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Whitehead Modules over Domains

Thomas Becker, Laszlo Fuchs[1], and Saharon Shelah[2]

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Abstract. Let R be a commutative domain with 1. By a Whitehead module is meant an R-module M satisfying $\mathrm{Ext}^1_R(M, R) = 0$. If R is such that RD-submodules of torsion-free Whitehead modules are again Whitehead, then the hypothesis $V = L$ makes it possible to reduce the problem of characterizing torsion-free Whitehead modules to Whitehead modules of cardinality $\leq |R|$. Proper Forcing is used to show that this criterion fails in ZFC.

Applications are given to P.I.D.s of cardinality $\aleph_1$, countable valuation domains and almost maximal valuation domains of cardinality $\aleph_1$.

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For about two decades, the Whitehead problem was one of the central open problems in abelian group theory. In 1952, J. H. C. Whitehead asked if an abelian group A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ (now called a Whitehead group) had to be free. (For the connection with other problems, see e.g. Nunke [10].) For countable A, the answer was already available in a paper by Stein [15]; unaware of this, Ehrenfeucht [1] published the (affirmative) solution for countable groups. Several authors were attracted by this problem, but could obtain only fragmentary results. The full answer was given by Shelah [11] in an unexpected claim: Whitehead's problem is undecidable in ZFC. More precisely, he proved that in L (the constructible universe) all Whitehead groups of cardinality $\aleph_1$ are free, while in the presence of Martin's Axiom and the denial of the CH (Continuum Hypothesis), there do exist non-free Whitehead groups of cardinality $\aleph_1$. For a more detailed presentation of the proof, see Eklof [2]. In a subsequent paper [12], Shelah proved that in L Whitehead groups of any cardinality are free. It is interesting to point out that the Whitehead problem stays undecidable even if CH is assumed; see Shelah [13] and Mekler [9].

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Now that we have a pretty good picture of how various set-theoretical axioms effect the structure of Whitehead groups, time has come to ask the analogous question by replacing $\mathbb{Z}$ by a P.I.D. or a Dedekind domain or more generally, by a domain R: *What are the Whitehead R-modules?* (Manifestly, by a Whitehead module is meant an R-module M such that $\mathrm{Ext}_R^1(M, R) = 0$.) In answering this question, we not only might learn more about modules over domains, but we could also understand better the case of abelian groups. The only cases discussed so far are concerned with slender P.I.D.'s (Gerstner-Kaup-Weidner [7] settled the case of countable rank) and with countable Dedekind domains (Eklof [4] showed that the situation is like for $\mathbb{Z}$).

Our purpose here is to investigate the problem of Whitehead modules over general domains R. Since for certain R's, there exist torsion Whitehead modules, and the field Q of quotients of R, or even all torsion-free R-modules can be Whitehead modules, it is clear that there is no hope for obtaining a description of their structures in such a general setting. We found it rather surprising that in L the problem of Whitehead modules can essentially be reduced to the case of R-modules up to the cardinality of R (see Theorem 3.1) whenever RD-submodules of Whitehead modules are again Whitehead. As expected, this reduction theorem is not a theorem in ZFC for lots of rings. In fact, by using PFA (Proper Forcing Axiom) it follows that for many rings of cardinality $\aleph_0$ the criterion of Theorem 3.1 is not a necessary condition (see Theorem 4.3).

We apply our general results to a couple of cases in which full information can be obtained about Whitehead modules of cardinalities not exceeding the cardinality of R. These cases include P.I.D.'s of cardinalities $\leq \aleph_1$ as well as countable valuation domains and almost maximal valuation domains of cardinality $\leq \aleph_1$. (An interested reader might find it useful to compare these results with those on Baer modules in Eklof-Fuchs [5].)

There are several questions left open. The most relevant one is concerned with the structure of torsion-free Whitehead modules of rank $\aleph_1$ over a P.I.D. or over a valuation domain of cardinality $\aleph_2$. In these cases, the methods available break down, and essentially new ideas seem to be required to deal with these situations.

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1. Preliminaries

R will denote throughout a domain (sometimes satisfying additional conditions stated explicitly), i.e. a commutative ring with 1 which contains no divisors of zero. Q will stand for its field of quotients; we always assume $Q \neq R$. All R-modules considered are unital.

As we have already stated, an R-module M is called a Whitehead module if $\mathrm{Ext}_R^1(M, R) = 0$. Free modules are trivially Whitehead modules. The class $\mathscr{W}$ of Whitehead modules is evidently closed under the formations of direct sums and direct summands. Under additional conditions on R, the class $\mathscr{W}$ may enjoy other closure properties.

1.1 Lemma. *If* i.d. $R = 1$, *then submodules of Whitehead modules are again Whitehead modules.*

Proof. $M \in \mathcal{W}$ and i.d. $R = 1$ imply that in the exact sequence $\mathrm{Ext}^1(M, R) \to \mathrm{Ext}^1(N, R) \to \mathrm{Ext}^2(L, R)$ (which is induced by the exact sequence $0 \to N \to M \to L \to 0$) the end terms vanish. Hence $N \in \mathcal{W}$, indeed. □

The same proof leads to

1.2 Lemma. *If* $M \in \mathcal{W}$ *with* p.d. $M \leq 1$, *then submodules* N *of* M *with* p.d. $M/N \leq 1$ *are likewise Whitehead modules.* □

Of course, it can very well happen that a domain admits torsion Whitehead modules (see e.g. 6.1). However, we wish to record the following simple observation.

1.3 Lemma. *If* $I \neq 0$ *is an ideal of* R *such that* $I \leq Ra < R$, *for some* $a \in R$, *then* R/I *is not a Whitehead module.*

Proof. The exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence $\mathrm{Hom}(R, R) \to \mathrm{Hom}(I, R) \to \mathrm{Ext}^1(R/I, R) \to 0$. Therefore, it suffices to show that under the stated hypotheses, there is a homomorphism $\varphi : I \to R$ which does not extend to one from R to R. Let φ be the restriction of $Ra \to R$ $(a \mapsto 1)$ to I. φ extends uniquely to a homomorphism $\psi : R \to Q$ which must carry 1 to $a^{-1} \notin R$, so the only candidate for the extension, the map ψ, has a larger image than R. □

Concentrating on torsion-free modules, we point out:

1.4 Lemma. $Q \in \mathcal{W}$ *if and only if* R *is* R*-complete. All torsion-free* R*-modules are Whitehead modules exactly if* R *is cotorsion in the sense of* [6], p. 243.

