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Prescribing endomorphism algebras of $\aleph_n$-free modules

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Abstract. It is a well-known fact that modules over a commutative ring in general cannot be classified, and it is also well-known that we have to impose severe restrictions on either the ring or on the class of modules to solve this problem. One of the restrictions on the modules comes from freeness assumptions which have been intensively studied in recent decades. Two interesting, distinct but typical examples are the papers by Blass [1] and Eklof [8], both jointly with Shelah. In the first case the authors consider almost-free abelian groups and assume the existence of large canonical, free subgroups. Nevertheless, there exist $\aleph_1$-separable torsion-free groups G of size $\aleph_1$ with a basic subgroup B of rank $\aleph_1$ such that all subgroups of G disjoint from B are also free, but the groups G are still not free. What else can we say about G? The other paper deals with Kaplansky's test problems (which are excellent indicators that the objects defy classification). The authors are able to construct very free abelian groups and verify the test problems for them by a careful choice of *particular* elements of their endomorphism rings.

Accordingly, we want to investigate and construct $\aleph_n$-free R-modules M (with n an arbitrary, but fixed natural number) over a domain R with $\mathrm{End}_R M = R$ for the first time more systematically and uniformly. Recall that M is $\aleph_n$-free if every subset of size $< \aleph_n$ is contained in a pure, free submodule of M. The requirement $\mathrm{End}_R M = R$ implies that M is indecomposable, hence complicated. (We will also allow that $\mathrm{End}_R M$ is a prescribed R-algebra, as in the title of this paper.)

By now it is folklore to construct such modules M using additional set-theoretic axioms, most notably Jensen's $\diamondsuit$-principle. In this case the freeness condition can even be strengthened (see [6] and many examples in [9]). However, if we insist on proving this result in ordinary ZFC, then the known arguments fail: The classical constructions from the fundamental paper by Corner [2] do not apply because they are based on pure submodules of p-adic completions of free A-modules, which are never even $\aleph_1$-free. If we use Shelah's Black Box instead of Jensen's $\diamondsuit$-principle, then the constructed modules M are still $\aleph_1$-free, but always fail to be even $\aleph_2$-free (see [4]). Thus we must develop new methods, which are presented for the first time in Sections 2 to 6, to achieve the desired result (Main Theorem 7.6). With these methods we provide a useful tool for a wide range of problems concerning $\aleph_n$-free structures which can then be attacked.

Keywords. Prediction principles, almost free abelian groups, endomorphism rings

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1. Introduction

A stimulating starting point for this investigation is Corner's fundamental realization theorem in [2], showing that any countable, reduced, torsion-free ring is the endomorphism ring of a countable, reduced torsion-free abelian group. Corner's theorem from 1963 has many applications in algebra. With regard to the next observation we would like to rephrase 'torsion-free' by $\aleph_0$-free, which (by Gauß's theorem about finitely generated abelian groups) is exactly the same requirement. Thus Corner's abelian groups are countable, reduced and $\aleph_0$-free. This first result was extended to larger cardinals in [6] and in a uniform way using Shelah's Black Box, which is designed for such constructions, in [4]. As a byproduct of the combinatorial arguments from the Black Box, it turns out that the abelian group (or more generally the R-module) is $\aleph_1$-free (of minimal size $2^{\aleph_0} = \beth_1$). Thus the problem of passing on to $\aleph_n$-free modules (of size $\beth_n$, i.e. taking n times the powerset of $\aleph_0$) with the same algebraic property, is in the air.

This question appeared in special cases even earlier; we will first describe some of its roots and indicate the difficulties in proving a parallel result.

In this section we will assume for simplicity that R is a countable principal ideal domain (a condition extended in Section 2.2). We will consider $\aleph_n$-free R-modules M with endomorphism algebra A. The oldest example is the Baer–Specker group (investigated in 1937) which is $\aleph_1$-free of cardinality $2^{\aleph_0}$ with $|A| = 2^{\aleph_0}$, hence definitely not free; see [10] for its properties and historical remarks. About 45 years later Griffith [18] and Hill [19] extended this result, showing for each natural number n the existence of $\aleph_n$-free abelian groups of cardinality $\aleph_n$ which are not free. Surprisingly, no further algebraic properties of these groups were shown. A first attempt to close this gap was Eda's paper [7] giving an example (using an idea from [22]) of an $\aleph_1$-free abelian group G of cardinality $\aleph_1$ with trivial dual, $G^* = \mathrm{Hom}(G, \mathbb{Z}) = 0$. Furthermore, inspired by work of Eklof and Mekler, it was shown, assuming Jensen's diamond principle, that any R-algebra A can be realized as the endomorphism algebra $\mathrm{End}_R\, M$ where M is an A-module with $|A| < \kappa = |M|$, κ is any infinite, regular, but not weakly compact cardinal and M is also κ-free (and more) (see [6] or also [9, 16]).

This stimulated the question of posing additional algebraic conditions on M. In [12, 13] we elaborate the 'case $\aleph_1$': *If R is a countable ring with free additive structure, then there exists an $\aleph_1$-free abelian group G of cardinality $\aleph_1$ with* $\mathrm{End}\, G = R$. There are also related results in [4], but restricted to $\aleph_1$. Moreover, a natural barrier appears: *the existence of indecomposable $\aleph_2$-free groups of cardinality $\aleph_2$ or the existence of such groups with endomorphism ring $\mathbb{Z}$ is undecidable*.

Despite this obstacle, Eklof and Shelah [8] found a way to realize certain subrings of a given ring A (which encodes Kaplansky's test problems) and were able to construct $\aleph_1$-separable abelian groups G of cardinality $\aleph_1$ which provide counter-examples to Kaplansky's test problems; see Section 9 and [16, pp. 603–606] for those rings.

Nevertheless, passing on under ZFC to $\aleph_n$-free abelian groups of size $\aleph_n$ with endomorphism ring $\mathbb{Z}$ and $n > 1$ is impossible, and we must relax our restrictions. We will replace the size of the $\aleph_n$-free module representing the endomorphism algebra by $\beth_n$ (or larger). Assuming GCH we have $\aleph_n = \beth_n$, and this illustrates that our assumptions about the size of the module are reasonable.

The reader may wonder why we restrict ourselves to $\aleph_n$-freeness for natural numbers n in this paper. As a test case for the present paper, in [14, 24] we first studied the existence of $\aleph_n$-free abelian groups with trivial dual, which basically clears the way for proceeding. But passing through $\aleph_\omega$, a new difficulty appears, which is Shelah's singular compactness theorem showing that λ-free modules of singular cardinality λ are free. Thus the inductive construction on n in this paper will break down at $\aleph_\omega$. Moreover, a theorem by Magidor and Shelah [21] presents another warning, that $\aleph_{\omega^2+1}$-free abelian groups G of size $|G| = \aleph_{\omega^2+1}$ are free in a suitable universe of set theory. Fortunately, this does not exclude the possibility of finding (in ordinary set theory) $\aleph_{\omega^2+1}$-free abelian groups G of cardinality $|G| > \aleph_{\omega^2+1}$. However, the tools must be much more refined and a construction of $\aleph_\omega$-free abelian groups G with trivial dual, as a natural test case, might need weak additional set-theoretic axioms. (Note that this is not in conflict with the singular compactness theorem, because $|G| > \aleph_\omega$.) This study is still in progress [25].

Finally, we want to discuss our main results (see Theorems 7.6 and 8.1). For simplicity we consider a special case. Let A be an R-algebra with free R-module structure A_R of cardinality $|A| \le \mu$, where R is a domain with a distinguished element $p \in R$ such that R is p-reduced ($\bigcap_{n<\omega} p^n R = 0$) and $\mathrm{Hom}(\widehat{R}, R) = 0$, where $\widehat{R}$ is the p-adic completion of R. (We then say that R is *p-cotorsion-free*; cf. [16].) In order to control the size of the constructed $\aleph_k$-free A-modules, we define inductively a (modified) $\beth$-sequence: put $\beth_0^+(\mu) = \mu$ and $\beth_{n+1}^+(\mu) = (2^{\beth_n^+(\mu)})^+$, which is the successor cardinal of the powerset of $\beth_n^+(\mu)$. We put $\beth_k^+(\aleph_0) = \beth_k^+$. Then we are ready to state our final result.

Main Theorem 1.1. *Let R be a p-cotorsion-free domain and A an R-algebra with free R-module A_R and $|A| \le \mu$ as above. If $\lambda = \beth_k^+(\mu)$ for some positive integer k, then we can construct an $\aleph_k$-free A-module G of cardinality λ with R-endomorphism algebra* $\mathrm{End}_R\, G = A$.

Thus clearly we get a proper class of $\aleph_k$-free A-modules G with $\mathrm{End}_R\, G = A$. The idea which leads to $\aleph_k$-freeness comes from the classical Black Box (prediction), where we get $\aleph_1$-freeness for the constructed modules automatically, due to a support argument on branches of the trees involved (see e.g. [4]). This support will be refined (in Section 2), and the old arguments must be modified by an elementary-closure condition (from model theory, hidden in [24]) which, in the Freeness Proposition 3.6 and the Freeness Lemma 3.7, will show that unions of suitable ascending chains of submodules of length $\aleph_k$ are free. The remaining steps of this paper are arguments to control endomorphisms by two prediction principles, the Easy Black Box (Proposition 6.1) and the (older) Strong Black Box 7.5. The repeated application of the Strong Black Box requires the cardinal sequence $\beth_k^+(\mu)$, which explains $|G| = \beth_k^+(\mu)$ in Theorem 1.1. And clearly we find $\aleph_k$-free indecomposable R-modules of any size $\beth_k^+(\mu)$ for $|R| \le \mu$. The problem of reducing the size of the modules to the ordinary $\beth$-sequence $\beth_k(\mu)$ (defined by $\beth_0 = \mu$ and $\beth_{n+1}(\mu) = 2^{\beth_n(\mu)}$) is left open. It seems plausible that this could follow by an improved prediction principle replacing the Strong Black Box 7.5.

It will follow immediately in the proof that the existence of G can be extended to the existence of a fully A-rigid family of such $\aleph_k$-free A-modules G, as explained in Theorem 8.1. The free choice of an algebra A allows us to prescribe Corner's list of finite groups (see Section 9) as exactly those finite groups which appear as automorphism groups of $\aleph_k$-free abelian groups. Moreover, realizing the appropriate R-algebras mentioned in Section 9, we also obtain in this situation counter-examples for Kaplansky's test problems, showing that decompositions of $\aleph_k$-free R-modules behave badly. In addition (again using the appropriate R-algebras), we find superdecomposable $\aleph_k$-free R-modules which have no indecomposable summands different from 0.

2. The basics for the new combinatorial Black Box

2.1. Set-theoretic preliminaries

The new Black Box depends on a finite sequence of cardinals satisfying some cardinal conditions. Thus we will fix a positive integer k and a sequence $\overline{\lambda} = \langle \lambda_1, \dots, \lambda_k \rangle$ of cardinals such that:

(i) $\lambda_\ell := \mu_\ell^+$ for $1 \leq \ell \leq k$.
(ii) $\mu_1 = \mu_1^{|A|}$.
(iii) $\mu_{\ell+1} = \mu_{\ell+1}^{\lambda_\ell}$ for $1 \leq \ell < k$.

This implies that $\lambda_1 = \lambda_1^{|A|}$ and $\lambda_{\ell+1} = \lambda_{\ell+1}^{\lambda_\ell}$; see the Hausdorff formula [20, p. 57, (5.22)].

If λ is a cardinal, then ${}^{\omega\uparrow}\lambda$ will denote all *order preserving* maps $\eta : \omega \to \lambda$, which we also call *infinite branches* on λ, while ${}^{\omega\uparrow>}\lambda$ denotes the family of all order preserving *finite branches* $\eta : n \to \lambda$ on λ, where the natural number n, λ and ω (the first infinite ordinal) are considered as sets, e.g. $n = \{0, \dots, n-1\}$, thus the finite branch η has length n.

Moreover, we associate with $\overline{\lambda}$ two sets Λ and Λ_*. First, let

$$\Lambda = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow}\lambda_k. \tag{2.1}$$

For the second set we replace the m-th (and only the m-th) coordinate ${}^{\omega\uparrow}\lambda_m$ by the finite branches ${}^{\omega\uparrow>}\lambda_m$, thus we let

$$\Lambda_m = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_k \text{ for } 1 \leq m \leq k \quad \text{and} \quad \Lambda_* = \dot{\bigcup_{1 \leq m \leq k}} \Lambda_m. \tag{2.2}$$

The elements of Λ, Λ_* will be written as sequences $\overline{\eta} = (\eta_1, \dots, \eta_k)$ with $\eta_\ell \in {}^{\omega\uparrow}\lambda$ or $\eta_\ell \in {}^{\omega\uparrow>}\lambda$ (for $1 \leq \ell \leq k$), respectively.

With each member of Λ we can associate a subset of Λ_*:

Definition 2.1. If $\overline{\eta} = (\eta_1, \ldots, \eta_k) \in \Lambda$ and $1 \leq m \leq k$, $n < \omega$, then let $\overline{\eta}\upharpoonleft\langle m, n\rangle$ be the following element in Λ_m (thus in Λ_*)

$$(\overline{\eta}\upharpoonleft\langle m, n\rangle)_\ell = \begin{cases} \eta_\ell & \text{if } 1 \leq \ell \neq m \leq k, \\ \eta_m \upharpoonright n & \text{if } \ell = m. \end{cases}$$

We associate with $\overline{\eta}$ its *support* $[\overline{\eta}] = \{\overline{\eta}\upharpoonleft\langle m, n\rangle \mid 1 \leq m \leq k, n < \omega\}$, which is a countable subset of Λ_*. For $N < \omega$ let $[\overline{\eta}]_N = \{\overline{\eta}\upharpoonleft\langle m, n\rangle \mid 1 \leq m \leq k, N \leq n < \omega\}$ be the N-*support of* $\overline{\eta}$. If $S \subseteq \Lambda$, then the support of S is the set $[S] = \bigcup_{\overline{\eta}\in S}[\overline{\eta}] \subseteq \Lambda_*$.

2.2. *Algebraic preliminaries for $\aleph_n$-free modules*

Let R be a commutative ring with $\mathbb{S}$ a countable multiplicatively closed subset containing 1 such that:

(i) The elements of $\mathbb{S}$ are not zero-divisors, i.e. if $s \in \mathbb{S}$, $r \in R$ and $sr = 0$, then $r = 0$.
(ii) $\bigcap_{s\in\mathbb{S}} sR = 0$.

We also say that R is an $\mathbb{S}$-*ring*. If (i) holds, then R is $\mathbb{S}$-*torsion-free*, and if (ii) holds, then R is $\mathbb{S}$-*reduced* (see [16]). To ease notations we use the letter $\mathbb{S}$ only if we want to emphasize that the argument depends on it. If M is an R-module, then these definitions naturally carry over to M. Finally, we enumerate $\mathbb{S} = \{s_n \mid n < \omega\}$ and put $q_n = \prod_{i<n} s_i$, thus $q_{n+1} = q_n s_n$.

If $G \subseteq M$, then G is $\mathbb{S}$-*pure* in M if $G \cap sM \subseteq sG$ for all $s \in \mathbb{S}$. If $G \subseteq M$ are torsion-free R-modules, then G_* denotes the smallest, unique $\mathbb{S}$-pure submodule of M containing G, and we write $G \subseteq_* M$ if G is $\mathbb{S}$-pure in M.

We also fix an R-algebra A and consider A-modules. Slightly strengthening [9] (by $\mathbb{S}$-purity) we call an A-module M κ-*free* if there is a family $\mathcal{C}$ of $\mathbb{S}$-pure A-submodules of M satisfying the following conditions:

(i) Every element of $\mathcal{C}$ is a $<\kappa$-generated free A-submodule of M.
(ii) Every subset of M of cardinality $<\kappa$ is contained in an element of $\mathcal{C}$.
(iii) $\mathcal{C}$ is closed under unions of well-ordered chains of length $<\kappa$.

We say that $\mathcal{C}$ is $<\kappa$-*closed*.

This definition applies for regular cardinals, in particular for $\kappa = \aleph_n$, which is the case we are interested in. Purity refers to $\mathbb{S}$-pure A-submodules of M as above.

The $\mathbb{S}$-*topology* of an $\mathbb{S}$-reduced R-module M is generated by the basis sM ($s \in \mathbb{S}$) of neighbourhoods of 0. It is Hausdorff on M and we consider the $\mathbb{S}$-completion $\widehat{M}$ of M; see [16] for elementary facts on the elements of $\widehat{M}$. The R-module M is *cotorsion-free* (with respect to $\mathbb{S}$) if M is $\mathbb{S}$-reduced and $\mathrm{Hom}_R(\widehat{R}, M) = 0$.

Given a cotorsion-free R-algebra A, we first define (similar to the Black Box in [4]), the basic, free A-module B, which is

$$B = \bigoplus_{\overline{\nu}\in\Lambda_*} Ae_{\overline{\nu}}.$$

Definition 2.2. If $U \subset \Lambda_*$, then we get a *canonical summand* $B_U = \bigoplus_{\overline{\nu} \in U} Ae_{\overline{\nu}}$ of B, and in particular, let $B_{\overline{\eta}} = B_{[\overline{\eta}]}$ for $\overline{\eta} \in \Lambda$ be the *canonical summand of* B.

Every element $b \in \widehat{B}$ has a natural (Λ_*-)support $[b] \subseteq \Lambda_*$, which are those $\overline{\nu} \in \Lambda_*$ contributing to the canonical sum representation $b = \sum_{\overline{\nu} \in \Lambda_*} b_{\overline{\nu}} e_{\overline{\nu}}$ with coefficients $0 \neq b_{\overline{\nu}} \in \widehat{A}$. Thus let $[b] = \{\overline{\nu} \in \Lambda_* \mid b_{\overline{\nu}} \neq 0\}$. Note that $[b]$ is at most countable. If $S \subseteq \widehat{B}$, then the Λ_*-*support* of S is the set $[S] = \bigcup_{b \in S} [b]$. As in the earlier Black Boxes (see [16]), we use conditions on the support (given by the prediction) to select particular elements from $\widehat{B}$ added to B to get the final structure M such that

$$B \subseteq M \subseteq_* \widehat{B}.$$

We will use B, Λ_*, Λ to define the Strong Black Box for $\aleph_n$-free A-modules in Section 7.

3. $\aleph_n$-free A-modules

Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring, A a cotorsion-free R-algebra, and let $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}$ be the A-module freely generated by $\{e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*\}$ with $\Lambda_* = \bigcup_{\overline{\eta} \in \Lambda} [\overline{\eta}]$ and $[\overline{\eta}] = \{\overline{\eta} \restriction \langle m, n\rangle \mid 1 \leq m \leq k,\ n < \omega\}$.

Next we choose particular elements from $\widehat{B}$. If $\overline{\eta} \in \Lambda$ and $i < \omega$, then we call

$$y_{\overline{\eta} i} = \sum_{n=i}^{\infty} \frac{q_n}{q_i} \Big(\sum_{m=1}^{k} e_{\overline{\eta} \restriction \langle m,n\rangle} \Big)$$

a *branch element associated with* $\overline{\eta}$. In particular let

$$y_{\overline{\eta}} = y_{\overline{\eta} 0} = \sum_{n=0}^{\infty} q_n \Big(\sum_{m=1}^{k} e_{\overline{\eta} \restriction \langle m,n\rangle} \Big).$$

Given $\overline{\eta} \in \Lambda$, we also choose $b_{\overline{\eta}} \in B$, $\pi_{\overline{\eta}} = \sum_{n=0}^{\infty} q_n r_n \in \widehat{R}$ and let $\pi_{\overline{\eta} i} = \sum_{n=i}^{\infty} (q_n/q_i) r_n$. Then we define *branch-like elements* by

$$y'_{\overline{\eta} i} = \pi_{\overline{\eta} i} b_{\overline{\eta}} + y_{\overline{\eta} i}.$$

In particular we have $y'_{\overline{\eta}} = y'_{\overline{\eta} 0} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}}$.

Definition 3.1. Suppose $Y_* \subseteq \Lambda_*$.

(i) Y_* is *almost tree-closed* if there is a finite set $E_* \subseteq \Lambda_*$ such that for any $\overline{\eta} \in \Lambda$ with $1 \leq m \leq k$, $n_1 \leq n_2 < \omega$ and $\overline{\eta} \restriction \langle m, n_2\rangle \in Y_*$ we have $\overline{\eta} \restriction \langle m, n_1\rangle \in Y_* \cup E_*$.

(ii) In particular X_* ($\subseteq Y_*$) $\subseteq \Lambda_*$ is *tree-closed* (with respect to Y_*) if for any $\overline{\eta} \in \Lambda$ with $1 \leq m \leq k$, $n_1 \leq n_2 < \omega$ and $\overline{\eta} \restriction \langle m, n_2\rangle \in X_*$ (and $\overline{\eta} \restriction \langle m, n_1\rangle \in Y_*$) we have $\overline{\eta} \restriction \langle m, n_1\rangle \in X_*$.

Thus Y_* is tree-closed if and only if Y_* is almost tree-closed with $E_* = \emptyset$.

Definition 3.2. A pair (Y_*, Y) is called Λ-*closed* (over N) if:

(i) $Y \subseteq \Lambda$ and $Y_* \subseteq \Lambda_*$.
(ii) There exists $N < \omega$ such that $[\overline{\eta}]_N \subseteq Y_*$ for all $\overline{\eta} \in Y$.
(iii) Y_* is almost tree-closed.

Definition 3.3 (The construction of the A-module G_{Y_*Y}). If (Y_*, Y) is Λ-closed (over N) and we have a family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B_{Y_*}, \overline{\eta} \in Y\}$ of branch-like elements $y'_{\overline{\eta}}$, then we let

$$G_{Y_*Y} = \langle B_{Y_*}, Ay'_{\overline{\eta}i} \mid \overline{\eta} \in Y, N \leq i < \omega\rangle = \langle B_{Y_*}, Ay'_{\overline{\eta}N} \mid \overline{\eta} \in Y\rangle_* \subseteq \widehat{B}.$$

Observation 3.4. *If (Y_*, Y) is Λ-closed (over N), then it is Λ-closed (over N') for $N < N'$, and the A-modules G_{Y_*Y} defined by N or by N' (as in Definition 3.3), are the same.*

Proof. Trivial. □

In this paper we mainly consider $\aleph_n$-free A-modules (for $1 \leq n < \omega$). Thus the following observation is interesting for us. If the ring R is sufficiently special and the algebra A is a free R-module, then any $\aleph_1$-free A-module G is cotorsion-free. If we want to show cotorsion-freeness for more general rings R, then G must be more special. In particular, if $G = G_{Y_*Y}$ this will follow with a support argument from the classical Black Box (see [16, pp. 447–448]).

Observation 3.5. *Let A be a cotorsion-free R-algebra.*

(a) *If the $\mathbb{S}$-ring R is a countable principal ideal domain, and G is $\aleph_1$-free, then G is cotorsion-free.*
(b) *If R is an $\mathbb{S}$-ring and G is the R-module G_{Y_*Y} as in Definition 3.3, then G is cotorsion-free.*

The final R-modules in Theorems 7.6 and 8.1 are of the form described in Observation 3.5, thus cotorsion-free.

Proof. (a) In this case we can apply [16, p. 426, Proposition 12.3.2], replacing the $R\setminus\{0\}$-topology by any $\mathbb{S}$-topology. Thus G is $\mathbb{S}$-cotorsion-free if and only if the quotient field $Q(R)$ the modules R/pR and $\widehat{R}_p$ for primes p with $pR \cap \mathbb{S} \neq \emptyset$ do not embed into G. Assuming that G is $\aleph_1$-free as an R-module, since $|R/pR|, |Q(R)| < \aleph_1$ it remains to show that $\widehat{R}_p$ does not embed into G. We can choose $\pi \in \widehat{R}$ which is transcendental over R (see [16, p. 16, Theorem 1.1.20] or [11]) and consider the R-submodule $\langle 1R, \pi R\rangle_* \subseteq \widehat{R}$, which has rank 2 and is indecomposable by Baer's theorem (see [10, Vol. 2, p. 123, Theorem 88.1]). If $\widehat{R}$ embeds into G, then so does $\langle 1R, \pi R\rangle_*$. Since G is $\aleph_1$-free, the countable R-module $\langle 1R, \pi R\rangle_*$ would be a free R-module of rank 2, which is a contradiction.

(b) Suppose $\varphi : \widehat{R} \to G$ is a nontrivial R-homomorphism. Then $1\varphi \neq 0$ because G is reduced and φ is continuous. Choose

$$n < \omega \quad \text{with} \quad q_n(1\varphi) = b + \sum_{\overline{\eta} \in I} a_{\overline{\eta}} y'_{\overline{\eta}N} \tag{3.1}$$

such that $b \in B$ and $0 \neq a_{\overline{\eta}} \in A$ for all $\overline{\eta} \in I \subseteq Y$. Moreover, I is finite. For $\pi \in \widehat{R}$ with $\pi\varphi \in G$ we also have

$$n \leq n' < \omega \quad \text{with} \quad q_{n'}(\pi\varphi) = b' + \sum_{\overline{\eta} \in I'} a'_{\overline{\eta}} y'_{\overline{\eta}N} \tag{3.2}$$

such that $b' \in B$ and $0 \neq a'_{\overline{\eta}} \in A$ for all $\overline{\eta} \in I'$. Moreover, I' is finite.

