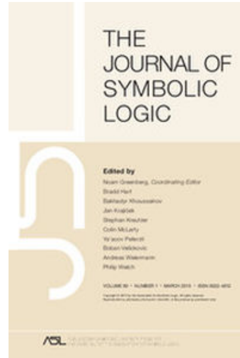

Isomorphic but not lower base-isomorphic cylindric set algebras

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ISOMORPHIC BUT NOT LOWER BASE-ISOMORPHIC CYLINDRIC SET ALGEBRAS

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Abstract. This paper belongs to cylindric-algebraic model theory understood in the sense of algebraic logic. We show the existence of isomorphic but not lower base-isomorphic cylindric set algebras. These algebras are regular and locally finite. This solves a problem raised in [N 83] which was implicitly present also in [HMTAN 81]. This result implies that a theorem of Vaught for prime models of countable languages does not continue to hold for languages of any greater power.

In this paper we deal with algebraic logic, in particular with special isomorphisms of cylindric set algebras (Cs's). We use the terminology and notation of the basic monographs [HMT 71] and [HMT 85] on cylindric algebras. The base set base $(\mathfrak{A})$ of a Cs $\mathfrak{A}$ was defined therein in Definition 1.1.5 or 3.1.1. (base $(\mathfrak{A})$ is that set from which $\mathfrak{A}$ is built up: i.e., $\mathrm{base}(\mathfrak{A}) = \bigcup\{\mathrm{Rng}\, f: f \in \bigcup A\}$, where A is the universe of $\mathfrak{A}$). Isomorphisms of Cs's which are induced by a bijection between their bases are called *base-isomorphisms.* These are very rare because if $\mathfrak{A}$ and $\mathfrak{B}$ are base-isomorphic then $|\mathrm{base}(\mathfrak{A})| = |\mathrm{base}(\mathfrak{B})|$ must hold, and isomorphic Cs's may have bases of almost arbitrarily different cardinalities. To get rid of this cardinality restriction, *lower base-isomorphisms* were introduced in [HMTAN 81] generalizing base-isomorphisms. (Using the terminology of [HMT 85], a lower base-isomorphism is a composition of a strong ext-isomorphism with a sub-base-isomorphism. So two Cs's are lower base-isomorphic if they both are ext-base-isomorphic to a third one.) If an injection $\mathrm{base}(\mathfrak{A}) \supseteq \mathrm{Dom}\, f \to \mathrm{base}(\mathfrak{B})$ induces an isomorphism $h: \mathfrak{A} \to \mathfrak{B}$ then h is a lower base-isomorphism.

Throughout, α is an ordinal. Cs_α and CA_α are the classes of α-dimensional Cs's and α-dimensional cylindric algebras, respectively. The subclasses $\mathrm{Cs}_\alpha^{\mathrm{reg}} \subset \mathrm{Cs}_\alpha$ and $\mathrm{Lf}_\alpha \subset \mathrm{CA}_\alpha$ were introduced in [HMT 85] and [HMT 71], respectively. To every $\mathfrak{A} \in \mathrm{CA}_\alpha$ and $n < \alpha$ the natural n-dimensional part $\mathfrak{Nr}_n\mathfrak{A}$ of $\mathfrak{A}$ was defined in 2.6.28 of [HMT 71].

Several results in the literature deal with conditions under which every isomorphism on a Cs_α $\mathfrak{A}$ is a lower base-isomorphism. Such conditions are given in Theorem I.3.6 of [HMT 81], in Lemma 4.5 of [L 85] and in Proposition 3.4.3 of

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[AN 81]. For countably generated algebras in $\mathrm{Cs}_\alpha^{\mathrm{reg}} \cap \mathrm{Lf}_\alpha$ the following result was given in [S 83].

THEOREM 1 (SERÉNY). *If* $\alpha \geq \omega$, $\mathfrak{A} \in \mathrm{Lf}_\alpha$ *is countably generated, for every* $n \in \omega$ $\mathfrak{Nr}_n\mathfrak{A}$ *is atomic,* $\mathfrak{B}_i \in \mathrm{Cs}_\alpha^{\mathrm{reg}}$, *and* $f_i \in \mathrm{Is}(\mathfrak{A}, \mathfrak{B}_i)$ $(i \in 2)$, *then* $f_1 \circ f_0^{-1}: B_0 \to B_1$ *is a lower base-isomorphism.*

Concerning Serény's result, the following questions were raised in [N 83]; cf. also Proposition 3.5(3), Problem 3 and Problem 4 of [AN 81].

