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STABILITY, THE f.c.p., AND SUPERSTABILITY; MODEL THEORETIC PROPERTIES OF FORMULAS IN FIRST ORDER THEORY*

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Abstract: We investigate in detail stable formulas, ranks of types and their definability, the f.c.p., some syntactical properties of unstable formulas, indiscernible sets and degrees of types in superstable theories. There is a list of all results connected with those properties, or whose proof use them.

Contents

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§0. Introduction

As, on the one hand this paper is only defining some concepts and invertigating them; and on the other hand those investigations help in solving some problems not mentioning them; instead of abstracting the content of this paper, we shall give a list of all theorems and problems connected to them. Those whose proof depends on this paper are denoted by *.

Some of the material in §2 and some other, is a repetition on Shelah [B] because it is improved here, for completeness, and as also there some proofs were only hinted. As in the introduction we do not review the paper, this is done for every section separately in its beginning. The paper is self-contained.

The list is divided to the following topics: (for completeness, each part contain also results unconnected to stability)

(Definitions)

A) *Stability*

B) *Saturativity and universality*

C) *Categoricity of theories*

D) *Homogeneity*

E) *The number of non-isomorphic models of a theory*

F) *Categoricity of pseudo elementary classes*

G) *The number of non-isomorphic models of a pseudo-elementary class*

H) *Keisler's order and saturation of ultrapowers*

I) *Other results*

[Results not attributed to anybody neither in the list nor in the historical remarks, are unpublished results of the author. Usually for every result there is a reference, whereas the historical remarks are concentrated at the end of each part].

We should first define some concepts (for further explanation see §1).

T is a complete first order theory in $L = L(T)$. M is *λ-saturated* if every (consistent) type over a set $A \subset |M|$, $|A| < \lambda$ is realized in M. M is *λ-compact* if every type p over $|M|$, $|p| < \lambda$ is realized in M. M is *max-λ-saturated* if it is λ-saturated but not λ^+-saturated, similarly for compact

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M is μ-*universal* if every model of $\mathrm{Th}(M) = T$ of cardinality $\leq \mu$ is isomorphic to an elementary submodel of M. Notice, that for $\lambda > |L|$, M an L-model, M is λ-saturated iff it is λ-compact.

M is saturated if it is $\|M\|$-saturated.

For $T_1 \supset T$, $PC(T_1, T)$ is the class of reducts of models of T_1 to the language of T. $I(\lambda, T_1, T)$ is the (maximal) number of non-isomorphic models in $PC(T_1, T)$ of cardinality λ; and $I(\lambda, T) = I(\lambda, T, T)$.

T is stable in λ if for every model M of T, $A \subset |M|$, $|A| \leq \lambda$, the number of types elements of M realized over A (in M) is $\leq \lambda$. T is stable if it is stable in at least one cardinality, superstable if it is stable in every $\lambda \geq 2^{|T|}$.

T is categorical in λ if it has, up to isomorphism, exactly one model of cardinality λ.

$T_1 \triangleleft_\lambda T_2$ provided that: *if* M_1 is a model of T_1, M_2 a model of T_2, D a $(\aleph_0, \lambda)$-regular ultrafilter over λ, M_2^λ/D is λ^+-compact *then* M_1^λ/D is λ^+-compact. This is Keisler's order from [A]. D will always be a non-principal ultrafilter. Let $T_1 \triangleleft T_2$ hold when for every λ, $T_1 \triangleleft_\lambda T_2$.

T has the *f.c.p.*, if there is a formula $\varphi(x, \bar{y})$, such that for every $n < \omega$, T has a model M, sequences $\bar{a}^0, \ldots, \bar{a}^m$, $(\omega > m \geq n)$ such that $\{\varphi(x, \bar{a}^i) : i \leq m\}$ is not realized in M, but for every $j \leq m$, $\{\varphi(x, \bar{a}^i) : i \leq m, i \neq j\}$ is realized.

$\mu(\lambda)$ is the first cardinality such that: if T is a theory, p a type, $|T|$, $|p| \leq \lambda$, and T has a model omitting p of cardinality $\geq \mu(\lambda)$, then it has such models of arbitrarily high power. (See e.g. Chang [A] for the values of $\mu(\lambda)$).

$D_n(T)$ is the set of complete and consistent types with the variables $x_0, \ldots, x_{n-1}$ only (consistent - that is, consistent with T.) Let $D(T) = \bigcup_{n<\omega} D_n(T)$.

A. Stability

1) For every stable theory T there are cardinals $\kappa(T) \leq |T|^+$, $\lambda(T) \leq 2^{|T|}$ such that: T is stable in λ iff $\lambda = \lambda(T) + \Sigma_{\kappa<\kappa(T)} \lambda^\kappa$. We stipulate that for unstable T, $\kappa(T) = \infty$.

There is a syntactical condition $(C * \lambda)$ which is equivalent to $\kappa < \kappa(T)$. [See Shelah [D] Th. 4.4], *2) If T is superstable $\lambda(T) \leq |D(T)| + 2^{\aleph_0}$.

3) *Conjecture*: For every stable T $\lambda(T) = |D(T)|$ or $\lambda(T) = |D(T)| + 2^{\aleph_0}$. (i.e., $\lambda(T)$ can be chosen in this way.) Proved in Shelah [N*]

4) If T is stable in λ, M a model of T, $A \subset B \subset |M|$ $|A| \leq \lambda < |B|$, then B has a subset of cardinality $\geq \lambda^+$ which is indiscernible over A. (See Def. 5.2.) [See Shelah [D] Th. 3.1; [F] Th. 2.2, and here 5.8 are generalizations.]

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There are some definitions of prime model; and proofs it exists under certain stability conditions. For simplicity, assume all models we deal with are elementary submodels of some very saturated model $\overline{M}$ of T, and $A \subset |\overline{M}|$, $|A|$ is small.

Definition. M is λ-*prime* over A if: $A \subset M$, M is λ-compact, and if $A \subset N$, N is λ-compact, then there is an elementary embedding of M into N, which is the identity on A. The type p is λ-isolated, if there is $q \subset p$, $|q| < \lambda$, such that every element realizing q realizes p. [We can consistently replace compact by saturated, and $|q| < \lambda$ by q is a type on some A, $|A| < \lambda$; then we get $(\lambda, 1)$-prime models and $(\lambda, 1)$-isolated type. Note that every model is $\aleph_0$-compact.

5) Suppose T is stable in some $\mu < 2^\lambda$, or T is stable, $\lambda > |T|$.

(A) On every A, there is λ-prime model (and also a $(\lambda, 1)$ prime model). (See Ressayre [A] Th. 4.3, Shelah [B] Th. 3.5.)

(B) This model is unique, i.e., if M, N are λ-prime models over A, then there is an isomorphism from M onto N, which is the identity over A.

(C) If M is a λ-prime [$(\lambda, 1)$-prime] model over A, every finite sequence in M realize over A a λ-isolated [$(\lambda, 1)$-isolated] type provided that cf $\lambda \geq \kappa(T)$. [See Ressayre [A¼ Th. 4.3.]

(D) If M is a $(\lambda, 1)$-prime (or λ-prime) model over A, then in M there is no indiscernible set (Def. 5.2) over A of cardinality $> \lambda$.

(E) If T is superstable $A \subset M$, M is λ-compact [λ-saturated] and satisfies the conclusions of (C), (D) *then* it is a λ-prime [$(\lambda, 1)$-prime] model over A. For stable T there is a characterization of prime models. See Shelah [N*], [O*].

6) If T is stable, $\lambda \geq \kappa(T) + \aleph_1$, then over every A there is a $(\lambda, 1)$-prime model (which seemingly does not satisfy 5C, but satisfies 5D). *Conjecture*: See Shelah [N*], the model is unique.

7) Suppose T is stable and countable. Then for every A there is a model M, $A \subset M$ such that: if $\bar{c} \in M$, $\varphi(\bar{x}, \bar{y})$ is a formula, then there is $\theta(\bar{x}, \bar{b})$, $\bar{b} \in A$ which satisfies, for every $\bar{a} \in A$, $\models \varphi[\bar{c}, \bar{a}] \Rightarrow \models (\forall \bar{x})[\theta(\bar{x}, \bar{b}) \rightarrow \varphi(\bar{x}, \bar{a})]$, and $\models \theta[\bar{c}, \bar{b}]$.

8) *Question*: Is $\kappa(T) < \aleph_1$ sufficient in (7)? Is there a stable T which does not satisfy the conclusion of (7)?

9) If T is stable, M a model of T, $\aleph_0 \leq |Q^M| < \|M\|$, then for every $\lambda \geq \mu \geq |T|$, T has a model N, $\|N\| = \lambda$, $|Q^N| = \mu$. [See Shelah [A] Th. 6.3.]

10) If T is stable and countable, N an elementary extension of M which is a model of T, $N \neq M$, $Q^M = Q^N$ then there is an elementary extension N_1 of N, $N_1 \neq N$, $Q^{N_1} = Q^N = Q^M$. [See Lachlan [D*]]

11) If T is stable in λ, then it has a model M, $\|M\| = \lambda$, $A \subset |M|$, $|A| = \lambda$, and every permutation of A can be extended to an automorphism of M, and A is a maximal indiscernible set in M. Also if $\lambda = \Sigma_{i<\mu}\lambda_i$; $\mu, \lambda_i < \lambda$; T stable in λ_i, this holds. [See Harnik and Ressayre [B] 1.11.]

*12) If T is unstable, then it has the f.c.p. [Here Th. 3.8.].

*13) T is unstable *iff* it has a formula $\varphi(\bar{x}, \bar{y})$, a model M, and sequences $\{\bar{a}^n : n < \omega\}$ from M such that $M \models \varphi[\bar{a}^n, \bar{a}^m] \Leftrightarrow n < m$ [Here, Th. 2.13.]. [A generalization - Shelah [F] §2.]

Remark: For other kinds of prime models see Shelah [D] §5; [D] proof of 7.10; [C]; [K]. On totally transcendental theories (= $\aleph_0$-stable) see also Blum [A], Lachlan [B], [C], Baldwin [E].

Historical remarks

1) In Morley [A] it is proved that if T is countable and $\aleph_0$-stable, then it is λ-stable for every λ. Rowbottom [A], and Ressayre [A] prove

(indep.) that if T is stable in λ, $\lambda^{\aleph_0} > \lambda$, then T is stable in every $\mu \geq \lambda$. Rowbottom depend on G.C.H. In Shelah [B], (1) is proved for countable T, and partially for every T, and Shelah [D] gives 1) and [N*] is the full solution. A further conjecture appears here, §4 (conjecture 4E).

4) Morley [A] proves it for $\aleph_0$-stable, countable theories. Rowbottom has an unpublished weaker result than (4). In Shelah [B] it is proved for $\lambda > |T|$, and in [D] it is proved.

5) (A) For the case T stable in $\mu < 2^\lambda$, it was proved (indep. and successively) by Rowbottom, Ressayre [A], Shelah [A], [B]. The case $\lambda > |T|$ (which is new only if $2^\lambda = 2^{|T|}$) is of Shelah [A], [B].

(B) For $|A| = |T| = \lambda = \aleph_0$, this is due to Vaught [A]. This answers a question of Sacks in Chang and Keisler [B].

(C) Due indep. to Rowbottom and Ressayre, for regular λ. For singular λ it follows in fact from B3.

7) This is due indep. to Lachlan and Shelah (about the same time).

11) Morley [A] p. 537, (5) asked whether for every $\aleph_0$-stable T, $\kappa \geq \aleph_0$, T has a model M, $A \subset |M|$, $|A| = \|M\| = \kappa$, and every permutation of A can be extended to an automorphism of M. Silver answers affirmatively the question, for regular λ, using saturated models. By B2, it follows this is true for every λ if T is stable in λ. By Harnik and Ressayre [B], we can take A as a maximal indiscernible set for regular λ. In fact it is a particular case of their theorem.

12) Keisler [A], 5.1 proves a little weaker theorem: the property (E) implies the f.c.p. where (E) T has a model M, $A \subset |M|$ is infinite, φ a formula, $\varphi = \varphi(x_1, ..., x_n)$, and for every n different elements of A, $a_1, ..., a_n$ there is a permutation θ such that $M \models \varphi[a_{\theta(1)}, ..., a_{\theta(n)}]$, and a permutation θ such that $M \models \neg\varphi[a_{\theta(1)}, ..., a_{\theta(n)}]$.

This property (E) was first defined by Ehrenfeucht [A]. Here 4.7B, it is proved that there is unstable T which do not satisfy (E).

B. Saturativity and universality

1) If M is an L-model, $\|M\| \leq \mu = \Sigma_{\kappa<\lambda}\mu^{\kappa} + 2^{|L|}$, λ regular, then M has an elementary extension of cardinality μ which is λ-saturated. [See Morley and Vaught [D].]

*2) If T is stable in λ, then it has a saturated model of cardinality λ.

*3) If M_i $i < \delta$ is an increasing elementary chain of models of T, T is stable $\lambda > |T|$, every M_i is λ-saturated, cf $\delta \geq \kappa(T)$ (see (a)) then $\bigcup_{i<\infty} M_i$ is also λ-saturated.

4) *Conjecture*: We can replace $\lambda > |T|$ by $\lambda \geq \kappa(T)$. Proved in Shelah [N*].

*5) If M_i $i < \delta$ is an increasing elementary chain of models of T, T is superstable, and every M_i is λ-saturated, *then* $\bigcup_{i<\delta} M_i$ is λ-saturated.

6) If T is unstable, T has a saturated model of cardinality $\lambda > |T|$ iff $\lambda = \Sigma_{\kappa<\lambda}\lambda^{\kappa}$. [See Shelah [D] claim 6.5.2.]

7) If $\lambda = \beth_\delta > |T|$, cf $\delta < \kappa(T)$ (unstability is sufficient) then T has a model of cardinality λ which has no elementary extension of power λ which is $(\text{cf }\delta)^+$-saturated. [See Shelah [D] Th. 6.4.]

8) *Conjecture*: We can replace cf δ by κ, and $\lambda = \beth_\delta$ by $\lambda^\kappa > \lambda$.

9) For every T and regular λ, T has a max λ-saturated model. [See Shelah [D], 6.3, 6.6.]

10) If T has a $\kappa(T)$-saturated model M which is not λ^+-saturated, $\lambda^{|T|} < \|M\|$ *then* for every regular λ, T has max λ-saturated models of arbitrarily great powers. [See Shelah [D], 6.9 [N*]. The condition $\lambda^{|T|} < \|M\|$ cannot be weakened to $\lambda < \|M\|$.]

*11) If T has a $|T|^+$-saturated model, not λ^+-saturated; $(\lambda^+)^{|T|}$-universal *then* for every $\mu \geq \lambda$, λ regular T has a μ-universal, max λ-saturated model. (If T is unstable, or even if it has the f.c.p., the conclusion holds.)

*12) Every T satisfies exactly one of the following:

(i) For every $\mu \geq \lambda \geq 2^{|T|}$, T has a λ^+-universal model of cardinality $\geq \mu$ which is not $(2^\lambda)^+$-universal,

(ii) There is a model of T of cardinality $2^{|T|}$, every elementary extension of which is saturated.

13) If T has a model M, and there is a type p on $|M|$, $p = \{\varphi_i(x, \bar{a}^i): i < i_0 < \|M\|\}$, p is omitted, and $\{\varphi_i(x, \bar{y}^i): i < i_0\}$ is finite *then* the conclusion of (10) holds.

14) *Conjecture*: In (12) (i) we can say that for every $\mu \geq \lambda \geq |T|$ T has a λ-universal, not λ^+-universal model of cardinality $\geq \mu$.

15) *Conjecture*: If T has the independence p. (def. 4.1) $\lambda \geq 2^{|T|}$, then T has a universal model of cardinality λ iff $\lambda = \Sigma_{\mu<\lambda} 2^{\mu}$.

Note that

16) If T has the independence $p.$, $R \subset \lambda \times \lambda$, *then* T has a model M, formula $\varphi(\bar{x}, \bar{y})$ and sequences $\bar{a}^i$ $i < \lambda$ from M, *such that* $M \models \varphi[\bar{a}^i, \bar{a}^j]$ iff $\langle i, j\rangle \in R$.

Historical remarks

2) For $\lambda > |T|$, Harnik proves it using the method of the proof of F5 (or Theorems 5.16, 5.12 here) and Al and the definition of $(C * \lambda)$ (See Shelah [D]). The author completes it for $\lambda = |T|$.

3) It is implicit in the proof of (2).

10) This answers affirmatively question 4C, Keisler [A], p. 41 (one version), and implicitly answers 4A. Moreover, we do not need G.C.H.

Before (10) was proved (indep.); Harnik [C] proved: (G.C.H.) Every T satisfies exactly one of the following: (i) For every $\mu > \lambda \geq |T|$, λ regular, T has a max λ-saturated model of cardinality $\geq \mu$.

(ii) There is λ_0 such that every λ_0-saturated model of T of power $> \mu^{|T|}$ is μ^+-saturated.

11) This answers affirmatively questions 4C, Keisler [A] p. 41 (second version), and implicitly answers to 4B, and as in (10), without G.C.H. Keisler [A], 4.2b (ii) proved with G.C.H. that when T has the f.c.p., $\aleph_0 \leq \lambda \leq \mu$, then T has a μ^+-universal model which is max λ^+-saturated.

C. Categoricity of theories

*1) If T is categorical in λ, $\lambda > |T|$, then T is categorical in every $\mu > |T|$; and every model of T of cardinality $> |T|$ is saturated. (For countable T see Morley [A], for uncountable Shelah [K].)

2) *Conjecture*: (Morley) If T is categorical in $|T|$, $|T| > \aleph_0$, then T has a model of cardinality $< |T|$. Moreover T is a definitional extension of some $T' \subset T$, $|T'| < |T|$.

3) This conjecture was affirmed for the following cases:
(1) $\aleph_0 < |T| < 2^{\aleph_0}$, $|T|$ regular, [Keisler [G]].
(2) $\beth_\delta < |T| < \beth_{\delta+1}$, cf $\delta = \omega$, $|T|$ is regular.
(3) $|T| = |T|^{\aleph_0}$ [see Shelah [C]].
(4) By A6, in all cases in which by G), unsuperstability of T implies it has in $|T|$ many models. For example, assuming G.C.H., (2) holds. In fact if (2) fails, $\kappa(T) \geq |T|$, or $\kappa(T)^+ = |T|$, $\kappa(T)$ is singular.

4) If T is a complete theory in the language $L(Q_{eq})$ and is categorical in $\lambda > |T|^+$, and $\lambda_n < \lambda \Rightarrow \Pi_{n<\omega}\lambda_n < \lambda$ or $\lambda = \beth_\delta$, *then* T is categorical in every $\mu \geq \lambda$, and in some $\mu < \mu(|T|)$ [$L(Q_{eq})$ is the language with the added quantifier "there exist x's of the cardinality of the model".] (Perhaps we can improve the conclusion to "every $\mu > |T|^+$", by the methods of the proof of (1), but this was not checked yet.) [See Shelah [A] §6, [J].]

5) *Conjecture*: If T is a complete theory in the language $L(Q_{eq})$ and is categorical in one $\lambda > |T|^+$, then it is categorical in every $\lambda > |T|^+$. If T is categorical in $\lambda = |T|^+ > \aleph_1$, then T is a definitional extension of $T' \subset T$, $|T'| < |T|$.

6) Let T be categorical in $|T|^+$. Then over every A, $|A| > |T|$, there is a minimal and unique prime model M; and every elementary permutation of A can be extended to an automorphism of M. Also over every model M there is a minimal unique prime model. If M is a model of T, $\|M\| > |T|$, $A \subset |M|$, every elementary permutation of A can be extended to an automorphism of M. [See Harnik and Ressayre [B].]

7) If every model of T of power λ is $|T|^+$-universal, then T is categorical in $|T|^+$. [Shelah [N]].

8) *Conjecture* (Keisler): If Ψ is a sentence in $L_{\omega_1\omega}$, and it is categorical in some $\lambda \geq \beth_{\omega_1}$, then it is categorical in every $\lambda \geq \beth_{\omega_1}$. Similarly for $\aleph_1$ instead of $\beth_{\omega_1}$ (Compare with D5).

9) If T is countable, T is categorical in $\aleph_0$ iff for every n, $D_n(T)$ is finite. [See Ryll-Nardzewski [A], or Svenonius [A].]

10) *Conjecture*: 1) If T is finitely axiomatizable and complete, then it is not categorical both in $\aleph_1$ and $\aleph_0$.
2) If T is categorical in $\aleph_1$ but not in $\aleph_0$, then it is not finitely axiomatizable.

For simplifications of the proof of Morley's categoricity theorem see Chang and Keisler [B], Baldwin and Lachlan [B], Ressayre [A] and Keisler [I*]. Marsh [A] is repeated and completed in Baldwin and Lachlan [B]. See also Baldwin [A], [C], [D], [E], [F*], Glashmire [A], Harnik [D], Mcintyre [A*], Ash [A*], Keisler [I*]. Baldwin [C] can be generalized to uncountable T.

Historical remarks

1) This is Łoś' conjecture. In fact mainly from the attempts to solve it, the theory of stability was developed.

Ehrenfeucht [A] proved that if T is countable and categorical in $\mu = 2^\lambda$, then it does not have the property (E). (See historical remarks to A12.) Scott improved it to the case $\mu = \kappa^\lambda > \kappa$ and Morley [A] to any $\mu > \aleph_0$. Later Keisler [A] shows (E) can be replaced by the f.c.p.

Morley [A] proved (1) for countable T. For uncountable theories, successive and independent approximations were Rowbottom [A], Ressayre [A] and Shelah [B]. Assuming G.C.H. Rowbottom proved that if T is categorical in $\mu > \chi = \inf\{\chi : \chi \geq |T|, \chi^{\aleph_0} > \chi\}$, then T is categorical in every $\lambda \geq \mu$. Ressayre eliminated G.C.H. and proved that also for $|T|^+ < \mu < \chi$ and also mostly for $|T|^+ = \mu$; and shows that T is categorical in some $\lambda < \beth[(2^{|T|})^+]$. He also gave a unified and simplified proof for countable and uncountable T. Shelah shows it for every $\mu > |T|$, $\mu \neq \chi$; and shows that T is categorical in some $\lambda < \mu(|T|)$. In Shelah [K], (1) is fully proved.

2, 3) The conjecture appears in Morley [A], and he said it is not due to somebody in particular. The conjecture is partially verified by (3). The case $|T| < 2^{\aleph_0}$ is due to Keisler, the case $\beth_\delta < |T| < \beth_{\delta+1}$ is a refinement of the author.

7) This affirms a conjecture of Keisler.

9) This was proved independently by Engeler [A], Ryll-Nardzewski [A], Svenonius [A].

10) It is not clear to whom to attribute this conjecture. Part (2) is question (2), Morley [A], p. 537.

D. Homogeneity

Let $H(T)$ be the class of cardinalities in which every model of T is homogeneous.

*1) If there is λ, $|T| + \aleph_1 \leq \lambda \in H(T)$, then every μ, $\mu \geq \lambda$ or $\mu > |D(T)|$, belongs to $H(T)$. But $\aleph_0 < \lambda \leq |D(T)|$ implies $\lambda \notin H(T)$. [See Shelah [D], 7.6.] [See example in Keisler [A], p. 41. 4A.]

2) If $|T| = \aleph_0$, and T is not categorical in $\aleph_1$, then in every λ, $\aleph_0 < \lambda \leq 2^{\aleph_0}$, T has a model which is not $\aleph_1$-homogeneous. Hence if $|T| = \aleph_0$, $\aleph_1 \in H(T)$ then T is categorical in $\aleph_1$. [See Shelah [D] Th. 7.9.]

3) If $\aleph_0 < |T| \in H(T)$, then T is a definitional extension of some $T' \subset T$, $|T'| < |T|$.

4) Let Ψ be a sentence in $L_{\omega_1,\omega}$. If $\aleph_1 \in H(\Psi)$, then every $\lambda \in H(\Psi)$. for $\lambda > \aleph_0$. (See Keisler [B] §3. Also for the class of reducts of models of Ψ.)

5) Let Ψ be a sentence in $L_{\omega_1,\omega}$, $\lambda > \aleph_0$. Suppose (α) Ψ is categorical in λ, (β) its models of cardinality λ are $\aleph_1$-homogeneous (γ) every countable model of Ψ has elementary extensions of arbitrarily high power which is a model of Ψ. *Then* Ψ is categorical in every $\lambda > \aleph_0$. (See Keisler [G].)

Let T be a theory in L, $P = \{p_i : i < i_0\}$ a set of types in L, $H(T, P)$ the class of cardinals $\lambda \geq |L| + \aleph_1$ in which every model of T which omits every $p \in P$ is homogeneous.

6) If there is $\lambda \in H(T, P)$, $\lambda > |T|$, then there is $\mu_0 < \beth[(2^{|L|})^+]$ such that: every $\mu \geq \mu_0$ belongs to $H(T, P)$, and every μ, $|L| < \mu < \mu_0$ does not belong to $M(T, P)$ except possibly two, if there are two exceptions, then $\mu_0 \leq (2^{|T|})^+$. If κ is such an exception then: (1) $(2^{\kappa})^+ \geq \mu_0$ (2) cf $\kappa > \omega$, or $\mu_0 \leq (2^{|T|})^+$ (3) if $\lambda = \Sigma_{\chi<\lambda} \lambda^{\chi}$, $\kappa \neq \lambda^+$, [See Shelah [D], Th. 7.10, 6.7, 6.8, 7.5.]

7) (G.C.H.) Let $h_T(\lambda)$ be the number of homogeneous models of power λ (with a small change in the definition for singular λ). Then $|T| \leq \aleph_0 < \lambda < \mu$ implies $h_T(\mu) \leq h_T(\lambda)$, and $h_T(\aleph_0) \leq \aleph_0$, $\aleph_0 < \lambda$ implies $h_T(\lambda) \leq h_T(\aleph_0)$. (See Keisler and Morley [F].)

8) *Conjecture* (Keisler and Morley): (G.C.H.) For every $\lambda \geq \aleph_2$, $h_T(\lambda) = h_T(\aleph_2)$ (for countable T).

9) *Conjecture* (Keisler and Morley): Suppose M is an $(2^{\aleph_0})^+$-homogeneous model of cardinality $> 2^{\aleph_0}$, $|\mathrm{Th}(M)| = \aleph_0$. Then for every λ, M has an elementary extension N, which realizes no new type, is λ-homogeneous, and is of power $\geq \lambda$.

10) *Conjecture*: If M is a λ^+-homogeneous model of T of cardinality λ, then $\lambda \leq 2^{|L|}$.

Historical remarks

1) The example appearing in Keisler [A] is due to Morley, and it is of a countable T, for which $H(T) = \{\lambda : \lambda > 2^{\aleph_0}\}$. Keisler [B] proves for countable T, that $\lambda \geq \aleph_1 \in H(T)$ implies $\lambda \in H(T)$. Question B of Keisler [B] p. 260 is partially answered by (1), which is due to the author. Partial result is Shelah [D] Th. 7.6.

2) This answers affirmatively question D, Keisler [B] p. 260.

6) This partially answers question B, Keisler [B] p. 260.

E. The number of non-isomorphic models of a theory

(Remember T is always complete.) (Notice G.10.)

1) If T is countable, categorical in $\aleph_1$ but not in $\aleph_0$, then $I(\aleph_0, T) = \aleph_0$ [see Baldwin and Lachlan [B]. If T is countable and superstable, $I(\aleph_0, T) > 1$ *then* $I(\aleph_0, T) \geq \aleph_0$. [Lachlan [D*]]

2) *Conjecture*: If T is categorical in $|T|^+$ but not in $|T|$, $|T| = \aleph_\alpha$ then $I(|T|, T) = |\alpha| + \aleph_0$.

