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WHITEHEAD GROUPS MAY NOT BE FREE EVEN ASSUMING CH, II

BY

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ABSTRACT

We prove some theorems on uncountable abelian groups, and consistency results promised in the first part, and also that a variant of $\diamondsuit_{\omega_1}$ called $\clubsuit$ (club), is consistent with $2^{\aleph_0} > \aleph_1$.

§0. Introduction

§§2, 4 and 5 assume knowledge of forcing whereas §§1, 3 and 6 do not.

In §1 we define uniformization properties, and show some easy properties.

In §2 we prove that for a stationary set S, if for one $\Phi = \{\eta_\delta : \delta \in S\}$, $(\Phi, 2)$ has the uniformization property (see the definition at the beginning of §1), then this does not necessarily hold for every $(\Phi', 2)$, $\Phi' = \{\eta'_\delta : \delta \in S\}$. So the question whether $(\Phi, 2)$ has the uniformization property does not depend on S only. By §3 this means that the question whether a group is a Whitehead group is delicate, and apparently minor changes in the definition may change this property. In §2 we show also that the weak diamond from Devlin and Shelah [3] is not equivalent to natural strengthening of it, and that the union of two Φ's with the uniformization property does not necessarily have this property.

In §3 we indicate the connection between uniformization properties and Whitehead groups. (Essentially they are equivalent, so we translate the (partially) Whitehead problem to a purely combinatorial one.)

In §4 we show that it is consistent that some $(\Phi, 2)$ has the uniformization property, Φ a set of sequences of natural numbers. Remember that in [7], $\aleph_1$-free groups of cardinality $\aleph_1$ were partitioned into three cases; case III is the free one.

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It was proved: (a) [CH] if G is of case I, it is not Whitehead; (b) $[V=L]$ if G is of case I or II, it is not Whitehead. By [8], it is consistent with ZFC + G.C.H. that there are Whitehead groups of case II, and by [3] CH implies there are always non-Whitehead groups of case II. By [9] $\mathrm{MA}+2^{\aleph_0}>\aleph_1$ implies G ($\aleph_1$-free, $|G|=\aleph_1$) is Whitehead iff it is of case II or III. Now we can complete the picture by showing it is consistent with ZFC that there are Whitehead groups of case I. (Always there are non-Whitehead groups of case I, see [9].) So the result is a conistency of a statement which at first glance should follow from $\mathrm{MA}+2^{\aleph_0}>\aleph_1$, but not only does it not follow, it contradicts $\mathrm{MA}+2^{\aleph_0}>\aleph_1$.

In §5 we prove the consistency of ZFC + $\clubsuit_{\omega_1}+2^{\aleph_0}=2^{\aleph_1}$. This is dual to [8] as we show that even though CH fails, $\diamondsuit_{\omega_1}$ almost holds.

In §6, we prove another result.

Consequences and more results on abelian groups appear in [9].

Remarks. (1) In [8] theorem 2.4 $\diamondsuit_S$ holds for $S\in V^P$, $S\neq\emptyset \bmod D$ too. (This is because if $A\subseteq\omega_1$, $A\in V^P$, as P satisfies the $\aleph_2$-C.C. for some $\alpha(0)<\omega_2$, $A\in V^{P_{\alpha(0)}}$, then we make the forcing in two stages. The first is $P_{\alpha(0)}$, after which $\diamondsuit^*_{\omega_1}$ holds, hence $\diamondsuit_S$. The rest of the forcing behaves just like P itself, so we finish.

(2) It is not hard to check that if $V\models(\forall S\subseteq\omega_1)(\exists S_0\subseteq S)$ $\mathrm{Stat}(S)\rightarrow\mathrm{Stat}(S_0)\wedge(\omega_1-S_0)\in D]$ then this holds in V^P too. (It suffices to check this in $V^{P_{\alpha(0)}}$, as in the previous remarks, because the forcing by P preserves stationarity (see [8] 1.8).) Let S^* be the diagonal union of the S_α, $\alpha<\alpha(0)$. We can assume $S\in V^{P_{\alpha(0)}}$, $S\cap S^*=\emptyset$. Let $\mathbf{S}$ be a $P_{\alpha(0)}$-name of S. For each $p\in P_{\alpha(0)}$, and working in V, let $S_p=\{\delta<\omega_1$: for some $q, p\leqq q\in P_{\alpha(0)}$, $q\Vdash$"$\delta\in\mathbf{S}\}\in V$. Clearly S_p is stationary, so choose $S^0_p\subseteq S_p$, $\omega_1-S^0_p\in D$, S^0_p stationary. Now let S^0 be a diagonal union of the S^0_p's (as $|P_{\alpha(0)}|=\aleph_1$), so $\omega_1-S^0\in D$, and it is clear that $\Vdash^{P_{\alpha(0)}}$"$\mathbf{S}\cap S^0$ is stationary", as required.

Notation as in [8]; in particular k, l, m, n are natural numbers, λ, μ, χ, κ cardinals (usually infinite) and i, j, α, β, γ, δ, ξ, ζ ordinals; δ is a limit ordinal.

§1. Uniformization properties

Definition 1.1. Let A be a set, h be a function from A to the cardinals >1, and $\Phi=\{\eta^*_i: i<i^*\}$, a family of one-to-one sequences from A. (Essentially, A is defined by Φ and by h, so we write $A=\mathrm{Dom}\,\Phi$; if h is constant, we replace it by its value.)

(1) Φ is of type (λ,μ,δ) if $i^*=\lambda$, $|A|=\mu$, and $l(\eta^*_i)=\delta$ for each i. We say Φ

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is continuous if for $i < j < i^*$, ζ limit $\eta_i^*(\zeta) = \eta_j^*(\zeta)$ if for all large enough $\beta < \zeta$, $\eta_i^*(\beta) = \eta_j^*(\beta)$. We say Φ is (forms) a tree if

$$\eta_i^*(\alpha) = \eta_j^*(\beta) \Rightarrow \alpha = \beta \wedge (\forall \gamma < \alpha)[\eta_i^*(\gamma) = \eta_j^*(\gamma)].$$

We say Φ has splitting $< \kappa$ if for each a, α

$$|\{\eta(\alpha + 1) : \eta \in \Phi, \eta(\alpha) = a\}| < \kappa.$$

We say Φ is almost disjoint if for $i, j < i^*$, $i \neq j$ there are $\alpha < l(\eta_i^*)$, $\beta < l(\eta_j^*)$ such that

$$\alpha \leqq \alpha' < l(\eta_i^*) \wedge \beta \leqq \beta' < l(\eta_j^*) \Rightarrow \eta_i^*(\alpha') \neq \eta_j^*(\beta').$$

(Remember, λ and μ are cardinals, but not necessarily δ.)

(2) We call $\bar{f}$ a candidate, if $\bar{f} = \{f_i : i < i^*\}$, $\operatorname{Dom} f_i = \operatorname{Range} \eta_i$, and for each $\alpha < l(\eta_i)$, $f_i[\eta_i^*(\alpha)] \in h[\eta_i^*(\alpha)]$.

(3) We say the candidate $\bar{f}$ can be uniformized if there is a uniformizing function g, i.e., $\operatorname{Dom} g = A$, and for each $i < i^*$ there is $\alpha < l(\eta_i^*)$ such that

$$\alpha \leqq \beta < l(\eta_i^*) \Rightarrow g[\eta_i^*(\beta)] = f_i[\eta_i^*(\beta)].$$

Let $\alpha = F(i)$, then F determines, essentially, g; we call F a compressing function (for $\bar{f}$).

(4) We say (Φ, h) has the uniformization property, if each candidate $\bar{f}$ can be uniformized.

(5) G is a homomorphism from (Φ^1, h^1) to (Φ^2, h^2) if (a) G maps $\operatorname{Dom} \Phi^1$ into $\operatorname{Dom} \Phi^2$, $\eta \in \Phi^1 \Rightarrow \langle G(\eta(\alpha)) : \alpha < l(\eta) \rangle \in \Phi^2$ or is an unbounded subsequence of a member of Φ^2, and (b) $h^2(G(a)) \geqq h^1(a)$.

REMARK. (1) Those notions have obvious monotonicity properties which we do not bother to mention.

(2) If $\eta_0, \eta_1 \in \Phi$ and

$$(\forall i_0, i_1)(\exists j_0, j_1)[i_0 < l(\eta_0^*) \wedge i_1 < l(\eta_1^*) \rightarrow i_0 \leqq j_0 < l(\eta_0^*)$$
$$\wedge\, i_1 \leqq j_1 < l(\eta_1^*) \wedge \eta_0^*(j_0) = \eta_1^*(j_1)]$$

then clearly Φ does not have the uniformization property. So the "natural" Φ's are those with Φ almost disjoint.

CLAIM 1.1. (1) If Φ has type $(\aleph_1, \aleph_0, \omega)$ then $(\Phi, \aleph_0)$ does not have the uniformization property.

(2) If Φ has type (λ, μ, κ), and is continuous, $\lambda > \mu$, *then* (Φ, μ) does not have the uniformization property.

(3) If Φ has splitting $<\kappa^+$, is continuous, and $|\Phi|>|\mathrm{Dom}\,\Phi|$, then (Φ,κ) does not have the uniformization property.

(4) If there is a homomorphism from (Φ^1,h^1) to (Φ^2,h^2), (Φ^2,h^2) has the uniformization property, then so does (Φ^1,h^1).

PROOF. (1), (2) follow from (3).

(3) For each a let

$$\{b^{a,\alpha}_\beta:\beta<\beta_a\leqq\kappa\}=\{\eta(\alpha+1):\eta(\alpha)=a,\ \eta\in\Phi\}$$

and for each i, and $\alpha<l(\eta^*_i)$, let $f_i(\eta^*_i(\alpha))$ be the ordinal such that $b^{\eta_i(\alpha),\alpha}_{\beta}=\eta^*_i(\alpha+1)$. So $\bar f=\{f_i:i<i^*\}$ is defined, and suppose it can be uniformized and let F be a compressing function. As $|\Phi|>|\mathrm{Dom}\,\Phi|$ for some $i\neq j$, and $\eta^*_i(F(i))=\eta^*_j(F(j))$. Now we prove by induction on $\alpha\geqq F(i)$, $\eta_i(\alpha)=\eta_j(F(j)+(\alpha-F(i)))$. For $\alpha=F(i)$ we have just proved it; for α limit it follows by the continuity of Φ, and for α successor by the definition of $\bar f$. Now clearly η_i, η_j contradict uniformization (as they are almost disjoint).

(4) Easy.

CLAIM 1.2. Suppose $\mathrm{MA}+2^{\aleph_0}>\aleph_1$.

(1) If Φ has type $(\aleph_1,\aleph_0,\omega)$ and is a tree, then it has a subset Φ' of the same type which is a tree of splitting $\leqq 2$. By 1.1(3), $(\Phi',2)$, hence $(\Phi,2)$, does not have the uniformization property.

(2) If Φ has type $(\aleph_1,\aleph_0,\omega)$, then there are Φ', a tree with splitting <3, and a homomorphism from $(\Phi',2)$ to $(\Phi,2)$, provided that Φ is almost disjoint.

(3) For Φ of type $(\aleph_1,\aleph_0,\omega)$, $(\Phi,2)$ does not have the uniformization property.

PROOF. (1) W.l.o.g. let $\omega=\mathrm{Dom}\,\Phi$. Let $\Phi_1=\{\eta\in\Phi$: for every $n<\omega$, there are uncountably many $\nu\in\Phi$, such that $\eta\restriction n<\nu\}$. Clearly $|\Phi-\Phi_1|\leqq\aleph_0$, hence $|\Phi_1|=\aleph_1$.

Let $P=\{t:t$ a finite subset of Φ_1, which is a tree of splitting $\leqq 2\}$.

We consider P as a partially ordered set ordered by inclusion.

We first show P satisfies the countable chain condition. Let t_i $(i<\omega_1)$ be $\aleph_1$ elements of P. For each i there is $n=n(i)$ such that $\eta\neq\nu\in t_i\Rightarrow\eta\restriction n\neq\nu\restriction n$. So for some uncountable $S_0\subseteq\omega_1$, and $n<\omega$, $i\in S_0\Rightarrow n(i)=n$. Let $t_i=\{\eta^i_l:l<m_i\}$. As $(\forall k)[\eta^i_l(k)<\omega]$ and $m_i<\omega$, there is an uncountable $S_1\subseteq S_0$ and $m<\omega$, and $\nu_0,\cdots,\nu_{m-1}$ such that for every $i\in S_1$, $m_i=m$, $\eta^i_l\restriction n=\nu_l$ $(l=0,1,\cdots,m-1)$. Clearly $i,j\in S_1\Rightarrow t_i\cup t_j\in P$, so all t_i $(i\in S_1)$ are pairwise compatible.

Let $\Phi_1=\{\eta^\alpha:\alpha<\omega_1\}$ be an enumeration with no repetitions; $D^\alpha=\{t\in p:(\exists\beta>\alpha)\eta^\beta\in t\}$ for $\alpha<\omega_1$.

Each D^α is dense (use Φ_1's definition). Hence there is a (directed) $G \subseteq P$, intersecting each D^α. Now $\Phi' = \bigcup\{t : t \in G\}$ clearly is a tree of splitting $\leqq 2$ (as each $t \in G$ is; and G is closed under finite union). Φ' has cardinality $\aleph_1$ because it intersects each D_α.

(2) Similar.

(3) By (2), 1.1(4).

