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MODELS WITH SECOND ORDER PROPERTIES I. BOOLEAN ALGEBRAS WITH NO DEFINABLE AUTOMORPHISMS

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We prove that there are models with few automorphisms, and give alternative proof to Chang's two-cardinal theorem.

Introduction

This is (the first) part of a series of papers, in which we try to build models with various second order properties.

Examples of such properties are:

(a) Let T be a (first order) theory, and we want a model M, such that $M \models T$ and every Boolean algebra definable in M, all its automorphism are definable in M.

(b) We extend first order logic by allowing quantification over automorphism of Boolean algebras, and want to show this logic is compact (see [3, p. 356] for a definition of second order quantification) and this amounts to (a) assuming T satisfies various schemes, e.g. T has a model expanding $(H(\kappa^+), \in)$.

(c) We have a model M of T in which a (definable) Boolean algebra is rigid, and we want to build other such models.

We want more concretely to build those models in specific cardinals, and replace "automorphisms of Boolean algebras" by automorphisms of other structures (e.g. ordered fields) branches of trees etc.

Note that (a) is stronger than (b) which is stronger than (c); and in (b), (c), we can expand the language, so w.l.o.g. the theory has Skolem functions, and has enough build in set theory. But in (a) this is forbidden. Note that if we e.g. want to get a (λ^+, λ)-model in which a tree of height λ^+ has only definable branches, Chang's original proof of his two cardinal theorem is not appropriate, as he extends the language (so to encode finite sets); see Section 2. Note also that (b) is good for giving examples of compact logic. For example, allowing quantification over automorphisms of Boolean algebras, gives us a logic stronger than first order even on finite models (we can say a Boolean algebra is atomic, and has an automorphism of order two, which moves every atom. This distinguishes among the finite Boolean algebras between those with even number of atoms. As we shall prove

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this logic is compact (assuming GCH) this partially solves a problem of Freedman (see [3], p. 343, and 1).

Let us return to this paper.

This is a continuation of Rubin and Shelah [2]. There we prove (assuming $\diamondsuit_{\aleph_1}$ or even CH) that for a Boolean algebra B, the (two-sorted) model $(B, \mathrm{Aut}\, B)$ (with the Boolean operations, and the operations of applying an automorphism where $\mathrm{Aut}\, B$ is the group of automorphism of B) is elementarily equivalent to $(B', \mathrm{Aut}\, B')$ for some B' of cardinality $\aleph_1$. We can add more relations and make other strengthening. Our first aim was to generalize this to uncountable cardinalities. This is done in Section 1, for an appropriate λ our results are stronger; because we prove a theory T has a model in λ^+ in which every automorphism of $(P(M) \cup Q(M), R^M)$ is inner (i.e. definable is the model) and we do not expand the language to have, essentially, some replacement axioms. Here the requirements on λ are severe; but this paper was written in order to exemplify the technique developed for solving the problem, and better results will appear in [5].

In Section 2 we use our technique to prove (a slightly stronger version of) Chang two cardinal theorem without expanding the language.

The result was announced in [4].

Remarks. (1) In Section 1 we can easily omit hypothesis (E) by amalgamating the conditions as done in Section 2.

(2) We can replace "λ measurable" by "λ strongly inaccessible". The only change is in 1.9.2C(2). We redefine the equivalence relation S as: $a\,S\,b$ *if* $a = b$ or there is a 2-indiscernible sequence in N of power $\geqslant \mu(|A|+|T|)$ in which a and b appear (where $\mu(\kappa)$ is the Hanf member of omitting a type for theories of cardinality κ).

1. A model with only definable automorphisms

1.1 Assumption.

(A) $\lambda = \lambda^{<\lambda}$ (hence λ is regular).

(B) $\diamondsuit(S^*)$ where $S^* = \{\delta < \lambda^+ \colon \mathrm{cf}\ \delta = \lambda\}$), and let $\pi_\delta : \delta \to \delta$ be functions (for $\delta \in S^*$) such that for every $\pi : \lambda^+ \to \lambda^+$ $\{\delta \in S^* \colon \pi \restriction \alpha = \pi_\alpha\}$ is stationary; this implies $2^\lambda = \lambda^+$.

(C) T will be a complete, (first order) theory, $|T| < \lambda$, P, Q one place predicate, R a two-place predicate (the interpretation is: Q is a family of subsets of P, and R the membership relation). We suppose T contains the sentences saying P, Q are disjoint, $x\,R\,y \to P(x) \wedge Q(y)$ and R satisfies the extensionality axiom; *and*: *the strong independence property*

$$(\forall x_1, \ldots, x_n \in P)(\forall y_i, \ldots, y_n \in P)\Big[\bigwedge_{i,j} x_i \neq y_1 \to$$
$$(\exists z \in Q)\Big(\bigwedge_{i=1}^{n} x_i R z \wedge \bigwedge_{j=1}^{n} \neg y_j R z\Big)\Big]$$

(*note* in a λ-saturated model the strong independence property is equivalent to the corresponding property with $<\lambda$ variables).

(D) λ measurable, is particular λ is strongly inaccessible, and $\lambda \to (\omega)^{<\omega}_{\aleph_0}$, hence for every $\mu < \lambda$, $\lambda \to (\omega)^{<\omega}_{\mu}$.

(E) There is a family $\{S_\alpha : \alpha < \lambda^+\}$ of stationary, almost disjoint subsets of λ (i.e. for $\alpha \neq \beta$, $|S_\alpha \cap S_\beta| < \lambda$).

1.2. Conclusion. T has a λ-saturated model M in λ^+, such that every automorphisms of $(P \cup Q, R, Q)$ is definable in M (with parameters).

1.3. Remark. Conditions (D) and (E) are not necessary, but clarify the exposition.

We look at $a \in Q(M)$ (M a model of T) as a subset of $P(M)$.

1.4. Sketch of the Construction. We shall build an elementary chain of models M_α ($\alpha < \lambda^+$) $M_{\alpha+1}$ is λ-saturated, $\|M_\alpha\| = \lambda$ and $\bigcup_\alpha M_\alpha$ is the desired model; for the notational simplicity M_α has universe $\lambda(1+\alpha)$. The chain will be continuous and for $\alpha \in S^*$ (i.e. cf $\alpha = \lambda$) we shall "kill" the automorphism π_α, except when it is definable. To kill means that we prevent $\bigcup_\alpha M_\alpha$ from having an automorphism π such that $\pi \restriction M_\alpha = \pi_\alpha$. The murder is done in the construction of $M_{\alpha+1}$, by adding an element $a^\alpha \in Q(M_{\alpha+1})$ such that the sets $\pi(a^\alpha_{M_\alpha})$, and its complemented in $P(M_\alpha)$ cannot be separated in $M_{\alpha+1}$ by a formula; and we shall demand that in every M_β, $\beta > \alpha$ those sets cannot be separated.

The main difficulty will be preserving the non-separability in limit stages and for overcoming this difficulty we shall strengthen our demands.

1.5. Definition. Let A, B be disjoint subsets of a model M, λ a cardinal, D a filter over λ. The sets are *D-strongly unseparated* if there are sequences $\langle a_{\alpha,n} : n < \omega\rangle$ for $\alpha < \lambda$ such that:

(a) $a_{\alpha,0} \in A$, $a_{\alpha,1} \in B$ for each α;

(b) for every (finite sequence) $\bar{c} \in M$, the set $\{\alpha : \langle a_{\alpha,n} : n < \omega\rangle$ is an indiscernible sequence over $\bar{c}\}$ belongs to D.

We say that $\langle\langle a_{\alpha,n} : n < \omega\rangle : \alpha < \lambda\rangle$ exemplify this; and such a sequence satisfying (b) will be called D-nice.

