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A NOTE ON CANONICAL FUNCTIONS

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ABSTRACT
We construct a generic extension in which the $\aleph_2$nd canonical function on $\aleph_1$ exists.

Introduction

For ordinal functions on ω_1, let $f < g$ if $\{\xi < \omega_1 : f(\xi) < g(\xi)\}$ contains a closed unbounded set. By induction on α, the *αth canonical function f_α* is defined (if it exists) as the least ordinal function greater than each f_β, $\beta < \alpha$ (i.e. if h is any other function greater than all f_β, $\beta < \alpha$, then $f_\alpha \leqq h$). If f_α exists then it is unique up to the equivalence

$$\{\xi < \omega_1 : f(\xi) = g(\xi)\} \text{ contains a closed unbounded set.}$$

It is well known [2] that for each $\alpha < \omega_2$, the αth canonical function exists. A. Hajnal has shown (private communication) that if $V = L$ then the $\aleph_2$nd canonical function does not exist. In this note we show that it is consistent that the $\aleph_2$nd canonical function exists. We prove a somewhat more general result:

THEOREM. *Assume that $2^{\aleph_0} = \aleph_1$, and let ϑ be an ordinal. There is a cardinal preserving generic extension in which for every $\alpha < \vartheta$ the αth canonical function exists.*

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REMARKS. (1) In the model of the theorem, each f_α, $\alpha < \vartheta$, is a function into ω_1; thus $2^{\aleph_1} \geqq |\vartheta|$.

(2) In the model of [4], canonical functions exist for all ordinals α. The model is constructed under the assumption of a measurable cardinal; that assumption is necessary since if all canonical functions exist then the closed unbounded filter is precipitous.

(3) Consider the statement

"the constant function ω_1 is a canonical function".

Its consistency implies the consistency of set theory with predication [3], and hence of various mildly large cardinals.

(4) The theorem generalizes, in the obvious way, to ordinal functions on any regular uncountable cardinal.

The proof of the theorem uses iterated forcing. We use the standard terminology of forcing; see e.g. [1] for iterated forcing. A notion of forcing is *ω-distributive* if it adds no new countable sequences of ordinals; it is $\aleph_2$-c.c. if it has no antichain of size $\aleph_2$. A set $S \subseteq \omega$ is *costationary* if $\omega_1 - S$ is stationary.

By a *countable model N* we mean a countable elementary submodel of $(V_\kappa, \in)$ where κ is a sufficiently large cardinal. A sequence $\{p_n\}_{n\in\omega}$ of conditions in P is *generic* for a countable model N if $P \in N$, $\{p_n : n \in \omega\} \subset N$, and if $\{p_n\}_n$ meets every dense set $D \subseteq P$ such that $D \in N$.

Construction of the model

We construct the forcing P in two stages: first we adjoin generically a ϑ-sequence of functions $f_i : \omega_1 \to \omega_1$, $i < \vartheta$, such that $f_i < f_j$ whenever $i < j$. The forcing P_0 that does it is ω-closed and satisfies the $\aleph_2$-chain condition. The second stage is an iteration, with countable support, of length $\lambda = (2^{\aleph_1} \cdot |\vartheta|)^+$ that successively destroys all stationary sets which witness that the functions f_i are not canonical. We will prove that the iteration forcing is ω-distributive and $\aleph_2$-c.c. Hence P preserves cardinals, and one can arrange all the names for subsets of ω_1 in a sequence $\{\dot{S}_\alpha : 1 \leqq \alpha < \lambda\}$ such that for each α, $\dot{S}_\alpha$ is in $M_\alpha = V^{P|\alpha}$. Moreover, this can be done in such a way that each $\dot{S}$ appears in the sequence cofinally often. We remark that if $S \subseteq \omega_1$ is in M_λ then $S \in M_\alpha$ for some $\alpha < \lambda$; if $M_\lambda \vDash S$ is stationary then $M_\alpha \vDash S$ is stationary; if $M_\alpha \vDash f < g$ then $M_\lambda \vDash f < g$, and if $M_\lambda \vDash f < g$ then for all sufficiently large $\alpha < \lambda$, $M_\alpha \vDash f < g$.