3) If T is countable, $I(\aleph_0, T) \neq 2$. [Vaught [A]]

4) For every $n \neq 2$, there is a countable T, $I(\aleph_0, T) = n$. [See Vaught [A], (the result is due to Ehrenfeucht)].

5) If T is $\aleph_0$-stable but not $\aleph_1$-categorical then for every α, $I(\aleph_\alpha, T) \geq |\alpha + 1|$ (see Shelah [D], 7.9, and 6.9, Rosental [A*].

*6) If T is countable, and not categorical in $\aleph_1$, then for every $\alpha \geq \beth_2$, $I(\aleph_\alpha, T) \geq \beth_2$. [compare with Gl].

*7) If T is superstable $\lambda = \lambda(T) > |T|$, $\mu \geq |T|$, $2^\mu > 2^{\aleph_0}$, then $I(\mu, T) \geq \min(2^\mu, 2^\lambda)$.

8) *Conjecture*: If T is stable $\lambda = \lambda(T) > |T|$, (see Al) then for every $\mu \geq |T| + \aleph_1$, $I(\mu, T) \geq \min(2^\mu, 2^\lambda)$. False.

9) If $|D(T)| > |T|$, $\lambda \geq |T|$, then $I(\lambda, T) \geq |D(T)|$. [See Ehrenfeucht [B], in fact.]

*10) If $|T| \leq \aleph_\alpha \leq |D(T)|$ T is superstable, then $I(\aleph_\alpha, T) \geq |\alpha + 1|$.

11) If T is superstable, and it has an $\aleph_0$-saturated model M, in which there are two maximal indiscernible sets of different infinite cardinalities, *then* for every $\aleph_\alpha \geq |T|$, $I(\aleph_\alpha, T) \geq |\alpha|$.

12) *Conjecture*: Every T satisfies one of the following:
(i) for every $\lambda \geq |T|$, $I(\lambda, T) \leq 2^{2^{|T|}}$.
(ii) for every $\aleph_\alpha \geq |T|$, $I(\aleph_\alpha, T) \geq |\alpha + 1|$.

This is a special case of a mode general conjecture, which for simplicity we phrase for countable T, and large enough α.

13) *Conjecture*: Every T, satisfies exactly one of the following (for every $\alpha \geq \beth_2$).

I) $I(\aleph_\alpha, T) = 1$
II) $I(\aleph_\alpha, T) = \beth_2$
III) $I(\aleph_\alpha, T) = |\alpha|$

IV) $I(\aleph_\alpha, T) = |\alpha|^{\aleph_0}$

V) $I(\aleph_\alpha, T) = |\alpha|^{\beth_1}$

VI) $I(\aleph_\alpha, T) = 2^{|\alpha|}$

VII)$_i$ $I(\aleph_\alpha, T) = \min[2^{\aleph_\alpha}, \beth(|\alpha|, i+1)]$ (where $1 < i < \aleph_1$) [probably $i \leq \mathrm{Deg}\, x = x$]

VIII) $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$

$[\beth(\lambda, i) = \lambda + \Sigma_{j<i} 2^{\beth(\lambda, j)}]$

(case VII is $\aleph_1$ cases, in fact). (Each of the cases is realized by some T).

This will, essentially, affirm

14) *Conjecture* (Morley): $I(\lambda, T)$ is a non-decreasing function, for $\lambda > |T|$.

15) For every $\lambda, \mu \leq 2^\lambda$ there is T, $|T| = 2^\lambda$, such that: $I(\kappa, T) = 2^\mu$ for every $\kappa \geq 2^\lambda$. Also for every λ there is T, $|T| = \lambda$ such that for every $\kappa \geq 2^\lambda$, $I(\kappa, T) = 2^{2^\lambda}$. Also there is a countable theory T, such that for every $\lambda > |T|$, $I(\lambda, T) = 2^\lambda$, but every $2^{|T|}$-universal model of T is saturated.

16) *Conjecture* (Vaught): $|T| = \aleph_0$, $I(\aleph_0, T) > \aleph_0$, implies $I(\aleph_0, T) = 2^{\aleph_0}$. (See G21).

Historical remarks

1) This was Vaught's conjecture. Morley [B] proves that $I(\aleph_0, T) \leq \aleph_0$. Baldwin and Lachlan [B] proved also the other inequality. This answers affirmatively question (1), Morley [A], p. 537.

2) I thank Chang for suggesting to conjecture something of this form.

5) This was a conjecture of Harnik, in [A]. The proof is based on a result from Morley [B].

7) For $|T| = \aleph_0$, $\mu = \beth_1$ this affirms a conjecture of Keisler (Chang and Keisler [B] open problem 13).

13) In Harnik [A] appears an example of T, $\beth_1 \leq I(\lambda, T) \leq \beth_2$. The example is due to Rabin, and the proof of Harnik. Gaifman [A] showed that the theory of numbers satisfies VIII.

15) The construction can be based on the example of Morley, appearing in Keisler [A], p. 41 (or the one of Harnik [A]). The last example is a construction uniting the previous construction with that of Baldwin and Lachlan [B], §4; but then new phenomena arise.

F. Categoricity of pseudo-elementary classes

*1) Let χ, λ, μ be infinite cardinals, $\chi < \lambda$, $\chi < \mu$. The following statements are equivalent.

(i) If $|T_1| \leq \chi$, $PC(T_1, T)$ is categorical in λ *then* it is categorical in μ.

(ii) If $|T_1| \leq \chi$, p a type $|p| \leq \chi$, Q a one-place predicate, and T_1 has a model M, $\mu = \|M\| > |Q^M|$ which omits p, *then* T_1 has a model N, $\lambda = \|N\| > |Q^N|$, which omits p.

(iii) If $|T_1| \leq \chi$, $PC(T_1, T)$ has only homogeneous models in λ, *then* it has only homogeneous models in μ. [See Shelah [B] p. 200, [H].] (Here we can replace T_1 by a sentence of $(L_1)_{\chi^+,\omega}$.)

2) If $PC(T_1, T)$ is categorical in $\aleph_1$, $|T_1| = \aleph_0$, *then* it is categorical in every $\lambda > \aleph_0$. [See Keisler [B], §3 for even a more general result.] (C.H. is eliminated by 1)).

*3) If $PC(T_1, T)$ is categorical in λ, $\lambda > |T_1|$, $\mu(|T_1|) = \beth_\delta$ δ divides γ, *then* $PC(T_1, T)$ is categorical in $\beth_\gamma$ [See Shelah [B] Th. 4.5 and G.10.]

*4) If $PC(T_1, T)$ is categorical in $\lambda > |T_1|$ then:

I) T is superstable,

II) stable in every $\lambda \geq |T_1|$,

III) has not the f.c.p.;

IV) is categorical in $\mu \geq |T_1|$ iff every model in it of cardinality μ is saturated. [See Shelah [H]; Shelah [B] Th. 4.2; Keisler [A] 4.2 (i); Shelah [B] 4.5.3c, [H].]

*5) If T is countable, superstable [$\aleph_0$-stable] and without the f.c.p., then there exists T_1, $T \subset T_1$, $|T_1| = 2^{\aleph_0}$, [$|T_1| = \aleph_0$] such that $PC(T_1, T)$ is categorical in every cardinality $\geq 2^{\aleph_0}$ [$\geq \aleph_0$]. [See Shelah [G].]

*6) If T is countable and superstable, then there exists T_1, $T \subset T_1$, $|T_1| = 2^{\aleph_0}$, such that $PC(T_1, T)$ is categorical in $2^{\aleph_0}$ [See Shelah [G].] (Compare 5, 6 to G10, G14.)

7) *Question*: If T is uncountable, what are the necessary and sufficient conditions for the existence of T_1, $T \subset T_1$, such that $PC(T_1, T)$ is categorical in every $\lambda > |T_1|$?

8) *Conjecture*: Suppose $PC(T_1, T)$ is categorical in $\lambda = |T_1|$. Then $\lambda > \aleph_0$ implies I, II, IV of (4), and $\lambda > \beth_1$ implies III of (4). *Remark:* For $\lambda \geq 2^{\aleph_0}$, it suffices to prove T is superstable. For partial results see G7, G12, G13, G15, G23, A2. So if $\lambda = \lambda^{\aleph_0}$, (8) holds.

Historical remarks

1, 2, 3) From the proof of Th. 3.3, Keisler [B], p. 256 it is trivial that (ii) → (iii). Keisler [B] 4.2 (i.e. 2) assuming C.H.) is a particular case of (ii) → (i).For countable T, (ii) → (i) and (3) are due, independently, to Choodnovski, Keisler [E] p. 18.2 (for $\lambda = \lambda^{\aleph_0}$ only) Shelah [I], [B]; and generally to Shelah [I], [B], [H]. The direction (i) → (ii) is due to the author, and also (iii) → (ii). I don't know whether Choodnovski uses the restriction $\lambda = \lambda^{\aleph_0}$. This answers partially questions from Keisler [B].

4) I, II, IV can be seen quite easily from the proofs in the case $T_1 = T$; the history of which appears in C1; but seemingly this was first noted in Shelah [A], [B], [P*].

III was proved by Keisler [A] 4.2b (i) p. 41 for countable T. By IV the generalized case has the same proof.

G. The number of non-isomorphic models of a pseudo elementary class

1) If $|T_1| = \aleph_0$, $|D(T)| > \aleph_0$, then $I(\aleph_1, T_1, T) = 2^{\aleph_1}$. [See Keisler [C] Th. 5.6.] (In fact, by using Ehrenfeucht-Mostowski models, for every $\lambda > \aleph_0$, $I(\lambda, T_1, T) \geq 2^{\aleph_1}$).

2) If $|T_1| = \aleph_0$, T unstable in $\aleph_0$, $2^{\aleph_1} > 2^{\aleph_0}$, then for every $\lambda > \aleph_0$, $I(\lambda, T_1, T) \geq 2^{\aleph_1}$. (This follows from (1) by (3).)

Sh:10

3) If $T(A)$ is the complete diagram of $A \subset |M|$, $T \subset T^* \subset T(A)$ M a model of T, and $I(\lambda, T_1 \cup T^*, T^*) \geq \mu$, $\mu > \lambda^{|A|}$ then $I(\lambda, T_1, T) \geq \mu$.

4) *Conjecture* (Keisler): If $|T_1| = \aleph_0$, $|D(T)| > \aleph_0$, then $I(\beth_1, T_1, T) = \beth_2$.

5) *Conjecture*: In (2), $2^{\aleph_1} > 2^{\aleph_0}$ is not needed. False.

*6) If T is unstable, $\lambda > |T_1|$, then $I(\lambda, T_1, T) = 2^\lambda$. [See Shelah [E].]

*7) If T is unstable, $\lambda = |T_1| \geq \aleph_1$, then $I(\lambda, T_1, T) = 2^\lambda$, except in the very rare case that: there is a family of cardinality 2^λ, of subsets of λ each of cardinality λ, the intersection of any two of which is finite. [See Shelah [E].]

8) *Question*: Is it consistent that such $\lambda > \aleph_0$ exists? (This is a set-theoretic question, of course.) Baumgartner said yes.

9) *Conjecture*: If T is unstable, $\lambda \geq |T_1| + \aleph_1$, then $I(\lambda, T_1, T) = 2^\lambda$.

*10) If T is not superstable, $\lambda > |T_1|$ is regular, then $I(\lambda, T_1, T) = 2^\lambda$. Moreover, for every $\mu > \lambda, I(\mu, T_1, T) \geq 2^\lambda$. [See Shelah [H].]

*11) If T is unsuperstable but stable, $\lambda > |T_1|$ is regular, then in $PC(T_1, T)$ there are 2^λ models M_i of cardinality λ, such that if $i \neq j$, M_i cannot be elementarily embedded in M_j. [See Shelah [H].]

*12) If T is unsuperstable, $\lambda = \lambda^{\aleph_0} \geq |T_1|$ then $I(\lambda, T_1, T) > \lambda$. If $\lambda = \Sigma_{n<\omega} \lambda_n, \lambda_n = \lambda_n^{\aleph_0}, \lambda \geq |T_1|$ then $I(\lambda, T_1, T) \geq \lambda$. [See Shelah [H].]

*13) If T is unsuperstable, $\lambda \geq |T_1|, \mu < \lambda \leq \mu^{\aleph_0}, 2^\mu < 2^\lambda$, then $I(\lambda, T_1, T) = 2^\lambda$. In fact suppose there is a tree with μ nodes and $\geq \lambda$ branches of height $\kappa < \kappa(T)$, $|T_1| \leq \lambda$, $2^\mu < 2^\lambda$, *then* $I(\lambda, T_1, T) = 2^\lambda$. Also if there is such a tree, $\kappa < \kappa(T)$, $\aleph_{\alpha+\beta} \geq |T_1|, \chi < \aleph_\alpha \Rightarrow \chi^\kappa < \aleph_\alpha$, $2^{\mu+|\beta|} < 2^\lambda$, then $I(\aleph_{\alpha+\beta}, T_1, T) \geq 2^\lambda$. [See Shelah [H] and also [M], Th. 2.]

14) *Conjecture*: If T is unsuperstable, $\lambda \geq |T_1| + \aleph_1$, then $I(\lambda, T_1, T) = 2^\lambda$. Moreover, if $\lambda > |T_1|$, in $PC(T_1, T)$ there are 2^λ models of cardinality λ, no one of them can be elementarily embedded in any other.

*15) Suppose T is stable and has the f.c.p., $\aleph_\beta = \min(2^{\aleph_0}, |T_1|)$. Then for every $\aleph_\alpha \geq |T_1|$, $I(\aleph_\alpha, T_1, T) \geq 2^{|\alpha - \beta|}$. [See Shelah [G].] (Compare F6)

Definition: $m(\lambda, \mu, T) = \min\{I(\lambda, T_1, T): T \subset T_1, |T_1| = \mu\}$.

16) For every α there is a complete theory T_α, $|T_\alpha| = |\alpha| + \aleph_0$, such that for every $\aleph_\gamma \geq \mu \geq |\alpha| + \beth_1$, $m(\aleph_\gamma, \mu, T) = \beth(|\gamma - \beta|, \alpha)$ where $\aleph_\beta = 2^{\aleph_0}$, $\gamma \geq \beta + \omega$.

17) There is a countable complete T, which is superstable, and for every $\lambda \geq \mu + (2^{\aleph_0})^+$, $m(\lambda, \mu, T) = 2^\lambda$.

18) *Conjecture*: If $\aleph_\beta = 2^{\aleph_0}$, and for some $\aleph_\alpha = \lambda > \mu \geq 2^{|T|}$, $m(\lambda, \mu, T) \geq \beth(|\alpha - \beta|, |T|^+)$ *then* for every $\lambda > \mu \geq 2^{|T|}$, $m(\lambda, \mu, T) = 2^\lambda$.

19) *Conjecture*: For simplicity let T be countable, $\lambda > \mu \geq 2^{\aleph_0}$. Then one of the following occurs:

(i) for every $\lambda > \mu \geq 2^{\aleph_0}$, $m(\lambda, \mu, T) = 2^\lambda$

$(\mathrm{ii})_\gamma$ for every $\aleph_\alpha > \mu \geq 2^{\aleph_0} = \aleph_\beta$, $m(\aleph_\alpha, \mu, T) = \beth(|\alpha - \beta|, \gamma)$ if $\alpha \geq \beta + \omega$ $[\gamma < |T|^+]$.

(iii) for every $\lambda > \mu \geq 2^{\aleph_0}$, $m(\lambda, \mu, T) = 1$.

*20) Suppose T is countable, p a type in its language. If T has a model omitting p in $\beth_\omega$ but not in $\beth_{\omega_1}$, *then* for every $\alpha \geq \omega$, $I(\aleph_\alpha, T) \geq |\alpha|^{\aleph_0}$, and for every n, $I(\aleph_n, T) \geq 2^n$.

21) If $|T_1| = \aleph_0$, $I(\aleph_0, T_1, T) > \aleph_1$ then $I(\aleph_0, T_1, T) = 2^{\aleph_0}$. This is true also if T_1 is in $L_{\omega_1, \omega}$. [See Morley [C].]

22) There are countable T_1, T $I(\aleph_0, T_1, T) = \aleph_1$ (T-the theory of linear order, T_1 saying for every two elements there is an order-automorphism taking the first to the second. We can complete the theories w.l.o.g.) [Kunen, unpublished].

23) If $|D(T)| > |T_1|$, $\lambda \geq |T_1|$ then $I(\lambda, T_1, T) \geq |D(T)|$ [See Ehrenfeucht [B] in fact].

See also Shelah [C].

Historical remarks

1) This improves Ehrenfeucht [B], which proves, in fact, that for every $\lambda \geq \aleph_0$, $I(\lambda, T_1, T) \geq 2^{\aleph_0}$.

6) Gaifman [A] proves it for the theory of numbers. Ehrenfeucht proves, assuming T has the property (E) (see A12), that $I(2^\lambda, T_1, T) > 1$. Shelah [A] Th. 2.13 [B] 4.51 proves $I(\aleph_\alpha, T_1, T) \geq |\alpha - \beta|$ where $\aleph_\beta = |T_1|$ and T is unstable.

10) In Shelah [B] 4.5.2, it was proved that if T is unsuperstable, $\aleph_\alpha \geq \aleph_\beta = |T_1|$, then $I(\aleph_\alpha, T_1, T) \geq |(\alpha - \beta)/\omega|$.

II. Keisler's order and saturation of ultrapowers

1) $\triangleleft$ is a reflective order. [See Keisler [A].]

2) T is $\triangleleft$-minimal iff for every model M of T, and $(\aleph_0, \lambda)$-regular ultrafilter D over λ, M^I/D is λ^+-compact. There are minimal theories which are countable. [See Keisler [A] p. 32.]

3) T is $\triangleleft$-maximal iff: for every model M of T, and $(\aleph_0, \lambda)$-regular ultrafilter D over λ, M^I/D is λ^+-compact iff D is λ^+-good. There are maximal theories which are countable. For $\lambda > \aleph_0$, no theory is $\triangleleft_\lambda$-minimal and $\triangleleft_\lambda$-maximal. [See Keisler [A] p. 32.]

4) The following condition is sufficient for $\triangleleft$-maximality: there is $\varphi(\bar{x}, \bar{y})$, such that for any $n < \omega$, $w \subset n \times n$, $\Psi \in T$ where

$$\Psi = (\exists \bar{y}^1, \ldots, \bar{y}^n) \bigwedge_{\substack{\langle i,j\rangle \in w \\ i<j}} (\exists \bar{x})(\varphi(\bar{x}, \bar{y}^i) \wedge \varphi(\bar{x}, \bar{y}^j))$$

$$\bigwedge_{\substack{\langle i,j\rangle \notin w \\ i<j}} \neg (\exists \bar{x})(\varphi(\bar{x}, \bar{y}^i) \wedge \varphi(\bar{x}, \bar{y}^j)) \quad .$$

[See Keisler [A] 3.1 Benda [A], part II Th. 9.]

The following results will appear in Shelah [G], sometimes using Kunen [A].

*5) If $\lambda \geq 2^{\aleph_0}$, T is countable, then T is $\lhd_\lambda$-minimal iff T has not the f.c.p. (by A12, this implies T is stable).

*6) If $\aleph_0 < \lambda < 2^{\aleph_0} < 2^\lambda$, T is countable, then T is $\lhd_\lambda$-minimal iff T is stable.

*7) T is $\lhd$-minimal iff T has not the f.c.p.

In order to show that in (6), $2^{\aleph_0} < 2^\lambda$ is superfluous, it suffices to prove

8) *Conjecture*: If M is a λ^+-universal model of T, T unstable, D $(\aleph_0, \lambda)$-regular ultrafilter on λ, *then* M^λ/D is not λ^+-compact.

9) If T has the strict order property (8) holds [for definition see 4.2]. Also if $2^{(\lambda^+)} > 2^\lambda$, (8) holds.

*10) If T_1, T_2 are countable stable and with the f.c.p., then $T_1 \lhd T_2 \lhd T_1$. If T_3 is unstable $T_1 \lhd T_3$.

*11) Suppose there is $\lambda > \aleph_0$, $\lambda^+ < 2^\lambda < 2^{\lambda^+}$, T_1 is countable and stable, T_2 unstable. Then $T_1 \lhd T_2$ but not $T_2 \lhd T_1$.

*12) There is a countable T (not minimal nor maximal) iff there is a $(\aleph_0, \lambda)$-regular non-good ultrafilter D on a cardinal μ such that $\Pi n_i/D \geq \aleph_0$ implies $\Pi n_i/D > \mu$. So if G.C.H. fails, this holds.

*13) There are two unstable countable theories, T_{ord}, T_{ind}, (defined here, Th. 4.7) such that for every unstable T, $T_{\text{ord}} \lhd T$, or $T_{\text{ind}} \lhd T$ (or both).

14) *Conjecture*: Those theories are incomparable.

*15) If M^λ/D is $(2^\lambda)^+$-saturated, D any ultrafilter, M is μ-saturated, *then* M^λ/D is μ^λ/D-saturated.

*16) If T is countable and without the f.c.p., D an ω_1-incomplete ultrafilter over I, M a model of T, then M^I/D is $\aleph_0^I/D$-saturated. Moreover, if λ is the least cardinal such that there is $\varphi(x, \bar{y})$ and $p = \{\varphi(x, \bar{a}^i) : i < \lambda\}$ ($\bar{a}^i \in |M|$) which is a consistent type over $|M|$, but omitted, *then* M^I/D is maximally λ^I/D-saturated.

*17) If T is countable stable and with the f.c.p., D an ω_1-incomplete ultrafilter over λ, $\mu = \min\{\Pi_{i<\lambda} n_i/D\colon \Pi n_i/D \geq \aleph_0,\ n_i < \aleph_0\}$ *then* M^I/D is max μ-saturated.

18) There are theories T_α, $|T_\alpha| = \aleph_\alpha$, such that if $\alpha < \beta$, $T_\alpha \triangleleft T_\beta$ but not $T_\beta \triangleleft T_\alpha$. If there is a countable T, which is not minimal nor maximal, then there are incomparable theories.

*19) If, vaguely speaking, we replace in the definition of $\triangleleft$, ultrapower by limit ultrapower; then by the resulting order $\leq$ the countable stable theories are divided into four clases: K_1-superstable without f.c.p., K_2-unsuperstable without the f.c.p., K_3-superstable with the f.c.p.; K_4-unsuperstable with the f.c.p. K_1 is minimal (among them); K_3, K_2-incomparable. If $T_1 \leq T_2$, then $\kappa(T_1) \leq \kappa(T_2)$.

*20) If T is not $\triangleleft_\lambda$-minimal then it is not $\triangleleft_\mu$-minimal for any $\mu \geq \min(\lambda, 2^{|T|})$.

21) *Conjecture*: If T is not $\triangleleft_\lambda$-minimal, then it is not $\triangleleft_\mu$-minimal ($\mu = |T| + 2^{\aleph_0}$).

22) *Conjecture*: If T_1, T_2 are countable, unstable and do not have the independence property (Def. 4.1) then $T_1 \triangleleft T_2 \triangleleft T_1$.

23) *Conjecture*: If T_1, T_2 are countable, unstable and do not satisfy (3) from Th. 4.8 for $\mu = \aleph_1$ then $T_1 \triangleleft T_2 \triangleleft T_1$. Perhaps instead of not (3), it suffices T_1, T_2 has not the strict order p (see Def. 4.2).

Historical remarks

7) This answers affirmatively a question of Keisler [A] p. 40. He proved ([A], 4.20) that if T is minimal, then it has not the f.c.p.

9) This was proved independently by Keisler and the author, for the theory of linear order; hence to every theory with the strict order p.

12) The notice in Notices of the A.M.S. vol. 16 (1969) p. 501 claiming to prove the existence of such D was an error.

15) This answers affirmatively question 4D, Keisler [A] p. 41.

18) This answers negatively question 2E Keisler [A] p. 32; and answers positively question 2C (but the theory is not the one he suggested and is uncountable).

I. Other results

Definition: Let M be a model of T, $A \subset M$, S a set of types over A (in the language L of T), and $T \subset T_1$. S is called (T_1, T)-independent if: for every $S_1 \subset S$ there are models M_1, M_2 of T_1 of arbitrarily high power such that: M_1 is a elementary extension of M_2; the reduct of M_1 to L is an elementary extension of M; and $p \in S$ is realized is M_2 iff $p \in S_1$. (Of course $A \subset |M_2|$).

1) If $|S|^{|A|} + |T_1|^{|A|} < 2^{|S|}$, S is (T_1, T)-independent, then for every $\lambda \geq |T_1| + |S|$, $I(\lambda, T_1, T) \geq 2^{|S|}$ (notice G3).

2) If T is unstable, J is an ordered set with a dense subset J_1, $|J| = \mu$, $|J_1| = \lambda$, then there are A, $S \subset S(A)$, $|A| = \lambda$, $|S| = \mu$ such that for every T_1, S is (T_1, T)-independent.

3) If $\kappa < \kappa(T)$, λ a cardinal, then there are A, $S \subset S(A)$, $|A| = \Sigma_{\mu<\kappa} \lambda^\mu$, $|S| = \lambda^\kappa$, such that for every T_1, S is (T_1, T)-independent (notice G13).

4) If T has the independence p (Def. 4.1) λ a cardinal, then there are A, $S \subset S(A)$, $|A| = \lambda$, $|S| = 2^\lambda$, such that for every T_1, S is (T_1, T)-independent.

5) If T is superstable, $|S(A)| > |A| + |T|$, then there are B, $|B| \leq |A| + |T|$ and $S \subset S(B)$, $|S| = |S(A)|$ such that S is (T, T)-independent.

Remark: A related result is Keisler [C] 5.4.

6) Let T be a superstable theory, $\aleph_\beta = |T|^+$. Let for $\aleph_\alpha > 2^{|T|}$, $I^*(\aleph_\alpha, T)$ be the number of non-isomorphic $|T|^+$-saturated models of T of cardinality $\aleph_\alpha$. Then exactly one of the following holds:

(i) $I^*(\aleph_\alpha, T) = 1$

(ii) $I^*(\aleph_\alpha, T) = |\alpha - \beta|$

(iii) $I^*(\aleph_\alpha, T) = |\alpha - \beta|^\lambda$ for $\alpha \geq \beta + \lambda$, where $\lambda \leq 2^{|T|}$

(iv) $I^*(\aleph_\alpha, T) \geq 2^{|\alpha - \beta|}$ for $\aleph_\alpha > \alpha \geq \beta + |T|^+$

There are equivalent syntactical conditions, and the structure of the models in the first three cases is characterized. For example, if (iv) holds, there is a formula $\varphi(x, \bar{y})$ such that for every $\{\lambda_i : i \in I\}$, $(\lambda_i > 2^{|T|} + |I|)$

T has a model M, and there are sequences $\bar{a}^i$ $i \in I$ from M such that $|\{b \in |M|: M \models \varphi[b, \bar{a}^i]\}| = \lambda_i$.

7) *Conjecture*: We can generalize (6) to stable theories.

8) *Question*: Can we in (6) (iv) omit the condition $\aleph_\alpha > \alpha$? Yes.

9) T has a saturated model of cardinality λ iff $\lambda = \sum_{\mu < \lambda} \lambda^\mu + |D(T)|$ or T is stable in λ [Follows from various results mentioned in B].