1.6. Definition. D_λ is the filter generated by the closed unbounded subsets of λ. In 1.5 if $D = D_\lambda$ we omit it. If $S \subseteq \lambda$ is stationary $D(S)$ is the filter generated by the members of D and S.

Now we prove the theorem. The proof is divided into stages.

1.7. Stage. We define by induction on $i < \lambda^+$ models M_i such that for $\beta < i$ $M_\beta <$ M_i (so we build an elementary chain). M_i has the universe $\lambda(1+i)$ (so necessarily the chain is continuous) and for i successor, M_i is λ-saturated. (This implies for $i \in S^*$ this holds too, but if $1 < \mathrm{cf}\ i < \lambda$, not necessarily). During the induction we define for some of the β's sets $A^\beta, B^\beta \subseteq M_\beta$ and sequences $\boldsymbol{a}^\beta_\zeta = \langle a^\beta_{\zeta,n} : n < \omega \rangle$ for $\zeta < \lambda$, and demand that for each $\beta \le i$, $\langle \boldsymbol{a}^\beta_\zeta : \zeta < \lambda \rangle$ exemplify A^β, B^β are $D(S_\beta)$-strongly unseparated in M_i. So for $\beta < i$ we have to assure only $\langle \boldsymbol{a}^\beta_\zeta : \zeta < \lambda \rangle$ is $D(S_\beta)$-nice.

We let M_0 be any saturated model of T of cardinality λ, whose universe is λ.

For $i = \delta$ a limit ordinal we let $M_\delta = \bigcup_{j<\delta} M_j$. It is easy to see all the induction hypothesis from 1.7 still holds.

1.8. Stage. Suppose $i = j + 1$, cf $j \neq \lambda$, M_j is defined and we have to define M_i.

M_i should be an elementary extension of M_j, be λ-saturated, and have universe $\lambda(1+i)$, so the cardinality of $|M_i| - |M_j|$ should be λ.

In this stage we do not add new sequences $\langle \boldsymbol{a}_\zeta : \zeta < \lambda \rangle$ but we have to fulfill our old obligations: $\langle \boldsymbol{a}^\alpha_\zeta : \zeta < \lambda \rangle$ should be$D(S_\alpha)$-nice in M_i whenever defined, $\alpha < j$.

1.8.1. Notation. Let $|M_j| = \{b_\xi : \xi < \lambda\}$. Now we have at most λ α's smaller than j for which $\langle \boldsymbol{a}^\alpha_\zeta : \zeta < \lambda \rangle$ is defined, so we list them without repetitions: $\alpha(\rho) : \rho < \rho(0) \le \lambda$. We let $S'_{\alpha(\rho)} = S_{\alpha(\rho)} - \bigcup_{\xi<\rho} S_{\alpha(\xi)}$; so the sets $S'_{\alpha(\rho)} (\rho < \rho(0))$ are pairwise disjoint, and $S'_{\alpha(\rho)} = S_{\alpha(\rho)} \mod D_\lambda$ (in fact $S'_{\alpha(\rho)} \subseteq S_{\alpha(\rho)}$, $|S_{\alpha(\rho)} - S'_{\alpha(\rho)}| < \lambda$) so $S'_{\alpha(\rho)} \in D(S_{\alpha(\rho)})$.

Now we define by induction on $\gamma < \lambda$, types $p_\gamma = p_\gamma\ (x_0, \ldots, x_\beta, \ldots)_{\beta<\gamma}$ in M_j (i.e. set of formulas with parameters from M_j, finitely satisfied in M_j, with the free variables $\{x_\beta : \beta < \gamma\}$). We define them such that:

(a) $|p_\gamma| < \lambda$, $p_\gamma (\gamma < \lambda)$ is increasing and continuous;

(b) even $p_\gamma \cup \{x_\beta \neq b_\xi : \beta < \gamma,\ \xi < \lambda\}$ is a type in M_j (at the end $\bigcup_{\gamma<\lambda} p_\gamma$ is essentially, the diagram of M_i).

For constructing them we let $\{q_\beta : \beta < \lambda\}$ be a list of all types in M_j, of cardinality $< \lambda$, and free variables among $\{y\} \cup \{x_\beta : \beta < \lambda\}$ only, each type appearing λ times. By a possible renaming we can assume $q_\beta = q_\beta (y, x_0, \ldots, x_\xi, \ldots b_0, \ldots, b_\xi, \ldots)_{\xi<\beta}$ (i.e. the free variables of q_β are among $\{y\} \cup \{x_\xi : \xi < \beta\}$, and the parameters appearing in it are among $\{b_\xi : \xi < \beta\}$).

All this is possible as $\lambda = \lambda^{<\lambda}$ (assumption A). (This will be used to assure M_i is λ-saturated.) (Note that many times q_β will have much fewer variables and parameters.)

We define p_γ by induction on γ:

first case $\gamma = 0$: $p_0 = \emptyset$;

second case γ *limit*: let $p_\gamma = \bigcup_{\beta<\gamma} p_\beta$.

1.8.2. Substage third case $\gamma = \beta + 2$.

(Here we take care of the saturativity of M_i.) So $p_{\beta+1}$ is already defined, we separate three cases.

(a) If $p_{\beta+1} \cup \{x_\zeta \neq b_\xi: \zeta < \beta+1, \xi < \lambda\} \cup q_\beta \cup \{y \neq b_\xi: \xi < \lambda\}$ is consistent (i.e. finitely satisfiable in M_j), then we let

$$p_\gamma = p_{\beta+1} \cup q_\beta(x_{\beta+1}, x_0, \ldots, b_0, \ldots).$$

(b) If not (a) but

$$p' = p_{\beta+1} \cup \{x_\zeta \neq b_\xi: \zeta < \beta+1, \xi < \lambda\} \cup q_\beta.$$

is consistent, then as not (a), for some $n > \omega$, and $b^0, \ldots, b^n \in M_j$, $p' \cup \{y \neq b^l : l \leq n\}$ is inconsistent, hence for some l, $p' \cup \{y = b^l\}$ is consistent, so let

$$p_\gamma = p_{\beta+1} \cup q(b^l, x_0, \ldots, b_0, \ldots).$$

(c) Not (a) nor (b), then let $p_\gamma = p_{\beta+1}$

1.8.3 Substage fourth case $\gamma = \beta+1$, β limit.

In this substage we take care of the niceness of the appropriate sequences. Now we separate to two cases:

Case (α): for some $\alpha < \lambda$, the following holds

(i) $\beta \in S'_\alpha$;

(ii) p_β is a type over $B_\beta = \{b_\xi : \xi < \beta\}$ (i.e. all parameters appearing in p_β are from B_β);

(iii) $\boldsymbol{a}^\alpha_\beta$ is indiscernible over B_β.

In this case we let

$$p_\gamma = p_\beta \cup \{\text{the formulas saying } \boldsymbol{a}^\alpha_\beta \text{ is an indiscernible sequence over } B_\beta \cup \{x_\xi : \xi < \beta\}\}$$

(see below for the proof p_γ is appropriate).

Case (β): there is no such α.

Then let $p_\gamma = p_\beta$.

1.8.3A. Claim. $p_\gamma \cup \{x_\beta \neq b_\xi : \beta < \gamma, \xi < \lambda\}$ is consistent.

We should check any finite subset is consistent. Suppose Γ is a finite inconsistent subset.