LEMMA 1.3. *Let $\Phi = \{\eta_i : i < \lambda\}$ have type (λ, μ, κ).*

(1) *If $\lambda = 2^\mu$ then $(\Phi, 2)$ does not have the uniformization property. In fact it suffices to assume there is $A \subseteq \mu$, $2^{|A|} = |\{\eta \in \Phi : |A \cap \text{Range}\,\eta| = \kappa\}|$.*[†]

(2) *If* MA *holds, and $\lambda < 2^{\aleph_0}$, $\kappa = \omega$, and for each countable $A \subseteq \text{Dom}\,\Phi$*

$$|\{\eta \in \Phi : A \cap \text{Range}\,\eta \textit{ is infinite}\}| \leqq \aleph_0$$

and Φ is almost disjoint then $(\Phi, \aleph_0)$ has the uniformization property.

(3) *Suppose $S \subseteq \omega_1$ is stationary, and $\delta \in S \Rightarrow (\exists \alpha)\delta = \omega^2\alpha$. Then we can find for each $\delta \in S$ an increasing sequence of ordinals η_δ of length ω^2 converging to δ, such that $(\{\eta_\delta : \delta \in S\}, 2)$ does not have the uniformization property. Moreover there are $c_\delta = {}^{(\omega^2)}2$ for $\delta \in S$ such that for any $f : \omega_1 \to 2$*

$$(*) \quad \{\delta \in S : (\exists n < \omega)(\forall m \geqq n)(\exists k)(\forall l \geqq k)[f(\eta_\delta(\omega m + l)) = c_\delta(\omega m + l)]\}$$

is not stationary.

(4) *If $2^{\aleph_0} = \aleph_1$, $S \subseteq \omega_1$ stationary then for any choice of η_δ as in* (3) *we can find $c_\delta \in {}^{(\omega^2)}2$ such that* $(*)$ *holds.*

PROOF. (1) Let $\eta_{i(\alpha)}$ $(\alpha < 2^{|A|})$ be distinct, and $|A \cap \text{Range}\,\eta_{i(\alpha)}| = \kappa$. Let $\{f^0_\alpha : \alpha < 2^{|A|}\}$ be a list of the functions $f : A \to 2$. We define f_i $(i < \lambda)$, $f_i : \text{Range}\,\eta_i \to 2$ by $f_{i(\alpha)}(a) = 1 - f^0_\alpha(a)$ for $a \in \text{Range}\,\eta_i$.

(2) Let $\{f_i : i < \lambda\}$ $(f_i : \text{Range}\,\eta_i \to \omega)$ be given, and we shall prove they can be uniformized. Let

$$P = \{h : h \text{ a finite function from } \lambda \text{ to } \omega, \text{ and:}$$

$$\text{if } \eta_i(\alpha) = \eta_j(\beta),\ \alpha \geqq h(i),\ \beta \geqq h(j) \text{ then } f_i(\eta_i(\alpha)) = f_j(\eta_j(\beta))\}.$$

P is partially ordered by inclusion, and we shall first prove P satisfies the countable chain condition. Let h_α $(\alpha < \omega_1)$ be $\aleph_1$ pairwise incompatible members of P.

Let $\text{Dom}\,h_i = \{\alpha^i_l; l < n(i)\}$, g_i the function $g_i(\gamma) = m$ *iff* for some $l < n(i)$, $h_i(\alpha^i_l) \leqq n < \omega$, $\gamma = \eta_{\alpha^i_l}(n)$, $f_{\alpha^i_l}(\eta_{\alpha^i_l}(n)) = m$. We can assume w.l.o.g. $n(i) = n$,

[†] In fact, in 1.3(1) it suffices to assume $2^\lambda < 2^\mu$.

and $h \subseteq h_i$, $h = h_i \restriction \operatorname{Dom} h$, and $\operatorname{Dom} h_i - \operatorname{Dom} h$ $(i < \omega_1)$ are pairwise disjoint, so w.l.o.g. $h = \emptyset$ (as $h_i \restriction (\operatorname{Dom} h_i - \operatorname{Dom} h) \in P$ are pairwise incompatible). So clearly $\operatorname{Dom} g_i$ are pairwise almost disjoint, and for $i \neq j$, $g_i \cup g_j$ is not a function.

Let $A_i = \bigcup_{j<i} \operatorname{Dom} g_j$; by a hypothesis, the definition of the g_i's, and as $\operatorname{Dom} h_\alpha$ $(\alpha < \omega_1)$ are pairwise disjoint clearly for every i, for some $j(i)$,

$$j(i) \leqq \alpha < \omega_1 \Rightarrow |A_i \cap \operatorname{Dom} g_\alpha| < \aleph_0.$$

So for each limit $\delta < \omega_1$, there is $\alpha(\delta) < \delta$ such that $A_\delta \cap \operatorname{Dom} g_{j(\delta)} \subseteq A_{\alpha(\delta)}$. By several applications of the Fodour theorem, there is a stationary $S \subseteq \omega_1$, such that for $\delta \in S$, $g_{j(\delta)} \restriction (A_\delta \cap \operatorname{Dom} g_{j(\delta)})$ is constant. So choose $\delta_1, \delta_2 \in S$, $j(\delta_1) < \delta_2$, so $g_{\delta_1} \cup g_{\delta_2}$ is a function, contradiction.

(3) Let, for each $\alpha < \omega_1$, $\{\eta_i^\alpha : i < \omega_1\}$ be a tree of type $(\aleph_1, \aleph_0, \omega)$ and of splitting < 3 with domain $[\alpha, \alpha + \omega]$. We define η_δ, so that for some increasing sequence of limit ordinals $\alpha(n) = \alpha(n, \delta)$ converging to δ, $\operatorname{Range} \eta_\delta = \bigcup_{n<\omega} \operatorname{Range} \eta_\delta^{\alpha(n)}$.

By the proof of 1.1(3) for each α there are functions f_i^α $(i < \omega_1)$ so that not only $\langle f_i^\alpha : i < \omega_1 \rangle$ cannot be uniformized, but even any uncountable subsequence cannot be uniformized. Define c_δ as follows:

$$c_\delta(\eta_\delta(\omega n + k)) = f_\delta^{\alpha(n,\delta)}(\eta_\delta^{\alpha(n,\delta)}(k)).$$

(4) For $\delta \in S$ let $\alpha(\delta, n) = \bigcup \{\eta_\delta(i) : i < \omega(n+1)\}$ (so it is a limit ordinal). For each limit α let $\{f_i^\alpha : i < \aleph_1\}$ be a list of all functions $f : \alpha \to 2$. Now we define c_δ such that:

$$\text{for each } n < \omega, \text{ and } i < \delta,\ \alpha = \alpha(\delta, n),$$

$$\{k < \omega : f_i^\alpha(\eta_\delta(\omega n + k)) \neq c_\delta(\omega n + k)\} \text{ is infinite.}$$

Lemma 1.4. (1) *Let S be a stationary subset of ω_1, and for simplicity $\alpha \in S$ implies $\omega\alpha = \alpha$. Let η_δ denote always an increasing ω-sequence converging to δ.*

If $(\{\eta_\delta : \delta \in S\}, 2)$ has the uniformization property for every η_δ, then $(\{\eta_\delta : \delta \in S\}, \aleph_0)$ has the uniformization property for every η_δ.

(2) *$(\{\eta_\delta : \delta \in S\}, \aleph_0)$ has the uniformization property (where* Range η_δ *is a set of ordinals) if for every $\{c^\delta : \delta \in S)$ $(c_\delta \in \omega)$, $(\{\eta_\delta^{c^\delta} : \delta \in S\}, 2)$ has the uniformization property where for each δ, $k_n = \Sigma_{l<n}(c^\delta(l) + 2)$, and for $k_n \leqq i < k_{n+1}$, $\eta_\delta^{c^\delta}(i) = \omega\eta_\delta(i) + i$.*

Remark. Notice this proof does not work if we restrict ourselves to trees.

Proof. (1) By (2).

(2) It suffices to prove that any candidate $\{c^\delta : \delta \in S\}$ can be uniformized for

$(\{\eta_\delta : \delta \in S\}, \aleph_0)$. Define, for each $\delta \in S$, c_δ as follows: $k_n = \Sigma_{l<n}(c^\delta(l)+2)$, $c_\delta(i) = 0$ if for some $n, k_n \leqq i < k_{n+1} - 1$, and $c_\delta(i) = 1$ otherwise. Now $\{c_\delta : \delta \in S\}$ can be uniformized for $(\{\eta_\delta^{c^\delta} : \delta \in S\}, 2)$, and its compressing function is sufficient for the uniformization we need.

Definition 1.2. We call $\Phi = \{\eta_i^* : i < i^*\}$ free if there is a function F, $\operatorname{Dom} F = i^*$, $F(i) < l(\eta_i^*)$, and for any distinct $i, j < i^*$

$$[F(i) \leqq \alpha < l(\eta_i^2)] \wedge [F(j) \leqq \beta < l(\eta_j^*)] \Rightarrow \eta_i^*(\alpha) \neq \eta_j^*(\beta).$$

Claim 1.5. If Φ is free then (Φ, κ) has the uniformization property, for any κ.

Proof. Trivial; we use F as a compressing function.

Lemma 1.6. *If there is a non-free* Φ, $\eta \in \Phi \Rightarrow l(\eta) = \omega$, *such that* $(\Phi, 2)$ *has the unifiormization property*, then *for some regular* $\lambda > \aleph_0$ *and stationary* $S \subsetneqq \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0\}$, *and a tree* $\Phi = \{\eta_\delta : \delta \in S\}$, η_δ *an increasing* ω*-sequence converging to* δ *(for* $\delta \in S$*)*, $(\Phi, 2)$ *has the uniformization property. It is obviously not free by the Fodour Lemma.*

Proof. Easy, by induction on $|\Phi|$.

Case I. If $|\Phi| \leqq \aleph_0$, it is easy to prove it is free, contradiction.

Case II. If $|\Phi|$ is a singular cardinal, then by [10] for some $\Phi' \subseteq \Phi$, $|\Phi'| < |\Phi|$, Φ' is not free. Clearly Φ' too has the uniformization property, hence we can use the induction hypothesis.

Case III. We are left with the case $\lambda = |\Phi|$ is a regular cardinal and w.l.o.g. every $\Phi' \subseteq \Phi$, $|\Phi'| < |\Phi|$ is free. Clearly $|\operatorname{Dom} \Phi| \leqq \lambda$.

We know that Φ is almost disjoint (as $(\Phi, 2)$ has the uniformization property).

Case IIIa. For some $A \subseteq \operatorname{Dom} \Phi$, $|A| < \operatorname{Dom} \Phi$ but the cardinality of $\Phi_1 = \{\eta \in \Phi : (\operatorname{Range} \eta) \cap A \text{ is infinite}\}$ is λ.

For every $\eta \in \Phi_1$ let η^1 be an ω-subsequence of η, $\operatorname{Range} \eta \subseteq A$ (i.e. $\eta^1 = \langle \eta(l_k) : k < \omega \rangle, \cdots, l_k < l_{k+1}, \cdots)$.

Let $\Phi^1 = \{\eta^1 : \eta \in \Phi_1\}$. Clearly there is a homomorphism from $(\Phi^1, 2)$ into $(\Phi, 2)$, hence by 1.1 also $(\Phi^1, 2)$ has the uniformization property. As $\lambda = |\Phi_1|$ also $\lambda = |\Phi^1|$ (as Φ is almost disjoint) so $|\Phi^1| = \lambda > |A| \supseteq \operatorname{Dom} \Phi^1$; so trivially Φ^1 is not free. Let $\mu = |\operatorname{Dom} \Phi^1| < \lambda$, so w.l.o.g. $\mu = \operatorname{Dom} \Phi^1$, and let

$$\Phi^1 = \{\nu_\delta : \delta \in S\}, \quad S = \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0, \delta \text{ divisible by } \mu\omega \text{ and by } |\delta| \cdot \omega\}.$$

We can choose $\alpha(\delta, n)$ for $\delta \in S$, $n < \omega$ such that $\delta = \bigcup_{n<\omega} \alpha(\delta, n)$ and $\alpha(\delta, n) = \alpha(\delta', n') \Rightarrow n = n'$, and $\alpha(\delta, n+1) = \alpha(\delta', n+1) \Rightarrow \alpha(\delta, n) = \alpha(\delta', n)$ and $\alpha(\delta, n)$ is divisible by μ.

We let $\eta_\delta(n) = \alpha(\delta, n) + \nu_\delta(n)$ for $\delta \in S$, $n < \omega$, and $\{\eta_\delta : \delta \in S\}$ is as required.

Case IIIb. There is no A as above. As $|\operatorname{Dom} \Phi| \leqq |\Phi| + \aleph_0 \leqq \lambda$, w.l.o.g. $\operatorname{Dom} \Phi \subseteq \lambda$, and let $\Phi = \{\nu_i : i < \lambda\}$. As Case IIIa fails, for every $i < \lambda$, for some $f(i) < \lambda$, $f(i) \leqq j < \lambda$ implies $(\operatorname{Range} \eta_j) \cap i$ is finite.

Let $S_0 = \{\delta < \lambda : \delta$ a limit ordinal, and for every $i < \delta$, $f(i) < \delta\}$. Clearly S_0 is closed unbounded. Let

$$S = \{\delta \in S_0 \text{: there is } i \geqq \delta \text{ such that } (\operatorname{Range} \eta_i) \cap \delta \text{ is infinite}\}.$$

We now show that S is stationary. Otherwise there is a closed unbounded set $S_1 \subseteq S_0 - S$, and let $S_1 \cup \{0\} = \{\alpha(i) : i < \lambda\}$, $\alpha(i)$ increasing continuous. Let $\Phi_i = \{\eta_j : \alpha(i) \leqq j < \alpha(i+1)\}$, so $\Phi = \bigcup_{i<\lambda} \Phi_i$. By the choice of S_1, there is a function $F^* : \lambda \to \omega$ such that $\eta_j \in \Phi_i$; $k \geqq F(j)$ implies $\eta_j(k) \geqq \alpha(i)$. Now by the choice of $|\Phi|$ as minimal each Φ_i is free (as $|\Phi_i| \leqq |\alpha(i+1)| < \lambda$) so there is a function F_i, $\operatorname{Dom} F_i = \{\alpha : \alpha(i) \leqq \alpha < \alpha(i+1)\}$ exemplifying it. Define $F : F(\alpha) = \operatorname{Max}\{F^*(\alpha), F_i(\alpha)\}$ when $\alpha(i) \leqq \alpha < \alpha(i+1)$. Clearly F exemplify Φ is free, contradiction. So S is stationary and working a little as in Case IIIa we can finish (using 1.1(4)).