§ 1. Notations

We shall use $\alpha, \beta, \gamma, i, j, k, l$ for ordinals; m, n, r for finite ordinals (= natural numbers); $\kappa, \lambda, \chi, \mu$, for infinite cardinals (= initial ordinals); and δ for a limit ordinal. The cardinality of a set A will be $|A|$. A sequence $\bar{s}$ is a function from a ordinal, which will be called the length of the sequence, and denoted by $l(\bar{s})$. Let $\bar{s}(i) = s_i$ be the ith element of the sequence. $\bar{s} \upharpoonright k = \langle \bar{s}(i): i < k \rangle$; and $\operatorname{Rang} \bar{s} = \{\bar{s}(i): i < l(\bar{s})\}$. Let ${}^{\alpha}A = \{\bar{s} : l(\bar{s}) = \alpha, \operatorname{Rang} \bar{s} \subset A\}$, and ${}^{\alpha>}A = \bigcup_{\beta<\alpha} {}^{\beta}A$. Sequences of ordinals will be denoted by η, τ; usually they will be sequences of zeroes and ones. So writing only $l(\eta) = \alpha$, we mean $\eta \in {}^{\alpha}2$. T will be a fixed, first-order, complete theory in a language L. Formulas of L will be denoted by φ, Ψ, θ. $\varphi(\bar{x})$ is a pair $\langle \varphi, \bar{x} \rangle$ where every free variable of φ belongs to $\operatorname{Rang} \bar{x}$. We shall not differentiate strictly between φ and $\varphi(\bar{x})$; and the exact meaning will be clear from the context. Variables will be denoted by x, y, z; and finite sequences of variables - by $\bar{x}, \bar{y}, \bar{z}$. Let $\bar{\kappa}$ be a cardinal greater than the cardinalities of all the models and sets we shall speak about. By Morley and Vaught [D] T has a $\bar{\kappa}$-saturated model $\bar{M}$, and every model of T of cardinality $< \bar{\kappa}$ is isomorphic to an elementary submodel of $\bar{M}$. So for simplicity, saying a model of T we shall mean an elementary submodel of $\bar{M}$ of cardinality $< \bar{\kappa}$. M, N shall denote models of T, $|M|$ - the set of elements of M, and hence $\|M\|$ - the cardinality of M. A, B, C shall denote sets included in some $|M|$. Let a, b, c denote elements of $\bar{M}$; $\bar{a}, \bar{b}, \bar{c}$ finite sequences of such elements. We shall write $\bar{a} \in A$ instead of $\bar{a} \in {}^{\omega>}A$; and $a \in M$ instead of $a \in |M|$. For "$\varphi[\bar{a}]$ is satisfied in M" we shall write $M \models \varphi[\bar{a}]$. But as M is an elementary submodel of $\bar{M}$, $M \models \varphi[\bar{a}]$ iff $\bar{M} \models \varphi[\bar{a}]$; hence we shall omit M; and sometimes say only $\varphi[\bar{a}]$ holds. Note that M is an elementary submodel of N iff $|M| \subset |N|$.

Let Δ denote a set of formulas $\varphi(\bar{x})$ (not φ). We say p is a Δ-m-type over (or on) A if:

(1) its elements are of the form $\varphi(\bar{x}; \bar{a})$ where $a \in A$, $\bar{x} = \langle x_0, \ldots, x_{m-1} \rangle$, and $\varphi(\bar{x}; \bar{y})$ is Ψ or $\neg\Psi$ for some $\Psi(\bar{x}; \bar{y}) \in \Delta$.

(2) p is consistent; that is, for every finite $q \subset p$, $\models (\exists \bar{x}) \bigwedge_{\Psi \in q} \Psi$.

Types will be denoted by p, q. If Δ is the set of all formulas of L, then we omit it; and if $m = 1$ we omit it. If $\Delta = \{\varphi\}$, we shall write φ instead of Δ. p is a complete Δ-m-type over A if it is maximal; that is if $\bar{a} \in A$, $\varphi \in \Delta$ then $\varphi(\bar{x}\,;\bar{a}) \in p$ or $\neg\varphi(\bar{x}\,;\bar{a}) \in p$. Let $S^m_\Delta(A)$ be the set of complete Δ-m-types over A. We say $\bar{c}$ realizes p if $\varphi(\bar{x}\,;\bar{a}) \in p$ implies $\models \varphi[\bar{c}\,;\bar{a}]$. As $\bar{M}$ is $\bar{\kappa}$-saturated, every type is realized by some $\bar{c}$. The type $\bar{c}$ realized over A is $\{\varphi(\bar{x}\,;\bar{a}) : \bar{a} \in A, \models \varphi[\bar{c}\,;\bar{a}]\}$. Let p restricted to A be $p|A = \{\varphi(\bar{x}\,;\bar{a}) \in p : \bar{a} \in A\}$.

Let φ^i be φ if $i = 0$ and $\neg\varphi$ if $i = 1$. Let $\varphi^{\text{if(st.)}}$ where st. is a statement be φ if st. is true and $\neg\varphi$ if st. is false.

Writing $\varphi = \varphi(\bar{x}\,;\bar{y})$ we shall always mean $l(\bar{x}) = m$, and see Rang $\bar{x}$ as a set of variables, and Rang $\bar{y}$ as a set of parameters. (The meaning shall become clear in the usage, as we deal with types $\{\varphi(\bar{x}\,;\bar{a}^k) : k < k_0\}$.) In fact sometimes when we say a formula φ, we mean $\varphi(\bar{x}\,;\bar{y})$ or $\varphi(\bar{x})$, or a formula obtained from φ by a suitable change of variables. Note that $\bar{x}$ is not uniquely fixed - we can add to it dummy variables or change the order of the variables. For simplicity we assume $\bar{x} = \langle x_0, \ldots, x_{m-1} \rangle$, $x_0 = x$.

M is λ-saturated, if every type p on $|M|$, $p \in S(A)$, $|A| < \lambda$ is realized in M. M is λ-compact if every type p on $|M|$, $|p| < \lambda$, is realized in M. Note that for $\lambda > |T|$, those two concepts are the same.

§2. Properties equivalent to unstability

We shall define some properties (A-F) of formula $\varphi(\bar{x};\bar{y})$ and prove their equivalence by a series of lemmas. They are

(A) $\varphi(\bar{x};\bar{y})$ is unstable; i.e., for every λ there is an A, $|S_\varphi^m(A)| > \lambda \geq |A|$.

(B) $\varphi(\bar{x};\bar{y})$ is unstable in at least one λ.

(C) $\varphi(\bar{x};\bar{y})$ has the order property; i.e., there are $\bar{a}^0, \bar{a}^1, \ldots$ such that for every n,

$$\{\neg\varphi(\bar{x};\bar{a}^0), \ldots, \neg\varphi(\bar{x};\bar{a}^n), \varphi(\bar{x};\bar{a}^{n+1}), \varphi(\bar{x};\bar{a}^{n+2}), \ldots\}$$

is consistent.

(D) $\Gamma = \{\varphi(\bar{x}_\eta;\bar{y}_{\eta|n})^{\eta(n)} : \eta \in {}^\omega 2, n < \omega\}$ is consistent (with T).

(E) It is false that:

we can define for every φ-m-type p a Rank, which is a natural number $< n(\varphi, m) < \omega$ such that (1) $p \subset q$ implies Rank $q \leq$ Rank p (2) every φ-m-type has a finite subtype of the same rank (3) For every p and $\varphi(\bar{x}, \bar{a})$.

$$\text{Rank}\,[p \cup \{\varphi(\bar{x}, \bar{a})\}] < \text{Rank}\, p$$

or

$$\text{Rank}\,[p \cup \{\neg\varphi(\bar{x}, \bar{a})\}] < \text{Rank}\, p.$$

We also prove

Theorem 2.13. *The following properties of T are equivalent:*

(1) *T is unstable*

(2) *T is unstable in at least one λ, $\lambda = \lambda^{|T|}$*

(3) *some formula $\varphi(x;\bar{y})$ is unstable*

(4) *some formula $\varphi(\bar{x};\bar{y})$ is unstable*

(5) *there are a formula $\varphi(\bar{x};\bar{y})$ and sequences $\bar{a}^n$, $n < \omega$ of a fixed length such that for $n, r < \omega$*
$\models \varphi[\bar{a}^n, \bar{a}^r] \Leftrightarrow n < r$.

By 5.3 we can add to those properties

(6) *There is an infinite set of sequences of the same length and a formula $\varphi = \varphi(\bar{x}^0, \ldots, \bar{x}^{n-1}, \bar{a})$ such that the formula and its negation are connected over the set.*

(7) *There is an infinite indiscernible sequence (of sequences) which is not an indiscernible set.*

In all the theorems, we can replace φ-m-types by Δ-m-types, and $S^m_\varphi(A)$ by $S^m_\Delta(A)$; for any finite set of formulas Δ. This can be done directly by repeating the proofs, or by using the following formula:

$$\text{if } \Delta = \{\varphi_k(\bar{x}\,;\bar{y}^k) \colon k < n < \omega\}$$

let

$$\Psi(\bar{x}\,;\bar{y}^0, \bar{y}^1, \ldots, \bar{y}^{n-1}, z, z_0, \ldots, z_{n-1}) =$$

$$= \bigwedge_{k<n} [z = z_k \to \varphi(\bar{x}\,;\bar{y}^k)].$$

This is true also for other sections. So we prove some theorems for φ, but use them for any finite Δ.

Definition 2.1.

A) $\varphi(\bar{x}\,;\bar{y})$ is *stable in* λ iff for every A, $|A| \leq \lambda$ implies $|S^m_\varphi(A)| \leq \lambda$.

B) $\varphi(\bar{x}\,;\bar{y})$ is *stable* iff it is stable in at least one cardinal.

C) T is stable in λ if $|A| \leq \lambda$ implies $|S(A)| \leq \lambda$.

D) T is stable if it is stable in at least one cardinal.

Property A. $\varphi(\bar{x}\,;\bar{y})$ is unstable.
Property B. $\varphi(\bar{x}\,;\bar{y})$ is unstable in at least one λ.

Definition 2.2. $\varphi(\bar{x}\,;\bar{y})$ has the order property (or order p) if there are $\bar{a}^0, \ldots, \bar{a}^n, \ldots$ such that for every r, the set

$$\{\varphi(\bar{x}\,;\bar{a}^n)^{\text{if } (n \geq r)} \colon n < \omega\}$$

is consistent.

Property C. $\varphi(\bar{x}\,;\bar{y})$ has the order p.

Definition 2.3. Let

$$\Gamma(\varphi, m, \alpha) = \{\varphi(\bar{x}_\eta\,;\bar{y}_{\eta|k})^{\eta(k)} \colon \eta \in {}^\alpha 2, k < \alpha\} \cup T$$

(since what m is clear, we shall usually omit it.)

Property D. $\Gamma(\varphi, \omega)$ is consistent.

Before we continue to define the other properties we shall prove that A, B, C, D are equivalent, by proving A → B → C → D → A.

It is self-evident that A implies B

Lemma 2.1. *If for some A, $|S_\varphi^m(A)| > |A| \geq \aleph_0$, then $\varphi(\bar{x};\bar{y})$ has the order p. (i.e. property* B *implies property* C*).*

Remark. In Shelah [F] a stronger theorem appears.

Proof. Let $\varphi = \varphi(\bar{x};\bar{y})$, $l(\bar{y}) = n$. Clearly, $|{}^nA| = |A|^n = |A| \geq \aleph_0$. For every $p \in S_\varphi^m(A)$ let $w(p) = \{\bar{a} \in {}^nA : \varphi(\bar{x};\bar{a}) \in p\}$, and $W = \{w(p) : p \in S_\varphi^m(A)\}$. Clearly if $p, q \in S_\varphi^m(A)$ then $p = q$ iff $w(p) = w(q)$. Hence $|W| = |S_\varphi^m(A)| > |A| = |{}^nA| \geq \aleph_0$; and by definition, W is a family of subsets of ${}^nA$. By a theorem of Erdös and Makkai [A] this implies that there are $\bar{a}^0, ..., \bar{a}^r, ... \in {}^nA$; $w(p_0), ..., w(p_r), ... \in W$ such that:

either (1) for every $k, l < \omega$ $\bar{a}^l \in w(p_k)$ iff $l < k$

or (2) for every $k, l < \omega$ $\bar{a}^l \in w(p_k)$ iff $k \leq l$.

By the definition of $w(p_k)$ we can conclude that:

either (1) for every $k < \omega$, $\{\varphi(\bar{x};\bar{a}^l)^{\text{if } (l<k)} : l < \omega\}$ is consistent.

or (2) for every $k < \omega$, $\{\varphi(\bar{x};\bar{a}^l)^{\text{if } (k \leq l)} : l < \omega\}$ is consistent.

In both cases, clearly φ has the order property. [In case (1), we should reverse the order of the first $l(< \omega)$ $\bar{a}^r$'s, and use the compactness theorem.]

Lemma 2.2. *If $\varphi(\bar{x};\bar{y})$ has the order p, then $\Gamma(\varphi, \omega)$ is consistent (i.e. property* C *implies property* D*).*

Proof. Let us define an order on ${}^{\omega\geq}2$: if $\eta|k = \tau|k$, $\eta(k) = 0$, $\tau(k) = 1$ then $\eta < \tau$; if $\eta|k = \tau|k$, $l(\eta) = k$, $\tau(k) = 1$ then $\eta < \tau$; if $\eta|k = \tau|k$, $l(\eta) = k$, $\tau(k) = 0$ then $\tau < \eta$.

By the compactness theorem, as $\varphi(\bar{x};\bar{y})$ has the order property, the set

$$T \cup \{\varphi(\bar{x}_\eta;\bar{y}_\tau)^{\text{if } (\eta<\tau)} : l(\eta) = \omega, l(\tau) < \omega\}$$

is consistent. If $\bar{a}_\tau$ realizes $\bar{y}_\tau$, $\bar{c}_\eta$ realizes $\bar{x}_\eta$, then for every $\eta \in {}^\omega 2$, $p_\eta = \{\varphi(\bar{x};\bar{a}_{\eta|r})^{\eta(r)}: r < \omega\}$ is consistent. Hence $\Gamma(\varphi, \omega)$ is consistent.

Lemma 2.3. *If $\Gamma(\varphi, \omega)$ is consistent, then $\varphi(\bar{x};\bar{y})$ is unstable (i.e. property* D *implies property* A*).*

Proof. Let λ be any infinite cardinal, and we shall prove that φ is unstable in λ.

Let $\mu = \inf\{\mu: 2^\mu > \lambda\}$. As $\Gamma(\varphi, \omega)$ is consistent, clearly also $\Gamma(\varphi, \mu)$ is consistent. So let M be a model of it, let $\bar{a}_\tau$ realize $\bar{y}_\tau$, and $\bar{c}_\eta$ realize $\bar{x}_\eta$. Let $A = \bigcup\{\operatorname{Rang} \bar{a}_\eta : l(\eta) < \mu\}$. Clearly $|A| \leq (\Sigma_{\alpha<\mu} 2^\alpha)\cdot\aleph_0 \leq \mu\cdot\lambda\cdot\aleph_0$ λ (as $\lambda < 2^\lambda$, $\mu \leq \lambda$ by definition). For $\eta \in {}^\mu 2$, let p_η be the φ-m-type that $\bar{c}_\eta$ realizes over A. Clearly if $l(\eta) = l(\tau) = \mu$, $\eta \neq \tau$, and k is the first ordinal such that $\eta(k) \neq \tau(k)$, then

$$\varphi(\bar{x};\bar{a}_{\eta|k})^{\eta(k)} \in p_\eta$$

and

$$\neg\varphi(\bar{x};\bar{a}_{\eta|k})^{\eta(k)} = \varphi(\bar{x};\bar{a}_{\tau|k})^{\tau(k)} \in p_\tau,$$

hence $p_\eta \neq p_\tau$. So

$$|S_\varphi^m(A)| \geq |\{p_\eta : l(\eta) = \mu\}| = |\{\eta : l(\eta) = \mu\}| = 2^\mu > \lambda$$

or

$$|S_\varphi^m(A)| > \lambda \geq |A|.$$

Hence we have proved the equivalence of properties A, B, C, D.

Now we shall define ranks of φ-m-types.

Definition 2.4. For every A, α, and $\varphi(\bar{x};\bar{y})$ we define $S^m_{\varphi,\alpha}(A)$ by induction on α:

(1) $S^m_{\varphi,0}(A)$ is the set of φ-m-types over A (not necessarily complete).

(2) If $S^m_{\varphi,\alpha}(A)$ has been defined then $S^m_{\varphi,\alpha+1}(A)$ will be the set of types $p \in S^m_{\varphi,\alpha}(A)$ such that for every finite $q \subset p$ there are B, $A \subset B$, and $\bar{b} \in B$ such that:

$$q \cup \{\varphi(\bar{x};\bar{b})\} \in S^m_{\varphi,\alpha}(B); \quad q \cup \{\neg\varphi(\bar{x};\bar{b})\} \in S^m_{\varphi,\alpha}(B).$$

(3) $S^m_{\varphi,\delta}(A) = \bigcap_{k<\delta} S^m_{\varphi,k}(A)$.

Lemma 2.4. *If p is a φ-m-type over A, $A \subset B$, then for every α, $p \in S^m_{\varphi,\alpha}(B)$.*

Proof. Immediate, by induction on α.

Definition 2.5. If p is a φ-m-type over A, then $\mathrm{Rank}^m_\varphi p$ will be the greatest ordinal α for which $p \in S^m_{\varphi,\alpha}(A)$. (As $S^m_{\varphi,\delta}(A) = \bigcap_{k<\delta} S^m_{\varphi,k}(A)$, always there exists such α; and by Lemma 2.4 it does not depend on A, but only on p.)

If $p \in S^m_{\varphi,\alpha}(A)$ for every α, then $\mathrm{Rank}^m_\varphi p = \infty$; and we stipulate $\alpha < \infty$ for every ordinal α. To inconsistent set of formulas we give the rank -1.

Remark. When φ and m are clear, we shall omit them and write Rank p.

Property E. There is a φ-m-type whose rank is ∞.
Property F. For every n there is a φ-m-type whose rank is $\geq n$.

We have proved that the properties A-D are equivalent. Now we shall prove D $\to$ E, F $\to$ D, and as E $\to$ F is self-evident we finish the proof of the main theorem of this section. Before proving this we shall prove a lemma on ranks.

Lemma 2.5. (A) *If p, q are φ-m-types, and $p \subset q$ then* Rank $p \geq$ Rank q.

(B) *Every φ-m-type p has a finite subtype q, such that* Rank p = Rank q.

(C) *Every φ-m-type of rank $<\infty$ has no two extensions q_1, q_2 such that* Rank p = Rank q_1 = Rank q_2, *and for some $\bar{a}$, $\varphi(\bar{x}, \bar{a}) \in q_1$, $\neg\varphi(\bar{x}, \bar{a}) \in q_2$.*

(D) *If p is a φ-m-type on A of rank α, $\alpha < \infty$ then it has no more than one extension in $S^m_\varphi(A)$ of rank α.*

Proof. (A) Suppose $p \subset q \in S^m_{\varphi,0}(A)$, and α = Rank $p <$ Rank q. Then $p \in S^m_{\varphi,\alpha}(A)$, $p \notin S^m_{\varphi,\alpha+1}(A)$, but $q \in S^m_{\varphi,\alpha+1}(A)$. Let p_1 be any finite subtype of p; then p_1 is also a finite subtype of q; hence, by definition 2.4, as $q \in S^m_{\varphi,\alpha+1}(A)$, there exists B, $A \subset B$, and $\bar{b} \in B$, such that $p_1 \cup \{\varphi(\bar{x}, \bar{b})\}, p_1 \cup \{\neg\varphi(\bar{x}, \bar{b})\} \in S^m_{\varphi,\alpha}(B)$. As this is true for every finite subtype of p, $p \in S^m_{\varphi,\alpha+1}(A)$, a contradiction.

(B) Suppose $p \in S^m_{\varphi,0}(A)$, Rank $p = \alpha$. Hence $p \in S^m_{\varphi,\alpha}(A)$, but $p \notin S^m_{\varphi,\alpha+1}(A)$. By the definition of $S^m_{\varphi,\alpha+1}(A)$, p has a finite subtype q such that for every $\bar{b}$, B; $A \subset B$, $\bar{b} \in B$, $q \cup \{\varphi(\bar{x}, \bar{b})\} \notin S^m_{\varphi,\alpha}(B)$ or $q \cup \{\neg\varphi(\bar{x}, \bar{b})\} \notin S^m_{\varphi,\alpha}(B)$. Hence, as q is a finite φ-m-type, Rank $q \leq \alpha$. On the other hand by (A) as $q \subset p$, Rank $q \geq$ Rank $p = \alpha$. Hence Rank q = Rank $p = \alpha$. (If Rank $p = \infty$, any finite subtype of p will be O.K.)

(C) Suppose $p, q_1, q_2 \in S^m_{\varphi,\alpha}(A)$, Rank p = Rank q_1 = Rank $q_2 = \alpha$. Then $q_1 \in S^m_{\varphi,\alpha}(A)$, and as $p \cup \{\varphi(\bar{x}, \bar{a})\} \subset q_1$, Rank$(p \cup \{\varphi(\bar{x}, \bar{a})\}) \geq$ Rank $q_1 = \alpha$, [by (A)] hence $p \cup \{\varphi(\bar{x}, \bar{a})\} \in S^m_{\varphi,\alpha}(A)$. Similarly, using q_2, we can show that $p \cup \{\neg\varphi(\bar{x}, \bar{a})\} \in S^m_{\varphi,\alpha}(A)$. By the definition of $S^m_{\varphi,\alpha+1}(A)$, $p \in S^m_{\varphi,\alpha+1}(A)$. A contradiction to Rank $p = \alpha$.

(D) An immediate corollary of (C).

Lemma 2.6. *If* $\Gamma(\varphi, \omega)$ *is consistent, then there is a* φ-m-*type with rank* ∞. (Hence property D implies property E).

Proof. As $\Gamma(\varphi, \omega)$ is consistent, it has a model, and let $\bar{a}_\eta$ realize $\bar{y}_\eta$ for each η, $l(\eta) < \omega$, and $\bar{b}_\eta$ realizes $\bar{x}_\eta$ for each η, $l(\eta) = \omega$.

As $\Gamma(\varphi, \omega) = \{\varphi(\bar{x}_\eta, \bar{y}_{\eta|n})^{\eta(n)} : l(\eta) = \omega, n < \omega\}$. Clearly for each η of length $< \omega$, $p_\eta = \{\varphi(\bar{x}, \bar{a}_{\eta|n})^{\eta(n)} : n < l(\eta)\}$ is a consistent φ-m-type. As the ranks are well ordered, there is among the types $\{p_\eta : l(\eta) < \omega\}$ one with minimal rank, say p_τ. Hence Rank $p_\tau \leq$ Rank $p_{\tau\frown\langle 0\rangle}$ but as $p_\tau \subset p_{\tau\frown\langle 0\rangle}$ also Rank $p_\tau \geq$ Rank $p_{\tau\frown\langle 0\rangle}$. So Rank p_τ = Rank $p_{\tau\frown\langle 0\rangle}$ and similarly Rank p_τ = Rank $p_{\tau\frown\langle 1\rangle}$. As $\varphi(\bar{x}, \bar{a}_\tau) \in p_{\tau\frown\langle 0\rangle}$, $\neg\varphi(\bar{x}, \bar{a}_\tau) \in p_{\tau\frown\langle 1\rangle}$ we get, by Lemma 2.5C a contradiction.

Lemma 2.7. *If p is a φ-m-type, then* Rank $p \geq n$ *iff*

$$\Gamma_{p,n} = \{\varphi(\bar{x}_\eta, \bar{a})^i : \varphi(\bar{x}, \bar{a})^i \in p,\ i \in z,\ l(\eta) = n\} \cup$$

$$\cup\ \{\varphi(\bar{x}_\eta, \bar{y}_{\eta|k})^{\eta(k)} : l(\eta) = n,\ k < n\}$$

is consistent, i.e., realized in $\overline{M}$. ($\Gamma_{p,n}$ depends, in fact, also on φ.)

Proof. Suppose Rank $p \geq n$, hence $p \in S^m_{\varphi,n}(A)$, and we shall prove that $\Gamma_{p,n}$ is consistent. By Lemma 2.5A and the compactness theorem it suffices to prove this for finite p. We shall now define by induction on k,

$k \leq n$, A_k, $\bar{a}_\eta$ for $\eta; l(\eta) < k$, such that for each η, $l(\eta) = k$, $p_\eta = p \cup \{\varphi(\bar{x};\bar{a}_{\eta|l})^{\eta(l)}: l < k\} \in S^m_{\varphi,n-k}(A_k)$.

For $k = 0$, $A_0 = A$, and $p_{\langle\,\rangle} = p$; hence $p_{\langle\,\rangle} \in S^m_{\varphi,n-0}(A_0)$ as $\operatorname{Rank} p \geq n$.

Suppose we have defined for k, and we shall define for $k+1$. Let $l(\eta) = k$, then as $p_\eta \in S^m_{\varphi,n-k}(A_k)$, $|p_\eta| < \aleph_0$, by the definition of $S^m_{\varphi,n-k}(A_k)$ there exists B_η, $A_k \subset B_\eta$ and $\bar{a}_\eta \in B_\eta$ such that $p_\eta \cup \{\varphi(\bar{x}, \bar{a}_\eta)\}, p_\eta \cup \{\neg\varphi(\bar{x}, \bar{a}_\eta)\} \in S^m_{\varphi,n-k-1}(B_\eta)$. Let $p_{\eta\frown\langle 0\rangle} = p_\eta \cup \{\varphi(\bar{x}, a_\eta)\}$, $p_{\eta\frown\langle 1\rangle} = p_\eta \cup \{\neg\varphi(\bar{x}, a_\eta)\}$, and $A_{k+1} = \bigcup \{B_\eta : l(\eta) = k\}$. Clearly the conditions for $k+1$ are satisfied, hence we can finish the definition by induction. So for η, $l(\eta) = n$, $p_\eta \in S^m_{\varphi,0}(A_n)$, hence there is $\bar{c}_\eta$ which realizes p_η. Taking $\bar{a}_\eta$ for $\bar{y}_\eta$ and $\bar{c}_\eta$ for $\bar{x}_\eta$ we see that $\Gamma_{p,n}$ is consistent.

So we have proved one direction of the equivalence. Let us prove the other direction. We assume that $\Gamma_{p,n}$ is consistent, and we should prove that $\operatorname{Rank} p \geq n$. Let M be a model of $\Gamma_{p,n}$ and let $\bar{a}_\eta$ realize $\bar{y}_\eta$, and $\bar{c}_\eta$ realize $\bar{x}_\eta$. Define $p_\eta = p \cup \{\varphi(\bar{x};\bar{a}_{\eta|k})^{\eta(k)}: k < l(\eta)\}$. We shall prove by induction on $k \leq n$ that if $l(\eta) = n - k \leq n$ then $\operatorname{Rank} p_\eta \geq k$. For $k = 0$ this holds as $\bar{c}_\eta$ realizes p_η, so clearly p_η is consistent, and hence belongs to $S^m_{\varphi,0}(|M|)$. Suppose we have proved it for k, $k < n$, and we shall prove it for $k+1$. Let $l(\eta) = n - k - 1$, then, by the induction hypothesis, the ranks of $p_{\eta\frown\langle 0\rangle} = p_\eta \cup \{\varphi(\bar{x};\bar{a}_\eta)\}$ and $p_{\eta\frown\langle 1\rangle} = p_\eta \cup \{\neg\varphi(\bar{x};\bar{a}_\eta)\}$ are $\geq k$; hence by Lemma 2.5C $\operatorname{Rank} p_\eta \geq k+1$. So we proved that $\operatorname{Rank} p_\eta \geq n - l(\eta)$ and hence $\operatorname{Rank} p = \operatorname{Rank} p_{\langle\,\rangle} \geq n$. So we prove the second direction in Lemma 2.7 and thus prove it.