Let

$$\varphi^{dj}_{m,n}(x_0, \ldots, x_{m-1}; y_0, \ldots, y_{n-1}) = \bigwedge_{l,k} x_l \neq y_k.$$

But for notational simplicity we write φ^{dj} instead $\varphi^{dj}_{m,n}$. We can assume p_β is closed under conjunctions so we can assume this finite subset has the form

$$\{\psi(\bar{x}; \bar{b})\} \cup \{\varphi_l(a^\alpha_{\beta,n(1,l)}, a^\alpha_{\beta,n(2,l)}, \ldots, a^\alpha_{\beta,n(k(l),l)}; \bar{x}, \bar{b}^l) \equiv \varphi_l(a^\alpha_{\beta,m(1,l)}, a^\alpha_{\beta,m(2,l)}, \ldots, a_{\beta,m(k(l),l)}; \bar{x}, \bar{b}^l) : l < l(0)\} \cup \{\varphi^{dj}(\bar{x}; \bar{c})\}$$

where $n(1, l) < n(2, l) < n(3, l) < \dots, m(1, l) < m(2, l) < m(3, l) < \dots, \bar{x} = \langle x_{\xi(0)}, \dots, x_{\xi(l(1)-1)} \rangle$, $i(l) < \beta$, $\bar{b}, \bar{b}^l \in B_\beta$ and $\bar{c} \in M$. Let the conjunction of the second sets be $\theta(\bar{x}, a^\alpha_{\beta,0}, \dots, a^\alpha_{\beta,l(2)}; \bar{b}^*)$ (so $\bar{b}^* \in B_\beta$). As Γ is inconsistent

$$M \models (\forall \bar{x})(\varphi(\bar{x}, \bar{b}) \wedge \theta(\bar{x}, a^\alpha_{\beta,0}, \dots, \bar{b}^*) \rightarrow \neg\varphi^{dj}(\bar{x}; \bar{c})).$$

Let

$$\bar{x}^l = \langle x^l_{\xi(0)}, \dots, x^l_{\xi(l(1)-1)} \rangle, \qquad \bar{c} = \langle c_0, \dots, c_{l(2)-1} \rangle$$

$$M \models (\forall \bar{x}^0, \dots, \bar{x}^{l(2)}) \Big[\bigwedge_{l < l(2)} (\varphi(\bar{x}^l, \bar{b}) \wedge \theta(\bar{x}^l, \bar{a}^\alpha_{\beta,0}, \dots, \bar{b}^*) \rightarrow \neg \bigwedge_{l \neq k} \varphi^{dj}(\bar{x}^l; \bar{x}^k) \Big].$$

As $\varphi(\bar{x}; \bar{b}) \in p_\gamma$, there are $\bar{d}^l \in M (l \leq l(2))$, such that

$$M \models \bigwedge_l \varphi[\bar{d}^l, \bar{b}] \wedge \bigwedge_{l \neq k} \varphi^{dj}[\bar{d}^l; \bar{d}^k]$$

(define by induction on l). By Ramsey theorem there are natural numbers $n(0) < n(1) < n(2) < \cdots$ such that $\{a^\alpha_{\beta,n(0)}, a^\alpha_{\beta,n(1)}, \dots\}$ is an indiscernible sequence for $\varphi_l(y_1, \dots, y_{k(l)}, \bar{d}^k, \bar{b}^l)$ for each $l < l(0)$, $k \leq l(2)$ and is disjoined to $\{c_k : < l(2)\}$ Hence

$$M \models \bigwedge_l [\varphi(\bar{d}^l_l, \bar{b}) \wedge \theta(\bar{d}^l, a^\alpha_{\beta,n(0)}, a^\alpha_{\beta,n(1)}, \dots, \bar{b}^*)] \wedge \bigwedge_{l \neq k} \varphi^{dj}(\bar{d}^l; \bar{d}^k).$$

so

$$M \models \neg (\forall \bar{x}^0, \dots, \bar{x}^{l(2)}) \Big[\bigwedge_{l < l(2)} (\varphi(\bar{x}^l; \bar{b}) \wedge \theta(\bar{x}^l, a^\alpha_{\beta,n(0)}, a^\alpha_{\beta,n(1)}, \dots, \bar{b}^*)) \rightarrow \neg \bigwedge_{l \neq k} \varphi^{dj}(x^l; \bar{x}^k) \Big].$$

But $\bar{b}, \bar{b}^* \in B_\beta$, and $\langle a^\alpha_{\beta,m} : m < \omega \rangle$ is indiscernible over B_β. Hence

$$M \models \neg (\forall \bar{x}^0, \dots, x^{l(1)} \Big[\bigwedge_{l < l(2)} (\varphi(\bar{x}^l, \bar{b}) \wedge \theta(\bar{x}^1, a^\alpha_{\beta,0} a^\alpha_{\beta,1}, \dots; \bar{b}^*) \rightarrow \neg \bigwedge_{l \neq k} \varphi^{dj}(\bar{x}^l; \bar{x}^k) \Big]$$

contradicting a previous statement. So Γ, hence p_γ is consistent, and we finish subcase 1.8.3A.

1.8.3B. The model.

Now we define $p_\lambda = \bigcup_{\gamma < \lambda} p_\gamma$, clearly $p_\lambda \cup \{x_\beta \neq b_\xi : \beta < \lambda, \xi < \lambda\}$ is consistent, hence M_j has an elementary extension N, such that some assignment $x_\xi \mapsto c_\xi$

satisfies p. By the choice of $p_{\beta+2}$, $|M_j| \cup \{c_\xi : \xi < \lambda\}$ is the universe of an elementary submodel of N. Because for every formula $\varphi(y, c_{\xi(1)}, \ldots, b_{\zeta(1)}, \ldots)$ for some β $q_\beta = \{\varphi(y; x_{\xi(1)}, \ldots; b_{\zeta(1)}, \ldots)\}$. Look now at 1.8.2, if (a) holds, then $\varphi(x_{\beta+1}; x_{\xi(1)}, \ldots; b_{\zeta(1)}, \ldots) \in p_\gamma$ so $N \models \varphi[c_{\beta+1}; c_{\xi(1)}, \ldots; b_{\zeta(1)}, \ldots]$; if (b) holds, for some $b \in M_j$ $\varphi(b; x_{\xi(1)}, \ldots; b_{\zeta(1)}, \ldots)$ so $N \models \varphi[b; c_{\xi(1)}, \ldots; b_{\zeta(1)}, \ldots]$; and if (c) holds clearly $N \models \neg(\exists y)\varphi(y; c_{\xi(1)}, \ldots; b_{\zeta(1)}, \ldots)$. So by Tarski–Vaught test, M_i the submodel of N with universe $|M_j| \cup \{c_\xi : \xi < \lambda\}$, is elementary submodel of N. By notational changes we can assume $|M_i| = \lambda(1+i)$, and clearly $M_j \prec M_i$.

Let us check the more serious conditions. M_i is λ-saturated by the definition of $p_{\beta+2}$ in 1.8.2. But the main point is why for every $\alpha < i$ for which $\langle a^\alpha_\zeta : \zeta < \lambda \rangle$ is defined, it is nice in M_i. Clearly 1.8.3 is intended to secure this, and it is clear from it that it suffices to prove $\{\beta < \lambda$: case (α) from 1.8.3 hold$\} \in D(S_\alpha)$. For this (as $D(S_\alpha)$ is a filter) it suffices to check that for each of three conditions in case (α) of 1.8.3, the set of β's satisfying them is in $D(S_\alpha)$. (For the implicite condition "β limit" this is trivial.) For (i), as mentioned after 1.8.1, $S'_\alpha \in D(S_\alpha)$, so for (ii); define for every $\gamma < \lambda$

$$g(\gamma) = \min\{\zeta < \lambda : \text{for any } b_\xi \text{ appearing in } p_\gamma,\ \xi < \zeta\}$$

as $|p_\gamma| < \lambda$ clearly $g(\gamma) < \lambda$, hence

$$\{\gamma : (\forall \zeta < \gamma) g(\zeta) < \gamma\} \in D_\lambda$$

hence by p_γ's continuity, as $D_\lambda \subseteq D(S_\alpha)$ $\{\gamma : p_\gamma$ is a type over $B_\gamma\} \in D(S_\alpha)$. We are left with (iii). As $\langle a^\alpha_\zeta : \zeta < \lambda \rangle$ is $D(S_\alpha)$-nice in M_j for some closed unbounded subset $C_{\bar b}$ of λ, $\{\zeta < \lambda : a^\alpha_\zeta$ is an indiscernible sequence over $\bar b\} \subseteq S_\alpha \cap C_{\bar b}$;

Now for each $\beta < \lambda$, $C^*_\beta = \bigcap\{C_{\bar b} : \bar b \in B_\beta\}$ is closed unbounded hence $C^{**} = \{\beta : (\forall \zeta < \beta)(\beta \in C^*_\zeta)\}$ is closed unbounded and on $C^{**} \cap S'_\alpha$ (iii) holds. So we finish the construction of M_i, $i = j+1$ (i.e. 1.8).