Comparing (3.1) and (3.2) we get

$$q_{n'}(\pi\varphi) = \frac{q_{n'}}{q_n}\pi\Big[b + \sum_{\overline{\eta}\in I} a_{\overline{\eta}} y'_{\overline{\eta}N}\Big] = b' + \sum_{\overline{\eta}\in I'} a'_{\overline{\eta}} y'_{\overline{\eta}N}.$$

If $\overline{\eta} \neq \overline{\eta}' \in \Lambda$, then $[\overline{\eta}]_N \cap [\overline{\eta}']_N$ is finite, and equating coefficients gives $I = I'$, $(q_{n'}/q_n)\pi a_{\overline{\eta}} = a'_{\overline{\eta}}$ for all $\overline{\eta} \in I$ and therefore also $(q_{n'}/q_n)\pi b = b'$.

Using the $\mathbb{S}$-purity of $A \subseteq_* \widehat{A}$ it is immediate that $(q_{n'}/q_n)\widehat{A} \cap A = (q_{n'}/q_n)A$, hence $\pi a_{\overline{\eta}} \in A$ $(\overline{\eta} \in I)$ and $\pi b \in B$.

If $I \neq \emptyset$, then we can choose a homomorphism $\varphi_1 : \widehat{R} \to A$ $(\pi \mapsto \pi a_{\overline{\eta}})$ which is not the zero homomorphism, a contradiction (because A is cotorsion-free).

If $I = \emptyset$, then $b \neq 0$, $b = \sum_{\overline{\nu}\in J} a_{\overline{\nu}} e_{\overline{\nu}}$ $(J \neq \emptyset)$ and $a_{\overline{\nu}} \neq 0$ $(\overline{\nu} \in J \subseteq Y_*)$. Choose any $\overline{\nu} \in J$. Similarly we get a homomorphism $\varphi_2 : \widehat{R} \to A$ $(\pi \mapsto \pi a_{\overline{\nu}})$ which is not the zero homomorphism, which is a final contradiction showing that G is cotorsion-free. □

If X is any set, then $\mathfrak{P}_{\text{fin}}(X)$ denotes the collection of all finite subsets of X.

Freeness Proposition 3.6. *Let $F : \Lambda \to \mathfrak{P}_{\text{fin}}(\Lambda_*)$ be any function, $1 \leq f \leq k$ and Ω a subset of Λ of cardinality $\aleph_{f-1}$ with a family of sets $u_{\overline{\eta}} \subseteq \{1, \ldots, k\}$ satisfying $|u_{\overline{\eta}}| \geq f$ for all $\overline{\eta} \in \Omega$. Then we can find an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_{f-1}\rangle$ of Ω, $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha < \omega$ $(\alpha < \aleph_{f-1})$ such that*

$$\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F \quad \textit{for all } n \geq n_\alpha,$$

where $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta \leq \alpha\}$.

Proof. The proof is by induction on f. We begin with $f = 1$, hence $|\Omega| = \aleph_0$. Let $\Omega = \{\overline{\eta}^\alpha \mid \alpha < \omega\}$ be an enumeration without repetitions. From $1 = f \leq |\overline{u}_{\overline{\eta}}|$ it follows that $\overline{u}_{\overline{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ for all $\alpha < \omega$. If $\alpha \neq \beta < \omega$, then $\overline{\eta}^\alpha \neq \overline{\eta}^\beta$ and there is $n_{\alpha,\beta} \in \omega$ so that $\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \neq \overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle$ for all $n \geq n_{\alpha\beta}$. Since $\bigcup \Omega_\alpha F$ is finite, we may enlarge $n_{\alpha,\beta}$, if necessary, so that $\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \bigcup \Omega_\alpha F$ for all $n \geq n_{\alpha,\beta}$. If $n_\alpha = \max_{\beta<\alpha} n_{\alpha,\beta}$, then $\overline{\eta}^\alpha \restriction \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \restriction \langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence the case $f = 1$ is settled and we let $f' = f + 1$ and assume that the proposition holds for f.

Let $|\Omega| = \aleph_f$ and choose an $\aleph_f$-filtration $\Omega = \bigcup_{\delta<\aleph_f} \Omega_\delta$ with $\Omega_0 = \emptyset$ and $|\Omega_1| = \aleph_{f-1}$. The next crucial idea comes from [24] based on the construction of elementary submodels: We can also assume that the chain $\{\Omega_\delta \mid \delta < \aleph_f\}$ is closed, meaning that for any $\delta < \aleph_f$, $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$ and $\overline{\eta} \in \Omega$ with

$$\{\eta_m \mid 1 \leq m \leq k\} \subseteq \{\nu_m, \nu'_m, \nu''_m \mid \overline{\nu}'' \in \overline{\nu}F \cup \overline{\nu}'F,\ 1 \leq m \leq k\}$$

we have $\overline{\eta} \in \Omega_\delta$. Thus, if $\overline{\eta} \in \Omega_{\delta+1} \setminus \Omega_\delta$, then the set

$$u^*_{\overline{\eta}} = \{1 \leq \ell \leq k \mid \exists n < \omega, \overline{\mu} \in \Omega_\delta \text{ such that } \overline{\eta}\upharpoonright\langle \ell, n\rangle = \overline{\mu}\upharpoonright\langle \ell, n\rangle \text{ or } \overline{\eta}\upharpoonright\langle \ell, n\rangle \in \overline{\mu}F\}$$

is empty or a singleton. Otherwise there are $n, n' < \omega$ and distinct $1 \leq \ell, \ell' \leq k$ with $\overline{\eta}\upharpoonright\langle \ell, n\rangle \in \{\overline{\nu}\upharpoonright\langle \ell, n\rangle\} \cup \overline{\nu}F$ and $\overline{\eta}\upharpoonright\langle \ell', n'\rangle \in \{\overline{\nu}'\upharpoonright\langle \ell', n'\rangle\} \cup \overline{\nu}'F$ for certain $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$. Hence $\{\eta_m \mid 1 \leq m \leq k\} \subseteq \{\nu_m, \nu'_m, \nu''_m \mid \nu''_m \in \overline{\nu}F \cup \overline{\nu}'F,\ 1 \leq m \leq k\}$, and the closure property implies the contradiction $\overline{\eta} \in \Omega_\delta$.

If $\delta < \aleph_f$, then let $D_\delta = \Omega_{\delta+1} \setminus \Omega_\delta$, and $u'_{\overline{\eta}} := u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ must have size $\geq f' - 1 = f$. Thus the induction hypothesis applies to $f, \{u'_{\overline{\eta}} \mid \overline{\eta} \in D_\delta\}$ for each $\delta < \aleph_f$ and we find an enumeration $\overline{\eta}^{\delta\alpha}$ $(\alpha < \aleph_{f-1})$ of D_δ as in the proposition. Finally, we put these chains for each $\delta < \aleph_f$ together with the induced ordering to get an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_f\rangle$ of Ω satisfying the proposition. ◻

Freeness Lemma 3.7. *The module G_{Y_*Y} from Definition 3.3 is $\aleph_k$-free as an A-module.*

Proof. Besides the Λ_*-support $[g]$ any element g of the module $G_{Y_*Y} = \langle B_{Y_*}, Ay'_{\overline{\eta}N} \mid \overline{\eta} \in Y\rangle_*$ has a refined natural finite support $[g]_{Y_*Y}$ arriving from Definition 3.3. It consists of all those elements of Y_* and Y, respectively, contributing to g. We observe that g is generated by elements $y'_{\overline{\eta}N}$ and $e_{\overline{\nu}}$, and simply collect the branches $\overline{\eta} \in Y$ and $\overline{\nu} \in Y_*$ needed. Clearly $[g]_{Y_*Y}$ is a finite subset of $Y_* \cup Y$.

Hence any subset H of G_{Y_*Y} has a natural support $[H]_{Y_*Y}$ taking the union of the supports of its elements, and if $|H| \geq \aleph_0$, then there are subsets $\Omega_* \subseteq Y_*$ and $\Omega \subseteq Y$ of size $|\Omega_*|, |\Omega| \leq |H|$ such that H is a subset of the pure A-submodule

$$G_{\Omega_*\Omega} = \langle Ae_{\overline{\nu}}, Ay'_{\overline{\eta}N} \mid \overline{\nu} \in \Omega_*,\ \overline{\eta} \in \Omega\rangle_* \subseteq G_{Y_*Y}.$$

Without loss of generality we may assume $\Omega_* = \bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}]_N \cup \bigcup_{\overline{\eta}\in\Omega}[b_{\overline{\eta}}]$ and write

$$\begin{aligned} G_{\Omega_*\Omega} = G_\Omega = \langle Ae_{\overline{\eta}\upharpoonright\langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}N} \mid \overline{\eta} \in \Omega,\ \overline{\nu} \in [b_{\overline{\eta}}],\ 1 \leq m \leq k,\ N \leq n < \omega\rangle_* \\ \subseteq G_{Y_*Y} \end{aligned}$$

as $Ae_{\overline{\nu}}$ is a direct summand of $G_{\Omega_*\Omega}$ for all $\overline{\nu} \in \Omega_* \setminus (\bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}]_N \cup \bigcup_{\overline{\eta}\in\Omega}[b_{\overline{\eta}}])$.

Thus, in order to show $\aleph_k$-freeness of G_{Y_*Y}, we will consider any $\Omega \subseteq Y$ of size $|\Omega| < \aleph_k$ and show the freeness of the module G_Ω. We may assume that $|\Omega| = \aleph_{k-1}$. Let $F : \Lambda \to \mathfrak{P}_{\text{fin}}(\Lambda_*)$ be any map which assigns to $\overline{\eta} \in Y$ the set $\overline{\eta}F = [b_{\overline{\eta}}]$.

By Proposition 3.6 (putting simply $u_{\overline{\eta}} = \{1, \ldots, k\}$ for all $\overline{\eta} \in \Omega$) we can express

$$G_\Omega = \langle e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha n} \mid \alpha < \aleph_{k-1},\ \overline{\nu} \in \overline{\eta}^\alpha F,\ 1 \leq m \leq k,\ N \leq n < \omega\rangle_A,$$

where $\langle \ldots\rangle_A$ denotes the A-module generated by $\langle \ldots\rangle$, and we find a sequence of pairs (ℓ_α, n_α) with $1 \leq \ell_\alpha \leq k$ and $N \leq n_\alpha < \omega$ such that for $n \geq n_\alpha$,

$$\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta\upharpoonright\langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F. \tag{3.3}$$

Let $G_\alpha = \langle e_{\overline{\eta}^\gamma \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\gamma n} \mid \gamma < \alpha, \overline{\nu} \in \overline{\eta}^\gamma F, 1 \le m \le k, N \le n < \omega\rangle_A$ for any $\alpha \le \aleph_{k-1}$; thus $G_{\aleph_{k-1}} = G_\Omega$, and if $\alpha < \aleph_{k-1}$, then

$$\begin{aligned} G_{\alpha+1} &= G_\alpha + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha n} \mid \overline{\nu} \in \overline{\eta}^\alpha F,\ 1 \le m \le k,\ N \le n < \omega\rangle_A \\ &= G_\alpha + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} \mid N \le n < n_\alpha\rangle_A + \langle y'_{\overline{\eta}^\alpha n} \mid n \ge n_\alpha\rangle_A \\ &\quad + \langle e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}} \mid \overline{\nu} \in \overline{\eta}^\alpha F,\ 1 \le \ell_\alpha \ne m \le k,\ N \le n < \omega\rangle_A. \end{aligned}$$

Hence any element in $G_{\alpha+1}$ can be represented as a sum of the form

$$g + \sum_{N \le n < n_\alpha} a_n e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} + \sum_{n \ge n_\alpha} a_n y'_{\overline{\eta}^\alpha n} + \sum_{N \le n < \omega} \sum_{\ell_\alpha \ne m \le k} a_{mn} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle} + \sum_{\overline{\nu} \in \overline{\eta}^\alpha F} a_{\overline{\nu}} e_{\overline{\nu}},$$

where $g \in G_\alpha$ and all coefficients $a_n, a_{mn}, a_{\overline{\nu}}$ are from A.

Moreover, the summands involving the $e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$s have disjoint supports. Now condition (3.3) applies recursively. Hence, assuming the above sum is zero, by disjointness of supports (identifying $e_{\overline{\nu}}$ ($\overline{\nu} \in \overline{\eta}^\alpha F$) with one of the $e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$s if possible and merging all $e_{\overline{\eta}^\alpha} \upharpoonright \langle m,n\rangle \in G_\alpha$ and $e_{\overline{\nu}} \in G_\alpha$ into g), it also follows that all the coefficients $a_n, a_{mn}, a_{\overline{\nu}}$ and consequently also g must be zero. This shows that $G_{\alpha+1} = G_\alpha \oplus \bigoplus_{b \in B_\alpha} Ab$ for

$$\begin{aligned} B_\alpha = \{ & e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,k\rangle}, y_{\overline{\eta}^\alpha \ell}, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}} \mid \\ & N \le k < n_\alpha,\ \ell \ge n_\alpha,\ 1 \le \ell_\alpha \ne m \le k,\ N \le n < \omega,\ \overline{\nu} \in \overline{\eta}^\alpha F\} \setminus G_\alpha. \end{aligned}$$

Thus $G_\Omega = \bigoplus_{\alpha < \aleph_{k-1}} \bigoplus_{b \in B_\alpha} Ab$ is a free A-module. The $\aleph_k$-freeness of $G_{Y_* Y}$ is now immediate from the existence of the $<\aleph_k$-closed family $\mathfrak{F} = \{G_{\Omega_* \Omega} \mid |\Omega_*|, |\Omega| < \aleph_k\}$ of free, pure submodules of $G_{Y_* Y}$. $\square$

4. The triple-homomorphism ρ and freeness

Definition 4.1. (a) For each triple (Y_*, Y, X_*) with $X_*, Y_* \subseteq \Lambda_*$ and $\overline{\eta} \in Y \subseteq \Lambda$ let $u_{\overline{\eta}}(X_*) = \{1 \le m \le k \mid \exists n_0 < \omega$ such that $\overline{\eta} \upharpoonright \langle m,n\rangle \notin X_*$ for all $n \ge n_0\}$. If X_* is clear from the context, then we will write $u_{\overline{\eta}}$ for $u_{\overline{\eta}}(X_*)$. We put $Y_{X_*} = \{\overline{\eta} \in Y \mid [\overline{\eta}]_n \subseteq X_*$ for some $n < \omega\}$.

(b) Let $1 \le f \le k$. Then a triple (Y_*, Y, X_*) is called *f-closed* if:

- (i) (Y_*, Y) is Λ-closed.
- (ii) $X_* \subseteq Y_*$.
- (iii) X_* is almost tree-closed.
- (iv) If $\overline{\eta} \in Y$, then either $|u_{\overline{\eta}}| \ge f$ or $[\overline{\eta}]_n \subseteq X_*$ for some $n < \omega$.

Observation 4.2. (a) For every f-closed triple (Y_*, Y, X_*), $\overline{\eta} \in Y$ and $1 \le m \le k$ there is $n_0 < \omega$ such that either $\overline{\eta} \upharpoonright \langle m,n\rangle \in X_*$ for all $n \ge n_0$ or $\overline{\eta} \upharpoonright \langle m,n\rangle \notin X_*$ for all $n \ge n_0$, because X_* is almost tree-closed.

(b) For k-closed triples (Y_*, Y, X_*) Definition 4.1(b)(iv) is equivalent to the following condition: If $\overline{\eta} \in Y$ and there is $1 \le m \le k$ such that $\overline{\eta} \upharpoonright \langle m,n\rangle \in X_*$ for arbitrarily large $n < \omega$, then $[\overline{\eta}]_{n'} \subseteq X_*$ for some $n' < \omega$.

(c) Since X_* is almost tree-closed, we have

$$[\overline{\eta}]_n \subseteq X_* \text{ for some } n < \omega \ \Leftrightarrow\ [\overline{\eta}]_N \subseteq X_*,$$

where $N = \max\{n \mid \exists 1 \leq m \leq k \text{ and } \overline{\eta} \in \Lambda \text{ with } \overline{\eta}\restriction\langle m, n\rangle \in E_*\} + 1$ (see Definition 3.1(i) for E_*).

Next we define the natural projection ρ.

Definition 4.3. Let (Y_*, Y, X_*) be a triple with $X_* \subseteq Y_* \subseteq \Lambda_*$, $Y \subseteq \Lambda$ and let

$$\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y \text{ and } b_{\overline{\eta}} \in B_{Y_*}\}$$

be a family of branch-like elements from $\widehat{B}$.

(a) We say that the family $\mathfrak{F}$ is (Y_*, Y, X_*)-*suitable* (or just *suitable*) if $[b_{\overline{\eta}}] \subseteq X_*$ for each $\overline{\eta} \in Y_{X_*}$.
(b) Let the homomorphism $\rho = \rho_{Y_* Y X_*} : G_{Y_* Y} \to \widehat{B}$ be defined in two steps. Put

$$e_{\overline{\nu}}\rho = \begin{cases} 0 & \text{if } \overline{\nu} \in X_*, \\ e_{\overline{\nu}} & \text{if } \overline{\nu} \in Y_* \setminus X_*. \end{cases}$$

and extend ρ by linearity and continuity with domain $G_{Y_* Y}$. This homomorphism $\rho = \rho_{Y_* Y X_*}$ will be called a *triple homomorphism.*
(c) Let $G_{Y_* Y X_*} = G_{Y_* Y}\rho_{Y_* Y X_*}$ be the *triple module* for (Y_*, Y, X_*).

Notation 4.4. If $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y \text{ and } b_{\overline{\eta}} \in B_{Y_*}\}$ is a family of branch-like elements from $\widehat{B}$ and $X \subseteq Y$, then we will write $\mathfrak{F}_X = \{y'_{\overline{\eta}} \mid \overline{\eta} \in X\}$.

Triple modules satisfy important freeness conditions.

Theorem 4.5. *Let (Y_*, Y, X_*) be an f-closed triple for some $1 \leq f \leq k$ and suppose that $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y\}$ is a suitable family of branch-like elements. Then:*

(a) *If $X = Y_{X_*}$, then (X_*, X) is Λ-closed.*
(b) *The subfamily $\mathfrak{F}_X$ of $\mathfrak{F}$ of branch-like elements generates a well-defined A-module $G_{X_* X}$ (as given in Definition 3.3).*
(c) *$G_{X_* X} \subseteq G_{Y_* Y}$ canonically.*
(d) *$G_{X_* X}$ and $G_{Y_* Y}$ are $\aleph_k$-free.*
(e) *$G_{Y_* Y X_*} \cong G_{Y_* Y}/G_{X_* X}$ is $\aleph_f$-free.*

Proof. (a) We must verify Definition 3.2. Since (i) and (iii) are obvious, we only consider (ii): If $\overline{\eta} \in X$, then Observation 4.2(c) yields $[\overline{\eta}]_N \subseteq X_*$ for some fixed $N < \omega$, and (X_*, X) is Λ-closed (over N).

(b) If $\overline{\eta} \in X$, then $[b_{\overline{\eta}}] \subseteq X_*$ as $\mathfrak{F}$ is suitable. Moreover $b_{\overline{\eta}} \in B_{Y_*}$ because $\overline{\eta} \in Y$. Thus $b_{\overline{\eta}} \in B_{X_*}$.

(c) From (a) we know that (X_*, X) is Λ-closed (over N), while (Y_*, Y) is Λ-closed (over N') and we may assume that $N \geq N'$. Hence (c) is obvious, because $G_{X_* X}$ and $G_{Y_* Y}$ are canonical A-submodules of $\widehat{B}$ with $X \subseteq Y$, and $X_* \subseteq Y_*$.

(d) is immediate from Lemma 3.7.

(e) Next we claim that $\ker\rho = G_{X_*X}$.

If $\overline{\nu} \in X_*$, then $e_{\overline{\nu}}\rho = 0$, and if $\overline{\eta} \in X$, then $y_{\overline{\eta}N}\rho = 0$ by Definition 3.2 and continuity of ρ. Since $[b_{\overline{\eta}}] \subseteq X_*$, also $\pi_{\overline{\eta}}b_{\overline{\eta}}\rho = 0$ and thus $y'_{\overline{\eta}N}\rho = 0$. It follows that $G_{X_*X} \subseteq \ker\rho$.

For the converse inclusion we apply a support argument. If $x \in G_{Y_*Y}$ with $x\rho = 0$, then we must show that $x \in G_{X_*X}$. Replacing x by $q_n x$ with a suitable $q_n \in \mathbb{S}$ it is enough to show that $q_n x\rho = 0$. Thus we may assume that

$$x = \sum_{\overline{\nu}\in Y_*} a_{\overline{\nu}}e_{\overline{\nu}} + \sum_{\overline{\eta}\in Y} a_{\overline{\eta}}y'_{\overline{\eta}N}$$

and almost all coefficients $a_{\overline{\nu}}, a_{\overline{\eta}}$ from A are zero.

Case 1: If $a_{\overline{\eta}} \neq 0$ for some $\overline{\eta} \in Y \setminus X$, then $|u_{\overline{\eta}}| \geq f \geq 1$, and there are some $1 \leq m \leq k$ and $n_0 < \omega$ with $\overline{\eta}\restriction\langle m,n\rangle \notin X_*$ for all $n \geq n_0$. Recall that $[\overline{\eta}']_N \cap [\overline{\eta}]_N$ is finite for distinct branches $\overline{\eta}' \neq \overline{\eta}$, thus enlarging n_0 we may assume that $\overline{\eta}\restriction\langle m,n\rangle \notin X_*$ for all $n \geq n_0$ and moreover (by the choice of ρ) the $e_{\overline{\eta}\restriction\langle m,n\rangle}$-component of x is $a_{\overline{\eta}}e_{\overline{\eta}\restriction\langle m,n\rangle} \neq 0$ and remains invariant under ρ. So $x \notin \ker\rho$, a contradiction, and so $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in Y \setminus X$.

Case 2: If now $a_{\overline{\nu}} \neq 0$ for some $\overline{\nu} \in Y_* \setminus X_*$, then the $e_{\overline{\nu}}$-component of x is nonzero, and invariant under ρ, a contradiction.

Hence $x \in G_{X_*X}$ and the claim $\ker\rho = G_{X_*X}$ follows.

We have shown that $G_{Y_*YX_*} = \operatorname{Im}\rho \cong G_{Y_*Y}/G_{X_*X}$ and it remains to show that $\operatorname{Im}\rho$ is $\aleph_f$-free. We choose an arbitrary subset $H \subseteq G_{Y_*Y} \setminus G_{X_*X}$ of cardinality $\aleph_{f-1}$ and will show that $H\rho$ can be embedded into a free, pure submodule of $\operatorname{Im}\rho$.

As in the proof of Lemma 3.7 we can find $\Omega_* \subseteq Y_*$ and $\Omega \subseteq Y$ with $|\Omega_*|, |\Omega| \leq |H|$ such that

$$H \subseteq G_{\Omega_*\Omega} = \langle Ae_{\overline{\nu}}, Ay'_{\overline{\eta}N}, G_{X_*X} \mid \overline{\nu} \in \Omega_*, \overline{\eta} \in \Omega\rangle_* \subseteq G_{Y_*Y}.$$

Moreover, let $\Delta = \Omega_* \setminus (\bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}]_N \cup \bigcup_{\overline{\eta}\in\Omega}[b_{\overline{\eta}}] \cup X_*)$. Then B_Δ is a free direct summand of $G_{\Omega_*\Omega}$, $B_\Delta\rho$ is a free direct summand of $G_{\Omega_*\Omega}\rho$, and we may assume that $\Omega_* \subseteq \bigcup_{\overline{\eta}\in\Omega}[\overline{\eta}]_N \cup \bigcup_{\overline{\eta}\in\Omega}[b_{\overline{\eta}}] \cup X_*$. We get

$$\begin{aligned} G_{\Omega_*\Omega} \subseteq G_\Omega := \langle &Ae_{\overline{\eta}\restriction\langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}N}, G_{X_*X} \mid \\ &\overline{\eta} \in \Omega \setminus X,\ \overline{\nu} \in [b_{\overline{\eta}}],\ 1 \leq m \leq k,\ N \leq n < \omega\rangle_*, \end{aligned}$$

which is a pure submodule of G_{Y_*Y}.

Clearly $H\rho \subseteq_* G_\Omega\rho$ and $G_\Omega\rho \subseteq_* G_{Y_*Y}/G_{X_*X} = \operatorname{Im}\rho$ is pure by Prüfer (see [10, p. 115, Lemma 26.1(ii)]) because $\ker\rho = G_{X_*X} \subseteq G_\Omega$.

By Proposition 3.6 (applied to $\Omega \setminus X$ with $|u_{\overline{\eta}}| \geq f$ and $u_{\overline{\eta}}$ given by Definition 4.1) we can express

$$\begin{aligned} G_\Omega = \langle &e_{\overline{\eta}^\alpha\restriction\langle m,n\rangle}, e_{\overline{\nu}}, e_{\overline{\nu}'}, y'_{\overline{\eta}^\alpha n}, y'_{\overline{\eta}'n} \mid \\ &\alpha < \aleph_{f-1},\ 1 \leq m \leq k,\ N \leq n < \omega,\ \overline{\nu} \in \overline{\eta}^\alpha F,\ \overline{\nu}' \in X_*,\ \overline{\eta}' \in X\rangle_A \end{aligned}$$

and there are pairs (ℓ_α, n_α) with $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $N \leq n_\alpha < \omega$ such that

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\beta, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F \quad \text{for all } n \geq n_\alpha.$$

By Definition 4.1 and $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ we also get

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\beta, n\rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F \cup X_*.$$

As in Lemma 3.7 we choose an $|\Omega \setminus X|$-filtration of G_Ω, so let

$$G_\alpha = \langle e_{\overline{\eta}^\gamma \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, e_{\overline{\nu}'}, y'_{\overline{\eta}^\gamma n}, y'_{\overline{\eta}' n} \mid \gamma < \alpha,\ 1 \leq m \leq k,\ N \leq n < \omega,\ \overline{\nu} \in \overline{\eta}^\gamma F,\ \overline{\nu}' \in X_*,\ \overline{\eta}' \in X\rangle_A.$$

Thus it is immediate that $G_0 = G_{X_*X}, G_{|\Omega\setminus X|} = G_\Omega$ and the arguments of Lemma 3.7 show that $G_\Omega = G_{X_*X} \oplus F$ for some free A-module F. This and $\ker \rho = G_{X_*X}$ imply that $H\rho$ can be embedded into a pure, free A-submodule $G_\Omega \rho$, and (e) holds.