Lemma 2.8. *If there is no finite upper bound of the rank of φ-m-types, then $\Gamma(\varphi, \omega)$ is consistent (i.e. property* F *implies property* D*). Moreover,* $\operatorname{Rank}^m_\varphi p \geq \omega$ *implies* $\operatorname{Rank}^m_\varphi p = \infty$.

Proof. Let p be the empty φ-m-type. As for any n, there is a φ-m-type p_n of rank $\geq n$, and $p \subset p_n$; $\operatorname{Rank} p \geq \operatorname{Rank} p_n \geq n$. So $\operatorname{Rank} p \geq \omega$. Hence by Lemma 2.7, $\Gamma_{p,n}$ is consistent for any n. It is clear by definition that as p is empty, $\Gamma_{p,n} = \Gamma(\varphi, n)$. Hence $\Gamma(\varphi, n)$ is consistent for every n. Hence $\Gamma(\varphi, \omega)$ is consistent. The proof of the second phrase is similar. *So we finish the proof of the main theorem.*

Theorem 2.9. *Let $\varphi(\bar{x};\bar{y})$ be a formula. Then the following statements about $\varphi(\bar{x};\bar{y})$ are equivalent.*

(A) *$\varphi(\bar{x};\bar{y})$ is unstable in every infinite cardinal.*

(B) *$\varphi(\bar{x};\bar{y})$ is unstable in at least one infinite cardinal.*

(C) *$\varphi(\bar{x};\bar{y})$ has the order property.*

(D) *$T \cup \Gamma(\varphi, m, \omega)$ is consistent.*

(E) *Not every φ-m-type has a rank $<\infty$.*

(F) *There does not exist a finite upper bound on the ranks of the φ-m-types.*

Proof. This follows from Lemmas 2.1-2.8.

Now we shall apply this theorem to theories.

Lemma 2.10. *The following statements are equivalent*

(A) *T is stable in λ*

(B) *for every m, $|A| \leq \lambda$, implies $|S^m(A)| \leq \lambda$.*

Proof. As (A) is a particular case of (B) (for $m = 1$) clearly (B) implies (A).

So suppose (A) holds, and we shall prove (B). Let $m < \omega$ and $|A| \leq \lambda$. We define by induction A_n such that: $|A_n| \leq \lambda$, and $A_n \subset A_{n+1}$. Let $A_0 = A$. If A_n is defined, as $|S(A_n)| \leq \lambda$, there is A_{n+1}, $A_n \subset A_{n+1}$, $|A_{n+1}| = \lambda$, such that every type in $S(A_n)$ is realized in A_{n+1}. It is easy to show that every type in $S^m(A)$ is realized in A_m, hence $|S^m(A)| \leq |A_m|^m = \lambda^m = \lambda$. So we prove the theorem

Lemma 2.11. *If $\varphi(\bar{x};\bar{y})$ is unstable in λ, then T is unstable in λ. Moreover $|S^m(A)| \geq |S^m_\varphi(A)|$ always.*

Proof. As $\varphi(\bar{x};\bar{y})$ is unstable in λ, there exists A such that $|S^m_\varphi(A)| > \lambda \geq |A|$. Every $p \in S^m_\varphi(A)$ has an extension p^* in $S^m(A)$. Clearly $p \neq q; p, q \in S^m_\varphi(A)$ implies $p^* \neq q^*$. Hence $|S^m(A)| \geq |\{p^* : p \in S^m_\varphi(A)\}| = |S^m_\varphi(A)| > \lambda \geq |A|$. By the previous lemma this implies that T is unstable in λ.

Definition 2.6. If $\varphi = \varphi(\bar{x};\bar{y})$, p an m-type, then
$p|\varphi = \{\Psi(\bar{x};\bar{a}): \Psi(\bar{x};\bar{a}) \in p,\ \Psi = \varphi \text{ or } \Psi = \neg\varphi\}$.

Lemma 2.12. *If T is unstable in λ, $\lambda = \lambda^{|T|}$, then there are φ, A such that $|S_\varphi(A)| > |A| + \aleph_0$. Moreover, $|S(A)| \leq \Pi_\varphi\ |S_\varphi(A)|$.*

Proof. As T is unstable in λ, there exists an A, $|S(A)| > \lambda \geq |A|$. If for some φ, $|S_\varphi(A)| > \lambda$, we prove the lemma. So suppose that for every φ, $|S_\varphi(A)| \leq \lambda$. Now if $p, q \in S(A)$ $p \neq q$ then for some φ, $p|\varphi \neq q|\varphi$, hence

$$|S(A)| = |\{p : p \in S(A)\}| \leq |\{\langle p|\varphi : \varphi \in \mathrm{L}\rangle : p \in S(A)\}|$$
$$\leq \Pi_\varphi\ |S_\varphi(A)| \leq \Pi_\varphi \lambda = \lambda^{|T|} = \lambda,$$

a contradiction. So we have proved the lemma.

Theorem 2.13. *The following properties of T are equivalent:*
(1) *T is unstable.*
(2) *T is unstable in a cardinal $\lambda = \lambda^{|T|}$.*
(3) *there is an unstable formula $\varphi(x;\bar{y})$.*
(4) *there is an unstable formula $\varphi(\bar{x};\bar{y})$.*
(5) *there is a formula $\Psi(\bar{y}^1, \bar{y}^2)$, $l(\bar{y}^1) = l(\bar{y}^2) = n$, and sequences $\bar{a}^r$ $r < \omega$ of length n such that for $k, l < \omega$*

$$\models \Psi[\bar{a}^k, \bar{a}^l] \text{ iff } k < l.$$

Proof. By the definition of unstability, (1) implies (2). By Lemma 2.12 (2) implies (3). It is self-evident that (3) implies (4). If $\varphi(\bar{x};\bar{y})$ is unstable, then it is unstable in every λ, and hence by Lemma 2.11, T is unstable in every λ. Hence (4) implies (1). So (1) $\to$ (2) $\to$ (3) $\to$ (4) $\to$ (1).

Now suppose (3) holds. Then by Theorem 2.9, $\varphi(x;\bar{y})$ has the order property. Hence there are $\bar{c}^n$ such that for every $k < \omega$, $p_k = \{\varphi(x;\bar{c}^n)^{\text{if } (n \geq k)} : n < \omega\}$ is consistent. Let b^k realize p_k. Let $\bar{a}^n = \langle b^n\rangle \frown \bar{c}^n$, then clearly $\models \varphi[b^k, \bar{c}^n]$ iff $n \geq k$. By adding dummy variables to φ, we get a formula

$$\Psi = \Psi(x^1, \bar{y}^1; x^2, \bar{y}^2) = \varphi(x^1, \bar{y}^2) \wedge x^1 \neq x^2$$

and clearly $\models \Psi[\bar{a}^k;\bar{a}^n]$ iff $n > k$. So (5) holds.

Suppose (5) holds. Define $\bar{c}^n = \bar{a}^{2n+1}$. Then clearly for every k, $\{\Psi(\bar{y};\bar{c}^n)^{\text{if }(n\geq k)}: n < \omega\}$ is consistent as it is realized by $\bar{a}^{2k}$. Hence Ψ has the order p, and so by Theorem 2.9 it is unstable. So (4) holds.

As we have shown (1) → (2) → (3) → (4) → (1), (3) → (5) → (4), clearly the theorem holds.

§3. Properties of stable formulas and the finite cover property

We say that $\varphi = \varphi(\bar{x};\bar{y})$ has the finite cover property (f.c.p.) if for every n there is a set of $\geq n$ formulas $\varphi(\bar{x};\bar{a})$ which is inconsistent, but every subset of it is consistent. We prove that if some $\varphi(\bar{x};\bar{y})$ has the f.c.p., then some $\varphi(x;\bar{y})$ has the f.c.p. We generalize Keisler [A] Th. 5.1 to: unstability implies the f.c.p. We also show that if T has not the f.c.p., then for every $\varphi(\bar{x};\bar{y})$ there is n_0, such that every φ-m-type has a subtype q of the same rank, $|q| < n_0$. We also prove that if T is stable, every type $p \in S^m(A)$ is definable within A, that is: for every $\varphi(\bar{x};\bar{y})$ there is $\Psi(\bar{y};\bar{z})$ and $\bar{c} \in A$ such that $\varphi(\bar{x};\bar{a}) \in p$ iff $\bar{a} \in A$, $\models \Psi[\bar{a};\bar{c}]$. Moreover, if $|A| \geq 2$, the choice of Ψ depend only on φ. There are some other results.

Definition 3.1. (A) The φ-m-type p is $\Psi(\bar{y};\bar{c})$-*defined* if

$$\varphi(\bar{x};\bar{a}) \in p \Rightarrow \models \Psi[\bar{a};\bar{c}]$$

and

$$\neg\varphi(\bar{x};\bar{a}) \in p \Rightarrow \models \neg\Psi[\bar{a};\bar{c}] .$$

(B) The φ-m-type p is $\Psi(\bar{y};\bar{z})$-A-*definable* if there exists $\bar{c} \in A$ such that p is $\Psi(\bar{y};\bar{c})$-defined.

Theorem 3.1. (A) *Suppose $\varphi(\bar{x};\bar{y})$ is stable. Then there is a formula $\Psi(\bar{y};\bar{z})$ such that:*

every $p \in S^m_\varphi(A)$, ($|A| \geq 2$) is Ψ-A-definable.

Also every $p \in S^m_\varphi(A)$, is Ψ_1-A-definable for some Ψ_1; and we can demand Rank $p < \omega$, *instead $\varphi(\bar{x},\bar{y})$ is stable.*

(B) *For every $\varphi(\bar{x};\bar{y})$, r, n and $\eta \in {}^n2$, there is a formula Ψ^r_η such that: for every $\bar{a}^0, \ldots, \bar{a}^{n-1}$,*

$\models \Psi^r_\eta[\bar{a}^0, \ldots, \bar{a}^{n-1}]$ *iff* Rank$\{\varphi(\bar{x};\bar{a}^l)^{\eta(l)} : l < n\} \geq r$.

Remark. Usually, using 3.1A we shall ignore the restriction $|A| \geq 2$. We also prove, in fact, that for every φ, m there is a finite Δ, such that every $p \in S^m_\varphi(A)$ is Ψ-A-definable for some $\Psi \in \Delta$, and by 3.1A we can choose Δ, $|\Delta| = 2$.

Proof. (A) First let A be any set, $p \in S_\varphi^m(A)$ be any type; and we shall prove it is Ψ-A-defined for some Ψ. By Lemma 2.5B p has a finite subtype q such that $\mathrm{Rank}\, p = \mathrm{Rank}\, q = r$, and let $q = \{\varphi(\bar{x}; \bar{a}^l)^{\eta(l)} : l < n\}$. For every $\bar{a} \in A$, let $q(\bar{a}) = q \cup \{\varphi(\bar{x}; \bar{a})\}$. Clearly, if $\varphi(\bar{x}; \bar{a}) \in p$ then $q(\bar{a}) \subset p$, hence $\mathrm{Rank}\, q(\bar{a}) \geq \mathrm{Rank}\, p = r$, but as $q \subset q(\bar{a})$, $r = \mathrm{Rank}\, q \geq \mathrm{Rank}\, q(\bar{a})$; hence $\mathrm{Rank}\, q(\bar{a}) = r$ (we use Lemma 2.5A). Now if $\neg\varphi(\bar{x}; \bar{a}) \in p$, then similarly $\mathrm{Rank}\,[q \cup \{\neg\varphi(\bar{x}; \bar{a})\}] \leq \mathrm{Rank}\, p = r$, hence by Lemma 2.5C, $\mathrm{Rank}\, q(\bar{a}) = \mathrm{Rank}\,[q \cup \{\varphi(\bar{x}; \bar{a})\}] < r$.

So for any $\bar{a} \in A$ $\varphi(\bar{x}; \bar{a}) \in p$ iff $\mathrm{Rank}\,[q(\bar{a})] \geq r$. By Lemma 2.7 $\mathrm{Rank}\,[q(\bar{a})] \geq r$ iff $\Gamma_{q(\bar{a}),r}$ is consistent. Let $\theta(\bar{x}; \bar{c}) = \theta(\bar{x}; \bar{a}^0, ..., \bar{a}^{n-1}) = \wedge_{l<n} \varphi(\bar{x}, \bar{a}^l)^{\eta(l)}$. Clearly $\Gamma_{q(\bar{a}),r}$ is consistent iff $\models \Psi[\bar{a}; \bar{c}]$, where

$$\Psi[\bar{a}; \bar{c}] = (\exists \dots \bar{y}_\tau \dots)_{l(\tau)<r} \bigwedge_{l(\eta)=r} (\exists \bar{x})\, [\theta(\bar{x}; \bar{c}) \wedge$$

$$\varphi(\bar{x}; \bar{a}) \wedge \bigwedge_{l<r} \varphi(\bar{x}; \bar{y}_{\eta|l})^{\eta(l)}]$$

Hence, for $\bar{a} \in A$, $\varphi(\bar{x}; \bar{a}) \in p$ iff $\models \Psi[\bar{a}; \bar{c}]$, where $\bar{c} \in A$. So we should prove only that the choice of Ψ depends only on φ, and not on A. In fact, it suffices to prove that there is a finite Δ, such that every $p \in S_\varphi^m(A)$ is Ψ-A-definable for some $\Psi \in \Delta$. For if $\Delta = \{\Psi^k(\bar{y}; \bar{z}^k) : k < n\}$ then

$$\Psi(\bar{y}; \bar{z}) = \Psi(\bar{y}; \bar{z}^0, ..., \bar{z}^{n-1}, z, z_0, ..., z_{n-1})$$

$$= \bigwedge_{l<n} [z \neq z_l \rightarrow \Psi^l(\bar{y}; \bar{z}^l)]$$

is the required formula.

Suppose there is no such finite Δ; we get a contradiction. Let P be a new one-place predicate and $b_0, ..., b_{m-1}$ new individual constants. For every Δ let

$$T_\Delta = T \cup \{\neg(\exists z_0, z_1, ...)\, [\bigwedge_l P(z_l) \wedge$$

$$(\forall y_0, y_1, ...)\, [\bigwedge_k P(y_k) \rightarrow$$

$$\varphi(\bar{b}, y_0, ...) \equiv \Psi(y_0, ...; z_0, ...)]\,] : \Psi \in \Delta\}$$

where $b = \langle b_0, ..., b_{m-1} \rangle$.

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Clearly if $p \in S^m_\varphi(A)$, for no $\Psi \in \Delta$ p is Ψ-A-definable, $A \subset M$, M is a model of T, then $(M, A, b_0, ..., b_{m-1})$ is a model of T_Δ where $\langle b_0, ..., b_{m-1}\rangle$ realizes p in M. Hence for every finite Δ, T_Δ is consistent. Let Δ_0 be the set of all formulas of L. Clearly T_{Δ_0} is consistent; and let $(M, P^M, b_0, ..., b_{m-1})$ be a model of it. Let $A = P^M$, and p be the φ-m-type $\langle b_0, ..., b_{m-1}\rangle$ realizes over A. Clearly for no Ψ p is Ψ-A-definable. This contradicts what we first proved. So 3.1A is proved.

(B) It is immediate by Lemma 2.7; and in fact we prove it in the proof of (A).

Corollary 3.2. *Suppose T is stable, $p \in S^m(A)$, and for every φ,* $\mathrm{Rank}(p|\varphi) = \mathrm{Rank}[(p|\varphi)|B_\varphi]$, $|B_\varphi| \geq 2$. *Then there are Ψ_φ, $\bar{c}_\varphi \in B_\varphi$ for every φ, such that:*

$$\bar{a} \in A, \ \vDash \Psi_\varphi[\bar{a}; \bar{c}_\varphi] \ \textit{ iff } \ \varphi(\bar{x}; \bar{a}) \in p$$

Proof. Immediate from the proof of the previous theorem.

Theorem 3.3. *If $p \in S^m_\varphi(|M|)$, $|M| \subset A$, then p can be extended to an equi-rank type in $S^m_\varphi(A)$.*

Proof. We shall prove it only for stable, as this is the only case we need.

By Theorem 3.1 there is a formula $\Psi(\bar{y}; \bar{z})$ and a sequence $\bar{c} \in |M|$ such that

$$\varphi(\bar{x}; \bar{a}) \in p \ \text{ iff } \ \bar{a} \in |M|, \vDash \Psi[\bar{a}; \bar{c}] \ .$$

Let us define $q = \{\varphi(\bar{x}; \bar{a})^{\mathrm{if}(\Psi[\bar{a};\bar{c}])} : \bar{a} \in A\}$. Clearly $p \subset q$ hence $\mathrm{Rank}\, q \leq \mathrm{Rank}\, p = n$. Suppose $\mathrm{Rank}\, q < n$, and we get a contradiction, hence prove the theorem. (The case q is not consistent, can be considered as a particular case; and anyway the proof is the same.)

As $\mathrm{Rank}\, q < n$, there is a finite $q_1 \subset q$, $\mathrm{Rank}\, q_1 < n$, and let

$$q_1 = \{\varphi(\bar{x}; \bar{a}^l)^{\eta(l)} : l \leq r\}$$

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hence by 3.1B, $\models \neg \Psi^n_\eta [\bar{a}^0, ..., \bar{a}^r]$. By the definition of q, for every $l \leq r$, $\models \Psi [\bar{a}^l; \bar{c}]^{\eta(l)}$. Hence

$$\models (\exists \bar{y}^0, ..., \bar{y}^r) [\neg \Psi^n_\eta (\bar{y}^0, ..., \bar{y}^r) \wedge \bigwedge_{l \leq r} \Psi(\bar{y}^l; \bar{c})^{\eta(l)}].$$

As $\bar{c} \in M$, there are $\bar{b}^0, ..., \bar{b}^r \in |M|$ such that

$$\models \neg \Psi^n_\eta [\bar{b}^0, ..., \bar{b}^r] \wedge \bigwedge_{l \leq r} \Psi [\bar{b}^l; \bar{c}]^{\eta(l)}.$$

By the definition of $\bar{c}$ and Ψ, for every $l \leq r$, $\varphi(\bar{x}; \bar{b}^l)^{\eta(l)} \in p$, and let

$$p_1 = \{\varphi(\bar{x}; \bar{b}^l)^{\eta(l)} : l \leq r\}.$$

So $p_1 \subset p$, and hence $n = \text{Rank}\, p \leq \text{Rank}\, p_1$. But by the definition of Ψ^n_η, and of $\bar{b}^0, ..., \bar{b}^r$, $\text{Rank}\, p_1 < n$; a contradiction.

Theorem 3.4. *If* $p \in S^m(|M|)$, $|M| \subset A$, *then there is* $q \in S^m(A)$, *such that for every* $\varphi(\bar{x}; \bar{y})$, $\text{Rank}^m_\varphi(p|\varphi) = \text{Rank}^m_\varphi(q|\varphi)$, *and* $p \subset q$.

Proof. The same as the proof of the previous theorem.

Theorem 3.5. *Suppose* $p \in S^m_\varphi(A)$, $\text{Rank}\, p < \omega$, *and there is no finite* $p_0 \subset p$ *such that p is the only extension of* p_0 *in* $S^m_\varphi(A)$. *Then*

(A) *There is a finite* $q \subset p$ *and* $r < \text{Rank}\, p$ *such that:*

(1) *q has infinitely many extensions of rank r in* $S^m_\varphi(A)$.

(2) *if* $\Psi \in p$ *then for only finitely many extensions* q_1 *of q of rank r in* $S^m_\varphi(A)$, $\neg \Psi \in q_1$.

(B) *There are types* $p^n \in S^m_\varphi(A)$, $\text{Rank}\, p^n = r < \text{Rank}\, p$ *such that* $\Psi \in p$ *implies* $|\{n : \neg\Psi \in p^n\}| < \aleph_0$.

Remark. Theorem 5.14 has a stronger hypothesis and nicer conclusion than 3.5B.

Proof. Clearly (A) implies (B), so it suffices to prove (A).

By Lemma 2.5B there is a finite $p_0 \subset p$. $n_0 = \text{Rank}\, p_0 = \text{Rank}\, p$. Clearly if $p_0 \subset q \in S^n_{\varphi,0}(A)$ then $\text{Rank}\, q \leq n_0$, and if in addition $q \neq p$,

$q \in S_\varphi^m(A)$ then Rank $q < n_0$ (by Lemma 2.5C). So $r(q) < n_0$ for $p_0 \subset q \subset p$, where

$$r(q) = \max\{\text{Rank}\, q_1 : q \subset q_1 \in S_\varphi^m(A), q_1 \neq p\}.$$

(As Rank q_1 can be one of n_0 values only, this maximum exists.) Now let

$$r_0 = \min\{r(q) : p_0 \subset q \subset p, |q| < \aleph_0\}.$$

Clearly the minimum exists, and $r_0 < n_0$. So let q_0 be such that $r(q_0) = r_0$, $p_0 \subset q_0 \subset p$, $|q_0| < \aleph_0$. By the definition of $r(q_0)$, q_0 has at least one extension $q_1 \in S_\varphi^m(A)$, $q_1 \neq p$ of rank r_0. We shall now prove that q_0 has infinitely many such extensions. Otherwise suppose $q^1, ..., q^r$ are all such extensions. For each k, $1 \leq k \leq r$ as $q^k \neq p$, q^k, $p \in S_\varphi^m(A)$, there is $\varphi(\bar{x}; \bar{a}^k)^{\eta(k)} \in p$ such that $\neg\varphi(\bar{x}; \bar{a}^k)^{\eta(k)} \in q^k$. Let

$$q_1 = q_0 \cup \{\varphi(\bar{x}; \bar{a}^k)^{\eta(k)} : 1 \leq k \leq r\}.$$

Clearly $|q_1| < \aleph_0$, $p_0 \subset q_1 \subset p$. Also as

$$\{q_2 : q_1 \subset q_2 \in S_\varphi^m(A), q_2 \neq p\} \subset \{q_2 : q_0 \subset q_2 \in S_\varphi^m(A),$$
$$q_2 \neq p\}$$

it is clear that $r(q_1) \leq r(q_0) = r_0$. But by definition q_1 has no extension in $S_\varphi^m(A)$ of rank $r_0 = r(q_0)$, as it contradicts $q^1, ..., q^r$; so $r(q_1) < r(q_0) = r_0$. This contradicts the definition of r_0.

We can conclude that

$$P = \{p^* \in S_\varphi^m(A) : q_0 \subset p^*, \text{Rank}\, p^* = r_0\}$$

is infinite. (The condition $p^* \neq p$ is superfluous, as Rank $p = n_0 > r_0 =$ Rank p^*.)

We shall prove now that q_0, r_0 satisfy the conditions mentioned in Theorem 3.5A, and so prove the theorem. Condition (1) (that P is infinite) has already been proved. So we should prove only that for every

$\Psi \in p$, $\{p^*: p^* \in P, \neg \Psi \in p^*\}$ is finite. Suppose it is not true, then there are different p^n, $n < \omega$, such that $\neg \Psi \in p^n$. For simplicity let $\Psi = \varphi(\bar{x}; \bar{a})$ for some $\bar{a} \in A$ (and not $\neg \varphi(\bar{x}; \bar{a})$). Let D be a (non-principal) ultrafilter over ω. Define

$$q = \{\theta : \{n : \theta \in p^n\} \in D\}.$$

Clearly $\neg \varphi(\bar{x}; \bar{a}) \in q$, and hence $q \neq p$. Moreover, for every $\bar{b} \in A$, $\varphi(\bar{x}; \bar{b}) \in q$ or $\neg \varphi(\bar{x}; \bar{b}) \in q$.

Now we shall prove that $q \in S^m_{\varphi, r_0}(A)$ (and hence is also consistent). Suppose this is not true, then by Lemma 2.5B (or the compactness theorem if q is not consistent) q has a finite subset which does not belongs to $S^m_{\varphi, r_0}(A)$. Let this subset be $\{\Psi_0, ..., \Psi_l\}$ where $l < \omega$.

By the definition of q, for every $k \leq l$, $\{n : \Psi_k \in p^n\} \in D$. Hence, by the definition of the ultrafilter

$$\{n : \Psi_0 \in p^n, ..., \Psi_l \in p^n\} = \bigcap_{k \leq l} \{n : \Psi_k \in p^n\} \in D.$$

As the empty set $\notin D$, there exists p^n such that $\{\Psi_0, ..., \Psi_l\} \subset p^n$, and hence by Lemma 2.5A, Rank$\{\Psi_0, ..., \Psi_l\} \geq$ Rank $p^n = r_0$, and so $\{\Psi_0, ..., \Psi_l\} \in S^m_{\varphi, r_0}(A)$, a contradiction. Hence $q \in S^m_{\varphi, r_0}(A)$, or in other words Rank $q \geq r_0$. As $q \in S^m_\varphi(A)$, $q \neq p$, $q_0 \subset q$, clearly Rank $q \leq r_0$ [by the definition of $r_0 = r(q_0)$]. So Rank $q = r_0$. By Lemma 2.5B q has a finite subtype q^0 of rank r_0. As before $\{n : q^0 \subset p^n\} \in D$. Hence q^0 has in $S^m_\varphi(A)$ infinitely many extensions of rank r_0, a contradiction to 2.5D. So we prove Theorem 3.5.

Corollary 3.6. *If $p \in S^m_\varphi(A)$, then there are $n \leq$* Rank p *and types p_η, $\eta \in {}^{n\geq}\omega$ such that*

(1) $p_{\langle\rangle} = p$, and every $p_\eta \in S^m_\varphi(A)$.

(2) if $l(\eta) < n$, $\Psi \in p_\eta$, then $\{k < \omega : \neg \Psi \in p_{\eta \frown \langle k \rangle}\}$ is finite.

(3) if $l(\eta) = n$, then p_η has a finite subtype q_η, such that p_η is the only extension of q_η in $S^m_\varphi(A)$.

Proof. By iterating Theorem 3.5.

Definition 3.2. (A) $\varphi(\bar{x};\bar{y})$ has the *finite cover property* if for arbitrarily large natural numbers n there are $\bar{a}^0, \ldots, \bar{a}^{n-1}$ such that:

$$\models \neg(\exists \bar{x}) \bigwedge_{k<n} \varphi(\bar{x}, \bar{a}^k)$$

but for every $l < n$

$$\models (\exists \bar{x}) \bigwedge_{\substack{k<n\\ k\neq l}} \varphi(\bar{x};\bar{a}^k).$$

(B) T has the f.c.p. if there exists a formula $\varphi(x;\bar{y})$ which has the f.c.p.

Lemma 3.7. *The formula $\varphi(\bar{x};\bar{y})$ does not have the f.c.p. iff there is a natural number n such that: Γ is a set of formulas of the form $\varphi(\bar{x};\bar{a})$, and every subset of Γ of cardinality $< n$ is consistent, then Γ is consistent.*

Proof. Immediate, by the definition.

Theorem 3.8. (A) *If T has not the f.c.p. then T is stable. (In other words, every unstable theory has the f.c.p.)*

(B) *If $\varphi(\bar{x};\bar{y})$ is unstable then the formula*

$$\Psi(\bar{x};\bar{z}) = \Psi(\bar{x};\bar{y}^1, \bar{y}^2, \bar{y}^3, \bar{y}^4) =$$

$$= [\varphi(\bar{x};\bar{y}^1) \equiv \neg\varphi(\bar{x};\bar{y}^2)] \wedge [\varphi(\bar{x};\bar{y}^3) \equiv \varphi(\bar{x};\bar{y}^4)]$$

has the f.c.p.