PROBLEM 2. (a) Can the condition "$\mathfrak{A}$ is countably generated" be omitted from Theorem 1; i.e., for every $\alpha \geq \omega$ does there exist an Lf_α $\mathfrak{A}$ such that, for every $n \in \omega$, $\mathfrak{Nr}_n\mathfrak{A}$ is atomic, and, for $i = 0$ and $i = 1$, $\mathfrak{B}_i \in \mathrm{Cs}_\alpha^{\mathrm{reg}}$ and $f_i \in \mathrm{Is}\ (A, \mathfrak{B}_i)$ such that $f_1 \circ f_0^{-1}$ is not a lower base-isomorphism?

(b) Is it possible with the above conditions that $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are not lower base-isomorphic?

The answers to both 2(a) and 2(b) are shown below to be affirmative. Though 2(a) is a special case of 2(b), we deal first with the former.

To begin with we prove a lemma which shows that *a theorem of Vaught on (elementary) prime models in countable languages* (Theorem 2.3.4 of [ChK 73] or Theorem 27.10 of [M 76]) *cannot be extended to languages of any greater power.*

Notation. If $\mathscr{L}$ is a first-order language then $F_{\mathscr{L}}$ is the set of all formulas of $\mathscr{L}$, while if $n \in \omega$, $F^n_{\mathscr{L}}$ denotes the set of those formula of $\mathscr{L}$ in which the free variables are taken from the set $\{v_0, \ldots, v_{n-1}\}$.

LEMMA 3. *There exist a language* $\mathscr{L}$ *of power* $\aleph_1$, *and a complete theory* Th *in* $\mathscr{L}$ *such that, for any* $n \in \omega$, $F^n_{\mathscr{L}}/\mathrm{Th}$ *is atomic and there exist two models of* Th, $\mathfrak{T}^0$ *and* $\mathfrak{T}^1$, *without isomorphic elementary submodels.*

PROOF. Let $\mathfrak{T}'^0 = \langle T^0, <^0 \rangle$ be a normal Aronszajn tree; i.e.,

(1) $\mathfrak{T}'^0$ is a poset, and

(2) for every $a \in T^0$, $T^0|_a \stackrel{\mathrm{d}}{=} \{x \in T^0 : x <^0 a\}$ is well-ordered by $<^0$.

If for any ordinal β

$$T^0_\beta \stackrel{\mathrm{d}}{=} \{a \in T^0 : \mathrm{Typ}(T^0|_a, <^0 \restriction T^0|_a) = \beta\}$$

then the following assertions hold true:

(3) $|T^0_0| = 1$.

(4) There are no ordered subsets of $\mathfrak{T}'^0$ of order type ω_1.

(5) For every $\beta < \omega_1$, $|T^0_\beta| \leq \aleph_0$.

(6) If $\beta < \omega_1$ is a limit ordinal, $a, b \in T^0_\beta$ and $T^0|_a = T^0|_b$, then $a = b$.

(7) Each $a \in T^0$ has $\aleph_0$ immediate successors.

(8) If $\beta < \gamma < \omega_1$ and $x \in T^0_\beta$, then there is a $y \in T^0_\gamma$ such that $x <^0 y$.

(As is well known, ZFC implies the existence of normal Aronszajn trees.)

Let our language $\mathscr{L}$ have one binary relation symbol $<$ and $\aleph_1$ unary relation symbols T_β $(\beta < \omega_1)$. Let $\mathfrak{T}^0 = (\mathfrak{T}'^0, T^0_\beta)_{\beta < \omega_1} \in \mathrm{Md}_{\mathscr{L}}$. Let $\mathrm{Th} = \mathrm{Th}(\mathfrak{T}^0)$, the full first order theory of $\mathfrak{T}^0$. Define $\mathfrak{T}^1$ as follows.

$$T^1 \stackrel{\mathrm{d}}{=} \{f : f \text{ is a function from some } \beta < \omega_1 \text{ into } \omega \text{ with } |\{\beta : f(\beta) > 0\}| < \aleph_0\}.$$

Let $\mathfrak{T}'^1 = (T^1, <^1)$, where $f <^1 g$ iff $f \subseteq g$; i.e., $f <^1 g$ iff f is a restriction of g.