The $\aleph_f$-freeness of $G_{Y_*YX_*}$ is now immediate from the existence of the $<\aleph_f$-closed family $\mathcal{C} = \{(G_\Omega \oplus B_\Delta)\rho \mid |\Delta|, |\Omega| < \aleph_f\}$ of free, pure submodules of $G_{Y_*YX_*}$ (cf. Section 2.2). $\square$

Next we prove

Transitivity Lemma 4.6. (a) *Given two f-closed triples (Z_*, Z, Y_*) and (Y_*, Y, X_*) such that $Y = Z_{Y_*}$, the triple (Z_*, Z, X_*) is also f-closed.*

(b) *Given also a (Z_*, Z, Y_*)-suitable family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z,\ b_{\overline{\eta}} \in B_{Z_*}\}$ such that $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable, the following holds:*

(i) $Z_{X_*} = Y_{X_*}$.
(ii) *$\mathfrak{F}$ is (Z_*, Z, X_*)-suitable.*
(iii) *$G_{Y_*YX_*} \subseteq G_{Z_*ZX_*}$ with $G_{Z_*ZX_*}/G_{Y_*YX_*} \cong G_{Z_*ZY_*}$*

Observation 4.7. With Theorem 4.5 it follows from the Transitivity Lemma 4.6 that $G_{Y_*YX_*} \subseteq G_{Z_*ZX_*}$ and $G_{Z_*ZX_*}/G_{Y_*YX_*}$ is $\aleph_f$-free.

Proof of the Transitivity Lemma. (a) Note that (Z_*, Z) is Λ-closed, because (Z_*, Z, Y_*) is f-closed and X_* is almost tree-closed because (Y_*, Y, X_*) is f-closed. Now we continue to exploit the f-closedness of (Y_*, Y, X_*) (see Definition 4.1(b)). First we find that $X_* \subseteq Y_*$, hence if $\overline{\eta} \in Z$ with $|u_{\overline{\eta}}(Y_*)| \geq f$, then also $|u_{\overline{\eta}}(X_*)| \geq f$. Secondly, if $[\overline{\eta}]_n \subseteq Y_*$ for some $n < \omega$, then $\overline{\eta} \in Y$, and therefore either $|u_{\overline{\eta}}(X_*)| \geq f$ or $[\overline{\eta}]_{n'} \subseteq X_*$ for some $n' < \omega$.

(b) (i) From $Y \subseteq Z$ it follows that $Y_{X_*} \subseteq Z_{X_*}$. Conversely, if $\overline{\eta} \in Z$ and $[\overline{\eta}]_n \subseteq X_*$ for some $n < \omega$, then $[\overline{\eta}]_n \subseteq Y_*$ (from $X_* \subseteq Y_*$) and $\overline{\eta} \in Y$ by the definition of Y. Now it follows that $Z_{X_*} \subseteq Y_{X_*}$, and therefore $Z_{X_*} = Y_{X_*}$.

(ii) If $\overline{\eta} \in Z_{X_*}$, then (i) yields $\overline{\eta} \in Y_{X_*}$, and therefore $[b_{\overline{\eta}}] \subseteq X_*$, because $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable.

(iii) Clearly $\rho_{Y_*YX_*} \subseteq \rho_{Z_*ZX_*}$ and $G_{Y_*Y} \subseteq G_{Z_*Z}$ (see Theorem 4.5). Hence $G_{Y_*YX_*} = G_{Y_*Y}\rho_{Y_*YX_*} \subseteq G_{Z_*Z}\rho_{Z_*ZX_*} = G_{Z_*ZX_*}$.

Next we calculate with the help of (i) and Theorem 4.5(e):

$$\begin{aligned}
&\mathfrak{G}_{Z_*ZX_*}/G_{Y_*YX_*} = (G_{Z_*Z})\rho_{Z_*ZX_*}/(G_{Y_*Y})\rho_{Y_*YX_*}\\
&\quad\cong (G_{Z_*Z}/\ker(\rho_{Z_*ZX_*}))/(G_{Y_*Y}/\ker(\rho_{Y_*YX_*})) = (G_{Z_*Z}/G_{X_*Z_{X_*}})/(G_{Y_*Y}/G_{X_*Y_{X_*}})\\
&\quad= (G_{Z_*Z}/G_{X_*Z_{X_*}})/(G_{Y_*Y}/G_{X_*Z_{X_*}}) \cong G_{Z_*Z}/G_{Y_*Y} \cong G_{Z_*ZY_*}. \qquad \square
\end{aligned}$$

Definition 4.8. A triple (Y_*, Y, X_*) is $\mathfrak{F}$-*closed* for a family

$$\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y,\ b_{\overline{\eta}} \in B_{Y_*}\}$$

if:

(i) (Y_*, Y) is Λ-closed.
(ii) $X_* \subseteq Y_*$.
(iii) X_* is almost tree-closed.
(iv) If $\overline{\eta} \in Y$ and there is $1 \le m \le k$ such that $\overline{\eta}\restriction\langle m, n\rangle \in X_*$ for arbitrarily large $n < \omega$, then $[\overline{\eta}]_N \subseteq X_*$ for some $N < \omega$.
(v) If $\overline{\eta} \in Y$ and $[\overline{\eta}]_N \subseteq X_*$ for some $N < \omega$, then also $[b_{\overline{\eta}}] \subseteq X_*$.

It is clear from the definition and Observation 4.2 that (Y_*, Y, X_*) is $\mathfrak{F}$-closed for a family $\mathfrak{F}$ as above if and only if (Y_*, Y, X_*) is k-closed and $\mathfrak{F}$ is (Y_*, Y, X_*)-suitable.

Observation 4.9. *Let* $\mathfrak{F}=\{y'_{\overline{\eta}}=\pi_{\overline{\eta}}b_{\overline{\eta}}+y_{\overline{\eta}} \mid \overline{\eta}\in Y,\ b_{\overline{\eta}}\in B_{Y_*}\}$ *and* (Y_*, Y, X_*^1), (Y_*, Y, X_*^2) *be* $\mathfrak{F}$*-closed. Then:*

(a) $(Y_*, Y, X_*^1 \cup X_*^2)$ *is* $\mathfrak{F}$*-closed.*
(b) $(Y_*, Y, X_*^1 \cap X_*^2)$ *is* $\mathfrak{F}$*-closed.*

Proof. Trivial. $\square$

Theorem 4.10. (a) *If* (Y_*, Y, X_*) *is* $\mathfrak{F}$*-closed, then* (Y_*, Y, X_*) *is* k*-closed.*
(b) *If* (Y_*, Y) *is* Λ*-closed and* $\Omega_* \subseteq Y_*$*, then there exists* $X_* \subseteq Y_*$ *such that* (Y_*, Y, X_*) *is* k*-closed,* $\Omega_* \subseteq X_*$ *and* $|X_*| \le |\Omega_*|^{\aleph_0}$*. Moreover, there is a unique, minimal tree-closed* $X_* = \overline{\Omega_*} = \overline{\Omega_*}(Y_*, Y)$ *with respect to* Y_* *such that for all* $\overline{\eta} \in Y$ *with* $[\overline{\eta}]_n \subseteq X_*$ *for some* $n < \omega$ *we have* $[\overline{\eta}] \cap Y_* \subseteq X_*$*.*
(c) *If* (Y_*, Y) *is* Λ*-closed,* $\Omega_* \subseteq Y_*$ *and* $\mathfrak{F} = \{y'_{\overline{\eta}} \mid \overline{\eta} \in Y,\ b_{\overline{\eta}} \in B_{Y_*}\}$*, then there is* $X_* \subseteq Y_*$ *with* (Y_*, Y, X_*) $\mathfrak{F}$*-closed,* $\Omega_* \subseteq X_*$ *and* $|X_*| \le |\Omega_*|^{\aleph_0}$*. There is a unique, minimal tree-closed* $X_* = \overline{\Omega_*} = \overline{\Omega_*}(Y_*, Y, \mathfrak{F})$ *with respect to* Y_* *such that for all* $\overline{\eta} \in Y$ *with* $[\overline{\eta}]_n \subseteq X_*$ *for some* $n < \omega$ *we have* $[\overline{\eta}] \cap Y_* \subseteq X_*$*.*

Proof. (a) We must show that $\overline{\eta} \in Y$ with $[\overline{\eta}]_n \not\subseteq X_*$ for any $n < \omega$ implies $|u_{\overline{\eta}}| = k$.

If $|u_{\overline{\eta}}| < k$, then we can choose $1 \le m \le k$ with $m \notin u_{\overline{\eta}}$. Thus (by definition of $u_{\overline{\eta}}$) it follows that $\overline{\eta}\restriction\langle m, n\rangle \in X_*$ for arbitrarily large $n < \omega$. And from Definition 4.8(iv) it also follows that $[\overline{\eta}]_{n'} \subseteq X_*$ for some $n' < \omega$, which is a contradiction.

(b) $\overline{\Omega_*}$ is uniquely determined by the closure of Ω_* under Definition 3.1(ii) and Definition 4.8(iv).

(c) follows similarly to (b) using Definition 4.8(v). $\square$

Notation 4.11. Let (Y_*, Y) be Λ-closed and $\Omega_* \subseteq Y_*$ as in Theorem 4.10. If $\Omega_* \subseteq \overline{\Omega_*} \subseteq Y_*$, then we call $\overline{\Omega_*}$ $\mathfrak{F}$-*closed* in Y_* if:

- $\overline{\Omega_*} \subseteq Y_*$ is tree-closed with respect to Y_*, i.e. for any $\overline{\eta} \in \Lambda$ with $1 \le m \le k$, $n_1 \le n_2 < \omega$ and $\overline{\eta}\restriction\langle m, n_2\rangle \in \overline{\Omega_*}$ and $\overline{\eta}\restriction\langle m, n_1\rangle \in Y_*$ we have $\overline{\eta}\restriction\langle m, n_1\rangle \in \overline{\Omega_*}$.
- If $\overline{\eta} \in Y$ and there is $1 \le m \le k$ such that $\overline{\eta}\restriction\langle m, n\rangle \in \overline{\Omega_*}$ for arbitrarily large $n < \omega$, then $[\overline{\eta}] \cap Y_* \subseteq \overline{\Omega_*}$.
- If $\overline{\eta} \in Y$ and $[\overline{\eta}]_n \subseteq \overline{\Omega_*}$ for some $n < \omega$, then also $[b_{\overline{\eta}}] \subseteq \overline{\Omega_*}$.

Moreover, we call $\overline{\Omega_*} = \overline{\Omega_*}(Y_*, Y, \mathfrak{F})$ the $\mathfrak{F}$-*closure* of Ω_* in Y_* if $\overline{\Omega_*}$ is $\mathfrak{F}$-closed in Y_*, and $\overline{\Omega_*}$ is minimal with $\Omega_* \subseteq \overline{\Omega_*} \subseteq Y_*$.

Remark 4.12. Given a Λ-closed pair (Y_*, Y), $\Omega_* \subseteq Y_*$, and a family $\mathfrak{F} = \{y'_{\overline{\eta}} \mid \overline{\eta} \in Y,\ b_{\overline{\eta}} \in B_{Y_*}\}$, by Theorem 4.10(c) there is a triple (Y_*, Y, X_*) for Ω_* such that X_* is the $\mathfrak{F}$-closure of Ω_* in Y_*. The set $X = Y_{X_*}$ has the following properties (due to Theorems 4.5 and 4.10(a)):

$$B_{\Omega_*} \subseteq G_{X_* X} \subseteq_* G_{Y_* Y}, \qquad G_{Y_* Y}/G_{X_* X} \text{ are } \aleph_k\text{-free with } |X_*| \le |\Omega_*|^{\aleph_0}.$$

Note that $G_{X_* X}$ is inspired by the concept of elementary submodels.

Observation 4.13. *For any Λ-closed (Y_*, Y) and $\Omega_*^1, \Omega_*^2 \subseteq Y_*$ and $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y,\ b_{\overline{\eta}} \in B_{Y_*}\}$ the following holds:*

(i) $\overline{\Omega_*^1 \cup \Omega_*^2} = \overline{\Omega_*^1} \cup \overline{\Omega_*^2}$.
(ii) $\overline{\Omega_*^1 \cap \Omega_*^2} \subseteq \overline{\Omega_*^1} \cap \overline{\Omega_*^2}$.

A similar statement holds for the k-closures of subsets of Y_*.

Proof. Trivial. □

Lemma 4.14. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be f-closed triples with $Y = Z_{Y_*}$, $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z,\ b_{\overline{\eta}} \in B_{Z_*}\}$ being (Z_*, Z, Y_*)-suitable, $\mathfrak{F}_Y$ being (Y_*, Y, X_*)-suitable and $U \subseteq G_{Z_* Z X_*}$. Then there exist $\Omega_* \subseteq Z_*$ and $\Omega \subseteq Z$ such that:*

(a) $|\Omega_*|, |\Omega| \le |U| \cdot \aleph_0$.
(b) $(Y_* \cup \Omega_*, Y \cup \Omega, Y_*)$ *is f-closed and* $Y = (Y \cup \Omega)_{Y_*}$.
(c) $\mathfrak{F}_{Y \cup \Omega}$ *is* $(Y_* \cup \Omega_*, Y \cup \Omega, Y_*)$*-suitable.*
(d) $U \subseteq G_{Y_* \cup \Omega_*, Y \cup \Omega, X_*} \subseteq G_{Z_* Z X_*}$.
(e) *If* (Z_*, Z, X_*) *is f'-closed, then so is* $(Y_* \cup \Omega_*, Y \cup \Omega, X_*)$.

Proof. Choose a minimal family $U' \subseteq G_{Z_* Z}$ of preimages of elements of U under $\rho = \rho_{Z_* Z X_*}$ with $U'\rho = U$ and let Ω be the family of all $\overline{\eta} \in Z$ such that $y_{\overline{\eta}}$ contributes to the representation of some $u \in U'$. Moreover, let Ω_* be the tree-closure (under Definition 3.1(ii)) of $([U'] \cup [\Omega]) \cap Z_*$ with respect to Z_*. Hence (a) obviously holds.

Recall that Ω_* and Y_* are almost tree-closed. Hence also $Y_* \cup \Omega_*$ is almost tree-closed. If $\overline{\eta} \in \Omega$, then $[\overline{\eta}] \cap Z_* \subseteq \Omega_*$. Now it is clear that $(Y_* \cup \Omega_*, Y \cup \Omega, Y_*)$ is f-closed (because (Z_*, Z, Y_*) is). If $\overline{\eta} \in (Y \cup \Omega)_{Y_*}$, then $[\overline{\eta}]_n \subseteq Y_*$ for some $n < \omega$, hence $\overline{\eta} \in Z_{Y_*} = Y$, so $(Y \cup \Omega)_{Y_*} \subseteq Y$ and the converse inclusion is trivial, thus (b) holds.

Since $\mathfrak{F}$ is (Z_*, Z, Y_*)-suitable, so is $\mathfrak{F}_{Y \cup \Omega}$, which shows (c).

For (d) recall that (Z_*, Z, X_*) is f-closed due to the Transitivity Lemma 4.6. We note that $U \subseteq G_{Y_* \cup \Omega_*, Y \cup \Omega, X_*}$ through our choice of U', Ω_* and Ω.

Moreover, $G_{Y_* \cup \Omega_*, Y \cup \Omega, X_*} \subseteq G_{Z_* Z X_*}$ follows from $\rho_{Y_* \cup \Omega_*, Y \cup \Omega, X_*} \subseteq \rho_{Z_* Z X_*} = \rho$, $G_{Y_* \cup \Omega_*, Y \cup \Omega, X_*} = G_{Y_* \cup \Omega_*, Y \cup \Omega} \rho$ and $G_{Z_* Z X_*} = G_{Z_* Z} \rho$ with $G_{Y_* \cup \Omega_*, Y \cup \Omega} \subseteq G_{Z_* Z}$ as required. Now (e) holds trivially. □

Observation 4.15. (a) The proof of Lemma 4.14 applies for arbitrary almost tree-closed sets $\Omega_* \subseteq Z_*$ with $([U'] \cup [\Omega]) \cap Z_* \subseteq \Omega_*$. In particular, this is the case when $\Omega_* = \overline{([U'] \cup [\Omega]) \cap Z_*}(Z_*, Z, \mathfrak{F})$; however, the cardinal condition (a) becomes $|\Omega_*| \leq |U|^{\aleph_0}$.

(b) The proof of Lemma 4.14 also applies if we replace Ω by the larger family $\Omega' = Z_{\Omega_*}$ with $\Omega_* = \overline{([U'] \cup [\Omega]) \cap Z_*}(Z_*, Z, \mathfrak{F})$.

Observe that $\Omega_* = \overline{([U'] \cup [\Omega']) \cap Z_*}(Z_*, Z, \mathfrak{F})$ and $|\Omega'| \leq |U|^{\aleph_0}$.

(c) Note that by the construction of Ω_* and Ω, the following holds: If $U^\flat \subseteq U \subseteq G_{Z_* Z X_*}$ and U, Ω_*, Ω are as described in the lemma, then we can choose $\Omega_*^\flat \subseteq \Omega_*$ and $\Omega^\flat \subseteq \Omega$ so that also $U^\flat$, $\Omega_*^\flat$, $\Omega^\flat$ are as described in the lemma.

Lemma 4.16. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be f-closed triples such that $Y = Z_{Y_*}$, $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z, b_{\overline{\eta}} \in B_{Z_*}\}$ is (Z_*, Z, Y_*)-suitable, $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable and $U \subseteq G_{Y_* Y X_*}$. Then there exists $\Omega_* \subseteq Y_*$ such that:*

(a) $|\Omega_*| \leq |U|^{\aleph_0}$.
(b) $(Z_*, Z, X_* \cup \Omega_*)$ *and* $(X_* \cup \Omega_*, Y', X_*)$ *are f-closed with* $Y' = Z_{X_* \cup \Omega_*} = Y_{X_* \cup \Omega_*}$.
(c) $\mathfrak{F}$ *is* $(Z_*, Z, X_* \cup \Omega_*)$*-suitable.*
(d) $\mathfrak{F}_{Y'}$ *is* $(X_* \cup \Omega_*, Y', X_*)$*-suitable.*
(e) $U \subseteq G_{X_* \cup \Omega_*, Y', X_*} \subseteq G_{Y_* Y X_*} \subseteq G_{Z_* Z X_*}$.

Proof. Let $U' \subseteq G_{Y_* Y}$ again be a minimal family of preimages of elements of U under $\rho = \rho_{Z_* Z X_*}$ with $U' \rho = U$ and put $\Omega'_* = [U']$ and $\Omega_* = \overline{\Omega'_*}(Y_*, Y, \mathfrak{F}_Y)$. Then (a) holds automatically. Moreover, $\Omega_* \subseteq Y_*$ is almost tree-closed and hence so is $X_* \cup \Omega_*$.

If $\overline{\eta} \in Z$ with $|u_{\overline{\eta}}(Y_*)| \geq f$, then $|u_{\overline{\eta}}(X_* \cup \Omega_*)| \geq f$ follows from $X_* \cup \Omega_* \subseteq Y_*$. For otherwise $\overline{\eta} \in Z$ with $[\overline{\eta}]_n \subseteq Y_*$ for some $n < \omega$, so $\overline{\eta} \in Y$. If now $|[\overline{\eta}]_n \cap \Omega_*| = \aleph_0$, then $[\overline{\eta}]_n \subseteq \Omega_* \subseteq X_* \cup \Omega_*$ because $\Omega_* = \overline{\Omega'_*}(Y_*, Y, \mathfrak{F}_Y)$ is $\mathfrak{F}_Y$-closed. If $|[\overline{\eta}]_n \cap \Omega_*| < \aleph_0$, then $u_{\overline{\eta}}(X_* \cup \Omega_*) = u_{\overline{\eta}}(X_*)$ and $|u_{\overline{\eta}}(X_* \cup \Omega_*)| = |u_{\overline{\eta}}(X_*)| \geq f$ for some $n' < \omega$ or $[\overline{\eta}]_{n'} \subseteq X_* \subseteq X_* \cup \Omega_*$, respectively from the f-closedness of (Y_*, Y, X_*). This is half of (b).

If $\overline{\eta} \in Z$ with $[\overline{\eta}]_n \subseteq X_* \cup \Omega_*$ for some $n < \omega$, then similarly $\overline{\eta} \in Y$ and $[b_{\overline{\eta}}] \subseteq X_* \cup \Omega_*$ because $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable and $\Omega_* = \overline{\Omega'_*}(Y_*, Y, \mathfrak{F}_Y)$ is $\mathfrak{F}_Y$-closed. Hence $\mathfrak{F}$ is $(Z_*, Z, X_* \cup \Omega_*)$-suitable. From the above we also have $Y' = Z_{X_* \cup \Omega_*} = Y_{X_* \cup \Omega_*}$.

For $\overline{\eta} \in Y'$, by definition we have $[\overline{\eta}]_n \subseteq X_* \cup \Omega_*$ for some $n < \omega$, hence $(X_* \cup \Omega_*, Y')$ is Λ-closed. Clearly $(X_* \cup \Omega_*, Y', X_*)$ is f-closed, because (Y_*, Y, X_*) is.

If $\overline{\eta} \in Y' \subseteq Y$ and $[\overline{\eta}]_n \subseteq X_*$ for some $n < \omega$, then $[b_{\overline{\eta}}] \subseteq X_*$ because $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable. Hence $\mathfrak{F}_{Y'}$ is $(X_* \cup \Omega_*, Y', X_*)$-suitable and (d) holds.

For (e) we have $U' \subseteq G_{X_* \cup \Omega_*, Y'}$ from the first line of the proof: For each $u \in U'$ we have $[u] \subseteq [U'] \subseteq \Omega_*$, while for each $\overline{\eta} \in Y$ with $y_{\overline{\eta}}$ used in the representation of u we have $|[\overline{\eta}] \cap [U']| = \aleph_0$, hence $[\overline{\eta}] \cap Y_* \subseteq \Omega_*$ and $\overline{\eta} \in Y'$ because $\Omega_* = \overline{\Omega'_*}(Y_*, Y, \mathfrak{F}_Y)$ is $\mathfrak{F}_Y$-closed. Thus $U \subseteq G_{X_* \cup \Omega_*, Y', X_*}$. The proof of the remaining inclusions of (e) follows as in Lemma 4.14. □

Remark 4.17. (i) The family $\Omega_* = \overline{\Omega'_*}(Y_*, Y, \mathfrak{F}_Y)$ is $\mathfrak{F}_Y$-closed by construction.

(ii) The construction of U', Ω'_* and Ω_* depends only on (Y_*, Y, X_*), $\mathfrak{F}_Y$ and U. By $Y' = Z_{X_* \cup \Omega_*} = Y_{X_* \cup \Omega_*}$ also the triple $(X_* \cup \Omega_*, Y', X_*)$ depends only on (Y_*, Y, X_*), $\mathfrak{F}_Y$ and U.

The following theorem is the main result of this section. It provides the possibility of concentrating on those particular triple submodules mentioned below of relatively small size when proving the principal theorem of this paper.

Main Theorem 4.18. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be f-closed triples such that $Y = Z_{Y_*}$, $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z,\ b_{\overline{\eta}} \in B_{Z_*}\}$ is (Z_*, Z, Y_*)-suitable, $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable and $H \subseteq G_{Y_* Y X_*}$, $K \subseteq G_{Z_* Z X_*}$ with $|H|, |K| \leq \kappa$. Then there exist triples (Z'_*, Z', Y'_*), (Y'_*, Y', X'_*) such that:*

(a) $Z'_* \subseteq Z_*$, $Y'_* \subseteq Y_*$, $X'_* \subseteq X_*$, $Z' \subseteq Z$, $Y' \subseteq Y$.
(b) (Z'_*, Z', Y'_*) *and* (Y'_*, Y', X'_*) *are f-closed, and* $Y' = Z'_{Y'_*}$.
(c) $\mathfrak{F}_{Z'}$ *is* (Z'_*, Z', Y'_*)*-suitable.*
(d) $\mathfrak{F}_{Y'}$ *is* (Y'_*, Y', X'_*)*-suitable.*
(e) $H \subseteq G_{Y'_* Y' X'_*} \subseteq G_{Y_* Y X_*} \subseteq G_{Z_* Z X_*}$.
(f) $K \subseteq G_{Z'_* Z' X'_*} \subseteq G_{Z_* Z X_*}$.
(g) $|Z'_*|, |Y'_*|, |X'_*|, |Z'|, |Y'| \leq \kappa^{\aleph_0}$.
(h) $Z' \subseteq Z \setminus Z_{X_*}$, $Y' = Z_{X_* \cup Y'_*} \setminus Z_{X_*}$, *and* $Z'_{X_*} = Y'_{X_*} = \emptyset$.
(i) *The sets Y', Y'_* and X'_* depend only on the choice of Y, Y_*, X_*, $\mathfrak{F}_Y$ and H.*
(j) *If (Z_*, Z, X_*) is f'-closed, then so is (Z'_*, Z', X'_*).*

Proof. First we apply Lemma 4.16 and Remark 4.17 to H and we find an $\mathfrak{F}_Y$-closure $\Omega^1_* \subseteq Y_*$ of size $|\Omega^1_*| \leq |H|^{\aleph_0}$ such that

$$(Z_*, Z, X_* \cup \Omega^1_*), (X_* \cup \Omega^1_*, Y_1, X_*) \quad \text{with} \quad Y_1 = Z_{X_* \cup \Omega^1_*} = Y_{X_* \cup \Omega^1_*} \tag{4.1}$$

are f-closed. Moreover:

(I) $\mathfrak{F}$ is $(Z_*, Z, X_* \cup \Omega^1_*)$-suitable.
(II) $\mathfrak{F}_{Y_1}$ is $(X_* \cup \Omega^1_*, Y_1, X_*)$-suitable.
(III) $H \subseteq G_{X_* \cup \Omega^1_*, Y_1, X_*} \subseteq G_{Y_* Y X_*}$.
(IV) The sets Ω^1_* and Y_1 depend only on (Y_*, Y, X_*), $\mathfrak{F}_Y$ and H.