Remark. This strengthens Theorem 5.1, Keisler [A], p. 42; and simplifies the proof.

Proof. (A) By Theorem 2.13, as T is unstable, some $\varphi(x;\bar{y})$ is unstable, hence (A) follows from (B).

(B) As $\varphi(\bar{x};\bar{y})$ is unstable, by Theorem 2.9 it has the order p. So by the definition (2.2) there are $\bar{a}^0, \ldots, \bar{a}^l, \ldots l < \omega$ such that for every $k < \omega$

$$p_k = \{\varphi(\bar{x};\bar{a}^l)^{\text{if } (k \leq l)} : l < \omega\}$$

is consistent.

Let n be any natural number. For any $k < n$ define $\bar{c}^k = \bar{a}^0 \frown \bar{a}^n \frown \bar{a}^k \frown \bar{a}^{k+1}$. We claim that $q = \{\Psi(\bar{x}; \bar{c}^k) : k < n\}$ is inconsistent, but for every $l < n$, $q_l = \{\Psi(\bar{x}; \bar{c}^k) : k < n, k \neq l\}$ is consistent. As n is arbitrary, by Lemma 3.7, clearly this will prove the theorem.

Let us first prove that q is inconsistent. Suppose $\bar{b}$ realizes q. Then as $\models \Psi(\bar{b}; \bar{c}^0)$, also $\models \varphi[\bar{b}; \bar{a}^0] \equiv \neg\varphi[\bar{b}; \bar{a}^n]$. On the other hand for every $k < n$, as $\models \Psi[\bar{b}; \bar{c}^k]$; clearly $\models \varphi[\bar{b}; \bar{a}^k] \equiv \varphi[\bar{b}; \bar{a}^{k+1}]$.

Hence

$$\models \varphi[\bar{b}; \bar{a}^0] \iff \models \varphi[\bar{b}; \bar{a}^1] \iff \models \varphi[\bar{b}; \bar{a}^2] \iff \ldots \iff \models \varphi[\bar{b}; \bar{a}^n]$$

$$\iff \models \neg\varphi[\bar{b}; \bar{a}^0]$$

a contradiction. Therefore q is inconsistent.

Now if $k < n$, then clearly a sequence realizing p_{k+1} realizes q_k; hence q_k is consistent. So we prove the theorem.

Theorem 3.9. (A) *There is a stable theory with the f.c.p.; and there is a stable theory without the f.c.p.*

(B) *There are stable theories which are*

(1) *superstable and without the f.c.p.; which are also stable in $\aleph_0$.*

(2) *superstable with the f.c.p.; which are also stable in $\aleph_0$.*

(3) *unsuperstable without the f.c.p.*

(4) *unsuperstable, with the f.c.p.; but stable.*

Remark. (A) was proved in Keisler [A], p. 44.

Proof. Clearly (B) implies (A), so we prove (B) only.

We shall describe the examples but shall not prove their properties.

(1) The theory of a model whose only relation is the equality.

(2) Let M be a model with the equality relation, and an equivalence relation; such that for every n there is an equivalence class of cardinality n. Clearly its theory satisfies our demands. (This is the example of Keisler.)

(3) Let M be a model such that $|M| = {}^{\omega}\omega$; the relations of M are the equality and for every n the equivalence relation E_n defined as

$$E_n = \{\langle \eta, \tau \rangle : \eta, \tau \in {}^{\omega}\omega, \eta | n = \tau | n\}.$$

The theory of M is the required theory.

(4) By combining the two previous examples, we can easily construct such a theory.

Lemma 3.10. (A) *If T has not the f.c.p., then every formula $\varphi(\bar{x};\bar{y})$ has not the f.c.p.*

(B) *If T has not the f.c.p., for every finite Δ, m, there is $r = r(\Delta, m) < \omega$ such that: If Γ is a set of formulas $\varphi(\bar{x};\bar{a})$, $l(\bar{x}) = m$, $\varphi(\bar{x};\bar{y}) \in \Delta$, and every subset of Γ of cardinality $< r$ is consistent then Γ is consistent.*

Proof. (A) We shall prove it by induction on $m = l(\bar{x})$. For $m = 1$ this is self-evident. Hence assume we have proved it for m and we shall prove it for $m + 1$. Suppose it is not true. Then there is $\varphi = \varphi(\bar{x};\bar{y}) = \varphi(x_0, ..., x_m;\bar{y})$ and for every $n < \omega$ a set Γ_n of formulas of the form $\varphi(\bar{x};\bar{a})$ such that: Γ_n is inconsistent, but every subset of Γ_n of cardinality $< n$ is consistent. By Lemma 3.7 there is $r < \omega$ such that: *if* Γ is a set of formulas of the form $\varphi(x_0, c_1, ..., c_m, \bar{a})$, and every subset of Γ of cardinality $\leq r$ is consistent; *then* Γ is consistent. Let

$$\Psi(x_1, ..., x_m;\bar{y}^0, ..., \bar{y}^{r-1}) = (\exists x_0)[\bigwedge_{k<r} \varphi(x_0, x_1, ..., x_m;\bar{y}^k)]$$

and let for every n

$$\Gamma_n^* = \{\Psi(x_1, ..., x_m;\bar{a}^0, ..., \bar{a}^{r-1}) : \text{for}$$

$$k,\ 0 \leq k < r;\ \varphi(\bar{x};\bar{a}^k) \in \Gamma_n\}.$$

Clearly every subset of Γ_n^* of cardinality $< n/r$ is consistent. Hence by the induction hypothesis, for some sufficiently large n, Γ_n^* is consistent. Hence there are $c_1, ..., c_m$ which realizes Γ_n^*. Let

$$\Gamma = \{\varphi(x_0, c_1, ..., c_m, \bar{a}) : \varphi(x_0, ..., x_m, \bar{a}) \in \Gamma_n\}.$$

Clearly every subset of Γ of cardinality $\leq r$ is consistent; and so by the definition of r, Γ is consistent; hence there is c_0 which realizes it. So $\langle c_0, c_1, ..., c_n\rangle$ realizes Γ_n; a contradiction. So the theorem is proved.

(B) The proof is a variation of (A).

Lemma 3.11. *If T has not the f.c.p. then for every formula $\varphi(\bar{x};\bar{y})$ there is a natural number n such that:*

if p is a φ-m-type, then there is $q \subset p$, $|q| < n$ such that $\mathrm{Rank}_{\varphi}^{m} q = \mathrm{Rank}_{\varphi}^{m} p$.

Proof. As there exists a natural number n_0 such that every φ-m-type has rank $\le n_0$; it is clearly sufficient to prove that: for every $k \le n_0$ there is $n(k) < \omega$ such that φ-m-type of rank k, has a subtype of cardinality $\le n(k)$ of the same rank.

Define for $i \in z$

$$\Psi^i(\ldots, \bar{x}_\eta, \ldots, \bar{y}, \ldots, \bar{z}_\tau, \ldots)_{\substack{\eta \in {}^k 2 \\ \tau \in {}^{k>}2}} =$$

$$= \bigwedge_{\eta \in {}^k 2} \varphi(\bar{x}_\eta, \bar{y})^i \wedge \bigwedge_{\substack{\eta \in {}^k 2 \\ l < k}} \varphi(\bar{x}_\eta, \bar{z}_{\eta | l})^{\eta(l)} .$$

For simplicity let $\Psi^i = \Psi^i(\bar{x}^*, \bar{y}, \bar{z})$. By Lemma 2.7 it is clear that for every $q \subset p$, $\mathrm{Rank}_{\varphi}^{m} q = k$ iff $\{\Psi^i(\bar{x}^*, \bar{a}, \bar{z}) : \varphi(\bar{x};\bar{a})^i \in q\}$ is consistent. By the previous lemma it is clear that this lemma follows.

§4. Unstable formulas; the independence and strict order property

We can consider as the center of this section the investigation of the function

$$K_\varphi^m(\lambda) = \mathrm{LUB}\{\,|S_\varphi^m(A)| : |A| \le \lambda\}.$$

From Theorem 2.9 (A), (B) it follows that if for one λ, $K_\varphi^m(\lambda) > \lambda^+$, then for every μ, $K_\varphi^m(\mu) > \mu^+$. Hence if G.C.H. holds, $\varphi(\bar{x};\bar{y})$ is unstable then $K_\varphi^m(\lambda) = \lambda^{++}$ for every λ. Without this assumption we get:

if $\mathrm{Ded}(\lambda)$ is always regular, then $K_\varphi^m(\lambda)$ can be only one of the following functions: $(2^\lambda)^+$, $\mathrm{Ded}(\lambda)$, λ^+, n ($n > 1$).

($\mathrm{Ded}(\lambda)$ is the first cardinal μ such that there is no ordered set of cardinality μ, with a dense subset of cardinality λ.)

We prove also that if $(2^\lambda)^+ \ge \mu^+ > \mu \ge \mathrm{Ded}(\lambda)$, μ is regular; then $K_\varphi^m(\lambda) \ge \mu^+$ implies $\varphi(\bar{x};\bar{y})$ has a syntactical property which implies that for every μ, $K_\varphi^m(\mu) = (2^\mu)^+$. The property is the independence property: there are $\bar{a}^0, \dots, \bar{a}^n, \dots$ such that for every $w \subset \omega$, $\{\varphi(\bar{x};\bar{a}^n)^{\text{if } (n\in w)} ; n < \omega\}$ is consistent. Like Theorem 2.13 we prove here that if $\varphi(\bar{x};\bar{y})$ has the independence p, then some $\Psi(x;\bar{y})$ has the independence p (Theorem 4.6).

We define also a syntactical property which can be considered as complementary to the independence p: the strict order p. It appears that $\varphi(\bar{x};\bar{y})$ is unstable iff it has the independence p or some Boolean combination of it has the strict order p. Also there are unstable theories where some formulas $\varphi(x;\bar{y})$ has the independence p, but no formula $\Psi(\bar{x};\bar{y})$ has the strict order p; and conversely. By this we prove there is an unstable T without the property (E) (see §0 A12).

We end the section by a list of open problems; and a discussion on them.

Definition 4.1. (A) A formula $\varphi(\bar{x};\bar{y})$ has the independence p if for every n there are sequences $\bar{a}^0, \dots, \bar{a}^{n-1}$ such that:

for every $w \subset n$, $\models (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x};\bar{a}^k)^{\text{if } (k\in w)}]$

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(B) T has the *independence p* if some formula $\varphi(x;\bar{y})$ has the independence p.

Definition 4.2. (A) A formula $\varphi(\bar{x};\bar{y})$ has the *strict order p* if for every n there are $\bar{a}^0, \ldots, \bar{a}^{n-1}$ such that:

$$\text{if } k, l < n, \text{ then } \vDash (\exists \bar{x})\, [\neg\varphi(\bar{x};\bar{a}^k) \wedge \varphi(\bar{x};\bar{a}^l)] \iff k < l.$$

(B) T has the *strict order p* if some formula $\varphi(\bar{x};\bar{y})$ has the strict order p.

Remark. Note that in 4.2B we say $\varphi(\bar{x};\bar{y})$ and not $\varphi(x;\bar{y})$ as in Definitions 4.1B and 3.2B.

Theorem 4.1. (A) *T is unstable iff T has the independence p or the strict order p. Moreover T is unstable iff some $\varphi(x;\bar{y})$ has the independence p or the strict order p.*

(B) *$\varphi(\bar{x};\bar{y})$ is unstable iff it has the independence p, or for some n, $\eta \in {}^n2$*

$$\Psi_\eta(\bar{x};\bar{y}^0, \ldots, \bar{y}^{n-1}) = \bigwedge_{k<n} \varphi(\bar{x};\bar{y}^k)^{\eta(k)}$$

has the strict order p.

Proof. (A) This follows from (B) by Theorem 2.13. If T is unstable, some $\varphi(x;\bar{y})$ is unstable; hence by (B) $\varphi(x;\bar{y})$ has the independence p or $\Psi_\eta(x;\bar{y}^0, \ldots, \bar{y}^{n-1})$ has the strict order p (for some η). So, by definition, the conclusion follows.

Suppose, on the other hand, that $\varphi(\bar{x};\bar{y})$ has one of those properties. Then by (B), $\varphi(\bar{x};\bar{y})$ is unstable, and so by Theorem 2.13 T is unstable.

(B) Suppose $\varphi(\bar{x};\bar{y})$ has the independence p. Then, by the definitions, $\varphi(\bar{x};\bar{y})$ has the order p; hence by Theorem 2.9 $\varphi(\bar{x};\bar{y})$ is unstable.

Next suppose that for some η, $\Psi_\eta = \Psi_\eta(\bar{x};\bar{y}^0, \ldots, \bar{y}^{n-1})$ has the strict order p. Then from the definitions (2.2, 4.2A) and Th. 2.9 it follows that Ψ_η is unstable. As $l(\bar{x}) = m$, clearly there is an A, $|S^m_{\Psi_\eta}(A)| > |A| \geq \aleph_0$. For every $q \in S^m_{\Psi_\eta}(A)$ let $\bar{a}_q$ be a sequence realizing q, and

$$q^* = \{\varphi(\bar{x};\bar{a})^i : i \in \{0, 1\}, \bar{a} \in A, \vDash \varphi[\bar{a}_q;\bar{a}]^i\}.$$

Clearly $q^* \in S^m_\varphi(A)$; and for $p, q \in S^m_{\Psi_\eta}(A)$, $q^* = p^*$ implies $q = p$; hence $q \neq p$ implies $q^* \neq p^*$, so:

$$|S^m_\varphi(A)| \geq |\{q^* : q \in S^m_{\Psi_\eta}(A)| > |A| \geq \aleph_0 .$$

This means that $\varphi(\bar{x};\bar{y})$ is unstable. So the strict order p of Ψ_η implies the unstability of φ; and also the independence p of $\varphi(\bar{x};\bar{y})$ implies $\varphi(\bar{x};\bar{y})$ is unstable.

So it remains to be proved that if $\varphi = \varphi(\bar{x};\bar{y})$ is unstable, then it has the independence p or for some η, Ψ_η has the strict order p. By Theorem 2.9 $\varphi(\bar{x};\bar{y})$ has the order p, so there are sequences $\bar{a}^0, ..., \bar{a}^n, ...$ such that for every $r < \omega$ there exists $\bar{c}^r$ such that: $\models \varphi[\bar{c}^r;\bar{a}^n]$ iff $r \leq n$. By the compactness theorem and Ramsey theorem ([A]); as proved in Ehrenfeucht and Mostowski [C], we can assume that $\langle \bar{a}^n : n < \omega \rangle$ is an indiscernible sequence (of sequences); where

Definition 4.3. The sequence $\langle \bar{a}^k : k < \alpha \rangle$ is an indiscernible sequence if for every

$$r < \omega, k_0 < ... < k_r < \alpha, \; l_0 < l_1 < ... < l_r < \alpha$$

the following sequences realize the same type:

$$\bar{a}^{k_0} \frown \bar{a}^{k_1} \frown ... \frown \bar{a}^{k_r}; \; \bar{a}^{l_0} \frown \bar{a}^{l_1} \frown ... \frown \bar{a}^{l_r} .$$

Remark. For details see the beginning of Section 5.

If for every $n < \omega$, $w \subset n$,

$$\models (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x};\bar{a}^k)^{\text{if } (k \in w)}]$$

then clearly $\varphi(\bar{x},\bar{y})$ has the independence p and so we finish. So we assume that there are $n < \omega$, $w \subset n$ such that

$$\models \neg (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x};\bar{a}^k)^{\text{if } (k \in w)}]$$

Let $|w| = r$. We can easily define $w_0, ..., w_\alpha$ $\alpha < \omega$ such that:

(1) $w_0 = w$, $w_\alpha = \{n - r, n - r + 1, ..., n - 1\}$

(2) for every $l \leq \alpha$, $|w_l| = r$, $w_l \subset n$

(3) for every $l < \alpha$ there is $k_l < n$ such that $w_{l+1} = w_l \cup \{k_l + 1\} - \{k_l\}$ (and so $k_l \in w_l$, $k_l \notin w_{l+1}$, $k_l + 1 \notin w_l$, $k_l + 1 \in w_{l+1}$) (we step by step raise $w = w_0$ to w_α).

We assume $\models \neg (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x}; \bar{a}^k)^{\text{if } (k \in w_0)}]$.

On the other hand by the definition of the $\bar{a}^k$'s and w_α

$$\models (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x}; \bar{a}^k)^{\text{if } (k \in w_\alpha)}].$$

Hence there is $l < \alpha$ such that

$$\models \neg (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x}; \bar{a}^k)^{\text{if } (k \in s)}] \text{ , where } s = w_l$$

and

$$\models (\exists \bar{x}) [\bigwedge_{k<n} \varphi(\bar{x}; \bar{a}^k)^{\text{if } (k \in t)}] \text{ where } t = w_{l+1}.$$

Let $\beta = k_l$, and

$$\Psi = \Psi(\bar{x}; \bar{y}, \bar{y}^0, \ldots, \bar{y}^{\beta-1}, \bar{y}^{\beta+2}, \ldots, \bar{y}^{n-1}) =$$

$$= \bigwedge_{\substack{k<n \\ k \neq \beta, \beta+1}} \varphi(\bar{x}; \bar{y}^k)^{\text{if } (k \in s)} \wedge \varphi(\bar{x}; \bar{y})$$

We shall prove that Ψ has the strict order p. Let $\gamma < \omega$.

Define $\bar{b} = \bar{b}^0 \frown \ldots \frown \bar{b}^{\beta-1} \frown \bar{b}^{\beta+2+\gamma} \frown \ldots \frown \bar{b}^{n-1+\gamma}$. By the indiscernibility of $\langle \bar{a}^k : k < \omega \rangle$, and the definition of Ψ, if $\beta \leq k < l < \beta + \gamma$, then (by the indiscernability of $\{\bar{a}^i : i < \omega\}$ and as this holds for $k = \beta$, $l = \beta + 1$, $\gamma = 0$).

$$\models (\exists \bar{x}) [\Psi(\bar{x}; \bar{a}^l, \bar{b}) \wedge \neg \varphi(\bar{x}; \bar{a}^k)]$$

but (by the same argument)

$$\models \neg (\exists \bar{x}) [\Psi(\bar{x}; \bar{a}^k, \bar{b}) \wedge \neg \varphi(\bar{x}; \bar{a}^l)].$$

Hence, observing again the definition of Ψ

$$\models (\exists \bar{x})\, [\Psi(\bar{x}; \bar{a}^{l}, \bar{b}) \wedge \neg \Psi(\bar{x}; \bar{a}^{k}, \bar{b})]$$

$$\models \neg (\exists \bar{x})\, [\Psi(\bar{x}; \bar{a}^{k}, \bar{b}) \wedge \neg \Psi(\bar{x}; \bar{a}^{l}, \bar{b})].$$

As this is true for every $\gamma < \omega$, clearly Ψ has the strict order p; and it is also clear that Ψ is of the required form. As this was proved under an assumption that $\varphi(\bar{x}; \bar{y})$ does not have the independence p, we have proved the theorem.

Definition 4.4. Ded(λ) is the first cardinal μ such that there is no ordered set of cardinality μ, with a dense subset of cardinality λ.

Remark. It is known that for every λ, $\lambda^{+} < \mathrm{Ded}(\lambda) \leq (2^{\lambda})^{+}$; and it is consistent with ZFC that $\mathrm{Ded}(\aleph_1) < (2^{\aleph_1})^{+}$. See Baumgartner [A], [B], Mitchell [A].

Theorem 4.2. *If $\varphi(\bar{x}; \bar{y})$ is unstable, $\lambda < \kappa < \mathrm{Ded}(\lambda)$, then there is an A such that $|A| \leq \lambda$, $|S_{\varphi}^{m}(A)| \geq \kappa$.*

Proof. By the definition of Ded(λ), there are an ordered set J, $|J| \geq \kappa$, with a dense subset I, $|I| = \lambda$. As $\varphi(\bar{x}; \bar{y})$ in unstable, by Theorem 2.9 φ has the order p. Hence by the compactness theorem there are $\bar{a}_s$, $s \in J$ such that:

$$\text{for every } t \in J, \quad \{\varphi(\bar{x}; \bar{a}_s)^{\text{if } (t \leq s)} : s \in J\}$$

is consistent.

Let $A = \bigcup \{\mathrm{Rang}(\bar{a}_s) : s \in I\}$. Clearly $|A| \leq |I| \cdot \aleph_0 = \lambda$. For every $t \in J$ let

$$p_t = \{\varphi(\bar{x}; \bar{a}_s)^{\text{if } (t < s)} : s \in I\}.$$

Clearly p_t is a consistent φ-m-type over A. Let $p_t \subset q_t \in S_{\varphi}^{m}(A)$. Now if $s_1, s_2 \in J$, $s_1 < s_2$ then there is $t \in I$, $s_1 < t < s_2$, so $\varphi(\bar{x}; \bar{a}_t) \in q_{s_1}$, $\neg \varphi(\bar{x}; \bar{a}_t) \in q_{s_2}$. Hence $s_1 \neq s_2$ implies $q_{s_1} \neq q_{s_2}$. So

$$|S_{\varphi}^{m}(A)| \geq |\{q_s : s \in J\}| = |J| = \kappa > \lambda = |A|.$$

Theorem 4.3. (A) *If $\varphi = \varphi(\bar{x};\bar{y})$ has the independence p, then for every λ there is an A such that: $|A| \leq \lambda$, $|S^m_\varphi(A)| = 2^\lambda$.*

(B) *If for some infinite A there is a regular cardinal λ, such that $|S^m_\varphi(A)| \geq \lambda \geq \mathrm{Ded}(|A|)$ then $\varphi(\bar{x};\bar{y})$ has the independence property.*

Remark. It can also be shown that:

(1) if for every n there is a finite A such that $|S^m_\varphi(A)| \geq |A|^n$, *then* $\varphi(\bar{x};\bar{y})$ has the independence p.

(2) if $\varphi(\bar{x};\bar{y})$ has the independence p, $l(\bar{y}) = r$, *then* for every n there is A, $|A| \leq nr$, $|S^m_\varphi(A)| \geq 2^n$.

Proof. (A) Let

$$\Gamma = \left\{ (\exists x) \left[\bigwedge_{k\in w} \varphi(\bar{x};\bar{y}^k) \wedge \bigwedge_{k\in u} \neg\varphi(\bar{x};\bar{y}^k) \right] : \right.$$
$$\left. : w \subset \lambda, u \subset \lambda, w \cap u = 0; \quad w, u \text{ are finite} \right\}$$

As φ has the independence p, clearly $\Gamma \cup T$ is consistent, and hence has a model M. Let $\bar{a}^k$ realizes $\bar{y}^k$, and $A = \bigcup\{\mathrm{Rang}\, \bar{a}^k : k < \lambda\}$. Clearly $|A| = \lambda$. For every $w \subset \lambda$ let $p_w = \{\varphi(\bar{x};\bar{a}^k)^{\mathrm{if}\,(k\in w)} : k < \lambda\}$. By the definition of Γ, p_w is consistent, and so there is $q_w \in S^m_\varphi(A)$, $p_w \subset q_w$. Clearly $w \neq u$ implies $q_w \neq q_u$ so

$$|S^m_\varphi(A)| \geq |\{q_w : w \subset \lambda\}| = |\{w : w \subset \lambda\}| = 2^\lambda.$$

So we prove 4.3A.

(B) Suppose A is infinite and $|S^m_\varphi(A)| \geq \lambda \geq \mathrm{Ded}(|A|)$ where A is infinite and λ regular. Let μ be the first cardinal such that there exists $B \subset A$, $|B| = \mu$ and $|S^m_\varphi(B)| \geq \lambda$; and let B be such set. Let $n = l(\bar{y})$ and ${}^nB = \{\bar{a}^k : k < \mu\}$. Clearly B is infinite and hence $|{}^nB| = |B|^n = \mu^n = \mu$.

For $k \leq \mu$, $p \in S^m_{\varphi,0}(B)$ let

$$p|k = \{\Psi : \Psi \in p, \Psi = \varphi(\bar{x};\bar{a}^l) \text{ or } \Psi = \neg\varphi(\bar{x};\bar{a}^l)$$
$$\text{where } l < k\}.$$

For $k < \mu$ let

$$S_k = \{p|k : p \in S^m_\varphi(B), \text{ and } p|k \text{ has} \geq \lambda \text{ extensions in } S^m_\varphi(B)\}$$

$$S_\mu = \{p : p \in S^m_\varphi(B), \text{ for every } k < \mu, p|k \in S_k\}.$$

Let $S = \bigcup_{k \leq \mu} S_k$, $S^* = \bigcup_{k < \mu} S_k$. We define an order on S:
(1) if $p|k = q|k$, $\varphi(\bar{x};\bar{a}^k) \in p$, $\neg\varphi(\bar{x};\bar{a}^k) \in q$ then $q < p$
(2) if $p \in S_k$, $q|k = p$, and $\neg\varphi(\bar{x};\bar{a}^k) \in q$ then $q < p$
(3) if $q \in S_k$, $p|k = q$, $\varphi(x;a^k) \in p$ then $q < p$.
Clearly this is a total ordering of S, and S^* is a dense subset of S.
Now we shall show

$$(*) \quad |S_\varphi^m(B) - S_\mu| < \lambda \; .$$

$$S^m(B) - S_\mu = \{p : p \in S_\varphi^m(B), \text{ for some } k < \mu, p|k \notin S_k\}$$

$$= \bigcup \{\{q \in S_\varphi^m(B) : p|k \subset q\} : p \in S_\varphi^m(B), p|k \notin S_k\}.$$

Let $B_k = \bigcup\{\operatorname{Rang} \bar{a}^l : l < k\}$. Clearly $k < \mu$ implies $|B_k| < \mu$, hence by the definition of μ, as $B_k \subset B \subset A$, $|S_\varphi^m(B_k)| < \lambda$. Hence

$$|\{p|k : p \in S_\varphi^m(B), p|k \notin S_k\}| \leq |\{p|k : p \in S_\varphi^m(B_k), p|k \notin S_k\}|$$

$$\leq |\{p|k : p \in S_\varphi^m(B_k)\}|$$

$$\leq \sum_{k<\mu} |S_\varphi^m(B_k)| < \lambda$$

(The last inequality holds as $\mu \leq |A| < \operatorname{Ded}(|A|) \leq \lambda$, and λ is regular.)
So

$$|\{p|k : p \in S_\varphi^m(B), k < \mu, p|k \notin S_k\}| < \lambda.$$

On the other hand, by the definition of S_k, if $p \in S_\varphi^m(B)$, $p|k \notin S_k$, $k < \mu$, *then* $|\{q \in S_\varphi^m(B) : p|k \subset q\}| < \lambda$. So $S_\varphi^m(B) - S_\mu$ is the union of less than λ sets, each of cardinality $< \lambda$. As λ is regular, we prove (*).

Hence if $q \in S_k$ then

$$|\{p \in S_\mu : q \subset p\}| \geq \lambda$$

as otherwise

$$|\{p \in S_\varphi^m(B) : q \subset p\}| = |\{p \in S_\mu : q \subset p\}| +$$

$$+ |\{p : p \in [S_\varphi^m(B) - S_\mu],\; q \subset p\}| < \lambda$$

and so $q \notin S_k$, a contradiction.

