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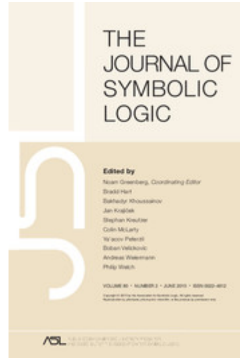

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ON THE ELEMENTARY EQUIVALENCE OF AUTOMORPHISM GROUPS OF BOOLEAN ALGEBRAS; DOWNWARD SKOLEM LÖWENHEIM THEOREMS AND COMPACTNESS OF RELATED QUANTIFIERS

MATATYAHU RUBIN AND SAHARON SHELAH

Abstract.

THEOREM 1. ($\diamondsuit_{\aleph_1}$) *If B is an infinite Boolean algebra (BA), then there is B_1 such that* $|\mathrm{Aut}(B_1)| \leq |B_1| = \aleph_1$ *and* $\langle B_1, \mathrm{Aut}(B_1)\rangle \equiv \langle B, \mathrm{Aut}(B)\rangle$.

THEOREM 2. ($\diamondsuit_{\aleph_1}$) *There is a countably compact logic stronger than first-order logic even on finite models.*

This partially answers a question of H. Friedman. These theorems appear in §§1 and 2.

THEOREM 3. (a) ($\diamondsuit_{\aleph_1}$) *If B is an atomic $\aleph_1$-saturated infinite BA, $\psi \in L_{\omega_1\omega}$ and* $\langle B, \mathrm{Aut}(B)\rangle \models \psi$ *then there is B_1 such that* $|\mathrm{Aut}(B_1)| \leq |B_1| = \aleph_1$ *and* $\langle B_1, \mathrm{Aut}(B_1)\rangle \models \psi$. *In particular if B is 1-homogeneous so is B_1.* (b) (a) *holds for $B = P(\omega)$ even if we assume only CH.*

Introduction. The basic constructions in this paper appear in Theorems 1.2, 2.5 and 3.1; they have the following aim: given a complete theory T, which contains a certain small part of set theory, construct a model M of T of power $\aleph_1$, such that every function from $|M|$ to $|M|$ with certain properties is definable in M.

In applying these constructions we have three options: (1) Apply the construction to a theory T that has exactly the required properties, and then if T does not have definable automorphisms we can obtain a rigid model of T.

(2) Apply the construction to a theory T, which in addition to the set theory required for the construction, contains also some comprehension axioms. Here we obtain a model of T in which every automorphism is "inner". For example, let H_{λ^+} be the set of all sets which are hereditarily of cardinality $\leq \lambda$. We obtain a model $M \equiv \langle H_{\lambda^+}, \in\rangle$ of power $\aleph_1$, such that every permutation π of λ^M, π belongs to $|M|$, provided it has the following property: for every a: $|M| \ni a \subseteq \lambda^M$ implies $\pi(a) \in |M|$.

(3) We can consider of course theories T which do not have the required set theory for the construction. In such a case we will first expand T to include the needed set theory and then apply the construction.

Typical examples of this application are Theorems 1 and 3 in the abstract.

In fact it is this direction in which we obtain more interesting results.

Theorems 1 and 3 can be regarded as a counterpart of [R1], [R2], [M] and [S2].

In [M] and further in [S2] it is shown that many symmetric groups are categorical in the class of symmetric groups. Every symmetric group is of course the automorphism group of a BA. Our theorems show under $\diamondsuit_{\aleph_1}$ that no infinite symmetric

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group is categorical in the class of automorphism groups of BA's; moreover, Sym($\aleph_0$), which is an automorphism group of a 1-homogeneous BA, is not even categorical in the class of automorphism groups of all 1-homogeneous BA's.

In [R2] assuming $V = L$ it was shown that every automorphism group of a countable BA is categorical in the class of automorphism groups of countable BA's. This paper shows, on the other hand, that no automorphism group of an infinite BA is categorical in the class of automorphism groups of all BA's.

In [R1] it is shown that if B is an atomic BA that has some homogeneity properties then in Aut (B) one can interpret e.g. the sets of finite sets of atoms of B.

Let B_0 be the BA of finite and cofinite subsets of ω. Let $B_1 \not\cong B_0$ be a BA such that $\text{Aut}(B_1) \equiv \text{Aut}(B_0)$; the existence of such a BA follows from this paper; then in Aut (B_1) the set of finite sets of atoms cannot be interpreted.

A method for constructing models with certain second-order properties usually gives rise to a compactness theorem of a certain generalized quantifier. This is also the case here.

Suppose χ is a sentence in the language of $\langle B, \text{Aut}(B); \subseteq, \text{Op}\rangle$. Let $(Q_\chi xy)$ $(\psi(x, y, \bar{u}))$mean that $\{\langle x, y\rangle \mid \models \psi(x, y, \bar{u})\}$ is a partial ordering $\subseteq$ of its domain A such that $\langle A, \subseteq\rangle$ is a BA, and

$$\langle A, \text{Aut}(\langle A, \subseteq\rangle); \subseteq, \text{Op}\rangle \models \chi.$$

In 2.9 we prove under $\diamondsuit_{\aleph_1}$ that the language containing all the above quantifiers is countably compact.

In fact, Shelah [S3] proved that in §§1 and 2, $\diamondsuit_{\aleph_1}$ can be replaced by CH. Assuming $\lambda^{\lambda} = \lambda$ and $\diamondsuit_{\lambda^+}$ Shelah [S1] also proved the analogues of the theorems in §§1 and 2 gotten by replacing $\aleph_1$ by λ^+. This yields, e.g. under $V = L$, that the above language is compact.

Let χ be the sentence saying that B is atomic, and there is $f \in \text{Aut}(B)$ such that f moves every atom of B and $f^2 = \text{Id}$; then Q_χ is stronger than first-order logic on finite models. This partially answers a question by H. Friedman (see [F] and [S1]).

THEOREM (SHELAH). ($\diamondsuit_{\aleph_1}$) *Let F be an ordered field then there is an ordered field* F_1 *such that* $|\text{Aut}(F_1)| \le |F_1| = \aleph_1$ *and* $\langle F_1, \text{Aut}(F_1); +, \cdot, <, \text{Op}\rangle \equiv \langle F, \text{Aut}(F); +, \cdot, <, \text{Op}\rangle$.

The proof of this theorem appears in [S3].

The questions about Boolean algebras answered in this paper were raised by Rubin, and these questions initiated this work.

§1, in which the case of atomic BA's is dealt with, is essentially due to Rubin who proved it for the algebra $P(\omega)$. Shelah noted the generality of the proof. §2, dealing with general BA's, and §3, about the second category method, are due to Shelah. Lemma 2.3 is due to Rubin.

§0. Notations. *Set theory.* $\bar{a}, \bar{b}$ denote finite sequences. $|A|$ is the cardinality of A. $S_\lambda(A) = \{B \mid B \subseteq A \text{ and } |B| < \lambda\}$, so $S_\omega(A)$ is the set of finite subsets of A. 2^A denotes the power set of A, regarded as a topological space, with the product topology; that is, a base for the topology is $\{V_{\sigma_1\sigma_2} \mid \sigma_1, \sigma_2 \in S_\omega(A) \text{ and } \sigma_1 \cap \sigma_2 \neq \emptyset\}$, where $V_{\sigma_1\sigma_2} = \{B \mid B \subseteq A, \sigma_1 \subseteq B \text{ and } B \cap \sigma_2 = \emptyset\}$.

Model theory. M, N denote models, always with equality. $|M|$ is the universe of M. If M is a model then $L(M)$ denotes the language of M; if T is a theory i.e. a set of sentences, then $L(T)$ is the language of T. $\mathrm{Th}(M)$ is the set of all sentences of $L(M)$ true in M. $\mathrm{CD}(M)$ is the complete diagram of M, i.e. add constants to represent all the elements of $|M|$, and $\mathrm{CD}(M)$ is the set of all sentences in this expanded language true in M. If $R \in L(M)$ is a relation or function symbol then R^M is the interpretation of R in M. So for example if $< \in L(M)$ then a $<^M b$ means $\langle a, b\rangle \in <^M$, if $\varepsilon \in L(M)$ then $a\ \varepsilon^M b$ means $\langle a, b\rangle \in \varepsilon^M$. Let $\varphi(x, x_1, \ldots, x_n) \in L(M)$, $\bar{a} = \langle a_1, \ldots, a_n\rangle \in |M|^n$ then $\varphi(x, \bar{a})^M = \{b \mid M \models \varphi[b, \bar{a}]\}$. Let $0 < k < \omega$ and $A \subseteq |M|^k$. A is said to be a definable set in M if there is a formula $\varphi(\bar{x}, \bar{y})$ in $L(M)$ and $\bar{a} \in |M|^n$ such that $A = \{\bar{b} \mid M \models \varphi[\bar{b}, \bar{a}]\}$. In this notation it is implied that $\bar{a}$ and $\bar{y}$ have the same length n and the length of $\bar{x}$ is k. So when we say "definable" we mean definable with the aid of parameters from M. $M \equiv N$ means $\mathrm{Th}(M) = \mathrm{Th}(N)$, that is, M and N are elementary equivalent. $M \prec N$ means M is an elementary submodel of N. $M \cong N$ means M is isomorphic to N. $\mathrm{Aut}(M)$ denotes the automorphism group of M, as well as its universe.

If $P \subseteq |M|^k$ then (M, P) denotes the expansion of M in which a relation symbol is added to represent P. Usually P is going to be just a subset of M.

Let $M_1, \ldots, M_k$ be models in pairwise disjoint languages and $R_1, \ldots, R_l$ are relations or functions on $\bigcup_{i=1}^k |M_i|$ then $\langle M_1, \ldots, M_k; R_1, \ldots, R_l\rangle$ is the model with universe $\bigcup_{i=1}^k |M_i|$, and whose relations and functions are those which appear in M_i, $i = 1, \ldots, k$, and in addition: $R_1, \ldots, R_i$, equality and unary predicates to represent each $|M_i|$, $i = 1, \ldots, k$. Sometimes some of the M_i's or all of them are replaced by sets U_i, then we regard U_i as a model with the only relation being the equality relation, and the previous notation is still valid.

Suppose for example that $N = \langle M, U; R\rangle$ and $N_1 \equiv N$. Then M^{N_1} denotes the model whose universe is the interpretation in N_1 of the unary predicate that represents $|M|$ in N; the relations and functions of M^{N_1} are the interpretations in N_1 of the symbols of $L(M)$. U^{N_1} is the interpretation in N_1 of the unary predicate that represents U in N.

$\mathrm{Aut}(M)$ is regarded as a model with the binary operation of composition $\circ$. Let us define the relation Op on $|M| \cup \mathrm{Aut}(M)$: $\langle f, a, b\rangle \in \mathrm{Op}$ iff $f \in \mathrm{Aut}(M)$, $a, b \in |M|$ and $f(a) = b$. The notation $\langle M, \mathrm{Aut}(M)\rangle$ always abbreviates the longer notation $\langle M, \mathrm{Aut}(M); \mathrm{Op}\rangle$. The same abbreviation is used when instead of $\mathrm{Aut}(M)$ we have $\mathrm{En}(M)$ the set of endomorphisms of M or other sets of functions from $|M|$ to $|M|$.

Boolean algebras. A "BA" is an abbreviation for a Boolean algebra. B, C denote BA's as well as their universes. We regard BA's as models of the type: $\langle B, \subseteq\rangle$ where $\subseteq$ is the partial ordering of B. $0, 1, \cup, \cap, -$ are the constants and operations of a BA which are defined by means of its partial ordering $\subseteq$. $\mathrm{At}(B)$ denotes the set of atoms of B. If B is an atomic BA and $b \in B$, let $\bar{b}$ denote $\{a \mid a \in \mathrm{At}(B)$ and $a \subseteq b\}$, so $b \mapsto \bar{b}$ is the natural embedding of B in the power set of $\mathrm{At}(B)$.

§1. The case of atomic BA's.