Now we apply Lemma 4.14 and Observation 4.15 to K and to $(Z_*, Z, X_* \cup \Omega^1_*)$, $(X_* \cup \Omega^1_*, Y_1, X_*)$ to get the following facts.

(V) There are an $\mathfrak{F}$-closure $\Omega^2_* \subseteq Z_*$ and $\Omega^2 = Z_{\Omega^2_*} \subseteq Z$ with $|\Omega^2_*|, |\Omega^2| \leq |K|^{\aleph_0}$.

(VI) $(X_* \cup \Omega^1_* \cup \Omega^2_*, Y_1 \cup \Omega^2, X_* \cup \Omega^1_*)$ and $(X_* \cup \Omega^1_*, Y_1, X_*)$ are f-closed with $Y_1 = (Y_1 \cup \Omega^2)_{X_* \cup \Omega^1_*}$.
(VII) $\mathfrak{F}_{Y_1 \cup \Omega^2}$ is $(X_* \cup \Omega^1_* \cup \Omega^2_*, Y_1 \cup \Omega^2, X_* \cup \Omega^1_*)$-suitable.
(VIII) $K \subseteq G_{X_* \cup \Omega^1_* \cup \Omega^2_*, Y_1 \cup \Omega^2, X_*} \subseteq G_{Z_* Z X_*}$.
(IX) If (Z_*, Z, X_*) is f'-closed, then so is $(X_* \cup \Omega^1_* \cup \Omega^2_*, Y_1 \cup \Omega^2, X_*)$.

We want to show that the sets

$$Z'_* = \Omega^1_* \cup \Omega^2_*, \qquad Y'_* = \Omega^1_*, \qquad X'_* = X_* \cap \Omega^1_* \tag{4.2}$$

and

$$Z' = (Y_1 \cup \Omega^2) \setminus (Y_1 \cup \Omega^2)_{X_*}, \qquad Y' = Y_1 \setminus (Y_1)_{X_*} \tag{4.3}$$

satisfy the conditions of Theorem 4.18.

If $\overline{\eta} \in Y_1$, then $[\overline{\eta}]_N \subseteq X_* \cup \Omega^1_*$ for some $N < \omega$ through (VI), and

$$[\overline{\eta}]_n \subseteq X_* \text{ for some } n < \omega \quad \text{or} \quad [\overline{\eta}]_N \subseteq \Omega^1_* \tag{4.4}$$

follows from either $|[\overline{\eta}]_N \cap \Omega^1_*| = \aleph_0$ and the $\mathfrak{F}_Y$-closedness of Ω^1_*, or $u_{\overline{\eta}} = \emptyset$ together with (VI); see Notation 4.11. Hence (by definition of Y') $[\overline{\eta}]_N \subseteq \Omega^1_*$ for any $\overline{\eta} \in Y'$.

Similarly for $\overline{\eta} \in Y_1 \cup \Omega^2$ we have $[\overline{\eta}]_N \subseteq X_* \cup \Omega^1_* \cup \Omega^2_*$ for some $N < \omega$ from (VI), and

$$[\overline{\eta}]_n \subseteq X_* \text{ for some } n < \omega \quad \text{or} \quad [\overline{\eta}]_{N'} \subseteq \Omega^1_* \text{ for some } N' < \omega \quad \text{or} \quad [\overline{\eta}]_N \subseteq \Omega^2_* \tag{4.5}$$

follows from either $|[\overline{\eta}]_N \cap \Omega^2_*| = \aleph_0$ and the $\mathfrak{F}$-closedness of Ω^2_*, or $u_{\overline{\eta}}(X_* \cup \Omega^1_*) = \emptyset$ and $\overline{\eta} \in Y_1$ with the help of (VI). Hence (by definition of Z') $[\overline{\eta}]_{N'} \subseteq \Omega^1_*$ or $[\overline{\eta}]_N \subseteq \Omega^2_*$ for any $\overline{\eta} \in Z'$.

Using (4.5) and (4.4) we see that (Z'_*, Z') and (Y'_*, Y') are Λ-closed, because for any $\overline{\eta} \in Z'$ we have $[\overline{\eta}]_{N''} \subseteq \Omega^1_* \cup \Omega^2_* = Z'_*$ for some $N'' < \omega$, and for any $\overline{\eta} \in Y'$ we have $[\overline{\eta}] \subseteq \Omega^1_* = Y'_*$. With X_* and Ω^1_* also $X'_* = X_* \cap \Omega^1_*$ is almost tree-closed.

Next we show (b) and begin with the f-closedness of (Z'_*, Z', Y'_*). Let $\overline{\eta} \in Z' \subseteq Y_1 \cup \Omega^2$. If $|u_{\overline{\eta}}(X_* \cup \Omega^1_*)| \geq f$, then also $|u_{\overline{\eta}}(Y'_*)| \geq f$ by $Y'_* = \Omega^1_* \subseteq X_* \cup \Omega^1_*$. But if $|u_{\overline{\eta}}(X_* \cup \Omega^1_*)| < f$, then $[\overline{\eta}]_n \subseteq X_* \cup \Omega^1_*$ for some $n < \omega$ and $\overline{\eta} \in Y_1$ by (VI). From (4.4) it follows that $[\overline{\eta}]_{n'} \subseteq X_*$ for some $n' < \omega$ or $[\overline{\eta}]_n \subseteq \Omega^1_*$, hence $[\overline{\eta}]_n \subseteq \Omega^1_* = Y'_*$, resulting from the definition of Z'.

Now we show the f-closedness of (Y'_*, Y', X'_*). Let $\overline{\eta} \in Y' \subseteq Y_1$. If $|u_{\overline{\eta}}(X_*)| \geq f$, then also $|u_{\overline{\eta}}(X'_*)| \geq f$ by $X'_* = X_* \cap \Omega^1_* \subseteq X_*$. But if $|u_{\overline{\eta}}(X_*)| < f$, then $[\overline{\eta}]_n \subseteq X_*$ for some $n < \omega$ and $\overline{\eta} \in Y_1$ by (VI), which contradicts $\overline{\eta} \in Y'$.

Next we want show that $Y' = Z'_{Y'_*}$. Since $Y_1 = (Y_1 \cup \Omega^2)_{X_* \cup \Omega^1_*}$, by (VI) and (4.4) we see for any $\overline{\eta} \in Y_1 \cup \Omega^2$ that $\overline{\eta} \in Y'$ if and only if $[\overline{\eta}]_n \subseteq X_* \cup \Omega^1_*$ for some $n < \omega$ and $[\overline{\eta}]_{n'} \not\subseteq X_*$ for any $n' < \omega$. This is the case if and only if $[\overline{\eta}]_n \subseteq \Omega^1_*$ for some $n < \omega$ and $[\overline{\eta}]_{n'} \not\subseteq X_*$ for any $n' < \omega$.

Using that $Z' = (Y_1 \cup \Omega^2) \setminus (Y_1 \cup \Omega^2)_{X_*}$ we find for $\overline{\eta} \in Y_1 \cup \Omega^2$ that

$$\overline{\eta} \in Z'_{Y'_*} \Leftrightarrow [\overline{\eta}]_n \subseteq Y'_* = \Omega^1_* \text{ for some } n < \omega \text{ and } [\overline{\eta}]_{n'} \not\subseteq X_* \text{ for any } n' < \omega.$$

Hence $Y' = Z'_{Y'_*}$, and (b) is established.

For (c) we also consider $\overline{\eta} \in Z'$ with $[\overline{\eta}]_n \subseteq Y'_* = \Omega^1_*$ for some $n < \omega$. Then $\overline{\eta} \in Y_1$ by (VI) and $[b_{\overline{\eta}}] \subseteq \Omega^1_*$, using the $\mathfrak{F}_Y$-closedness of Ω^1_*.

For (d) we let $\overline{\eta} \in Y'$. Then $[\overline{\eta}]_n \subseteq X'_* \subseteq X_*$ for some $n < \omega$ contradicts the definition of Y' and the claim of Definition 4.3(a) is empty, thus (d) holds trivially.

Next we show (e). Obviously $G_{Y'_*Y'} \subseteq G_{X_*\cup\Omega^1_*,Y_1}$, and also $\rho_{Y'_*Y'X'_*} \subseteq \rho_{X_*\cup\Omega^1_*,Y_1,X_*} = \rho$ satisfies $e_{\overline{\nu}}\rho = 0$ for all $\overline{\nu} \in (X_* \cup \Omega^1_*) \setminus Y'_* \subseteq X_*$ as well as $y_{\overline{\eta}}\rho = 0$ for all $\overline{\eta} \in Y_1 \setminus Y' = (Y_1)_{X_*}$. In particular $G_{Y'_*Y'X'_*} = G_{Y'_*Y'}\rho = G_{X_*\cup\Omega^1_*,Y_1}\rho = G_{X_*\cup\Omega^1_*,Y_1,X_*}$, and (III) implies (e).

Condition (f) follows similarly by using $G_{Z'_*Z'X'_*} = G_{X_*\cup\Omega^1_*\cup\Omega^2_*,Y_1\cup\Omega^2,X_*}$ and (VIII).

The first part of (g) is clear from the choice of Ω^1_* and Ω^2_*: $|Z'_*|, |Y'_*|, |X'_*| \leq \kappa^{\aleph_0}$. From (4.4) it follows that $|Y'| \leq |\Omega^1_*|^{\aleph_0} \leq |H|^{\aleph_0}$ and with (4.5) we also infer that $|Z'| \leq |\Omega^1_* \cup \Omega^2_*|^{\aleph_0} \leq \kappa^{\aleph_0}$.

(i) follows by the definition of Y'_*, Y' and X'_* with (IV).

For (j) we must show that (Z'_*, Z', X'_*) is also f'-closed. If $\overline{\eta} \in Z'$ with $|u_{\overline{\eta}}(X_*)| \geq f'$, then also $|u_{\overline{\eta}}(X'_*)| \geq f'$ from $X'_* = X_* \cap \Omega^1_* \subseteq X_*$. But if $|u_{\overline{\eta}}(X_*)| < f'$, then $[\overline{\eta}]_n \subseteq X_*$ for some $n < \omega$ by (IX) contradicts $\overline{\eta} \in Z'$. In particular, also (h) follows due to the definition of Z' and Y'. □

Remark 4.19. (i) From the definitions of Y' and Z', (V) and (4.1) it follows that

$$Y' = Z_{X_*\cup\Omega^1_*} \setminus Z_{X_*} \quad \text{and} \quad Z' = (Z_{X_*\cup\Omega^1_*} \cup Z_{\Omega^2_*}) \setminus Z_{X_*} = Y' \cup (Z_{\Omega^2_*} \setminus Z_{X_*}).$$

In particular, $Z_{\Omega^2_*} \setminus Z_{X_*} \subseteq Z'$.

(ii) Observe that $|\Omega^2_*| \leq |K|^{\aleph_0}$ and $|Z_{\Omega^2_*} \setminus Z_{X_*}| \leq |K|^{\aleph_0}$.

(iii) If for the tuple $(H, K, \Omega^1_*, \Omega^2_*, Y', Z')$ the theorem holds and $K' \subseteq K \subseteq G_{Z_*ZX_*}$, then by Observation 4.15 we can choose $\Omega^{2'}_* \subseteq \Omega^2_*, Z'' \subseteq Z'$ such that the tuple $(H, K', \Omega^1_*, \Omega^{2'}_*, Y', Z'')$ also fulfills the conclusion of the theorem.

5. Chains of triples

First we define closure properties ‘preserving freeness’, which are also important in [14, 24]; compare Proposition 3.6.

Definition 5.1. Let $Y_* \subseteq \Lambda_*$. Then $X_* \subseteq Y_*$ is *pairwise closed* (*for* Y_*) if from $\overline{\eta}\restriction\langle m, n\rangle, \overline{\eta}\restriction\langle m', n'\rangle \in X_*$ with $1 \leq m < m' \leq k$, $n, n' < \omega, \overline{\eta} \in \Lambda$ it follows that $[\overline{\eta}] \cap Y_* \subseteq X_*$.

Lemma 5.2. *Let $X_* \subseteq Y_* \subseteq \Lambda_*$. Then there is a minimal set $PC(X_*, Y_*)$ with $X_* \subseteq PC(X_*, Y_*) \subseteq Y_*$, and $PC(X_*, Y_*)$ is pairwise closed (for Y_*). Moreover, $|PC(X_*, Y_*)| \leq |X_*| \cdot \aleph_0$.*

Proof. Trivial. □

If Y_* is clear from the context, we will replace $PC(X_*, Y_*)$ by $PC(X_*)$.

Theorem 5.3. *Let* (Z_*, Z, Y_*) *and* (Y_*, Y, X_*) *be* f*-closed* (*for some* $f \geq 2$) *with* $Y = Z_{Y_*}$, $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z,\ b_{\overline{\eta}} \in B_{Z_*}\}$ (Z_*, Z, Y_*)*-suitable and* $\mathfrak{F}_Y$ (Y_*, Y, X_*)*-suitable. Moreover, let* $\Omega^1_* \subseteq Y_*$, $\Omega^2_*, \Omega^{2n}_* \subseteq Z_*$, $Z', Z'_n \subseteq Z$ $(n < \omega)$ *have the following properties:*

(A) Ω^1_* *is* $\mathfrak{F}_Y$*-closed and* $\Omega^2_*, \Omega^{2n}_*$ $(n < \omega)$ *are* $\mathfrak{F}$*-closed.*
(B) $PC(\Omega^{2n}_*, Z_*) \subseteq \Omega^{2,n+1}_*$.
(C) $(\Omega^1_* \cup \Omega^2_*, Z', \Omega^1_*)$, $(\Omega^1_* \cup \Omega^{2n}_*, Z'_n, \Omega^1_*)$, $(\Omega^1_*, Y', X_* \cap \Omega^1_*)$ *are* f*-closed, with* $Y' = Z'_{\Omega^1_*} = (Z'_n)_{\Omega^1_*}$ *for all* $n < \omega$.
(D) $\mathfrak{F}_{Z'}$ *is* $(\Omega^1_* \cup \Omega^2_*, Z', \Omega^1_*)$*-suitable,*
$\mathfrak{F}_{Z'_n}$ *is* $(\Omega^1_* \cup \Omega^{2n}_*, Z'_n, \Omega^1_*)$*-suitable and*
$\mathfrak{F}_{Y'}$ *is* $(\Omega^1_*, Y', X_* \cap \Omega^1_*)$*-suitable.*
(E) $G_{\Omega^1_* \cup \Omega^2_*, Z', X_* \cap \Omega^1_*} \subseteq G_{Z_* Z X_*}$, $G_{\Omega^1_* \cup \Omega^{2n}_*, Z'_n, X_* \cap \Omega^1_*} \subseteq G_{Z_* Z X_*}$ *and*
$G_{\Omega^1_*, Y', X_* \cap \Omega^1_*} \subseteq G_{Y_* Y X_*} \subseteq G_{Z_* Z X_*}$.
(F) $Z'_{X_*} = Y'_{X_*} = (Z'_n)_{X_*} = \emptyset$.
(G) $Z_{\Omega^2_*} \setminus Z_{X_*} \subseteq Z' \subseteq Z \setminus Z_{X_*}$ *and* $Z_{\Omega^{2n}_*} \setminus Z_{X_*} \subseteq Z'_n \subseteq Z \setminus Z_{X_*}$.

We define the following subsets of Λ *and* Λ_**, respectively:*

(i) $Z''_* = \Omega^1_* \cup \Omega^2_* \cup \bigcup_{n<\omega} \Omega^{2n}_*$,
(ii) $Y''_* = \Omega^1_* \cup \bigcup_{n<\omega} \Omega^{2n}_*$,
(iii) $X''_* = X_* \cap \Omega^1_*$,
(iv) $Z'' = Z' \cup \bigcup_{n<\omega} Z'_n$,
(v) $Y'' = Y' \cup \bigcup_{n<\omega} Z'_n$.

Then:

(a) (Z''_*, Z'', Y''_*) *is* $(f-1)$*-closed,* (Y''_*, Y'', X''_*) *is* f*-closed with* $Y'' = Z''_{Y''_*}$.
(b) $\mathfrak{F}_{Z''}$ *is* (Z''_*, Z'', Y''_*)*-suitable.*
(c) $\mathfrak{F}_{Y''}$ *is* (Y''_*, Y'', X''_*)*-suitable.*
(d) $G_{Y''_* Y'' X''_*} = G_{\Omega^1_*, Y', X_* \cap \Omega^1_*} + \sum_{n<\omega} G_{\Omega^1_* \cup \Omega^{2n}_*, Z'_n, X_* \cap \Omega^1_*} \subseteq G_{Z_* Z X_*}$,
(e) $G_{Z''_* Z'' X''_*} = G_{\Omega^1_* \cup \Omega^2_*, Z', X_* \cap \Omega^1_*} + \sum_{n<\omega} G_{\Omega^1_* \cup \Omega^{2n}_*, Z'_n, X_* \cap \Omega^1_*} \subseteq G_{Z_* Z X_*}$.
(f) $Z'' \subseteq Z \setminus Z_{X_*}$ *and* $Z''_{X_*} = Y''_{X_*} = \emptyset$.
(g) *If* (Z_*, Z, X_*) *is* f'*-closed, then so is* (Z''_*, Z'', X''_*).

Proof. (a) Observe that $\bigcup_{n<\omega} \Omega^{2n}_* \subseteq Z_*$ is almost tree-closed, because Ω^{2n}_* is tree-closed for Z_*. Hence also Z''_*, Y''_* and X''_* are almost tree-closed and (Z''_*, Z''), (Y''_*, Y'') are Λ-closed.

Now we show that (Z''_*, Z'', Y''_*) is $(f-1)$-closed. Indeed, if $\overline{\eta} \in Z'' \subseteq Z$ and $|u_{\overline{\eta}}(\Omega^1_*)| < f$, then $[\overline{\eta}]_{n'} \subseteq \Omega^1_* \subseteq Y''_*$ for some $n' < \omega$ due to the definition of Z'' and (C).

Conversely, if $|u_{\overline{\eta}}(\Omega^1_*)| \geq f$, then $u_{\overline{\eta}}(Y''_*) \subseteq u_{\overline{\eta}}(\Omega^1_*)$ because $\Omega^1_* \subseteq Y''_*$. If $|u_{\overline{\eta}}(\Omega^1_*) \setminus u_{\overline{\eta}}(Y''_*)| > 1$, then there are $1 \leq m_1 < m_2 \leq k$ and $n_1, n_2 < \omega$ such that $m_1, m_2 \in u_{\overline{\eta}}(\Omega^1_*) \setminus u_{\overline{\eta}}(Y''_*)$, $\overline{\eta} \upharpoonright \langle m_1, n_1 \rangle, \overline{\eta} \upharpoonright \langle m_2, n_2 \rangle \in Y''_* \setminus \Omega^1_* \subseteq \bigcup_{n<\omega} \Omega^{2n}_*$. Hence $\overline{\eta} \upharpoonright \langle m_1, n_1 \rangle \in \Omega^{2n'_1}_*$ and $\overline{\eta} \upharpoonright \langle m_2, n_2 \rangle \in \Omega^{2n'_2}_*$ for some $n'_1, n'_2 < \omega$. If $N = \max\{n'_1, n'_2\}$, then $\overline{\eta}\restriction\langle m_1, n_1\rangle, \overline{\eta}\restriction\langle m_2, n_2\rangle \in \Omega_*^{2N}$ and $[\overline{\eta}]_{n'} \subseteq \Omega_*^{2,N+1} \subseteq Y''_*$ for some $n' < \omega$, as required.

If, however, $|u_{\overline{\eta}}(\Omega_*^1)| \geq f$ and $|u_{\overline{\eta}}(\Omega_*^1) \setminus u_{\overline{\eta}}(Y''_*)| \leq 1$, then clearly $|u_{\overline{\eta}}(Y''_*)| \geq f - 1$.

Next we show that (Y''_*, Y'', X''_*) is f-closed. Recall that (Z_*, Z, X_*) is f-closed by the Transitivity Lemma 4.6 and the assumptions of the theorem. If $\overline{\eta} \in Y'' \subseteq Z$ and $|u_{\overline{\eta}}(X_*)| \geq f$, then $|u_{\overline{\eta}}(X''_*)| \geq f$ from $X''_* \subseteq X_*$. But if $|u_{\overline{\eta}}(X_*)| < f$, then $[\overline{\eta}]_{n'} \subseteq X_*$ for some $n' < \omega$ by the f-closedness of (Z_*, Z, X_*), and the branch $\overline{\eta}$ belongs to either Y'_{X_*} or $(Z'_n)_{X_*}$ for some n, which contradicts (F).

Finally, we must show $Y'' = Z''_{Y''_*}$. The inclusion $\subseteq$ is obvious. Conversely, let $\overline{\eta} \in Z''_{Y''_*}$. Then $[\overline{\eta}]_{n'} \subseteq \Omega_*^1 \cup \bigcup_{n<\omega} \Omega_*^{2n}$ for some $n' < \omega$. If $|u_{\overline{\eta}}(\Omega_*^1)| < f$, then $[\overline{\eta}]_{n''} \subseteq \Omega_*^1$ for some $n'' < \omega$ and $\overline{\eta} \in Y' \subseteq Y''$ by definition of Z'' and (C). If, however, $|u_{\overline{\eta}}(\Omega_*^1)| \geq f \geq 2$, then again there are $1 \leq m_1 < m_2 \leq k$ and $n_1, n_2 < \omega$ such that $m_1, m_2 \in u_{\overline{\eta}}(\Omega_*^1)$ and $\overline{\eta}\restriction\langle m_1, n_1\rangle, \overline{\eta}\restriction\langle m_2, n_2\rangle \in \bigcup_{n<\omega} \Omega_*^{2n}$. As above there is $N < \omega$ such that $\overline{\eta}\restriction\langle m_1, n_1\rangle, \overline{\eta}\restriction\langle m_2, n_2\rangle \in \Omega_*^{2N}$. By (B) we have $[\overline{\eta}] \cap Z_* \subseteq \Omega_*^{2,N+1}$, and $\overline{\eta} \in Z_{\Omega_*^{2,N+1}}$. Using (G) and the definition of Z'', we also have $Z'' \subseteq Z \setminus Z_{X_*}$, and thus $\overline{\eta} \notin Z_{X_*}$. Finally $\overline{\eta} \in Z_{\Omega_*^{2,N+1}} \setminus Z_{X_*} \subseteq Z'_{N+1} \subseteq Y''$, and so $Y'' = Z''_{Y''_*}$. Thus (a) holds.

(b) If $\overline{\eta} \in Z''$ and $[\overline{\eta}]_{n'} \subseteq Y''_*$ for some $n' < \omega$, then (using the arguments above) $[\overline{\eta}]_{n''}$ is a subset of either Ω_*^1 or Ω_*^{2n} for some $n, n'' < \omega$. If $[\overline{\eta}]_{n''} \subseteq \Omega_*^1$, then due to (D) and the definition of Z'' we have $[b_{\overline{\eta}}] \subseteq \Omega_*^1 \subseteq Y''_*$. For $[\overline{\eta}]_{n''} \subseteq \Omega_*^{2n}$ we find that $[b_{\overline{\eta}}] \subseteq \Omega_*^{2n} \subseteq Y''_*$, because Ω_*^{2n} is $\mathfrak{F}$-closed.

(c) If $\overline{\eta} \in Y''$ and $[\overline{\eta}]_{n'} \subseteq X''_* \subseteq X_*$ for some $n < \omega$, then $\overline{\eta}$ belongs to either Y'_{X_*} or $(Z'_n)_{X_*}$ for some n, which contradicts (F), so (c) follows.

(d) Clearly $\rho_{\Omega_*^1, Y', X_* \cap \Omega_*^1} \subseteq \rho_{Y''_* Y'' X''_*} = \rho$ and $\rho_{\Omega_*^1 \cup \Omega_*^{2n}, Z'_n, X_* \cap \Omega_*^1} \subseteq \rho_{Y''_* Y'' X''_*} = \rho$, and hence

$$G_{\Omega_*^1, Y', X_* \cap \Omega_*^1} + \sum_{n<\omega} G_{\Omega_*^1 \cup \Omega_*^{2n}, Z'_n, X_* \cap \Omega_*^1} = \Big(G_{\Omega_*^1 Y'} + \sum_{n<\omega} G_{\Omega_*^1 \cup \Omega_*^{2n}, Z'_n}\Big)\rho = G_{Y''_* Y''}\rho$$
$$= G_{Y''_* Y'' X''_*}.$$

The inclusion $G_{Y''_* Y'' X''_*} \subseteq G_{Z_* Z X_*}$ follows from (E).