Sh:290

For $\beta < \omega_1$, define T^1_β as follows: $T^1_\beta = \{f \in T^1: \mathrm{Dom}(f) = \beta\}$. Let $\mathfrak{T}^1 = (\mathfrak{T}'^1, T^1_\beta)_{\beta<\omega_1} \in \mathrm{Md}_{\mathscr{L}}$.

Trivially, Th is a complete theory. We prove the following facts:

(9) $\mathfrak{T}^1 \models \mathrm{Th}$; i.e., $\mathfrak{T}^0 \equiv_{ee} \mathfrak{T}^1$.

(10) For every $n \in \omega$, $F^n_{\mathscr{L}}/\mathrm{Th}$, the Boolean algebra of the n-ary formulas in $F_{\mathscr{L}}$ modulo Th, is atomic.

(11) Every elementary submodel of $\mathfrak{T}^0$ satisfies (4), but no elementary submodels of $\mathfrak{T}^1$ satisfy (4).

In fact, by (2), (3) and (6), $\mathfrak{T}'^0$ is a meet semilattice. It is also easy to see that $\mathfrak{T}'^1$ is a meet semilattice too. Therefore from now on, throughout, we shall work in a definitional expansion of $(\mathscr{L}, \mathrm{Th})$ with the binary operation symbol $\cdot$ with definition

$$(\forall v_0 \forall v_1 \forall v_2)(v_2 = v_0 \cdot v_1 \leftrightarrow v_2 \le v_0 \;\&\; v_2 \le v_1 \;\&\; (\forall v_3)(v_3 \le v_0 \;\&\; v_3 \le v_1 \rightarrow v_3 \le v_2)).$$

If, for an $i \in \omega$, $\mathfrak{T}^i$ is a model of $\mathscr{L}$ and it is a meet semilattice, then the interpretation of $\cdot$ in $\mathfrak{T}^i$ is denoted by $\cdot^i$.

(12) $\mathfrak{T}^1$ satisfies (1)–(3) and (5)–(8).

For k, $m \in \omega$ and $\beta \in \omega_1$ let the formula $\varphi^{k,m,\beta} \in F_{\mathscr{L}}$ be defined as $\varphi^{k,m,\beta} = T_\beta(v_k \cdot v_m)$. (In our models this above formula means that the meet of v_k and v_m belongs to the βth level of the tree.) For $n \in \omega$, $i < 2$, and $q \in^n T^i$ define Φ_q (a finite subset of $F_{\mathscr{L}}$) as follows:

$$\Phi_q = \{\varphi^{k,m,\beta}: k,m < n \text{ and } \mathfrak{T}^i \models \varphi^{k,m,\beta}[q]\}.$$

For any $q \in^n T^i$ let $\varphi_q = \bigwedge \Phi_q$. For $i, j < 2$, $n \in \omega$, $q \in^n T^i$ and $r \in^n T^j$ define

$$qI_n r \quad \text{iff} \quad \models \varphi_q \leftrightarrow \varphi_r.$$

By (7), (8) and (12),

(13) If $q \in {}^nT^i$, $r \in {}^nT^j$, $qI_n r$ and $a \in T^i$ then there is a $b \in T^j$ such that $\langle q_0, \ldots, q_{n-1}, a\rangle I_{n+1} \langle r_0, \ldots, r_{n-1}, b\rangle$.

Using (13) for $i \ne j$, one can easily show (9) (see [M 76, Lemmas 26.5 and 26.8]). Using (13) for $i = j = 0$, we can see that, for every $q \in^n T^0$, φ_q is an atomic formula in $F^n_{\mathscr{L}}/\mathrm{Th}$, and hence, $F^n_{\mathscr{L}}/\mathrm{Th}$ is atomic (see [M 76, Chapters 26 and 27]).

Next we prove (11). Trivially, every elementary submodel of $\mathfrak{T}^0$ satisfies (1)–(8). Let

$$\mathfrak{T}^* = (T^*, <^*, T^*_\beta)_{\beta<\omega_1} \preccurlyeq \mathfrak{T}^1$$

(where $\preccurlyeq$ means elementary substructure). For any $\beta < \omega_1$, choose $f_\beta \in T^*_\beta$. Define $H: \omega_1 \to \omega_1$ as follows:

$$H(\beta) = \begin{cases} \text{the greatest } \gamma \text{ such that } f_\beta(\gamma) \ne 0, & \text{if such a } \gamma \text{ exists,} \\ 0 \quad \text{otherwise.} \end{cases}$$