§2. Consistency results

In this section η_δ will always be an increasing sequence of length ω converging to δ, δ will be limit ordinal $< \omega_1$ and S a stationary set of limit ordinals $< \omega_1$, and Φ be $\{\eta_\delta : \delta \in S\}$ (with a common superscript attached to each of them, if necessary). We let f be a function from ω_1 (usually to $\omega + 1$).

The question we deal with is "if (Φ, h) has the uniformization property, does (Φ^1, h^1) have it too?" and our results are the consistency of negative answers.

We first deal with the model V^P constructed in [8] 1.1 (and its notations) with only one change: h will be a fixed function from ω_1 to ω, and instead of demanding $c_\delta \in {}^\omega 2$ we demand everywhere

$$c_\delta \in {}^\omega\omega \wedge (\forall n)[c_\delta(n) < h(\eta_\delta(n))]$$

hence in the proof of 1.8 from [8] $Q(W, h)$ will change accordingly, but remains finite. So P depends on the choice of (Φ, h) and in V^P, (Φ, h) has the uniformization property.

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This is a totally inessential change.

THEOREM 2.1. $(V = L)$ *In the Model V^P described above, let (Φ^*, h^*) be another pair and*

(A) *$S^* - S$ is stationary,*

or

(B) *for every closed unbounded $C \subseteq \omega_1$, there are $\delta \in S^* \cap S \cap C$, such that for every $\alpha < \delta$, $n < \omega$ there is β, $\alpha < \beta < \delta$, $\beta \in C$, and*

$$\left[\prod \{h(\eta_\delta(m)) : m < \omega, \eta_\delta(m) < \beta\}\right]^n$$

$$< \prod \{h^*(\eta^*_\delta(m)) : m < \omega, \alpha < \eta^*_\delta(m) < \beta\}.$$

Then *in V^P, (Φ^*, h^*) does not have the uniformization property* (*whereas (Φ, h) has*).

REMARK. If $h^*(i) = \omega$ for every i, condition (B) always holds. But even if we demand Range $h^* \subseteq \omega$ there is no problem to construct examples: if $[\eta_{\delta_0}(n) = \eta_{\delta_1}(m) \rightarrow n = m \wedge \eta_{\delta_0} \restriction n = \eta_{\delta_1} \restriction m]$ then let $\Phi^* = \Phi$, $h^*(\eta_\delta(n)) = 1 + \Pi_{l \leq n} h[\eta_\delta(l)]^n$.

PROOF. Case A is trivial as $\diamondsuit_{S^* - S}$ holds in this case so we concentrate on case B, first assuming $\Phi^*, h^* \in V$. As $V = L$, $\diamondsuit^*_{S^* \cap S}$ holds; we shall use this to define in V appropriate c_δ's at the end of the proof (from the proof we shall see the demands on them).

So suppose $\bar{c} = \langle c_\delta : \delta \in S^* \rangle$ is given, and we suppose for simplicity $\phi \Vdash^P$ "$\bar{c}$ can be uniformized (for Φ^*)", and we shall get a contradiction.

As P satisfies the $\aleph_2$-C.C. we can replace P by some $P_{\alpha(0)}$, $\alpha(0) < \omega_2$, and let τ be a name of such a uniformization. Let

$$N = (H(\omega_2), \varepsilon, P_{\alpha(0)}, \Vdash, \tau)$$

and $N^1 < N$ be an elementary submodel of N, of cardinality $\aleph_1$, such that $\bigwedge_n [a_n \in N^1] \Rightarrow \{a_n : n < \omega\} \in N^1$ and $a \in b \in N^1 \Rightarrow a \in N^1$. Let $N^1 = \bigcup_{\alpha < \omega_1} N_\alpha$, N_α countable, increasing and continuous. We code N^1 as a subset A of ω_1, so that $A \cap \delta$ codes $\langle N_\alpha : \alpha < \delta \rangle$. Now as $\diamondsuit^*_{S^*}$ holds (in V) we are given for each $\delta \in S^*$ possible $\langle N_\alpha : \alpha < \delta \rangle : \langle N^{\delta,n}_\alpha : \alpha < \delta \rangle$, so that for each $\langle N_\alpha : \alpha < \omega_1 \rangle$ as we get above $\{\delta : (\exists n)(\forall \alpha < \delta) N^{\delta,n}_\alpha = N_\alpha\}$ contains a closed unbounded set, hence for some δ and n, $(\forall \alpha < \delta) N^{\delta,n}_\sigma = N_\alpha$ and the condition from **B** holds. We want to define c_δ such that if at last $N^{\delta,n}_\alpha = N_\alpha$ for $\alpha < \delta$, and the condition from B holds then there will be $p \in P_{\alpha(0)}$, $p \Vdash^{P_{\alpha(0)}}$ "for infinitely many $n < \omega$, $\tau(\eta_\delta(n)) \neq c_\delta(n)$". We should remember that even if $p_n \leqq p_{n+1}$ in $P_{\alpha(0)}$, $\operatorname{Dom} p_n(\zeta) = \alpha_n$ for each $\zeta \in \operatorname{Dom} p_n$, $\bigcup_{n<\omega} \alpha_n = \delta$, not necessarily $\bigcup_{n<\omega} p_n$ is included in a condition of $P_{\alpha(0)}$. Hence we should repeat the trick of [8] 1.8, that is, we define by induction on $k < \omega$, $\alpha_{k+1} < \omega$, W_k and T_k, satisfying (i), (ii), (iv), (v) there with $N^{\delta,m}_{\alpha_{k+1}}$ for N_k there (and we forget to say there that T_k, $W_k \in N_k$). At the end we shall get a set of conditions $P_{\bar{c}}$ (see last paragraph of [8] §1) such that for at least one $\bar{c}$ there is $q \in P_{\alpha(0)}$, $p_{\bar{c}} \leqq q$; now for each k and $p_{\bar{c}}$ there is $t^{\bar{c}}_k \in Q(W_k, k)$ with maximal domain such that $T(t^{\bar{c}}_k) \leqq p_{\bar{c}}$. Hence it suffices that for each $m < \omega$, if $\langle N^{\delta,m}_\alpha : \alpha < \delta \rangle$ satisfy the condition from (B), then there are k, l such that: for each $t \in Q(W_k, k)$ with maximal domain for some l

$$\bigcup_{\alpha<\delta} N^{\delta,m}_\alpha \models \text{“} T_k(t) \Vdash^{P_{\alpha(0)}} \text{“} \tau(\eta_\delta(l)) \neq c_\delta(l) \text{”}.$$

More exactly T_k, W_k depends on m, so we should have written α^m_k, T^m_k, W^m_k; and we define T^m_k, W^m_k, α^m_k by induction on $m + k$, and a finite information on c_δ.

The point is that we have $\aleph_0$ assignments and on each stage we have one assignment and have defined already $c_\delta(l)$ for finitely many l's only. So suppose T_k, $W_k \in N^{\delta,m}_{\alpha_k}$, $\alpha = \delta\omega$, by (B) we find an appropriate $\beta = N^{\delta,m}_\beta \cap \omega_1$, and we can find T', $T_k \leqq T' \in N^{\delta,m}_\beta$, such that for each $t \in Q(W_k, k)$ of maximal domain, $T'(t)$ determines (by $\Vdash^{P_{\alpha(0)}}$) what is $\tau(\eta_\delta(l))$ when $\eta_\delta(n) < \beta$. Now we can define $c_\delta(l)$ for the l's satisfying $\alpha < \eta_\delta(l) < \beta$, to contradict this.

We demanded Φ^*, $h^* \in V$, but this is not necessary, for each Φ^*, h^*, as $P = P_{\omega_2}$ satisfies the $\aleph_2$-C.C. for some $\alpha(0)$, h^*, $\Phi^* \in V^{P_{\alpha(0)}}$, so we can make the forcing in two steps: first by $P_{\alpha(0)}$, and then it is known that $V^{P_{\alpha(0)}} \models \diamondsuit^*_{\omega_1}$ (which is what we really demand from V in 2.1), and then the rest of the forcing, where the proof of 2.1 works. Note also that by Theorem 2.3, it may occur that (A) and (B) fail, but still (Φ, h) has the uniformization property but (Φ^*, h^*) does not.

Claim 2.2. If $(\exists n)(\forall i < \omega_1)(h_1(i) \leqq h_2(i)^n)$ and (Φ, h_2) has the uniformization property then so does (Φ, h_1).

Proof. Trivial.

Theorem 2.3. *For $\kappa \leqq \aleph_0$ it is consistent with* ZFC + G.C.H., *that for some stationary $S \subseteq \omega_1$, the assertion (Φ^κ_S) fails but (Φ^k_S) holds for each $k < \kappa$, where*

(Φ^κ_S) for every function $F : {}^{\alpha>}2 \to \kappa$ ($\alpha = \bigcup S$) there is $g \in {}^\alpha\kappa$ such that for every $f \in {}^\alpha 2$, $\{\beta \in S : F(f \restriction \beta) = g(\beta)\}$ is stationary.

Remark. By Devlin and Shelah [3] $(\Phi^2_{\omega_1})$ follows from $2^{\aleph_0} < 2^{\aleph_1}$ clearly for $k < \kappa$, $(\Phi^\kappa_S) \Rightarrow (\Phi^k_S)$.

PROBLEM. Does $(\Phi^{\aleph_0}_{\omega_1})$ or even $\Phi^3_{\omega_1}$ follows from CH?[†]

PROOF. Start with $V = L$, choose stationary costationary $S \subseteq \omega_1$, and for simplicity let $\kappa = k + 1 < \aleph_0$, and like [8] §1, we defined P_α $(\alpha \leqq \omega_2)$, $P = P_{\omega_2}$ so that in V^P, the statement $(\Phi^\kappa_{\omega_1})$ fails for some specific $F^* \in V$. More exactly, for a given $F^* : {}^{\omega_1 >}2 \to \kappa$, and $g : \omega_1 \to \kappa$, let $Q^{F^*}_g = \{f : \operatorname{Dom} f$ an ordinal $\alpha < \omega_1$, $\operatorname{Range} f \subseteq 2$ and for every limit $\delta \leqq \alpha$, $\delta \in S$ implies $F^*(f \restriction \delta) \neq g(\delta)\}$. Now F^* shall be chosen later and g_α is a P_α-name for a function from ω_1 to κ, and $P_\alpha = \{p : \operatorname{Dom} p$ a countable subset of ω_1 for $\zeta \in \operatorname{Dom} p$, $p \restriction \zeta \vDash^{P_\zeta}$ "$p(\zeta) \in Q^{F^*}_{g_\alpha}$"$\}$. The proof that forcing by P_{ω_2} does not change cofinalities, preserve stationarity, does not add reals, and in V^P F^* exemplify Φ^κ_S fails, is just like [8]. Now suppose $k < \kappa$, and we want to show that (Φ^k_S) holds. So let $F \in V^P$, $F : {}^{\omega_1 >}2 \to k$, hence (as P satisfies the $\aleph_2$-chain condition) for some $\xi < \omega_2$, $F \in V^{P_\xi}$ and let $\boldsymbol{F}$ be a name for it. We have to define $g \in V^{P_\xi}$ which exemplify (Φ^k_S) for F. Also let $f \in V^P$ be $f : \omega_1 \to 2$, and $\boldsymbol{g}$, $\boldsymbol{f}$ their respective names.

Remember we have to define $F^* \in V$ and $g \in V^{P_\xi}$. As in V, $\diamondsuit_S$ holds, let $\langle S_\delta : \delta \in S \rangle$ examplify it. If $S_\delta \subseteq \delta$ encode an appropriate model $(N_\delta, P^\alpha, \Vdash, q, f)$ (as in the proof of 2.1), let A be the set of "ordinals" of N_δ, and as in [8] 1.8 we define a function $p_{\bar c}$, $\bar c \in {}^A\kappa$ such that:

(1) $\operatorname{Dom} p_{\bar c} = A$, $\delta = \operatorname{Dom} p_{\bar c}(\zeta)$ for each $\zeta \in A$, each $p_{\bar c}$ is a union of a generic (for N_δ) set of conditions in P^α, $[\zeta \in A, \bar c_1 \restriction \zeta = \bar c \restriction \zeta] \Rightarrow [p_{\bar c_1} \restriction \zeta = p_{\bar c_2} \restriction \zeta]$. So $p_{\bar c}$ determine $f \restriction \delta$ or $f_{\bar c} \restriction \delta$.

(2) The functions $p_{\bar c}(\zeta)$ $(\zeta \in A,\ \bar c \in {}^A\kappa)$ are distinct (members of ${}^\delta 2$).

So we define $F^* \restriction {}^\delta 2$ such that $F^*(p_{\bar c}(\zeta)) = \bar c(\zeta)$ (the range of F^* is $\subseteq \kappa$ as $\bar c$ was a function from A to κ). As $\langle S_\delta : \delta \in S \rangle \in V$ clearly $F^* \in V$.