Proof. For the first statement, see Matlis ([8], p. 25) or Fuchs-Salce ([6], p. 97). The second part is nothing more than the definition of cotorsionness (which occurs e.g. if R is a maximal valuation domain). □

Finally, we note that the class $\mathcal{W}$ is closed under extensions; moreover,

1.5 Lemma. *Suppose that* M *is the union of a continuous well-ordered ascending chain* $\{M_\nu \mid \nu < \tau\}$ *of submodules such that* $M_0 = 0$ *and, for each* $\nu + 1 < \tau$, *the module* $M_{\nu+1}/M_\nu$ *is Whitehead. Then* M *itself is a Whitehead module.*

Proof. See Eklof ([4], p. 27) or Fuchs-Salce ([6], p. 74). □

2. V = L: the case of regular cardinals

We start our study with the case V = L.

The following lemma is crucial in handling the case of regular cardinals in L. It is a modified version of 1.4 of Eklof [3]. In this version, rank argument replaces cardinality arguments which allows us to descend below the cardinality of the ring R with the ranks of torsion-free R-modules. Recall that $|Q| = |R|$ and a torsion-free R-module M of rank $\leqq |R|$ is contained in $\bigoplus_{|R|} Q$, so $|M| \leqq |R|$. If rank $M \geq |R|$, then clearly $|M| = \operatorname{rank} M$. (An RD-submodule of M is a submodule N satisfying $rN = N \cap rM$ for all $r \in R$.)

2.1 Lemma (V = L). *Let R be a domain, κ a regular cardinal, and $|R| \leq \kappa$. If M is a torsion-free R-module of rank κ, and $\{M_\nu | \nu < \kappa\}$ is a continuous well-ordered ascending chain of RD-submodules of M such that*

(i) $\bigcup_{\nu<\kappa} M_\nu = M$,

(ii) $\operatorname{rank} M_\nu < \kappa$ *(for all $\nu < \kappa$)*,

(iii) $\operatorname{Ext}^1_R(M_\nu, R) = 0$ *(for all $\nu < \kappa$)*,

(iv) *the set $E = \{\nu < \kappa | \operatorname{Ext}^1_R(M_{\nu+1}/M_\nu, R) \neq 0\}$ is stationary in κ,*

then $\operatorname{Ext}^1_R(M, R) \neq 0$.

Proof. Let $\{I_\nu | \nu < \kappa\}$ be a continuous well-ordered ascending chain of subsets of M such that I_ν is a maximal independent set for M_ν. Then $I = \bigcup_{\nu<\kappa} I_\nu$ is a maximal independent set for M. Furthermore, let $\{C_\nu | \nu < \kappa\}$ be a continuous well-ordered ascending chain with $C_\nu \subseteq M_\nu \times R$ such that $|C_\nu| < \kappa$ for all $\nu < \kappa$ and $\bigcup_{\nu<\kappa} C_\nu = M \times R$. By a version of Jensen's diamond principle (see e.g. Eklof [3], 0.2) there is a sequence of functions $\{f_\nu : I_\nu \to C_\nu | \nu \in E\}$ such that for every function $f: I \to M \times R$ there is a $\nu \in E$ with $f | I_\nu = f_\nu$. Note that a homomorphism from M_ν to a torsion-free R-module is uniquely determined by its values on I_ν.

We will now construct a non-split exact sequence E_*: $0 \to R \to N \xrightarrow{\pi} M \to 0$ as the direct limit of exact sequences $E_\nu: 0 \to R \to N_\nu \to M_\nu \to 0$ $(\nu < \kappa)$ such that the underlying set of N_ν is $M_\nu \times R$, and whenever $\delta < \nu < \kappa$, there is a commutative diagram

$$\begin{array}{ccccccccc} E_\delta : 0 & \longrightarrow & R & \to & N_\delta & \xrightarrow{\pi_\delta} & M_\delta & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ E_\nu : 0 & \longrightarrow & R & \to & N_\nu & \xrightarrow{\pi_\nu} & M_\nu & \longrightarrow & 0 \end{array} \qquad (1)$$

where the vertical maps are inclusions. Let $\nu < \kappa$ and assume that E_μ has been defined for all $\mu < \nu$. If ν is a limit ordinal, let $E_\nu = \varinjlim E_\mu (\mu < \nu)$. If $\nu = \delta + 1$, we distinguish between two cases:

Case I. $\delta \notin E$, or there is no splitting homomorphism $g_\delta : M_\delta \to N_\delta$ for π_δ such that $g_\delta | I_\delta = f_\delta$. Then let E_ν be any extension of E_δ such that (1) commutes and the underlying set of N_ν is $M_\nu \times R$. E_ν exists since all sequences split by assumption (iii).

Case II. $\delta \in E$, and there is a splitting homomorphism $g_\delta : M_\delta \to N_\delta$ for π_δ such that $g_\delta | I_\delta = f_\delta$. Then by Eklof ([3], 1.3) there is an extension E_ν of E_δ such that g_δ does not extend to a splitting homomorphism for π_ν. By a cardinality argument, we may assume that $N_\nu = M_\nu \times R$ as sets.

Let $E_* = \varinjlim E_\nu (\nu < \kappa)$, $E_* : 0 \to R \to N \xrightarrow{\pi} M \to 0$, where now $N = M \times R$ as sets.

Assume, by way of contradiction, that there exists a splitting homomorphism g for π. By the choice of the f_ν, there exists $\delta \in E$ with $g | I_\delta = f_\delta$, so $E_{\delta+1}$ has been constructed according to Case II above. The desired contradiction will follow if we can show that $g(M_{\delta+1}) \subseteq N_{\delta+1}$, since then $g | M_{\delta+1}$ is an extension of $g | M_\delta$ to a splitting homomorphism for $\pi_{\delta+1}$. We will actually show that $g(M_\nu) \subset N_\nu$ for all $\nu < \kappa$. Assume that $g(a) \in N \setminus N_\nu$ for some $a \in M_\nu$. Let $b \in N_\nu$ with $\pi_\nu(b) = a$. Then $g(a) - b \in \ker \pi \subseteq N_\nu$, a contradiction. □

The following theorem and proof closely parallel the abelian group case as presented in Eklof [2]. The only assumption we need on R is that RD-submodules of torsion-free Whitehead modules are Whitehead again (see 1.1).