1.9. Stage. We construct M_i, when $i = j+1$, cf $j = \lambda$. During this stage we denote M_i by M π_i by π.

We now assume π is an automorphism of $(P^M \cup Q^M, P, R)$ (otherwise we behave as in 1.8). So M is a saturated model of cardinality λ and a permutation π mapping P^M onto P^M, Q^M onto Q^M, preserving R, but not definable in M (i.e. by a first order formula with parameters). We want to "kill" π by constructing a model N, $M \prec N$, N λ-saturated, satisfying previous obligations (on niceness of sequences) and such that for some $a_i \in Q^N$

$$A_i = \{\pi b : b \in P^M, bRa_i\}, \qquad B_i = \{\pi b : b \in P^M, \neg bRa_i\}$$

are not separated in N by any formula from $L_{\lambda,\lambda}$ with parameters. Later we prove this implies they are $D(S_i)$-strongly inseparated in N. Together with $\diamondsuit(S^*)$ this will assure that in the end we get a model with only definable automorphisms.

1.9.1. Claim. π is not definable in M by an $L_{\lambda,\lambda}$-formula with parameters.

Proof. In a λ-saturated model, for a formula $\varphi(x_0,\ldots)\in L_{\lambda,\lambda}$ the holding of $M\models\varphi[a_0,\ldots]$ depends only on the first order type of $\langle a_0,\ldots\rangle$. Now suppose $\varphi(x,y)=\varphi(x,y,a_0,a_1,\ldots)$ is a formula from $L_{\lambda,\lambda}$ with parameters $a_0,a_1,\ldots$ from M defining π, then the question whether $M\models\varphi[a,b]$ depends only on $\text{tp}(\langle a,b\rangle,A)$(= the type $\langle a,b\rangle$ realise in M over A) where A is the set of parameters appearing in φ, so $A\subseteq M$, $|A|<\lambda$.
Now for each $a\in P\cup Q$ there should be a b such that $M\models\varphi[a,b]$. If for some $b'\neq b$, $\text{tp}(\langle a,b\rangle,A)=\text{tp}(\langle a,b'\rangle,A)$ we would get $b'=\pi(a)=b$ contradiction. So $\text{tp}(b,A\cup\{a\})$ is realized by a unique element. Also the parallel assertion we get by interchanging a and b holds. So for some $\varphi_a(x,y;\bar c_a)$

$$(\bar c_a\in A,\ \varphi_a(x,y,\bar z)\in L)\qquad M\models(\forall x)(\exists! y)\varphi_a(x,y,\bar c_a)$$
$$M\models(\forall y)(\exists! x)\varphi_a(x,y,\bar c_a)$$

and

$$M\models\varphi_a[a,\pi a,\bar c_a].$$

We want to show $\varphi_a(x,y,\bar c_a)$ can be chosen independently of a.

1.9.1A. First we show that if $\text{tp}(a_1,A)=\text{tp}(a_2,A)$, then we could have chosen the $\varphi_a(x,y,\bar c_a)$'s such that $\varphi_{a_1}=\varphi_{a_2}$, $\bar c_{a_1}=\bar c_{a_2}$. This is because for a fixed a_1, for all a_2 realizing $\text{tp}(a_1,A)$, by the λ-saturativity of M, there is a b uch that $\text{tp}(\langle a_1,\pi a_1\rangle,A)=\text{tp}(\langle a_2,b\rangle,A)$. Hence (as $\varphi(x,y)$ define π) $b=\pi(a_2)$, so $M\models\varphi_{a_1}(a_2,b,\bar c_{a_1})$. So for every complete type r over A (i.e. a type of the form $\text{tp}(a_1,A)$), $P(x)\vee Q(x)\in r$, there are $\varphi_r,\bar c_r$ such that for every $a\in M$ realizing r, $\varphi_a=\varphi_r$, $\bar c_a=\bar c_r$.

1.9.1B. Secondly we show that for every complete type q over A, $P(x)\in q$ there is a formula $\psi(x)$, such that for every $a\in|M|-A$, realizing ψ, we can choose $\varphi_a=\varphi_q$, $\bar c_a=\bar c_q$. Suppose not, w.l.o.g. A is algebraically closed. (i.e. every type over A realized in M by an element not in A, is realized by infinitely many such elements) and A is closed under π and π^{-1} (note that every type over A realized in M by only finitely many elements not in A, has a finite subtype with this property, so if A is the universe of an elementary submodel of (M,π) it is as desired).

Now let

$$\theta(x,\bar c_q)=(\exists! y)\varphi_a(x,y,\bar c_q)\wedge(\forall y)[\varphi_q(x,y,\bar c_q)\rightarrow(\exists! z)\varphi_a(z,y,\bar c_q)].$$

It is easy to check $\theta(x)=\theta(x,\bar c_q)\in q$, so it follows that if $\psi(x)\in q$, $M\models(\forall x)[\psi(x)\rightarrow\theta(x)]$, then [as we assume no ψ satisfies our conclusion] for some a_ψ, $M\models\psi[a_\psi]\wedge\neg\varphi_q[a_\psi,\pi a_\psi,\bar c_q]$, and there is a unique d_ψ such that $M\models\varphi_q[a_\psi,d_\psi,\bar c_q]$. Let Ψ be the set of such $\psi(x)$.

Now we show that by appropriate choice of the a_ψ's for some $a^* \in Q^M$:

(*) if $\psi(x) \in q$, $M \models (\forall x)[\psi(x) \to \theta(x)]$, then $M \models a_\psi R a^*$, $M \models \neg \pi^{-1} d_\psi R a^*$.

As M is λ-saturated, and T satisfies the strong independence property, for every disjoint $A, B \subseteq P^M, |A|+|B| < \lambda$ there is $c \in Q^M$ such that $(\forall a \in A) bRc, (\forall b \in B) \neg bRc$. So what we have to check is $\{a_\psi: \psi \in \Psi\} \cap \{\pi^{-1} d_\psi: \psi \in \psi\} = \emptyset$. Remember that $M \models \neg \varphi_q[a_\psi, \pi a_\psi, \bar{c}_q] \wedge \varphi_q[a_\psi, d_\psi, \bar{c}_q]$ by the choice of a_ψ, d_ψ; hence $a_\psi \neq \pi^{-1} d_\psi$. So we have to show that for $\psi_1 \neq \psi_2$, $a_{\psi_1} \neq \pi^{-1} a_{\psi_2}$.