For notational simplicity let $\xi = 0$ so $F \in V$. Analysing for which sets $T \subseteq {}^A\kappa$, there is always $\bar c \in {}^A\kappa$ and $p \in P$, $p_{\bar c} \leqq p$; we see that a sufficient condition is:

(*) in the following game $\operatorname{Gm}(T)$ player I has no winning strategy: by induction on $\zeta \in A$, player I chooses $i_\zeta < \kappa$, and $\alpha_\zeta < \omega_1$, $\alpha_\zeta > \bigcup_{\xi < \zeta} \beta_\xi$, and then player II chooses $c_\zeta < \kappa$, $c_\zeta \neq i_\zeta$ and $\beta_\zeta < \omega_1$, $\beta_\zeta > \bigcup_{\xi < \zeta} \alpha_\xi$. Player II wins if $\langle c_\zeta : \zeta \in A \rangle \in T$ and for every limit $\delta \in A$, $\bigcup_{\zeta < \delta} \alpha_\zeta$ $(= \bigcup_{\zeta \in \delta} \beta_\zeta)$ does not belong to S.

PROOF OF THE SUFFICIENCY OF (*)

We describe a strategy for player I: in addition to choosing i_ζ, α_ζ he chooses $p_\zeta \in P_\zeta$ such that $\langle c'_\xi : \xi \in A \rangle \in {}^A\kappa$; $c'_\xi = c_\xi$ for $\xi < \zeta$ implies $p_{\bar c} \restriction \zeta \leqq p_\zeta$, and $\bigcup_{\xi < \zeta} \alpha_\xi \leqq \operatorname{Dom} p_\zeta(i) \leqq \alpha_\zeta$, and $p_\zeta \Vdash$ "$g_\zeta(\delta) = i_\zeta$".

[†] We now know that the answer is negative.

As player I has no winning strategy there is a play in which he uses this strategy, but player II wins. Now $\bigcup_{\zeta \in A} p_\zeta$ is the required p.

Now ${}^A\kappa = \bigcup_{l<k} T_l$, where for each $\bar{c} \in T_l$, $F(f_{\bar{c}} \restriction \delta) = l$, where $f_{\bar{c}} \restriction \delta$ is the value of $f \restriction \delta$ as forced by $P_{\bar{c}}$ (remember we have assumed w.l.o.g. $F \in V$). Now player I cannot have winning strategy for all the game $\mathrm{Gm}(T_l)$ $(l < k)$.

Otherwise, let γ be the order type of A. We can prove by induction on γ, that there is an increasing and continuous sequence $\langle N_i : i \leqq \gamma + 1 \rangle$ of countable elementary submodels of $(H(\aleph_2), \in)$, such that A, $\langle p_{\bar{c}} : \bar{c} \in A \rangle$ and the k strategies belong to N_0; and $N_i \in N_{i+1}$ and for every limit $\delta \leqq \gamma$, $\langle N_i : i \leqq \delta \rangle \in N_{\delta+1}$ and $N_\delta \cap \omega_1 \notin S$. Now there are k plays, the lth one $(l < k)$ being

$$\langle i_0^l, \alpha_0^l \rangle, \langle c_0^l, \beta_0^l \rangle; \langle i_1^l, \alpha_1^l \rangle, \langle c_1^l, \beta_1^l \rangle; \cdots; \langle i_\zeta^l, \alpha_\zeta^l \rangle, \langle c_\zeta^l, \beta_\zeta^l \rangle, \cdots$$

such that

(1) the moves of player II do not depend on l, i.e.

$$\langle c_\zeta^l, \beta_\zeta^l \rangle = \langle c_\zeta^0, \beta_\zeta^0 \rangle.$$

(2) In the lth play, player I uses his winning strategy for $\mathrm{Gm}(T_l)$.

(3) For each ζ, $\langle \cdots ; \langle i_\xi^l, \alpha_\xi^l \rangle, \langle c_\xi^l, \beta_\xi^l \rangle; \cdots \rangle_{\xi < \zeta}$ belong to $N_{\zeta+1}$.

(4) Player II chooses $\beta_\zeta^l = (N_{\zeta+1} \cap \omega_1) - N_\zeta$, and $c_\zeta^l = c_\zeta^0 = \min\{i \in \kappa :$ for every $l < k$, $i \neq i_\zeta^l\}$.

It is easy to check all requirements, so we get that $\langle c_\xi^0 : \xi \in A \rangle \in {}^A\kappa$ does not belong to T_l (by the lth play) for each $l < k$. So player II constructs $\bar{c} \in {}^A\kappa$ outside $\bigcup_{l<k} T_l$, contradiction. So we define $g(\delta)$ as an $l < k$ such that in $\mathrm{Gm}(T_l)$ player I has no winning strategy.

THEOREM 2.4. *Suppose, for simplicity, $V = L$, S is a stationary, costationary subset of ω_1, $\Phi = \{\eta_\delta : \delta \in S\}$, $h : \omega_1 \to \omega + 1$ are given. We can chose stationary disjoint $S_0, S_1 \subseteq S$ such that the following will hold.*

We define $\bar{c}^\alpha$, P_α as in [8] 1.1, *but the domain of $\bar{c}^\alpha$ is not necessarily S but a subset of it (so we have more conditions), and for $\alpha = 2\beta + l$ $(l = 0, 1)$ $\phi \Vdash^{P_\alpha}$ "$\bar{c}^\alpha$ has the form $\langle c_\delta^\alpha : \delta \in S_l \rangle$, $c_\delta^\alpha \in \Pi_n h(\eta_\delta(n)$. All the work of* [8] §1 *holds, in V^P (Φ_l, h) has the uniformization property $(\Phi_l = \{\eta_\delta : \delta \in S_l\})$ but*

(*) *not only (Φ, h) does not have the uniformization property, but Φ_0, Φ_1 cannot be separated, i.e. (in V^P) for no $A \subseteq \omega_1$ does*

$$(\forall l \in 2)(\forall \delta \in S_l)[\text{for all large enough } n,\ \eta_\delta(n) \in A \Leftrightarrow l = 0].$$

REMARK. We can strengthen (*) to: for no $l_0 \in 2$, $A \subseteq \omega_1$ does

$$l = l_0 \Leftrightarrow (\forall \delta \in S_l)(\exists n_0)(\forall n \geqq n_0)[\eta_\delta(n) \in A].$$

This does not require essential changes in the proof.

PROOF OF 2.4. For simplicity let $h(i)=\omega$ for every i. For each limit $\delta<\omega_1$, the diamond sequence "guess" for us models $\langle N_i : i<\delta\rangle$, $N_i \prec (H(\aleph_i), \in, P, \Vdash, P, f)$, $p\in P$, $p \Vdash$ "f a function from ω_1 to $\{0,1\}$ which separate Φ_1 from Φ_2", which are as usual.

As in the proof of 2.1, 2.3, we let A be the set of "ordinals" of $N_\delta = \bigcup_{i<\delta} N_i$, and we can find $p_{\bar c}$ ($\bar c \in T \overset{\text{def}}{=} \Pi_{n<\omega} h(\eta_\delta(n))$),

$$[\bar c_0 \restriction \zeta = \bar c_1 \restriction \zeta] \Rightarrow [p_{\bar c_0}\restriction \zeta = p_{\bar c_1} \restriction \zeta], \quad \operatorname{Dom} p_{\bar c} = A, \quad \operatorname{Dom} p_{\bar c}(\zeta) = \delta,$$

$p_{\bar c}$ determines (through $\Vdash$) $f\restriction\delta$ as $f_{\bar c}$ and we know that for some $\bar c \in T$, $p_{\bar c} \leqq p$ for some $p\in P$.

The main point is that a large number of $p_{\bar c}$'s are not necessary: if $\delta\in S_l$, the splitting is necessary only for $\alpha + l$ even. So we have a free choice to determine to which l, $\delta \in S_l$, and in what way to reduce the set of $p_{\bar c}$. Let

$$T_0 = \{\bar c \in T : \text{for every } n \text{ large enough, } f_{\bar c}(\eta_\delta^{(n)}) = 0\}, \qquad T_1 = T - T_0.$$

As in the proof of 2.3, we define games $\mathrm{Gm}(l)$, $l=0,1$ in a play of $\mathrm{Gm}(l)$, in the ζ-th move: if $\zeta = l \bmod 2$ then player I chooses $c_\zeta<\omega$, $\alpha_\zeta<\omega_1$, $\alpha_\zeta > \bigcup_{\xi<\zeta}\alpha_\xi$, and if $\zeta \neq l \bmod 2$ then player II chooses $c_\zeta<\omega$, $\alpha_\zeta<\omega_1$, $\alpha_\zeta>\bigcup_{\xi<\zeta}\alpha_\xi$ (so unlike 2.3, each time only one of the players moves).

In the end player II wins if $\langle c_\zeta : \zeta\in A\rangle \in T_l$, and for each limit $\delta\in A\cup \{\operatorname{Sup} A\}$, $\bigcup_{\zeta\in A} \notin S$.

As in the proof of 2.3, for some l player I has no winning strategy, and this implies that for some $\bar c \in T_l$, $p_{\bar c} \leqq p \in P$ for some p.

THEOREM 2.5. *In 2.4, if $h:\omega_1\to\omega$ we can give a priori stationary S_0, $S_1 \subseteq S$, and then define appropriate η_δ ($\delta\in S_0\cup S_1$) so the conclusion holds.*

PROOF OF 2.5. Define η_δ ($\delta \in S_0$) arbitrarily. Now for every $\delta\in S_1$, the diamond sequence gives us $\delta_n \in S_0$, $\delta_n<\delta_{n+1}$, $\bigcup_n \delta_n = \delta$, and $\langle N_i : i \leqq \omega^2\rangle$ increasing continuous sequence of countable models, which are (up to isomorphism) elementary submodels of $(H(\aleph_2), \in, P, \Vdash, p, f)$ as in 2.4, $N_{\omega n}\cap\omega_1 = \delta_n$, $\langle N_i : i \leqq j\rangle \in N_{j+1}$. We have to define η_δ.

We define by induction on k, T_k, W_k, $\eta_\delta(k)$ as in the proof of 1.1 [8] (or 2.11 with $N_{\omega k+k}$ for N_k but the domain of W_k consists of odd ordinals only (because $\delta\in S_1$) and

(*) $\qquad$ for every maximal $t\in Q(W_k, k)$, $T_k(c) \Vdash$ "$f(\eta_\delta(k)) = 0$".

If we succeed — fine; otherwise we use δ_{k+1} to show f is not a counterexample. We define $T_{k,l}$, $W_{k,l}$ $(l<\omega)$ as in 1.1 [8] (or 2.1) but with the models $N_{\omega k+k+l}$ and $\eta_{\delta_{k+1}}$, such that $\mathrm{Dom}\, W_{k,l} - \mathrm{Dom}\, W_k$ consists of even ordinals only, $W_{k,0}=W_k$, $T_{k,0}=T_k$. In the end we use the following Claims 2.5(1), (2).

Claim 2.5(1). Suppose I is a tree, $I=\bigcup_{n\le k} I_n$, I_n the nth level of I, $|I_0|=1$. Let $I_k=A_0\cup A_1$, $A_0\cap A_1=\emptyset$ and $k=V_0\cup V_1$. $V_0\cap V_1=\emptyset$. *Then* we can find $l\in\{0,1\}$ and $J\subseteq I$ such that

(i) $I_0\subseteq J$,

(ii) if $a\in I_n\cap J$, $n\in V_l$, $n<k$ *then* each immediate successor of a is in J,

(iii) if $a\in I_n\cap J$, $n\in V_{1-l}$, $n<k$ *then* at least one immediate successor of a is in J,

(iv) $I_k\cap J\subseteq A_l$.

Proof. We prove by downward induction on $m\leqq k$ that for each $a\in J_m$ there is $J_a\subseteq I$ and $l_a\in\{0,1\}$ satisfying (ii), (iii), (iv) when $m\leqq n\leqq k$ and $a\in J_a$.

For $n=k$ let $J_a=\{a\}$,

$$l_a=\begin{cases}0 & a\in A_0,\\ 1 & a\in A_1.\end{cases}$$

Suppose we define J_a, l_a for every $a\in J_{n'}$, $n'>n$; and w.l.o.g. $n\in V_0$. If for each immediate successor b of a, $l_b=0$, let $l_a=0$ and $J_a=\{a\}\cup\{J_b \mid a<b\in I_{n+1}\}$. Otherwise a has an immediate successor b, $l_b=1$, and let $l_a=1$, $J_a=\{a\}\cup J_b$.

Clearly for the $a\in J_0$, J_a is a J as required.

Claim 2.5(2). Let γ, ξ be ordinals, I a tree, $I=\bigcup_{\alpha\leqq\gamma} I_\alpha$, I_α the αth level, $|I_0|=1$, with unique limits.

Let A_ζ $(\zeta<\xi)$ be a partition of I_γ, and V_ζ $(\zeta<\xi)$ a partition of γ, and each V_ζ is the union of a finite number of closed intervals. Then there are $J\subseteq I$, $\zeta<\xi$ such that (i)–(iv) of the previous claim holds, with ζ replacing l.

Proof. By 2.5(1).

Theorem 2.6. *Suppose, e.g., $V=L$, and S is a stationary costationary subset of ω_1, $\Phi=\{\eta_\delta:\delta\in S\}$, $h:\omega_1\to\omega$ and P are defined as in 2.1 such that in V^P, (Φ,h) has the uniformization property.*

Then we can define $\{\eta^*_\delta:\delta\in S\}$, such that: for every $A\subseteq\omega_1$, $A\in V^P$ for a stationary set of $\delta\in S$, for every n large enough, $(\eta^*_\delta(2n)\in A)\equiv(\eta^*_\delta(2n+1)\in A)$.

Proof. Left to the reader.