In this section we will prove the following theorem.

THEOREM 1.1. ($\diamondsuit_{\aleph_1}$) *If B is an infinite atomic BA, then there is a BA, B_1, such that* $|\mathrm{Aut}(B_1)| \leq |B_1| = \aleph_1$ *and* $\langle B_1, \mathrm{Aut}(B_1)\rangle \equiv \langle B, \mathrm{Aut}(B)\rangle$.

In fact we will prove stronger and more general theorems; they all follow from Theorem 1.2 or from its proof. Theorem 1.2 is, thus, the essential theorem in this section.

THEOREM 1.2. ($\diamondsuit_{\aleph_1}$) *Let T be a countable theory which has an infinite model, P be a unary predicate, ε and* Cr *be binary predicates. Suppose T has the following properties*: (1) *T implies that every finite subset of P is represented by ε, that is for every n*

$$T \vdash \forall x_1 \cdots x_n \Big(\bigwedge_{i=1}^{n} P(x_i) \to \exists y \forall z (z \varepsilon y \leftrightarrow \bigvee_{i=1}^{n} (z = x_i)) \Big).$$

(2) *T implies that* Cr *is the ordering of finite cardinalities of subsets of P, that is for every $m < n < \omega$*

$$T \vdash \forall x_1 \cdots x_m y_1 \cdots y_n x y \Big(\Big(\bigwedge_{i=1}^{m} P(x_i) \wedge \bigwedge_{i=1}^{n} P(y_i) \wedge \bigwedge_{1 \le i < j \le m} (x_i \neq x_j) \wedge \bigwedge_{1 \le i < j \le n} (y_i \neq y_j) \wedge \forall z (z \varepsilon x \leftrightarrow \bigvee_{i=1}^{m} (z = x_i)) \wedge \forall z (z \varepsilon y \leftrightarrow \bigvee_{i=1}^{n} (z = y_i)) \Big) \to \mathrm{Cr}(x, y) \Big) \wedge \forall x \forall y \neg (\mathrm{Cr}(x, y) \wedge \mathrm{Cr}(y, x)).$$

Then there is a model M of T such that:

(a) $\|M\| = \aleph_1$.

(b) *If π is a permutation of P^M such that for every $x \in |M|$ there is $y \in |M|$ such that $\forall z (z \varepsilon^M x$ iff $\pi(z) \varepsilon^M y)$; then π is definable (possibly by parameters) in M.*

(b^*) *If A is a definable subset of M, $\pi: A \to P^M$, and for every $x \in |M|$ the set $\pi^{-1}(x) =^{\mathrm{def}} \{y \mid y \in A$ and $\pi(y) \varepsilon^M x\}$ is definable, then π is definable.*

Obviously (b^*) implies (b).

We first state the corollaries of 1.2.

DEFINITION. Let $h: B \to C$ be a homomorphism from B to C; we say that h is complete if for every $A \subseteq B$ and $a \in B$ if a is the supremum of A, then $h(A)$ has a supremum, and $h(a)$ is the supremum of $h(A)$. Note that every onto endomorphism is complete. Let End(B) be the semigroup of complete endomorphisms B.

Theorem 1.1 is a corollary of both 1.3 and 1.4.

COROLLARY 1.3. ($\diamondsuit_{\aleph_1}$) *If B is atomic and infinite, then there is B_1 such that $|\mathrm{End}(B_1)| \le |B_1| = \aleph_1$, and $\langle B, \mathrm{End}(B)\rangle \equiv \langle B_1, \mathrm{End}(B_1)\rangle$.*

PROOF. We show how 1.3 follows from 1.2. Let $M_0 = \langle B, \mathrm{End}(B), \mathrm{Cr}, P, \varepsilon\rangle$ where $\mathrm{Cr} \subseteq B \times B$ and $\langle a, b\rangle \in \mathrm{Cr}$ iff $|\bar{a}| < |\bar{b}|$, $P = \mathrm{At}(B)$ and $a \varepsilon b$ iff $a \in \mathrm{At}(B)$, $b \in B$ and $a \subseteq b$. Certainly Th(M_0) satisfies the assumptions of 1.2. So let M be a model of Th(M_0) as in 1.2. Denote $B_1 = B^M$, $H = (\mathrm{End}(B))^M$. We show that $H = \mathrm{End}(B_1)$.

Clearly there is a sentence in Th(M_0) saying that every element of End(B) is an endomorphism of B, so since $M \equiv M_0$ every element of H is an endomorphism of B_1. For atomic BA's there is also a formula $\varphi(h)$ saying that h is complete, namely

$$\varphi(h) \equiv (\forall a \in B)\, (\forall x \in \mathrm{At}(B))\, (x \in \overline{h(a)} \leftrightarrow (\exists y \in \bar{a})\, (x \in \overline{h(y)})).$$

So $M_0 \models (\forall h \in \mathrm{End}(B))\varphi(h)$; so this holds in M, too; so every element of H is a complete homomorphism. Suppose $h \in \mathrm{End}(B_1)$. We show that $h \in H$. Let $\tilde{h}: \mathrm{At}(B_1) \to \mathrm{At}(B_1)$ be defined as follows: $\tilde{h}(x) = y$ iff $\overline{h(y)} \ni x$. Since h is complete, $1 = h(1) = \bigcup h(\mathrm{At}(B_1))$; that is $h(\mathrm{At}(B_1))$ is a partition of B_1, so $\mathrm{Dom}(\tilde{h}) = \mathrm{At}(B_1)$. Let $A = \mathrm{At}(B_1)$, $\pi = \tilde{h}$; then $\tilde{h}$ satisfies the conditions of (b^*) in 1.2. So $\tilde{h}$ is definable in M; so h is definable in M, say $\varphi(x, y, \bar{c})$ defines h. In M_0 the following schema holds: "for every $\bar{u}$: if $\psi(x, y, \bar{u})$ defines a complete homomorphism of B then there is $h \in \text{End}(B)$ such that $\forall xy(h(x) = y \leftrightarrow \psi(x, y, \bar{u}))$" $\psi(x, y, \bar{u})$ ranges over all formulas in the language of M_0. So this schema holds in M and in particular it assures that there is $h_1 \in H$ such that $M \models \forall xy(h_1(x) = y \leftrightarrow \varphi(x, y, \bar{c}))$ so $h_1 = h$. So $H = \text{End}(B_1)$. So $\langle B, \text{End}(B)\rangle \equiv \langle B_1, \text{End}(B_1)\rangle$. Q.E.D.

Theorem 1.1 is a special case of the following corollary.

COROLLARY 1.4. $(\diamondsuit_{\aleph_1})$ *Let* $N = \langle B, \subseteq, R_1, \ldots, R_n\rangle$ *where* $\langle B, \subseteq\rangle$ *is an infinite atomic BA, and N is an expansion of* $\langle B, \subseteq\rangle$ *in a finite language; then there is* $M \equiv N$ *such that* $|\text{Aut}(M)| \leq \|M\| = \aleph_1$ *and* $\langle M, \text{Aut}(M), \text{Aut}(B^M)\rangle \equiv \langle N, \text{Aut}(N), \text{Aut}(B)\rangle$.

PROOF. Repeat the proof of 1.3 replacing everywhere End(B) by Aut(N).

DEFINITION. If $A_1, A_2 \subseteq |M|^k$, we say that A_1 and A_2 are separable in M if there is a formula $\varphi(\bar{x}, \bar{c})$, such that for every $\bar{a} \in A_1$, $M \models \varphi[\bar{a}, \bar{c}]$, and for every $\bar{a} \in A_2$, $M \models \neg\varphi[\bar{a}, \bar{c}]$; otherwise A_1, A_2 are called inseparable in M.

Explanation of the proof of 1.2. We will construct an elementary continuous chain of countable models of T, $\{M_i \mid i < \aleph_1\}$. At the ith step of the construction we will be given a definable set A of M_i and an undefinable $\pi: A \to P^{M_i}$ which fulfills the condition of 1.2(b*). We wish that in M_{i+1} it will be impossible to extend π to a function that still fulfills the conditions of 1.2(b*). This will be accomplished in the following way: in M_{i+1} there will be a new element a, which will represent a subset D of $P^{M_{i+1}}$. We will take care that $\pi^{-1}(D \cap |M^i|)$ and $\pi^{-1}(P^{M_i} - D)$ will be inseparable; more precisely, the sets $D_1 = \{y \mid y \in A \text{ and } \pi(y)\ \varepsilon^{M_{i+1}}\ a\}$ and $D_2 = \{y \mid y \in A \text{ and } \pi(y) \not\varepsilon^{M_{i+1}}\ a\}$ will be inseparable. This of course will assure that if $\tilde{\pi} \supseteq \pi$ then $\tilde{\pi}^{-1}(a)$ will not be definable, since if $\varphi(x, \bar{c})$ defines $\tilde{\pi}^{-1}(a)$ then it also separates D_1 and D_2. Of course in order that in later stages in the construction it will be impossible to extend π to a function that satisfies (b*), we have to keep D_1 and D_2 inseparable forever. So in every step of the construction we will have also a countable set of previous obligations of the type: "for some inseparable D_1, D_2 of M_i keep D_1, D_2 inseparable in M_{i+1}". Now in order that finally all the undefinable π's will be "killed" we will use $\diamondsuit_{\aleph_1}$.

The key Lemma 1.5 describes the construction of M_{i+1} from M_i.

LEMMA 1.5. *Let M be a countable model such that* Th(M) *satisfies the assumptions of* 1.2. *Suppose that* $\{\langle D_i^1, D_i^2\rangle \mid i \in \omega\}$ *is a set of inseparable pairs in M, A is a definable subset of* $|M|$, $\pi: A \to P^M$ *is undefinable, but for every* $x \in P^M$, $\pi^{-1}(x)$ *is definable; then there is N and* $a \in |N|$, *such that* $M \prec N$, $\|N\| = \aleph_0$, *for every* $i \in \omega$, D_i^1, D_i^2 *are inseparable in N and* $\{x \mid x \in A \text{ and } \pi(x)\varepsilon^N a\}$ *and* $\{x \mid x \in A \text{ and } \pi(x) \not\varepsilon^N a\}$ *are inseparable in N.*

PROOF. We will regard the elements of $|M|$ as representing subsets of P^M, so we will use the usual set-theoretic operation symbols for elements of $|M|$. For example: if $b_1, b_2 \in |M|$ and $\sigma \subseteq P^M$, $\sigma \subseteq b_1$ will mean $\sigma \subseteq \{x \mid x \in P^M \text{ and } x\varepsilon^M b_1\}$, and $b_1 \cap b_2 = \emptyset$ will mean $\{x \mid P^M \ni x\varepsilon^M b_1\} \cap \{x \mid P^M \in x\varepsilon^M b_2\} = \emptyset$, etc.

Let T_0 be the complete diagram of M. Suppose $\psi(x, \bar{a}) \in L(T_0)$ and σ is a subset of P^M; we say that $\psi(x, \bar{a})$ is independent outside σ, if for every pair $\langle\sigma^1, \sigma^2\rangle$ of finite disjoint subsets of $P^M - \sigma$ there is $c \in |M|$ such that $M \models \psi[c, \bar{a}]$ and $c \supseteq \sigma^1$ and $c \cap \sigma^2 = \emptyset$.

We say that $\psi(x, \bar{a})$ is nowhere independent if it is not independent outside any finite set.