Let $\Psi = \{\psi_\zeta: \zeta < \zeta_0\}$, and let us define $a'_{\psi_\zeta}, d'_{\psi_\zeta}$ by induction on ζ: if we have defined for each $\xi < \zeta$. Choose a'_{ψ_ζ} realizing

$$\operatorname{tp}(a_{\psi_\zeta}, A) \cup \{x \neq a_{\psi_\xi} \wedge (\exists y)[\varphi_q(x, y; \bar{c}_q) \wedge y \neq d_{\psi_\xi}]: \xi < \zeta\}.$$

This is possible as $\operatorname{tp}(a_{\psi_\zeta}, A)$ is realized by $\geq \aleph_0$ elements of M, hence by $\lambda > 2|\zeta|$ elements (as M is λ-saturated) and all but $2|\zeta|$ of them are as desired. Now choose d'_{ψ_ζ} such that $M \models \varphi_q(a'_{\psi_\zeta}, d'_{\psi_\zeta}, \bar{c}_q)$. (Notice that as $\operatorname{tp}(a_{\psi_\zeta}, A) = \operatorname{tp}(a'_{\psi_\zeta}, A)$ $M \models (\forall y)([\varphi_q(a_{\psi_\zeta}, y, \bar{c}_q) \to \neg \varphi_a(a, y, \bar{c}_q)]$ where $a = a'_{\psi_\zeta}$, hence $a'_{\omega_\zeta}, d'_{\psi_\zeta}$ are as good as $a_{\psi_\zeta}, d_{\psi_\zeta}$.

So we can assume a^* satisfying (*) exists. Now as π is an automorphism, for every $a \in P^M$ $M \models aRA^* \equiv (\pi a)R(\pi a^*)$, hence for every a realizing $q: M \models aRA^* \equiv (\exists y)[\varphi_q(x, y, \bar{c}_q) \wedge yR(\pi a^*)]$ but for every $\psi_1(x) \in q$, letting $\psi(x) = \psi_1(x) \wedge \theta(x) \in q$, there is an $a (= a_\psi)$ such that $M \models \neg[aRa^* \equiv (\exists y)(\varphi_q(x, y, \bar{c}_q) \wedge yR(\pi a^*))]$ and $M \models \psi[a]$ hence $M \models \psi_1[a]$. By the λ-saturativity of M we get a contradiction, hence prove 1.9.1B.

Remark. Note that as the above mentioned formulas are in L, π is not a function symbol, but πa^* is a parameter.

1.9.1C. Now we prove 1.9.1.

By 1.9.1B for every complete type q over A, $P(x) \in q$, there is $\psi_q(x) \in q$ such that for every $a \notin A$ satisfying ψ, we could have chosen $\varphi_a = \varphi_q$, $\bar{c}_a = \bar{c}_q$. Let $\{\psi_\xi: \xi < \xi_0\}$ be a list of the suitable ψ's (i.e. of the form ψ_q). Then by our proof $\{\psi_\xi(x): \xi < \xi_0\} \cup \{P(x) \wedge x \neq a: a \in A\}$ is inconsistent, hence has an inconsistent finite subset, so w.l.o.g. for some finite n, and $a_l \in A$ $\{\psi_\xi(x): \xi < n\} \cup \{P(x) \wedge x \neq a_l: < n\}$ is inconsistent. If $\theta_\xi(x, y, \bar{c}_\xi)$ is appropriate for ψ_ξ, i.e. define a one-to-one function on $\psi_\xi(M) \cap P(M)$ which is π on $(\psi_\xi(M) - A) \cap P(M)$, let $\theta(x, y, \bar{c})$ be

$$P(x) \wedge \bigwedge_{m<n} \Big[\bigwedge_{l<n} x \neq a_l \wedge \psi_m(x) \wedge \bigwedge_{k<m} \neg \psi_k(x) \to \theta_m(x; y; \bar{c}_k) \Big]$$
$$\wedge \bigwedge_{l<n} (x = a_l \Rightarrow y = \pi a_l).$$

Clearly $\theta(x, y, \bar{c})$ defines a function on $P(M)$, and call it π'. Clearly $\pi \restriction (P(M) - A) = \pi' \restriction (P(M) - A)$. If $\{a \in P(M): \pi(a) \neq \pi'(a)\}$ is finite, clearly as π' is definable also $\pi \restriction P(M)$ is definable, so by the extensionality of R π is definable. So suppose there are distinct $a_n \in P(M)(n < \omega)$ such that $\pi(a_n) \neq \pi'(a_n)$. w.l.o.g. for $n \neq m$ $\pi'(a_m) \neq a_n \neq \pi(a_m)$ and so (remembering $\pi \restriction P(M)$ is one-to-one and onto $P(M)$) clearly $\pi^{-1}\pi'(a_n) \neq a_m$.

We now (by the strong independence property) choose $a^* \in Q(M)$ such that

$$M \models a_n R a^* \wedge \neg(\pi^{-1}\pi'(a_n)) R a^* \quad \text{for each} \quad n < \omega;$$
$$\text{so} \quad M \models a_n R a^* \wedge \neg(\pi' a_n)(R(\pi a^*))$$

(as π is an automorphism of $(P(M) \cup Q(M), R\, P)$. Let $B = \{b \in P(M): bRa^* \equiv (\exists y)[\theta(b, y, \bar{c}) \wedge y R \pi a^*]\}$. Now if $b \notin A$, $M \models (\forall y)[\theta(b, y, \bar{c}) \equiv (y = \pi b)]$, so as $M \models bRa^* \equiv (\pi b) R \pi a^*)$, clearly $b \notin B$. So $B \subseteq A$, but for $n < \omega$, $a_n \in B$ (as $M \models a_n R a^*$ and $M \models \neg(\pi' a_n) R(\pi a^*)$). So $B \subseteq M$ is definable, of infinite cardinality $< \lambda$, contradicting the λ-saturativity of M.

So π should be definable, contradiction, so we prove 1.9.1—π is not definable even by a formula in $L_{\lambda,\lambda}$.

9.2. N's construction.

So we have a saturated model M of cardinality λ and an undefinable (in M) automorphism π of $(P(M) \cup Q(M), R)$. Let $\{\varphi_\gamma(y; \dots; x_\xi, \dots; \dots, b_\xi, \dots)_{\xi<\gamma} : \gamma < \lambda\}$ be a list of all $L_{\lambda,\lambda}$-formulas with parameters from M. We define by induction on γ a type $p_\gamma = p_\gamma(x_0, \dots, x_i, \dots; b_0, \dots, b_i, \dots)_{i<\gamma}$ in M, such that:

(a) $|p_\gamma| < \lambda$, p_γ increasing and continuous, and w.l.o.g. closed under (finite) conjunctions and existential quantifications;

(b) $Q(x_0) \in p_\gamma$;

(c) for every formula $\psi(x_0) \in p_\gamma$ and $n < \omega$ there is a finite $\bar{a}^n_\psi$ outside of which x_0 is n-independent, i.e.

$$M \models (\forall y_1, \dots, y_n, z_1, \dots, z_n) \Big(\varphi^{dj}(y_1, \dots, y_n; z_1, \dots, z_n) \wedge \varphi^{dj}(y_1, \dots, y_n; \bar{a}^n_\psi) \wedge \varphi^{dj}(z_1, \dots, z_n; \bar{a}^n_\psi) \rightarrow (\exists x_0)\Big[\psi(x_0) \wedge \bigwedge_{k<n} y_k R x_0 \wedge \bigwedge_{k\leq n} \neg z_k R x_0\Big]\Big).$$

Note that (c) is equivalent to

(c′) there is $C^\gamma \subseteq M$, $|C^\gamma| < \lambda$, such that for every disjoint $C_0, C_1 \subseteq P(M) - C^\gamma$, $p_\gamma \cup \{c_0 R x_0 \wedge \neg c_1 R x_0: c_0 \in C_0, c_1 \in C_1\}$ is consistent. At the end $\bigcup_\gamma p_\gamma$ will be the complete diagram of N, $A^i = \{\pi a : a R x_0 \in \bigcup_i p_\gamma\}$, $B^i = \{\pi a : \neg a R x_0 \in \bigcup_\gamma p_\gamma\}$, and we want that no formula of $L_{\lambda,\lambda}$ with parameters from N will separate between A^i and B^i (later we shall define $\langle \boldsymbol{a}^i_\zeta : \zeta < \lambda \rangle$). (If $|N| - |M|$ will have cardinality $< \lambda$ we repeat 1.8.)