(e) follows by the same arguments as (d).

(f) From (G) and the definition of Z'' it follows that $Z'' \subseteq Z \setminus Z_{X_*}$. Similarly $Z''_{X_*} = Y''_{X_*} = \emptyset$ is a consequence of (F).

(g) If $\overline{\eta} \in Z''$ and $|u_{\overline{\eta}}(X_*)| \geq f'$, then also $|u_{\overline{\eta}}(X''_*)| \geq f'$ because $X''_* \subseteq X_*$. But if $|u_{\overline{\eta}}(X_*)| < f'$, then $[\overline{\eta}]_n \subseteq X_*$ for some $n < \omega$ because (Z_*, Z, X_*) is f'-closed, which contradicts (f), showing (g). □

6. The Step Lemma

If δ is an ordinal with $\mathrm{cf}(\delta) = \omega$, then let

$$\Gamma_\delta = \{\eta \in {}^{\omega\uparrow}\delta \mid \sup \eta = \delta\}, \quad \text{and if } \eta \in {}^{\omega\uparrow}\delta, \text{ then} \quad [\eta] = \{\eta\restriction n \mid n < \omega\} \subseteq {}^{\omega\uparrow>}\delta.$$

Proposition 6.1 (The Easy Black Box). *For each cardinal $\lambda \geq \aleph_0$ and set Ξ of cardinality $\leq \lambda^{\aleph_0}$ there is a family $\langle g_\eta \mid \eta \in {}^{\omega\uparrow}\lambda\rangle$ with the following properties:*

(i) $g_\eta : [\eta] \to \Xi$.
(ii) *For each map $g : {}^{\omega\uparrow>}\lambda \to \Xi$ there exists some $\eta \in {}^{\omega\uparrow}\lambda$ with $g_\eta \subseteq g$.*

Proof (see [14, p. 55, Lemma 2.3], which we outline for the convenience of the reader). Since $|\Xi| \leq \lambda^{\aleph_0} = |{}^{\omega}\lambda|$, we can fix an embedding $\pi : \Xi \hookrightarrow {}^{\omega}\lambda$. And since $|{}^{\omega>}\lambda| = \lambda$, there is also a list ${}^{\omega>}\lambda = \langle \mu_\alpha \mid \alpha < \lambda\rangle$ with enough repetitions for each $\eta \in {}^{\omega>}\lambda$, i.e. $\{\alpha < \lambda \mid \mu_\alpha = \eta\} \subseteq \lambda$ is unbounded. Moreover, we define for each $n < \omega$ a coding map

$$\pi_n : {}^{n}\Xi \to {}^{n^2}\lambda \subseteq {}^{\omega>}\lambda \;\; (\overline{\varphi} = \langle \varphi_0, \dots, \varphi_{n-1}\rangle \mapsto \overline{\varphi}\pi_n = (\varphi_0\pi\restriction n)^\wedge \dots {}^\wedge(\varphi_{n-1}\pi\restriction n)).$$

Finally, let $X \subseteq {}^{\omega\uparrow}\lambda$ be the collection of all order preserving maps $\eta : \omega \to \lambda$ such that

$$\exists\overline{\varphi} = \langle \varphi_i \mid i < \omega\rangle \in {}^{\omega}\Xi \quad \text{with} \quad (\overline{\varphi}\restriction n)\pi_n = \mu_{n\eta} \text{ for all } n < \omega. \tag{6.1}$$

By definition of π_n it follows that $\overline{\varphi}$ is uniquely determined by (6.1). (Just note that $\mu_{n\eta}$ determines $\varphi_m\pi\restriction n$ for all $m < n$.)

We now prove the two statements of the proposition. For (i) we consider any $\eta \in {}^{\omega\uparrow}\lambda$. If $\eta \notin X$, then we can choose arbitrary elements $g_\eta(\eta\restriction n) \in \Xi$, and if $\eta \in X$, then we choose the uniquely determined sequence $\overline{\varphi}$ from (6.1) and let $g_\eta(\eta\restriction n) = \varphi_n$.

For (ii) we consider some $g : {}^{\omega\uparrow>}\lambda \to \Xi$. In this case we must define $\eta = \langle \alpha_n \mid n < \omega\rangle \in {}^{\omega\uparrow}\lambda$. Since the list of μ_αs is unbounded, we can choose inductively $\alpha_n > \alpha_{n-1}$ with $\langle g(\eta\restriction m) \mid m < n\rangle\pi_n = \mu_{\alpha_n}$ for all $n < \omega$.

Finally, we check (ii). Using (6.1) we will find that the sequence η belongs to X:

If $\overline{\varphi} = \langle g(\eta\restriction i) \mid i < \omega\rangle \in {}^{\omega}\Xi$, then $(\overline{\varphi}\restriction n)\pi_n = \langle g(\eta\restriction m) \mid m < n\rangle\pi_n = \mu_{\alpha_n} = \mu_{n\eta}$ for all $n < \omega$, and $g_\eta(\eta\restriction n) = \varphi_n = g(\eta\restriction n)$ for all $n < \omega$ is immediate. □

Definition 6.2. If $0 \leq f < k$ and $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \dots \times {}^{\omega\uparrow}\lambda_k$, then we put

- $\Lambda^{\overline{\xi}} = \{\overline{\eta} \in \Lambda \mid \overline{\eta}\restriction(f,k] = \overline{\xi}\}$.
- $\Lambda^{\overline{\xi}}_* = \{\overline{\nu} \in \Lambda_* \mid \overline{\nu}\restriction(f,k] = \overline{\xi}\}$.
- If $f < i \leq k$, then $\Lambda^{\overline{\xi}i}_* = \{\overline{\nu} \in \Lambda_* \mid \nu_i \trianglelefteq \xi_i \neq \nu_i,\ \nu_m = \xi_m \text{ for all } 1 < m \neq i \leq k\}$.
- $\Lambda_{\overline{\xi}*} = \dot{\bigcup}_{f<i\leq k} \Lambda^{\overline{\xi}i}_*$.

Lemma 6.3. *If $1 \leq f < k$, $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \dots \times {}^{\omega\uparrow}\lambda_k$ and $E_* \subseteq \Lambda_*$ is a finite subset, then (J_*, J, I_*) is f-closed for $J_* = I_* \cup \Lambda^{\overline{\xi}}_*$, $J = \Lambda^{\overline{\xi}}$ and $I_* = \Lambda_{\overline{\xi}*} \cup E_*$.*

Proof. From the hypothesis and Definition 6.2 it follows that $\Lambda^{\overline{\xi}}_*, \Lambda^{\overline{\xi}i}_*, \Lambda_{\overline{\xi}*}$ are tree-closed, hence I_* and J_* are almost tree-closed. If $\overline{\eta} \in J$, then $[\overline{\eta}] \subseteq \Lambda^{\overline{\xi}}_*\dot{\cup}\Lambda_{\overline{\xi}*} \subseteq J_*$. Moreover, (J_*, J) is Λ-closed. Hence we must only check Definition 4.1(b)(iv). If $\overline{\eta} \in J$, then $\{1, \dots, f\} \subseteq u_{\overline{\eta}}(I_*)$ follows from $\Lambda^{\overline{\xi}}_* \cap \Lambda_{\overline{\xi}*} = \emptyset$ and finiteness of E_*. Thus $|u_{\overline{\eta}}| \geq f$ as required. □

Definition 6.4. (a) For $\overline{\nu} \in \Lambda_*$ we define the *ordinal content* $\operatorname{orco} \overline{\nu} = \bigcup\{\operatorname{Im} \nu_m \mid 1 \leq m \leq k\}$.
(b) If $Y_* \subseteq \Lambda_*$, then $\operatorname{orco} Y_* = \bigcup_{\overline{\nu} \in Y_*} \operatorname{orco} \overline{\nu}$.
(c) If $S, T \subseteq \lambda_k$ and $\tau : S \to T$ is a bijection, then τ extends canonically to a bijection $\tau : {}^{\omega\geq}S \to {}^{\omega\geq}T$ and for $\overline{\eta} \in \Lambda_* \cup \Lambda$ we define $\overline{\eta}\tau = (\eta_1\tau, \dots, \eta_k\tau)$.
(d) If $X_* \subseteq \Lambda_*$, then we call a bijection $\tau : S \to T$ X_*-*admissible* if $\operatorname{orco} X_* \subseteq S$ and $X_*\tau \subseteq \Lambda_*$.
(e) If $\tau : S \to T$ is an X_*-admissible bijection, then τ extends canonically to an A-module monomorphism $\tau : \widehat{B_{X_*}} \to \widehat{B_{\Lambda_*}} = \widehat{B}$, which we call the *shift isomorphism* (onto its image).

We want to show that X_*-admissible maps are compatible with the notions of triple modules etc. from the last sections.

Observation 6.5. (i) *If $X \subseteq \Lambda$ and $\tau : S \to T$ is an $[X]$-admissible bijection, then $X\tau \subseteq \Lambda$.*
(ii) *If (Y_*, Y, X_*) is f-closed and $\tau : S \to T$ is a Y_*-admissible bijection, then $(Y_*, Y, X_*)\tau := (Y_*\tau, Y\tau, X_*\tau)$ is f-closed as well.*
(iii) *If $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y\}$ is (Y_*, Y, X_*)-suitable and τ is Y_*-admissible, then $\mathfrak{F}_\tau := \{y'_{\overline{\eta}}\tau = \pi_{\overline{\eta}}(b_{\overline{\eta}}\tau) + y_{\overline{\eta}\tau} \mid \overline{\eta} \in Y\} = \{y'_{\overline{\eta}'} = \pi_{\overline{\eta}'\tau^{-1}}(b_{\overline{\eta}'\tau^{-1}}\tau) + y_{\overline{\eta}'} \mid \overline{\eta}' \in Y\tau\}$ is $(Y_*, Y, X_*)\tau$-suitable.*
(iv) *If $G_{Y_*YX_*}$ is the triple module from Theorem 4.5 generated by the triple (Y_*, Y, X_*) and the family $\mathfrak{F}$ of branches, and if τ is Y_*-admissible, then $G_{(Y_*YX_*)\tau} := G_{Y_*\tau, Y\tau, X_*\tau}$ (for $(Y_*, Y, X_*)\tau$ and $\mathfrak{F}_\tau$) is a well-defined A-module and $(G_{Y_*YX_*})\tau = G_{(Y_*YX_*)\tau}$.*
(v) *If τ is Y_*-admissible, then $(\overline{\Omega_*}(Y_*, Y, \mathfrak{F}))\tau = \overline{\Omega_*\tau}(Y_*\tau, Y\tau, \mathfrak{F}_\tau)$.*
(vi) *If τ is Y_*-admissible, then $PC(X_*, Y_*)\tau = PC(X_*\tau, Y_*\tau)$.*

Proof. Since all statements are obvious, for illustration we only show that $X\tau \subseteq \Lambda$ for $X \subseteq \Lambda$, which is part of (i).

If $\overline{\eta} \in X$, then $[\overline{\eta}] \subseteq [X]$ and $[\overline{\eta}]\tau \subseteq \Lambda_*$ In particular, $\overline{\eta}\restriction\langle m, n\rangle\tau \in \Lambda_*$ for any $1 \leq m \leq k$ and $n < \omega$. Thus $(\eta_m\restriction n)\tau \in {}^{\omega\uparrow>}\lambda_m$ and $\eta_m\tau \in {}^{\omega\uparrow}\lambda_m$, and so $\overline{\eta}\tau \in \Lambda$. ◻

We now prove the central step lemma. Step lemmas are designed to kill unwanted homomorphisms. It is critical that the construction takes place in the category we are interested in, in this paper $\aleph_n$-free A-modules. The preparation for this is the work in the preceding sections.

Step Lemma 6.6. *Using the notation from Section 4 and above, assume that the following parameters are given:*

(i) $0 \leq f < k$ *and* $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$.
(ii) $E_* \subseteq \Lambda_*$ *is a finite set.*
(iii) (J_*, J, I_*) *is a triple such that*

$$I_* = I_*(\overline{\xi}) = \Lambda_{\overline{\xi}*} \cup E_*, \quad J = J(\overline{\xi}) = \Lambda^{\overline{\xi}}, \quad J_* = J_*(\overline{\xi}) = I_* \cup \Lambda_*^{\overline{\xi}}.$$

(iv) *$G_1 = G_1(\overline{\xi}) = B_{I_*}$ is a free A-module.*
(v) *(V_*, V, U_*) is $(f+1)$-closed.*
(vi) *$\mathfrak{G} = \{y''_{\overline{\eta}} = \pi'_{\overline{\eta}} b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V\}$ is (V_*, V, U_*)-suitable.*
(vii) *$G = G_{V_* V U_*}$ and $\varphi : G_1 \to G$ is a homomorphism with $z\varphi \neq 0$ for some $z \in B_{E_*} \subseteq G_1$.*

Then there are $\pi_{\overline{\eta}} \in \widehat{R}$ $(\overline{\eta} \in J)$ such that $G_2 = G_{J_* J}$ with $\mathfrak{H} = \{x_{\overline{\eta}} = \pi_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in J\}$ has the following property.

If (Z_*, Z, Y_*) and (Y_*, Y, X_*) are $(f+1)$-closed with $Y = Z_{Y_*}$, $\mathfrak{F} = \{y'_{\overline{\eta}} = \rho_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z\}$ is (Z_*, Z, Y_*)-suitable, $\mathfrak{F}_Y$ is (Y_*, Y, X_*)-suitable and τ is a V_*-admissible bijection with $(V_*, V, U_*)\tau = (Y_*, Y, X_*)$, $\mathfrak{G}\tau = \mathfrak{F}_Y$ and $G_4 = G_{Z_* Z X_*}$, $G_3 = G_{Y_* Y X_*}$, then $G_1 \subseteq G_2$, $G\tau = G_3 \subseteq G_4$ and

$$\varphi\tau : G_1 \to G_3 \text{ does not extend to a homomorphism } G_2 \to G_4.$$

Remark 6.7. The $\pi_{\overline{\eta}}$'s can be chosen to depend only on G, or on $\mathfrak{G} \setminus \mathfrak{G}_{V_{U_*}}$, respectively, but not on $\mathfrak{G}_{V_{U_*}}$.

The mappings in the step lemma can be visualized by the following diagram, where arrows without a name are inclusions.

$$\begin{tikzcd}
G_2 \arrow[rr, "\nexists"] & & G_4 \\
& G \arrow[dr, "\tau"] & \\
G_1 \arrow[uu] \arrow[ur, "\varphi"] \arrow[rr, "\varphi\tau"] & & G_3 \arrow[uu]
\end{tikzcd}$$

Proof. The step lemma is shown by induction on f.

The case $f = 0$

If $f = 0$, then the basic sets satisfy

$$\overline{\xi} \in \Lambda, \quad \Lambda_*^{\overline{\xi}} = \emptyset, \quad J = \Lambda^{\overline{\xi}} = \{\overline{\xi}\}, \quad \Lambda_{\overline{\xi}*} = [\overline{\xi}],$$

$$I_* = [\overline{\xi}] \cup E_* \ (E_* \subseteq \Lambda_* \text{ finite}), \quad J_* = I_* \cup \Lambda_*^{\overline{\xi}} = I_*,$$

and the corresponding A-modules are

$$G_1 = B_{I_*} \text{ (which is free)}, \quad G_2 = G_{J_* J} = \langle B_{J_*}, Ax_{\overline{\xi}} \rangle_* = \langle B_{I_*}, Ax_{\overline{\xi} i} \mid i < \omega \rangle \subseteq_* \widehat{B}.$$

Hence $G_2/G_1 \cong \mathbb{S}^{-\infty}A$ is an $\mathbb{S}$-divisible, $\mathbb{S}$-torsion-free A-module of A-rank 1. So the $\mathbb{S}$-adic closure $\overline{G_1}$ of G_1 is $\overline{G_1} = G_2$. Moreover, $G = G_{V_* V U_*}$ is $\aleph_1$-free by Theorem 4.5 because (V_*, V, U_*) is 1-closed and $\mathfrak{G}$ is (V_*, V, U_*)-suitable. In particular G is $\mathbb{S}$-cotorsion-free by Observation 3.5(b) and $0 \neq z\varphi \in G$. Thus we find $\pi \in \widehat{R}$ (the $\mathbb{S}$-completion of R) such that $\pi z\varphi \notin G$.

The choice of π depends only on G: Using that $G_{V_*,V\setminus V_{U_*}} \subseteq G_{V_*V}$ with the associated family $\mathfrak{G} \setminus \mathfrak{G}_{V_{U_*}}$ of branch elements and $\rho_{V_*,V\setminus V_{U_*},U_*} \subseteq \rho_{V_*VU_*} = \rho$ with $y''_{\overline{\eta}}\rho = 0$ for all $\overline{\eta} \in V_{U_*}$, we find that $G_{V_*VU_*} = G_{V_*,V\setminus V_{U_*},U_*}$. Hence the choice of π does not depend on $\mathfrak{G}_{V_{U_*}}$. This explains Remark 6.7.

Next we consider two extensions of G_1, namely $G'_2 = \langle B_{J_*}, Ay_{\overline{\xi}}\rangle_*$ and $G''_2 = \langle B_{J_*}, A(\pi z + y_{\overline{\xi}})\rangle_*$, and claim that φ cannot extend to a homomorphism $\widehat{\varphi}$ of both with image in G. Otherwise $\pi z\varphi = (\pi z + y_{\overline{\xi}})\widehat{\varphi} - y_{\overline{\xi}}\widehat{\varphi} \in G$ is a contradiction. So we can choose $\pi_{\overline{\xi}} \in \{0, \pi\}$ such that $\varphi : G_1 \to G$ does not extend to a homomorphism $\widehat{\varphi} : G_2 \to G$, where $G_2 = \langle B_{J_*}, A(\pi_{\overline{\xi}} z + y_{\overline{\xi}})\rangle_*$. In particular,

$$x_{\overline{\xi}}\widehat{\varphi} \notin G, \quad \text{where} \quad x_{\overline{\xi}} = \pi_{\overline{\xi}} z + y_{\overline{\xi}}. \tag{6.2}$$

If there are (Z_*, Z, Y_*), (Y_*, Y, X_*), $\mathfrak{F}$ and τ contradicting the step lemma, then by the Transitivity Lemma 4.6(b)(iii) we have $G_4/G_3 \cong G_{Z_*ZY_*}$, and $G_{Z_*ZY_*}$ is $\aleph_1$-free by Observation 4.7 because (Z_*, Z, Y_*) is 1-closed and $\mathfrak{F}$ is (Z_*, Z, Y_*)-suitable. Hence G_3 is $\mathbb{S}$-adically closed in G_4 by Observation 3.5. The homomorphism $\varphi : G_1 \to G$ extends uniquely (by continuity) to $\widehat{\varphi} : G_2 \to \widehat{G}$, and the shift isomorphism $\tau : G \to G_3$ also extends uniquely to $\widehat{\tau} : \widehat{G} \to \widehat{G}_3$.

If the composition map $\varphi\tau : G_1 \to G_3$ extends to $\psi : G_2 \to G_4$, then by uniqueness $\psi = \widehat{\varphi}\widehat{\tau}$ and $x_{\overline{\xi}}\psi = x_{\overline{\xi}}\widehat{\varphi}\widehat{\tau} \in G_4 \cap \widehat{G}_3 = G_3 = G\tau$. We get $x_{\overline{\xi}}\widehat{\varphi} \in G$, which is a contradiction.

The case $f > 0$

Now suppose that $f > 0$ and the lemma is already shown for $f - 1$. Let $\lambda = \lambda_f$ and $\theta = \lambda_{f-1}$ (setting $\lambda_0 = |A|$), hence $\theta < \lambda$. The A-modules G_1 and G are given. In particular, $G_1 = B_{I_*}$ is free and (J_*, J, I_*) is f-closed by Lemma 6.3. Moreover, $\{1, \dots, f\} \subseteq u_{\overline{\eta}}(I_*)$ for each $\overline{\eta} \in J$, hence $|u_{\overline{\eta}}(I_*)| \geq f$, and in particular $[\overline{\eta}] \nsubseteq I_*$, hence $I := J_{I_*} = \emptyset$, and $\mathfrak{H} = \{x_{\overline{\eta}} = \pi_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in J\}$ is (J_*, J, I_*)-suitable. (Observe that the factors $\pi_{\overline{\eta}}$ $(\overline{\eta} \in J)$ are not yet known, but $[\pi_{\overline{\eta}} z] \subseteq I_*$, which suffices here to see that $\mathfrak{H}$ is (J_*, J, I_*)-suitable.)

So $G_1 = B_{I_*} = G_{I_*I}$ is as stated in Theorem 4.5 and

$$G_1 \subseteq G_2 = G_{J_*J}, G_1, G_2 \text{ are } \aleph_k\text{-free}, \quad \text{and} \quad G_2/G_1 \cong G_{J_*JI_*} \text{ is } \aleph_f\text{-free}.$$

By construction, $|I_*| = |J_*| = |J_*|^{\aleph_0} = \lambda^{\aleph_0} = \lambda$. Since $f > 0$ and $|A| \leq \lambda_1 \leq \lambda_f = \lambda$ we may also assume that $|G_1| = |G_2| = \lambda$, and using the assumption that (V_*, V, U_*) is $(f + 1)$-closed and $\mathfrak{G}$ is (V_*, V, U_*)-suitable, we see that the module $G = G_{V_*VU_*}$ is also $\aleph_{f+1}$-free by Theorem 4.5.

Preparing the predictions on G_1 for the step lemma

For the next steps we recall (from above) the definition of $\Gamma = {}^{\omega\uparrow}\lambda = \dot{\bigcup}_{\delta\in\lambda^o}\Gamma_\delta$ with $\lambda^o = \{\alpha \in \lambda \mid \mathrm{cf}(\alpha) = \omega\}$. From our choice of cardinals in Section 2.1 (i)–(iii), for any $\delta \in \lambda^o$ we have $|\delta| \leq \mu_f$, and thus $|\Gamma_\delta| \leq \mu_f^{\aleph_0} = \mu_f < \lambda$. We can first well-order each Γ_δ for $\delta \in \lambda^o$ and then extend the ordering lexicographically using that $\lambda = \mu_f^+$ is regular. This gives an enumeration $\langle \eta_\alpha \mid \alpha < \lambda \rangle$ of Γ without repetitions and a monotonic *norm function* $\| \cdot \| : \lambda \to \lambda^o$ $(\alpha \mapsto \|\alpha\|)$ satisfying $\eta_\alpha \in \Gamma_{\|\alpha\|}$ for all $\alpha \in \lambda$, which we fix for the rest of this investigation.

For $\nu \in {}^{\omega\uparrow>}\lambda$ and $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$, we define

$$\Lambda_*^{\nu\overline{\xi}} := \{\overline{\nu} \in \Lambda_* \mid \overline{\nu}_f \trianglelefteq \nu,], \overline{\nu}\restriction(f,k] = \overline{\xi}\}, \qquad G_{1\nu} = B_{\Lambda_*^{\nu\overline{\xi}}}, \qquad \mathcal{G} = \{G_{1\nu} \mid \nu \in {}^{\omega\uparrow>}\lambda\}.$$

Clearly $|G_{1\nu}| = \theta$ and $|\mathcal{G}| = \lambda$.

Let $(V'_*, V', U'_*) = (\Omega^1_*, V', U_* \cap \Omega^1_*)$ be the triple defined in Theorem 4.18 and in (4.2) by (V_*, V, U_*) and $\operatorname{Im}\varphi \subseteq G_{V_* V U_*}$ with the associated family $\mathfrak{G}_{V'}$ of branches. From Theorem 4.18 it follows that (V'_*, V', U'_*) is $(f+1)$-closed and $|V'_*|, |V'|, |U'_*| \leq |\operatorname{Im}\varphi|^{\aleph_0} \leq |G_1|^{\aleph_0} = \lambda$. In particular $|\operatorname{orco} V'_*| \leq \lambda = \lambda_f$, and we can find $\Delta \subseteq \lambda_{f+1} \setminus \operatorname{orco} V'_*$ with $|\Delta| = \lambda$.

Until now we have used sequences $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ (as in Section 2.1) based on cardinals λ_ℓ (which are ordinals and hence particular sets). In order to have room for the construction of A-modules, we must now pass to sets of ordinals. Extending Section 2.1 we define a sequence $\overline{\lambda'} = \langle \lambda'_1, \ldots, \lambda'_k \rangle$ of sets of ordinals by

$$\lambda'_\ell = \begin{cases} \lambda_\ell & \text{if } 1 \leq \ell \leq f, \\ \Delta \mathbin{\dot\cup} \operatorname{orco} V'_* & \text{if } f < \ell \leq k. \end{cases}$$

Similarly to the old definition for Λ we now set $\Lambda' = {}^{\omega}\lambda'_1 \times \cdots \times {}^{\omega}\lambda'_k$ and $\Lambda'_m = {}^{\omega}\lambda'_1 \times \cdots \times {}^{\omega>}\lambda'_m \times \cdots \times {}^{\omega}\lambda'_k$ for any $1 \leq m \leq k$. In contrast to the definition of Λ we do not utilize the ordering on λ'_ℓ (as a set of ordinals). Again put $\Lambda'_* = \dot{\bigcup}_{1 \leq m \leq k} \Lambda'_m$. Now we are ready to define a relatively small A-module $\mathbb{V}$ into which we send interesting submodules by shift isomorphisms for their predictions. Let $\mathbb{V} = \widehat{\bigoplus_{\overline{\nu} \in \Lambda'_*} A e'_{\overline{\nu}}}$, which is the $\mathbb{S}$-adic completion of the free A-module $\bigoplus_{\overline{\nu} \in \Lambda'_*} A e'_{\overline{\nu}}$, thus a canonical $\widehat{A}$-module. Moreover, let $\mathcal{H} = \{H \subseteq \mathbb{V} \mid H \text{ is an } A\text{-submodule}, |H| \leq \theta\}$. The cardinalities of these new structures are immediate due to Section 2.1(iii). We have

$$|\Lambda'| = |\Lambda'_*| = \lambda^{\aleph_0} = \lambda, \qquad |\mathbb{V}| = \lambda^{\aleph_0} = \lambda, \qquad |\mathcal{H}| = \lambda^\theta = \lambda_f^{\lambda_{f-1}} = \lambda.$$

Now we can also give the exact definition of a trap. This notion comes from [4]; it is designed to 'catch' small unwanted homomorphisms and is derived from particular elementary submodels.