We will construct N by a Henkin type construction. Let a and $\{a_i \mid i \in \omega\}$ be constants not in $L(T_0)$ and $L = L(T_0) \cup \{a\} \cup \{a_i | i \in \omega\}$. Let $\psi(\bar{y}, a, a_{i_1}, \ldots, a_{i_k}, c_1, \ldots, c_n)$ be a formula in L where $c_1, \ldots, c_n \in L(T_0)$ and $a_{i_1}, \ldots, a_{i_k}$ is the list of all elements of $\{a_i | i \in \omega\}$ occuring in ψ; define $\psi^*(x) = \exists x_{i_1}, \ldots, x_{i_k} \psi(\bar{y}; x, x_{i_1}, \ldots, x_{i_k}, c_1, \ldots, c_n)$. We define by induction theories T_n and finite subsets of P^M, σ_n, such that $T_n = T_0 \cup \{\psi_n(a, \bar{a}, \bar{c})\}$ and ψ_n^* is independent outside σ_n. We will take care that $\Sigma = \bigcup_{n \in \omega} T_n$ will be a complete diagram of a model whose universe is $|M| \cup \{a\} \cup \{a_i | i \in \omega\}$. We will have a countable list of tasks to be accomplished along the construction of the T_n's. There will be four kinds of tasks: (I) given a sentence φ in L, decide whether $\varphi \in \Sigma$ or $\neg \varphi \in \Sigma$; (II) given a sentence $\exists x \varphi(x)$ in L, if it is provable from T_n, choose an a_i which does not occur in T_n and add to T_n the sentence $\varphi(a_i)$; (III) given $\langle D_i^1, D_i^2 \rangle$ and a formula $\chi(y, a, \bar{a}, \bar{c})$, take care that $\chi(y)$ will not separate D_i^1 and D_i^2; (IV) given $\chi(y, a, \bar{a}, \bar{c})$ take care that $\chi(y)$ will not separate $\{x | A \ni x \text{ and } \pi(x) \varepsilon a\}$ and $\{x | x \in A \text{ and } \pi(x) \not\varepsilon a\}$.

So let $\{s_i | i \in \omega\}$ be a list of all objects of the following forms: (I) φ where φ is a sentence of L; (II) $\langle \exists x \varphi(x), \exists \rangle$ where $\exists x \varphi(x)$ is a sentence in L; (III) $\langle i, \chi(y, a, \bar{a}, \bar{c}) \rangle$ where $i \in \omega$ and χ is a formula in L with one free variable (this will designate the task: "take care that $\chi(y)$ will not separate D_i^1 and D_i^2"); (IV) $\chi(y, a, \bar{a}, \bar{c})$ where χ is as in (III). We can regard $T_0 = \text{Th}(M) \cup \{a = a\}$ and $\sigma_0 = \emptyset$, so the induction hypothesis holds. Suppose $T_n = T_0 \cup \{\psi_n(a, \bar{a}, \bar{c})\}$ and σ_n have been defined so that $\sigma_n \subseteq P^M$ is finite and ψ_n^* is independent outside σ_n. Suppose that $s_n = \varphi(a, \bar{a})$ is of type (I). If $(\psi_n \wedge \varphi)^*$ is independent outside some finite set σ, define $T_{n+1} = T_0 \cup \{\psi_n \wedge \varphi\}$ and $\sigma_{n+1} = \sigma$, otherwise we define $T_{n+1} = T_0 \cup \{\psi_n \wedge \neg\varphi\}$ and $\sigma_{n+1} = \sigma_n$. We show that $(\psi_n \wedge \neg\varphi)^*$ is independent outside σ_n. If not then there are disjoint finite subsets of $P^M - \sigma_n$, σ^1 and σ^2, such that for no $b \in |M|$: $M \models (\psi_n \wedge \neg \varphi)^*[b]$, and $\sigma^1 \subseteq b$ and $\sigma^2 \cap b = \emptyset$. Notice that ψ_n^* is logically equivalent to $(\psi_n \wedge \varphi)^* \vee (\psi_n \wedge \neg\varphi)^*$. Let σ^3, σ^4 be disjoint finite subsets of $P^M - \sigma_n - \sigma^1 - \sigma^2$, so by the independence of ψ_n^* outside σ_n, there is $b \in |M|$ such that $M \models \psi_n^*[b]$ and $\sigma^1 \cup \sigma^3 \subseteq b$ and $(\sigma^2 \cup \sigma^4) \cap b = \emptyset$; by our assumption on σ^1 and σ^2, $M \not\models (\psi_n \wedge \neg\varphi)^*[b]$, so $M \models (\psi_n \wedge \varphi)^*[b]$, hence $(\psi_n \wedge \varphi)^*$ is independent outside $\sigma_n \cup \sigma^1 \cup \sigma^2$, contrary to our assumption. So $(\psi_n \wedge \neg\varphi)^*$ is independent outside σ_n, and the induction hypothesis holds. We have actually proved:

(*) if σ is finite, ψ^* is independent outside σ and $(\psi \wedge \varphi)^*$ is nowhere independent, then $(\psi \wedge \neg\varphi)^*$ is independent outside σ.

It is easy to define T_{n+1}, σ_{n+1} when s_n is of type (II).

Suppose now that $s_n = \langle i, \chi(y, a, \bar{a}, \bar{c}) \rangle$. We say that a pair of elements of $|M|$, $\langle b_1, b_2 \rangle$, is a "prevention" of $\psi(x, \bar{c}) \in L(T_0)$ outside σ, if b_1, b_2 and σ are pairwise disjoint, and for no $b \in |M|$: $M \models \psi[b, \bar{c}]$ and $b_1 \subseteq b$ and $b \cap b_2 = \emptyset$. We say that a pair of elements of $|M|$, $\langle b_1, b_2 \rangle$, is finite if both b_1 and b_2 represent finite subsets of P^M (relative to ε^M), otherwise we say that $\langle b_1, b_2 \rangle$ is infinite. It is easy to see that there is a formula $\text{Cr}_1(x_1, x_2, y_1, y_2)$ in $L(T)$ such that $\text{Cr}_1[b_1, b_2, c_1, c_2]$ holds in M whenever $\langle b_1, b_2 \rangle$ is finite and $\langle c_1, c_2 \rangle$ is infinite, and it does not hold in M whenever $\langle b_1, b_2 \rangle$ is infinite and $\langle c_1, c_2 \rangle$ is finite.

Using the fact that D_i^1, D_i^2 are inseparable in M, we will show that: either (i) there is $d_1 \in D_i^1$ such that $(\psi_n \wedge \neg\chi(d_1, a, \bar{a}, \bar{c}))^*$ is independent outside some finite set σ^1; or (ii) there is $d_2 \in D_i^2$ such that $(\psi_n \wedge \chi(d_2, a, \bar{a}, \bar{c}))^*$ is independent outside some finite set σ^2.

If (i) holds, define $T_{n+1} = T_0 \cup \{\psi_n \wedge \neg\chi(d_1)\}$ and $\sigma_{n+1} = \sigma^1$; if (ii) holds, define T_{n+1} and σ_{n+1} analogously. This obviously assures that $\chi(y)$ will not separate D_i^1 and D_i^2 in N.

Suppose neither (i) nor (ii) hold. Let $\alpha(y)$ be the formula saying: $(\psi_n \wedge \chi)^*$ has prevention $\langle b_1, b_2\rangle$ outside σ_n, such that for every prevention $\langle c_1, c_2\rangle$ of $(\psi_n \wedge \neg\chi)^*$ outside σ_n $\mathrm{Cr}_1(b_1, b_2, c_1, c_2)$. (Of course $\alpha(y)$ has parameters.) We show that $\alpha(y)$ separates D_i^2 and D_i^1. Suppose $d \in D_i^2$. Then $(\psi_n \wedge \chi(d))^*$ is nowhere independent, in particular it has a finite prevention $\langle b_1, b_2\rangle$ outside σ_n. On the other hand, by (*) every prevention $\langle c_1, c_2\rangle$ of $(\psi_n \wedge \neg\chi(d))^*$ outside σ_n is infinite so $M \models \mathrm{Cr}_1[b_1, b_2, c_1, c_2]$, hence $M \models \alpha[d]$. A similar argument shows that if $d \in D_i^1$ then $M \models \neg\alpha\,[d]$. This contradicts the assumption that D_i^1, D_i^2 are inseparable in M, so either (i) or (ii) hold.

Suppose now that $s_n = \chi(y, a, \bar{a}, \bar{c})$ and we want to take care that $\chi(y)$ will not separate $\{x | x \in A$ and $\pi(x)\varepsilon^N a\}$ and $\{x | x \in A$ and $\pi(x)\not\varepsilon^N a\}$. Clearly it suffices to show that either (i) there is $b \in A$ such that $(\psi_n \wedge \pi(b)\varepsilon a \wedge \neg\,\chi(b, a, \bar{a}, \bar{c}))^*$ is independent outside some finite set σ^1; or (ii) there is $b \in A$ such that $(\psi_n \wedge \pi(b)\not\varepsilon a \wedge \chi(b, a, \bar{a}, \bar{c}))^*$ is independent outside some finite set σ^2.

Suppose neither (i) nor (ii) hold. We show that π is definable. Let $\beta(u, v)$ be the following formula:

$$\beta(u, v) \equiv (u \in P - \sigma_n) \wedge (v \in A - \pi^{-1}(\sigma_n)) \wedge (\forall l \in P - \sigma_n - \{u\})$$

(there is a prevention $\langle b_1, b_2\rangle$ of $(\psi_n \wedge u\varepsilon a \wedge l\not\varepsilon a \wedge \neg\chi(v, a, \bar{a}, \bar{c}))^*$ outside $\sigma_n \cup \{u, l\}$, such that for every prevention $\langle d_1, d_2\rangle$ of $(\psi_n \wedge u\varepsilon a \wedge l\not\varepsilon a \wedge \chi(v, a, \bar{a}, \bar{c}))^*$ outside $\sigma_n \cup \{u, l\}$ $\mathrm{Cr}_1(b_1, b_2, d_1, d_2)$).

Remembering that A is definable, σ_n is finite, and for every $x \in P^M$, $\pi^{-1}(x)$ is definable, it is easy to see that there is really a first-order formula in $L(T_0)$ which expresses β. We will show that $M \models \beta[m, b]$ iff $m \in P^M - \sigma_n$ and $\pi(b) = m$. Suppose $m \in P^M - \sigma_n$ and $\pi(b) = m$. Let $l \in P^M - \sigma_n - \{m\}$. By the independence of ψ_n^*:

$$(**) \qquad \psi_n^* \wedge m\varepsilon x \wedge l\not\varepsilon x \equiv (\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a)^*$$

is independent outside $\sigma_n \cup \{m, l\}$. By $\neg$(i), $(\psi_n \wedge m\varepsilon a \wedge \neg\chi(b))^*$ is nowhere independent, so also

$$(***) \qquad (\psi_n \wedge m\varepsilon a \wedge \neg\chi(b))^* \wedge l\not\varepsilon x \equiv (\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \neg\chi(b))^*$$

is nowhere independent, in particular it has a finite prevention $\langle b_1, b_2\rangle$ outside $\sigma_n \cup \{m, l\}$; on the other hand by (**), (***) and (*) every prevention $\langle d_1, d_2\rangle$ of $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \chi(b))^*$ outside $\sigma_n \cup \{m, l\}$ is infinite, so $M \models \mathrm{Cr}_1[b_1, b_2, d_1, d_2]$ so $M \models \beta[m, b]$.

Now suppose $m \in P^M - \sigma_n$, $b \in A - \pi^{-1}(\sigma_n)$ and $\pi(b) \neq m$. Let $l = \pi(b)$ so $l \in P^M - \sigma_n - \{m\}$. By $\neg$(ii), $(\psi_n \wedge l\not\varepsilon a \wedge \chi(b))^*$ is nowhere independent so $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \chi(b))^*$ is nowhere independent, in particular it has a finite prevention $\langle d_1, d_2\rangle$ outside $\sigma_n \cup \{m, l\}$. As in the previous case $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a)^*$ is independent outside $\sigma_n \cup \{m, l\}$, so by (*) every prevention $\langle b_1, b_2\rangle$ of $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \neg\,\chi(b))^*$ outside $\sigma_n \cup \{m, l\}$ is infinite, so $M \not\models \mathrm{Cr}_1[b_1, b_2, d_1, d_2]$ so $M \not\models \beta[m, b]$. So $\beta(u, v)$ defines $\pi \restriction (A - \pi^{-1}(\sigma_n))$; since σ_n is finite and for every $x \in \sigma_n$, $\pi^{-1}(x)$ is definable, β can be "corrected" to define π, contrary to our assumption so either (i) or (ii) hold.