THEOREM 2.7. *Suppose $V = L$, $S \subsetneq \omega_1$ stationary and costationary, and $\Phi = \{\eta_\delta : \delta \in S\}$. Suppose S_δ is a countable family of functions from* Range η_δ *to δ.*

Then for some forcing notion P, it does not change cofinality, stationarity and does not add ω-sequences, and

(1) *every $\langle c_\delta : \delta \in S\rangle$, $c_\delta \in S_\delta$, can be uniformized but*

(2) *$(\Phi, 2)$ does not have the uniformization property.*

PROOF. Left to the reader.

So as usual we are given δ, $\langle N_i : < \delta\rangle$, p, f and we want to define the right p_ε's. Let $S_\delta = \{c_n : n < \omega\}$, so we can (in [8], 1.1, more exactly 1.8, p. 199) redefine:

(i) $Q(W, k) = \{\tau : \mathrm{Dom}\,\tau$ is an initial segment of $W\}$, we let $\tau \in Q(W, k)$, we define $t(\tau)$ as a function with domain Dom W,

$$(t(\tau))(\zeta) = c^{\delta}_{\tau(\zeta)} \restriction \{\eta_\delta(i) : \alpha_{W(\zeta)} \leqq \eta_\delta(i) < \alpha_k\};$$

(ii) we say p is consistent with $\tau \in Q(W, k)$ if p is consistent with $t(\tau)$.

Now we define T_k, $W_k \in N_{\alpha_k}$, $\alpha_k < \alpha_{k+1}$ (and $\bigcup_k \alpha_k = \delta$). The point is that though eventually Dom W_k has to grow, we can hold it fixed, for "a long time", by computation (as $|\mathrm{Dom}\, W_k|$ is smaller than 2^l for big enough l). We can define c_δ so that f will be forced by some condition not to be eventually equal to it on η_δ.

We leave the details to the reader.

THEOREM 2.8. *Suppose $V = L$, $S \subsetneq \omega_1$ stationary, costationary and $\Phi = \{\eta_\delta : \delta \in S\}$, η_δ increasing sequence converging to δ, t a two-place function on ω. We can find a forcing notion P, as in the previous theorems, such that*

(1) *in V^P, for every $h : \omega_1 \to \omega$, (Φ, h) has the uniformization property; moreover, for every $h : \omega_1 \to \omega$, we can uniformize*

$$\langle c_\delta : \delta \in S\rangle, \qquad \text{if } c_\delta \in {}^\omega\omega, \quad c_\delta(n) < t(h(\eta_\delta(n)), n),$$

(2) *$(\Phi, \aleph_0)$ does not have the uniformization property.*

PROOF. Again as in [8] §1, this time the trees are finite though we do not have an *a priori* bound on the size of the tree after n stages; so (2) is easy as in 2.1.

§3. The uniformization property and Whitehead groups

Let a fixed triple $(\Phi, \bar{d}, G)$ be given (for this section). $\Phi = \{\eta_\delta : \delta < \lambda\}$, $\eta_\delta = \langle \eta_\delta(n) : n < \omega\rangle$; $n \neq m \Rightarrow \eta_\delta(n) \neq \eta_\delta(m)$; $\eta_\delta(n)$ a successor ordinal $< \delta$, and the η_δ's are almost disjoint. Also $\bar{d} = \langle \bar{d}_\delta : \delta < \lambda\rangle$, $\bar{d}_\delta = \langle d_\delta(n) : n < \omega\rangle$, each $d_\delta(n)$ is a natural number $> I$ and $d_\delta(n)$ divides $d_\delta(n+1)$, $d_\delta(n) \neq d_\delta(n+1)$.

Let $d^*_\delta(n) = d_\delta(n)/d_\delta(n-1)$ where we stipulate $d_\delta(-1) = 1$. G is an abelian group, generated by x_{i+1} $(i < \lambda)$, $y_\delta = y_{\eta_\delta}$ $(\delta < \lambda$, δ always limit) and z^n_δ $(\delta < \lambda$, $n < \omega)$ with the only relations:

$$d_\delta(n) z^n_\delta = y_\delta - \sum_{l \leqq k_\delta(n)} b^l_\delta x_{\eta_\delta}(l)$$

where $b^l_\delta \in \mathbf{Z}$ (the integers) and the greatest common divisor of $d_\delta(n)$, b_l $(k_\delta(n-1) < l \leqq k_\delta(n))$ is $d_\delta(n-1)$. We write $G = G(\Phi, \bar{d})$ if $b^{l-1}_\delta = d_\delta(l)$, $k_\delta(n) = n$. Our sysem is a tree if Φ is a tree, and in addition $\eta_{\delta(0)}(l) = \eta_{\delta(1)}(l)$; $k_{\delta(0)}(n-1) < l \leqq k_{\delta(0)}(n)$ implies $k_{\delta(0)}(m) = k_{\delta(1)}(m)$ for $m = 0, \cdots, n$, $\eta_\delta(0) \restriction (k_{\delta(0)}(n) + 1) = \eta_{\delta(1)} \restriction (k_{\delta(0)}(n) + 1)$ and $b^m_{\delta(0)} = b^m_{\delta(1)}$ for $m \leqq k_{\delta(0)}(n)$ and $d_{\delta(0)}(m) = d_{\delta(1)}(m)$ for $m \leqq n$.

A set $S \subseteq \lambda$ is *closed* if $\delta \in S \wedge n < \omega \Rightarrow \eta_\delta(n) \in S$ and for a closed S let $G(S)$ be the subgroup of G generated by x_{i+1}, y_δ, z^n_δ $(i + 1 \in S, \delta \in S, n < \omega)$.

We call Φ *free* if there is $\mathscr{G} : \lim \lambda \to \omega$, s.t. $\mathscr{G}(\delta_l) \leqq n_l < \omega$, $\delta_1 \neq \delta_2 \in \lim \lambda$ implies $\eta_{\delta_1}(n_1) \neq \eta_{\delta_2}(n_2)$, where $\lim \lambda = \{\delta : \delta < \lambda \text{ limit}\}$. We call Φ λ-free if every $\Phi' \subseteq \Phi$, $|\Phi'| < \lambda$, is free.

Explanation. If we would omit the generators z^n_δ, we would get a free group. But as we have defined, for each δ, we make it somewhat more difficult for G to be free. However, as the η_δ are almost disjoint those reasons are unrelated. Note that if G_0 is the subgroup generated by the $\{x_{i+1} : i + 1 < \lambda\}$, then in G/G_0, y_δ is divisible by infinitely many integers: the $d_\delta(n)$ $(n < \omega)$. So if $d_\delta(n) = n!$ then G/G_0 is, essentially, a vector space over the rationals.

Claim 3.1. Suppose κ is an (infinite) cardinal. If Φ is κ-free then G is κ-free. If Φ is a tree, G κ-free then Φ is κ-free.

Proof. As the condition for $\kappa = \aleph_0$ implies the condition for $\kappa = \aleph_1$, we can assume $\kappa > \aleph_0$.

Let $H \subseteq G$, $|H| < \kappa$, then for some closed $S \subseteq \lambda$, $|S| < \kappa$, and $H \subseteq G(S)$. As every subgroup of a free group is free it suffices to prove $G(S)$ is free. Let $g : S \cap \lim \lambda \to \omega$ be as mentioned in the definition of freeness. Let

$$S^1 = \{\eta_\delta(l) : \delta \in S, l \geqq g(\delta)\},$$

$$S_0 = S - S^1 - S \cap \operatorname{Lim} \lambda.$$

We define S_i by induction on i such that:

(a) if $\cap \operatorname{Lim} \lambda$, $g(\delta) \leqq n$, and $\eta_\delta(n) \in S_i$ then $\delta \in S_i$,

(*) (b) if $\cap \operatorname{Lim} \lambda$, then for every $l < \omega$, $\eta_\delta(l) \in S_i$,

(c) S_i is increasing and continuous.

Now S_0 is defined: for i limit there is no problem; for $i = j+1$, choose $\delta \in S - S_i$ with minimal $g(\delta)$ and let

$$S_{i+1} = S_i \cup \{\delta\} \cup \{\eta_\delta(l) : l < \omega\}.$$

So clearly for some α, $S_\alpha = S$, and it suffices to prove that $G(S_0)$, $G(S_{i+1})/G(S_i)$ are free. This is quite easy.

We have proved one implication, the "only if" part. Now we prove the "if" part. So let $S \subseteq \lambda$, $|S| < \kappa$, and we should prove $\{\eta_\delta : \delta \in S\}$ is free. For this end we prove a somewhat stronger assertion:

(*) suppose $S_0 \subseteq S_1 \subseteq \lambda$, $|S_1| < \kappa$, S_0 and S_1 are closed and $\delta \in S_1 - S_0$ implies $\{\eta_\delta(n) : n < \omega\} \cap S_0$ is finite and $G(S_1)/G(S_0)$ is free (abelian group), then $\{\eta_\delta : \delta \in S_1 - S_0\}$ is free.

Why is (*) enough? For a given S, we let $S_1 = S \cup \{\eta_\delta(n) : n < \omega, \delta \in S\}$, $S_0 = \emptyset$ then $G(S_0) = \{0\}$, $G(S_1)$ is free by the hypothesis (as $|G(S_1)| \leqq |S_0| + \aleph_0 < \kappa$), so (*) gives the required conclusion.

Now we prove (*) by induction on $|S_1 - S_0|$. If $|S_1 - S_0| \leqq \aleph_0$, then $\{\eta_\delta : \delta \in S_1 - S_0\}$ is free because it is countable and $\{\eta_\delta : \delta \in \lim \lambda\}$ is almost disjoint. Suppose now $\mu = S_1 - S_0$ is uncountable. Note that $G(S_1)/G(S_0)$ is generated by μ element: $x_{i+1} + G(S_0)$, $y_0 + G(S_0)$, $z_\delta^n + G(S_0)$ ($i+1 \in S_1 - S_0$, $\delta \in S_1 - S_0$, and $n < \omega$); let $\{\tau_i + G(S_0) : i < \mu\}$ be a free basis of $G(S_1)/G(S_0)$. Let

$$K = \{S : S_0 \subseteq S \subseteq S_1,\ S \text{ is closed and } G(S)/G(S_0) \text{ is generated by } \{\tau_i + G(S_0) : \tau_i \in G(S)\} \text{ and } i \in S - S_0,\ n < \omega, \text{ and if } (\exists \delta)\ \eta_\delta(n) = i \text{ then } (\exists \delta \in S - S_0)\ \eta_\delta(n) = i\}.$$

Clearly $(\forall S \subseteq S_1)(\exists S' \in K)(S \subseteq S' \wedge |S' - S_0| = |S - S_0| + \aleph_0)$ and K is closed under increasing chains. Hence we can find T_i ($i \leqq \mu$) increasing continuous, $T_0 = S_0$, $T_\mu = S_1$, $|T_i - S_0| < \mu$ for $i < \mu$, and $T_i \in K$.

FACT. If $T \in K$, $\delta \in S_1 - T$ then $\{\eta_\delta(n) : n < \omega\} \cap T$ is finite.

We delay the proof of the Fact. Meanwhile, clearly T_{i+1}, T_i satisfies the requirements on S_1, S_0 in (*) so by the induction hypothesis on μ, $T_{i+1} - T_i$ is free hence some f_i exemplify it. Define f, $\mathrm{Dom} f = (S_1 - S_0) \cap \lim \lambda$:

If $\delta \in T_{i+1} - T_i$ (this holds for one and only one i) $f(\delta)$ is the maximal element of $\{f_i(\delta)\} \cup \{n+1 : \eta_\delta(n) \in T_i\}$; there is a maximal element by the Fact.

So we have just to prove the Fact.

PROOF OF THE FACT. If S, $\delta \in S_1 - S$, are a counterexample, as Φ is a tree, $\eta_\delta(n) \in S$ for every n. So in $G(S_1)/G(S)$, y_δ is divisible by $d_\delta(n)$ for every n, and this shows $G(S_1)/G(S)$ is not free, contradiction.

CLAIM 3.2. If λ is regular and $\{\delta : (\forall n)[\eta_\delta(n) < \delta]\}$ is a stationary subset of λ then Φ is not free.

REMARK. Claim 3.2 indicates a way to produce many non-free Φ; and 3.1 gives the expected translation of properties of G to those of Φ.

PROOF. (1) Suppose $f : \lim \lambda \to \omega$ exemplify Φ is free and we shall get a contradiction. Let $f' : \lim \delta \to \lambda$ be defined by $f'(\delta) = \eta_\delta(f(\delta))$, so clearly $\delta \in S \Rightarrow f'(\delta) < \delta$, hence by the Fodour theorem f' is constant on some stationary set $S' \subseteq S$. But any distinct $\delta_1, \delta_2 \in S'$ contradict the choice of f.

REMARK. In fact we can devise a necessary and sufficient criterion for the freeness of G.

DEFINITION 3.1. Let H be a torsion-free group.

(1) For $c_1, c_2 \in H$, $d \in \mathbf{Z}$ we say $c_1 \equiv c_2 \operatorname{mod}_H d$ if for some $x \in H$, $dx = c_1 - c_2$. This is equivalent to saying $c_1/dH = c_2/dH$ where dH is the subgroup $\{dx : x \in H\}$.

(2) $E_0(H)$ is the group[†] consisting of the sequences $\bar{c} = \langle c_\delta : \delta < \lambda \rangle$ where $c_\delta = \langle c_\delta(n) : n < \omega \rangle$, $c_\delta(n) \in H$, and $c_\delta(n+1) \equiv c_\delta(n) \operatorname{mod}_H d_\delta(n)$. We let $c^*_\delta(n)$ be the unique solution of $d_\delta(n-1)x = c_\delta(n) - c_\delta(n-1)$.

We call $\bar{c}$ appropriate for $\bar{d}$ (and H).

