zeta_n}$.

(e′) α_{n+1} is good for (ζ_n, α, k).

By the induction hypothesis that the lemma holds for the ζ_n's there are indeed such sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$. Let $T = \bigcap_{n<\omega} T_{\alpha_n}$. As above, T is a perfect tree and $T = T_{\gamma(\alpha,k)}$, $T_{\gamma(\alpha,k)} \subseteq T_\alpha$ and $T_{\gamma(\alpha,k)} \cap {}^k 2 = T_\alpha \cap {}^k 2$.

We shall now see that for both cases of ζ with which we are dealing, (iii) holds for $\xi < \zeta$. If ζ is a successor then $\xi \le \zeta - 1$ and, since α_0 is, by (a), good for $(\zeta - 1, \alpha, k)$, there is a $\beta < \delta_\zeta$ such that $T_{\alpha_0} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Note that if $\alpha < \delta_{\zeta-1}$ then, by the induction hypothesis, we have a $\gamma < \delta_\zeta$ which is good for (ζ, α, k), and if $\zeta = 0$ then (iii) holds vacuously, hence we may assume that $\zeta > 0$ and $\alpha \in [\delta_{\zeta-1}, \delta_\zeta)$. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_0}$, β is as required by (iii). If ζ is a limit ordinal, then $\xi \le \zeta_n$ for some $n < \omega$. Since α_{n+1} is good for ζ_n, then there is a $\beta < \delta_\xi$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{n+1}}$, β is as required by (iii).

The only case left is that where ζ is a limit ordinal and $\xi = \zeta$ in (iii). Since $\alpha, \epsilon < \zeta$ also $\alpha, \epsilon < \zeta_n$ for some $n < \omega$. α_{n+1} is good for ζ_n, hence there is a $\beta < \delta_{\zeta_n}$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{n+1}}$ and $\zeta_n < \zeta$, β is as required by (iii).

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