If (i) holds define $T_{n+1} = T_0 \cup \{\psi_n \wedge \neg\, \chi(b) \wedge \pi(b)\varepsilon a\}$ and $\sigma_{n+1} = \sigma^1$, otherwise (ii) holds and then define T_{n+1}, σ_{n+1} analogously.

$\Sigma = \bigcup_{n\in\omega} T_n$ is the complete diagram of a model N which satisfies the requirements of the lemma. Q.E.D.

REMARKS 1.6. (a) The construction assures that N is a proper extension of M, since if for $n \in \omega$ and $b \in |M|$, $T_n \vdash a = b$, then ψ_n^* is nowhere independent. Also the construction assures that we can get a proper elementary extension of M, when the $\langle D_i^1, D_i^2\rangle$'s are given but no π is given; we then confine ourselves to the tasks of types (I), (II), (III) only.

(b) If in addition to the $\langle D_i^1, D_i^2\rangle$'s and π we are given $\aleph_0$ infinite subsets $E_i \subseteq P^M$, it is easy to construct N and a in such a way that in addition to the previous requirements N and a will have the following property: for every $i \in \omega$, $E_i \cap a$ and $E_i - a$ are infinite. This will yield the following additional property of M in 1.2: for every countable family $\{E_i | i \in \omega\}$ of infinite subsets of P^M there is an uncountable subset A of $|M|$ such that for every $a, b \in A$ and for every $i \in \omega$, $E_i \cap a$, $E_i - a$ are infinite and $E_i \cap a - E_i \cap b$ is infinite.

(c) Though in most cases our construction will yield that $P^N \supsetneqq P^M$, it does not assure it. However this can be accomplished by applying the following lemma.

LEMMA. *Suppose M is a countable model in a countable language L, P, $\varepsilon \in L$ and every finite subset of P^M is represented by some element of M (relative to ε). Suppose that $\{\langle D_i^1, D_i^2\rangle \mid i \in \omega\}$ is a set of inseparable pairs in M then there is $N \succ M$ such that $P^N \supsetneqq P^M$ and for every $i \in \omega$, D_i^1, D_i^2 are inseparable in N.*

This lemma can yield that in Theorem 1.2, for every $a_1, \dots, a_k \in |M|$ either $\bigcap_{i=1}^k a_i$ is finite or uncountable.

On the other hand in Theorem 1.2 it is not always possible to construct M in such a way that $|P^M| = \aleph_0$, even if we know that T has a two cardinal model; however Theorem 3.1 shows that this is possible in some cases.

PROOF OF THEOREM 1.2. Let T be as in Theorem 1.2, and let $\{S_\alpha | \alpha < \aleph_1\}$ be a sequence such that for every $\pi \subseteq \aleph_1 \times \aleph_1$, $\{\alpha | S_\alpha = \pi \cap \alpha \times \alpha\}$ is stationary.

We construct an elementary chain $\{M_\delta | \delta \in \aleph_1$ and δ is a limit$\}$ and a set $\{\langle D_\delta^1, D_\delta^2\rangle \mid \delta < \aleph_1$ and δ is a limit$\}$ such that: (1) $|M_\delta| = \delta$; (2) $M_\omega \models T$; (3) for every $\delta_1 < \delta_2$, $\langle D_{\delta_1}^1, D_{\delta_1}^2\rangle$ is an inseparable pair in M_{δ_2}.

Let M_ω be some model of T whose universe is ω. If δ is a limit of limit ordinals define $M_\delta = \bigcup \{M_i | i < \delta, i \text{ is a limit}\}$. Suppose M_δ and $\{\langle D_i^1, D_i^2\rangle \mid i < \delta\}$ have been defined, we define $M_{\delta+\omega}$ and $\langle D_\delta^1, D_\delta^2\rangle$.

Case I. Suppose for some $\bar{a} \in |M_\delta|^k$ and $\varphi(x, \bar{y})$, $S_\delta = \Pi$ is an undefinable function in M_δ from $\{b | M_\delta \models \varphi[b, \bar{a}]\}$ to P^{M_δ}, and for every $x \in P^{M_\delta}$, $\Pi^{-1}(x)$ is definable in M_δ; construct $M_{\delta+\omega}$ and $a \in |M_{\delta+\omega}|$ with universe $\delta + \omega$ as in Lemma 1.5 for $M = M_\delta$, $A = \{b | M_\delta \models \varphi[b, \bar{a}]\}$, $\{\langle D_i^1, D_i^2\rangle \mid i \in \omega\} = \{\langle D_i^1, D_i^2\rangle \mid i < \delta\}$ and $\pi = \Pi$, and define $D_\delta^1 = \{x \mid x \in A \text{ and } \pi(x)\ \varepsilon^{M_{\delta+\omega}}\ a\}$ and $D_\delta^2 = \{x | x \in A \text{ and } \pi(x)\ \not\varepsilon^{M_{\delta+\omega}}\ a\}$. By Remark 1.6(a) this is possible.

If for some $c \in P^M$, $\Pi^{-1}(c)$ is undefinable in M_δ define $M_{\delta+\omega}$ with universe $\delta + \omega$ as in Remark 1.6(a) for $\{\langle D_i^1, D_i^2\rangle \mid i \leq \delta\}$, where $D_\delta^1 = \Pi^{-1}(c)$ and $D_\delta^2 = \Pi^{-1}(P^{M_\delta} - \{c\})$.

Case II. If the conditions of case I do not hold, define $M_{\delta+\omega}$ with universe $\delta + \omega$ according to 1.6(a) and $D_\delta^1 = D_\delta^2 = \delta + \omega$.

Let $M = \bigcup \{M_\delta \mid \delta < \aleph_1, \delta \text{ is a limit}\}$. We show that M satisfies the requirements of 1.2. Let $A = \{b \mid M \models \varphi[b, \bar{a}]\}$ be a definable subset of M and $\pi: A \to P^M$ such that for every $a \in |M|$, $\{b \mid b \in A \text{ and } \pi(b)\varepsilon^M a\}$ is definable. Let $F = \{\delta \mid (M_\delta, \pi \restriction \delta) \prec (M, \pi)\}$. Then F is closed and unbounded. Let $S = \{\alpha \mid S_\alpha = \pi \cap \alpha \times \alpha\}$. Then S is stationary, so $S \cap F \neq \emptyset$, let $\delta \in S \cap F$. If $\pi \restriction \delta$ is not definable, then there is $a \in |M_{\delta+\omega}|$ such that $\{x \mid x \in A \cap \delta \text{ and } \pi(x)\varepsilon^{M_{\delta+\omega}} a\}$ and $\{x \mid x \in A \cap \delta \text{ and } \pi(x)\not\varepsilon^{M_{\delta+\omega}} a\}$ are inseparable in M, so certainly $\pi^{-1}(a)$ is undefinable in M, contradicting our assumption. So $\pi \restriction \delta$ is definable in M_δ and since $(M_\delta, \pi \restriction \delta) \prec (M, \pi)$, π is definable in M. Q.E.D.

Theorem 1.2 yields essentially two kinds of results, however all the natural results of the first kind that we know of are seemingly weaker than known theorems.

COROLLARY 1.7. ($\diamondsuit_{\aleph_1}$) *Suppose* Th(M) *satisfies the conditions of* 1.2, $L(M)$ *is finite, every model N of* Th(M) *is rigid over P^N (that is there is just one automorphism which is the identity on P^N) and M has no nontrivial definable automorphisms; then there is an $N \equiv M$ such that $\|N\| = \aleph_1$ and N is rigid; and if M is countable then N can be chosen such that $M \prec N$.*

PROOF. Let us first see that if T is a theory, P a unary predicate in $L(T)$ and every model M of T is rigid over P^M, then there is a formula $\varphi(F, x, y)$ in $L(T) \cup \{F\}$ where F is a new binary predicate (to be interpreted as a partial function) such that for every $M \models T$ and for every $f \in \text{Aut}(M)$:

$$(M, f \restriction P^M) \models \varphi[F, a, b] \text{ iff } f(a) = b.$$

By compactness we may assume that $L(T)$ is finite. Regard T as a theory in the language $L(T) \cup \{F\}$, let H be a unary function symbol, and ψ be the sentence in $L(T) \cup \{F\} \cup \{H\}$ saying H is an automorphism of all the relations except F, and H extends F. By Beth theorem, H is explicitly definable, and our claim is proved.

Let us return to our original T. Let N be as constructed in 1.2, and let $f \in \text{Aut}(N)$; in particular $f \restriction P^N$ has the properties of π in 1.2, so it is definable. So by our previous claim f is definable. Since there is a schema in Th(M) saying that there is no definable automorphism other than the identity $f = \text{Id}$ so N is rigid. If M is countable we can start the construction of N from M, so N can be chosen so that $M \prec N$.

EXAMPLES. (a) Let T be any theory in a finite language L such that $L \supseteq \{\in, =\}$ and T contains the extensionality axiom, the regularity schema for all formulas of L, and the set of sentences saying that every finite set is represented, then every model of T satisfies the requirements of 1.7. (Here P is the whole universe.)

(b) Let T be a theory in a finite language L such that $L \supseteq \{0, 1, +, \cdot\}$ and T contains Peano arithmetic, the induction schema ranges over all formulas of L, then every model of T satisfies the requirements of 1.7.

The other application of 1.2 is formulated in 1.8.

COROLLARY 1.8. ($\diamondsuit_{\aleph_1}$) *Let T be a theory in a finite language L, $P, \varepsilon \in L$ and satisfy the requirements of* 1.2 (Cr *need not appear in* L), *and every model M of T is rigid over P^M, then for every $M \models T$ there is N such that $|\text{Aut}(N)| \leq \|N\| = \aleph_1$ and* $\langle M, \text{Aut}(M)\rangle \equiv \langle N, \text{Aut}(N)\rangle$.

PROOF. Apply 1.2 and Beth theorem as in 1.7 to Th $(\langle M, \mathrm{Aut}(M), \mathrm{Cr}\rangle)$.

EXAMPLES. (a) $|L(T)| < \aleph_0$, $L(T) \supseteq \{=, \in\}$, T contains extensionality and the axioms saying that every finite set is represented.

(b) $|L(T)| < \aleph_0$, $L(T) \supseteq \{P, \varepsilon\}$, T contains the axioms that say that every finite subset of P is represented, and the axiom $\forall x \forall y(\forall z(P(z) \to z\varepsilon x \leftrightarrow z\varepsilon y) \to x = y)$. This includes of course 1.1 and 1.4.

(c) Th(M), M is a Frankel-Mostowski model.

(d) T the theory of free infinite semigroups with more than one generator.

(d) is a special case of (e).

(e) Let A be a set of more than one element, and α be an ordinal closed under addition and let $M_{A,\alpha} = \langle {}^{\alpha>}A, {}^\frown\rangle$ where ${}^{\alpha>}A$ is the set of all functions from all $\beta<\alpha$ to A, and ${}^\frown$ is the operation of concatenation. Let T be the set of all sentences true in every such $M_{A,\alpha}$, then T satisfies the requirements of 1.8.