We shall now prove by induction on r that:
(**) For every $q \in S^*$, and for every natural number r there are $a_q^0, \ldots, a_q^{r-1} \in {}^nB$ such that for every $w \subset r$,

$$q \cup \{\varphi(\bar{x}; \bar{a}_q^k)^{\text{if } (k \in w)} : k < r\}$$

is consistent.

Clearly if we prove (**) then it follows that φ has the independence p, and so we finish the proof.

For $r = 0$, (**) is trivial.

Suppose we have proved (**) for r, and we shall prove it for $r + 1$.

For $q \in S^*$ let

$$S_q^* = \{p \in S^* : q \subset p\}, S_q = \{p \in S : q \subset p\},$$

$$S_{q,k} = \{p \in S_k : q \subset p\}.$$

Before starting (**) we have proved that $|S_{q,\mu}| \geq \lambda$. It is clear that $S_{q,\mu} \subset S_q$, hence $|S_q| \geq \lambda$. Also it is clear that S_q^* is a dense subset of S_q; and as $|S_q| \geq \lambda \geq \operatorname{Ded}(|A|) \geq \operatorname{Ded}(\mu)$, clearly $|S_q^*| > \mu$. (This is by the definition of $\operatorname{Ded}(\mu)$.) As $S_q^* = \bigcup_{k<\mu} S_{q,k}$, there is $k < \mu$ such that $|S_{q,k}| > \mu$. For every $p \in S_{q,k}$, by the induction hypothesis, there are $\bar{a}_p^0, \ldots, \bar{a}_p^{r-1} \in {}^nB$ such that for every $w \subset r$, $p \cup \{\varphi(\bar{x}; \bar{a}_p^l)^{\text{if } (l \in w)} : l < r\}$ is consistent. Now there are only μ such r sequences, hence there are $p_1, p_2 \in S_{q,k}, p_1 \neq p_2$ such that $\bar{a}_{p_1}^0 = \bar{a}_{p_2}^0, \ldots, \bar{a}_{p_1}^{r-1} = \bar{a}_{p_2}^{r-1}$. As $p_1 \neq p_2$, $p_1, p_2 \in S_k$ there is $\bar{a}^l \in {}^nB$ such that $\varphi(\bar{x}; \bar{a}^l) \in p_1$, $\neg\varphi(\bar{x}; \bar{a}^l) \in p_2$ (or conservely, and then we can interchange p_1 and p_2). Define

$$\bar{a}_q^0 = \bar{a}_{p_1}^0, \ldots, \bar{a}_q^{r-1} = \bar{a}_{p_1}^{r-1}, \bar{a}_q^r = \bar{a}^l.$$

As $p_1, p_2 \in S_q^*$, clearly $q \subset p_1, q \subset p_2$, and so we prove (**) for $r + 1$.

Thus we prove (**) and hence prove the theorem.

Definition 4.5. (A) $K_\varphi^m(\lambda) = \text{LUB}\{|S_\varphi^m(A)| : |A| \leq \lambda\}$.

(B) $K^m(\lambda) = \text{LUB}\{|S^m(A)| : |A| \leq \lambda\}$.

Theorem 4.4. *Suppose* Ded(λ) *is regular for every* λ. *Then* $K^m_\varphi(\lambda)$ *can be only one of the following functions:*

$$n\ (>1),\ \lambda^+,\ \mathrm{Ded}(\lambda),\ (2^\lambda)^+.$$

Moreover, each of these functions is $K^1_\varphi(\lambda)$ *for some* T, φ.

Proof. First suppose that $\varphi(\bar{x};\bar{y})$ is unstable. Then by Theorem 4.2, $\lambda < \kappa < \mathrm{Ded}(\lambda)$ implies there is an A, $|S^m_\varphi(A)| \geq \kappa$. Hence $K^m_\varphi(\lambda) \geq \mathrm{Ded}(\lambda)$. If for every λ, $K^m_\varphi(\lambda) = \mathrm{Ded}(\lambda)$, the conclusion of the theorem holds. So Suppose for at least one λ, $K^m_\varphi(\lambda) \neq \mathrm{Ded}(\lambda)$, hence $K^m_\varphi(\lambda) > \mathrm{Ded}(\lambda)$. So by Definition 4.5, there is A, $|A| \leq \lambda$, $|S^m_\varphi(A)| \geq \mathrm{Ded}(\lambda)$. As $\mathrm{Ded}(\lambda)$ is regular, by Theorem 4.3B, $\varphi(\bar{x};\bar{y})$ has the independence p. So by Theorem 4.3A for every λ there is an A, $|A| \leq \lambda$, $|S^m_\varphi(A)| \geq 2^\lambda$, hence $K^m_\varphi(\lambda) > 2^\lambda$. But always $|S^m_\varphi(A)| \leq 2^{|A|+\aleph_0}$, hence $K^m_\varphi(\lambda) \leq (2^\lambda)^+$. So if φ is unstable, $K^m_\varphi(\lambda) = \mathrm{Ded}(\lambda)$ (for every λ) or $K^m_\varphi(\lambda) = (2^\lambda)^+$ (for every λ).

So suppose $\varphi(\bar{x};\bar{y})$ is stable. For every n, if for one λ, $K^m_\varphi(\lambda) > n$, then there is A, $|A| \leq \lambda$, $|S^m_\varphi(A)| \geq n$. It is easy to find $B \subset A$, $|B| \leq \aleph_0$, such that $|S^m_\varphi(B)| \geq n$; hence for every μ, as $\mu \geq \aleph_0 \geq |B|$, $K^m_\varphi(\mu) > n$. Hence if for some λ, $K^m_\varphi(\lambda) = n$, then for every μ, $K^m_\varphi(\mu) = n$. As for every $A, \bar{a}, \bar{a}$ realize a type over A, clearly $|S^m_\varphi(A)| \geq 1$, so $n > 1$.

So suppose $K^m_\varphi(\lambda) \geq \aleph_0$ for some λ. Then for any n there is A_n, $|S^m_\varphi(A_n)| \geq 2^n$. We can define by induction on $k < n$, $\bar{a}^k$, $\eta(k)$ such that:

$$|\{p \in S^m_\varphi(A_n) : \varphi(\bar{x};\bar{a}^l)^{\eta(l)} \in p \text{ for every } l < k\}| \geq 2^{n-k}$$

and $\{\varphi(\bar{x};\bar{a}^l)^{\eta(l)} : l < k-1\} \cup \varphi(\bar{x};\bar{a}^{k-1})^{1-\eta(k-1)}$ is consistent. For simplicity suppose every $\eta(l)$ is zero. Hence $\{\varphi(\bar{x}^n;\bar{y}^l)^{\text{if }(l<n)} : l \leq n < \omega\}$ is consistent. So for every λ, $\{\varphi(\bar{x}^k;\bar{y}^l)^{\text{if }(l<k)} : l \leq k < \lambda\}$ is consistent. Let $\bar{a}^l$ realize $\bar{y}^l$ and $\bar{c}^k$ realize $\bar{x}^k$, $A = \bigcup\{\mathrm{Rang}\,\bar{a}^l : l < \lambda\}$, and p_k the φ-m-type $\bar{c}^k$ realize over A. Clearly

$$|A| = \lambda,\ |S^m_\varphi(A)| \geq |\{p_k : k < \lambda\}| = \lambda.$$

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Hence for every λ, $K^m_\varphi(\lambda) \geq \lambda^+$. As $\varphi(\bar{x};\bar{y})$ is stable, $|A| \leq \lambda$ implies $|S^m_\varphi(A)| \leq \lambda$, so $K^m_\varphi(\lambda) \leq \lambda^+$. So $K^m_\varphi(\lambda) = \lambda^+$.

Let us prove that each of the mentioned functions is $K^1_\varphi(\lambda)$ for some φ and T.

(1) By Theorem 4.7 there are T and $\varphi(x;y)$ such that $\varphi(x;y)$ has the independence p. Hence $K^1_\varphi(\lambda) = (2^\lambda)^+$.

(2) Let T be the theory of the order of the rationals, and $\varphi(x;y) = x < y$. Clearly $K^m_\varphi(\lambda) = \mathrm{Ded}(\lambda)$.

(3) If T is the theory of equality, $\varphi(x;y) = [x = y]$, then $K^m_\varphi(\lambda) = \lambda^+$.

(4) Let T be a theory with an equivalence relation E with $n \geq 1$ equivalence classes, and $\varphi(x;y) = [x\, E\, y]$. Clearly $K^1_\varphi(\lambda) = n + 1$.

Lemma 4.5. *If* $\mu < \lambda$. λ *is regular and for every* A, $|A| \leq \mu$ *implies* $|S(A)| < \lambda$, *then for every* A, m; $|A| \leq \mu$ *implies* $|S^m(A)| < \lambda$.

Remark. This strengthens Lemma 2.10.

Proof. We shall prove it by induction on m. For $m = 1$ this is self-evident. Suppose it is true for m, and we shall prove it for $m + 1$. Let $|A| \leq \mu$, and for every $q \in S^{m+1}(A)$ define:

$$q^* = \{(\exists x_m)\, \Psi(x_0, ..., x_m, \bar{a}) : \Psi(x_0, ..., x_m, \bar{a}) \in q\}.$$

It is easily seen that q^* is a (consistent) m-type over A, and has a unique extension in $S^m(A)$ which we shall denote by q^+. Clearly

$$|\{q^+ : q \in S^{m+1}(A)\}| \leq |S^m(A)| < \lambda.$$

By the regularity of λ if suffices to prove that for every $p \in S^m(A)$,

$$|\{q \in S^{m+1}(A) : q^+ = p\}| < \lambda.$$

Let $\langle c_0, ..., c_{m-1}\rangle$ realize p. As every $q \in S^{m+1}(A)$ is closed under finite conjunction, clearly if $q^+ = p$ then

$$q' = \{\Psi(c_0, ..., c_{m-1}, x, \bar{a}) : \Psi(x_0, ..., x_m, \bar{a}) \in q\}$$

is consistent. Moreover $q_1' = q_2'$ implies $q_1 = q_2$. Hence

$$|\{q : q \in S^{m+1}(A), q^+ = p\}| \leq |S(A \cup \{c_0, ..., c_{m-1}\})| < \lambda.$$

As said before, this implies $|S^{m+1}(A)| < \lambda$. So we prove the induction step, and so also the lemma.

Theorem 4.6. *The following statements are equivalent:*

(A) *there is a formula $\varphi(x, \bar{y})$ which has the independence p.*

(B) *there is a formula $\varphi(\bar{x}; \bar{y})$ which has the independence p.*

In the case there are μ, λ; $\mathrm{Ded}(\mu) = \lambda^+$, $\lambda = \lambda^{|T|} < 2^\mu$ *also the statements.*

(C) *for every μ there is an A such that $|A| = \mu$, $|S(A)| = 2^\mu$.*

(D) *for some A, m, $|S^m(A)| > \lambda^{|T|} = \lambda$, $\lambda^+ = \mathrm{Ded}(|A|)$.*

Remark. In the proof we assume there are such λ, μ, and through this proves the equivalence of (A) and (B). By set theoretic consideration this assumption can be removed. (See Baumgartner [A], [B] and Mitchell [A]). (This is done like many removings of G.C.H. from proofs.) We can eliminate the use of set theory by using $K_\varphi(\lambda)$ for finite λ. That is by using the remark to Theorem 4.3, and replacing (C) and (D) by

(C*) there are $\varphi = \varphi(x; \bar{y})$ and n such that for every r there is an A, $|A| = r$, $|S_\varphi(A)| \geq 2^{r/n}$.

(D*) there are $\varphi = \varphi(\bar{x}; \bar{y})$ and $m = l(\bar{x})$, such that for every n there is an A, $|A| < \aleph_0$, $|S^m_\varphi(A)| \geq |A|^n$.

Proof of theorem 4.6. Clearly (A) implies (B).

By Theorem 4.3A, (B) implies that for every μ, for some m, A, $|S^m_\varphi(A)| = 2^{|\mu|}$, $|A| = \mu$. Hence by Lemma 2.11 $|S^m(A)| \geq |S^m_\varphi(A)| = 2^\mu$, $|A| = \mu$. So (B) implies (D).

Suppose (D) holds, i.e., $|S^m(A)| > \lambda^{|T|} = \lambda$, $\lambda^+ = \mathrm{Ded}(|A|)$. By Lemma 4.5, there is B, $|B| \leq |A|$, such that $|S(B)| > \lambda$. By Lemma 2.12

$$\prod_\varphi |S_\varphi(B)| \geq |S(B)| \geq \lambda^+ > \lambda = \lambda^{|T|}.$$

Hence for some φ, $|S_\varphi(B)| \geq \lambda^+$. By Theorem 4.3B, $\varphi = \varphi(x; \bar{y})$ has the independence p. So at least, (D) implies (A). So (A) $\to$ (B) $\to$ (D) $\to$ (A). As it is easy to see that (A) implies (C) by Theorem 4.3A; and that (C) implies (D) we have proved the theorem.

Theorem 4.7. (A) *There is a theory T_{ord} with the strict order p and without the independence p. Moreover, some formula $\varphi(x;y)$ has the strict order p.*

(B) *There is a theory $T = T_{\mathrm{ind}}$ with the independence p and without the strict order p. Moreover there is no infinite A, and a formula $\varphi(x_1, \ldots, x_n; \bar{a})$ which is antisymmetric and connected over A* [This is the property (E) of Ehrenfeucht [A]. See §0 A/2]. (T_{ind} also does not satisfy (3) from 4.8, for every μ.)

Proof. (A) Let $T = T_{\mathrm{ord}}$ be the theory of dense order without first and last element. Clearly the formula $\varphi(x;y) = x < y$ has the strict order p. It is also clear that for any infinite A, $|S(A)| < \mathrm{Ded}(|A|)$. Hence by Theorem 4.6 T has not the independence property. (This can also be shown depending on the fact that for finite A, $|S(A)| \leq |A| + 1$.)

(B) In the language of T there will be only the equality sign, a one place predicate $P(x)$, and a two place predicate $x\,E\,y$. Its axioms will be:

(1) $x\,E\,y$ implies $\neg P(x), P(y)$; that is

$$(\forall xy)\,[x\,E\,y \rightarrow \neg P(x) \wedge P(y)].$$

(2) if $P(y)$ then y is uniquely determined by $\{x : x\,E\,y\}$, and conversely; i.e.,

$$(\forall y_1 y_2)[P(y_1) \wedge P(y_2) \wedge (\forall x)(xEy_1 \equiv xEy_2) \rightarrow y_1 = y_2]$$

$$(\forall x_1 x_2)[\neg P(x_1) \wedge \neg P(x_2) \wedge (\forall y)(x_1\,Ey \equiv x_2\,Ey) \rightarrow x_1 = x_2]$$

$(3)_n$ For every $2n$ different elements in $\neg P$, $x_1, \ldots, x_n, x^1, \ldots, x^n$ there is a y such that $x_1\,Ey, \ldots, x_n\,Ey, \neg x^1\,Ey, \ldots, \neg x^n\,Ey$. That is,

$$(\forall x_1, \ldots, x_n)(\forall x^1, \ldots, x^n)\left[\bigwedge_{\substack{0<k\leq n\\ 0<l\leq n}} [x_k \neq x^l \wedge \neg P(x_k) \wedge \neg P(x^l)] \rightarrow (\exists y) \bigwedge_{0<k\leq n} (x_k\,Ey \wedge \neg x^k\,Ey)\right]$$

$(4)_n$ The same as $(3)_n$, interchanging P and $\neg P$, xEy and yEx.

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That is,

$$(\forall y_1, ..., y_n)(\forall y^1, ..., y^n) \left[\bigwedge_{\substack{0<k\le n\\ 0<l\le n}} [y_k \neq y^l \wedge P(y_k) \wedge P(y^l)]\right.$$

$$\left. \rightarrow (\exists x) \bigwedge_{0<k\le n} (xEy_k \wedge \neg xEy^k)\right]$$

It is not hard to prove that T is consistent, by building a model for it. It is also easy and standard to prove it has elimination of quantifiers, and is complete. By this it can be shown that no formula $\varphi = \varphi(\bar{x};\bar{y})$ has the strict order p. On the other hand, clearly $\varphi(x;y) = xEy$ has the independence p.

Discussion of some open problems

Clearly this section leaves some natural problems unsolved.

We have proved that if some formula $\varphi(\bar{x};\bar{y})$ has the f.c.p., then some formula $\varphi(x;\bar{y})$ has the f.c.p.; and if some formula $\varphi(\bar{x};\bar{y})$ has the independence relation, then some formula $\varphi(x;\bar{y})$ has the independence relation. But we do not prove

Conjecture 4A. If some formula $\varphi(\bar{x};\bar{y})$ has the strict order p, then some formula $\varphi(x;\bar{y})$ has the strict order p.

Remark: Lachlan proved this. in [F*].

If a formula $\varphi(\bar{x};\bar{y})$ is stable, or equivalently has not the order p, we succeed in exploiting it to prove something about it: its types have rank; and they are describable (Theorems 2.9, 3.1, 3.2). Naturally we asked the vague questions:

Question 4B. What can we say about the φ-m-types, when T has not the independence p?

Question 4C. What can we say about the φ-m-types, when T has not the strict order p?

Remark. By the way, a result connected to this and to 6.4 is

Theorem 4.8. *For any μ, m, the following condition are equivalent:*

(1) *For every* $\lambda = \Sigma_{\kappa<\mu}\lambda^\kappa$ *there are* A, $|A| = \lambda$ *and* λ^μ *m-types on* A, *contradictory in pairs, each of them of cardinality* μ.

(2) *There are a set* A, *a set* S *of m-types on* A, *contradictory in pairs,* $|S| > \lambda \geq \Sigma_{\kappa<\mu}(|A| + |T|)^\kappa$, *and for every* $p \in S$, $|p| \leq \chi$, *and* $2^{|T|+\chi} \leq \lambda$ *for some* χ.

(3) *There are* $\varphi_\alpha(\bar{x}, \bar{y}^\alpha)$ $\alpha < \mu$, *and* $\bar{a}_\eta$ $\eta \in {}^{\mu>}\omega$ *such that:* (A) *for every* $\eta \in {}^\mu\omega$, $\{\varphi_\alpha(\bar{x}, \bar{a}_{\eta|\alpha}) : \alpha < \mu\}$ *is consistent* (B) *for every* $\eta \in {}^{\mu>}\omega$, $k < l < \omega$, $l(\eta) + 1 = \alpha$, $\{\varphi_\alpha(\bar{x}, \bar{a}_{\eta^\frown\langle k\rangle}), \varphi_\alpha(\bar{x}, \bar{a}_{\eta^\frown\langle l\rangle})\}$ *is inconsistent.* $(l(\bar{x}) = m)$.

In the case $\mu = \aleph_0$, *we can add* (see Definition 6.2)

(4) $\mathrm{Deg}^m(x = x) = \infty$.

It is clear that if (3) holds for one $\mu > |T|$, we can take all the φ_α equal, and so (3) holds for every μ. It should be noted that T_{ind} does not satisfy (3) for any $\mu \geq \aleph_0$; but we can find a theory which satisfies (3) for every μ (hence is unstable) but has not the strict order p. If T is stable, μ regular, then $\mu < \kappa(T)$ iff (3) holds for μ. If T has the strict order p, then for every μ (3) holds. Perhaps in Question 4C, we should add that T does not satisfy (3) for $|T|^+$.

Unfortunately, in the case $\mathrm{Ded}(\lambda)$ is singular for some λ, we do not know what $K_\varphi^m(\lambda)$ can be.

Conjecture 4D. If for at least one infinite A, $|S_\varphi^m(A)| \geq \mathrm{Ded}(|A|)$ then $\varphi = \varphi(\bar{x}; \bar{y})$ has the independence p.

Possibly this conjecture can be affirmed by answering Question 4B.
In Shelah [D] Theorem 4.4 it was proved:

Theorem 4.9. *If* T *is stable, then there are cardinals* $\lambda \leq 2^{|T|}$, $\kappa \leq |T|^+$ *such that:*

T *is stable in* μ *iff* $\mu = \lambda + \mu^{(\kappa)}$ $(= \lambda + \Sigma_{\chi<\kappa}\mu^\chi)$.

It is natural to ask what are the possible functions $K^m(\lambda)$ and how they are fixed by syntactical properties of T. A natural conjecture is (for simplicity we take $m = 1$):

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Conjecture 4E. $K^1(\lambda)$ is determined by the holding of the following properties of T:

(1) The stability or unstability of T.

(2) The holding of the independence p.

(3) The values of $K^1(\lambda)$ for $\lambda \leq |T|$.

$(4)_\lambda$ There are formulas $\varphi_k(x;\bar{y}^k)$ for $k < \lambda$ and sequences $\bar{a}^{k,n}$ $k < \lambda, n < \omega$ such that: for every $\langle n^k : k < \lambda\rangle$ the type

$$\{\varphi_k(x;\bar{a}^{k,n})^{\text{if } (n > n_k)} : n < \omega, k < \lambda\}$$

is consistent.

$(5)_\lambda$ There are formulas $\varphi_k(x;\bar{y}^k)$ $k < \lambda$, and sequences $\bar{a}^{k,n}$ $k < \lambda$, $n < \omega$ such that for every $\langle n^k : k < \lambda\rangle$ the type

$$\{\varphi_k(x, \bar{a}^{k,n})^{\text{if } (n = n_k)} : n < \omega, k < \lambda\}$$

is consistent.

$(6)_\lambda$ There are formulas $\varphi_k(x;\bar{y}^k)$ $k < \lambda$, and sequences $\bar{a}_\tau$ $\tau \in {}^{\lambda >}\omega$ such that for every $\tau \in {}^\lambda\omega$ the type

$$\{\varphi_k(x, \bar{a}_\eta)^{\text{if } (\eta = \tau | k)} : \eta \in {}^k\omega, k < \lambda\}$$

is consistent.

Remark. Condition $(6)_\lambda$ is a reformulation of $(C * \lambda)$ from Shelah [D] Definition 4.2.

Clearly if $\lambda < \mu$ then $(4)_\mu \to (4)_\lambda$, $(5)_\mu \to (5)_\lambda$, $(6)_\mu \to (6)_\lambda$; Also if T is unstable, it satisfies $(6)_\lambda$ for every λ; and if T has the independence p, then it satisfies $(4)_\lambda$, $(5)_\lambda$ for every λ. On the other hand $(6)_{|T|^+}$ implies T is unstable, $(4)_\lambda$ implies $(5)_\lambda$ and $(5)_{|T|^+}$ implies T has the independence p. So instead of all of $(4)_\lambda$, $[(5)_\lambda]$, $[(6)_\lambda]$ we can take the first cardinality for which they are not satisfied. [Also $(5)_\lambda$ implies $(6)_\lambda$.]

Clearly this conjecture may depend on Conjecture 4D.

It is known that if T is stable, and it has a model M, $\aleph_0 \leq |P^M| < \|M\|$, then for every $\lambda \geq \mu \geq |T|$, T has a model N, $|P^N| = \mu$, $\|M\| = \lambda$. (See

Shelah [A]. For $\aleph_0$-stable theories see Shelah [D], the proof of Theorem 7.9, and Baldwin and Lachlan [B] proof of Theorem 3). Now what will occur if we replace "T is stable" by "T has not the independence p" or "T has not the strict order p". Perhaps an advance in Questions 4B, 4C will help here.

Question 4F. What are the classes K of pairs of cardinals such that for some theory T, and perdicate P, where T has not the independence p:

$$\langle \lambda, \mu \rangle \in K \text{ iff } T \text{ has a model } N,\ \|N\| = \lambda,\ |P^N| = \mu.$$

Question 4G. The same as 4F, but where T does not have the strict order p.

§5. On indiscernible sets and sequences

First, let us mention a result which does not mention indiscernibility: (Theorem 5.14).

Let T be stable, and p a Δ-m-type over M, Δ be finite [countable]. Then there is $n(\Delta, m) < \omega$ [$= \aleph_0$]; and sequences $\bar{a}^k$ $k < \omega$ from M such that:

$$\varphi(\bar{x}; \bar{a}) \in p \text{ implies } |\{k < \omega : \models \neg\varphi[\bar{a}^k; \bar{a}]\}| < n(\Delta, m).$$

We say that a set I of sequences is an indiscernible set (over A) if every n distinct sequences taken from I realize the same type (over A). (In fact, we deal more with a finite version of this concept which will be used in investigating saturation of ultraproducts, Keisler's order, and related topics.)

We associate with every indiscernible set I in a model M a cardinal $\mu = \dim(I, M)$ such that:

(1) I can be extended in M to a maximal indiscernible set of cardinality μ (Definition 5.4).

(2) I cannot be extended in M to an indiscernible set of cardinality $\mu^+ + |T|^+$ (Corollary 5.12) (we can replace $\mu^+ + |T|^+$ by $\mu^+ + \kappa$, where $\kappa \geq \kappa(T)$ is regular).

(3) If $|I_1 \cap I_2| \geq \aleph_0$, then $\dim(I_1, M) + \kappa(T) = \dim(I_2, M) + \kappa(T)$ (Theorem 5.12).

(4) If M is λ-saturated but not λ^+-saturated $\lambda > |T|$, then there is in M an indiscernible set I such that $\dim(I, M) = \lambda$ (Theorem 5.16). [For superstable T, $\lambda > \aleph_0$ suffice].

Remark. Define $p(I) = \{\varphi(x_1, ..., x_n) : n < \omega, a_1, ..., a_n \in I$ $i \neq j \Rightarrow a_i \neq a_j, \models \varphi[a_1, ..., a_n]\}$. If T is superstable for every $p(I)$ there is $r(p(I)) < \omega$ such that $|I_1 \cap I_2| > r(p(I_1)), r(p(I_2))$ implies $\dim(I_1, M) = \dim(I_2, M)$.

We also discuss in this section the connection between the existence of indiscernible sequences which are not indiscernible sets; the existence of connected antisymmetric relations over an infinite set; and unstability (in suitable formulation those concepts are equivalent).

Remember: members of Δ are $\varphi(\bar{x})$ and not φ.

Definition 5.1. (A) $\{\bar{a}^k : k < \alpha\}$ is a Δ-n-indiscernible sequence over A if $k < l < \alpha$ implies $\bar{a}^k \neq \bar{a}^l$ and:

$$\text{for every } \varphi(\bar{x}^0, \ldots, \bar{x}^{n-1}; \bar{y}) \in \Delta, \bar{c} \in A,$$

$k_0 < \ldots < k_{n-1} < \alpha$, $l_0 < \ldots < l_{n-1} < \alpha$ and permutation θ of $\{0, \ldots, n-1\}$

$$\models \varphi[\bar{a}^{k_{\theta(0)}}, \ldots, \bar{a}^{k_{\theta(n-1)}}; \bar{c}] \text{ iff } \models \varphi[\bar{a}^{l_{\theta(0)}}, \ldots, \bar{a}^{l_{\theta(n-1)}}; \bar{c}] .$$

(B) If Δ is the set of all formulas we omit it. If $\{\bar{a}^k : k < \alpha\}$ is Δ-n-indiscernible for every n, then we say it is Δ-indiscernible. If we omit the words "over A" we mean over the empty set. If $\Delta = \{\varphi\}$, then we write φ instead of Δ.

Remark. For simplicity, we treat $\{\bar{a}^k : k < \alpha\}$ as a sequence also. Also we shall always assume $l(\bar{a}^k) = l(\bar{a}^0) = m$. An I is Δ-($< n$)-indiscernible iff it is Δ-r-indiscernible for any $r < n$.