We repeat the construction of Stage 1.8. Note that requirement (c) does not make much difference, for if φ is a formula, and $p_\gamma \cup \{\varphi\}$ fail to satisfy (c′), this is exemplified by some finite C_0, C_1; but now $p_\gamma \cup \{\neg\varphi\}$, $C^\gamma \cup C_0 \cup C_1$ satisfies (c′).

The new point is that when defining $p_{\gamma+1}$ we also add to it a formula of the form $aRx_0 \wedge \neg bRx_0$, so that $\varphi_\gamma(y, x_0, \ldots, b_0, \ldots)$ will not separate between A^i and B^i, more exactly, its intersection with M (which is not necessarily definable) will not define A^i.

1.9.2A. This is possible.

Suppose p_γ is defined, and we want to find suitable a, b as above, and $B \subseteq |M|$ have cardinality $<\lambda$, be such that all parameters of p_γ are in B, and also C^γ from (c′) is $\subseteq B$, and we can assume it is closed under π, and p_γ is a complete type over B. Suppose there are no appropriate a, b, note that for every distinct $a, b \in P(M) - C^\gamma$, $p_\gamma \cup \{aRx_0, \neg bRx_0\}$ satisfies (a), (b), (c). Clearly every extension of $p_\gamma \cup \{aRx_0,\}$ complete over $B \cup \{a, \pi a\}$ satisfying (a) (b) (c) shows $\varphi_\gamma(\pi a, x_0, \ldots, b_0, \ldots)$ holds (as N will be λ-saturated, the satisfaction of φ_γ depends only on the satisfaction of first order formulas). But for every $b \neq \pi a$, $p_\gamma \cup \{aRx_0\}$ has an extension, complete over $B \cup \{a, b\}$, satisfying (a), (b), (c) which shows $\neg\varphi_\gamma(b, x_0, \ldots)$ holds (start with $p_\gamma \cup \{aRx_0, \neg(\pi^{-1}b)Rx_0\}$ and complete it, and now b will belong to B^i, $\neg\varphi_\gamma(b, x_0, \ldots)$ will hold). We have shown that if there are no suitable a, b then $b \neq \pi(a)$ implies $\mathrm{tp}(b, B \cup a\} \neq \mathrm{tp}(\pi a, B \cup a\}$ so π is definable in $L_{\lambda\lambda}$ over M contradicting 1.9.1.

1.9.2B. $\langle a^i_\zeta : \zeta < \lambda\rangle$'s definition.

Above we have constructed N, $M < N$, N saturated, $\|N\| = \lambda$, and disjoint A^i, $B^i \subseteq M$ not separated by any $L_{\lambda,\lambda}$-formula with parameters from N. This shows π cannot be extended to N, as if a^* corresponds to x_0, and π^* extend π, x_0Ra^* will be a formula separating A^i and B^i. So we "kill" π. But for this to continue to hold we have to show:

1.9.2C. Claim. (1) A^i, B^i are $D(S_i)$-strongly unseparated in N.

(2) For every set $A \subset M$, $|A| < \lambda$ there is an indiscernible sequence $\langle a_n : n < \omega\rangle$ over A, such that $a_0 \in A^i$, $a_1 \in B^i$. The claim holds for every saturated M of cardinality λ and two L_λ-inseparable sets.

Proof. (1) Let $|N| = \{c_\xi : \xi < \lambda\}$, and for every ζ, use (2) for $A = \{c_\xi : \xi < \zeta\}$, and get a sequence $\langle a_{\zeta,n} : n < \omega\rangle$, indiscernible over A, $a_{\zeta,0} \in A^i$, $a_{\zeta,1} \in B^i$.

(2) Define a two-place relation S on $P(M)$: aSb *iff* $a = b$ or there is an infinite indiscernible sequence in N over A in which a and b appear.

Now we choose inductively a_ζ such that $a_\zeta \in A^i \cup B^i$ and $(\forall \xi < \zeta) \neg a_\xi S a_\zeta$.

If we can define a_ζ for $\zeta < \lambda$, as $\lambda \to (\omega)^{<\omega}_\mu$ $(\mu = 2^{|A|} < \lambda)$ then $\langle a_\zeta : \zeta < \lambda\rangle$ contains an infinite subsequence indiscernible over A, a contradiction. (see hypothesis (D)).

So a_ζ is defined for $\zeta<\zeta(0)<\lambda$ only, and let $A^*=A\cup\{a_\zeta: \zeta<\zeta(0)\}$. Now we assume $a\in A^i, b\in B^i$ but $\mathrm{tp}(a,A^*)=\mathrm{tp}(b,A^*)$ and get a contradiction, so for every $a\in A^i$, $b\in B^i$, $\mathrm{tp}(a,A^*)\neq\mathrm{tp}(b,A^*)$ thus $\bigvee_{a\in A^i}(\bigwedge\{\varphi(x)\}: \varphi(x)\in\mathrm{tp}(a,A^*))$ separates A^i, B^i; this is a formula in $L_{\gamma\gamma}$ contradicting our assumption. Thus we finish 1.9.2C (2)'s proof.

As $a_{\zeta(0)}$ is not defined, for some $\xi<\zeta(0)$ $a_\xi S a$ holds, so in some infinite indiscernible sequence of A^*, a_ξ and a appear, so (as $a, a_\xi\in A^i\cup B^i$) $a\in A^i\Leftrightarrow a_\xi\in A^i$. But as $\mathrm{tp}(a, A^*)=\mathrm{tp}(b, A^*)$ also $\mathrm{tp}(\langle a, a_\xi\rangle, A)=\mathrm{tp}(\langle b, a_\xi\rangle, A)$ hence (remember N is λ-saturated) there is an infinite indiscernible sequence over A in which, b, a_ξ appear so again $b\in A^i\Leftrightarrow a_\xi\in A^i$. Thus $a\in A^i\Leftrightarrow b\in A^i$, contradiction. So 1.9.2C is proved.

1.10. The end

So we have defined inductively $M_i (i<\lambda^+)$ and let $M=\bigcup_{i<\lambda^+}M_i$. Clearly this is a λ-saturated model of T, of cardinality λ^+.

Now suppose π is an automorphism of $(P(M)\cup Q(M), R)$.
Let

$$C=\{i<\lambda^+: (M_i, \pi\restriction M_i)<(M,\pi)\},$$

$$S=\{i<\lambda^+: \pi\restriction M_i=\pi_i \quad \text{and} \quad \mathrm{cf}\, i=\lambda\}.$$

Clearly C is a closed unbounded subset of λ^+, and S is a stationary set (by π_i's choice, see hypothesis (B)). Hence there is $j\in S\cap C$. If $\pi_j=\pi\restriction M_j$ is definable in M_j, it is definable in M by the same formula (as $j\in C$). If π_j is not definable in M_j then we have "killed" π_j, i.e. ensured no extension of it will be an automorphism of $(P(M)\cup Q(M), R)$.

2. Chang two-cardinal theorem revisited

In this section we shall give a new proof of the two-cardinal theorem of Chang; whose novelty is that we do not expand the language. Our aim is to exemplify a technique of proof used later. Remember that the problem in the proof of Chang theorem is that *if* $M_i (i<\delta)$ is an increasing elementary chain of λ-saturated models, $P(M_i)=P(M_0)$ and p is a type in $\bigcup_{i<\delta}M_i$, $P(x)\in p$, $|p|<\lambda$, *then* not necessarily p is realized in $\bigcup_{i<\delta}M_i$. The solution of Chang was that if the theory is rich enough (e.g. we can encode the finite subsets of P by elements of P) the above mentioned assertion holds; and that the two-cardinal model we started with can be expanded so that this condition holds. But sometimes we do not want to expand the language.