THEOREM 1.9. ($\diamondsuit_{\aleph_1}$) *Let T be as in* 1.2, *for every $M \models T$ let $S(M) = \{D | D \subseteq P^M$ and there is a $\in |M|$ such that for every $d \in P^M$, $d \in D$ iff $d\varepsilon^M a\}$, that is $S(M)$ is the set of all subsets of P^M which are represented in M. Let $N(M) = \langle P^M, S(M); \in\rangle$. Then there are $\{M_i | i < 2^{\aleph_1}\}$ such that for every $i \neq j < 2^{\aleph_1}$, M_i satisfies the conclusion of* 1.2 *and $N(M_i) \not\equiv N(M_j)$.*

PROOF. Let $\{S_\alpha | \alpha < \aleph_1\}$ have the following property: $S_\alpha = \langle A^1_\alpha, A^2_\alpha, \Pi_\alpha\rangle$, $A^i_\alpha \subseteq \alpha$, $i = 1, 2$, and $\Pi_\alpha \subseteq \alpha \times \alpha$; for every $A^1 \subseteq \aleph_1$, $A^2 \subseteq \aleph_1$ and $\Pi \subseteq \aleph_1 \times \aleph_1$, $\{\alpha | A^1 \cap \alpha = A^1_\alpha$ and $A^2 \cap \alpha = A^2_\alpha$ and $\Pi \cap \alpha \times \alpha = \Pi_\alpha\}$ is stationary. We now construct by induction a binary tree of models, that is, for every $\eta \in {}^{\aleph_1>}2$ we define M_η and $\langle D^1_\eta, D^2_\eta\rangle$ such that $|M_\eta| = \omega \cdot \mathrm{length}(\eta)$, for every $\nu < \eta$, $\langle D^1_\nu, D^2_\nu\rangle$ is an inseparable pair in M_η and $M_\nu \prec M_\eta$ and if $\eta = \bigcup \eta_i$ then $M_\eta = \bigcup M_{\eta_i}$. In stage α we have already defined M_η for all η's whose length is $\leq \alpha$ and all $\langle D^1_\eta, D^2_\eta\rangle$ for all η's whose length is $< \alpha$. If δ is limit we define $M_\eta = \bigcup_{i<\delta} M_{\eta \restriction i}$ for every η whose length is δ and no new inseparable pairs $\langle D^1_\eta, D^2_\eta\rangle$ are defined. Suppose that the α's stage of the construction has been carried out and we define the models of the $\alpha + 1$ stage. If $A^1_\alpha = A^2_\alpha$ and η is the characteristic function of A^1_α, let $M_{\eta^\frown\langle 0\rangle} = M_{\eta^\frown\langle 1\rangle}$ and $\langle D^1_\eta, D^2_\eta\rangle$ be constructed for M_η, $\{\langle D^1_{\eta\restriction i}, D^2_{\eta\restriction i}\rangle | i < \alpha\}$ and Π_α as in Theorem 1.2 and for every other sequence ν of length α define $M_{\nu^\frown\langle 0\rangle} = M_{\nu^\frown\langle 1\rangle} \succ M_\nu$ with universe $\omega \cdot (\alpha + 1)$ in which $\langle D^1_{\nu\restriction i}, D^2_{\nu\restriction i}\rangle$ is inseparable for every $i < \alpha$ and define $D^1_\nu = D^2_\nu = \omega \cdot (\alpha + 1)$. Suppose $A^1_\alpha \neq A^2_\alpha$ and let the characteristic function of A^j_α be η_j, $j = 1, 2$. If Π_α is not a 1-1 function from $P^{M_{\eta_1}}$ to $P^{M_{\eta_2}}$, define $\alpha + 1$'s stage so that it will satisfy the induction hypothesis (possible by Remark 1.6(a)). Otherwise define $N = M_{\eta_1{}^\frown\langle 0\rangle} = M_{\eta_1{}^\frown\langle 1\rangle} \succ M_{\eta_1}$ and $a \in |N|$ such that $D^1_{\eta_2} = \{\Pi_\alpha(x) | P^{M_{\eta_1}} \ni x\varepsilon^N a\}$ and $D^2_{\eta_2} = \{\Pi_\alpha(x) | P^{M_{\eta_1}} \ni x\not\varepsilon^N a\}$ are inseparable in M_{η_2} and extend all the other M_ν's (that is length (ν) $= \alpha$, $\nu \neq \eta_1$) and define D^i_ν's in such a way that the induction hypothesis will hold.

For every $\eta \in {}^{\aleph_1}2$ let $M_\eta = \bigcup_{i<\aleph_1} M_{\eta\restriction i}$. It is easily seen that $\{M_\eta | \eta \in {}^{\aleph_1}2\}$ is as required.

REMARK. In [Rl] it is proved that if a BA, B, has certain weak homogeneity properties, then the model $\langle \mathrm{At}(B), S_\omega(\mathrm{At}(B)), \mathrm{Aut}(B); \mathrm{Op}, \in\rangle$ can be interpreted in Aut(B).

On the other hand, BA's that are constructed using the method of this section do not have this property, since ω of their imitated second-order logic is nonstandard. It shows that in [Rl] some homogeneity properties should have been required.

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§2. The case of a general BA. In this section we will prove the analogue of Corollary 1.4 for arbitrary infinite BA's, that is, we prove:

THEOREM 2.1. ($\diamondsuit_{\aleph_1}$) *Let* $N = \langle B, \subseteq, R_1, \ldots, R_n\rangle$ *where* $\langle B, \subseteq\rangle$ *is an infinite BA and N is an expansion of* $\langle B, \subseteq\rangle$ *in a finite language; then there is M such that* $|\mathrm{Aut}(M)| \leq \|M\| = \aleph_1$ *and* $\langle M, \mathrm{Aut}(M)\rangle \equiv \langle N, \mathrm{Aut}(N)\rangle$.

DEFINITION. Let B be a BA. A set $E \subseteq B$ is called a partition of unity (PU), if for every distinct $e_1, e_2 \in E$, $e_1 \cap e_2 = 0$, and for every nonzero $b \in B$ there is $e \in E$ such that $b \cap e \neq 0$.

We first prove an analogue of 1.5.

LEMMA 2.2. *Let* T, P, ε, Cr *be as in* 1.2, S, $\subseteq$ *are unary and binary predicates, respectively, such that for every* $M \models T$: $\subseteq^M \subseteq S^M \times S^M$ *and* $\langle S^M, \subseteq^M\rangle$ *is a BA.*

Let M be a countable model of T and denote $B = \langle S^M, \subseteq^M\rangle$; *let E be a PU of B,* $0 \notin E$, $\pi: E \to P^M$, π *is* 1-1, π *is undefinable; let* $\{\langle D_i^1, D_i^2\rangle \mid i \in \omega\}$ *be a set of inseparable pairs in M. Then there is a countable N and* $a \in |N|$ *such that* $M \prec N$, *for every* $i \in \omega$, $\langle D_i^1, D_i^2\rangle$ *are inseparable in N, and the sets* $D^1 = \{b \mid 0 \neq b \in B \wedge (\exists c \in E)(b \subseteq^M c \wedge \pi(c)\varepsilon^N a)\}$ *and* $D^2 = \{b \mid 0 \neq b \in B \wedge (\exists c \in E)(b \subseteq^M c \wedge \pi(c) \not\varepsilon^N a)\}$ *are inseparable in N.*

PROOF. The proof is very similar to the proof of 1.5. So we use the same notations. The induction hypotheses are as in 1.5. The tasks (I), (II), (III) are as in 1.5 and task (IV) is slightly changed in the following way: (IV) given $\chi(y, a, \bar a, \bar c)$ take care that $\chi(y)$ will not separate D^1 and D^2. So let $\{s_i \mid i < \omega\}$ be as in 1.5. It is clear that if s_n is of the kind (I), (II) or (III), then it can be carried out exactly as in 1.5. Suppose $s_n = \chi(y, a, \bar a, \bar c)$ is of type (IV). Clearly it suffices to show that either (i) there is $e \in E$ and $0 \neq e_1 \subseteq e$ such that $(\psi_n \wedge \pi(e)\varepsilon a \wedge \neg\chi(e_1, a, \bar a, \bar c))^*$ is independent outside some finite σ^1; or (ii) there is $e \in E$ and $0 \neq e_1 \subseteq e$ such that $(\psi_n \wedge \pi(e)\not\varepsilon a \wedge \chi(e_1 a, \bar a, \bar c))^*$ is independent outside some finite σ^2. Suppose neither (i) nor (ii) hold. We show that π is definable. Let $\beta(u, v)$ be the following formula:

$$\beta(u,v) \equiv (u \in P - \sigma_n) \wedge (0 \neq v \subseteq 1 - \bigcup \pi^{-1}(\sigma_n)) \wedge (\forall v_1 \subseteq v, v_1 \neq 0)(\forall l \in P - \sigma_n - \{u\})$$

(there is a prevention $\langle b_1, b_2\rangle$ of $(\psi_n \wedge u\varepsilon a \wedge l\not\varepsilon a \wedge \neg\chi(v_1, a, \bar a, \bar c))^*$ outside $\sigma_n \cup \{u, l\}$ such that for every prevention $\langle d_1, d_2\rangle$ of $(\psi_n \wedge u\varepsilon a \wedge l\not\varepsilon a \wedge \chi(v_1, a, \bar a, \bar c))^*$ outside $\sigma_n \cup \{u, l\}$, $\mathrm{Cr}_1(b_1, b_2, d_1, d_2)$).

As in 1.5 it is easy to see that there is really a first-order formula in $L(T_0)$ which expresses β.

We will show that $M \models \beta[m, b]$ iff $m \in P^M - \sigma_n$ and there is $e \in E$ such that $\pi(e) = m$ and $0 \neq b \subseteq e$. Suppose $m \in P^M - \sigma_n$, $0 \neq b \subseteq e \in E$ and $\pi(e) = m$. Let $0 \neq b_1 \subseteq b$ and $l \in P^M - \sigma_n - \{m\}$. By the independence of ψ_n^*:

$$(**) \qquad \psi_n^* \wedge m\varepsilon x \wedge l\not\varepsilon x \equiv (\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a)^*$$

is independent outside $\sigma_n \cup \{m, l\}$. By $\neg$(i), $(\psi_n \wedge m\varepsilon a \wedge \neg\chi(b_1))^*$ is nowhere independent so also

$$(***) \qquad (\psi_n \wedge m\varepsilon a \wedge \neg\chi(b_1))^* \wedge l\not\varepsilon x \equiv (\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \neg\chi(b_1))^*$$

is nowhere independent; in particular it has a finite prevention $\langle b^1, b^2\rangle$ outside $\sigma_n \cup \{m, l\}$. On the other hand, by (**), (***) and (*) (see 1.5) every prevention

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$\langle d^1, d^2\rangle$ of $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \chi(b_1))^*$ outside $\sigma_n \cup \{m, l\}$ is infinite, so $M \models \mathrm{Cr}_1[b^1, b^2, d^1, d^2]$, so $M \models \beta[m, b]$.

Now suppose $m \in P^M - \sigma_n$, $0 \neq b \subseteq 1 - \bigcup\pi^{-1}(\sigma_n)$ and there is no $e \in E$ such that $\pi(e) = m$ and $b \subseteq e$. Let $e \in E$ be such that $b \cap e \neq 0$. If $\pi(e) = m$, then $b \not\subseteq e$, so there is $e_1 \neq e$, $e_1 \in E$ such that $b \cap e_1 \neq 0$; since π is 1-1, $\pi(e_1) \neq m$. So, in any case, there is $e \in E$ such that $\pi(e) \neq m$ and $e \cap b = b_1 \neq 0$. Let $l = \pi(e)$. By $\neg$ (ii), $(\psi_n \wedge l\not\varepsilon a \wedge \chi(b_1))^*$ is nowhere independent; so $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \chi(b_1))^*$ is nowhere independent; in particular it has a finite prevention $\langle d^1, d^2\rangle$ outside $\sigma_n \cup \{m, l\}$. As in the previous case, $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a)^*$ is independent outside $\sigma_n \cup \{m, l\}$; so by (*) every prevention $\langle b^1, b^2\rangle$ of $(\psi_n \wedge m\varepsilon a \wedge l\not\varepsilon a \wedge \neg\chi(b_1))^*$ outside $\sigma_n \cup \{m, l\}$ is infinite, so $M \not\models \mathrm{Cr}_1[b^1, b^2, d^1, d^2]$, so $M \not\models \beta[m, b]$. Let $\beta'(u, v) \equiv \beta(u, v) \wedge \forall v_1(\beta(u, v_1) \rightarrow v_1 \subseteq v)$, then clearly β' defines $\pi \restriction (E - \pi^{-1}(\sigma_n))$, so π is definable; a contradiction. So the lemma is proved.