H is well-defined since $\{\gamma: f_\beta(\gamma) \ne 0\}$ is finite. H is a regressive function so, by Fodor's theorem, there exists a $\gamma < \omega_1$ such that $|H^{-1}\{\gamma\}| = \aleph_1$. T^*_γ is countable; so, by (8), there is an $h \in T^*_\gamma$ such that for each δ, if $\gamma < \delta < \omega_1$, then $h \cup \{(\kappa, 0): \gamma < \kappa \le \delta\}$

Sh:290

$\in T^*$. Then $\{h \cup \{(\kappa, 0): \gamma < \kappa \leq \delta\}: \gamma < \delta < \omega_1\}$ is an ordered subset of $(T^*, <^*)$ of order type ω_1.

(9), (10) and (11) prove Lemma 3. q.e.d.

DEFINITION 4. Let $\mathscr{L}$ be a first-order language with set of variable symbols $\{v_i: i < \alpha\}$, $\varphi \in F_{\mathscr{L}}$, and $\mathfrak{M}$ a model of $\mathscr{L}$. In this case $\varphi^{\mathfrak{M}\alpha} = \{v \in {}^{\alpha}M: \mathfrak{M} \models \varphi[v]\}$ and $\mathfrak{Cs}_{\alpha}^{\mathfrak{M}}$ is the cylindric set algebra associated with $\mathfrak{M}$; i.e., it is the cylindric set algebra with underlying set $\{\varphi^{\mathfrak{M}\alpha}: \varphi \in F_{\mathscr{L}}\}$ (cf. the bottom of p. 154 and Definition 4.3.3 in [HMT 85]). Sometimes, if there is no danger of confusion, the superscript α will be omitted.

THEOREM 5. *For every $\alpha \geq \omega$, there exists an $\aleph_1$-generated $\mathfrak{A} \in \mathrm{Lf}_{\alpha}$ such that, for every $n \in \omega$, $\mathfrak{Nr}_n\mathfrak{A}$ is atomic and for $i = 0$ and $i = 1$ there exist $\mathfrak{B}_i \in \mathrm{Cs}_{\alpha}^{\mathrm{reg}}$ and $f_i \in \mathrm{Is}(\mathfrak{A}, \mathfrak{B}_i)$ such that $f_1 \circ f_0^{-1}$ is not a lower base-isomorphism.*

PROOF. It is easily seen that if $\mathfrak{T}_0$ and $\mathfrak{T}_1$ satisfy the requirements of Lemma 3 then $\mathfrak{B}_i = \mathfrak{Cs}_{\alpha}^{\mathfrak{T}_i} (i < 2)$ satisfy the requirements of the theorem with $\mathfrak{A} = \mathfrak{B}_0$, $f_0 = \mathfrak{B}_0 \upharpoonright \mathrm{Id}$ and

$$f_1 = \{\langle \varphi^{\mathfrak{T}^0}, \varphi^{\mathfrak{T}^1}: \varphi \in F_{\mathscr{L}}\}. \qquad \text{q.e.d.}$$

With the aid of Theorem 5 and Lemma 3 we can solve Problem 2(b). To begin with we prove two lemmas.

LEMMA 6. *Let $\mathscr{L}$ and Th be the language and theory, respectively, which were defined in the proof of Lemma 3, and let $\mathfrak{T}^2 = (T^2, <^2, T_{\beta}^2)_{\beta < \omega_1}$ and $\mathfrak{T}^3 = (T^3, <^3, T_{\beta}^3)_{\beta < \omega_1}$ be two models of Th. For $i = 2, 3$ let $\mathfrak{B}_i = \mathfrak{Cs}_{\alpha}^{\mathfrak{T}^i}$. For $\beta < \omega_1$ let φ_{β} be the formula $T_{\beta}(v_0)$. Finally, let η be a bijection from T^2 onto T^3 such that $\tilde{\eta}: \mathfrak{B}_2 \to \mathfrak{B}_3$ is a base-isomorphism. Under these conditions, for any $\beta < \omega_1$*

$$\tilde{\eta}(\varphi_{\beta}^{\mathfrak{T}^2}) = \varphi_{\beta}^{\mathfrak{T}^3};$$

i.e., η preserves the relations T_{β} for any $\beta < \omega_1$.