Let T be a fixed complete first order theory in the language L.

2.1. Definition. (1) W will be called a (closed) set of cardinality witness if its elements are pairs $\langle\varphi(x;\bar{y}), \psi(\bar{y})\rangle$ of L-formulas (the intended meaning is "if $\psi(\bar{y})$ then $\{x: \varphi(x;\bar{y})\}$ has small cardinality") which satisfies

(i) $\langle\varphi(x,\bar{y}), (\exists^{\leq 1}x)\varphi(x,\bar{y})\rangle \in W$;

(ii) if $\langle\varphi_l(x,\bar{y}_l), \psi_l(\bar{y}_l)\rangle \in W$, $\varphi(x,\bar{y}) = \bigvee_l \varphi_l(x,\bar{y}_l)$, $(\bar{y}) = \bigwedge_l \psi_l(\bar{y}_l)$, then $\langle\varphi(x,\bar{y}), \psi(\bar{y})\rangle \in W$;

(iii) Suppose $\langle\varphi_1(x;y,\bar{z}), \psi_1(y,\bar{z})\rangle \in W$, $\langle\varphi_2(x,\bar{z}), \psi_2(\bar{z})\rangle \in W$ and T implies:

$$\varphi(x,\bar{z}) \to (\exists y)[\varphi_1(x;y,\bar{z}) \wedge \psi_1(y,\bar{z})],$$
$$(\exists x)\varphi_1(x;y,\bar{z}) \to \varphi_2(y,\bar{z}),\ \psi(\bar{z}) \to \psi_2(\bar{z}).$$

Then $\langle\varphi(x,\bar{z}), \psi(\bar{z})\rangle$.

(2) In a model M of T a formula, $\varphi(x,\bar{a})$ is called small if for some $\langle\varphi(x,\bar{y}), \psi(\bar{y})\rangle \in W$, $M \models \psi[\bar{a}]$. A 1-type p (i.e. in the variable x_0) is called small if for some finite $q \subseteq p$, $\bigwedge q$ is small.

(3) W is trivial for T if the formula $x = x$ is small (in some model of T).

2.2. Definition. Let M be a model, $A \subseteq M$, D an ultrafilter over $\boldsymbol{I}$, $\boldsymbol{I}$ a family of sequences of length m from $|M|$.

We then define $\mathrm{Av}(D, A, M)$ (Av for average) is

$$\{\varphi(\bar{x};\bar{a}): \bar{a} \in A, \varphi \in L(M), \quad \text{and} \quad \{\bar{b} \in \boldsymbol{I}: M \models \varphi[\bar{B};\bar{a}]\} \in D\}.$$

Notice that always $\mathrm{Av}(D, A, M)$ is a complete n-type over A in M. We can replace M, by any elementary extension, and when the meaning is clear from the text we omit M.

2.3. Theorem. *Suppose $|T| < \lambda = \lambda^{<\lambda}$, W a non-trivial set of cardinality witnesses. Then T has a λ-compact model of cardinality λ^+, such that for every 1-type p in M, p is realized by λ^+ elements of M iff it is not small.*

Proof. Let $\boldsymbol{C}$ be a λ^+-saturated model of T, and D'_μ (μ an (infinite) cardinal) an ultrafilter over $I_\mu = S_{\aleph_0}(\mu) =$ the family of finite subsets of μ, such that for every $w_o \in I_\mu$, $\{w \in I_\mu: w_0 \subseteq w\} \in D_\mu$.

We shall define by induction on $\alpha < \lambda^+$ models M_α such that

(a) M_α is λ-compact, $\lambda = \|M_\alpha\|$;

(b) for $\beta < \alpha$, $M_\beta < M_\alpha < \boldsymbol{C}$, and for limit α, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$;

(c) if $\beta < \alpha$, $\bar{a} \in M_\beta$, $\varphi(x,\bar{a})$ a small formula in M_β (hence in M_α), then $\varphi(M_\alpha, \bar{a}) \subseteq M_\beta$;

(d) if p is a non-small 1-type in M_α, $|p| < \lambda$, then for some $\gamma > \alpha$, there is an element of $|M_\gamma| - |M_\alpha|$ realizing p.

We now introduce some notation.

There is an enumeration of all indexed families of the form $\{a_w: w \in I_\mu\}$ for which $\mu < \lambda$, $a_w \in M_\alpha$: $\{\boldsymbol{a}^\xi : \zeta < \lambda(1+\alpha)\}$, $\boldsymbol{a}^\xi = \{a^\xi_w : w \in I_{\mu(\xi)}\}$ (so this just continues the enumeration for $\beta < \alpha$) and we let D_ξ be the following ultrafilter over $\boldsymbol{a}^\xi$: $\{A \subseteq \boldsymbol{a}^\xi: \{w: a^\xi_w \in A\} \in D'_{\mu(\xi)}\}$.

Now the main conditions are

(e) for each $\zeta<\lambda(1+\alpha)$ there are $\boldsymbol{b}^\zeta=\langle b^\zeta_\beta: \beta<\lambda\rangle$ (in M_α) such that for every $\bar c\in M_\alpha$ and $\varphi\in L$

$$\varphi(x,\bar c)\in \operatorname{Av}(D_\zeta,\bar c) \quad \text{implies} \quad \{\beta<\lambda: \boldsymbol{C}\models\varphi[b^\zeta_\beta,\bar c]\}\in D_\lambda.$$

Moreover

(f) if

$$\zeta(0)<\zeta(1)<\cdots<\zeta(n)<\lambda(1+\alpha)\varphi\in L,\ \bar c\in M_\alpha,$$

and $\quad \varphi(x_0,\ldots,x_n,\bar c)\in \operatorname{Av}(D_{\zeta(0)}\times D_{\zeta(1)}\times\cdots\times D_{\zeta(n)},\bar c)$

then

$$\{\beta<\lambda: \boldsymbol{C}\models\varphi[b^{\zeta(0)}_\beta, b^{\zeta(1)}_\beta,\ldots,b^{\zeta(n)}_\beta;\bar c]\}\in D_\lambda.$$

We now define by induction on α, first M_α then $\boldsymbol{a}^\zeta,\boldsymbol{b}^\zeta,D_\zeta$ for $\zeta<\lambda(1+\alpha)$

Case I: M_α, $\alpha=0$.

M_0 will be any λ-compact elementary submodel of $\boldsymbol{C}$ of cardinality λ.

Case II: $\boldsymbol{a}^\zeta,\boldsymbol{b}^\zeta,D_\zeta$ for $\zeta<(1+\alpha)$ after M_α is defined.

For α limit we have nothing to do, so assume $\alpha=0$ or α successor hence M_α is a λ-compact model of cardinality λ; and we shall define them for ζ, $\lambda\alpha\le\zeta<\lambda(1+\alpha)$ where $\beta=1+\alpha-1$. There are no problems in defining the $\boldsymbol{a}^\zeta$'s and then the D_ζ's are also defined. So we are left with $\boldsymbol{b}^\zeta$'s.