Definition 6.8. A tuple $(G, H, P, Q, \mathcal{R}, \psi)$ is a *trap* (for the step lemma) if $G \in \mathcal{G}$, $H \in \mathcal{H}$, $\psi : G \to H$ is an R-homomorphism, $P \subseteq \Lambda'_*$, $Q \subseteq \Lambda'$ and $\mathcal{R} \subseteq \mathbb{V}$ are subsets such that $|P|, |Q|, |\mathcal{R}| \leq \theta$. Let Θ be the family of all traps $(G, H, P, Q, \mathcal{R}, \psi)$.

Next we must determine the size of Θ, which is clearly $|\Theta| = |\mathcal{G}| \cdot |\mathcal{H}| \cdot |\Lambda'_*|^\theta \cdot |\Lambda'|^\theta \cdot |\mathbb{V}|^\theta \cdot \theta^\theta = \lambda \cdot \lambda^\theta \cdot \theta^\theta = \lambda$. Thus we can consider the easy black box stated as Proposition 6.1, but with the new crucial family Θ of traps:

The Easy Black Box 6.9. *There is a family $\langle g_\eta \mid \eta \in {}^{\omega\uparrow}\lambda \rangle$ with $g_\eta : [\eta] \to \Theta$ such that for each map $g : {}^{\omega\uparrow>}\lambda \to \Theta$ there exists some $\eta \in {}^{\omega\uparrow}\lambda$ with $g_\eta \subseteq g$.*

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The construction of G_2

First we would like to indicate our strategy: In order to construct the desired $\aleph_n$-free A-module G with $\operatorname{End} G = A$ we must find particular generators of G which will be branch-like elements involving a summand with a ring element $\pi \in \widehat{R}$ as factor which will prevent unwanted endomorphisms. The A-module G_2 is (a weak form of) an elementary submodel of G; thus it is not surprising that we must determine these factors first for G_2.

For $\alpha < \lambda$ and $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$ as in (i) of the Step Lemma 6.6 let $\overline{\xi}_\alpha \in {}^{\omega\uparrow}\lambda_f \times \cdots \times {}^{\omega\uparrow}\lambda_k$ be defined by $(\overline{\xi}_\alpha)_f = \eta_\alpha \in \Gamma$ and $(\overline{\xi}_\alpha)\restriction(f,k] = \overline{\xi}$.

Next we will choose recursively the elements $\pi_{\overline{\eta}} \in \widehat{R}$ for $\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$ and for each $\alpha < \lambda$. Since $J = \Lambda^{\overline{\xi}} = \dot{\bigcup}_{\alpha<\lambda} \Lambda^{\overline{\xi}_\alpha}$ (by the definition of Γ), in the end we will have constructed a family of ring elements $\pi_{\overline{\eta}}$ ($\overline{\eta} \in J$) from $\widehat{R}$ as needed for the triple (J_*, J, I_*) from above. Hence G_2 will be determined by $G_2 = G_{J_*J}$ and $\mathfrak{H} = \{x_{\overline{\eta}} = \pi_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in J\}$.

Let $\alpha < \lambda$ and $(G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, \mathcal{R}_{\alpha n}, \psi_{\alpha n}) := g_{\eta_\alpha}(\eta_\alpha\restriction n) \in \Theta$ be the traps given by the Easy Black Box 6.9. A special choice of $\pi_{\overline{\eta}}$ for $\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$ is only needed in particular situations of these traps, namely when they represent the local version of an unwanted endomorphism of G, and fortunately this will only be the case when we get support from the results of the last section. Otherwise we may put $\pi_{\overline{\eta}} = 0$.

Next we specify these conditions when $\pi_{\overline{\eta}} \in \widehat{R}$ must (seriously) be chosen (for killing maps):

We must work, i.e. do some book-keeping by using the results from Sections 4 and 5, if there are $(f+1)$-closed triples $(Z_*^\dagger, Z^\dagger, Y_*^\dagger)$, $(Y_*^\dagger, Y^\dagger, X_*^\dagger)$ with $Y^\dagger = Z^\dagger_{Y_*^\dagger}$ and there is an associated family $\mathfrak{F}^\dagger = \{y'^\dagger_{\overline{\eta}} = \rho^\dagger_{\overline{\eta}} b^\dagger_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z^\dagger\}$ of branch-like elements which is $(Z_*^\dagger, Z^\dagger, Y_*^\dagger)$-suitable, $\mathfrak{F}^\dagger_{Y^\dagger}$ is $(Y_*^\dagger, Y^\dagger, X_*^\dagger)$-suitable, and if there are

$$\begin{aligned}&\Omega_*^{1\dagger} \subseteq Y_*^\dagger, \quad \Omega_*^{2n\dagger} \subseteq Z_*^\dagger, \quad Y'^\dagger \subseteq Y^\dagger,\\ &Z_n'^\dagger \subseteq Z^\dagger (n < \omega) \text{ and } \tau^\dagger \text{ a } V_*\text{-admissible injective map,}\end{aligned}$$

and in addition there is a shift homomorphism $\sigma^\dagger$ with the following properties:

(A)† $\Omega_*^{1\dagger}$ is $\mathfrak{F}^\dagger_{Y^\dagger}$-closed and $\Omega_*^{2n\dagger}$ is $\mathfrak{F}^\dagger$-closed for $n < \omega$.
(B)† $PC(\Omega_*^{2n\dagger}, Z_*^\dagger) \subseteq \Omega_*^{2,n+1\dagger}$.
(C)† $(\Omega_*^{1\dagger} \cup \Omega_*^{2n\dagger}, Z_n'^\dagger, \Omega_*^{1\dagger})$ and $(\Omega_*^{1\dagger}, Y'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger})$ are $(f+1)$-closed with $Y'^\dagger = (Z_n'^\dagger)_{\Omega_*^{1\dagger}}$ for all $n < \omega$.
(D)† $\mathfrak{F}^\dagger_{Z_n'^\dagger}$ is $(\Omega_*^{1\dagger} \cup \Omega_*^{2n\dagger}, Z_n'^\dagger, \Omega_*^{1\dagger})$-suitable and $\mathfrak{F}^\dagger_{Y'^\dagger}$ is $(\Omega_*^{1\dagger}, Y'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger})$-suitable.
(E)† $G_{\Omega_*^{1\dagger} \cup \Omega_*^{2n\dagger}, Z_n'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}} \subseteq G_{Z_*{}^\dagger Z^\dagger X_*^\dagger}$ and $G_{\Omega_*^{1\dagger}, Y'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}} \subseteq G_{Y_*^\dagger Y^\dagger X_*^\dagger}$.
(F)† $(Y'^\dagger)_{X_*^\dagger} = (Z_n'^\dagger)_{X_*^\dagger} = \emptyset$.
(G)† $(Z^\dagger)_{\Omega_*^{2n\dagger}} \setminus (Z^\dagger)_{X_*^\dagger} \subseteq Z_n'^\dagger \subseteq Z^\dagger \setminus (Z^\dagger)_{X_*^\dagger}$.
(H)† $(V_*, V, U_*)\tau^\dagger = (Y_*^\dagger, Y^\dagger, X_*^\dagger)$ and $\mathfrak{G}\tau^\dagger = \mathfrak{F}^\dagger_{Y^\dagger}$.
(I)† $(\Omega_*^1, V', U_* \cap \Omega_*^1)\tau^\dagger = (\Omega_*^{1\dagger}, Y'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger})$ and $\mathfrak{G}_{V'}\tau^\dagger = \mathfrak{F}^\dagger_{Y'^\dagger}$.
(J)† $\sigma^\dagger : \operatorname{orco}(\Omega_*^{1\dagger} \cup \bigcup_{n<\omega} \Omega_*^{2n\dagger}) \to \Delta \cup \operatorname{orco} \Omega_*^1$ is injective.

(K)† $\sigma^\dagger \restriction \operatorname{orco} \Omega_*^{1\dagger} = (\tau^\dagger)^{-1} \restriction \operatorname{orco} \Omega_*^{1\dagger}$.
(L)† $P_{\alpha n} = \Omega_*^{2n\dagger} \sigma^\dagger$.
(M)† $Q_{\alpha n} = (Z_n'^\dagger \setminus Y'^\dagger)\sigma^\dagger$ and $\mathcal{R} = \mathfrak{F}^\dagger_{Z_n'^\dagger \setminus Y'^\dagger} \sigma^\dagger$.
(N)† $H_{\alpha n} \subseteq (G_{\Omega_*^{1\dagger} \cup \Omega_*^{2n\dagger}, Z_n'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}})\sigma^\dagger$.
(O)† $G_{\alpha n} = G_{1\eta_\alpha \restriction n}$.
(P)† The maps $\psi_{\alpha n} : G_{\alpha n} \to H_{\alpha n}$ $(n < \omega)$ extend each other, so that

$$\psi_\alpha = \bigcup_{n<\omega} \psi_{\alpha n} \text{ and } G_\alpha = G_{1\eta_\alpha} = \bigcup_{n<\omega} G_{1\eta_\alpha \restriction n} = \bigcup_{n<\omega} G_{\alpha n} \text{ are well-defined.} \tag{6.3}$$

Due to (6.3), (N)† and (P)† the map

$$\psi_\alpha (\sigma^\dagger)^{-1} : G_\alpha \to \sum_{n<\omega} G_{\Omega_*^{1\dagger} \cup \Omega_*^{2n\dagger}, Z_n'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}}$$

is also a well-defined homomorphism.

Next as in Theorem 5.3 we define unions of the above sets:

(ii)† $Y_*''^\dagger = \Omega_*^{1\dagger} \cup \bigcup_{n<\omega} \Omega_*^{2n\dagger}$.
(iii)† $X_*''^\dagger = X_*^\dagger \cap \Omega_*^{1\dagger}$.
(v)† $Y''^\dagger = Y'^\dagger \cup \bigcup_{n<\omega} Z_n'^\dagger$.

Hence $(Y_*''^\dagger, Y''^\dagger, X_*''^\dagger)$ is $(f+1)$-closed and in particular f-closed. Moreover, we have the map

(vi)† $\varphi\tau^\dagger : G_1(\overline{\xi}) \to G_{\Omega_*^{1\dagger}, Y'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}}$,

which is well-defined by the definition of (V_*', V', U_*') and (I)†. From $G_1(\overline{\xi}_\alpha) = G_1(\overline{\xi}) \oplus G_\alpha$ it follows that

(vii)† $\varphi^\dagger = \varphi\tau^\dagger \oplus \psi_\alpha (\sigma^\dagger)^{-1}$

is also a well-defined homomorphism

$$\varphi^\dagger : G_1(\overline{\xi}_\alpha) \to G_{\Omega_*^{1\dagger}, Y'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}} + \sum_{n<\omega} G_{\Omega_*^{1\dagger} \cup \Omega_*^{2n\dagger}, Z_n'^\dagger, X_*^\dagger \cap \Omega_*^{1\dagger}} = G_{Y_*''^\dagger Y''^\dagger X_*''^\dagger} \tag{6.4}$$

satisfying $z\varphi^\dagger = z\varphi\tau^\dagger \neq 0$. We now apply the induction hypothesis of the step lemma. Replace f, E_*, $G_1(\overline{\xi})$, $G_{V_* V U_*}$, φ, z respectively by

$$f-1,\ E_*,\ G_1(\overline{\xi}_\alpha),\ G_{Y_*''^\dagger, Y''^\dagger, X_*''^\dagger},\ \varphi^\dagger,\ z. \tag{6.5}$$

The Step Lemma 6.6 holds for $f-1$, and the existence of elements $\pi_{\overline{\eta}} \in \widehat{R}$ $(\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha})$ follows. Recall that now all of $\{\pi_{\overline{\eta}} \mid \overline{\eta} \in J\}$ and $\mathfrak{H} = \{x_{\overline{\eta}} = \pi_{\overline{\eta}} z_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in J\}$ are known. This finishes the construction of G_2.

G_2 satisfies the Step Lemma 6.6 for f.

We finally must show that the family $\mathfrak{H} = \{x_{\overline{\eta}} = \pi_{\overline{\eta}} z_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in J\}$ and thus G_2 is as required in the Step Lemma 6.6. We will prove this by contradiction.

Suppose that $(Z^\ddagger_*, Z^\ddagger, Y^\ddagger_*)$ and $(Y^\ddagger_*, Y^\ddagger, X^\ddagger_*)$ are $(f+1)$-closed triples with $Y^\ddagger = Z^\ddagger_{Y^\ddagger_*}$, $\mathfrak{F}^\ddagger = \{y'^\ddagger_{\overline{\eta}} = \rho^\ddagger_{\overline{\eta}} b^\ddagger_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z^\ddagger\}$ is $(Z^\ddagger_*, Z^\ddagger, Y^\ddagger_*)$-suitable, $\mathfrak{F}^\ddagger_{Y^\ddagger}$ is $(Y^\ddagger_*, Y^\ddagger, X^\ddagger_*)$-suitable and $\tau^\ddagger$ is a V_*-admissible bijection with

(H)‡ $(V_*, V, U_*)\tau^\ddagger = (Y^\ddagger_*, Y^\ddagger, X^\ddagger_*)$ and $\mathfrak{G}\tau^\ddagger = \mathfrak{F}^\ddagger_{Y^\ddagger}$,

but fails to satisfy the conclusion of the lemma. Thus the homomorphism

$$\varphi\tau^\ddagger : G_1 \to G_3 = G_{Y^\ddagger_* Y^\ddagger X^\ddagger_*} \quad \text{lifts to} \quad \psi^\ddagger : G_2 \to G_4 = G_{Z^\ddagger_* Z^\ddagger X^\ddagger_*} \tag{6.6}$$

(see the next diagram).

We now apply Theorem 4.18 to $(Z^\ddagger_*, Z^\ddagger, Y^\ddagger_*)$, $(Y^\ddagger_*, Y^\ddagger, X^\ddagger_*)$, $\operatorname{Im}\varphi\tau^\ddagger \subseteq G_3$ and $\operatorname{Im}\psi^\ddagger \subseteq G_4$ and get $(\Omega^{1\ddagger}_* \cup \Omega^{2\ddagger}_*, Z'^\ddagger, \Omega^{1\ddagger}_*)$ and $(\Omega^{1\ddagger}_*, Y'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*)$ as in (4.2). In particular we have:

(A)‡ $\Omega^{1\ddagger}_* \subseteq Y^\ddagger_*$ is $\mathfrak{F}^\ddagger_{Y^\ddagger}$-closed and $\Omega^{2\ddagger}_* \subseteq Z^\ddagger_*$ is $\mathfrak{F}^\ddagger$-closed, $Y'^\ddagger \subseteq Y^\ddagger$ and $Z'^\ddagger \subseteq Z^\ddagger$.
(C)‡ $(\Omega^{1\ddagger}_* \cup \Omega^{2\ddagger}_*, Z'^\ddagger, \Omega^{1\ddagger}_*)$ and $(\Omega^{1\ddagger}_*, Y'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*)$ are $(f+1)$-closed with $Y'^\ddagger = (Z'^\ddagger)_{\Omega^{1\ddagger}_*}$.
(D)‡ $\mathfrak{F}^\ddagger_{Z'^\ddagger}$ is $(\Omega^{1\ddagger}_* \cup \Omega^{2\ddagger}_*, Z'^\ddagger, \Omega^{1\ddagger}_*)$-suitable and $\mathfrak{F}^\ddagger_{Y'^\ddagger}$ is $(\Omega^{1\ddagger}_*, Y'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*)$-suitable.
(E)‡ $\operatorname{Im}\psi^\ddagger \subseteq G_{\Omega^{1\ddagger}_* \cup \Omega^{2\ddagger}_*, Z'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*} \subseteq G_{Z_*{}^\ddagger Z^\ddagger X^\ddagger_*}$ and $\operatorname{Im}\varphi\tau^\ddagger \subseteq G_{\Omega^{1\ddagger}_*, Y'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*} \subseteq G_{Y_*{}^\ddagger Y^\ddagger X^\ddagger_*}$.
(F)‡ $Y'^\ddagger{}_{X^\ddagger_*} = Z'^\ddagger{}_{X^\ddagger_*} = \emptyset$.
(G)‡ $Z^\ddagger{}_{\Omega^{2\ddagger}_*} \setminus Z^\ddagger{}_{X^\ddagger_*} \subseteq Z'^\ddagger \subseteq Z^\ddagger \setminus Z^\ddagger{}_{X^\ddagger_*}$. (Compare Remark 4.19.)
(I)‡ $(\Omega^1_*, V', U_* \cap \Omega^1_*)\tau^\ddagger = (\Omega^{1\ddagger}_*, Y'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*)$ and $\mathfrak{G}_{V'}\tau^\ddagger = \mathfrak{F}^\ddagger_{Y'^\ddagger}$. (Compare Observation 6.5.)
(Q)‡ $|\Omega^{1\ddagger}_*|, |\Omega^{2\ddagger}_*|, |Z'^\ddagger|, |Y'^\ddagger| \le \lambda$ (because $|G_1| = |G_2| = \lambda$).
(R)‡ $Y'^\ddagger$ and $\Omega^{1\ddagger}_*$ are uniquely determined by $Y^\ddagger, Y^\ddagger_*, X^\ddagger_*, \mathfrak{F}^\ddagger_{Y^\ddagger}$ and $\operatorname{Im}\varphi\tau^\ddagger$.

Next we choose an injection $\sigma^\ddagger$ with

(J)‡ $\sigma^\ddagger : \operatorname{orco}(\Omega^{1\ddagger}_* \cup \Omega^{2\ddagger}_*) \to \Delta \cup \operatorname{orco}\Omega^1_*$ such that
(K)‡ $\sigma^\ddagger\restriction \operatorname{orco}\Omega^{1\ddagger}_* = (\tau^\ddagger)^{-1}\restriction \operatorname{orco}\Omega^{1\ddagger}_*$.

This is possible, because $|\Omega^{1\ddagger}_*|, |\Omega^{2\ddagger}_*| \le \lambda = |\Delta|$. Also note $\Omega^1_*\tau^\ddagger = \Omega^{1\ddagger}_*$ by (I)‡.

Let us pause for a moment and describe the present situation of maps by a diagram. Recall that G is defined by $G = G_{V_* VU}$ together with $\mathfrak{G}$, G_2 comes from $G_{J_* J}$ with $\mathfrak{H}$, and $G_1 = B_{I_*}$ is a free A-module. Moreover, $G_3 = G_{Y^\ddagger_* Y^\ddagger X^\ddagger_*}$ and $G_4 = G_{Z^\ddagger_* Z^\ddagger X^\ddagger_*}$ above come with $\mathfrak{F}^\ddagger_{Y^\ddagger}$ and $\mathfrak{F}^\ddagger$, respectively. Naturally, we let $G' = G_{\Omega^1_*, V', U_* \cap \Omega^1_*}$, $G^\ddagger_3 = G_{\Omega^{1\ddagger}_*, Y'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*}$ and $G^\ddagger_4 = G_{\Omega^{1\ddagger}_* \cup \Omega^{2\ddagger}_*, Z'^\ddagger, X^\ddagger_* \cap \Omega^{1\ddagger}_*}$. Thus we have the following diagram (where arrows with no name are again inclusions):

$$\begin{tikzcd}
G_2 \arrow[rr, "\psi^\ddagger"] & & G_4^\ddagger \arrow[rr] & & G_4 \\
& G' \arrow[rr] \arrow[dr, shift left, "\tau^\ddagger"] & & G \arrow[dr, "\tau^\ddagger"] & \\
G_1 \arrow[uu] \arrow[ur] \arrow[rr, "\varphi\tau^\ddagger"] & & G_3^\ddagger \arrow[uu] \arrow[ul, shift left, "\sigma^\ddagger"] \arrow[rr] & & G_3 \arrow[uu]
\end{tikzcd}$$

We want to construct a function $g : {}^{\omega\uparrow>}\lambda \to \Theta$ for the use of Proposition 6.1. For this choose any $\nu \in {}^{\omega\uparrow>}\lambda$. The definitions show that $G_{1\nu} \subseteq G_2$, and $\psi^\ddagger \restriction G_{1\nu}$ is a well-defined homomorphism. Similarly to the first '‡-step' we now continue with a '${}^{\nu\ddagger}$-step'.

Using Theorem 4.18 let $(\Omega^{1\nu\ddagger}_* \cup \Omega^{2\nu\ddagger}_*, Z'^{\nu\ddagger}, \Omega^{1\nu\ddagger}_*)$ and $(\Omega^{1\nu\ddagger}_*, Y'^{\nu\ddagger}, X^{\nu\ddagger}_* \cap \Omega^{1\nu\ddagger}_*)$ be determined by the triples $(Z^\ddagger_*, Z^\ddagger, Y^\ddagger_*)$, $(Y^\ddagger_*, Y^\ddagger, X^\ddagger_*)$, $\operatorname{Im}\varphi\tau^\ddagger \subseteq G_3$ and $G_{1\nu}\psi^\ddagger \subseteq G_4$; compare also (4.2). In particular we have:

(A)$^{\nu\ddagger}$ $\Omega^{1\nu\ddagger}_* \subseteq Y^\ddagger_*$ is $\mathfrak{F}^\ddagger_{Y^\ddagger}$-closed and $\Omega^{2\nu\ddagger}_* \subseteq Z^\ddagger_*$ is $\mathfrak{F}^\ddagger$-closed, $Y'^{\nu\ddagger} \subseteq Y^\ddagger$ and $Z'^{\nu\ddagger} \subseteq Z^\ddagger$.

(C)$^{\nu\ddagger}$ $(\Omega^{1\nu\ddagger}_* \cup \Omega^{2\nu\ddagger}_*, Z'^{\nu\ddagger}, \Omega^{1\nu\ddagger}_*)$ and $(\Omega^{1\nu\ddagger}_*, Y'^{\nu\ddagger}, X^\ddagger_* \cap \Omega^{1\nu\ddagger}_*)$ are $(f+1)$-closed with $Y'^{\nu\ddagger} = (Z'^{\nu\ddagger})_{\Omega^{1\nu\ddagger}_*}$.

(D)$^{\nu\ddagger}$ $\mathfrak{F}^\ddagger_{Z'^{\nu\ddagger}}$ is $(\Omega^{1\nu\ddagger}_* \cup \Omega^{2\nu\ddagger}_*, Z'^{\nu\ddagger}, \Omega^{1\nu\ddagger}_*)$-suitable, and $\mathfrak{F}^\ddagger_{Y'^{\nu\ddagger}}$ is $(\Omega^{1\nu\ddagger}_*, Y'^{\nu\ddagger}, X^\ddagger_* \cap \Omega^{1\nu\ddagger}_*)$-suitable.

(E)$^{\nu\ddagger}$ $G_{1\nu}\psi^\ddagger \subseteq G_{\Omega^{1\nu\ddagger}_* \cup \Omega^{2\nu\ddagger}_*, Z'^{\nu\ddagger}, X^\ddagger_* \cap \Omega^{1\nu\ddagger}_*} \subseteq G_{Z^\ddagger_* Z^\ddagger X^\ddagger_*}$, $\operatorname{Im}\varphi\tau^\ddagger \subseteq G_{\Omega^{1\nu\ddagger}_*, Y'^{\nu\ddagger}, X^\ddagger_* \cap \Omega^{1\nu\ddagger}_*} \subseteq G_{Y_*{}^\ddagger Y^\ddagger X^\ddagger_*}$.

(F)$^{\nu\ddagger}$ $Y'^{\nu\ddagger}{}_{X^\ddagger_*} = Z'^{\nu\ddagger}{}_{X^\ddagger_*} = \emptyset$.

(G)$^{\nu\ddagger}$ $Z^\ddagger{}_{\Omega^{2\nu\ddagger}_*} \setminus Z^\ddagger{}_{X^\ddagger_*} \subseteq Z'^{\nu\ddagger} \subseteq Z^\ddagger \setminus Z^\ddagger{}_{X^\ddagger_*}$. (Compare again Remark 4.19.)

(I)$^{\nu\ddagger}$ $(\Omega^1_*, V', U_* \cap \Omega^1_*)\tau^\ddagger = (\Omega^{1\nu\ddagger}_*, Y'^{\nu\ddagger}, X^\ddagger_* \cap \Omega^{1\nu\ddagger}_*)$ and $\mathfrak{G}_{V'}\tau^\ddagger = \mathfrak{F}^\ddagger_{Y'^{\nu\ddagger}}$. (Compare also Observation 6.5.)

(Q)$^{\nu\ddagger}$ $|\Omega^{1\nu\ddagger}_*|, |Y'^{\nu\ddagger}| \le \lambda$ and $|\Omega^{2\nu\ddagger}_*|, |Z'^{\nu\ddagger} \setminus Y'^{\nu\ddagger}| \le \theta$ (because $|G_{1\nu}| = \theta$).