2.2 Theorem (V = L). *Let κ be a regular cardinal, R a domain with $|R| \leq \kappa$ such that RD-submodules of torsion-free Whitehead modules are again Whitehead. Then a torsion-free R-module M of rank κ is Whitehead if and only if there is a continuous well-ordered ascending chain $\{M_\nu | \nu < \kappa\}$ of submodules of M such that*

(i) $M_0 = 0$ *and* $M = \bigcup_{\nu<\kappa} M_\nu$;

(ii) rank $M_\nu < \kappa$ *for every* $\nu < \kappa$;

(iii) M_ν *is an RD-submodule in* M;

(iv) $M_{\nu+1}/M_\nu$ *is a Whitehead module for* $\nu < \kappa$.

Proof. It follows immediately from 1.5 that the stated condition is sufficient.
Conversely, assume $M \in \mathcal{W}$. We claim that M satisfies the following condition:

(*) Every RD-submodule A of M of rank $< \kappa$ is contained in an RD-submodule B of M of rank $< \kappa$ such that for every RD-submodule C of M above B with rank $C < \kappa$, C/B is Whitehead.

Assume for a contradiction that M fails to satisfy this condition. Then there exists an RD-submodule A of M with rank $A < \kappa$ such that for every RD-submodule B of M with rank $B < \kappa$ and $A \subseteq B$, we can find an RD-submodule C above B with rank $C < \kappa$ such that $C/B \notin \mathcal{W}$. We now define a continuous well-ordered ascending chain $\{A_\nu | \nu < \kappa\}$ of RD-submodules of M of ranks less than κ as follows. Let $A_0 = A$. Suppose $\nu < \kappa$ and A_μ has been defined for all $\mu < \nu$. Let $A_\nu = \bigcup_{\mu<\nu} A_\mu$ if ν is a limit ordinal. If $\nu = \delta + 1$, then take for A_ν an RD-submodule of M above A_δ such that rank $A_\nu < \kappa$ and $A_\nu / A_\delta \notin \mathcal{W}$. Now $\bigcup_{\mu<\kappa} A_\mu$ is Whitehead as an RD-submodule of M.

On the other hand, 2.1 applies with $E = \kappa$, showing that $\bigcup_{\mu<\kappa} A_\mu$ cannot be Whitehead.

We will now use (*) to represent M as the union of a continuous well-ordered ascending chain $\{\tilde{M}_\nu | \nu < \kappa\}$ of RD-submodules as follows. Let $\{a_\nu | \nu < \kappa\}$ be a maximal independent set of M. Set $\tilde{M}_0 = 0$. If $\nu < \kappa$ and $\tilde{M}_\mu$ has been defined for all $\mu < \nu$, then let $\tilde{M}_\nu = \bigcup_{\mu<\nu} \tilde{M}_\mu$ be for a limit ordinal ν. If $\nu = \delta + 1$, apply (*), taking for A the RD-closure of $\tilde{M}_\delta + Ra_\delta$, and for $\tilde{M}_\nu$ the B obtained from (*). Obviously, rank $\tilde{M}_\nu < \kappa$ for all $\nu < \kappa$. Since $M \in \mathscr{W}$, the set $E = \{\nu < \kappa | \tilde{M}_{\nu+1}/\tilde{M}_\nu \notin \mathscr{W}\}$ cannot be stationary in κ by 2.1. Let $f: \kappa \to \kappa$ be a strictly increasing continuous unbounded function whose image does not intersect E, and set $M_\nu = \tilde{M}_{f(\nu)}$. We finish the proof by showing that $M_{\nu+1}/M_\nu \in \mathscr{W}$ for all $\nu < \kappa$. In the exact sequence

$$0 \to \tilde{M}_{f(\nu)+1}/\tilde{M}_{f(\nu)} \to \tilde{M}_{f(\nu+1)}/\tilde{M}_{f(\nu)} \to \tilde{M}_{f(\nu+1)}/\tilde{M}_{f(\nu)+1} \to 0,$$

$\tilde{M}_{f(\nu)+1}/\tilde{M}_{f(\nu)} \in \mathscr{W}$ since $f(\nu) \notin E$ and $\tilde{M}_{f(\nu+1)}/\tilde{M}_{f(\nu)+1} \in \mathscr{W}$ since by construction $\tilde{M}_{f(\nu)+1}$ equals B in an application of (*). Hence $\tilde{M}_{f(\nu+1)}/\tilde{M}_{f(\nu)} = M_{\nu+1}/M_\nu$ must also be Whitehead. □

3. The main result for V = L

At this point, the question arises if the criterion of 2.2 can be extended to arbitrary cardinals. This amounts to asking whether or not there is an appropriate singular compactness property. This can indeed be obtained from a general theorem (see Shelah [14], introduction) by verifying certain axioms.

3.1 Theorem (V = L). *Let R be a domain with $|R| = \mu$ such that RD-submodules of torsion-free Whitehead R-modules are Whitehead again. A torsion-free R-module M of rank $\kappa \geq \mu$ is Whitehead if and only if it is the union of a continuous well-ordered ascending chain $\{M_\nu | \nu < \alpha\}$ of submodules ($M_0 = 0$) such that for all $\nu < \alpha$, $M_{\nu+1}/M_\nu$ is Whitehead and $|M_{\nu+1}/M_\nu| \leq \mu$.*

Proof. We stress again that for cardinals $> \mu$, rank and cardinality of torsion-free R-modules coincide. The proof of the theorem is by transfinite induction on κ. If $\kappa = \mu$, then the claim is trivial (no chain is needed at all). From now on assume that $\kappa > \mu$.

If κ is regular, first apply 2.2 to M and then the induction hypothesis to each of the factors $M_{\nu+1}/M_\nu$. It is then clear that the chain obtained in the first step can be refined to one with the desired properties.

If κ is singular, we will apply Shelah ([14], 0.4) to M and its RD-submodules of cardinalities $< \kappa$. For this purpose, we define, for submodules A, B of M: A is *free over* B (or $A|B$ is *free*) if there exists a continuous well-ordered ascending chain $\{A_\nu | \nu \leq \alpha\}$ of submodules of M such that 1) $A_0 = B$, $A_\alpha = A + B$; 2) for all $\nu < \alpha$, $|A_{\nu+1}/A_\nu| \leq \mu$; and 3) for each $\nu < \alpha$, $A_{\nu+1}/A_\nu$ is Whitehead. We say that $\{A_\nu | \nu \leq \alpha\}$ *witnesses the freeness of* $A|B$.

First of all, we have to verify that the following axioms of Shelah [14] are satisfied:

Axiom II. $A|B$ is free if $A+B|B$ is free; $A|A$ is free.
Axiom III. If $C \subseteq B \subseteq A$ and $A|B$, $B|C$ are free, then $A|C$ is free.
Axiom IV. If A_i $(i<\lambda)$ is increasing and $A_i|\bigcup_{j<i} A_j + B$ is free for $i<\lambda$, then $\bigcup_{i<\lambda} A_i|B$ is free.
Axiom VI. If $A|B+C$ is free, then for the μ-majority of $X \subseteq A+B+C$, $A \cap X|(B \cap X)+C$ is free.
Axiom VII. If $A|B$ is free, then for the μ-majority of $X \subseteq A+B$, $A|(A\cap X)+B$ is free.

Axioms II, III and IV are readily verified, using concatenation of chains for III and IV.

For axiom VI, assume that A is free over $B+C$, witnessed by $\{A_\nu | \nu \le \alpha\}$. For each $d \in A+B+C$, pick $a_d \in A$, $b_d \in B$, $c_d \in C$ with $d = a_d + b_d + c_d$ such that $a_d = 0$ whenever $d \in B+C$. Define functions f_A and f_B by setting $f_A(d) = a_d$ and $f_B(d) = b_d$ for all $d \in A+B+C$. For each $\nu < \alpha$, let $X_{\nu+1} \subseteq A_{\nu+1}$ be a complete set of representatives for $A_{\nu+1}/A_\nu$, and let $\{a^i_{\nu+1} | i < \mu\}$ be an enumeration of $X_{\nu+1}$. Since $A_0 = B+C$, we may assume that $X_{\nu+1} \subseteq A_{\nu+1} \cap A$. Now for each $d \in A+B+C$, let $\nu(d)$ be the minimal ν with $d \in A_\nu$, and define a set of functions $\{f_i | i<\mu\}$ on $A+B+C$ by setting $f_i(d) = 0$ if $d \in A_0$ and $f_i(d) = a^i_{\nu(d)}$ otherwise. This is possible since $\nu(d)$ (if not 0) is always a successor ordinal. Then the set of all submodules of $A+B$ which are closed under these functions forms a μ-majority. For the verification of axiom VI, we may assume that N is such a submodule, and we must show that $A\cap N$ is free over $(B\cap N)+C$. Now N obviously satisfies

(*) for each $\nu \le \alpha$, $A_{\nu+1} \cap N \nsubseteq A_\nu$ implies $X_{\nu+1} \subseteq A \cap N$.

We claim that $\{D_\nu | \nu \le \alpha\}$ with $D_\nu = (A_\nu \cap N) + (B \cap N) + C$ witnesses the freeness of $A \cap N$ over $(B\cap N)+C$. Using the fact that N is closed under f_A and f_B, it is easy to see that $(B\cap N)+C \subseteq [(B+C)\cap N] + (B\cap N) + C \subseteq (B\cap N)+C$, hence $D_0 = (B\cap N)+C$. Furthermore, $(A\cap N)+(B\cap N)+C \subseteq [(A+B+C)\cap N] + (B\cap N)+C \subseteq (A\cap N)+(B\cap N)+C$ whence $D_\alpha = (A\cap N)+(B\cap N)+C$.

It remains to show that $D_{\nu+1}/D_\nu$ is Whitehead and $|D_{\nu+1}/D_\nu| \le \mu$ for all $\nu<\alpha$. We distinguish between two cases.

Case I. $A_{\nu+1}\cap N \subseteq A_\nu$. Then $D_{\nu+1} = D_\nu$, and both claims are trivial.

Case II. $A_{\nu+1} \cap N \nsubseteq A_\nu$. Then by (*), $X_{\nu+1} \subseteq N$. Now every $a \in D_{\nu+1}$ is congruent to some $b \in A_{\nu+1}\cap N \bmod D_\nu$, and $b = x + c$ with $x \in X_{\nu+1}$ and $c \in A_\nu$. It follows that $c \in A_\nu \cap N \subseteq D_\nu$, and hence $a \equiv b \equiv x \bmod D_\nu$. We conclude that $X_{\nu+1}$ is a complete set of representatives for $D_{\nu+1}/D_\nu$. This together with $D_\nu \subseteq A_\nu$ implies that $\varphi: A_{\nu+1}/A_\nu \to D_{\nu+1}/D_\nu$ defined by $\varphi(x+A_\nu) = x + D_\nu (x \in X_{\nu+1})$ is an isomorphism, and $D_{\nu+1}/D_\nu$ inherits the desired properties from $A_{\nu+1}/A_\nu$.

For axiom VII, let A be free over B, witnessed by $\{A_\nu | \nu \le \alpha\}$. Working with the same notion of μ-majority as above, we let N be a submodule of $A+B$ such that (*) holds. We claim that $\{D_\nu | \nu\le\alpha\}$ with $D_\nu = A_\nu + (A\cap N)$ witnesses the freeness of A over $B+(A\cap N)$. Again it is obvious that $\{D_\nu|\nu\le\alpha\}$ is a continuous well-ordered ascending chain such that $D_0 = B + (A \cap N)$ and $D = A + B + (A \cap N)$. To verify the remaining two conditions, we proceed as above for axiom VI.

Case I. $A_{\nu+1} \cap N \nsubseteq A_\nu$. We claim that then $D_{\nu+1}/D_\nu = 0$. Indeed, if $a \in D_{\nu+1}$, then $a = b + c$ with $b \in A_{\nu+1}$, $c \in A \cap N$, and $b = x + d$ with $x \in X_{\nu+1}$, $d \in A_\nu$. Together, we obtain $a = x + d + c \in D_\nu$, since $X_{\nu+1}$, A_ν and $A \cap N$ are all contained in D_ν.

Case II. $A_{\nu+1} \cap N \subseteq A_\nu$. Since $X_{\nu+1}$ is obviously a complete set of representatives for $D_{\nu+1}/D_\nu$, the map $\varphi: A_{\nu+1}/A_\nu \to D_{\nu+1}/D_\nu$ given by $\varphi(x + A_\nu) = x + D_\nu (x \in X_{\nu+1})$ is an epimorphism. To see that it is injective, assume that $x \in X_{\nu+1} \cap D_\nu$. Then $x = a + b$ with $a \in A_\nu$, $b \in A \cap N$. It follows that $b \in A_{\nu+1}$, hence $b \in A_{\nu+1} \cap N \subseteq A_\nu$ so $x \in A_\nu$.

To see that the set of RD-submodules of M forms a μ-majority, we have to show that there is a set of at most μ functions such that RD-submodules are exactly those submodules which are closed under these functions. It is clear that such a set is the set of functions $\{f_r: M \to M \mid r \in R\}$ where $f_r(a) = b$ with $rb = a$ if such a b exists (it is then unique), and $f_r(a) = 0$ otherwise.

We have made all the preparations to apply Shelah ([14], 0.4) which states that, assuming A has singular cardinality κ, $A \mid B$ is free if and only if $A \mid B$ is κ-free, i.e. for the μ-majority of $X \subseteq A + B$, $|X| < \kappa$ implies $A \cap X \mid B$ is free. All what we have to do is to choose $A = M$, $B = 0$; then using the RD-submodules as μ-majority, induction hypothesis implies that M has a chain as stated in the theorem. □

Note that the hypothesis in the last theorem that RD-submodules of Whitehead R-modules be again Whitehead is satisfied by several domains R, e.g. if i.d. $R = 1$ then R has this property.

It is worthwhile pointing out that 3.1 can be generalized by replacing the cardinality conditions on the modules by milder restriction on their presentation. In this case, however, a more sophisticated proof is necessary.

4. Proper forcing

Our next purpose is to show that 3.1 is not a theorem in ZFC for lots of rings, including some to which our results will be applied. To this end, we need to find a universe in which 3.1 fails. The attempt to imitate the construction for abelian groups under the hypothesis of Martin's Axiom and the denial of CH encounters serious difficulties. If $|R| > \aleph_0$, the obstacles are of a rather obvious set-theoretical nature, whereas in the countable case, there are more subtle problems of algebraic character. We can, however, get the desired result with proper forcing as it was done for abelian groups by Shelah [13]. We refer the reader to Mekler [9] for a lucid presentation. We can, in fact, set things up in such a way that the proof of the main theorem carries over almost verbatim from Mekler ([9], 2.12), so that we will not get involved with the set theory here at all.

4.1 Theorem. *If* ZFC *is consistent, then* ZFC + GCH *is consistent with the existence of a stationary set* $S \subseteq \omega_1$ *such that the following holds:*

whenever R is a countable domain, M an R-module of cardinality $\aleph_1$, and $\{M_\nu \mid \nu < \omega_1\}$ a continuous well-ordered ascending chain of submodules of M such that

(i) $M_0 = 0$ *and* $\bigcup_{\nu<\omega_1} M_\nu = M$,
(ii) M_ν *is free of rank* $\leq \aleph_0$ *for all* $\nu < \omega_1$,
(iii) *the set* $E = \{\mu < \omega_1 \mid M_\nu/M_\mu$ *is not free for some* $\nu > \mu\}$ *contains only limit ordinals and is contained in* S,

then M is a Whitehead module.

Proof. The proof is essentially identical with the proof of "$\Leftarrow$" in Mekler ([9], 2.1.2) with the following observations to be kept in mind:

1) The given chain $\{M_\nu \mid \nu < \omega_1\}$ is already what Mekler calls nice, so it does not have to be thinned out as in [9].
2) For abelian groups, our definition of the set E is equivalent to the one usually given ($E = \{\nu < \omega_1 \mid M_\nu$ is not $\aleph_1$-pure$\}$). This equivalence fails, however, in our case, but (iii) above makes the proof work. □

To proceed, we require the following existence statement.

4.2 Lemma. *Let S be a stationary set in ω_1, R a domain with $|R| \leq \aleph_1$ which is not complete and Q is countably generated as an R-module. Then there exists an R-module M with $|M| = \aleph_1$ with admits a chain of submodules as described in* 4.1 (i)–(iii), *but no chain of the type described in* 2.2.

Proof. Using a well-known construction, one can define a continuous ascending chain $\{M_\nu \mid \nu < \omega_1\}$ of R-modules such that

(i) M_ν is free of countable rank for each $\nu < \omega_1$,
(ii) M_ν is RD- in $M_{\nu+1}$ for each $\nu < \omega_1$,
(iii) if we denote the set of all limit ordinals in S by E, then $\mu \in E$ implies $M_{\mu+1}/M_\mu$ contains a copy of Q, and if $\mu \notin E$, then M_ν/M_μ is free for all $\mu < \nu < \omega_1$.

Details of the construction for abelian groups may be found in Eklof ([2], 7.3). All that needs to be done to extend this to modules over domains with countably generated field of quotients is to pick $\{s_i \in R \mid i < \omega\}$ such that $\{s_i^{-1} \mid i < \omega\}$ generate Q and to replace in Eklof's construction, for $m \leq n < \omega$, $n!/m!$ by $s_m s_{m+1} \ldots s_n$. If we let $M = \bigcup_{\nu<\omega_1} M_\nu$, then $\{M_\nu \mid \nu < \omega_1\}$ will be a chain as described in 4.1 (i)–(iii).