Let $\operatorname{Av}(D_\zeta,M_\alpha)=\{\varphi^\zeta_i(x): i<\lambda\}$; so for satisfying (e) we have to choose $\boldsymbol{b}^\zeta$ such that b^ζ_i realizes $\{\varphi^\zeta_j(x): j<i\}$, but for ensuring (f) too we have to work more. Let $\lambda\beta=\{\zeta(i): i<\lambda(0)\}$, and $M_\alpha=\bigcup_{i<\lambda}A_i$, $A_i(i<\lambda)$ increasing and continuous, $|A_i|<\lambda$, and $L=\bigcup_{i<\lambda}L_i$, $L_i(i<\lambda)$ increasing and continuous, $|L_i|<\lambda$. Now we define $\boldsymbol{b}^\zeta$ for $\lambda\beta<\zeta<\lambda(1+\alpha)$ by induction of ζ: $\boldsymbol{b}^\zeta$ is chosen such that b^ζ_i realizes $\operatorname{Av}(D_\zeta, A_i\cup\{b^\xi_i:\lambda\beta\le\xi<\zeta\}, M_\alpha\restriction L_i)$. This is a 1-type in M_α of cardinality $<\lambda$, hence realized. It is easy to check our requirements are satisfied.

Case III: M_α, α limit,

$$M_\alpha=\bigcup_{\beta<\alpha}M_\beta.$$

Case IV: $M_{\alpha+1}$, after M_α, $\boldsymbol{a}^\zeta$, $\boldsymbol{b}^\zeta$ $(\zeta<\lambda(1+\alpha))$.

For eventually satisfying requirement (d), we have a non-small 1-type p_0 in M_α, $|p_0|<\lambda$, and we want some $b\in|M_{\alpha+1}|-|M_\alpha|$ will realize it.

So we define by induction on $i<\lambda$ elements $c_i\in\boldsymbol{C}$, such that, letting $A_i=|M_\alpha|\cup\{c_j: j<i\}$ the following conditions hold.

(i) A_i satisfies requirement (f) for $\zeta(0),\ldots,\zeta(n)<\lambda(1+\alpha)$;

(ii) $c_0\notin M_\alpha$, and c_0 realizes p_0;

(iii) for each i we are given a 1-type p_i over A_i, $|p_i|<\lambda$, such that for every j, and 1-type q over A_j, $|q|<\lambda$, for some i, $p_i=q$; *and* c_i realizes p_i;

(iv) if $\bar a\in M_\alpha$, $\varphi(x,\bar a)$ small, and $c_i\notin M_\alpha$, then $\boldsymbol{C}\models\neg\varphi(c_i,\bar a)$, moreover *if* $\bar b\in A_i$, $\boldsymbol{C}\models(\exists x)(\psi(x,\bar b)\wedge\varphi(x,\bar a))$, *then* for some $c\in M_\alpha$ $\boldsymbol{C}\models\psi[c,\bar b]\wedge\varphi[c,\bar a]$.

At the end, A_λ is the universe of $M_{\alpha+1}$, which will be as required. Notice that for $i=0$, and i limit there are no problems with condition (i). So suppose a_j are defined for $j<i$, and until this stage we have contradicted no condition.

For defining c_i, we define an increasing continuous sequence of 1-types $p^i_\gamma(\gamma<\lambda)$ in $\mathbf{C}$, $|p^i_\gamma|<\lambda$, and $c_i \in \mathbf{C}$ will realize $\bigcup_{\gamma<\lambda} P^i_\gamma$. For $i=0$, we demand in addition that each p^0_γ is not small.

We of course define $p^i_0 = p_i$. We have three assignments to ensure $\bigcup_\gamma p^i_\gamma$ will be complete, (iv) and (i).

Assignment I: Let $\{\varphi_\xi(x, \bar{a}_\xi): \xi<\lambda\}$ be a list of all L-formulas with parameters from A_i. So for $\gamma=\xi+2$ we let $p^i_{\gamma+1}$ be $p^i_\gamma \cup \{\varphi_\xi(x, \bar{a}_\xi)\}$ or $p^i_\gamma \cup \{\neg\varphi_\xi(x, \bar{a})\}$; clearly at least one of them is consistent, and for $i=0$, at least one of them is not small.

Assignment II: So suppose $\bar{a} \in M_\alpha$, $\varphi(y, \bar{a})$ is small, and $\bar{b} \in A_i$, and $(\exists y)[\psi(y, x, \bar{b}) \wedge \varphi(y, \bar{c})] \in p^i_\gamma$. Let us show that for some $a \in \varphi(M_\alpha, \bar{c})$ $p^i_\gamma \cup \{\psi(a, x, \bar{b})\}$ is consistent (and not small, $i=0$). First let $i>0$. If p^i_γ is finite—this follows as A_i satisfies (iv), so let $p^i_\gamma = \{\theta_\xi(x): i<\mu<\lambda\}$, and for each $w \in I_\mu = S_{\aleph_0}(\mu)$ choose $a_w \in \varphi(M_\alpha, \bar{c})$ such that $\{\theta_\xi(x): \xi \in w\} \cup \{\psi(a_w, x, \bar{b})\}$ is consistent. Let $a^\xi = \{a_w: w \in I_\mu\}$, So

$$\psi_w(y) = (\exists x)\Big(\bigwedge_{\xi\in w} \theta_\xi(x) \wedge \psi(y, x, \bar{b}) \wedge \varphi(y, \bar{c})\Big) \in \mathrm{Av}(D_\xi, |M_\alpha|),$$

hence $\{\xi<\lambda: \mathbf{C} \models \psi_w[b^\xi_\xi]\} \in D_\lambda$, but D_λ is λ-closed hence $\{\xi<\lambda$: for each $w \in I_\mu$, $\mathbf{C} \models \psi_w[b^\xi_\xi]\} \in D_\lambda$, so choose ξ_0 in it, and $b^\xi_{\xi_0}$ is the required a. For $i=0$, we should look not at formulas saying $p^i_\gamma \cup \{\psi(c, x, \bar{b})\}$ is consistent, but saying it is big. As their number is still $<|T|<\lambda$, this holds.

Now we can dedicate the definition of $p^i_{\delta+2}$ (δ is limit or zero) to deal with such specific assignments, so that each of them will hold.

Assignment III: Let $\{\zeta(j): j<\lambda\}$ enumerate $\lambda(1+\alpha)$, and $A_i = \bigcup_{j<\lambda} A^i_j (j<\lambda)$ increasing continuous and $|A^i_j|<\lambda$.

For each limit, $\delta<\lambda$, we define $p^i_{\delta+1}$ as follows: *If* p^i_δ is a type over A^i_δ, and for each $j<\delta$, $b^{\zeta(j)}_\delta$ realizes $\mathrm{Av}(D_{\zeta(j)}, A^i_\delta \cup \{b^{\zeta(\xi)}_\delta: \xi<\delta, \zeta(\xi)<\zeta(j)\})$, *then* we define $p^i_{\delta+1}$ such that for any c realizing it, for every $j<\delta$, $b^{\zeta(j)}_\delta$ will realize $\mathrm{Av}(D_{\zeta(j)}, A^i_\delta \cup \{c\} \cup \{b^{\zeta(\xi)}_\delta: \xi<\delta, \zeta(\xi)<\zeta(j)\})$. Otherwise $p^i_{\delta+1} = p^i_\delta$.

It is not hard to check the definition can be done.

Chang's theorem follows from Theorem 2.3: *Claim*: Let M be a model, μ a regular cardinal and W_0 a set of pairs $\langle\varphi(x, \bar{y}), \psi(\bar{y})\rangle$ such that $M \models \psi[\bar{a}] \Rightarrow |\varphi(M, \bar{a})|<\mu$, then each pair in W, the minimal set of cardinality witnesses including w_0, satisfies the same condition. This is easy to check.

Now given a predicate P such that $|P^M|<\|M\|$ we take $\mu=|P^M|^+$ and $W_0 = \{\langle P(x) \wedge y=y, y=y\rangle\}$, as $\|M\| \geq \mu$ we get $x=x \notin w$, so Theorem 2.3 gives the desired model for the two cardinal theorem.

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