(3) $E_1 = E_1(H)$ is the subgroup of E_0 consisting of those $\bar{c}$'s such that for some $h : \lambda \to H$,

$$c_\delta(n) \equiv h(\delta) - \sum_{l \leq k_\delta(n)} b^l_\delta h(\eta_\delta(l)) \operatorname{mod}_H d_\delta(n).$$

(4) We let $E = E(H)$ be E_0/E_1.

CLAIM 3.3. E is isomorphic to $\operatorname{Ext}(G, H)$.

PROOF. Check the (computational, not categorical) definition of Ext (see [4]).

DEFINITION 3.2. Let $\mathcal{D}^*$ be the set of finite sequences of non-zero natural numbers $\bar{d} = \langle d_0, \cdots, d_{n-1} \rangle$ such that d_l divides d_{l+1}. Writing $c_1 \equiv c_2 \operatorname{mod}_H \bar{d}$ we mean $c_1 \equiv c_2 \operatorname{mod}_H d_{n-1}$.

[†] Addition is coordinatewise.

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CLAIM 3.4. We can define for every $\bar{d} \in \mathscr{D}^*$ a set $H[\bar{d}] \subseteq H$ of representatives $\mathrm{mod}_H \bar{d}$, such that:

(0) $0 \in H[\bar{d}]$,

(1) $H[\bar{d} \restriction k] \subseteq H[\bar{d}]$,

(2) $H[\bar{d}]$ is a set of representatives $\mathrm{mod}_H \bar{d}$,

(3) if $a \in H[\bar{d}]$, $b \in H[\bar{d} \restriction k]$, $a \equiv b \mod \bar{d} \restriction k$ then $a - b \in H[\bar{d}]$ and $(a-b)/d(k-1) \in H(\langle d_k/d_{k-1}, d_{k+1}/d_{k-1}, \cdots, d_{n-1}/d_{k-1}\rangle)$ where $\bar{d} = \langle d_0, \cdots, d_{n-1}\rangle$.

PROOF. First define $H[\bar{d}]$ for every $\bar{d}$ of length $\leqq 1$. Now for $\bar{d} = \langle d_0, \cdots, d_{n-1}\rangle \in \mathscr{D}^*$ let

$$H(\bar{d}) = \left\{ \sum_{l=0}^{n-1} d_{l-1} x_l : x_l \in H(d_l/d_{l-1}) \right\}.$$

(We stipulate $d_{-1} = 1$.)

CLAIM 3.5. Let for $\bar{d} \in \mathscr{D}^*$ $H[\bar{d}]$ be a set of representatives $\mathrm{mod}_H \bar{d}$. Then for every $\bar{c} = \langle c_\delta : \delta < \lambda\rangle \in E_0$ there is $\bar{c}' = \langle c'_\delta : \delta < \lambda\rangle$, $c'_\delta(n) \in H[\bar{d}_\delta \restriction (n+1)]$, such that $\bar{c}' - \bar{c} \in E_0$.

PROOF. By induction on n.

Choose $c'_\delta(n) \in H[d_\delta \restriction (n+1)]$, $c'_\delta(n) \equiv c_\delta(n) \mathrm{mod}_H d_\delta \restriction (n+1)$. Now $h = 0$, show $\bar{c}' - \bar{c} \in E'_0$ (see Definition 3.1(3)).

CLAIM 3.6. Suppose (Φ, g) has the uniformization property, where

$$g(i) = \{x\colon \text{ for some } \delta,\ n \text{ and } m,\ x \in H[\bar{d}_\delta \restriction (m+1)),\ x \equiv 0 \mod \bar{d}_\delta \restriction m,\ \eta_\delta(n) = i \text{ and } k_\delta(m-1) < n \leqq k_\delta(m)\}$$

(remember $d^*_\delta(n) = d_\delta(n)/d_\delta(n-1)$; more formalistically, we should replace $g(i)$ by $|g(i)|$). Then $\mathrm{Ext}(G, H) = 0$.

PROOF. By 3.3 it suffices to prove that $E_0 = E_1$. So assume we are given an appropriate $\bar{c} \in E_0$, w.l.o.g. as $\bar{c}'$ in 3.5. Now we apply the uniformization property of (Φ, g) for the case. We attach to η_δ the sequence $e_\delta = \langle e_\delta(n) : n < \omega\rangle$ where $e_\delta(n)$ is defined as follows. First we define $e_\delta(l) \in H$, $l \leqq k_\delta(m)$ by induction on m such that $c_\delta(n) \equiv -\Sigma_{l \leqq k_\delta(n)} b^l_\delta e_\delta(l) \mathrm{mod}_H d_\delta(n)$.

For $n = 0$, we can first choose the $e_\delta(l)$'s as integral multiples of $c_\delta(0)$, and as the greatest common divisor of the b^l_δ $(l \leqq k_\delta(0))$ is $1 = d_\delta(-1)$, this is possible. Then we can replace them by equivalent members of H $(\langle d_\delta(0)\rangle)$. We can continue to define for $n+1$ such that

$$c_\delta(n+1) \equiv -\sum_{l \leqq k_\delta(n+1)} b^l_\delta e_\delta(l) \mathrm{mod}_H d_\delta(n+1)$$

and for $k_\delta(n) < l \leqq k_\delta(n+1)$, $e'_\delta(l) \equiv 0 \bmod_H d_\delta(n)$ (remember $c_\delta(n+1) \equiv c_\delta(n) \bmod_H d_\delta(n)$); let $e_\delta(l) = e'_\delta(e)/d_\delta(n) \in H[\langle d^*_\delta(n)\rangle]$. So $e_\delta = \langle e_\delta(l) : l < \omega\rangle$ are defined (by 3.4 they are suitable for the application of this uniformization). So there is a function $f^* : \lim\lambda \to \omega$, such that: $n_l \geqq f^*(\delta_l)$ $(l = 0, 1)$, $\eta_{\delta_0}(n_0) = \eta_{\delta_1}(n_1)$ (then $e_\delta(n_l)$ are equal (for $l = 0, 1$)). We can assume $f^*(\delta) = k_\delta(f^0(\delta))$ for some $f^0 : \lim\lambda \to \omega$.

We let $S_0 = \bigcup\{\eta_\delta(n) : n \geqq f(\delta)\}$. We now define an $h : \lambda \to H$ exemplifying $\bar{c} \in E_0$. On $\lambda - \lim\lambda - S_0$, h is constantly zero. We now define $h(\eta_\delta(n))$ $(\delta \in \lim\lambda)$ and $h(\delta)$ $(f(\delta) = n,\ \delta \in \operatorname{Lim}\lambda)$ as follows:

(i) if $n = f^*(\delta)$, $h(\delta) = c_\delta(n) + \Sigma_{l \leqq n} b^l_\delta h(\eta_\delta(l))$

and

(ii) for every δ, $n > f^*(\delta)$ let $h(\eta_\delta(n)) = e_\delta(n)$ (well defined as f^* uniformize).

There is no problem in the checking.

The following is a (one-sided) translation of the Whitehead problem to a combinatorial one.

Conclusion 3.7. If there is a non-free Φ, and $(\Phi, 2)$ has the uniformization property then there is a non-free Whitehead group.

Proof. By 1.6 w.l.o.g. Φ is a tree. By 3.1, $G = G(\Phi, d)$ (where $d_\delta(n) = 2^n$ for every n) is not free. By 3.2, G is a Whitehead group.

Remark. (1) There is no real difficulty in generalizing this section to not necessarily torsion free H. In such cases $c^*(n)$ is not uniquely defined.

A partial converse to 3.7 is:

Claim 3.8. Suppose Φ is a tree $G = G(\Phi, d)$ and let g be such that $g(i) = |H/d^*_\delta(n)H|$ whenever $\eta_\delta(n) = i$. Then $\operatorname{Ext}(G, H) = 0$ implies (Φ, g) has the uniformization property, provided that:

(i) $H = \mathbf{Z}$

or

(ii) $H = \mathbf{Z}_\omega$ = the direct sum of $\aleph_0$ copies of $\mathbf{Z}$.

Proof. Let $\langle c^*_\delta(n) : n < \omega\rangle$ $(\delta \in \lim\lambda)$ be given, $c^*_\delta(n) \in H[(d^*_\delta(n))]$ and we should find $f : \lim\lambda \to \omega$ such that $\delta_0 \neq \delta_1 \in \lim\lambda$, $n \geqq f(\delta_0)$, $n \geqq f(\delta_1)$, $\eta_{\delta_0}(n) = \eta_{\delta_1}(n)$ implies $c^*_{\delta_0}(n) = c^*_{\delta_1}(n)$.

Define $c_\delta(n)$ by

$$c_\delta(n) = c_\delta(n-1) + d_\delta(n-1)c^*_\delta(n) \qquad (c_\delta(-1) = 0),$$

$c_\delta = \langle c_\delta(n) : n < \omega \rangle$, $\bar{c} = \langle c_\delta : \delta \in \operatorname{Lim} \lambda \rangle$. As we assumed $\operatorname{Ext}(G, H) = 0$, there is h as in Definition 3.1 (3).

So clearly $\eta_{\delta_0}(n) = \eta_{\delta_1}(n)$ implies

$$c_{\delta_0}(n) - c_{\delta_1}(n) = h(\delta_0) - h(\delta_1) \bmod d_\delta(n).$$

We have to define f, and show that if in addition $n \geqq f(\delta_0)$, $n \geqq f(\delta_1)$ then $c^*_{\delta_0}(n) = c^*_{\delta_1}(n)$.

Case (a): $H = \mathbf{Z}$

We could have chosen $H[(d)] = [0, d)$ so $H[(d_0, \cdots, d_{n-1})] = [0, d_{n-1})$. We can assume for some $m \in \{0, 1, 2\}$ $c^*_{3n+m} = c^*_{3n+m+1} = 0$ (we just decompose our problem to three). So w.l.o.g. $m = 0$.

Now if $d_{\delta_0}(n) > 8|h(\delta_0)|$, $d_{\delta_1}(n) > 8|h(\delta_1)|$, then either (a) $c_{\delta_0}(n) - c_{\delta_1}(n) = h(\delta_0) - h(\delta_1)$ or (b) $c_{\delta_0}(n) - c_{\delta_1}(n) - (h(\delta_0) - h(\delta_1)) = \pm d_\delta(n)$ but if n is $\equiv 1 \operatorname{mod}_{\mathbf{Z}} 3$ then $c_{\delta_l}(n) = c_{\delta_l}(n-2)$, $0 \leqq c_{\delta_l}(n-2) < d_{\delta_l}(n-2)$ hence

$$|c_{\delta_0}(n) - c_{\delta_1}(n)| < 2d_{\delta_2}(n-2) < d_{\delta_l}(n)/2.$$

But remember

$$|h(\delta_0)| < d_{\delta_0}(n)/8,$$

$$|h(\delta_1)| < d_{\delta_1}(n)/8.$$

So clearly (b) cannot hold. Now it is easy to prove that if $d(n-3) > 8h(\delta_l)$ then $c^*_{\delta_0}(n-2) = c^*_{\delta_1}(n-2)$, so clearly defining $f(\delta)$ as the first n such that $d_\delta(n-3) > 8|h(\delta)|$ satisfies our requirement.

Case b: Let $\mathbf{Z}_\omega$ be freely generated by $\{x_n : n < \omega\}$. Now w.l.o.g. $c^*_\delta(n)$ is in the subgroup generated by $\{x_l : n \leqq l < \omega\}$, and choose f such that $h(\delta)$ is in the subgroup generated by $\{x_l : l < f(\delta)\}$.

Remark. In 3.8, the essential property of $\mathbf{Z}_\omega$ is the infiniteness of each $g(i)$.

Theorem 3.9. *There is a non-free Whitehead group of cardinality* $\aleph_1$ iff *some tree* (Φ, h) *has the uniformization property,* $|\Phi| = \aleph_1$, $h(a) > 1$ *for every* $a \in \operatorname{Dom} h$ *but* Φ *is not free.*

Proof. Let to the reader.

§4. The uniformization property for some $(\Phi, 2)$, Φ of type $(\aleph_1, \aleph_0, \omega)$ is consistent.

THEOREM 4.1. *Suppose* $2^{\aleph_0} = \aleph_1$, *and* T *is a tree* $h : T \to \omega$, $T = \bigcup_{n<\omega} T_n$, T_n — *the nth level, and for each* $l < \omega$, *for infinitely many* $n < \omega$, *for every* $a_0, \cdots, a_{l-1} \in T_n$, a_0 *has more than* $\Pi\{h(b) : b \leqq a_m, m < l\}$ *immediate successors.*

Then there is a Φ *of type* $(\aleph_1, \aleph_0, \omega)$ *such that*

$$(*) \qquad \text{for each } \eta \in \Phi, \quad \{\eta(n) : n < \omega\} \text{ is a branch of } T, \quad \eta(n) \in T_n$$

and if $k < \omega$, $\eta_i^l \in \Phi$ *for* $i < \omega_1$, $l < k$, then *for some* $n < \omega$, *and distinct* $a_l \in T_n$ $(l < k)$ *and* $w \subseteq \omega_1$

(i) $a_l = \eta_i^l(n)$ *for each* $i \in w$,

(ii) *for* $i \neq j \in w$, $\eta_i^l(n+1) \neq \eta_j^l(n+1)$,

(iii) $|w| > \Pi_{l<k} \Pi_{b \leqq a_l} h(b)$.

PROOF. Quite standard.

REMARK. Alternatively we can define Φ as a generic set of branches. For our purpose this forcing does not change, and we can have $|\Phi| > \aleph_1$. (The conditions have the form $\{a_l \in \eta_{i(l)} : l < k\}$, $a_l \in T$, $i(l) < \omega_1$, $i(l) = i(m) \Rightarrow a_l, a_m$ are comparable.) We can also in the main theorem make $2^{\aleph_0}$ any regular cardinal, and do not assume any instance of G.C.H.