PROOF. For $i = 2$, 3 and $\beta < \omega_1$ set $\bar{T}_{\beta}^i = \{x \in {}^{\alpha}T^i: x_0 \in T_{\beta}^i\}$. By the proof of Lemma 3 one can see that, for any $n \in \omega$ and $\varphi \in F_{\mathscr{L}}^n$, φ is n-atomic over Th iff there is a $q \in {}^nT^0$ such that Th $\models \varphi \leftrightarrow \varphi_q$. In particular,

(14) The l-atomic formulas over Th are exactly those unary formulas which are equivalent over Th to the formulas $T_{\beta}(v_0)$ for some $\beta < \omega_1$.

(15) The 2-atomic formulas over Th are those formulas that are equivalent over Th to one of the formulas $T_{\beta}(v_0)$ & $T_{\gamma}(v_1)$ & $T_{\delta}(v_0 \cdot v_1)$, where $\delta \leq \beta < \omega_1$ and $\delta \leq \gamma < \omega_1$.

Clearly if $\beta < \omega_1$ and $i < 4$, $\varphi_{\beta}^{\mathfrak{T}^i} = \bar{T}_{\beta}^i$.

Since $\tilde{\eta}$ is an isomorphism of the CA's, the image of an atom of $\mathfrak{Nr}_n\mathfrak{B}_2$ is an atom of $\mathfrak{Nr}_n\mathfrak{B}_3$; i.e.,

(16) If $x \in {}^{\alpha}T^2$ satisfies an n-atomic formula in $\mathfrak{T}^2$, then so does $\eta \circ x$ in $\mathfrak{T}^3$.

Assume that the statement of Lemma 6 is not true; i.e., η does not preserve T_{β} for some $\beta < \omega_1$. Let γ be the least β such that $\tilde{\eta}(\bar{T}_{\beta}^2) \neq \bar{T}_{\beta}^3$. By (14), (16) and the minimality of γ there is a $\delta > \gamma$ such that $\tilde{\eta}(\bar{T}_{\gamma}^2) = \bar{T}_{\delta}^3$. Let $x \in T_{\gamma}^2$. Since η is a bijection and $\mathfrak{T}^3 \models$ Th, there is a $y_1 \in T^2$ with $\eta y_1 <^3 \eta x$ and $\eta y_1 \in T_{\gamma}^3$. Since $(\eta x, \eta y_1)$ satisfies the 2-atomic formula $T_{\delta}(v_0)$ & $T_{\gamma}(v_1)$ & $T_{\gamma}(v_0 \cdot v_1)$, by (15) and (16), $(x, y_1) = \eta^{-1} \circ (\eta x, \eta y_1)$ satisfies a 2-atomic formula of the form $T_{\gamma}(v_0)$ & $T_{\varepsilon}(v_1)$ &