The proof of 2.1 will be along the same lines of the proof of 1.2. However, whereas in the construction of 1.2 we employ only steps of the kind assured by 1.5, in the present construction we will have to interlace two kinds of steps. One of them appears in 2.2 and the other one generalizes the construction of Kunen in [K].

LEMMA 2.3. *Let T be a theory in a countable language; D, $<$ are respectively unary and binary predicates in $L(T)$, and $T \vdash$ "$\langle D, <\rangle$ is a partially ordered directed set without a last element". Let $M \models T$ be countable, $\{\langle D_i^1, D_i^2\rangle \mid i \in \omega\}$ be a set of inseparable pairs in M. Then there is N and $d_0 \in D^N$ such that $\|N\| = \aleph_0$, $M \prec N$, for every $d \in D^M$, $d <^N d_0$, and for every $i \in \omega$, D_i^1, D_i^2 are inseparable in N.*

PROOF. Let d_0 be an individual constant which does not occur in $\mathrm{CD}(M)$, and let $T_0 = \mathrm{CD}(M) \cup \{d < d_0 \mid d \in D^M\}$. By the well-known omitting type condition, it is sufficient to show that for no $i \in \omega$, $\psi(\bar{x})$, $\chi(\bar{x}, y) \in L(T_0)$: $T_0 \vdash \exists\bar{x}\psi(\bar{x})$; for every $d \in D_i^1$, $T_0 \vdash \forall\bar{x}(\psi(\bar{x}) \rightarrow \chi(\bar{x}, d))$; and for every $d \in D_i^2$, $T_0 \vdash \forall\bar{x}(\psi(\bar{x}) \rightarrow \neg\chi(\bar{x}, d))$. Suppose by contradiction that $i \in \omega$, $\psi(\bar{x}) = \psi(d_0, \bar{x})$, $\chi(\bar{x}, y) = \chi(d_0, \bar{x}, y)$ are as above; then for every $d \in D_i^1$,

$$M \models \exists u(\forall v > u)[\exists\bar{x}\psi(v, \bar{x}) \wedge \forall\bar{x}(\psi(v, \bar{x}) \rightarrow \chi(v, \bar{x}, d))],$$

and for every $d \in D_i^2$,

$$M \models \exists u(\forall v > u)[\exists\bar{x}\psi(v, \bar{x}) \wedge \forall\bar{x}(\psi(v, \bar{x}) \rightarrow \neg\chi(v, \bar{x}, d))];$$

but, since $M \models$ "$\langle D, <\rangle$ is directed and without last element", the above formulas are contradictory in M; so D_i^1, D_i^2 are separable in M; a contradiction. Q.E.D.

If $\langle P, <\rangle$ is a partially ordered set, $A \subseteq P$ will be called a cofinal chain in P if every two elements of A are comparable and for every $b \in P$ there is $a \in A$ such that $b \leq a$. $B \subseteq P$ will be called a compatible subset of P if for every $b_1, b_2 \in B$ there is $a \in P$ such that $b_1, b_2 \leq a$.

COROLLARY 2.4 (A GENERALIZATION OF A THEOREM OF KUNEN [K]). ($\diamondsuit_{\aleph_1}$) *Suppose M is a countable model in a countable language; D, S, $<$, $\prec$ are 1-place, 2-place, 2-place predicates respectively, $<^M \subseteq D^M \times D^M$, $\prec^M \subseteq S^M \times S^M$, $\langle D^M, <^M\rangle$ is a partially ordered directed set, without a last element, $\langle S^M, \prec^M\rangle$ is a partially ordered set. Then there is N such that $\|N\| = \aleph_1$, $M \prec N$, $\langle D^N, <^N\rangle$ has a cofinal chain of type $\aleph_1$, and every maximal compatible subset of S^N which has a cofinal chain of type $\aleph_1$ is definable in N.*

PROOF. W.l.o.g. $|M| = \omega$; let us denote $M = M_\omega$. Let $\{S_\alpha \mid \alpha < \aleph_1\}$ as assured by $\diamondsuit_{\aleph_1}$. We define by induction a continuous elementary chain $\{M_\delta \mid \delta \leq \aleph_1$ and δ is a limit$\}$ such that for every δ, $|M_\delta| = \delta$, and simultaneously we define $\{\langle D_i^1, D_i^2\rangle \mid i < \delta,\ i \text{ limit}\}$ such that D_i^1, D_i^2 are inseparable in M_δ. Suppose M_δ, $\{\langle D_i^1, D_i^2\rangle \mid i < \delta\}$ have been defined. If $Q = S_\delta$ is a maximal compatible subset of S^{M_δ} which is undefinable in M_δ, then certainly Q, $S^{M_\delta} - Q$ are inseparable. Let $D_\delta^1 = Q$, $D_\delta^2 = S^{M_\delta} - Q$, and let $M_{\delta+\omega}$ be a model as in Lemma 2.3 for M_δ and $\{\langle D_i^1, D_i^2\rangle \mid i \leq \delta$ and i is a limit$\}$, such that $|M_{\delta+\omega}| = \delta + \omega$. Otherwise let $D_\delta^1 = D_\delta^2 = \delta$ and define $M_{\delta+\omega}$ similarly. If δ is a limit of limits, let M_δ be $\bigcup \{M_i \mid i < \delta$ and i is a limit$\}$. Let $N = M_{\aleph_1}$. Certainly D^N has a cofinal chain of type $\aleph_1$. Suppose B is a maximal compatible subset of S^N which has a cofinal chain E of type $\aleph_1$. Let α be such that $(M_\alpha, B \cap \alpha) \prec (N, B)$ and $S_\alpha = B \cap \alpha$. There is $e \in E$ such that for every $b \in B \cap \alpha$, $b <^N e$; so the formula $x < e$ separates $B \cap \alpha$, $S^{M_\alpha} - B$ in N. Also since B is a maximal compatible set in N and $(M_\alpha, B \cap \alpha) \prec (N, B)$, $B \cap \alpha = S_\alpha$ is a maximal compatible set in M_α so by the construction $B \cap \alpha$ is definable in M_α, so B is definable in N. Q.E.D.

In fact we will not use Corollary 2.4 in the sequel, but we will have to repeat the construction in 2.4, interlaced with the construction of 1.2. The following theorem is an analogue of 1.2, which interlaces the construction of 1.2 and 2.4.

THEOREM 2.5. ($\diamondsuit_{\aleph_1}$) *Let T be a theory in a countable language which has infinite models. Then T has a model N of power $\aleph_1$ with the following properties*:

(a) *If P, ε, Cr are respectively unary, binary and binary definable relations in N, (remember that always "definable" means possibly with parameters) and P, ε, Cr and CD(N) satisfy the conditions of 1.2, then*:

(I) *If $A \subseteq |N|$ is definable, $\pi: A \to P$, and for every $b \in |N|$, $\{y \mid \pi(y)\ \varepsilon\ b\}$ is definable, then π is definable in N.*

(II) *If $\langle B, \subseteq\rangle$ is a definable in N, Boolean algebra, then for every PU, E of B and for every* 1-1 π: *$E \to P$, if for every $b \in |N|$ the sets D_1^b, D_2^b are separable in N, then π is definable in N; where*

$$D_1^b = \{u \mid 0 \neq u \in B \wedge (\exists v \in E)\ (u \subseteq v \wedge \pi(v) \varepsilon b)\},$$

$$D_2^b = \{u \mid 0 \neq u \in B \wedge (\exists v \in E)\ (u \subseteq v \wedge \pi(v) \not\varepsilon b)\}.$$

(b) *If $\langle D, <\rangle$ is a definable in N, directed partially ordered set without a last element, then $\langle D, <\rangle$ has a cofinal chain of type $\aleph_1$.*

(c) *If $\langle P, \prec\rangle$ is a definable in N, partially ordered set, then every maximal compatible subset of P which has a cofinal chain of type $\aleph_1$, is definable in N.*

PROOF. Interlace the constructions in 1.7, 2.2 and 2.3 using ($\diamondsuit_{\aleph_1}$) in a similar way to 1.2 and 2.4.

COROLLARY 2.6. ($\diamondsuit_{\aleph_1}$) *Let $N = \langle B, \subseteq, R_1, \ldots, R_n\rangle$ be a model in a finite language such that $\langle B, \subseteq\rangle$ is an infinite BA. Then there is M such that $|\mathrm{Aut}(M)| \leq \|M\| = \aleph_1$ and $\langle M, \mathrm{Aut}(M)\rangle \equiv \langle N, \mathrm{Aut}(N)\rangle$.*

PROOF. Let N^{II} be the second-order model obtained from N; that is, $N^{\mathrm{II}} = \langle B, \mathfrak{R}_1(B), \mathfrak{R}_2(B), \ldots; \subseteq, R_1, \ldots, R_n, \varepsilon_1, \varepsilon_2, \ldots\rangle$ where $\mathfrak{R}_n(B)$ is the set of n-place relations on B and $\langle b_1, \ldots, b_n, r\rangle \in \varepsilon_n$ iff $r \in \mathfrak{R}_n(B)$ and $\langle b_1, \ldots, b_n\rangle \in r$. Let $M^* \equiv N^{\mathrm{II}}$ be as assured by Theorem 2.5, and let M be the relativized reduct of M^* such that $|M| = B^{M^*}$ and $L(M) = \{\subseteq, R_1, \ldots, R_n\}$. We will show that M is as required.

First we show that if $f \in \mathrm{Aut}(M)$ then f is definable in M^*. Let PU(B) denote the set of PU's of B, which do not contain 0; then $\mathrm{PU}(B) =^{\mathrm{def}} D$ is definable in N^{II}. We say that E_1 refines E_2, if $E_1, E_2 \in \mathrm{PU}(B)$ and every element of E_1 is a subelement of an element of E_2; let us denote "E_1 refines E_2" by $E_2 < E_1$. Certainly $<$ is definable in N^{II}, $\langle D, < \rangle$ is a directed partial ordering, and if B is not atomic $\langle D, < \rangle$ does not have a last element. Let $S = \{f \mid \mathrm{Dom}(f), \mathrm{Rng}(f) \in D$, and f is 1-1$\}$. We say that f_1 refines f_2 ($f_2 \prec f_1$), if $\mathrm{Dom}(f_2) < \mathrm{Dom}(f_1)$ and $f_1 \cup f_2$ is order preserving. So $\langle S, < \rangle$ is a partial ordering definable in N^{II}. Let $E \in D$. We define $x \varepsilon_E y$ iff $x \in E$, $y \in B$ and $x \subseteq y$; and $\mathrm{Cr}_E(y_1, y_2)$ iff there is 1-1 $g \in |N^{\mathrm{II}}|$ such that $\mathrm{Dom}(g) = \{x \mid x \varepsilon_E y_1\}$ and $\mathrm{Rng}(g) \subseteq \{x \mid x \varepsilon_E y_2\}$. Certainly E, ε_E and Cr_E are definable in N^{II} and satisfy the requirements of 1.2. So also $M^* \models \forall e(e \in D \to (e, \varepsilon_e, \mathrm{Cr}_e$ satisfy the requirements of 1.2)). (Notice that this is a schema rather than a single sentence.) Now let $f \in \mathrm{Aut}(M)$. For every $e \in D^{M^*}$, $(f \restriction e)^{-1}$ satisfies the condition on π in Theorem 2.5(II). Thus $(f \restriction e)^{-1}$ and so also $f \restriction e$ are definable in M^*, but in N^{II} as well as in M^* the following comprehension schema holds: "If $\varphi(x, y, \bar{a})$ defines a function g with both $\mathrm{Dom}(g) \subseteq B$ and $\mathrm{Rng}(g) \subseteq B$ then g is represented by an element of the model."