DEFINITION OF P_0. A condition consists of
(a) a countable ordinal γ,
(b) a countable set $A \subset \vartheta$,
(c) closed subsets c_{ij} of γ $(i, j \in A,\ i < j)$,
(d) functions $f_i : \gamma \to \omega_1$,
such that for all $i, j \in A$, $i < j$, $f_i(\xi) < f_j(\xi)$ for all $\xi \in c_{ij}$.

A stronger condition increases γ and A, exends the f_i, and end-extends the c_{ij}.

The forcing P_0 is ω-closed, and is $\aleph_2$-c.c. because $2^{\aleph_0} = \aleph_1$. Let $\dot{f}_i$, $i < \vartheta$, denote the names for the generic functions forced by P_0. Clearly, $M_0 \vDash \dot{f}_i < \dot{f}_j$ whenever $i < j$.

DEFINITION OF P. $P = P_\lambda$ is an iteration with countable support. For $1 \leqq \alpha \leqq \lambda$, P_α is the set of all α-sequences $\{p(\beta) : \beta < \alpha\}$ with countable support such that $p(0) \in P_0$ and such that $p(\beta) = \varnothing$ (trivial condition) unless the following is forced by $p \mid \beta$:

$$\text{(1)} \qquad \begin{gathered} \text{for some } i < \vartheta \text{ there exists a function } g \text{ such that} \\ g > \dot{f}_j \text{ for all } j < i, \text{ and } g(\xi) < \dot{f}_i(\xi) \text{ everywhere on } \dot{S}_\beta. \end{gathered}$$

In that case $p(\beta)$ is a countable closed set of countable ordinals that is forced by $p \mid \beta$ to be disjoint from $\dot{S}_\beta$.

A condition q is stronger than p if $q(0) \leqq p(0)$ and for all β, $1 \leqq \beta < \alpha$, $q(\beta)$ end-extends $p(\beta)$.

For every β that satisfies (1), the forcing produces a closed subset C of ω_1 disjoint from $\dot{S}_\beta$, and C is unbounded as long as $(\omega_1 - \dot{S}_\beta)$ is unbounded. We shall prove that P_λ is ω-distributive and $\aleph_2$-c.c., and that in M_λ the functions f_i are canonical.

LEMMA. *Let N be a countable model such that $P \in N$, and let $\delta = \omega_1 \cap N$. If $\{p_n\}_{n \in \omega}$ is a generic sequence for N, then there exists a q stronger than all the p_n, and such that for all $i \in N$, q forces*

$$\text{(2)} \qquad \dot{f}_i(\delta) = \sup\{\dot{f}_j(\delta) + 1 : j < i \text{ and } j \in N\}.$$

PROOF. Let X be the union of the supports of p_n, $n \in \omega$; note that $X \subset N$. We construct $q(\beta)$ by induction on β. If $\beta \notin X$ we let $q(\beta) = \varnothing$.

First let $\beta = 0$. Look at $\{p_n(0)\}_{n \in \omega}$. By the genericity of the sequence, the ordinals γ_n converge to δ, the countable sets A_n converge to $A = \vartheta \cap N$, the closed sets $(c_{ij})_n$ converge to $c_{ij} \subseteq \delta$ and the functions $(f_i)_n$ converge to functions $f_i : \delta \to \delta$ such that $f_i < f_j$ on c_{ij}.

Let $\gamma = \delta + 1$, let $\bar{c}_{ij} = c_{ij} \cup \{\delta\}$ and let $\bar{f}_i$ be the extensions of the f_i's that satisfy (2). Let $q(0)$ be the condition $(\delta, A, \bar{c}_{ij}, \bar{f}_i\}$; clearly, $q(0)$ forces (2).