(R)$^{\nu\ddagger}$ $Y'^{\nu\ddagger}$ and $\Omega^{1\nu\ddagger}_*$ are uniquely determined by $Y^\ddagger$, $Y^\ddagger_*$, $X^\ddagger_*$, $\mathfrak{F}^\ddagger_{Y^\ddagger}$ and $\operatorname{Im}\varphi\tau^\ddagger$. In particular, $Y'^{\nu\ddagger} = Y'^\ddagger$, $\Omega^{1\nu\ddagger}_* = \Omega^{1\ddagger}_*$ for all $\nu \in {}^{\omega\uparrow>}\lambda$. (Compare (R)‡.)

(S)$^{\nu\ddagger}$ $Z'^{\nu\ddagger} \subseteq Z'^\ddagger$, $\Omega^{2\nu\ddagger}_* \subseteq \Omega^{2\ddagger}_*$ for all $\nu \in {}^{\omega\uparrow>}\lambda$. (This follows from Remark 4.19.)

(B)$^{\nu\ddagger}$ $PC(\Omega_*^{2,\nu\restriction((\lg\nu)-1),\ddagger}, Z^\ddagger_*) \subseteq \Omega_*{}^{2\nu\ddagger}$ can be ensured by a recursive construction of $Z'^{\nu\ddagger}$, $\Omega^{2\nu\ddagger}_*$ along the length $\lg\nu$.

Now we describe the refinement of the last diagram by the last application of Theorem 4.18. Naturally we put

$$G_4^{\nu\ddagger} = G_{\Omega^{1\nu\ddagger}_* \cup \Omega^{2\nu\ddagger}_*, Z'^{\nu\ddagger}, X^\ddagger_* \cap \Omega^{1\nu\ddagger}_*} \subseteq G_4^\ddagger$$

and get the following diagram with the free A-modules

$$G_1 = B_{I_*} \quad \text{and} \quad G_{1\nu} = B_{\Lambda_*^{\nu\bar{\xi}}}$$

from above, where $\psi^\nu = (\psi^\ddagger \restriction G_{1\nu})\sigma^\ddagger$, otherwise restrictions of homomorphisms have the same name, and inclusions have no name.

$$\begin{tikzcd}
 & \mathbb{V} & & & \\
G_2 \arrow[ur, "\psi^\ddagger\sigma^\ddagger"] \arrow[rr, "\psi^\ddagger"] & & G_4^\ddagger \arrow[ul, "\sigma^\ddagger"'] \arrow[rr] & & G_4 \\
 & G' + G_{1\nu}\psi^\ddagger\sigma^\ddagger \arrow[uu] & & & \\
G_1 \oplus G_{1\nu} \arrow[uu] \arrow[ur, "\varphi\oplus\psi^\nu"] \arrow[rr, "\psi^\ddagger"] & & G_3^\ddagger + G_4^{\nu\ddagger} \arrow[ul, "\sigma^\ddagger"'] \arrow[uu] \arrow[rr] & & G_4 \arrow[uu] \\
 & G' \arrow[uu] \arrow[rr] \arrow[dr, shift left, "\tau^\ddagger"] & & G \arrow[dr, "\tau^\ddagger"] & \\
G_1 \arrow[uu] \arrow[ur, "\varphi"] \arrow[rr, "\varphi\tau^\ddagger"] & & G_3^\ddagger \arrow[ul, shift left, "\sigma^\ddagger"] \arrow[uu] \arrow[rr] & & G_3 \arrow[uu]
\end{tikzcd}$$

We now define the map $g : {}^{\omega\uparrow>}\lambda \to \Theta$ which we want to predict by

$$g(\nu) = (G^\nu, H^\nu, P^\nu, Q^\nu, \mathcal{R}^\nu, \psi^\nu)$$

and the following requirements:

$(\mathrm{L})^{\nu\ddagger}$ $P^\nu = \Omega_*^{2\nu\ddagger}\sigma^\ddagger$.
$(\mathrm{M})^{\nu\ddagger}$ $Q^\nu = (Z'^{\nu\ddagger} \setminus Y'^{\nu\ddagger})\sigma^\ddagger$ and $\mathcal{R}^\nu = \mathfrak{F}^\ddagger_{Z'^{\nu\ddagger}\setminus Y'^{\nu\ddagger}}\sigma^\ddagger$.
$(\mathrm{O})^{\nu\ddagger}$ $G^\nu = G_{1\nu}$.
$(\mathrm{T})^{\nu\ddagger}$ $H^\nu = G_{1\nu}\psi^\ddagger\sigma^\ddagger$.
$(\mathrm{U})^{\nu\ddagger}$ $\psi^\nu = (\psi^\ddagger\restriction G_{1\nu})\sigma^\ddagger : G^\nu \to H^\nu$.

From $(\mathrm{Q})^{\nu\ddagger}$ it follows that $|P^\nu|, |Q^\nu|, |\mathcal{R}^\nu| \le \theta$, and also $G^\nu \in \mathcal{G}, H^\nu \in \mathcal{H}$ and $P^\nu \subseteq \Lambda'_*$, $Q^\nu \subseteq \Lambda'$, $\mathcal{R}^\nu \subseteq \mathbb{V}$, and consequently $(G^\nu, H^\nu, P^\nu, Q^\nu, \mathcal{R}^\nu, \psi^\nu) \in \Theta$. The domain $\mathrm{orco}(\Omega_*^{1\ddagger} \cup \Omega_*^{2\ddagger})$ of $\sigma^\ddagger$ is 'large enough', in particular $\Omega_*^{2\nu\ddagger} \subseteq \Omega_*^{2\ddagger}$ due to $(\mathrm{S})^{\nu\ddagger}$, and following $(\mathrm{E})^{\nu\ddagger}$ we have

$(\mathrm{N})^{\nu\ddagger}$ $H^\nu \subseteq (G_{\Omega_*^{1\ddagger}\cup\Omega_*^{2\nu\ddagger}, Z'^{\nu\ddagger}, X_*^\ddagger\cap\Omega_*^{1\ddagger}})\sigma^\ddagger$.

Finally, the definition of ψ^ν in $(\mathrm{U})^{\nu\ddagger}$ yields

$(\mathrm{P})^{\nu\ddagger}$ $\psi^{\nu\restriction((\lg \nu)-1)} \subseteq \psi^\nu$.

Thus we can apply the Easy Black Box 6.9 and we find some $\eta \in \Gamma$ with $g_\eta \subseteq g$. There is some $\alpha < \lambda$ such that $\eta = \eta_\alpha$. Now for the construction of $\pi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha})$ the 'serious' case applies as it is witnessed by the choice of

$$(Z_*^{\ddagger}, Z^{\ddagger}, Y_*^{\ddagger}), (Y_*^{\ddagger}, Y^{\ddagger}, X_*^{\ddagger}), \mathfrak{F}^{\ddagger} \quad \text{and}$$
$$\Omega_*^{1\ddagger}, \Omega_*^{2n\ddagger} := \Omega_*^{2,\eta_\alpha \restriction n,\ddagger}, Y'^{\ddagger}, Z_n'^{\ddagger} := Z'^{\eta_\alpha \restriction n,\ddagger}, \tau^{\ddagger}, \sigma^{\ddagger}$$

as possible candidates for

$$(Z_*^{\dagger}, Z^{\dagger}, Y_*^{\dagger}), (Y_*^{\dagger}, Y^{\dagger}, X_*^{\dagger}), \mathfrak{F}^{\dagger} \quad \text{and} \quad \Omega_*^{1\dagger}, \Omega_*^{2n\dagger}, Y'^{\dagger}, Z_n'^{\dagger}, \tau^{\dagger}, \sigma^{\dagger}.$$

The necessary conditions $(\mathrm{A})^{\dagger}$ to $(\mathrm{P})^{\dagger}$ are satisfied by $(\mathrm{A})^{\ddagger}$ to $(\mathrm{P})^{\ddagger}$ and $(\mathrm{A})^{\nu\ddagger}$ to $(P)^{\nu\ddagger}$.

The concluding arguments of this proof are visualized in the following diagram: Similarly to the construction of the ring elements $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$) we define (as in Theorem 5.3)

(i)‡ $Z_*''^{\ddagger} = \Omega_*^{1\ddagger} \cup \Omega_*^{2\ddagger} \cup \bigcup_{n<\omega} \Omega_*^{2n\ddagger} = \Omega_*^{1\ddagger} \cup \Omega_*^{2\ddagger}$,
(ii)‡ $Y_*''^{\ddagger} = \Omega_*^{1\ddagger} \cup \bigcup_{n<\omega} \Omega_*^{2n\ddagger}$,
(iii)‡ $X_*''^{\ddagger} = X_*^{\ddagger} \cap \Omega_*^{1\ddagger}$,
(iv)‡ $Z''^{\ddagger} = Z'^{\ddagger} \cup \bigcup_{n<\omega} Z_n'^{\ddagger}$,
(v)‡ $Y''^{\ddagger} = Y'^{\ddagger} \cup \bigcup_{n<\omega} Z_n'^{\ddagger}$.

Hence $(Y_*''^{\ddagger}, Y''^{\ddagger}, X_*''^{\ddagger})$ is $(f+1)$-closed, and $(Z_*''^{\ddagger}, Z''^{\ddagger}, Y_*''^{\ddagger})$ is (only!) f-closed. As in the construction of the ring elements $\pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}_\alpha}$), we have

$$(G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, \mathcal{R}_{\alpha n}, \psi_{\alpha n}) := g_{\eta_\alpha \restriction n} \quad \text{and} \quad \psi_\alpha = \bigcup_{n<\omega} \psi_{\alpha n},$$

and let

(vii)‡ $\varphi^{\ddagger} = \varphi\tau^{\ddagger} \oplus \psi_\alpha(\sigma^{\ddagger})^{-1}$.

This is again a well-defined homomorphism

$$\varphi^{\ddagger} : G_1(\overline{\xi}_\alpha) \to G_{Y_*''^{\ddagger} Y''^{\ddagger} X_*''^{\ddagger}}.$$

By the prediction of the Easy Black Box 6.9 we get the following identities:

(L) $\Omega_*^{2n\dagger}\sigma^{\dagger} = P_{\alpha n} = \Omega_*^{2n\ddagger}\sigma^{\ddagger}$.
(M) $(Z_n'^{\dagger} \setminus Y'^{\dagger})\sigma^{\dagger} = Q_{\alpha n} = (Z_n'^{\ddagger} \setminus Y'^{\ddagger})\sigma^{\ddagger}$ and $\mathfrak{F}^{\dagger}_{Z_n'^{\dagger} \setminus Y'^{\dagger}}\sigma^{\dagger} = \mathcal{R}_{\alpha n} = \mathfrak{F}^{\ddagger}_{Z_n'^{\ddagger} \setminus Y'^{\ddagger}}\sigma^{\ddagger}$.
(O) $G_{\alpha n} = G_{1\eta_\alpha \restriction n} \subseteq G_{1\eta_\alpha} = G_\alpha$ (see (6.3)).
(U) $\psi_{\alpha n} = (\psi^{\ddagger} \restriction G_{\alpha n})\sigma^{\ddagger}$ and $\psi_\alpha = (\psi^{\ddagger} \restriction G_\alpha)\sigma^{\ddagger}$.

Now we consider the bijection $\sigma := \sigma^{\dagger}(\sigma^{\ddagger})^{-1}$ of ordinals. By definition $\Omega_*^{2n\dagger}\sigma = \Omega_*^{2n\ddagger}$, and $(\mathrm{I})^{\dagger}$, $(\mathrm{I})^{\ddagger}$, $(\mathrm{K})^{\dagger}$, $(\mathrm{K})^{\ddagger}$ yield

$$\Omega_*^{1\dagger}\sigma = (\Omega_*^{1\dagger}\sigma^{\dagger})(\sigma^{\ddagger})^{-1} = (\Omega_*^{1\dagger}(\tau^{\dagger})^{-1})(\sigma^{\ddagger})^{-1} = \Omega_*^{1}(\sigma^{\ddagger})^{-1} = \Omega_*^{1\ddagger}.$$

Since $Y_*''^{\dagger} = \Omega_*^{1\dagger} \cup \bigcup_{n<\omega} \Omega_*^{2n\dagger}$, from these definitions it follows that σ is a $Y_*''^{\dagger}$-admissible bijection with $Y_*''^{\dagger}\sigma = Y_*''^{\ddagger}$. Similarly, using the statements (M), $(\mathrm{I})^{\dagger}$, $(\mathrm{I})^{\ddagger}$, $(\mathrm{K})^{\dagger}$, $(\mathrm{K})^{\ddagger}$, also $(Z_n'^{\dagger} \setminus Y'^{\dagger})\sigma = Z_n'^{\ddagger} \setminus Y'^{\ddagger}$ and $Y'^{\dagger}\sigma = Y'^{\ddagger}$. Hence $Y''^{\dagger}\sigma = Y''^{\ddagger}$.

Similarly, $X_*''^{\dagger}\sigma = X_*''^{\ddagger}$, $(Y_*''^{\dagger}, Y''^{\dagger}, X_*''^{\dagger})\sigma = (Y_*''^{\ddagger}, Y''^{\ddagger}, X_*''^{\ddagger})$, and $\mathfrak{F}^{\dagger}_{Z_n'^{\dagger}\setminus Y'^{\dagger}}\sigma = \mathfrak{F}^{\ddagger}_{Z_n'^{\ddagger}\setminus Y'^{\ddagger}}$.

Using $(\mathrm{I})^{\dagger}$, $(\mathrm{I})^{\ddagger}$, $(\mathrm{K})^{\dagger}$, $(\mathrm{K})^{\ddagger}$, $(\mathrm{vii})^{\dagger}$ and $(\mathrm{vii})^{\ddagger}$, we get $(\varphi\tau^{\dagger})\sigma = \varphi\tau^{\ddagger}$ and thus finally

$$\varphi^{\dagger}\sigma = (\varphi\tau^{\dagger} \oplus \psi_{\alpha}(\sigma^{\dagger})^{-1})\sigma = (\varphi\tau^{\dagger})\sigma \oplus \psi_{\alpha}(\sigma^{\dagger})^{-1}\sigma = \varphi\tau^{\ddagger} \oplus \psi_{\alpha}(\sigma^{\ddagger})^{-1} = \varphi^{\ddagger}.$$

In view of (U) and (6.6) we get

$$\varphi^{\dagger}\sigma = \varphi\tau^{\ddagger} \oplus \psi_{\alpha}(\sigma^{\ddagger})^{-1} = \varphi\tau^{\ddagger} \oplus (\psi^{\ddagger}\restriction G_{1\eta_{\alpha}}) \subseteq \psi^{\ddagger}.$$

$$\begin{tikzcd}
 & & G_2 \arrow[rr, "\psi^{\ddagger}"] & & G_4^{\ddagger} \\
 & G' + \sum_{n<\omega} H_{\alpha n} \arrow[rr, equal] & & G' + \sum_{n<\omega} H_{\alpha n} & \\
G_{Y_*''^{\dagger} Y''^{\dagger} X_*''^{\dagger}} \arrow[ur, "\sigma^{\dagger}"] & & G_1(\overline{\xi}_{\alpha}) \arrow[ll, "\varphi^{\dagger}"'] \arrow[rr, "\varphi^{\ddagger}"] \arrow[uu] \arrow[ul, "\varphi\oplus\psi_{\alpha}"'] \arrow[ur, "\varphi\oplus\psi_{\alpha}"] & & G_{Y_*''^{\ddagger} Y''^{\ddagger} X_*''^{\ddagger}} \arrow[uu] \arrow[ul, "\sigma^{\ddagger}"'] \\
 & G' \arrow[uu] \arrow[rr, equal] \arrow[dl, tail, shift left, "\tau^{\dagger}"] & & G' \arrow[uu] \arrow[dr, tail, shift left, "\tau^{\ddagger}"] & \\
G_3^{\dagger} \arrow[uu] \arrow[ur, tail, shift left, "\sigma^{\dagger}"] & & G_1(\overline{\xi}) \arrow[ll, "\varphi\tau^{\dagger}"'] \arrow[rr, "\varphi\tau^{\ddagger}"] \arrow[uu] \arrow[ul, "\varphi"'] \arrow[ur, "\varphi"] & & G_3^{\ddagger} \arrow[uu] \arrow[ul, tail, shift left, "\sigma^{\ddagger}"]
\end{tikzcd}$$

The existence of $(Z_*''^{\ddagger}, Z''^{\ddagger}, Y_*''^{\ddagger})$, $(Y_*''^{\ddagger}, Y''^{\ddagger}, X_*''^{\ddagger})$, $\mathfrak{F}^{\ddagger}_{Z''^{\ddagger}}$, σ and $\psi^{\ddagger}$ with $\varphi^{\dagger}\sigma \subseteq \psi^{\ddagger}$ contradicts the statement of the step lemma, when we replace $f, E_*, G_1(\overline{\xi}), G_{V_* V U_*}, \varphi, z$ by $f-1, E_*, G_1(\overline{\xi}_{\alpha}), G_{Y_*''^{\dagger}, Y''^{\dagger}, X_*''^{\dagger}}, \varphi^{\dagger}, z$. In particular this contradicts the choice of $\pi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda^{\overline{\xi}_{\alpha}})$ at (6.5). Thus the step lemma follows. $\square$

7. Application of the Strong Black Box

In this section we want to construct $\aleph_k$-free R-modules with prescribed endomorphism R-algebras A using our preliminary work and the Strong Black Box as the prediction principle. The Strong Black Box comes from Shelah [23, Lemma 3.24, p. 28, Chapter IV], a model-theoretic version can be found in Eklof–Mekler [9], and a version adjusted to algebraic applications appears in Göbel–Wallutis [17]. We will apply [17], which is also outlined in [16]. As with other applications of the Strong Black Box, its setting has to fit its applications (cf. [16]): We must specify what we want to predict! Thus its formulation has to wait until we are ready for its use. We begin with the construction of an $\aleph_k$-free R-module related to the algebra A. Although it will be necessary and sufficient to assume that its R-module structure A_R is also $\aleph_k$-free, we will restrict ourselves for simplicity to the most interesting case when the R-algebra has a free R-module structure A_R. (The extension requires just a few technical changes.) Moreover, let $|A| < \lambda_1$. (Also here we could replace the size of the modules under study by their ranks and argue with cardinals of ranks; thus $\operatorname{rk} A \leq \lambda_1$ would be possible, which we however leave to the reader.)

Recall that $\langle \lambda_1, \dots, \lambda_k \rangle$ is the cardinal sequence from Section 2.1 satisfying the cardinal conditions (i)–(iii). We will fix in this section the cardinals $\lambda = \lambda_k$ and $\theta = \lambda_{k-1}$. The Strong Black Box will require $|R| \leq |A| \leq \theta$ and $\mu_k^\theta = \mu_k$, but this is no further restriction on μ_k due to assumptions (i)–(iii).

Also recall from Section 2.2 the definition of the free A-module $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}$ and its $\mathbb{S}$-adic completion $\widehat{B}$. Prediction principles, also the Strong Black Box, will need the notion of a trap, which are the objects to be predicted. This is intimately connected with an ordering which will tell us later which prediction comes first. Thus we define a very natural λ-norm on Λ and Λ_*.

Definition 7.1 (The *λ-norm function*).

(a) For $\eta \in {}^{\omega\geq}\lambda$ let $\|\eta\| = \sup_{\ell < \lg \eta}(\eta(\ell) + 1) \in \lambda$;
in particular $\|\alpha\| = \alpha + 1$ for $\alpha \in \lambda$.
(b) For $\overline{\eta} \in \Lambda$ let $\|\overline{\eta}\| = \|\eta_k\|$, and for $\overline{\nu} \in \Lambda_*$ let $\|\overline{\nu}\| = \|\nu_k\|$.
(c) For $Y \subseteq \Lambda$ put $\|Y\| = \sup_{\overline{\eta} \in Y} \|\overline{\eta}\|$ and note that $\|Y\| = \lambda$ if and only if $|Y| = \lambda$. Similarly $\|Y\| = \sup_{\overline{\nu} \in Y} \|\overline{\nu}\|$ if $Y \subseteq \Lambda_*$.
(d) If $b \in \widehat{B}$, then $\|b\| = \|[b]\|$, and for $S \subseteq \widehat{B}$, let $\|S\| = \sup_{b \in S} \|b\|$.

The black boxes also need a weak version of well-orderings, which reads as follows.

Definition 7.2. For $V \subseteq \Lambda$ the family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V\}$ of branch-like elements (from Section 3) is *regressive* if $\|b_{\overline{\eta}}\| < \|\overline{\eta}\| = \|y_{\overline{\eta}}\|$ for all $\overline{\eta} \in V$.

We are now ready to define the final version of a trap for the Strong Black Box. Note that we already used a different trap for the step lemma, which also needs a prediction. The crucial sets for this definition can be seen in Definition 6.2.

Definition 7.3. A quintuple $p = (\eta, V_*, V, \mathfrak{F}, \varphi)$ is a *trap* (*for the Strong Black Box*) if the following hold:

(i) $\eta \in {}^{\omega\uparrow}\lambda$.
(ii) $V_* \subseteq \Lambda_*$ and $V \subseteq \Lambda$ with $|V_*|, |V| \leq \theta$.
(iii) (V_*, V) is Λ-closed.
(iv) $\|\overline{\nu}\| < \|\eta\|$ for all $\overline{\nu} \in V_*$, and $\|\overline{\eta}\| < \|\eta\|$ for all $\overline{\eta} \in V$.
(v) $\Lambda_{\langle\eta\rangle *} \subseteq V_*$ (recall that by definition $\Lambda_{\langle\eta\rangle *} = \{\overline{\nu} \in \Lambda_* \mid \nu_k \lhd \eta, \nu_k \neq \eta\}$).
(vi) $\Lambda^{\langle\eta_k\rangle} \subseteq V$ for all $\overline{\eta} \in V$ (recall that $\Lambda^{\langle\eta_k\rangle} = \{\overline{\nu} \in \Lambda \mid \nu_k = \eta_k\}$).
(vii) For $\overline{\eta} \in \Lambda$ and $1 \leq m < k, n < \omega$ with $\overline{\eta}\upharpoonright\langle m, n\rangle \in V_*$ we have $[\overline{\eta}] \subseteq \Lambda_*^{\langle\eta_k\rangle} \cup \Lambda_{\langle\eta_k\rangle *} \subseteq V_*$ (recall that $\Lambda_*^{\langle\eta_k\rangle} = \{\overline{\nu} \in \Lambda_* \mid \nu_k = \eta_k\}$).
(viii) If $\overline{\eta} \in \Lambda$, $\|\overline{\eta}\| < \|\eta\|$ and $\overline{\eta}\upharpoonright\langle k, n\rangle \in V_*$ for infinitely many $n < \omega$, then $\overline{\eta} \in V$.
(ix) $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in B_{V_*}, \overline{\eta} \in V\}$ is regressive.
(x) $\varphi : P \to P$ is an R-endomorphism of the A-module $P = G_{V_* V}$ generated by V_* and $\mathfrak{F}$; compare Definition 3.3.

Convention 7.4. In the definition of a trap we put $\|p\| = \|\eta\| = \|V_*\|$, which is the *norm of the trap* p.

Recall that $\lambda^o = \{\alpha \in \lambda \mid \mathrm{cf}\,\alpha = \omega\}$.

The Strong Black Box 7.5. *Let $\theta \leq \lambda = \mu^+$ and $\mu^\theta = \mu$. If $E \subseteq \lambda^o$ is a stationary subset of λ^o, then there is a sequence $p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ $(\alpha < \lambda)$ of traps with the following properties:*

(i) $\|p_\alpha\| \in E$ *for all* $\alpha < \lambda$.
(ii) $\|p_\alpha\| \leq \|p_\beta\|$ *for all* $\alpha < \beta < \lambda$.
(iii) THE DISJOINTNESS CONDITION: *If $\alpha \neq \beta$ and $\|p_\alpha\| = \|p_\beta\|$, then $\|V_{\alpha *} \cap V_{\beta *}\| < \|p_\alpha\|$, in particular $\eta_\alpha \neq \eta_\beta$.*
(iv) THE PREDICTION: *For any $V_G \subseteq \Lambda$ with regressive family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V_G\}$, $G = G_{\Lambda_*, V_G}$ generated by Λ_* and $\mathfrak{F}$, $\varphi \in \mathrm{End}_R\, G$ and any set $S \subseteq \Lambda_*$ with $|S| \leq \theta$, the set*

$$\{\alpha \in E \mid \exists \beta < \lambda \text{ with } \|p_\beta\| = \alpha,\ V_\beta = (V_G)_{V_{\beta *}} \subseteq V_G,\ \mathfrak{F}_\beta = \mathfrak{F}_{V_\beta},\ \varphi_\beta \subseteq \varphi,\ S \subseteq V_{\beta *}\}$$

is stationary.

While in the earlier black boxes the prediction is about the existence of partial endomorphisms of $\widehat{B}$, the main point is that we now deal with homomorphisms which are related to a special class of submodules $G \subseteq \widehat{B}$. Indeed, by definition of the traps, this particular black box will fail to predict arbitrary endomorphisms of $\widehat{B}$.