Definition 5.2. (A) $\{\bar{a}^k : k < \alpha\}$ is a Δ-n-indiscernible set over A if $k < l < \alpha$ implies $\bar{a}^k \neq \bar{a}^l$ and:

$$\text{for every formula } \varphi(\bar{x}^0, \ldots, \bar{x}^{n-1}; \bar{y}) \in \Delta,$$

sequence $\bar{c} \in A$, and two sets of different ordinals $< \alpha$, $\{k^0, \ldots, k^{n-1}\}$, $\{l^0, \ldots, l^{n-1}\}$

$$\models \varphi[\bar{a}^{k^0}, \ldots, \bar{a}^{k^{n-1}}; \bar{c}] \text{ iff } \models \varphi[\bar{a}^{l^0}, \ldots, \bar{a}^{l^{n-1}}; \bar{c}].$$

(B) We adopt the same shortening as in Definition 5.1.

Lemma 5.1. (A) *If $\Delta \subset \Delta_1$, and $I = \{\bar{a}^k : k < \alpha\}$ is a Δ_1-n_1-indiscernible set [sequence] over A, then I is a Δ-n_1-indiscernible set [sequence] over A.*

(B) *If I is a φ-n-indiscernible set [sequence] over A for every $\varphi \in \Delta$, then it is a Δ-n-indiscernible set [sequence] over A.*

(C) *For every finite* n_i *and* Δ_i $i < r < \omega$, *there is a finite* Δ *such that:*

(1) *for any* $I = \{\bar{a}^k : k < \alpha\}$, I *is* Δ-n-*indiscernible set iff for every* $i < r$ *it is* Δ_i-n_i-*indiscernible set; where* $n = \max_{i<r} n_i$, $\alpha \geq n$.

(2) *the same as* (1) *for indiscernible sequences.*

Remark. Δ depends on m, which we considered fixed.

Definition 5.3. Suppose I is an infinite set of finite sequences of elements, all of the same length; $\varphi(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{y})$ a formula; and $\bar{c}$ a sequence. Then $\varphi(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{c})$ is connected and antisymmetric over I if for any n different sequences from I, $\bar{a}^0, ..., \bar{a}^{n-1}$ there are

(1) a permutation θ of n such that

$$\models \varphi[\bar{a}^{\theta(0)}, ..., \bar{a}^{\theta(n-1)}; \bar{c}]$$

(2) a permutation θ of n such that

$$\models \neg \varphi[\bar{a}^{\theta(0)}, ..., \bar{a}^{\theta(n-1)}; \bar{c}]$$

It can be easily seen that Ramsay theorem ([A]) implies

Theorem 5.2. (1) *If* I *is an infinite set of sequences of length* m; A, Δ, n *are finite, then* I *has an infinite subset* $\{\bar{a}^k : k < \omega\}$ *which is a* Δ-n-*indiscernible sequence over* A.

(2) *The analog of* (1) *using the finite version of Ramsey theorem.*

Theorem 5.3. (A) *Suppose* T *is stable,* $n < \omega$, Δ *finite, then there is* $n_0 = n_0(\Delta) < \omega$ *such that: every* Δ-n-*indiscernible sequence* I *over* A *is a* Δ-n-*indiscernible set over* A, *if* $|I| \geq n_0$.

(B) *If* T *is stable, for every formula* $\varphi = \varphi(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{y})$ *there is* $n(\varphi) < \omega$, *such that there is no set* I *of* $\geq n(\varphi)$ *sequences, each of length* $l(\bar{x}^0)$, *and sequence* $\bar{c}$, *and* $\varphi(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{c})$ *is antisymmetric and connected over* I.

(C) *Also the converses of* (A), (B) *hold. Moreover, if* T *is unstable, there is* $\Delta = \{\varphi(\bar{x}^0, \bar{x}^1)\}$ *and an indiscernible sequence* $I = \{\bar{a}^k : k < \omega\}$ *such that* $\varphi(\bar{x}^0, \bar{x}^1)$ *is connected and antisymmetric over* I, *and hence* I *is not* Δ-2-*indiscernible set.*

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Remark. Morley [A], Theorem 3.9 proves a similar thing for $\aleph_0$-stable T.

Proof of Theorem 5.3. (A) Suppose that the conclusion of (A) fails. Then clearly for every r there are A_r and I_r such that I_r is a Δ-n-indiscernible sequence over A_r, but not a Δ-n-indiscernible set over A, and $|I_r| \geq r$, and the elements of I_r are sequences of length m. By the definitions there are $\varphi_r(x^0, ..., x^{n-1}; \bar{y}) \in \Delta$ and $\bar{c}^r \in A$ such that $\varphi_r(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{c}^r)$ is antisymmetric and connected over I_r. As Δ is finite there is $\varphi \in \Delta$ such that for arbitrary large r $\varphi = \varphi_r$. This clearly contradicts the conclusion of 5.3B, so it suffices to prove part B of the theorem.

(B) Suppose the conclusion fails. By the compactness theorem there are a sequence $\bar{c}$ and an infinite set I of sequences of length $m = l(\bar{x}^0)$ such that $\varphi(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{c})$ is connected and antisymmetric over I. For simplicity let $I = \{\bar{a}^k : k < \omega\}$. By 5.2 and the compactness theorem we can assume w.l.o.g. that I is an indiscernible sequence over Rang $\bar{c}$. Let $\lambda = 2^{|T|}$, then we can find a dense ordered set J_1 with a dense subset J which is dense in J_1 such that $|J_1| > \lambda \geq |J|$, and, J_1 has no first or last element (if $\mu = \inf\{\kappa : 2^\kappa > \lambda\}$, then J_1 can be chosen as ${}^{\mu}2$, and J as the set of eventually constant sequences in ${}^{\mu}2$).

By the compactness theorem we can define $\{\bar{a}_s : s \in J_1\}$ such that: for every $s_1 < ... < s_r, r < \omega, \bar{c}^\frown\bar{a}_{s_1}{}^\frown ... {}^\frown\bar{a}_{s_r}, \bar{c}^\frown\bar{a}^1{}^\frown ... {}^\frown\bar{a}^r$ realize the same type. Now let $A = \bigcup\{\text{Rang}\,\bar{a}_s : s \in J\} \cup \text{Rang}\,\bar{c}$. Clearly $|A| \leq \lambda$. Let for $s \in J_1 - J$, p_s be the m-type which $\bar{a}_s$ realizes over A. If always $s \neq t \Rightarrow p_s \neq p_t$, then $|S^m(A)| \geq |\{p_s : s \in J_1 - J\}| = |J_1 - J| > \lambda \geq |A|$ a contradiction to the stability of T by Lemma 2.10, Th. 2.13.

So there are $s \neq t$, $s, t \in J_1 - J$, $p_s = p_t$ and w.l.o.g. $s < t$. Now without loss of generality assume $\vDash \varphi[\bar{a}^1, ..., \bar{a}^n; \bar{c}]$ (otherwise - replace φ by $\neg\varphi$). So there is a permutation θ of $\{1, ..., n\}$ such that $\vDash \neg\varphi[\bar{a}^{\theta(1)}, ..., \bar{a}^{\theta(n)}; \bar{c}]$. We can choose such θ with maximal $r = r(\theta) = \inf\{k : \theta(k) \neq k\}$. So $\vDash \neg\varphi[\bar{a}^1, ..., \bar{a}^{r-1}, \bar{a}^{\theta(n)}; \bar{c}]$.

As $s < t$, we can find $s_1, ..., s_n$ such that:

$$s_1 < ... < s_n \in J, \quad s_{r-1} < s < s_r, \quad s_{\theta(r)-1} < t < s_{\theta(r)} \,.$$

By the definition of the $\bar{a}$'s

$$\models \neg\varphi[\bar{a}_{s_1}, ..., \bar{a}_{s_{r-1}}, \bar{a}_{s_{\theta(r)}}, ..., \bar{a}_{s_{\theta(n)}}; \bar{c}]$$

and similarly, and by the indiscernibility

$$\models \neg\varphi[\bar{a}_{s_1}, ..., \bar{a}_{s_{r-1}}, \bar{a}_t, \bar{a}_{s_{\theta(r+1)}}, ..., \bar{a}_{s_{\theta(n)}}; \bar{c}]$$

as $p_s = p_t$

$$\models \neg\varphi[\bar{a}_{s_1}, ..., \bar{a}_{s_{r-1}}, \bar{a}_s, \bar{a}_{s_{\theta(r+1)}}, ..., \bar{a}_{s_{\theta(n)}}; \bar{c}]$$

again by the definition and the indiscernibility

$$\models \neg\varphi[\bar{a}_{s_1}, ..., \bar{a}_{s_{r-1}}, \bar{a}_{s_r}, \bar{a}_{s_{u(r+1)}}, ..., \bar{a}_{s_{u(n)}}; \bar{c}].$$

(Where u is a permutation of $\{r+1, ..., n\}$ such that $u(r+1), ..., u(n)$ and $\theta(r+1), ..., \theta(n)$ are ordered in the same way.)

So, by the definition of the $\bar{a}$'s

$$\models \neg\varphi[\bar{a}^1, ..., \bar{a}^r, \bar{a}^{u(r+1)}, ..., \bar{a}^{u(n)}; \bar{c}].$$

This contradicts the maximality of $r = r(\theta)$, hence we prove 5.3B.

(C) By Theorem 2.13, if T is unstable there are sequences $\{\bar{a}^k : k < \omega\}$ and a formula $\varphi(\bar{x}^0, \bar{x}^1)$ such that $\models \varphi[\bar{a}^k, \bar{a}^l]$ iff $k < l$. By using Theorem 5.2 as in the proof of (B), clearly (C) follows.

Lemma 5.4. (A) *If* $\{\bar{a}^k : k < \alpha\}$, $\alpha \geq \omega$ [α *a limit ordinal*] *is a* Δ-n-*indiscernible set [sequence], $\alpha < \beta$, then we can define* $\bar{a}^k$ *for* $\alpha \leq k < \beta$ *such that* $\{\bar{a}^k : k < \beta\}$ *is also a* Δ-n-*indiscernible set [sequence].*

(B) *We can omit* Δ *and/or* n *from* (A).

(C) *If* $\bar{a}^k \in |M|$ *for* $k < \alpha$, M *is* $(|\beta| + |\Delta|^+ + |\alpha|^+)$-*compact then we can choose the* $\bar{a}^k$, $\alpha \leq k < \beta$ *in* M.

Proof. Immediate.

Conjecture. The requirement "α a limit ordinal" is necessary.

Theorem 5.5. (A) *If T has not the f.c.p. and Δ is finite, then there is $n_1(\Delta) < \omega$ such that:*

if $\{\bar{a}^k : k < \alpha \geq n_1(\Delta)\}$ is a Δ-n-indiscernible set over A, $\beta > \alpha$, then we can define $\bar{a}^k$ for $\alpha \leq k < \beta$ such that $\{\bar{a}^k : k < \beta\}$ is a Δ-n-indiscernible set over A.

(B) *If $A \subset |M|$, $\bar{a}^k \in |M|$ for $k < \alpha$ and M is $(|A|^+ + |\Delta|^+ + |\beta| + |\alpha|^+)$-compact, then we can choose $\bar{a}^k \in |M|$ for $\alpha \leq k < \beta$.*

(C) *If A, Δ are finite $\alpha < \beta = \omega$, $A \subset |M|$, $\bar{a}^k \in |M|$ for $k < \alpha$ then we can choose $\bar{a}^k \in |M|$ for $\alpha \leq k < \beta$.*

Proof. (C) follows from (B), and (B) will be clear from the proof of (A). So we shall prove (A) only.

If n is too large, every set of different $\bar{a}^k$'s is a Δ-n-indiscernible set. Similarly if $m = l(\bar{a}^0)$ is too large. So we can prove the theorem for fixed n, m.

Let

$$\Delta^* = \{\varphi(\bar{x}^0, \ldots, \bar{x}^{n-1}; \bar{y}) : \varphi(\bar{x}^0, \ldots, \bar{x}^{n-1}; \bar{y}) =$$

$$= \Psi(\bar{x}^{\theta(0)}, \ldots, \bar{x}^{\theta(n-1)}; \bar{y}), \Psi(\bar{x}^0, \ldots, \bar{x}^{n-1}; \bar{y}) \in \Delta,$$

$$\text{and } \theta \text{ a permutation of } n\}$$

As T has not the f.c.p., by Theorem 3.10B there is a natural number $n^1 = n^1(\Delta)$ such that:

if Γ is a set of formulas of the form $\varphi(\bar{x}^0; \bar{a}^1, \ldots, \bar{a}^{n-1}, \bar{c})$, $\varphi \in \Delta^*$, and every subset of Γ of cardinality $< n^1$ is consistent *then* Γ is consistent.

It is clearly sufficient to prove that we can define $\bar{a}^\alpha$ such that $\{\bar{a}^k : k < \alpha + 1\}$ will be a Δ-n-indiscernible set over A. For this it is clearly sufficient to find $\bar{a}^\alpha$ which realizes

$$p_\alpha = \{\varphi(\bar{x}, \bar{a}^{k^{n-2}}, \ldots, \bar{a}^{k^0}, \bar{c}) : \bar{c} \in A, k^0 < \ldots < k^{n-2} < \alpha,$$

$$\varphi \in \Delta^* \text{ and } \models \varphi[\bar{a}^{n-1}, \bar{a}^{n-2}, \ldots, \bar{a}^0, \bar{c}]\}.$$

For this, it is clearly sufficient to prove that p_α is consistent. By the definition of n^1, it suffices to prove that every subset of p_α of cardinality $< n^1$ is consistent. Let q be a subset of p_α, $|q| < n^1$. Clearly in q appears $\leq (n-1)(n^1-1)$ $\bar{a}^k$'s. So if $\alpha > (n-1)(n^1-1)$, then there is $\bar{a}^k$ which does not appear in any of the formulas of q, hence $\bar{a}^k$ realize q, so q is consistent. So p_α is consistent; hence we can define $\bar{a}^\alpha$, hence we can define $\bar{a}^k$ for $\alpha \leq k < \beta$, by induction, as required.

Theorem 5.6. *If T is stable, Δ finite, then there are $r < \omega$, and finite Δ^* such that:*

if p_k is the Δ^*-m-type that $\bar{a}^k$ realizes over $A_k = A \cup \bigcup\{\operatorname{Rang} \bar{a}^l : l < k\}$, $p_0 \subset p_k$ and for every $\varphi \in \Delta^*$, $\operatorname{Rank}^m_\varphi(p_k | \varphi) = \operatorname{Rank}^m_\varphi(p_0 | \varphi)$ where $m = l(\bar{a}^0)$ then $\{\bar{a}^k : k < \alpha\}$ is a Δ-indiscernible sequence over A, and moreover, Δ^*-r-indiscernible sequence over A.

Proof. Clearly, as Δ is finite, there is an $r < \omega$ such that any sequence $\{\bar{b}^k : k < \beta\}$ is Δ-indiscernible sequence *iff* it is a Δ-r_1-indiscernible sequence for every $r_1 \leq r$. So by Lemma 5.1C there is a Δ^1 which is finite and any sequence $\{\bar{b}^k : k < \beta\}$ is Δ-indiscernible iff it is Δ^1-r-indiscernible.

By Theorem 3.1A (and see the remark for the case $|A| < 2$) we can define by downward induction the finite sets $\Delta_r, \Delta_{r-1}, \ldots, \Delta_0$; Δ^* such that:

(0) For any $\bar{b}^k$, $k < \beta$, and β, $I = \{\bar{b}^k : k < \beta\}$ is Δ^*-r-indiscernible over B, iff for every $i \leq r$, it is Δ_i-i-indiscernible over B.

(1) $\Delta^1 \subset \Delta_r$

(2) each Δ_k is closed under permutations of variables, i.e., *if* $\varphi(x_0, \ldots, x_{l-1}) \in \Delta_k$, θ a permutation of l *then* $\varphi(x_{\theta(0)}, \ldots, x_{\theta(l-1)}) \in \Delta_k$.

(3) if $\varphi = \varphi(\bar{x}^0, \ldots, \bar{x}^{k-1}; \bar{y}) \in \Delta_k$, $l(\bar{x}^0) = \ldots = l(\bar{x}^{k-1}) = m$, *then* every type p, $\operatorname{Rank}^m_\varphi p = \operatorname{Rank}^m_\varphi (p | A)$, is Ψ-A-definable for some $\Psi \in \Delta_{k-1}$ $[\Psi = \Psi(\bar{x}^1, \ldots, \bar{x}^{k-1}, \bar{y}; \bar{z})]$. [For the definition of Δ^* we use 5.1C.]

Now let A, $I = \{\bar{a}^k : k < \alpha\}$ be as required in the theorem, and we shall show $\{\bar{a}^k : k < \alpha\}$ is a Δ-indiscernible sequence over A. It clearly suffices to prove by induction on $l \leq r$ that I is a Δ_l-l-indiscernible sequence over A (as $\Delta^1 \subset \Delta_r$, and by the definition of Δ^1, Δ^*.)

Case I: $l = 0$. This is fulfilled trivially.
Case II: $l = 1$. As $p_0 \subset p_k$, for every k, clearly *if* $k^1, k^2 < \alpha$, *then* $\bar{a}^{k^1}$, $\bar{a}^{k^2}$ realize the same type over $A - p_0$.

Now notice that if $k < l < \alpha$, then for every φ, $p_0 | \varphi \subset p_k | \varphi$, $p_0 | \varphi \subset p_l | \varphi$ and $\mathrm{Rank}^m_\varphi (p_k | \varphi) = \mathrm{Rank}^m_\varphi (p_0 | \varphi) = \mathrm{Rank}^m_\varphi (p_l | \varphi)$. Now by Theorem 2.5D, $p_0 | \varphi$ has unique extension in $S^m_\varphi (A_k)$ of the same rank, and $p_k | \varphi \in S^m_\varphi (A_k)$. On the other hand $p_0 | \varphi \subset (p_l | \varphi) | A_k \subset p_l | \varphi$ hence $\mathrm{Rank}^m_\varphi [(p_l | \varphi) | A_k] = \mathrm{Rank}^m_\varphi (p_0 | \varphi)$. So $p_k | \varphi = (p_l | \varphi) | A_k$. As this is true for every $\varphi \in \Delta^*$, and $p_k \in S^m_{\Delta^*}(A_k)$, $p_l \in S^m_{\Delta^*}(A_l)$ clearly $p_l | A_k = p_k$.
Case III: It holds for l, and we should prove for $l + 1$.

As Δ_{l+1} is closed under permutations of variables it suffices to prove that, for any $\varphi(\bar{x}^0, \dots, \bar{x}^l; \bar{y}) \in \Delta_{l+1}$, $\bar{c} \in A$
(*) if $\alpha > k^0 > k^1 > \dots > k^l$, $\alpha > j^0 > j^1 > \dots > j^l$, then

$$\vDash \varphi[\bar{a}^{k^0}, \dots, \bar{a}^{k^l}; \bar{c}] \text{ iff } \vDash \varphi[\bar{a}^{j^0}, \dots, \bar{a}^{j^l}; \bar{c}]$$

By the symmetry in (*), we can assume $k^0 \geq j^0$. As $p_{j^0} \subset p_{k^0}$, claerly

$$\vDash \varphi[\bar{a}^{j^0}, \bar{a}^{j^1}, \dots, \bar{a}^{j^l}; \bar{c}] \text{ iff } \vDash \varphi[\bar{a}^{k^0}, \bar{a}^{j^1}, \dots, \bar{a}^{j^l}; \bar{c}].$$

By the definition of p_{k^0}, this implies it suffices to prove
(**) if $\alpha > i > k^1 > \dots > k^l$, $\alpha > i > j^1 > \dots > j^l$, then

$$\varphi(\bar{x}, \bar{a}^{k^1}, \dots, \bar{a}^{k^l}; \bar{c}) \in p_i \text{ iff } \varphi(\bar{x}, \bar{a}^{j^1}, \dots, \bar{a}^{j^l}; \bar{c}) \in p_i.$$

But now by the definition of Δ_l there is $\Psi \in \Delta_l$ and $\bar{b} \in A$ such that: for every $i > i^1 > \dots > i^l$

$$\varphi(\bar{x}, \bar{a}^{i^1}, \dots, \bar{a}^{i^l}; \bar{c}) \in p_i \text{ iff } \vDash \Psi[\bar{a}^{i^1}, \dots, \bar{a}^{i^l}, \bar{c}, \bar{b}]$$

But by the induction hypothesis

$$\vDash \Psi[\bar{a}^{k^1}, \dots, \bar{a}^{k^l}; \bar{c}] \text{ iff } \vDash \Psi[\bar{a}^{j^1}, \dots, \bar{a}^{j^l}; \bar{c}]$$

So Case III follows.

Corollary 5.7. *If T is stable and for every $k < \alpha \geq \omega$, p_k is the m-type $\bar{a}^k$ realizes over $A_k = \bigcup \{\mathrm{Rang}\, \bar{a}^l : l < k\} \cup A$ and for every φ, $\mathrm{Rank}^m_\varphi(p_k | \varphi) = \mathrm{Rank}^m_\varphi(p_0 | \varphi)$, and $p_0 \subset p_k$, then $\{\bar{a}^k : k < \alpha\}$ is an indiscernible set over A.*

Proof. By the previous theorem, it is an indiscernible sequence over A; and by Theorem 5.3A every indiscernible sequence over A is an indiscernible set over A.

Theorem 5.8. *Let T be stable, and I a set of sequences of length m.*

(A) *If $\lambda \leq |I|$ is a regular cardinal, n, Δ are finite $|A| < \lambda$, then there is $I_1 \subset I$, $|I_1| \geq \lambda$ which is Δ-n-indiscernible set over A.*

(B) *If $\lambda \leq |I|$ is regular, $\aleph_0 + |A| < \lambda$, and $|B| < \lambda$ implies $|S(B)| < \lambda$ then there is $I_1 \subset I$, $|I_1| \geq \lambda$ which is an indiscernable set over A.*

Remark. Similar theorems with similar proofs are Morley [A] for $\aleph_0$-stable theories, Shelah [B] Th. 3.1 p. 194 for stable theories, Shelah [D] Th. 3.1 p. 82 for stable diagrams; and Shelah [F] Th. 2.2. So we will not repeat the proof. The new part here is (A), which is necessary for proving the two-cardinal theorem for stable theories, and for Theorem 6.7.

Theorem 5.9. *Suppose $\Psi(\bar{y};\bar{x}) = \varphi(\bar{x};\bar{y})$ and $\Psi(\bar{y};\bar{x})$ has not the independence p, and let*

$$\Delta_n = \left\{ (\exists \bar{y}) \bigwedge_{r<n} \Psi(\bar{y};\bar{x}^r)^{\eta(r)} : \eta \in {}^n 2 \right\}.$$

Then there is $n = n(\varphi) < \omega$ such that

(A) *if $\{\bar{a}^k : k < \alpha\}$ is a Δ_n-n-indiscernible set, $\bar{c}$ a sequence then either*

$$|\{k < \alpha : \models \varphi[\bar{a}^k;\bar{c}]\}| \leq n$$

or

$$|\{k < \alpha : \models \neg\varphi[\bar{a}^k;\bar{c}]\}| \leq n.$$

(B) *if $\{\bar{a}^k : k < \alpha\}$ is a Δ_n-n-indiscernible sequence and $\bar{c}$ is a sequence, then there are $0 = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n = \alpha$ such that:*

if $i < n$, $\alpha_i \leq k^1$, $k^2 < \alpha_{i+1}$ *then*

$$\models \varphi[\bar{a}^{k^1}; \bar{c}] \text{ iff } \models \varphi[\bar{a}^{k^2}; \bar{c}] .$$

Remark. In Harnik and Ressayre [B] a similar theorem (1.3) is proved. Similar theorems are also 6.13 and Shelah [D] Th. 4.1. The theorems were proved independently.

Proof. (A) As $\Psi(\bar{y}; \bar{x})$ does not have the independence p there is $n_0 < \omega$ such that

$$\Gamma = \left\{ (\exists \bar{y}^w) \bigwedge_{r < n_0} \Psi(\bar{y}^w, \bar{x}^r)^{\text{if } (r \in w)} : w \subset n_0 = \{0, \ldots, n_0 - 1\} \right\}$$

is inconsistent. Let $n = n_0$.

Now suppose our conclusion is incorrect. Then there are different ordinals $k_0, \ldots, k_{n-1}, l_0, \ldots, l_{n-1} < \alpha$ such that

$$\models \varphi[\bar{a}^{k_0}; \bar{c}], \ldots, \models \varphi[\bar{a}^{k_{n-1}}; \bar{c}],$$

$$\models \neg \varphi[\bar{a}^{l_0}; \bar{c}], \ldots, \models \neg \varphi[\bar{a}^{l_{n-1}}; \bar{c}] .$$

Remembering that $\{\bar{a}^k : k < \alpha\}$ is a Δ_n-n-indiscernible set, we can see that taking $\bar{a}^r$ for $\bar{x}^r$ for $r < n$, Γ is satisfied. Hence Γ is consistent, contradiction.

(B) The proof is essentially the same.

Definition 5.4. (A) Let I be a Δ-n-indiscernible set in M (i.e., $\bar{a} \in I \Rightarrow \bar{a} \in |M|$). Then $\dim(I, \Delta, n, M)$ is the first cardinality μ such that there exists a maximal Δ-n-indiscernible set I_1, in M, $I \subset I_1$, $|I_1| = \mu$ (I_1 being maximal means that there is no Δ-n-indiscernible set I_2 in M, $I_1 \subset I_2, I_1 \neq I_2$).

(B) Similarly we define $\dim(I, \Delta, M)$, $\dim(I, n, M)$, $\dim(I, M)$ with Δ-indiscernibility, n-indiscernibility and indiscernibility instead of Δ-n-indiscernibility; and the same with $\dim(I, \Delta, < n, M)$.

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Theorem 5.10. *Suppose T does not have the independence p, Δ is finite. Then there are natural number $n_3 = n_3(\Delta)$, and a finite Δ^* such that: if I_1 is a Δ-n-indiscernible set in M, I_2 is a Δ^*-n_3-indiscernible set in M, $|I_1 \cap I_2| \geq n_3$ then*

$$n_3\,[\dim(I_1, \Delta, n, M)]^{n-1} \geq \dim(I_2, \Delta^*, n_3, M)$$

(so if one of the dimensions is infinite then $\dim(I_1, \Delta, n, M) \geq \dim(I_2, \Delta^*, n_3, M)$).

Proof. Notice that:

if $\Delta^1, ..., \Delta^r, n^1, ..., n^r$ are finite, $r < \omega$, then there are finite Δ, n such that if I is a Δ-n-indiscernible set, then it is Δ^i-n^i-indiscernible set for every i, $1 \leq i \leq r$ (by Lemma 5.1C).

Clearly we can prove the theorem for fixed n, (remembering that for sufficiently large n, every set of different sequences is a Δ-n-indiscernible set).

Now let $\Delta = \{\varphi_k(\bar{x}^0, ..., \bar{x}^{n-1}; \bar{y}^k) : k < k_0 < \omega\}$ where $l(\bar{x}^0) = ... = l(\bar{x}^{n-1}) = m$. Define, for $k < k_0$, θ a permutation of n

$$\Psi_{k,\theta}(\bar{x}^1, ..., \bar{x}^{n-1}, \bar{y}^k; \bar{x}^0) = \varphi_k(\bar{x}^{\theta(0)}, ..., \bar{x}^{\theta(n-1)}; \bar{y}^k)$$

denote $\bar{z}^k = \bar{x}^1 \frown ... \frown \bar{x}^{n-1} \frown \bar{y}^k$, so $\Psi_{k,\theta} = \Psi_{k,\theta}(\bar{z}^k; \bar{x}^0)$.