Now let $1 \leqq \beta < \lambda, \beta \in X$, and assume that we have already constructed $q \restriction \beta$ stronger than all the $p_n \restriction \beta$. As eventually all $p_n \restriction \beta$ force (1), it follows by their genericity that the countable sets $p_n(\beta)$ converge to a closed subset of δ. So we let $q(\beta) = \bigcup_n p_n(\beta) \cup \{\delta\}$, and in order that q be a condition, we have to verify that $q \restriction \beta \Vdash \delta \notin \dot{S}_\beta$.

Let q' be any condition in P_β stronger than $q \restriction \beta$. Since $\beta \in N$, we may assume that $\dot{S}_\beta \in N$, and N satisfies that for eventually all n, $p_n \restriction \beta$ forces (1). It follows that there exists a condition $r \leqq q'$, some $i \in N$ and some $\dot{g} \in N \cap M_\beta$ such that for all $j < i$ in N, $r \Vdash \dot{f}_j < \dot{g}$ and $r \Vdash (\forall \xi \in \dot{S}_\beta) \dot{g}(\xi) < \dot{f}_i(\xi)$.

For each $j < i$ in N, there exists an M_β-name $\dot{C}_j \in N$ such that every $p \in P_\beta$ forces that $\dot{C}_j$ is closed unbounded, and $r \Vdash (\forall \xi \in \dot{C}_j) \dot{f}_j(\xi) < \dot{g}(\xi)$. It follows, by the genericity of $\{p_n\}_n$, that $q \Vdash \dot{C}_j \cap \delta$ is cofinal in δ, and so $q \Vdash \delta \in \dot{C}_j$. Hence r forces that for all $j < i$ in N, $f_j(\delta) < \dot{g}(\delta)$. But since r also forces (2), it forces $\dot{f}_i(\delta) \leqq \dot{g}(\delta)$, and therefore $r \Vdash \delta \notin \dot{S}_\beta$. [Note that the proof also yields that $q \restriction \beta$ forces that $\dot{S}_\beta$ is costationary, as the argument above proves that $q \restriction \beta \Vdash \delta \in \dot{C}$ for every club name in N.] □

Corollary. *P is ω-distributive and $\aleph_2$-c.c.*

Proof. If $\dot{X} \in M_\lambda$ is a name for a countable set of ordinals and $p \in P_\lambda$, let N be a countable model such that $\dot{X} \in N$, $P_\lambda \in N$ and $p \in N$. Let $\{p_n\}_n$ be a generic sequence for N such that $p_0 = p$. By the Lemma, $\{p_n\}_n$ has a lower bound q, and by genericity, q decides each $\dot{X}(n)$. Hence P_λ is ω-distributive.

For each α, $M_{\alpha+1}$ is a forcing extensionof M_α via a set of conditions of size $\aleph_1$, therefore $\aleph_2$-c.c. As each P_α is an iterated forcing with countable support, it satisfies the $\aleph_2$-c.c. as well. □

We shall finish the proof of the Theorem by showing that in the generic extension by P, the functions f_i, $i < \vartheta$, are canonical. We show that for each $i < \vartheta$, f_i is the least function greater than all the f_j, $j < i$. We already know that $f_i > f_j$ for all $j < i$.

Let $g \in M_\lambda$ be any function such that $f_j < g$ for all $j < i$, and let $S = \{\xi : g(\xi) < f_i(\xi)\}$. We want to show that S is nonstationary. Let β be an ordinal such that $S_\beta = S$, sufficiently large so that all the clubs witnessing $f_j < g$ (all $j < i$) belong to M_β. Hence M_β satisfies (1), and so the forcing at stage β adjoins a closed unbounded set that is disjoint from S.

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REFERENCES

1. J. Baumgartner, *Iterated forcing*, in *Surveys in Set Theory*, London Math. Soc. Lecture Note Ser. **87** (1983), 1–59.
2. F. Galvin and A. Hajnal, *Inequalities for cardinal powers*, Ann. of Math. **101** (1975), 491–498.
3. T. Jech and W. Powell, *Standard models of set theory with predication*, Bull. Am. Math. Soc. **77** (1971), 808–813.
4. T. Jech, M. Magidor, W. Mitchell and K. Prikry, *Precipitous ideals*, J. Symb. Logic **45** (1980), 1–8.