So for every $e \in D^{M^*}$, $f \restriction e$ is represented by an element f_e of M^*. $A_f =^{\mathrm{def}} \{f_e \mid e \in D^{M^*}\} \subseteq S^{M^*}$ is a maximal compatible subset in $\langle S^{M^*}, \prec^{M^*} \rangle$, since for every $g \in S^{M^*} - A_f$, $f_{\mathrm{Dom}(g)}$ and g are incompatible. According to the construction, D^{M^*} has a cofinal chain $\{e_i \mid i < \aleph_1\}$ of type $\aleph_1$. Then if $e_i <^{M^*} e_j$ then $f_{e_i} \prec^{M^*} f_{e_j}$, since f is an automorphism. So A_f is definable, say by $\varphi(x, \bar{a})$. Then f is definable by the following formula $\psi(x, y, \bar{a})\ \exists h(\varphi(h, \bar{a}) \wedge x \in \mathrm{Dom}(h) \wedge h(x) = y)$.

By the comprehension schema mentioned above f is represented by an element of M^*. Let us identify f with the element in M^* representing it. Since $L(N) = L(M)$ is finite there is a formula $A(x)$ that defines $\mathrm{Aut}(N)$ in N^{II}. We have actually proved that $A(x)^{M^*} = \mathrm{Aut}(M)$. Since $\langle N, \mathrm{Aut}(N) \rangle$ and $\langle M, A(x)^{M^*} \rangle$ are relativized reducts of N^{II} and M^* respectively which are defined in the same way, and since $N^{\mathrm{II}} \equiv M^*$, $\langle N, \mathrm{Aut}(N) \rangle \equiv \langle M, A(x)^{M^*} \rangle = \langle M, \mathrm{Aut}(M) \rangle$. Q.E.D.

COROLLARY 2.8. ($\diamondsuit_{\aleph_1}$) *If B is an atomless BA and B is 1-homogeneous (that is for every $a, b \in B$ if $0 \neq a, b \neq 1$ then there is $f \in \mathrm{Aut}(B)$ such that $f(a) = b$), then there is a 1-homogeneous B_1 such that $|B_1| = \aleph_1$ and $\langle B, \mathrm{Aut}(B) \rangle \equiv \langle B_1, \mathrm{Aut}(B_1) \rangle$.*

PROOF. Let B_1 be as assured in the previous corollary. Since there is a sentence in $\langle B, \mathrm{Aut}(B) \rangle$ saying that B is 1-homogeneous and since $\langle B, \mathrm{Aut}(B) \rangle \equiv \langle B_1, \mathrm{Aut}(B_1) \rangle$, B_1 is 1-homogeneous.

REMARKS. (a) By [R1], Corollary 2.8 does not hold in general for BA's which have infinitely many atoms. E.g. 2.8 does not hold for the BA of finite and cofinite subsets of an infinite set A. However in the next section we will see a condition that implies 2.8, also for nonatomless BA's.

(b) Theorem 2.7 does not hold in general for linear orderings; more specifically, if φ is a sentence in the pure full second-order logic such that every model of φ is of cardinality $\geq \lambda$, then there is a linear ordering $M = \langle A, < \rangle$ such that for every linear ordering $N = \langle B, < \rangle$: if $\mathrm{Aut}(N) \equiv \mathrm{Aut}(M)$, then $\|N\| \geq \lambda$.

We now define a set of generalized quantifiers. Let ψ be a sentence in the language of $\langle B, \mathrm{Aut}(B); \subseteq, \mathrm{Op} \rangle$ where $\langle B, \subseteq \rangle$ is a BA. Let $Q_\psi x, y(\varphi(x, \bar{u}), \chi(x, y, \bar{u}))$ be the quantifier saying that:

(1) $\forall xy(\chi(x, y, \bar{u}) \to \varphi(x, \bar{u}) \wedge \varphi(y, \bar{u}))$,

(2) $\langle\{x \mid \varphi(x, \bar{u})\}, \{\langle x, y\rangle \mid \chi(x, y, \bar{u})\}\rangle =^{\text{def}} B$ is a BA,
(3) $\langle B, \text{Aut}(B)\rangle \models \psi$.

Note that χ and φ might be formulas in an arbitrary language.

H. Friedman [F] asked whether there is a compact language stronger than first-order logic on finite models. We will show a countably compact language stronger than first-order logic on finite models (assuming $\diamondsuit_{\aleph_1}$). In fact by Shelah [S1] this language is compact (assuming: "$\{\lambda \mid \lambda^{\lambda} = \lambda$ and $\diamondsuit_{\lambda^+}\}$ is unbounded in the class of ordinals").

THEOREM 2.9. ($\diamondsuit_{\aleph_1}$) *Let L^* be the language obtained from the first-order language L by adjoining all the quantifiers $\{Q_\psi \mid \psi$ is as above$\}$; then L^* is countably compact, and every finitely satisfiable countable set of sentences in L^* has a model of cardinality $\leq \aleph_1$.*

PROOF. Let Σ be a finitely satisfiable countable set of sentences in our language. For every $\sigma \in S_\omega(\Sigma)$, let $M_\sigma \models \sigma$. Every $\varphi \in \Sigma$ can be translated into a sentence φ^* in second-order logic. For every $\Gamma \subseteq \Sigma$ let $\Gamma^* = \{\varphi^* \mid \varphi \in \Gamma\}$. In M_σ^{II}, φ^* is interpreted as a first-order sentence. So $M_\sigma^{\text{II}} \models \sigma^*$. Let $\Sigma_1 = \Sigma^* \cup \{\varphi \mid$ for every $\sigma \in S_\omega(\Sigma)$, $M_\sigma^{\text{II}} \models \varphi\}$. Σ_1 is a finitely satisfiable countable set of first-order sentences, so it has a countable model M. Let N be a model as assured by Theorem 2.5 for $\text{Th}(M)$. The universe of N can be split into two parts: (1) the set of elements A; (2) "the set of relations on A". Let N_1 be the relativized reduct of N whose universe is A and whose language is $L(\Sigma)$. We will show that $N_1 \models \Sigma$. We prove by induction that for every $\varphi \in L^*$ and $\bar{a} \in |N_1|$, $N_1 \models \varphi[\bar{a}]$ iff $N \models \varphi^*[\bar{a}]$.

The only step in the proof that needs checking is when $\varphi \equiv Q_\psi xy(\alpha(x, \bar{u}), \beta(x, y, \bar{u}))$. Let $\bar{a} \in |N_1|$ by the induction hypothesis, $\alpha(x, \bar{a})^{N_1} = \alpha^*(s, \bar{a})^N$ and the same holds for β; so $B = \langle\alpha(x, \bar{a})^{N_1}, \beta(x, y, \bar{a})^{N_1}\rangle$ is a BA iff $B = \langle\alpha^*(z, \bar{a})^N, \beta^*(x, y, \bar{a})^N\rangle$ is a BA. By the construction of N, $\langle B, \text{Aut}(B)\rangle = \langle B, \text{Aut}(B)^N\rangle$, so $N_1 \models \varphi[\bar{a}] \Leftrightarrow B$ is a BA and $\langle B, \text{Aut}(B)\rangle \models \psi \Leftrightarrow B$ is a BA and $\langle B, \text{Aut}(B)^N\rangle \models \psi \Leftrightarrow N \models \varphi^*$. So the theorem is proved.

§3. The second category method. In this section we will deal with a more restricted class of BA's; on the other hand, the results that we will get will be stronger. It is possible to formulate the theorems in this section in the same general framework as in the previous section, however in order to simplify the discussion, we will deal just with BA's.

In the previous section we concluded that: assuming $\diamondsuit_{\aleph_1}$, there is a BA, B, nonisomorphic to $P(\omega)$, such that $\langle B, \text{Aut}(B)\rangle \equiv \langle P(\omega), \text{Aut}(P(\omega))\rangle$. Here we will e.g. conclude that: (CH) there is a 1-*homogeneous* BA, B, nonisomorphic to $P(\omega)$ such that $\langle B, \text{Aut}(B)\rangle \equiv \langle P(\omega), \text{Aut}(P(\omega))\rangle$. A. Litman proved the first result in any Cohen extension that adds a real.

DEFINITION. Let B an atomic infinite BA; every element of B can be regarded as an element in the topological space $2^{\text{At}(B)}$ (with the product topology) that is: if $a \in B$, let $\bar{a} = \{x \mid x \in \text{At}(B)$ and $x \subseteq a\}$. So $\bar{a} \in 2^{\text{At}(B)}$. We will say that B is of the second category (SC) if $\bar{B} =^{\text{def}} \{\bar{a} \mid a \in B\}$ is of the second category in $2^{\text{At}(B)}$ (that is, if it is not the union of countably many nowhere dense sets).

REMARKS. (a) By Baer category theorem, $P(\omega)$ is of course SC since $\overline{P(\omega)} = 2^{\text{At}(P(\omega))}$.

(b) If B is an infinite, atomic, $\aleph_1$-saturated BA then B is SC.

THEOREM 3.1. (a) ($\diamondsuit_{\aleph_1}$) *If* B *is* SC, $\psi \in L_{\omega_1\omega}$, $\langle B, \mathrm{Aut}(B)\rangle \models \psi$, *then there is* B_1 *such that* $\langle B_1, \mathrm{Aut}(B_1)\rangle \models \psi$ *and* $|\langle B_1, \mathrm{Aut}(B_1)\rangle| \leq \aleph_1$.

(b) *If in addition* $|\mathrm{At}(B)| = \aleph_0$, *then* $\diamondsuit_{\aleph_1}$ *can be replaced by CH.*

PROOF. (a) Let $M = \langle B, \mathrm{Aut}(B)\rangle \models \psi$. Let $M^1 = M^{\mathrm{II}}$ and let M^2 be an expansion of M^1 with the following properties: (1) M^2 has a unary predicate P, and individual constants $\{a_i \mid i < \omega\}$ such that $P^{M^2} \subseteq \mathrm{At}(B)$, $P^{M^2} = \{a_i^{M^2} \mid i < \omega\}$ and for $i \neq j$, $a_i^{M^2} \neq a_j^{M^2}$. (2) Function symbols are added to M^2 in such a way that whenever $N \equiv M^2$ and N omits $q = \{P(x)\} \cup \{x \neq a_i \mid i < \omega\}$, then $N \models \psi$. (3) A binary relation $<$ is added to M^2 which orders P^{M^2} in order type ω. (4) M^2 has Skolem functions.

Let M_ω be a countable elementary substructure of M^2. We will now construct an elementary chain $\{M_i \mid i < \aleph_1, i \text{ is a limit ordinal}\}$ in very much the same way as in Theorem 1.2. For this purpose we need a lemma analogous to 1.5, that will enable us to construct $M_{i+\omega}$ from M_i. In M^2 we have a unary predicate which is interpreted as B; let us denote this predicate also by B.

LEMMA 3.2. *Let* $N \equiv M^2$ *be a countable model which omits* q; *let* $\{\langle D_i^1, D_i^2\rangle \mid i \in \omega\}$ *be a set of inseparable sets in* N; *and let* $\pi: \mathrm{At}(B^N) \to \mathrm{At}(B^N)$ *be a permutation which is undefinable in* N; *then there is a proper elementary extension* N_1 *of* N *such that*: (a) N_1 *omits* q; (b) *for every* $i \in \omega$, D_i^1, D_i^2 *are inseparable in* N_1; *and* (c) *there is* $a \in B^{N_1}$ *such that* $D_1 = \{\pi(x) \mid x \in \mathrm{At}(B^N) \cap \bar{a}\}$ *and* $D_2 = \{\pi(x) \mid x \in \mathrm{At}(B^N) - \bar{a}\}$ *are inseparable in* N_1.

REMARK. Note that 3.2 is almost identical to 1.5, except of the omitting type requirement in (a). The proof will also be very similar.

PROOF. Let $T^0 = \mathrm{CD}(N)$ and let c be an individual constant not in $L(T^0)$.