Now each $\Psi_{k,\theta}(\bar{z}^k; \bar{x}^0)$ does not have the independence p (by Lemma 4.6, as T does not have the independence p). So there is $r = r(k, \theta) < \omega$ such that

$$\left\{(\exists \bar{z}^k) \bigwedge_{l<r} \Psi_{k,\theta}(\bar{z}^k, \bar{x}^{0,l})^{\text{if } (l \in w)} : w \subset r = r(k, \theta)\right\}$$

is inconsistent. Define

$$n_3 = \max\{r(k, \theta) + n : k < k_0, \theta \text{ a permutation of } n\}$$

$$\Delta^1 = \left\{(\exists \bar{z}^k) \bigwedge_{l<r} \Psi_{k,\theta}(\bar{z}^k; \bar{x}^{0,l})^{\text{if } (l \in w)} : w \subset r = r(k, \theta),\right.$$
$$\left. k < k_0,\ \theta \text{ is a permutation of } n\right\}$$

By Lemma 5.1C, and as increasing n_3 does not do any harm, it is clearly sufficient to prove:

if I_1 is a Δ-n-indiscernible set in M, I_2 is a Δ^1-$(< n_3)$-indiscernible set in M, $|I_1 \cap I_2| > n_3$ *then*

$$n_3 [\dim(I_1, \Delta, n, M)]^{n-1} \geq \dim(I_2, \Delta^1, < n_3, M)$$

By the definition of dimension we can assume I_1 is a maximal Δ-n-indiscernible set in M, of cardinality $\dim(I_1, \Delta, n, M)$, and similarly for I_2. So we should prove only that $n_3 |I_1|^{n-1} \geq |I_2|$.

Now there is an m-type q over $A = \bigcup\{\text{Rang}\, \bar{a} : \bar{a} \in I_1\}$ such that $\bar{c}$ realizes q iff $I_1 \cup \{\bar{c}\}$ is a Δ-n-indiscernible set. If $\varphi(\bar{x}, \bar{b}) \in q$ then clearly in it appear $n - 1$ sequences from I_1, hence, except $n - 1$ sequences, every $\bar{c} \in I_1 \cap I_2$ satisfies $\models \varphi[\bar{c}; \bar{b}]$. Hence

$$|\{\bar{c} \in I_2 : \models \varphi[\bar{c}, \bar{b}]\}| \geq |I_1 \cap I_2| - (n-1) \geq n_3 - (n-1).$$

By the definition of n_3 and Δ^1 clearly

$$|\{\bar{c} \in I_2 : \models \neg\varphi[\bar{c}, \bar{b}]\}| < n_3.$$

Now the number of formulas $\varphi(x, y)$ appearing in q is $\leq |\Delta^1| + 1$ (the $+1$ is for the formulas $\{\bar{x} \neq \bar{c} : \bar{c} \in I_1\}$).

Hence if

$$|I_2| > n_3 |q| = n_3 |I_1|^{n-1}(|\Delta^1| + 1) = n_4 |I_1|^{n-1}$$

then there is a $\bar{c} \in I_2$, such that for every $\varphi(\bar{x}, \bar{b}) \in q$, $\models \varphi[\bar{c}, \bar{b}]$, and hence, $I_1 \cup \{\bar{c}\}$ is a Δ-n-indiscernible set, and as $\bar{c} \in I_2$, $\bar{c} \in |M|$. This contradicts the choice of I_1 as maximal a Δ-n-indiscernible set in M. So $|I_2| \leq n_4 |I_1|^{n-1}$, hence we prove the theorem (as we can replace n_3 by n_4).

Theorem 5.11. (A) *If T has not the independence p and* $|I_1 \cap I_2| \geq \aleph_0$, *$I_1$, I_2 are indiscernible sets, then*

$$\dim(I_1, M) + |T| = \dim(I_2, M) + |T|.$$

(B) *If T is superstable,* $|I_1 \cap I_2| \geq \aleph_0$ *then* $\dim(I_1, M) = \dim(I_2, M)$.

(C) *If T is stable $\kappa = \kappa(T)$ is as defined on* 0.A1 $\lambda \geq \kappa$ *is regular* $|I_1 \cap I_2| \geq \aleph_0$, $\dim(I_1, M) \geq \lambda$, *then* $\dim(I_1, M) = \dim(I_2, M)$ [if T is superstable, $\kappa(T) = \aleph_0$].

Proof. (A) Exactly as the proof of the previous theorem.

(B, C) This follows from 6.13.

Corollary 5.12. *If T does not have the independence p,* $\mu = \dim(I, M) \geq |T|$ *(or $\mu^+ \geq \kappa(T)$), then I cannot be extended to a indiscernible set in M of cardinality $> \mu$; if $\mu > |T|$ or $\mu \geq \lambda \geq \kappa(T)$ λ is regular then every indiscernible set I_1 in M, $|I_1 \cap I| \geq \aleph_0$, can be extended to an indiscernible set in M of cardinality μ.*

Theorem 5.13. (A) *Suppose T is stable $p \in S^m(|M|)$, $B \subset |M|$, and for every φ,* $\text{Rank}^m_\varphi(p|\varphi) = \text{Rank}^m_\varphi[(p|\varphi)|B]$, *and for every $k < \alpha$, $\bar{a}^k$ realize over $B_k = \bigcup\{\text{Rank } \bar{a}^l : l < k\} \cup B$ the type $p|B_k$. Then $\varphi(\bar{x}; \bar{c}) \in p$ implies $|\{k < \alpha : \models \neg\varphi[\bar{a}^k; \bar{c}]\}| < \aleph_0$; and $\{\bar{a}^k : k < \alpha\}$ is an indiscernible set over B.*

(B) *Clearly always we can find the required B; and if M is $|T|^+$-saturated, we can also find suitable $\bar{a}^k$ for $k < |T|^+$.*

Proof. (B) is self-evident.

By Theorem 3.4 we can define $\bar{a}^k$ for $\alpha \leq k < \alpha + \omega$ such that the type $\bar{a}^k$ realizes over $|M| \cup \bigcup\{\text{Rank } \bar{a}^l : l < k\}$, which we name p_k satisfies:

(1) for every φ, $\text{Rank}^m_\varphi(p|\varphi) = \text{Rank}^m_\varphi(p_k|\varphi)$.

(2) $p \subset p_k$.

By Corollary 5.7, $\{\bar{a}^k : k < \alpha + \omega\}$ is an indiscernible set over B. By Theorem 5.9, for any $\varphi(\bar{x}, \bar{a}) \in p$ either $\{k < \alpha + \omega : \models \varphi[\bar{a}^k, \bar{a}]\}$ is finite or $\{k < \alpha + \omega : \models \neg\varphi[\bar{a}^k, \bar{a}]\}$ is finite. But $\varphi(\bar{x}, \bar{a}) \in p \subset p_k$ for $\alpha \leq k < \alpha + \omega$, hence $\models \varphi[\bar{a}^k, \bar{a}]$. So $\{k < \alpha + \omega : \models \neg\varphi[\bar{a}^k, \bar{a}]\}$ is finite, and so the conclusion follows.

Theorem 5.14. *Suppose T is stable. For any finite Δ there is $n = n(\Delta) < \omega$ such that: for any $p \in S^m_\Delta(|M|)$ there are sequences $\bar{a}^k \in |M|$ for $k < \omega$ for which $\varphi(\bar{x}, \bar{b}) \in p$ implies $|\{k < \omega : \models \neg\varphi[\bar{a}^k, \bar{b}]\}| \leq n$.*

Proof. As that of the previous theorem.

Theorem 5.15. *Suppose T is stable Δ is countable, $p \in S^m_\Delta(|M|)$. Then there are sequences $\bar{a}^k \in |M|$ for $k < \omega$ such that $\varphi(\bar{x}, \bar{b}) \in p$ implies $\{k < \omega : \models \neg\varphi[\bar{a}^k, \bar{b}]\}$ is smaller than $n_\varphi < \omega$.*

Proof. Clear from the proofs of the two previous theorems.

Theorem 5.16. *Suppose T is stable, M is λ-saturated but not λ^+-saturated, $\lambda > |T|$. Then there is in M a maximal indiscernible set $I = \{a_k : k < \lambda\}$.*

Proof. As M is not λ^+-saturated, it omits a 1-type p_0, $|p_0| \leq \lambda$, p_0 a type on $|M|$. Let $p \in S(|M|)$, $p_0 \subset p$. By Theorem 5.13A, 5.13B, there are $a^k \in |M|$, $k < \omega$ such that:

(1) $I = \{a^k : k < \omega\}$ is an indiscernible set.

(2) if $\varphi(x, a) \in p$, $\{k < \omega : \models \neg\varphi[a^k, \bar{a}]\}$ is finite.

Now we shall show that $\{a_k : k < \omega\}$ cannot be extended in M to an indiscernible set of cardinality λ^+. For suppose $\{a_l : l < \lambda^+\}$ is such a set. For every $\varphi(x, \bar{c}) \in p_0$, clearly $\varphi(x, \bar{c}) \in p$, hence by the definition of the a_k's

$$|\{l < \omega : M \models \neg\varphi[a_l, \bar{c}]\}| < \aleph_0$$

hence

$$|\{l < \omega : M \models \varphi[a_l, \bar{c}]\}| = \aleph_0$$

hence

$$|\{l < \lambda^+ : M \models \varphi[a_l, \bar{c}]\}| \geq \aleph_0$$

hence by Theorem 5.9

$$|\{l < \lambda^+ : M \models \neg\varphi[a_l, \bar{c}]\}| < \aleph_0.$$

Hence, as p_0 is not realized in M

$$\lambda^+ = |\{l < \lambda^+ : a_l \text{ does not realize } p_0\}|$$

$$\leq \sum_{\varphi(x, \bar{c}) \in p_0} |\{l < \lambda^+ : M \models \neg\varphi[\bar{a}_l, \bar{c}]\}| \leq |p_0|\aleph_0 \leq \lambda$$

contradiction. So $\{a_l : l < \omega\}$ cannot be extended in M to an indiscernible set of cardinality λ^+. As M is λ-saturated, it can be extended in M to an indiscernible set of cardinality λ.

Theorem 5.17. *Suppose T is stable and for every $i < \alpha$, p_i is the m-type $\bar{a}^i$ realize over $A_i = \bigcup\{\mathrm{Rang}\,\bar{a}^j : j < i\} \cup A$; $p_i \subset p_j$ for $i < j$. Then $\{\bar{a}^i : i < \alpha\}$ is the union of $\leq |T|$ sets, each of them an indiscernible set over A. Moreover, there are $\alpha_i < \alpha$ for $i < \beta < |T|^+$ such that: $i < j < \beta \Rightarrow \alpha_i < \alpha_j$ and for every $i < \beta$, $I_i = \{\bar{a}^j : \alpha_i \leq j < \alpha_{i+1}\}$ is an indiscernible set over A_{α_i}.*

Proof. Take $\{\alpha_i : i < \beta\} = \{\gamma < \alpha : \mathrm{Rank}^m(p_{\gamma+1} | \varphi) < \mathrm{Rank}^m(p_\gamma | \varphi)\}$. Then the theorem follows by 5.6. In fact there is I, $|I| \leq |T|$, such that $I_i - I$ is an indiscernible set over $A_{\alpha_i} \cup (A_\alpha - A_{\alpha_{i+1}})$.

Lemma 5.18. *In 5.17, we cannot improve the bound on β.*

Proof. Let $L(T)$ contain the equality, and the equivalence relations E_i, $i < \beta + 1$. T will consist of axioms saying that if $i < j$, E_j refines E_i, and every equivalence class of E_i, is the union of infinitely many E_j-equivalence classes; and that E_0 has infinitely many equivalence classes. Any E_β-equivalence class is infinite. Let A be empty, $\alpha_0 < \alpha_i < \ldots < \alpha_\beta < \alpha$, and a_i, $i < \alpha$ be such that: $a_i E_\gamma a_j$ iff $\alpha_\gamma \leq i, j$.

Remark. We can define dimension not only for indiscernible sets, but also for types p, provided that: if p is a 1-type over $|M|$, there is $q \in S(M)$, $p \subset q$ such that for every φ, $\mathrm{Rank}\, p | \varphi = \mathrm{Rank}\, q | \varphi$. Similarly for m-types, and this holds also for fixed types (Def. 6.5) when T is superstable.

§6. Degrees of types, and superstable theories

We define for every type a degree, which is an ordinal or ∞ such that

(1) every type has a finite subtype of the same degree. If $p \subset q$ then $\mathrm{Deg}\, q \le \mathrm{Deg}\, p$ (Lemma 6.2)

(2) if T is stable, T is superstable iff every type has degree $< \infty$ (Corollary 6.10).

(3) For every finite type p, there is a set A, $|A| \le |T|$ such that: p is a type on A, and if $q \in S(A)$, $p \subset q$, $A \subset B$, $\mathrm{Deg}\, p = \mathrm{Deg}\, q$, then q has a unique extension in $S(B)$ of the same degree (Corollary 6.8).

(4) If p is a type on A, then it has an extension in $S(A)$ of the same degree (Thoerem 6.6).

We also prove that if T is stable but not superstable, there are formulas $\varphi_n(x; \bar{y}^n)$, $n < \omega$, and sequences $\bar{a}_\eta$, $\eta \in {}^{\omega>}\omega$ such that

(1) for every $\eta \in {}^{\omega}\omega$, $\{\varphi_n(x, \bar{a}_{\eta \restriction n}) : n < \omega\}$ is consistent.

(2) for every $m, \eta \in {}^{m}\omega$, $n < r < \omega$, $\{\varphi_{m+1}(x, \bar{a}_{\eta^\frown\langle n\rangle}), \varphi_{m+1}(x, \bar{a}_{\eta^\frown\langle r\rangle})\}$ is inconsistent.

Definition 6.1. We define when $\mathrm{Deg}[\varphi(\bar{x}, \bar{a})] \ge \alpha$ holds by induction on α:

(1) $\mathrm{Deg}^m[\varphi(\bar{x}, \bar{a})] \ge 0$ iff $\models (\exists \bar{x})\varphi(\bar{x}, \bar{a})$;

(2) $\mathrm{Deg}^m[\varphi(\bar{x}, \bar{a})] \ge \delta$ iff for every $\alpha < \delta$, $\mathrm{Deg}^m[\varphi(\bar{x}, \bar{a})] \ge \alpha$

(3) $\mathrm{Deg}^m[\varphi(\bar{x}, \bar{a})] \ge \alpha + 1$, if there are $n < \omega$, formula $\Psi(\bar{x}; \bar{y})$, and sequences $\bar{a}^l$ $l < |T|^+$ such that:

(i) for every $l < |T|^+$, $\mathrm{Deg}[\varphi(\bar{x}, \bar{a}) \wedge \Psi(\bar{x}, \bar{a}^l)] \ge \alpha$

(ii) for every $w \subset |T|^+$, $|w| \ge n$,

$$\models \neg (\exists \bar{x}) \bigwedge_{l \in w} \Psi(\bar{x}, \bar{a}^l).$$

Definition 6.2. $\mathrm{Deg}^m[\varphi(\bar{x}, \bar{a})]$ is α if it is $\ge \alpha$ but not $\ge \alpha + 1$. It is ∞ if for every α, $\mathrm{Deg}^m[\varphi(\bar{x}, \bar{a})] \ge \alpha$. If $\models \neg(\exists \bar{x})\varphi(\bar{x}, \bar{a})$, then it is not defined and we shall ignore this case many times, or treat it as -1. If $m = l(\bar{x})$ is one, or it is obvious what it is, we omit it.
(Most of the time, except 6.10, it will be fixed but arbitrary.)

Lemma 6.1. (A) *If* $\models (\forall \bar{x})[\varphi(\bar{x}, \bar{a}) \to \Psi(\bar{x}, \bar{c})]$ *then* $\mathrm{Deg}\varphi(\bar{x}, \bar{a}) \leq \mathrm{Deg}\Psi(\bar{x}, \bar{c})$.

(B) *If* $\bar{a}, \bar{c}$ *realize the same type, then* $\mathrm{Deg}\varphi(\bar{x}, \bar{a}) = \mathrm{Deg}\varphi(\bar{x}, \bar{c})$.

(C) *If for no* $\varphi, \bar{a}$, $\mathrm{Deg}\varphi(\bar{x}, \bar{a}) = \alpha_0$, *then* $\mathrm{Deg}\Psi(\bar{x}, \bar{c}) \geq \alpha_0$ *implies* $\mathrm{Deg}\Psi(\bar{x}, \bar{c}) = \infty$

(D) *There is* $\alpha_0 < (2^{|T|})^+$, *such that* $\mathrm{Deg}\Psi(\bar{x}, \bar{c}) \geq \alpha_0$ *implies* $\mathrm{Deg}\Psi(\bar{x}, \bar{c}) = \infty$.

Proof. (A)–(B) Immediate; by induction.

(C) We prove by induction on $\beta \geq \alpha_0$ that $\mathrm{Deg}\Psi(\bar{x}, \bar{c}) \geq \alpha_0$ implies $\mathrm{Deg}\Psi(\bar{x}, \bar{c}) \geq \beta$. For $\beta = \alpha_0$, and for β a limit ordinal, it is immediate. Suppose it holds for β, and we shall prove for $\beta + 1$. So let $\gamma = \mathrm{Deg}\Psi(\bar{x}, \bar{c}) \geq \alpha_0$. Since for no $\varphi, \bar{a}$ $\mathrm{Deg}\varphi(\bar{x}, \bar{a}) = \alpha_0$, clearly $\gamma > \alpha_0$, or $\gamma \geq \alpha_0 + 1$. Hence by (3) from Definition 6.1, there are $n < \omega$, $\theta(\bar{x}, \bar{y})$ and $\bar{b}^l$ $l < |T|^+$ such that:

(i) for every $l < |T|^+$, $\mathrm{Deg}[\theta(\bar{x}, \bar{b}^l) \wedge \Psi(\bar{x}, \bar{c})] \geq \alpha_0$.

(ii) for every $w \subset |T|^+$, $|w| = n$, $\models \neg(\exists \bar{x}) \bigwedge_{l \in w} \theta(\bar{x}, \bar{b}^l)$.

By the induction hypothesis (i) implies

(i)′ for every $l < |T|^+$, $\mathrm{Deg}[\theta(\bar{x}, \bar{b}^l) \wedge \Psi(\bar{x}, \bar{c})] \geq \beta$.

Now by Definition 6.1 part (3), (i)′, (ii) implies $\mathrm{Deg}\,\Psi(\bar{x}, \bar{c}) \geq \beta + 1$. So we prove (C).

(D) Follows from (B) and (C).

Definition 6.3. For an m-type p, $\mathrm{Deg}^m\ p$ will be $\min\{\mathrm{Deg}^m\ [\bigwedge_{\varphi \in q} \varphi] :$ q a finite subtype of $p\}$.

Lemma 6.2. (A) *If* $p = \{\varphi_n(\bar{x}, \bar{a}^n) : n < r < \omega\}$ *then* $\mathrm{Deg}\,p = \mathrm{Deg}[\bigwedge_{n<r} \varphi_n(\bar{x}, \bar{a}^n)]$.

(B) *Every type p has a finite subtype of the same degree. Hence if* $i < j < \delta \Rightarrow p_i \subset p_j$, $\mathrm{Deg}\,p_0 = \mathrm{Deg}\,p_j$, δ *a limit ordinal, then* $\mathrm{Deg}[\bigcup_{i<\delta} p_i] = \mathrm{Deg}\,p_0$.

(C) *If* $p \subset q$ *then* $\mathrm{Deg}\,q \leq \mathrm{Deg}\,p$.

(D) *If every sequence realizing p realizes q then* $\mathrm{Deg}\,p \leq \mathrm{Deg}\,q$.

(E) $\mathrm{Deg}\,p = 0$ *iff p is algebraic (i.e., p is realized by a finite number of elements).*

(F) *If* $\alpha = \mathrm{Deg}\,p < \infty$, $n < \omega$, φ *is a formula* $\bar{a}^l$, $l < |T|^+$ *sequences and* *for every* $w \subset |T|^+$, $|w| = n$, $\{\varphi(\bar{x}, \bar{a}^l) : l \in w\}$ *is inconsistent, then for at least one* l, $\mathrm{Deg}[p \cup \{\varphi(\bar{x}, \bar{a}^l)\}] < \alpha$ *(or* $p \cup \{\varphi(x, \bar{a}^l)\}$ *is inconsistent.*

Proof. Immediate.

Definition 6.4. A type p splits strongly over A, if there is an indiscernible set over A, $\{\bar{a}^k : k < \omega\}$ such that for some φ, $\varphi(\bar{x}, \bar{a}^0) \in p$, $\neg\varphi(\bar{x}, \bar{a}^1) \in p$.

Lemma 6.3. *If p splits strongly over A,* $\mathrm{Deg}(p|A) < \infty$ *then* $\mathrm{Deg}\, p < \mathrm{Deg}(p|A)$; *provided that T has not the independence p.*

Proof. Let $q = p|A$, and let us define $\bar{b}^n = \bar{a}^{2n} \frown \bar{a}^{2n+1}$, and $\bar{b}^\alpha$ $\omega \leq \alpha < |T|^+$, such that $\{\bar{b}^\alpha : \alpha < |T|^+\}$ is an indiscernible set over A (clearly $\{\bar{b}^n : n < \omega\}$ is such a set, so by 5.4A it is possible). Let $\Psi(\bar{x}, \bar{y}^0, \bar{y}^1) = \varphi(\bar{x}, \bar{y}^0) \wedge \neg\varphi(\bar{x}, \bar{y}^1)$. By 5.9 there is $n < \omega$ such that for every $w \subset |T|^+$, $|w| = n$, $\{\Psi(x, \bar{b}^\alpha) : \alpha \in w\}$ is inconsistent. So by 6.2F, for some α $\mathrm{Deg}[q \cup \{\Psi(\bar{x}, b^\alpha)\}] < \mathrm{Deg}\, q$. As $\bar{b}^\alpha$, $\bar{b}^0$ realize the same type over A, by 6.1A, also $\mathrm{Deg}[q \cup \{\Psi(\bar{x}, \bar{b}^0)\}] < \mathrm{Deg}\, q$. By 6.2D $\mathrm{Deg}[q \cup \{\varphi(\bar{x}, \bar{a}^0), \neg\varphi(\bar{x}, a^1)\}] < \mathrm{Deg}\, q$. As $q \cup \{\varphi(\bar{x}, \bar{a}^0), \neg\varphi(x, \bar{a}^1)\}] \subset p$, by 6.2C $\mathrm{Deg}\, p < \mathrm{Deg}\, q = \mathrm{Deg}(p|A)$. So we prove the lemma.

Remark. Similarly if p is a type over A, $\bar{a}^n$, $n < \omega$ realizes the same type over A, and for some $r < \omega$, for every $w \subset \omega$, $|w| \geq r$, $\{\varphi(\bar{x}, \bar{a}^n) : n \in w\}$ is inconsistent, *then* $\mathrm{Deg}\, p > \mathrm{Deg}[p \cup \{\varphi(\bar{x}, \bar{a}^0)\}]$. (Assuming, of course, $\mathrm{Deg}\, p < \infty$.)

Theorem 6.4. *Let p be a finite m-type, then the following conditions are equivalent:*

(A) $\mathrm{Deg}\, p \geq |T|^+$

(B) *There are formulas* $\varphi_n(\bar{x}; \bar{y}^n)$, *natural numbers* r_n *for* $0 < n < \omega$ *and sequences* $\bar{a}_\eta$ *for* $\eta \in {}^{\omega>}(|T|^+)$ *such that*

(i) *for every* $\eta \in {}^{\omega}(|T|^+)$, $p_\eta = p \cup \{\varphi_n(\bar{x}, \bar{a}_{\eta|n}) : 0 < n < \omega$ *is consistent.*

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(ii) *for every* $\eta \in {}^n(|T|^+)$, $n < \omega$, $w \subset |T|^+$, $|w| \geq r_{n+1}$; $\{\varphi_{n+1}(\bar{x}; \bar{a}_{\eta\frown\langle l\rangle}) : l \in w\}$ *is inconsistent.*

(C) *The same as* (B) *with* $r_n = 2$ *for every n; and for every* $n > 0$, $\eta \in {}^{n+1}(|T|^+)$

$$\vDash (\forall \bar{x}) [\varphi_{n+1}(\bar{x}, \bar{a}_\eta) \to \varphi_n(\bar{x}, \bar{a}_{\eta|n})].$$

(D) $\mathrm{Deg}\, p = \infty$.

Proof. We shall prove A → B, B → C, C → D, and D → A, and this clearly is sufficient.

Remark. If we are interested only in the equivalence of A, B, D, then the proof of B → C can be skipped, as from the proof of C → D, it is clear that B → D. Indeed, C is needed mainly because of esthetic reasons.

Proof of A → B. Let us say that a type q satisfies $\langle \alpha; \varphi_1, r_1; ...; \varphi_k, r_k \rangle$ $(k < \omega)$ by $\bar{a}_\tau$ $\tau \in {}^{k\geq}(|T|^+)$ if:

(α) the degree of $q_\eta = q \cup \{\varphi_n(\bar{x}, \bar{a}_{\eta|n}) : 0 < n \leq k\}$ is $\geq \alpha$ for every $\eta \in {}^k(|T|^+)$.

(β) for every $\eta \in {}^n(|T|^+)$, $n < k$, $w \subset |T|^+$, $|w| \geq r_{n+1}$ the set $\{\varphi_{n+1}(\bar{x}; \bar{a}_{\eta\frown\langle l\rangle}) : l \in w\}$ is inconsistent.

We shall now define by induction on $n > 0$ φ_n, r_n such that for every $\alpha < |T|^+$, p satisfies $\langle \alpha; \varphi_1, r_1; ...; \varphi_n, r_n \rangle$. Clearly from this, B follows.

For $n = 0$ clearly the induction hypothesis holds. Suppose we have defined $\varphi_1, r_1, ..., \varphi_n, r_n$, such that p satisfies $\langle \alpha; \varphi_1, r_1; ...; \varphi_n, r_n \rangle$ by $\bar{a}^\alpha_\tau$ $\tau \in {}^{n\geq}(|T|^+)$ for every $\alpha < |T|^+$ and we shall define φ_{n+1}, r_{n+1} so that p satisfies $\langle \alpha; \varphi_1, r_1; ...; \varphi_{n+1}, r_{n+1} \rangle$ for every $\alpha < |T|^+$; and so prove the induction, hence prove A → B.

Let $q^\alpha_\eta = p \cup \{\varphi_k(\bar{x}, \bar{a}^\alpha_{\eta|k}) : 0 < k \leq l(\eta)\}$.

Now we shall prove by induction on i, $0 \leq i \leq n$ that:

(*) for every $\alpha < |T|^+$, $\eta \in {}^{n-i}(|T|^+)$, there are $\varphi^\alpha_\eta(\bar{x}, \bar{y})$, $r^\alpha_\eta < \omega$ such that $q^{\alpha+1}_\eta$ satisfies $\langle \alpha; \varphi_{n-i+1}, r_{n-i+1}; ...; \varphi_n, r_n; \varphi^\alpha_\eta, r^\alpha_\eta \rangle$ (if $i = 0$, this is $\langle \alpha; \varphi^\alpha_\eta, r^\alpha_\eta \rangle$).

For $i = 0$, this follows from Definition 6.1, (3). (remember p is finite, and by Lemma 6.2A there is in fact no differences between finite types and formulas.)

Suppose (*) holds for $i < n$, and we shall prove it for $i + 1$. