THE RANDOM GRAPH AND AUTOMORPHISMS OF THE RATIONAL WORLD

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ABSTRACT

It is shown that the group of almost automorphisms of Rado's 'random graph' cannot be embedded (via a permutation group embedding) into the group of homeomorphisms to itself of the space of rational numbers.

1. *Introduction*

The group of homeomorphisms of the space $\mathbb{Q}$ of rational numbers to itself, Homeo $\mathbb{Q}$, has a very rich structure, which has been explored in [**1**, **2**, **3**, **4**] and elsewhere. Of particular interest is the question of which permutation groups can be embedded in it. Here, by an 'embedding' we understand a 'permutation' embedding. That is, an *embedding* of a permutation group G_1 on a set Ω_1 into a permutation group G_2 on Ω_2 is a bijection from Ω_1 onto Ω_2 which induces a group monomorphism of G_1 into G_2. If G is a permutation group on a set Ω (for us always of cardinality $\aleph_0$), then a subset X of Ω is a *support* if for some $g \in G$, $X = \{x \in \Omega : gx \neq x\}$.

The principal permutation groups which we study here are the automorphism and almost automorphism groups of Rado's countable universal homogeneous graph (the 'random' graph), but as in [**5**] we find it a little easier to formulate the results in terms of the C-coloured version of the random graph, Γ_C, where C is a set (of 'colours') such that $2 \leqslant |C| \leqslant \aleph_0$. This may be characterized as follows: Γ_C is a set (of vertices) of cardinality $\aleph_0$ together with a 'colouring' function F from the set of two-element subsets of Γ_C into C, such that for any map α from a finite subset of Γ_C into C, there is $x \in \Gamma_C - \operatorname{dom} \alpha$ such that $(\forall y \in \operatorname{dom} \alpha)(\alpha(y) = F\{x, y\})$. A back-and-forth argument shows that Γ_C is then unique up to isomorphism, and the usual random graph may be identified with $\Gamma_{\{0,1\}}$ on joining x, y if and only if $F\{x, y\} = 1$. We write Γ for Γ_C when C is understood.

In addition to the automorphism group $\operatorname{Aut}\Gamma_C$ of Γ_C, we also study its 'almost automorphism' group, $\operatorname{AAut}\Gamma_C$. A permutation g of Γ_C is said to be *almost an automorphism* if the set of two-element subsets $\{x, y\}$ of Γ_C such that $F\{gx, gy\} \neq F\{x, y\}$ is finite. It can then be checked that the set $\operatorname{AAut}\Gamma_C$ of all almost automorphisms of Γ_C forms a highly transitive permutation group on Γ_C extending $\operatorname{Aut}\Gamma_C$.

The background to the question discussed here is briefly as follows. In [**2**], a necessary and sufficient condition, called the 'mimicking property', was given for a countable permutation group G to be embeddable in Homeo $\mathbb{Q}$. Another proof of this was given in [**4**], employing P. M. Neumann's reformulation of the mimicking property, there labelled (MC): any finite intersection of supports is empty or infinite. In addition, it was shown that $\operatorname{Aut}\Gamma$ and $\operatorname{AAut}\Gamma$ both fulfil (MC) in a strong form, namely that any finite intersection of *non-empty* supports is infinite. Since these are

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uncountable groups, one cannot immediately deduce from [2] that they are embeddable in Homeo $\mathbb{Q}$, but by a direct argument it was shown in [4] that in fact Aut Γ *can* be so embedded, with the situation for AAut Γ still being unresolved. We settle this here by showing that AAut Γ cannot be embedded in Homeo $\mathbb{Q}$. This applies to any C with $2 \leqslant |C| \leqslant \aleph_0$, though it would actually be enough to prove the result for $|C| = \aleph_0$, since AAut Γ_C is easily seen to be embeddable in AAut $\Gamma_{C'}$ whenever $|C'| \leqslant |C|$.

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2. *The non-embeddability result*

THEOREM 2.1. *For any C with $2 \leqslant |C| \leqslant \aleph_0$,* AAut Γ_C *cannot be embedded into* Homeo $\mathbb{Q}$.

Proof. Let us assume for the sake of a contradiction that AAut Γ *can* be embedded in Homeo $\mathbb{Q}$. This means that Γ can be identified with $\mathbb{Q}$ in such a way that all the members of AAut Γ are continuous.

By a *type* we shall understand a map α from a finite subset of Γ into C. (This is essentially a complete quantifier-free 1-type.) An element x *realizes* α if $(\forall y \in \operatorname{dom} \alpha)(F\{x, y\} = \alpha(y))$. The characterization of Γ_C then just amounts to saying that all types are realized. We let 1-1 maps p from subsets of Γ into Γ act on types in the natural way. Thus if α is a type, and $\operatorname{dom} \alpha \subseteq \operatorname{dom} p$, $p(\alpha)$ has domain $p(\operatorname{dom} \alpha)$, and $(p(\alpha))(p(x)) = \alpha(x)$. For any type α, let Y_α be the set of its realizations, and for any finite subset A of Γ and $x \in \Gamma - A$, let $\operatorname{tp}(x; A)$ be the (unique) type α with domain A such that $x \in Y_\alpha$. For $A, X \subseteq \Gamma$ with A finite, let $S(A, X)$ be the set of types α with domain A such that $X \cap Y_\alpha \neq \varnothing$. The following lemma is the main step in the proof.

LEMMA 2.2. *If X is a non-empty open subset of Γ, then there is a non-empty open subset Y of X and a finite subset A of Γ such that $\bigcup \{Y_\alpha : \alpha \in S(A, Y)\} \subseteq X$.*

Proof. Suppose otherwise. Thus for every non-empty open $Y \subseteq X$ and finite A, $\bigcup \{Y_\alpha : \alpha \in S(A, Y)\} \nsubseteq X$. By passing to a subset of X we may suppose that the complement of X contains a non-empty open set Z. Pick $x \in X$ and $z \in Z$.

We shall construct $g \in \operatorname{Aut} \Gamma$ by a back-and-forth argument. Let P be the family of all finite partial automorphisms of Γ taking z to x. Let Γ be enumerated as $\{u_n : n \in \omega\}$ and let $\{X_n : n \in \omega\}$, $\{Z_n : n \in \omega\}$ be sequences of open sets such that $X \supseteq X_0 \supseteq X_1 \supseteq \ldots$, $Z \supseteq Z_0 \supseteq Z_1 \supseteq \ldots$, $\bigcap \{X_n : n \in \omega\} = \{x\}$ and $\bigcap \{Z_n : n \in \omega\} = \{z\}$. We choose p_n in P by induction. Let $p_0 = \{(z, x)\}$. We ensure that the following hold:

(i) $u_n \in \operatorname{dom} p_{n+1} \cap \operatorname{range} p_{n+1}$;

(ii) for some $z_n \in Z_n$, $p_{n+1}(z_n) \notin X$.

Suppose p_n has been chosen. First extend p_n to p'_n so that $u_n \in \operatorname{dom} p'_n \cap \operatorname{range} p'_n$. Next we seek $z_n \in Z_n$ and $y_n \notin X$ such that $p_{n+1} = p'_n \cup \{(z_n, y_n)\} \in P$. For this it suffices that $p'_n(\operatorname{tp}(z_n; \operatorname{dom} p'_n)) = \operatorname{tp}(y_n; \operatorname{range} p'_n)$. Let $g_n \in \operatorname{Aut} \Gamma$ be an extension of p'_n. By assumption, for each $m \geqslant n$ there are $\alpha_m \in S(\operatorname{range} p'_n, X_m)$ and $x_m \in X_m \cap Y_{\alpha_m}$ such that $Y_{\alpha_m} \nsubseteq X$. Since $\lim_{m \to \infty} x_m = x$, and g_n^{-1} is continuous, $\lim_{m \to \infty} g_n^{-1}(x_m) = z$. Hence $g_n^{-1}(x_m) = z_n \in Z_n$ for some m. Let $y_n \in Y_{\alpha_m} - X$ for this m. Then

$$p'_n(\operatorname{tp}(z_n; \operatorname{dom} p'_n)) = \operatorname{tp}(x_m; \operatorname{range} p'_n) = \alpha_m = \operatorname{tp}(y_n; \operatorname{range} p'_n).$$

Hence $p_{n+1} \in P$.

Now letting $g = \bigcup_{n\in\omega} p_n$, we see from (i) that $g \in \mathrm{Aut}\,\Gamma$, and so it is continuous. Since $\lim_{n\to\infty} z_n = z$, it follows that $\lim_{n\to\infty} g(z_n) = x$, contrary to $g(z_n) \notin X$, all n.

We may now conclude the proof of Theorem 2.1. Let $X \subseteq \mathbb{Q}$ be a non-empty open set whose complement contains a non-empty open set Z, and let $Y \subseteq X$ and $A \subseteq \Gamma$ be as given by Lemma 2.2, Y non-empty open and A finite. Pick $x \in Y - A$ and $z \in Z - A$. Since $\mathrm{AAut}\,\Gamma$ is highly transitive, there is $g \in \mathrm{AAut}\,\Gamma$ fixing A pointwise and taking z to x. As g is a homeomorphism, there is an open set U with $z \in U \subseteq Z$ such that $gU \subseteq Y$. Since $F\{gu, gv\} \neq F\{u, v\}$ for only finitely many pairs $\{u, v\}$, there is $y \in U - A$ such that $F\{gy, gv\} = F\{y, v\}$, all $v \neq y$. Therefore $\mathrm{tp}(gy; A) = \mathrm{tp}(y; A) = \alpha$ say, so as $gy \in Y$, $\alpha \in S(A, Y)$ and so $Y_\alpha \subseteq X$. On the other hand, $y \in Y_\alpha$, and y lies in Z, which is disjoint from X, giving the desired contradiction.

3. *Further problems*

In [2] the question was raised as to whether the sufficiency of (MC) for the embeddability of G into Homeo $\mathbb{Q}$ could be established for a significantly larger class of groups than the countable ones. It was remarked that there is a Borel group for which the condition is *not* sufficient. The result of this paper indicates that a natural example of such a Borel group suffices for this, since it is clear that $\mathrm{AAut}\,\Gamma$ is actually F_σ. The obvious questions arising are as follows.

(1) Give a criterion for which Borel groups satisfying (MC) can be embedded in Homeo $\mathbb{Q}$. In particular, is it the case that every closed group fulfilling (MC) can be embedded in Homeo $\mathbb{Q}$? (We remark that the closed groups are precisely the automorphism groups of first-order structures.)

(2) The proof of Lemma 2.2 used only the embedding of $\mathrm{Aut}\,\Gamma$ into Homeo $\mathbb{Q}$, not that of $\mathrm{AAut}\,\Gamma$. Can one describe all the possible embeddings of $\mathrm{Aut}\,\Gamma$ into Homeo $\mathbb{Q}$?

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