We define by induction $T_n = T^0 \cup \{\psi_n(c, \bar{b})\}$. $T = \bigcup T_n$ will be a complete theory in the language $L_1 = L(T^0) \cup \{c\}$, and since N is Skolemized, T will describe the complete diagram of a model which is the Skolem hull of $|N| \cup \{c\}$; this model is going to be N_1. Along the construction of the T_n's we will have to fulfill the following types of tasks: (I) Given a sentence $\varphi \in L_1$, decide whether φ or else $\neg\varphi$ will belong to T. (II) Given a term $\tau(c, \bar{b})$ such that $T_n \vdash P(\tau(c, \bar{b}))$, for some a_i (which appears in the definition of M^2), add to T_n: $\tau(c, \bar{b}) = a_i$. (III) Given a formula $\chi(c, \bar{b}, x)$ and $i \in \omega$, take care that $\chi(c, \bar{b}, x)$ will not separate D_i^1 and D_i^2. (IV) Given χ as in (III), take care that χ will not separate D_1 and D_2. Denote tasks (I), (II), (III) and (IV) by $\langle\varphi\rangle$, $\langle\tau\rangle$, $\langle\chi, i\rangle$ and $\langle\chi\rangle$ respectively, and let $\{s_i \mid i \in \omega\}$ be a list of all tasks such that every tasks of type (II) appears in the list infinitely many times.

Now we have to formulate an induction hypothesis analagous to the independence of ψ_n in 1.5. Since M^2 expands $\langle B, \mathrm{Aut}(B)\rangle^{\mathrm{II}}$, for every formula $\psi(x, \bar{u})$ in $L(M^2)$ there is a formula $\mathrm{SC}_\psi(\bar{u})$ such that for every $\bar{d} \in |M^2|$, $M^2 \models \mathrm{SC}_\psi[\bar{d}]$ iff $\psi(x, \bar{d})$ defines a subset of B which is of the second category in $2^{\mathrm{At}(B)}$. Certainly the following sentence and the following schemas hold in M^2, and so in N:

(a) $\mathrm{SC}_{B(x)}$;

(b) $\mathrm{SC}_{\varphi\vee\psi}(\bar{u}) \to \mathrm{SC}_\varphi(\bar{u}) \vee \mathrm{SC}_\psi(\bar{u})$; and

(c) $[\mathrm{SC}_{\varphi(x,\bar{u})}(\bar{u}) \wedge \forall x(\varphi(x, \bar{u}) \to P(\tau(x, \bar{u})))] \to \exists y(P(y) \wedge \mathrm{SC}_{\varphi(x,\bar{u})\wedge\tau(x,\bar{u})=y}(\bar{u}, y))$.

Our induction hypothesis is that $N \models \mathrm{SC}_{\psi_n}[\bar{b}]$. Let $T_0 = T^0 \cup \{B(c)\}$; then by (a), T_0 satisfies the induction hypothesis. Schemas (b) and (c) are easily applied to show that tasks of types (I) and (II) can be accomplished. Suppose T_n has been defined and $s_n = \langle \chi(x, \bar{b}, y), i \rangle$. The formula $SC_{\psi_n \wedge \chi}(\bar{b}, y)$ does not separate D_i^1 and D_i^2 in N, so either for some $d \in D_i^1$, $N \models \neg SC_{\psi_n \wedge \chi}[\bar{b}, d]$, and then define $T_{n+1} = T^0 \cup \{\psi_n \wedge \neg\chi(c, \bar{b}, d)\}$, or for some $d \in D_i^2$, $N \models SC_{\psi_n \wedge \chi}[\bar{b}, d]$, and then define $T_{n+1} = T^0 \cup \{\psi_n \wedge \chi(c, \bar{b}, d)\}$; so T_{n+1} satisfies the induction hypothesis and s_n is accomplished.

Suppose now that T_n has been defined and $s_n = \langle \chi(c, \bar{b}, x) \rangle$ is of type (IV). We are looking for an $l \in \text{At}(B)$ such that either $\psi_{n+1} \equiv \psi_n \wedge l \subseteq c \wedge \neg\chi(c, \bar{b}, \pi(l))$ satisfies the induction hypothesis or $\psi_{n+1} \equiv \psi_n \wedge l \cap c = 0 \wedge \chi(c, \bar{b}, \pi(l))$ satisfies the induction hypothesis.

The formula $\text{Fin}(x)$ that says that there is a 1-1 function from a proper initial segment of $\langle P, < \rangle$ onto x is satisfied by an element of M^2 iff this element is a finite set. Since $\langle P^N, <^N \rangle$ is also of order type ω, (since q is omitted) $\text{Fin}(x)$ defines true finiteness also in N. The reader can easily check that if X is a topological space, $A \subseteq X$ is SC, then there is an open nonempty G such that A is hereditarily SC on G: that is for every G_1 nonempty open subset of G, $A \cap G_1$ is SC in X. This yields the following schema in M^2: Let $\psi(x, \bar{u})$ be a formula; then

$$(*) \quad M^2 \models SC_\psi \to \exists y(\text{Fin}(y) \wedge \forall z_1, z_2((z_1 \cup z_2 \subseteq \text{At}(B) - y \wedge z_1 \cap z_2 = \emptyset \wedge \text{Fin}(z_1) \wedge \text{Fin}(z_2)) \to SC_{\psi \wedge x \supseteq z_1 \wedge x \cap z_2} = 0)).$$

($x \supseteq z_1$ is an abbreviation of $(\forall t \in z_1)(t \subseteq x)$ similarly $x \cap z_2 = 0$.) Of course this schema holds in N too. Since Fin defines the finite sets also in N, this means that if $\psi(x, \bar{b})$ is SC then there is a finite $\sigma \subseteq \text{At}(B^N)$ such that for every finite, disjoint $\sigma_1, \sigma_2 \subseteq \text{At}(B^N) - \sigma$, $\{b \mid b \in B^N, N \models \psi[b, \bar{b}], b \supseteq \sigma_1 \text{ and } b \cap \sigma_2 = 0\}$ is SC.

Now suppose by contradiction that for every $l \in \text{At}(B)$

$$(**) \qquad \psi_n(x, \bar{b}) \wedge l \subseteq x \wedge \neg\chi(\pi(l), x, \bar{b}) \text{ is not SC; and}$$

$$(***) \qquad \psi_n(x, b) \wedge l \cap x = 0 \wedge \chi(\pi(l), x, \bar{b}) \text{ is not SC.}$$

Let σ be the finite set whose existence is assured by (*). Let

$$\alpha(u, v) \equiv (u \in (\text{At}(B) - \sigma)) \wedge (v \in (\text{At}(B) - \pi(\sigma))) \wedge (\forall l \in \text{At}(B) - \sigma - \{u\})(\neg SC_{\psi_n \wedge u \subseteq x \wedge l \cap x = 0 \wedge \chi(v, x, b)}).$$

By (**) and (***), $\alpha(u, v)$ defines $\pi \restriction \text{At}(B^N) - \sigma$, so π is definable in N, in contradiction to our assumption. So 3.2 is proved.

CONTINUATION OF THE PROOF OF 3.1(a). 3.1(a) is proved from this point on in exactly the same way as Theorem 1.2.

PROOF OF 3.1(b). If At(B) is countable in M, then in M^2 there is an element which is a 1-1 correspondence between P^{M^2} and At(B). Since in the construction, P remains unchanged, At(B) remains unchanged. In this case the reader can easily check that $\diamondsuit_{\aleph_1}$ is not needed and can be replaced by CH. Q.E.D.

COROLLARY 3.3. (a) ($\diamondsuit_{\aleph_1}$) *If B is 1-homogeneous atomic and SC, then there is B_1 such that B_1 is 1-homogeneous, $|\langle B_1, \mathrm{Aut}(B_1)\rangle| = \aleph_1$ and $\langle B, \mathrm{Aut}(B)\rangle \equiv \langle B_1, \mathrm{Aut}(B_1)\rangle$. We can also take care that B_1 will be SC.*

(b) (*CH*) *If B is 1-homogeneous atomic and SC, and $\mathrm{At}(B)$ is countable, then we can find B_1 as above such that in addition $\mathrm{At}(B_1)$ is countable and $B_1 \not\cong B$.*

ADDED IN PROOF.

REMARK. The results of this paper cannot be proved in ZFC, for if MA + $\aleph_1 < 2^{\aleph_0}$ holds, then every atomic BA of power $\aleph_1$ has continuum many automorphisms moving infinitely many atoms. This enables us to repeat at least some of the interpretation theorems of [R2]. So if B is an atomic BA such that every automorphism of B is induced by a permutation of the atoms moving only finitely many of them, then there is no B_1 of power $\aleph_1$ such that $\langle B_1, \mathrm{Aut}(B_1)\rangle \equiv \langle B, \mathrm{Aut}(B)\rangle$.

THEOREM (SHELAH). ($V = L$) *There is a compact generalized quantifier with a nonrecursively enumerable set of validities.*

PROOF. Let L be the language gotten from first-order language by adding the quantifier $Qxy\varphi(x, y)$. We will define a class of BA's K and define the semantics such that for every M: $M \models Qxy\varphi(x, y)$ iff $\varphi(x, y)$ defines on its domain a partial ordering of a BA belonging to K.

Let L^* be the first-order language gotten from L be adding an n-place relation symbol R_φ for every formula φ of L with n free variables. Let A be a non-r.e. subset of ω. We define a theory T^* in L^*. The following schemas are included in T^*:

(1) $R_{\neg\varphi} \leftrightarrow \neg R_\varphi$, $\exists x R_\varphi \leftrightarrow R_{\exists x\varphi}$, $R_\varphi \wedge R_\psi \leftrightarrow R_{\varphi\wedge\psi}$, etc. ... .

(2) $R_{Qxy\varphi} \rightarrow (x, y)$ defines on its domain a partial ordering of a BA.

(3) For every $n \in A$: If $R_\varphi(\bar{u}, x, y)$ defines the finite BA with n atoms then $R_{Qxy\varphi}(\bar{u})$.

(4) For every $n \notin A$: If $R_\varphi(\bar{u}, x, y)$ defines the finite BA with n atoms then $R_{\neg Qxy\varphi}(\bar{u})$.

(5) For every formula $\chi(x, y)$: $R_{Qxy\varphi}(\bar{u}) \wedge R_{\neg Qxy\psi}(\bar{v}) \rightarrow \chi(x, y)$ does not induce an isomorphism between the 2-place relations induced by φ, $\bar{u}$ and by ψ, $\bar{v}$.

For the sake of simplicity we will show how to define K so that the logic is just $\aleph_0$-compact. For every set of sentences S in L, let $S^* = \{R_\varphi | \varphi \in S\}$. Let $\{S_i | i < \aleph_1\}$ be an enumeration of all countable sets of sentences $S \subseteq L$ such that $S^* \cup T^*$ is consistent. We now define by induction two sets of BA's of power $\aleph_1$, K_i and $\bar{K}_i$. Suppose K_j, $\bar{K}_j$ have been defined for every $j < i$. Let M be a countable model of $T^* \cup S_i^*$, and let $N \succ M$ be a model of power $\aleph_1$ in which every definable BA is not isomorphic to any element of $\bigcup_{j<i} K_j \cup \bigcup_{j<i} \bar{K}_i$, and in which every isomorphism between two definable BA's is definable; such N exists by the methods of this paper. Let K_i be the set of all BA's definable in N by some $\varphi(\bar{a}, x, y)$ such that $N \models R_{Qxy\varphi}[\bar{a}]$ and let $\bar{K}_i$ be the set of all BA's definable in N by some formula $\varphi(\bar{a}, x, y)$ such that $N = R_{\neg Qxy\varphi}[\bar{a}]$.

Define K to be the class of all BA's isomorphic to some element of $\bigcup_{i<\aleph_1} K_i$. It is easy to see that the logic defined by K is $\aleph_0$-compact and has the downward Skolem Löwenheim theorem to $\aleph_1$. The set of validities of this logic is not r.e. because we included in T^* the axioms in group (3) and (4).

For full compactness one has to repeat the same argument for all sets of sentences in L. This necessitates universal choice and the methods of [S1].

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