Proof of 7.5. See the proof in Göbel–Wallutis [17] or in [16] with minor adjustments; note that $\lambda = \lambda_k$ satisfies the required cardinal conditions. For $V_\beta = (V_G)_{V_{\beta *}}$ observe that all traps p_β of the Strong Black Box (like the other Black Boxes [4, 16, 23]) are unions of admissible chains of partial traps $(p_\beta^n)_{n<\omega}$. At stage n we can also choose $(V_G)_{V_{\beta *}^n} \subseteq V_\beta^{n+1}$. This now implies $V_\beta = (V_G)_{V_{\beta *}}$, because any $\overline{\eta} \in (V_G)_{V_{\beta *}}$ satisfies $[\overline{\eta}] \subseteq V_{\beta *}$. Thus $\|\overline{\eta}\| = \|\overline{\eta}\upharpoonright\langle 1, 0\rangle\| < \|\eta\|$ by Definition 7.3(iv) and $\overline{\eta}\upharpoonright\langle k, n\rangle \in V_{\beta *}$

for infinitely many $n < \omega$. Definition 7.3(viii) implies $\overline{\eta} \in V_\beta$. The reverse inclusion is trivial. $\square$

We now want to apply the Strong Black Box 7.5 to derive the following main theorem. Here recall that the ring R has an $\mathbb{S}$-adic topology which is Hausdorff, hence the $\mathbb{S}$-completions $\widehat{R}$ and $\widehat{B}$ are well-defined, and $\mathrm{Hom}(\widehat{R}, R) = 0$, i.e. R (and thus also A) is cotorsion-free. See the definition of the $\beth^+$-sequence in Section 1.

Main Theorem 7.6. *If R is a cotorsion-free $\mathbb{S}$-ring and A an R-algebra with free R-module A_R, $|A| < \mu$, $k < \omega$ and $\lambda = \beth_k^+(\mu)$, then we can construct an $\aleph_k$-free A-module G of cardinality λ with R-endomorphism algebra* $\mathrm{End}_R\, G = A$.

Remark 7.7. Assuming that A is countable, the smallest examples of $\aleph_k$-free A-modules G in Theorem 7.6 have size $|G| = \beth_k^+$.

Proof of Theorem 7.6. We first construct the A-module G. We continue using the earlier notations $|A| < \lambda_1 < \cdots < \lambda_k$ from Section 2.1.

Thus we must construct a specific regressive family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V_G\}$ such that the A-module $G = G_{\Lambda_*, V_G}$ generated by Λ_*, $\mathfrak{F}$ satisfies the conclusion of Theorem 7.6 and in particular $\mathrm{End}_R\, G = A$.

Recall that $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}$ has cardinality λ and $\lambda = \mu_k^+$ is regular. By Solovay's decomposition theorem (see Jech [20, p. 433]) there is a decomposition $\lambda^o = \dot{\bigcup}_{z \in B} E_z$ into stationary sets E_z.

For all E_z ($z \in B$) with the help of the Strong Black Box 7.5 we choose a list of traps p_α^z ($\alpha < \lambda$) and relabel them (preserving the norms) to get a uniform sequence of traps

$$p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)\ (\alpha < \lambda) \quad \text{with} \quad \|p_\alpha\| \le \|p_\beta\| \quad \text{for all } \alpha < \beta < \lambda. \tag{7.1}$$

Put $V_G = \dot{\bigcup}_{\alpha < \lambda} \Lambda^{\langle \eta_\alpha \rangle}$. For each $\overline{\eta} \in V_G$ we must choose $\pi_{\overline{\eta}} \in \widehat{R}$ and $b_{\overline{\eta}} \in B$ for the definition of $y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}}$. We will choose recursively the pairs $(\pi_{\overline{\eta}}, b_{\overline{\eta}})$ for $\overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}$ and $\alpha < \lambda$. Thus we consider the trap $p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ and choose $z \in B$ with $\|p_\alpha\| \in E_z$. If $z \notin B_{V_{\alpha *}}$, then we do not work and put

$$\pi_{\overline{\eta}} = b_{\overline{\eta}} = 0 \quad \text{for all } \overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}. \tag{7.2}$$

Now let $z \in B_{V_{\alpha *}} \subseteq P_\alpha = \mathrm{Dom}\, \varphi_\alpha$, hence $z\varphi_\alpha \in P_\alpha$ is well-defined by Definition 7.3(x). We will distinguish three cases.

Case 1: Let $z = e_{\overline{\nu}}$ for some $\overline{\nu} \in \Lambda_*$. If $z\varphi_\alpha \in Az$, we do not work and choose the pair again trivially as in (7.2).

Otherwise $z\varphi_\alpha \notin Az$, and we arrive at the interesting case which needs work. We want to apply the Step Lemma 6.6 for $f = k - 1$, $\overline{\xi} = \langle \eta_\alpha \rangle$, $E_* = \{\overline{\nu}, \overline{\nu}'\}$ using some $\overline{\nu}'$ with $\overline{\nu} \ne \overline{\nu}' \in [z\varphi_\alpha]$, which exist as a result of the action of φ_α. Now we have

$$G_1(\overline{\xi}) = B_{\Lambda_{\overline{\xi}_*} \cup E_*} \subseteq P_\alpha,$$

because $E_* \subseteq V_{\alpha *}$ and $\Lambda_{\overline{\xi}*} \subseteq V_{\alpha *}$ by Definition 7.3(v).

In order to adjust our notations to the preliminaries of the Step Lemma 6.6 we put

$$V_* := V_{\alpha *} \,\dot\cup\, \Lambda_*^{\overline{\xi}}, \qquad V := V_\alpha \,\dot\cup\, \Lambda^{\overline{\xi}}, \qquad U_* := (\Lambda_{\overline{\xi}*} \setminus E_*) \cup \Lambda_*^{\overline{\xi}} \cup \{\overline{\nu}\}.$$

It is immediate that U_* is almost tree-closed and (V_*, V) is Λ-closed, and it follows that $u_{\overline{\eta}}(U_*) = \{1, \dots, k\}$ for $\overline{\eta} \in V_\alpha$ and $u_{\overline{\eta}}(U_*) = \emptyset$ for $\overline{\eta} \in \Lambda^{\overline{\xi}}$.

If $\overline{\nu}' = \overline{\eta}' {\restriction} \langle m, n\rangle$, then $[\overline{\eta}]_{n+1} \subseteq U_*$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}}$, hence $V_{U_*} = \Lambda^{\overline{\xi}}$ and the triple (V_*, V, U_*) is k-closed.

Put $\mathfrak{G} = \mathfrak{G}_{V_{\alpha *}} \,\dot\cup\, \mathfrak{G}_{\Lambda^{\overline{\xi}}}$ with $\mathfrak{G}_{V_{\alpha *}} := \mathfrak{F}_\alpha$ (given by the Strong Black Box 7.5), and $\mathfrak{G}_{\Lambda^{\overline{\xi}}} := \{y''_{\overline{\eta}} = \pi'_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^{\overline{\xi}}\} = \mathfrak{G}_{V_{U_*}}$.

We would like to point out that we have chosen the $\pi'_{\overline{\eta}}$s in $\mathfrak{G}_{\Lambda^{\overline{\xi}}}$ arbitrarily. This does not do any harm, as noted in Remark 6.7. Of course, the intended canonical choice is to set $\pi'_{\overline{\eta}} := \pi_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}}$), but these elements $\pi_{\overline{\eta}}$ are not yet known and will arrive at the final construction step. Due to this choice, $\mathfrak{G}$ is (V_*, V, U_*)-suitable, because from $[\overline{\eta}]_{n'} \subseteq U_*$ for some $n' < \omega$ it follows that $\overline{\eta} \in V_{U_*}$, and hence $[b_{\overline{\eta}}] = [z] = \overline{\nu} \in U_*$.

If $\psi_\alpha = (\varphi_\alpha {\restriction} G_1)\rho_{V_* V U_*}$ with $G_1 = G_1(\overline{\xi})$, then $\psi_\alpha : G_1 \to G = G_{V_* V U_*}$ and also $z\psi_\alpha \neq 0$ by $\overline{\nu}' \in [z\varphi_\alpha]$.

Now the assumptions of the Step Lemma 6.6 hold for

$$f = k - 1,\ \overline{\xi} = \langle \eta_\alpha \rangle,\ E_* = \{\overline{\nu}, \overline{\nu}'\},\ G_1(\overline{\xi}),\ G_{V_* V U_*},\ \psi_\alpha, z,$$

and by the step lemma we find elements $\pi_{\overline{\eta}} \in \widehat{R}$ ($\overline{\eta} \in \Lambda^{\overline{\xi}}$), while setting $b_{\overline{\eta}} = z$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}}$. From $z \in P_\alpha$ it also follows that $\|b_{\overline{\eta}}\| = \|z\| < \|\overline{\eta}\|$, and the related family $\mathfrak{F}$ is regressive.

Case 2: Let $z = e_{\overline{\nu}_1} - e_{\overline{\nu}_2}$ for distinct $\overline{\nu}_1, \overline{\nu}_2 \in \Lambda_*$. In this case we change the basis and let $e_{\overline{\nu}_1} = e'_{\overline{\nu}_1} + e'_{\overline{\nu}_2}$ and $e_{\overline{\nu}} = e'_{\overline{\nu}}$ for all $\overline{\nu} \neq \overline{\nu}_1$. Thus we have reduced Case 2 to Case 1, and the choice of the pairs $(\pi_{\overline{\eta}}, b_{\overline{\eta}} = z)$ for $\overline{\eta} \in \Lambda^{\langle \eta_\alpha \rangle}$ is as in Case 1.

Case 3: Now z is neither of the form $z = e_{\overline{\nu}}$ nor of the form $z = e_{\overline{\nu}_1} - e_{\overline{\nu}_2}$. In this case again we do not work and choose the pairs trivially as in (7.2).

Thus all pairs $(\pi_{\overline{\eta}}, b_{\overline{\eta}})$ ($\overline{\eta} \in V_G$) are constructed and the A-module G is defined by $G = G_{\Lambda_*, V_G}$ with the help of the family $\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V_G\}$.

It remains to show that

G is as required in Theorem 7.6.

Clearly $|G| = \lambda$, and G is an $\aleph_k$-free A-module by the Freeness Lemma 3.7. Since A acts faithfully on the A-module G, it is also clear that $A \subseteq \mathrm{End}_R\, G$, where we identify every $a \in A$ with its induced scalar multiplication on G. Thus it remains to show that $\mathrm{End}_R\, G \subseteq A$, and we let $\varphi \in \mathrm{End}_R\, G$.

First we want to show

Claim 1. *If $\overline{\nu} \in \Lambda_*$, then $e_{\overline{\nu}}\varphi \in Ae_{\overline{\nu}}$.*

Suppose for contradiction that there is $\overline{\nu} \in \Lambda_*$ with $e_{\overline{\nu}}\varphi \notin Ae_{\overline{\nu}}$. By construction the family $\mathfrak{F}$ is regressive, and applying the Strong Black Box 7.5 for the stationary set $E_{e_{\overline{\nu}}} \subseteq \lambda^o$ we see for G, $\mathfrak{F}$, φ and $S = \{\overline{\nu}\}$ that the set

$$\{\alpha \in E_{e_{\overline{\nu}}} \mid \exists\beta < \lambda \text{ with } \|p_\beta^{e_{\overline{\nu}}}\| = \alpha,\ V_\beta^{e_{\overline{\nu}}} = (V_G)_{V_{\beta*}^{e_{\overline{\nu}}}} \subseteq V_G, \\ \mathfrak{F}_\beta^{e_{\overline{\nu}}} = (\mathfrak{F})_{V_\beta^{e_{\overline{\nu}}}},\ \varphi_\beta^{e_{\overline{\nu}}} \subseteq \varphi,\ \overline{\nu} \in V_{\beta*}^{e_{\overline{\nu}}}\}$$

is stationary.

In particular there is $\alpha < \lambda$ such that

$$\|p_\alpha\| \in E_{e_{\overline{\nu}}}, \quad V_\alpha = (V_G)_{V_{\alpha*}}, \quad \mathfrak{F}_\alpha = \mathfrak{F}_{V_\alpha}, \quad \varphi_\alpha \subseteq \varphi, \quad \overline{\nu} \in V_{\alpha*}. \tag{7.3}$$

Hence $e_{\overline{\nu}} \in B_{V_{\alpha*}} \subseteq P_\alpha = \operatorname{Dom}\varphi_\alpha$ and by assumption $e_{\overline{\nu}}\varphi_\alpha \notin Ae_{\overline{\nu}}$. Now Case 1 of the construction applies and the $\pi_{\overline{\eta}} \in \widehat{R}$ ($\overline{\eta} \in \Lambda^{\langle\eta_\alpha\rangle}$) are chosen with the Step Lemma 6.6. In order to derive the desired contradiction, we denote the relevant sets similarly to Section 6. Put

$$Z_* := \Lambda_*, \quad Z := V_G, \quad Y_* := V_{\alpha*} \mathbin{\dot\cup} \Lambda_*^{\langle\eta_\alpha\rangle} \ (= V_*), \quad Y := V_\alpha \mathbin{\dot\cup} \Lambda^{\langle\eta_\alpha\rangle} \ (= V), \\ X_* := (\Lambda_{\langle\eta_\alpha\rangle_*} \setminus E_*) \cup \Lambda_*^{\langle\eta_\alpha\rangle} \cup \{\overline{\nu}\} \ (= U_*).$$

From the same argument as in the construction for V_G it follows that (Y_*, Y, X_*) is k-closed.

Next we show that also

$$(Z_*, Z, Y_*) \text{ is } k\text{-closed}. \tag{7.4}$$

If $\overline{\eta} \in Z$ and $\|\overline{\eta}\| > \|p_\alpha\|$, then $|u_{\overline{\eta}}(Y_*)| = k$ because $\|Y_*\| = \|p_\alpha\|$.

If $\|\overline{\eta}\| < \|p_\alpha\|$ and $|u_{\overline{\eta}}(Y_*)| < k$, then there is $1 \le m \le k$ with $m \notin u_{\overline{\eta}}(Y_*)$. If $m = k$, then $\overline{\eta}\restriction\langle k, n\rangle \in V_{\alpha*} \subseteq Y_*$ for infinitely many $n < \omega$. Definition 7.3(viii) implies $\overline{\eta} \in V_\alpha$, and $[\overline{\eta}]_N \subseteq V_{\alpha*} \subseteq Y_*$ for some $N < \omega$, as required. If $m < k$, then $\overline{\eta}\restriction\langle m, n\rangle \in V_{\alpha*}$ for some $n < \omega$, and Definition 7.3(vii) also yields $[\overline{\eta}] \subseteq \Lambda_*^{\langle\eta_k\rangle} \mathbin{\dot\cup} \Lambda_{\langle\eta_k\rangle*} \subseteq V_{\alpha*} \subseteq Y_*$, as required.

If $\|\overline{\eta}\| = \|p_\alpha\|$, then it follows from $\overline{\eta} \in Z = V_G$ that $\eta_k = \eta_\beta$ for some $\beta < \lambda$. If $\beta = \alpha$, then $[\overline{\eta}] \subseteq \Lambda_{\langle\eta_\alpha\rangle*} \cup \Lambda_*^{\langle\eta_\alpha\rangle} \subseteq Y_*$ by Definition 7.3(v). If finally $\beta \neq \alpha$, then clearly $\{1, \dots, k-1\} \subseteq u_{\overline{\eta}}(Y_*)$. If $k \notin u_{\overline{\eta}}(Y_*)$, then by $\eta_k = \eta_\beta$ we have $\overline{\eta}\restriction\langle k, n\rangle \in V_{\alpha*} \cap V_{\beta*}$ for infinitely many $n < \omega$. It follows that $\|V_{\alpha*} \cap V_{\beta*}\| = \|p_\alpha\| = \|p_\beta\|$, but this contradicts the disjointness condition of the Strong Black Box 7.5(iii). So (7.4) holds.

Next we show that $Y = Z_{Y_*}$: If $\overline{\eta} \in Z$, then $\overline{\eta} \in Z_{Y_*}$ if and only if $[\overline{\eta}]_n \subseteq Y_*$ for some $n < \omega$, hence $|u_{\overline{\eta}}(Y_*)| = 0$. This is equivalent to $[\overline{\eta}]_N \subseteq V_{\alpha*}$ for some $N < \omega$ or $\overline{\eta} \in \Lambda^{\langle\eta_\alpha\rangle}$, and also $\overline{\eta} \in (V_G)_{V_{\alpha*}} \cup \Lambda^{\langle\eta_\alpha\rangle} = V_\alpha \cup \Lambda^{\langle\eta_\alpha\rangle} = Y$ by (7.3).

It is easy to see that $\mathfrak{F}$ is also (Z_*, Z, Y_*)-suitable: If $[\overline{\eta}]_n \subseteq Y_*$ for some $\overline{\eta} \in Z$ and some $n < \omega$, then by definition $\overline{\eta} \in Z_{Y_*} = Y = V_\alpha \cup \Lambda^{\langle\eta_\alpha\rangle}$, and we distinguish two cases. If $\overline{\eta} \in V_\alpha$, then $y'_{\overline{\eta}} \in \mathfrak{F}_{V_\alpha} = \mathfrak{F}_\alpha$ (by (7.3)). Definition 7.3(ix) yields $b_{\overline{\eta}} \in B_{V_{\alpha*}}$ and $[b_{\overline{\eta}}] \subseteq V_{\alpha*} \subseteq Y_*$. If $\overline{\eta} \in \Lambda^{\langle\eta_\alpha\rangle}$, then by construction $b_{\overline{\eta}} = e_{\overline{\nu}} \in B_{V_{\alpha*}}$, and $[b_{\overline{\eta}}] \subseteq V_{\alpha*} \subseteq Y_*$, which we required.

$\mathfrak{F}_Y$ is also (Y_*, Y, X_*)-suitable. Note that $\mathfrak{F}_Y = \mathfrak{F}_{V_\alpha} \dot\cup \mathfrak{F}_{\Lambda^{\langle\eta_\alpha\rangle}} = \mathfrak{F}_\alpha \dot\cup \mathfrak{F}_{\Lambda^{\langle\eta_\alpha\rangle}}$. 'Suitable' then follows as shown in the construction of $\pi_{\overline{\eta}} \in \widehat{R}$ $(\overline{\eta} \in \Lambda^{\langle\eta_\alpha\rangle})$.

The construction implies $(V_*, V, U_*) = (Y_*, Y, X_*)$. In this case the triple (V_*, V, U_*) and $\mathfrak{F}_Y$ with the help of the Step Lemma 6.6 generates the same elements $\pi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda^{\langle\eta_\alpha\rangle})$ as with $\mathfrak{G}$ (and also the induced modules $G_{V_*VU_*}$ are the same) (see Remark 6.7). The homomorphism $\varphi\rho_{\Lambda_*VU_*}$ extends the homomorphism $\psi_\alpha = (\varphi_\alpha \restriction G_1)\rho_{V_*VU_*}$ to G, and hence to $G_2 \subseteq G$.

The existence of (Z_*, Z, Y_*), (Y_*, Y, X_*), $\mathfrak{F}$, $\tau = \mathrm{id}$ and $\psi_\alpha \subseteq \varphi\rho_{\Lambda_*VU_*}$ contradicts the Step Lemma 6.6 (and the choice of elements $\pi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda^{\langle\eta_\alpha\rangle})$ for $f = k-1$, $\overline{\xi} = \langle\eta_\alpha\rangle$, $E_* = \{\overline{\nu}, \overline{\nu}'\}$, $G_1(\overline{\xi})$, G_{V_*,V,U_*}, ψ_α, $z = e_{\overline{\nu}}$).

It remains to show

Claim 2. *If $\overline{\nu}_1 \neq \overline{\nu}_2 \in \Lambda_*$, then $(e_{\overline{\nu}_1} - e_{\overline{\nu}_2})\varphi \in A(e_{\overline{\nu}_1} - e_{\overline{\nu}_2})$.*

But this follows from the same arguments as in Case 2 in the construction.

From Claims 1 and 2 it is immediate that $\varphi \in A$. □

8. Fully rigid systems of $\aleph_k$-free R-modules with prescribed R-algebra A

Finally, we will use the arguments of Section 7 to extend Theorem 7.6 and show the existence of fully rigid families of A-modules. (See the definition of the $\beth^+$-sequence in Section 1.)

Theorem 8.1. *If R is a cotorsion-free ring and A an R-algebra with free R-module A_R and $|A| < \mu$, $k < \omega$ and $\lambda = \beth_k^+(\mu)$ (as in Section 2.1), then there is a family of $\aleph_k$-free A-modules $\langle G_u \mid u \subseteq \lambda\rangle$ of cardinality λ with the following properties for any $u, v \subseteq \lambda$:*

$$\mathrm{Hom}_R(G_u, G_v) = \begin{cases} A & \text{if } u \subseteq v, \\ 0 & \text{if } u \nsubseteq v. \end{cases}$$

Moreover, $G_u \subseteq G_v$ for all $u \subseteq v \subseteq \lambda$.

Proof. For the construction of the rigid family $\langle G_u \mid u \subseteq \lambda\rangle$ above we will modify the construction of the A-module G of Theorem 7.6 with $\mathrm{End}_R\, G = A$ slightly; so compare the first part of that proof. First we decompose $\lambda^o = \dot\bigcup_{\beta<\lambda} E_\beta$ and then $E_\beta = \dot\bigcup_{z\in B} E_{\beta z}$ into stationary sets $E_{\beta z}$ using Solovay's partition theorem (see Jech [20, p. 433]). The list of traps $p_\alpha = (\eta_\alpha, V_{\alpha*}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ $(\alpha < \lambda)$ needed here is a composition of traps $p_\alpha^{\beta z}$ $(\alpha < \lambda)$ from the Strong Black Box 7.5 for the stationary sets $E_{\beta z}$ $(\beta < \lambda, z \in B)$. As in the construction above we will find a family

$$\mathfrak{F} = \{y'_{\overline{\eta}} = \pi_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V_G\}$$

of branch-like elements with $V_G = \dot\bigcup_{\alpha<\lambda} \Lambda^{\langle\eta_\alpha\rangle}$, and $G = G_{\Lambda_* V_G}$ satisfies $\mathrm{End}_R\, G = A$, as seen from the second part of the proof of Theorem 7.6.

If $u \subseteq \lambda$, then put

$$V_u := \dot{\bigcup}\{\Lambda^{\langle \eta_\alpha \rangle} \mid \alpha < \lambda,\ p_\alpha = p_\alpha^{\beta z},\ \beta \in u,\ z \in B\}$$

and $G_u = G_{\Lambda_* V_u}$, which is generated by $\mathfrak{F}_{V_u}$. Then it is immediate that $G = G_\lambda$ and $G_u \subseteq G_v$ for all $u \subseteq v \subseteq \lambda$.

If $u, v \subseteq \lambda$ and $\varphi : G_u \to G_v$, then as in Section 7 it is clear that $\varphi \in A$, thus $\mathrm{Hom}_R(G_u, G_v) = A$ for $u \subseteq v$. For $u \nsubseteq v$ and $0 \neq \varphi \in A$, from $V_{u\setminus v} \subseteq V_u$ it follows that $V_{u\setminus v}\varphi \subseteq G_v$. Thus $u \setminus v \neq \emptyset$ and $V_{u\setminus v} \subseteq V_v$ are a contradiction. Hence $\mathrm{Hom}_R(G_u, G_v) = 0$ in this case. □

9. Applications of the Main Theorem

The applications of Theorem 7.6 are by now standard. All R-algebras A inserted into Theorem 7.6 and constructed earlier (see [16, Chapter 15]) have a free R-module structure. We assume that the ground ring R is a domain (thus has no nontrivial idempotents). Moreover, the algebras A are p-reduced by some element $p \in R$. Thus Theorem 7.6 applies. Under this hypothesis we can find R-algebras A which are countably generated over R with any of the following properties:

(i) A has no regular idempotents (see [16, p. 587, Example 15.1.1]).

(ii) Let $q > 0$ be an integer. A has free generators σ^i, σ_i $(0 \leq i \leq q)$ subject to the only relations $\sigma^i\sigma_j = \delta_{ij}$ and $\sum_{0\leq i\leq q} \sigma_i\sigma^i = 1$. Moreover, there is a 'trace'-homomorphism $T : A \to R/qR$ such that for any $\sigma, \varphi \in A$:

  (a) $(\sigma + \varphi)T = \sigma T + \varphi T$.
  (b) $(\sigma\varphi)T = (\varphi\sigma)T$.
  (c) $1T = 1 + qR$.

(iii) Let G be a finite group. Then G is a group of units of a domain R if and only if G is from Corner's list of subdirect products of primordial groups; see Corner [3] for these groups G.

Recall that the *primordial groups* are the cyclic groups Z_2, Z_4 and

$$G^{\epsilon\delta} = \langle a, b \mid a^{2+\epsilon} = b^{2+\delta} = (ab)^2\rangle \quad \text{for } \epsilon, \delta \in \{0, 1\}.$$

The latter groups are the quaternion group G^{00}, the dicyclic group G^{01} and the tetrahedral group G^{11}.

An immediate application (cf. [16, pp. 595–596, 603–606]) of these algebras establishes the following

Corollary 9.1. *Let R be a countable domain as above. Then (for each natural number k) there are $\aleph_k$-free R-modules G of cardinality $\beth_k^+$ with any of the following properties:*

(i) *G has no indecomposable summands different from 0, i.e. G is superdecomposable.*

(ii) *Let* $R = \mathbb{Z}$ *and* $q > 0$ *be an integer. Then* G *satisfies the Kaplansky test problem, i.e. for any* $r, s \in \mathbb{N}$,

$$G^r \cong G^s \;\Leftrightarrow\; r \equiv s \bmod q.$$

(iii) *Let* $R = \mathbb{Z}$. *A finite group* G *is the automorphism group of an* $\aleph_k$*-free abelian group if and only if it belongs to Corner's list of finite groups.*

(iv) G *is an indecomposable* R*-module.*

Clearly these applications can be extended to similar fully rigid systems of modules using Theorem 8.1.

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