MAIN THEOREM 4.2. *Suppose* G.C.H. *holds*; Φ *satisfies* $(*)$ *from Definition* 4.1. *Then there is a set of forcing conditions* $P = (P, \leqq)$ *such that*

(1) $|P| = \aleph_2$, *P satisfies the* $\aleph_1$-C.C.

(2) *In* V^P, (Φ, h) *has the uniformization property.*

PROOF. Let $\Phi = \{\eta_\zeta : \zeta < \omega_1\}$.

For each candidate $\bar{f}$ let $P_{\bar{f}}$ be a set of forcing conditions which will give a general compressive function. That is, $P_{\bar{f}}$ is the set of functions g, such that $\operatorname{Dom} g$ is a finite subset of ω_1, for $\zeta \in \operatorname{Dom} g$, $g(\zeta) < \omega$, and for $\zeta, \xi \in \operatorname{Dom} g$

$$(\forall n)[g(\zeta) \leqq n \wedge g(\xi) \leqq n \wedge \eta_\zeta(n) = \eta_\xi(n) \to f_\zeta(\eta_\zeta(n)) = f_\xi(\eta_\xi(n))]$$

the order is inclusion; trivially, $P_{\bar{f}}$ satisfies the $\aleph_1$-C.C. and the generic G is as required. But we have to iterate, in order to take care of all $\bar{f}$'s, including the new ones. On iterated forcing see, e.g., [6].

So we define by induction on $\alpha \leqq \omega_2$ a set of forcing conditions P_α, and carefully chosen names $\bar{f}^\alpha = \{f_\xi^\alpha : \xi < \omega_1\}$, such that $\Vdash^{P_\alpha}$ "$\bar{f}^\alpha$ is a candidate for (Φ, h)". The elements of P_α will be all finite functions p, $\operatorname{Dom} p \subseteq \alpha$, for each $\zeta \in \operatorname{Dom} p$ $p(\zeta)$ is a finite function from ω_1 to ω, and $p \restriction \zeta \Vdash^{P_\zeta}$ "$p(\zeta) \in P_{\bar{f}^\zeta}$". (So the elements of P_α are in V.)

The order is defined by $q \leqq p$ iff $\zeta \in \operatorname{Dom} q \Rightarrow q(\zeta) \subseteq p(\zeta)$. Now $P = P_{\omega_2}$; the only non-trivial point is to show P satisfies the $\aleph_1$-C.C. For this we prove by induction on $\alpha \leqq \omega_2$ the following stronger condition:

$(**)_\alpha$ If $k < \omega$, and for $i < \omega_1$, $l < k$, $p_i \in P_\alpha$ and $\eta_i^l \in \Phi$ and $n_1 < \omega$ then there are $n < \omega$, $n > n_1$, distinct $a_0, \cdots, a_{k-1} \in T$ and $w \subseteq \omega_1$ such that

(i) $a_l = \eta_i^l(n)$ for $l < k$, $i \in w$,

(ii) for $i \neq j \in w$, $\eta_i^l(n+1) \neq \eta_j^l(n+1)$ or $\eta_i^l = \eta_j^l$,

(iii) $|w| > \prod_{l<k} \prod_{b \leq a_l} h(b)$,

(iv) there is $q \in P_\alpha$ such that $p_i \leqq q$ for each $i \in w$.

Case I: $\alpha = 0$

There is nothing to prove.

Case II: $\alpha = \omega_2$

Then for some $\beta < \alpha$, $p_i \in P_\beta$ for every i, so $(**)_\beta$ gives the desired conclusion.

Case III: α limit, $\mathrm{cf}\,\alpha = \omega$

Let $\alpha = \bigcup_{n<\omega} \alpha_n$, then for each i, for some $n(i)$, $p_i \in P_{\alpha_{n(i)}}$, so for some n, $|\{i : n(i) = n\}| = \aleph_1$, so by renaming, $(**)_{\alpha_n}$ gives the conclusion.

Case IV: α limit, $\mathrm{cf}\,\alpha = \omega_1$

Let α_i $(i < \omega_1)$ be increasing and continuous, $\alpha = \bigcup_{i<\omega_1} \alpha_i$. For each i let $h(i) = \sup(\{0\} \cup (\mathrm{Dom}\, p_i \cap i))$; so for $i > 0$, $h(i) < i$, so for some $i(0)$, $S = \{i < \omega_1 : h(i) < i_0\}$ is stationary. W.l.o.g., $i, j \in S$, $i < j$ implies $p_i \in P_{\alpha_j}$. Now for $i, j \in S$, p_i, p_j are compatible iff $p_i \restriction \alpha_{i(0)}$, $p_j \restriction \alpha_{i(0)}$ are, so rename and use $(**)_{i(0)}$.

Case V: $\alpha = \beta + 1$

W.l.o.g. $|\mathrm{Dom}\, p_i(\beta)|$ is constant, so let $\mathrm{Dom}\, p_i(\beta) = \{\eta_i^l : k \leqq l < k(0)\}$ and w.l.o.g. $p_i(\beta)(\eta_i^l)$ depend on l only. Now we apply $(**)_\beta$ to $h(0)$, $p_i' = p_i \restriction \beta \in P_\beta$ and η_i^l $(i < \omega_1, l < k(0))$.

We get appropriate $n \geqq n_1 + n_0$, a_l $(l < k(0))$, w_0 and q_0 satisfying (i)–(iv) from $(**)_\beta$. Clearly we can find $q_1 \in P_\beta$, $q_0 \leqq q_1$, such that for each $i \in w_0$, $k \leqq l < k(0)$ and $m \leqq n$

$$q_1 \Vdash^{P_\beta} \text{“} f_\gamma^\alpha(m) = c_i(l, m) \text{”} \qquad \text{where } \eta_i^l = \eta_\gamma.$$

Clearly $c_i(l, m) < h(\eta_i^l(m))$, hence the number of possible functions c_i is $\leqq \prod_{k \leq l < k(0)} \prod_{b \leq a_l} h(b)$.

As $|w_0| > \prod_{l<k(0)} \prod_{b \leq a_l} h(b)$ clearly for some c, $w = \{i \in w_0 : c_i = c\}$ has cardinality $> \prod_{l<k} \prod_{b \leq a_l} h(b)$. Now it is easy to check that

$$q = q_1 \cup \left\{ \left\langle \beta, \bigcup_{i \in w_0} p_i(\beta) \right\rangle \right\} \in P_\alpha$$

and n, a_l $(l < k)$, w, q exemplify the conclusion of $(**)_\alpha$.

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CONCLUSION 4.3. It is consistent with ZFC that some Φ of type $(\aleph_1, \aleph_0, \omega)$ have the uniformization property (provided that ZFC is consistent). In this model there is a Whitehead group of cardinality $\aleph_1$ which satisfies Case I from [7].

§5. Club is not equivalent to diamond

For a stationary $S \subseteq \lambda$, λ regular $\diamondsuit_S$ means there are $S_\alpha \subseteq \alpha$ for $\alpha \in S$ such that for any $A \subseteq \lambda$, $\{\alpha \in S : A \cap \alpha = S_\alpha\}$ is stationary. ($\diamondsuit$ is $\diamondsuit_{\omega_1}$.) Jensen [6] introduces this principle and shows it holds if $V = L$; and it is widely used. Note that $\diamondsuit_S \Rightarrow \lambda^{<\lambda} = \lambda$ so $\diamondsuit_{\aleph_1} \Rightarrow 2^{\aleph_0} = \aleph_1$. This is discussed in Devlin [2].

Ostaszewski suggests a version called $\clubsuit = \clubsuit_{\omega_1}$, where for a stationary $S \subseteq \lambda$ (λ regular) $\clubsuit_S$ means: there are $S_\alpha \subseteq \alpha$ unbounded in α, for each $\alpha \in S$ such that for any unbounded $S \subseteq \lambda$, $S_\alpha \subseteq S$ for at least one $\alpha \in S$ (equivalently, for a stationary set of such α's). Our result may be helpful in proving consequences of the diamond are independent of CH. On forcing, see e.g. Jech [6].

Burgess and Devlin show CH + $\clubsuit \Rightarrow \diamondsuit$, and in fact $\lambda = \lambda^{<\lambda} + \clubsuit_S \Rightarrow \diamondsuit_S$ (if S_α exemplifies $\clubsuit_S$, $\{A_i : i < \alpha\}$ enumerate $\{A \subseteq \lambda : |A| < \lambda\}$, each appearing λ times). Let $B_\alpha = \bigcup_{i \in S_\alpha} A_i$ for $\alpha \in S$, so for each $A \subseteq \lambda$ let $j(\gamma)$ be the first $j > \bigcup_{\beta < \gamma} j(\beta)$ such that $A_j = A \cap (\bigcup_{\beta \leq \gamma} j(\beta))$. Now $J = \{j(\gamma) : \gamma < \lambda\}$ is unbounded, so for some α, $S_\alpha \subseteq J$ hence $B_\alpha = A \cap \alpha$.

In Devlin [2] and in a list of problems of Fleissner, it is asked whether $\clubsuit \Rightarrow$ CH (equivalently $\clubsuit \Rightarrow \diamondsuit$). The answer is negative. Baumgartner had proved years ago the consistency of a weaker assertion with $2^{\aleph_0} > \aleph_1$: there is a family of $\aleph_1$ countable subsets of $\aleph_1$, such that any uncountable subset of $\aleph_1$ contains one of them.

THEOREM. *It is consistent with* ZFC *that* $\clubsuit$ *whereas* CH *fails, and e.g.* $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ (*we can give* $2^{\aleph_0}$, $2^{\aleph_1}$ *any reasonable value*).

PROOF. Start with $V = L$. Use forcing. First add $\aleph_3$ subsets of ω_1, by the forcing: $P^0 = \{f : f$ a function from a countable $A \subseteq \omega_3$ to $\{0, 1\}\}$ ordered by inclusion (so in V^{P^0}, $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_3 = 2^{\aleph_2}$ and cardinalities are preserved). Next collapse $\aleph_1$ by the forcing $P^1 = \{f : f$ a function from a finite $A \subseteq \omega$ to $\omega_1\}$ (so P^1 collapse $\aleph_1$, and preserve cardinals $\neq \aleph_1$, and preserve 2^λ for $\lambda \neq \aleph_0$). $(V^{P^0})^{P^1}$ is as required by the following two facts:

FACT 1. If $\clubsuit_S$ holds, $\lambda = \sup S$ is regular, $|P| < \lambda$, P a set of forcing conditions then in V^P (i.e., any generic extension of V by P) $\clubsuit_S$ holds. This is because in V^P, any subset of λ is the union of $\leqq |P|$ substs of λ which belong to

V. Hence any unbounded subset of λ in V^P contains an unbounded subset of λ from V. So if $\langle S_\alpha : \alpha \in S\rangle \in V$ exemplifies $\clubsuit_S$ in V then it exemplifies in V^P too.

FACT 2. If $\diamondsuit_S$ holds in V, $\lambda = \sup S$ is regular, $S \subseteq \{\delta < \lambda : \mathrm{cf}\,\delta = \mu\}$, P is a μ^+-complete set of forcing conditions, then in V^P, $\clubsuit_S$ holds.

By $\diamondsuit_S$ we can define $M_\alpha = (\alpha, \leqq_\alpha, R_\alpha)$ for $\alpha \in S$ such that for any (partial) order $\leqq^*$ on λ, and two-place relation R on λ, for a stationary set of α's, $\leqq_\alpha = \leqq^* \restriction \alpha$, $R_\alpha = R \restriction \alpha$. For each $\alpha \in S$ choose $\xi^i_\alpha < \alpha$ (for $i < \mu$), $\alpha = \sup \xi^i_\alpha$; and choose inductively on $i < \mu$ if possible β^i_α, γ^i_α such that $\beta^0_\alpha = 0$, $\xi^i_\alpha \leqq \gamma^i_\alpha$, $R_\alpha(\beta^i_\alpha, \gamma^i_\alpha)$ and $\beta^i_\alpha (i < \mu)$ increase by $\leqq^*$ with i. If we succeed, let $S_\alpha = \{\gamma^i_\alpha; i < \mu\}$, and if we fail, let $S_\alpha = \{\xi^i_\alpha : i < \mu\}$. Now $\{S_\alpha : \alpha \in S\}$ exemplifies $\clubsuit_S$ even in V^P. For suppose $p \in P$, $p \Vdash$ "τ is an unbounded subset of λ". As $\diamondsuit_S$, clearly $\lambda^\mu = \lambda$, so we can choose $Q \subseteq P$, such that $|Q| = \lambda$, $p \in Q$, any chain in Q of length $\leqq \mu$ has an upper bound and for every $q \in Q$, $\alpha < \lambda$, for some $q' \in Q$, $\alpha' > \alpha$, $q \leqq q'$, $q' \Vdash$ "$\alpha' \in \tau$". Let $Q = \{q(i) : i < \lambda\}$, $q_0 = p$, and define $i \leqq^* j$ iff $q(i) \leqq q(j)$, $R = \{\langle i, j\rangle : q(i) \Vdash$ "$j \in \tau$"$\}$.

For some $\alpha \in S$, M_α is an elementary submodel of $(\lambda, \leqq^*, R)$, and any increasing chain (by $\leqq^*$) of length $< \mu$ has an upper bound in it. So we succeed in defining β^i_α, γ^i_α as required, hence $q(\beta^i_\alpha) \in Q$ $(i < \mu)$ is increasing, so it has a bound q. So as $\beta^0_\alpha = 0$, $p = q_0 \leqq q$; and as $q(\beta^i_\alpha) \leqq q$, $q \Vdash$ "$\gamma^i_\alpha \in \tau$". So $q \Vdash$ "$S_\alpha \subseteq \tau$" and $q_0 \leqq q$, hence we finish.