Now assume, by way of contradiction, that M admits a chain $\{N_\nu \mid \nu < \omega_1\}$ of submodules as described in 2.2. Since all submodules in question are of countable rank, we may apply the usual back and forth argument to conclude that there exists a club set $C \subseteq \omega_1$ such that $M_\nu = N_\nu$ for all $\nu \in C$. Since E is stationary as the set of all limit ordinals in a stationary set, we can pick $\nu \in C \cap E$. Let $\lambda < \omega_1$ be a limit ordinal such that $M_{\nu+1} \subseteq N_\lambda$. Then N_λ/N_ν contains a copy of Q and hence can not be Whitehead ($\mathrm{Ext}_R^1(Q, R) \neq 0$, and Q is a summand in N_λ/N_ν). On the other hand,

$N_\lambda/N_\nu = \bigcup_{\nu \le \mu < \lambda} N_\mu/N_\nu$, and for all μ with $\nu \le \mu < \lambda$, $(N_{\mu+1}/N_\nu)/(N_\mu/N_\nu) \simeq N_{\mu+1}/N_\mu$ is Whitehead, so N_λ/N_ν would have to be Whitehead by 1.5. □

Combining 4.1 and 4.2, and noting that a countable domain R cannot be R-complete, we now obtain:

4.3 Theorem. *If* ZFC *is consistent, then* ZFC + GCH *is consistent with the following: for each countable domain R there exists a torsion-free Whitehead module M over R, of cardinality* $\aleph_1$, *such that M does not allow a chain as described in* 2.2. □

5. Application to principal ideal domains

We wish to apply our results to domains of special kinds. The first case to be considered is when R is a P.I.D.

For countable P.I.D.'s our results in the preceding sections do not yield anything new. But if $|R| = \aleph_1$, then we can obtain full information about the Whitehead modules over R.

By Gerstner-Kaup-Weidner [7], if R is a slender P.I.D., then all Whitehead modules over R are torsion-free and those of countable rank are free. Using this, we can derive

5.1 Theorem (V = L). *Let R be a slender* P.I.D. *of cardinality* $\aleph_1$. *An R-module is Whitehead exactly if it is free.*

Proof. To verify necessity, observe that by 1.1, submodules of Whitehead modules are again Whitehead. We refer to 2.2 in order to derive that a Whitehead module M of rank $\aleph_1$ must be the union of a well-ordered continuous ascending chain of submodules $\{M_\nu \mid \nu < \omega_1\}$ such that $M_0 = 0$, rank $M_\nu \le \aleph_0$ and, for each $\nu < \omega_1$, $M_{\nu+1}/M_\nu$ is Whitehead. The quotients $M_{\nu+1}/M_\nu$ being of countable rank, they are, in view of the preceding remark, necessarily free R-modules. It follows that M itself has to be free.

If M is a Whitehead module of arbitrary cardinality, then by 3.1 it has a well-ordered continuous ascending chain of submodules $\{M_\nu \mid \nu < \kappa\}$ with Whitehead quotients of ranks $\leqq \aleph_1$. By the preceding paragraph, these quotients are free, so M itself is free. □

Examples of slender P.I.D.'s of cardinality $\aleph_1$ are abundant (e.g. the polynomial rings $F[x]$ over fields F of cardinality $\aleph_1$). On the other hand, the ring $\tilde{\mathbb{Z}}_p$ of the p-adic integers is not a slender P.I.D. and all torsion-free $\tilde{\mathbb{Z}}_p$-modules are Whitehead. (If V = L, $\tilde{\mathbb{Z}}_p$ has cardinality $\aleph_1$.)

6. Application to countable valuation domains

Our results will be applied to two special cases of valuation domains. First, to countable valuation domains, and secondly, to almost maximal valuation domains of cardinality $\aleph_1$. In both cases, the description of torsion-free Whitehead modules is as satisfactory as that of Whitehead groups.

We start with a few preliminary lemmas.

6.1 Lemma. *Let R be a valuation domain and A a cyclic torsion R-module, say, $A = R/I$ ($I \neq 0$ an ideal of R). A is a Whitehead module if and only if I is the maximal ideal of R and is not principal.*

Proof. The necessity follows at once from 1.3. To prove the converse, suppose that the maximal ideal P of R is not principal. The sequence $\mathrm{Hom}(R, R) \to \mathrm{Hom}(P, R) \to \mathrm{Ext}^1(R/P, R) \to 0$ (induced by the exact sequence $0 \to P \to R \to R/P \to 0$) shows that it suffices to prove that every homomorphism $\varphi : P \to R$ extends to a map $R \to R$. Evidently, φ extends to a homomorphism $\varphi^* : R \to Q$, and since $1 \in \mathrm{Im}\,\varphi$ is impossible, $\mathrm{Im}\,\varphi^* \leq R$ follows. □

6.2 Lemma. *Let R be a countable valuation domain and M a Whitehead R-module. Then both the torsion submodule tM of M and the torsion-free factor module M/tM are Whitehead.*

Proof. If R is countable, then all the ideals of R and Q are countably generated, and thus gl.d. $R \leqq 2$ and all torsion-free R-modules have projective dimensions $\leqq 1$. Therefore, in the exact sequence

$$0 = \mathrm{Hom}(tM, R) \to \mathrm{Ext}^1(M/tM, R) \to \mathrm{Ext}^1(M, R) \to$$
$$\to \mathrm{Ext}^1(tM, R) \to \mathrm{Ext}^2(M/tM, R)$$

the last term vanishes. Hence if M is Whitehead, then all the terms vanish. □

A similar argument applies to prove:

6.3 Lemma. *RD-submodules of torsion-free Whitehead modules over countable valuation domains are again Whitehead.* □

The following characterization of $\mathrm{Ext}^1_R(J, R)$ where J is a submodule of Q is crucial for our discussion. For any ring R, $\tilde{R}$ denotes the completion of R in the R-topology (see Matlis [8]). Furthermore, we let $R : J = \{r \in R \mid rJ \subseteq R\}$; this is an ideal of R.

6.4 Lemma. *Let R be a valuation domain and J a non-cyclic submodule of Q containing R. Then, as R-modules,*

$$\mathrm{Ext}^1_R(J, R) \simeq \tilde{S}/S$$

where S denotes the ring $R/(R : J)$.