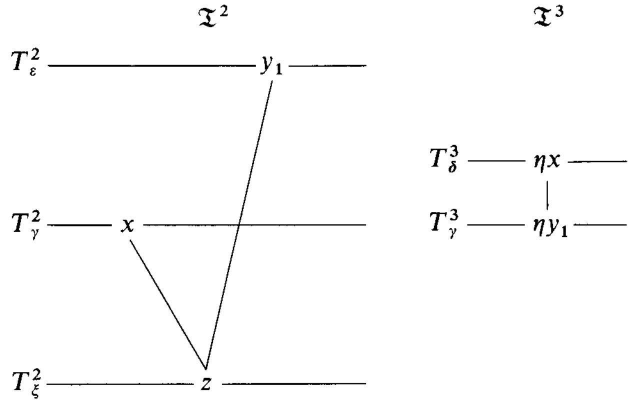

FIGURE 1/a

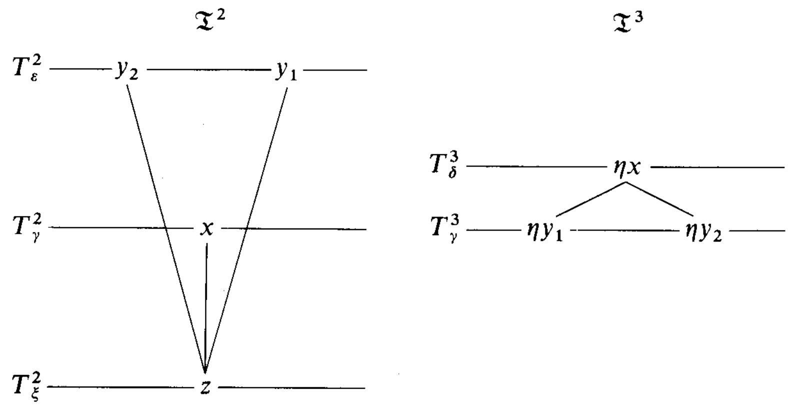

FIGURE 1/b

$T_\xi(v_0 \cdot v_1)$ for some $\xi \leq \gamma$ and $\xi \leq \varepsilon < \omega_1$. By $\tilde{\eta}(\bar{T}^2_\gamma) \neq \bar{T}^3_\gamma$ and the minimality of γ we have $\varepsilon > \gamma$. Let $x \cdot^2 y_1 = z$ (see Figure 1/a). By (7) and (8) there is a y_2 such that $y_1 \neq y_2 \in T^2_\varepsilon$ and $x \cdot^2 y_2 = z$ holds. Since (x, y_2) satisfies the above 2-atomic formula in $\mathfrak{T}^2$, $(\eta x, \eta y_2)$ satisfies the previous one in $\mathfrak{T}^3$ (see Figure 1/b).

However, this is impossible, as in the models of Th the statements $\eta y_1 <^3 \eta x$, $\eta y_2 <^3 \eta x$, $\eta y_1 \neq \eta y_2$ and "ηy_1 is not comparable with ηy_2" cannot hold simultaneously. q.e.d.

LEMMA 7. *Suppose the conditions of Lemma 6 hold. In addition assume that* $T^2 = \bigcup_{\beta < \omega_1} T^2_\beta$. *Under these conditions* $\tilde{\eta}(<^2) = <^3$.

PROOF. Assume to the contrary that there are x_1 and $y \in T^2$ such that $x_1 <^2 y$ but $\eta x_1 \not<^3 \eta y$. By the additional hypotheses of the lemma there exist $\beta < \gamma < \omega_1$ such that $x_1 \in T^2_\beta$ and $y \in T^2_\gamma$. By Lemma 6, $\eta x_1 \in T^3_\beta$ and $\eta y \in T^3_\gamma$. Since (x_1, y) satisfies the 2-atomic formula over Th

$$\chi(v_0, v_1) \stackrel{\mathrm{d}}{=} (T_\beta(v_0) \;\&\; T_\gamma(v_1) \;\&\; T_\beta(v_0 \cdot v_1)),$$

Sh:290

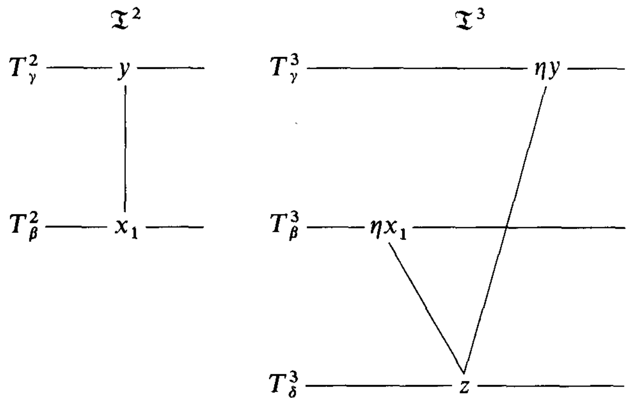

FIGURE 2/a

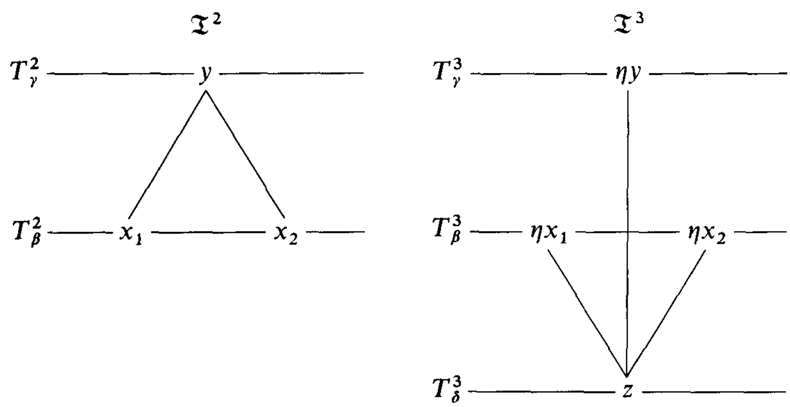

FIGURE 2/b

by (15), (16) and the indirect hypothesis, $(\eta x_1, \eta y)$ satisfies the formula

$$\psi(v_0, v_1) \stackrel{d}{=} (T_\beta(v_0) \;\&\; T_\gamma(v_1) \;\&\; T_\delta(v_0 \cdot v_1))$$

for a $\delta < \beta$ (see Figure 2/a).