§6. For many G, $|\mathrm{Ext}(G, \mathbf{Z})| \neq \aleph_0$

The motivation of the following theorem was whether for some abelian group G, $|\mathrm{Ex}(\mathbf{Z}, G)| = \aleph_0$ (see Hiller and Shelah [5] where it is proved that when $V = L$ there is no such G). The main point is that for $\aleph_1$-free G, $\mathrm{Ext}(G, \mathbf{Z})$ has cardinality 1 or $\geqq 2^{\aleph_1}$. By [5] this has consequences in algebraic topology. We want to prove this without the hypothesis $V = L$, but our result only implies this in many cases.

NOTATION 6.1. Let κ be a cardinal, $\{A_i : i \in S\}$ an indexed family of sets, and $\mathcal{D}_i$ a κ-complete filter over A_i, $A^* = \bigcup_{i \in S} A_i$. Let $\mathbf{P}(S)$ be the family of subsets of S. A colouring of A_i is a function $c : A_i \to \{0, 1\}$, a T-colouring (for $T \subseteq S$) is an indexed family $\{c_i : i \in T\}$, c_i a colouring of A_i. Let $\mathbf{0}_i$ be the constant function 0 on A_i, $\mathbf{0}_T = \{\mathbf{0}_i : i \in T\}$. We let, for $T \subseteq S$, $\{c^1_i : i \in T\} \approx_T \{c^2_i : i \in T\}$ if some $f : A^* \to \{0, 1\}$ exemplify it, i.e., for each $i \in T$, $\{a \in A_i : c^1_i(a) = c^2_i(a) + f(a) \bmod 2\} \in \mathcal{D}_i$. Clearly $\approx_T$ is an equivalence relation, and let $\mu(T)$ be the number of equivalence classes.

We call $f: A^* \to \{0,1\}$ a *solution* of $\{c_i : i \in T\}$ if it exemplifies its $\approx_T$-equivalence to $\mathbf{0}_T$.

We say $T \subseteq S$ is *separated* if for some $A \subseteq A^*$, for each $i \in T$, $A_i \cap A \in \mathcal{D}_i$, and for each $i \in S - T$, $A_i - A \in \mathcal{D}_i$.

THEOREM 6.2. *Suppose there is no measurable cardinal κ_1 and $i \in S$ such that $\kappa < \kappa_1 \leqq |A_i|$. If some $\{c_i : i \in S\}$ has no solution (or equivalently $\mu(S) > 1$) then $\mu(S) \geqq 2^{\aleph_0}$. Moreover $\mu(S) \geqq \kappa^+$ except, possibly, when there are infinitely many measurable cardinals $> \kappa$, $< |S|$. Also if $|S| = \kappa^+$ then $\mu(S) = 2^{\kappa^+}$.*

PROOF. Let E be the family of subsets T of S satisfying

(a) every $\langle c_i : i \in T\rangle$ has a solution,

(b) T is separated.

We now show E is an ideal (over S). For this we have to show:

(A) $S \notin E$.

This is so, as by hypothesis some $\langle c_i : i \in S\rangle$ has no solution, contradicting (a).

(B) If $T \in E$ and $T_1 \subseteq T$ then $T_1 \in E$.

T_1 satisfies (a) trivially, and as for (b) define c_i $(i \in T)$ as follows:

$$c_i(a) = \begin{cases} 0 & i \in T_1, \\ 1 & i \in T - T_1. \end{cases}$$

By (a) for T we have a solution f and by (b) a separating set $A \subseteq A^*$ for T. Now $A \cap \{\alpha : f(\alpha) = 0\}$ is a separating set for T_1.

(C) E is closed under union (of two).

If $T_1, T_2 \in E$, we can assume they are disjoint (by (B)), so if A_1, A_2 are separating sets for T_1, T_2 resp. then $A_1 \cup A_2$ is a separating set for $T_1 \cup T_2$ (as each $\mathcal{D}_i$ is a filter), so $T_1 \cup T_2$ satisfies (b). As for (a), let $\{c_i : i \in T_1 \cup T_2\}$ be given, then we can find solutions f_1, f_2 of $\{c_i : i \in T_1\}$, $\{c_i : i \in T_2\}$ resp. and then $f_1 \restriction A_1 \cup f_2 \restriction (A^* - A_1)$ is a solution for $\{c_i : i \in T_1 \cup T_2\}$.

(We remark that E is in fact κ-complete, but we do not need this.)

Now

CLAIM 6.3. S is not the union of κ members of E.

PROOF. Let $T_\alpha \in E$ for $\alpha < \kappa$, and suppose $\bigcup_{\alpha<\kappa} T_\alpha = S$, and we shall get a contradiction by showing every $\{c_i : i \in S\}$ has a solution.

An E is an ideal; we can assume the T_α's are pairwise disjoint.

For each $\alpha < \kappa$, as $T_\alpha \in E$ there is a separating set B_α for it. We can assume that also B_α $(\alpha < \kappa)$ are pairwise disjoint, for if $B'_\alpha = B_\alpha - \bigcup_{B<\alpha} B_\beta$ then for $i \notin T_\alpha$,

$$A_i - B'_\alpha \supseteq A_i - B_\alpha \in \mathcal{D}_i \qquad \text{hence } A_i - B'_\alpha \in \mathcal{D}_i$$

and for $i \in T_\alpha$

$$A_i \cap B'_\alpha = (A_i \cap B_\alpha) - \bigcup_{\beta<\alpha} (A_i - B_\beta) = \bigcap_{\beta<\alpha} [A_i \cap B_\alpha - (A_i \cap B_\beta)]$$

as $\beta \neq \alpha$, $i \notin T_\beta$, $[A_i \cap B_\alpha - (A_i \cap B_\beta)] \in \mathcal{D}_i$, but $\alpha < \kappa$, $\mathcal{D}_i$ is κ-complete, so $A_i \cap B'_\alpha \in \mathcal{D}_i$.

Now let us show each $\{c_i : i \in S\}$ has a solution, for let f_α be a solution of $\{c_i : i \in T_\alpha\}$, then $\bigcup_{\alpha<\kappa} (f_\alpha \restriction B_\alpha)$ is a solution for $\{c_i : i \in S\}$.

This contradicts a hypothesis, hence the claim holds.

Let E^c be the closure of E under unions of $\leqq \kappa$ sets, so E^c is a non-trivial κ^+-complete ideal over S.

Claim 6.4. If there are λ pairwise disjoint subsets of S not in E then $\mu(S) \geqq 2^\lambda$.

Proof. Let S_α $(\alpha < \lambda)$ be pairwise disjoint subsets of S which are not in E and suppose $\mu(S) < 2^\lambda$. For each $I \subseteq \lambda$ let $\bar{c}^I = \{c^I_i : i \in S\}$, where

$$c^I_i(a) = \begin{cases} 0 & i \in \bigcup_{\alpha \in I} S_\alpha, \\ 1 & \text{otherwise.} \end{cases}$$

So there are distinct $I, J \subseteq \lambda$ such that $\bar{c}^I \approx_S \bar{c}^J$; and let $f : A^* \to \{0, 1\}$ exemplify it. Let $K = \bigcup\{S_\alpha : \alpha \in I \equiv \alpha \notin J\}$, thus we can check that $\{\alpha : f(\alpha) = 1\}$ separates K. So for every family of λ disjoint subsets of S not in E, there is a non-empty subfamily whose union is separated. As we can partition $\{S_\alpha : \alpha < \lambda\}$ into λ pairwise disjoint families we have λ pairwise disjoint $S_\alpha \subseteq S$, each S_α is separated but does not belong to E. Hence each S_α fails to satisfy condition (a), so some $\{c^\alpha_i : i \in T_\alpha\}$ have no solution. Define for each $I \subseteq \lambda$, $\bar{c}_I = \{c^I_i : i \in S\}$ where

$$c^I_i = \begin{cases} c^\alpha_i & i \in T_\alpha, \quad \alpha \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{c}_I$ $(I \subseteq \lambda)$ is a family of 2^λ pairwise non $\approx_S$-equivalent S-colourings, so we prove the claim.

It remains to find those λ sets, so Theorem 6.2 follows from

Claim 6.5. (1) $\mathbf{P}(S)/E$ is infinite, hence there are in it $\mathbf{P}(S) - E$ $\aleph_0$ pairwise disjoint sets.

(2) Letting λ be the first λ such that in $\mathbf{P}(S)-E$ there are no λ pairwise disjoint elements, then λ is regular and $2^{<\lambda} \geqq \kappa^+$, except, maybe, when there are infinitely many measurables $\leqq |S|$, $>\kappa$.

(3) If $|S|=\kappa^+$, then $\lambda=\kappa^+$.

Proof. (1) Otherwise $T_1,\cdots,T_n$ are pairwise disjoint, $\bigcup_{l=1}^n T_l = S$, $T_l \notin E$, but $E \restriction T_l = \{A \cap T_l : A \in E\}$ is a prime ideal or equivalently $\mathcal{D}^l = \{T_l - A : A \in E\}$ is an ultrafilter over T_l. If no $\mathcal{D}^l$ is κ^+-complete each T_l is the union of $\leqq \kappa$ members of $E \restriction T$, hence $S = \bigcup_{l=1}^n T_l$ is the union of $\leqq \kappa$ members of E, contradicting 6.3. So assume $\mathcal{D}_l$ is κ^+-complete iff $l \leqq m$ where $1 \leqq m \leqq n$, and choose $T'_l \in \mathcal{D}^l$ of minimal cardinality. Let $\kappa_l = |T'_l|$, $T'_l = \{s^l_\alpha : \alpha < \kappa_l\}$, and $\mathcal{D}^l_1 = \{A \subseteq \kappa_l : \{s^l_\alpha : \alpha \in A\} \in \mathcal{D}^l\}$; clearly $\mathcal{D}^l_1$ is a uniform κ^+-complete ultrafilter over κ_l, for $l \leqq m$. Moreover, letting $\kappa_0 = \bigcup\{|A_i|^+ : i \in S\}$, each $\mathcal{D}^l_1$ is κ_0-complete (as if $\kappa(\mathcal{D})$ is the maximal κ for which $\mathcal{D}$ is κ-complete, $\kappa(\mathcal{D})$ is measurable, see e.g. [6]; and by a hypothesis in 6.2). Clearly for each $\alpha < \kappa_i$, $T^\alpha_l = \{s^l_\beta : \beta < \alpha\} \in E \restriction T_l$ (by the choice of T'_l). Let for every $\alpha_1 < \kappa_1, \cdots, \alpha_m < \kappa_m$,

$$T(\alpha_1,\cdots,\alpha_m) = \bigcup_{l=m+1}^{n} T_l \cup \bigcup_{l=1}^{m} (T_l - T'_l) \cup \bigcup_{l=1}^{m} T^{\alpha_l}_l.$$

It is easy to check $T(\alpha_1,\cdots,\alpha_m)$ is the union of $\leqq \kappa$ members of E. Let $\mathcal{D}^*$ be $\mathcal{D}^1_1 \times \cdots \times \mathcal{D}^m_1$, so it is a κ_0-complete ultrafilter over $\kappa_1 \times \cdots \times \kappa_m$.

Let $\{c_i : i \in S\}$ be any S-colouring, so for every $\alpha_1 < \kappa_1, \cdots, \alpha_m < \kappa_m$, $\{c_i : i \in T(\alpha_1,\cdots,\alpha_m)\}$ has a solution $f_{\alpha_1,\cdots,\alpha_m}$ (by the proof of 6.3). Let us define function $f : A^* \to \{0,1\}$. $f(a)$ is defined such that

$$\{\langle \alpha_1,\cdots,\alpha_m\rangle \in \kappa_1 \times \cdots \times \kappa_m : f_{\alpha_1,\cdots,\alpha_m}(a) = f(a)\} \in \mathcal{D}^*$$

(as $\mathcal{D}^*$ is an ultrafilter, $f(a)$ exists). As $\mathcal{D}^*$ is κ_0-complete, and for each i, $|A_i| < \kappa_0$, κ_0 measurable, so $2^{|A_i|} < \kappa_0$, clearly f is a solution for $\{c_i : i \in S\}$, contradiction. So we prove that in $\mathbf{P}(S)-E$ there are $\aleph_0$-disjoint elements.

(2) We define by induction on $\alpha \leqq \lambda$, for $\eta \in {}^\alpha 2$, sets $T_\eta \subseteq S$. $T_{\langle\ \rangle} = T$, if T_η is defined, $\notin E$, and $T_\eta \restriction E$ is not a prime ideal, we choose disjoint $T_{\eta \frown \langle 0\rangle}$, $T_{\eta\frown\langle 1\rangle} \in P(T_\eta) - E$ whose union is T_η, and for η of limit length δ such that $(\forall \alpha < \delta)\,(T_{\eta|\delta}$ is defined) let $T_\eta = \bigcap_{\alpha<\delta} T_{\eta|\alpha}$. Let $V = \{\eta : T_\eta$ defined$\}$; clearly if some $\eta \in V$ has length $\geqq \lambda$, there are λ pairwise disjoint sets in $\mathbf{P}(S)-E$, so suppose $\eta \in V \Rightarrow l(\eta) < \lambda$. We have a partition $\{T_\eta : \eta \in \mathbf{Q}_1\}$, $\mathbf{Q}_1 = \{\eta : T_{\eta\frown\langle i\rangle}$ not defined but T_η is defined$\}$ of S into $\leqq 2^{<\lambda}$ sets. Now for $\eta \in \mathbf{Q}_1$, $T_\eta \in E$ or $E \restriction T_\eta$ is a prime ideal. So if $2^{<\lambda} \leqq \kappa$ we can continue as in (1).

(3) By the Ulam theorem.

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