Proof. Using the exact sequences $0 \to R \to J \to J/R \to 0$ and $0 \to R \to Q \to Q/R \to 0$ we obtain the exact sequences

$$0 \to \operatorname{Hom}(J, R) \to \operatorname{Hom}(R, R) \to \operatorname{Ext}_R^1(J/R, R) \to \operatorname{Ext}_R^1(J, R) \to 0$$

and

$$0 \to \operatorname{Hom}(J/R, Q/R) \to \operatorname{Ext}_R^1(J/R, R) \to 0$$

By Fuchs-Salce ([6], VII. 3.1 and 2.4), we have $\operatorname{Hom}(J/R, Q/R) \simeq \operatorname{End}(J/R) \simeq \tilde{S}$, where $S = R/(R:J)$. Hence

$$\operatorname{Ext}_R^1(J/R, R) \simeq \tilde{S} \tag{2}$$

by the second exact sequence above. Since $\operatorname{Hom}(J, R) = R:J$ and $\operatorname{Hom}(R, R) = R$, the first sequence above implies that

$$\operatorname{Ext}_R^1(J, R) \simeq \operatorname{Ext}_R^1(J/R, R)/S. \tag{3}$$

The claim will follow from (2) and (3) if we can show that under the isomorphism (2) the image of an element $s \in S$ in $\operatorname{Ext}_R^1(J/R, R)$ is the same as its image as implied in (3).

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & J & \longrightarrow & J/R & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \longrightarrow & R & \longrightarrow & A & \longrightarrow & J/R & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & R & \longrightarrow & Q & \longrightarrow & Q/R & \longrightarrow & 0
\end{array}$$

which is constructed as follows. Starting with the bottom row, define the middle row by using $\alpha: J/R \to Q/R$ as the natural embedding followed by a multiplication by $s \in S$. The map $\beta: J \to A$ is obtained by applying the pullback property of A to the natural map $J \to J/R$ and to the multiplication by $r \in R$ in $J \to Q$ where $s = r + (R:J)$. An easy diagram chasing shows that γ (which is essentially the restriction of $J \to Q$) has to be multiplication by r. □

6.5 Lemma. *Let $J \neq Q$ be a non-cyclic submodule of Q containing the valuation domain R. If $R/(R:J)$ is countable, then* $\operatorname{Ext}^1(J, R)$ *is uncountable.*

Proof. Observe that the ring $S = R/(R:J)$ must have infinitely many ideals $\neq 0$ with intersection 0. In fact, otherwise either $R:J = P$ (which is easily ruled out as a possibility) or R is a discrete valuation ring (and J is cyclic). Therefore, the S-topology on S cannot be discrete. Consequently, the S-completion $\tilde{S}$ of S is uncountable. By 6.4, $\operatorname{Ext}^1(J, R) \cong \tilde{S}/S$ whence the claim follows. □

We are now in a position to prove that the classical results for uncountable abelian Whitehead groups hold over countable valuation domains.

6.6 Theorem. *Let R be a countable valuation domain, and M a countable, torsion-free R-module. Then M is Whitehead if and only if it is free.*

Proof. For the non-trivial direction, we first assume that M has finite rank k and induct on k. Let $k = 1$. Then M is isomorphic to a submodule of Q containing R. 6.5 implies that M is cyclic. Now let $k > 1$ and N a rank $k-1$ RD-submodule of M; by 6.3, N is again Whitehead. We claim that M/N is cyclic. Consider $0 \to N \to M \to M/N \to 0$ and the induced exact sequence

$$\mathrm{Hom}(N, R) \to \mathrm{Ext}^1_R(M/N, R) \to \mathrm{Ext}^1_R(M, R) = 0.$$

By induction hypothesis (applied to N), $\mathrm{Hom}(N, R)$ is finitely generated free, hence countable. If M/N were not cyclic, then $\mathrm{Ext}^1_R(M/N, R)$ would be uncountable by 6.5, which is impossible as it is an epimorphic image of $\mathrm{Hom}(N, R)$. Thus M/N is cyclic and $M \simeq N \oplus M/N$. This concludes the proof for the finite rank case.

If M is of countable rank, the claim follows immediately from the finite rank case together with 6.3 and the fact that a torsion-free R-module of countable rank is free exactly if all finite rank RD-submodules are free (Fuchs-Salce [6], IV. 3.6). □

We can now show that in the constructible universe, all torsion-free Whitehead modules over countable valuation domains are free.

6.7 Theorem (V = L). *Let R be a countable valuation domain. A torsion-free R-module is Whitehead if and only if it is free.*

Proof. By 3.5, a Whitehead R-module M admits a continuous well-ordered ascending chain $\{M_\nu | \nu < \kappa\}$ of submodules with $M_0 = 0$ and $M = \bigcup_{\nu<\kappa} M_\nu$ such that the quotients $M_{\nu+1}/M_\nu$ are Whitehead modules of countable rank. Let M^*_ν denote the RD-closure of M_ν in the torsion-free module M. Then $\{M^*_\nu | \nu < \kappa\}$ is likewise a continuous, well-ordered ascending chain with $M^*_0 = 0$ and $M = \bigcup_{\nu<\kappa} M^*_\nu$ where $M^*_{\nu+1}/M^*_\nu$ are torsion-free of countable rank. If we can show that they are Whitehead modules, then 6.6 will complete the proof at once.

From 1.5 we conclude that M/M_ν is Whitehead for each $\nu < \kappa$, while from 6.2 it follows that (M^*_ν/M_ν and) $M/M^*_\nu \in \mathcal{W}$. Hence, by 6.3, $M^*_{\nu+1}/M^*_\nu$ is again a Whitehead module. □

Once 6.7 has been established, we can argue with 6.2 to show that in L, an R-module M over a countable valuation domain R is Whitehead exactly if it is the direct sum of a free R-module and a torsion Whitehead module. (Unfortunately, we have no satisfactory results on torsion Whitehead modules; from 6.1 it is easy to derive that if the maximal ideal P of R is not principal, then direct sums of copies of R/P are in $\mathcal{W}$.)

We note that, in view of 4.3, the preceding theorem cannot be proved in ZFC + GCH.

7. Application to almost maximal valuation domains

In this section, we apply our general results to almost maximal valuation domains R; recall that in this case i.d. $R = 1$. We assume that R is not maximal (otherwise all torsion-free R-modules are Whitehead).

First, we deal with the non-torsion-free case.

7.1 Lemma. *Let R be an almost maximal valuation domain.*

(i) *If the maximal ideal P of R is principal, then there do not exist torsion Whitehead modules over R.*
(ii) *If P is not principal, then the torsion Whitehead modules are precisely the direct sums of copies of R/P.*

Proof. In view of almost maximality, 1.1 is applicable, and 6.1 implies (i). Again by 1.1 and 6.1, if P is not principal, then all cyclic submodules of a torsion Whitehead module M are isomorphic to R/P. Therefore, M is a semisimple R-module, and hence it has the indicated structure. □

7.2 Lemma. *If R is an almost maximal valuation domain, then every Whitehead R-module is the direct sum of a torsion and a torsion-free Whitehead module.*

Proof. By virtue of 6.3 and 7.1, the torsion part of a non-torsion-free Whitehead module M is, if not 0, a direct sum of copies of R/P. But such a module is readily seen to be pure-injective whence the claim is immediate. □

In view of these lemmas, the problem is reduced to the torsion-free case. We start with modules of finite and countable rank.

7.3 Theorem. *Let R be an almost maximal but not maximal valuation domain and M a torsion-free R-module of countable rank. Then M is Whitehead if and only if $M = \bigoplus_{j=1}^{k} I_j$, where the I_j are ideals of R ($k \leqq \omega$).*

Proof. If $M = \bigoplus_{j=1}^{k} I_j$ with ideals I_j of R, then $\mathrm{Ext}_R^1(M, R) = \prod_{j=1}^{k} \mathrm{Ext}_R^1(I_j, R) = 0$ since R is almost maximal (Fuchs-Salce [6], VI. 5.4).

Conversely, assume $M \in \mathscr{W}$ is of finite rank k. Induct on k. If $k = 1$, then $M \simeq Q$ being impossible (as R is not maximal), M is isomorphic to an ideal of R. Let $k > 1$ and $B = \oplus J_i$ a basic submodule of M. By [6, XIV. 1.4], M/B is divisible. Hence $\mathrm{Ext}^1(M/B, R)$ is torsion-free divisible. It is an epic image of a finite direct sum of ideals of R as is clear from the exact sequence

$$\oplus \mathrm{Hom}(J_i, R) = \mathrm{Hom}_R(B, R) \to \mathrm{Ext}_R^1(M/B, R) \to \mathrm{Ext}_R^1(M, R) = 0.$$

But this can happen only if $\mathrm{Ext}_R^1(M/B, R) = 0$ in which case $M/B = 0$, $M = B$, i.e. M is isomorphic to a direct sum of ideals.

If M is of countably infinite rank, then it is the union of a countable chain $0 = M_0 < M_1 < \dots < M_n < \dots$ of RD-submodules such that M_n has rank n. By 1.1, each $M_n \in \mathscr{W}$, so these M_n are, by what has been shown above, direct sums of ideals of R. Since rank one RD-submodules of completely decomposable torsion-free R-modules are summands (cf. [6], XIV. 2.2), we conclude by induction that M_n is a summand of M_{n+1} with rank one complement X_n. Now $M = \bigoplus_{n<\omega} X_n$ as is readily verified. □

The last theorem, unfortunately, does not extend to Whitehead modules of higher ranks, as is shown by the following example.

7.4 Example. Let R be an almost maximal but not maximal valuation domain of global dimension 2 such that Q is $\aleph_1$- but not $\aleph_0$-generated; such an R exists. Then p.d. $Q = 2$, so in a projective resolution $0 \to H \to F \to Q \to 0$ of Q where F is a free R-module, we have p.d. $H = 1$. As an RD-submodule of F, $H \in \mathscr{W}$. However, H is not completely decomposable, because its rank one summands are summands of F, and thus isomorphic to R, but H is not free.

By making use of 7.3, we wish to derive from 3.1 the following theorem.

7.5 Theorem (V = L). *Let R be an almost maximal, but not maximal valuation domain of cardinality $\aleph_1$. A torsion-free R-module M is Whitehead if and only if it is the union of a continuous well-ordered ascending chain of RD-submodules $\{M_\nu \mid \nu < \kappa\}$ such that $M_0 = 0$ and, for each $\nu < \kappa$, $M_{\nu+1}/M_\nu$ is isomorphic to an ideal of R.*

Proof. Sufficiency is obvious in view of 1.5 and $\mathrm{Ext}^1_R(J, R) = 0$ for ideals J of R.

Conversely, suppose $M \in \mathscr{W}$ has rank $\kappa \geqq \aleph_1$. By 3.1, M is the union of a continuous well-ordered ascending chain $\{N_\nu \mid \nu < \kappa\}$ such that $N_0 = 0$ and $N_{\nu+1}/N_\nu$ is Whitehead of cardinality $\leqq \aleph_1$, for each ν. The argument used at the end of the proof of 6.7 shows that without loss of generality we may assume that the N_ν are RD in M, i.e. $N_{\nu+1}/N_\nu$ are torsion-free. By making use of 2.2, along with the same RD-closure argument, each $N_{\nu+1}/N_\nu$ can be viewed as the union of such a chain with factors which are torsion-free Whitehead modules of countable ranks. Refining the chain $\{N_\nu \mid \nu < \kappa\}$ accordingly, and then again refining by using 7.3, we finally get a chain $\{M_\nu \mid \nu < \kappa\}$ as stated in the theorem. □

It is perhaps worthwhile pointing out that the main thrust of 7.5 is that a chain $\{M_\nu \mid \nu < \kappa\}$ can be found for M where $M_{\nu+1}/M_\nu$ are torsion-free of rank 1 and none is isomorphic to Q.

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Thomas Becker, Department of Mathematics, University of New Orleans, New Orleans, La 70148, USA
Laszlo Fuchs, Department of Mathematics, Tulane University, New Orleans, La 70118, USA
Saharon Shelah, Department of Mathematics, Hebrew University, Jerusalem, Israel