Let $z = \eta x_1 \cdot^3 \eta y$. By (16),

(17) $\tilde{\eta}^{-1}(\psi^{\mathfrak{T}^3}) = \chi^{\mathfrak{T}^2}$.

By (7) and (8) choose $\eta x_2 \in T^3_\beta$ so that $\eta x_2 \neq \eta x_1$ and $\eta x_2 \cdot^3 \eta y = z$, so $(\eta x_2, \eta y)$ satisfies ψ in $\mathfrak{T}^3$; hence, by (17), (x_2, y) satisfies χ in $\mathfrak{T}^2$ (see Figure 2/b).

However, this, similarly to the proof of Lemma 6, yields a contradiction. q.e.d.

THEOREM 8. *For any ordinal $\alpha \geq \omega$ and for $i \in 2$ there exists an $\aleph_1$-generated $\mathfrak{B}_i \in \mathrm{Lf}_\alpha \cap \mathrm{Cs}^{\mathrm{reg}}_\alpha$ such that, for any $n \in \omega$, $\mathfrak{Nr}_n \mathfrak{B}_i$ is atomic ($i = 0, 1$) and $\mathfrak{B}_0 \cong \mathfrak{B}_1$ but they are not lower base-isomorphic. Actually, $\mathfrak{B}_0$ and $\mathfrak{B}_1$ which were constructed in the proof of Theorem 5 satisfy the conditions.*

Sh:290

PROOF. We prove the theorem by showing that if $f \in \mathrm{Is}(\mathfrak{B}_0, \mathfrak{B}_1)$ is a lower base-isomorphism then $f = f_1$ (which was defined at the end of the proof of Theorem 5), which is not a lower base-isomorphism by Theorem 5. That is, we will show that if f is a lower base-isomorphism from $\mathfrak{B}_0$ onto $\mathfrak{B}_1$ then f preserves each relation of $\mathscr{L}$.

Now suppose that $F = k^{-1} \circ h \circ t$ is a lower base-isomorphism from $\mathfrak{B}_0$ onto $\mathfrak{B}_1$, where k and t are strong ext-isomorphisms and h is a base-isomorphism. (A strong ext-isomorphism is "an isomorphism that is obtained by relativization with respect to a Cartesian space". Cf. the definition of lower base-isomorphism in [HMTAN 81] or the Introduction to the present paper.)

For $i = 2, 3$ denote base$(\mathfrak{B}_i)$ by T^i; i.e., $k = \langle X \cap {}^\alpha T^3 : X \in B_1 \rangle$ and $t = \langle X \cap^\alpha T^2 : X \in B_0 \rangle$.

For $i < 2$ let $\mathfrak{T}^{i+2}$ be the substructure of $\mathfrak{T}^i$ with underlying set $\mathfrak{T}^{i+2}$. Let $h = \tilde{\eta}$ for an $\eta: T_2 \to T_3$. Since k and t are strong ext-isomorphisms, $\mathfrak{T}^2$, $\mathfrak{T}^3$ and η satisfy the conditions of Lemmas 6 and 7; hence, by these lemmas, η preserves all formulas of $F_{\mathscr{L}}$. Thus, trivially, $F = f_1$ (f_1 was defined in the proof of Theorem 5). However, by Theorem 5, f_1 is not a lower-base isomorphism. This contradiction proves Theorem 8. q.e.d.

We note that Theorems 5 and 8 are true not only for cylindric algebras of infinite dimension but for cyclindric algebras of any dimension greater than one (see [B 86]). We also note that none of the other conditions of Theorem 1 can be omitted. In more detail, it follows from the results of [B 85] that the condition "for every $n \in \omega$, $\mathfrak{Nr}_n\mathfrak{U}$ is atomic" cannot be omitted. It is shown in [B 86a] and [B 87] that the same holds for the condition "$\alpha \geq \omega$" and the conditions "$\mathfrak{A}$ is regular" and "$\mathfrak{A} \in \mathrm{Lf}_\alpha$"; moreover, Lf cannot be replaced by Dc.

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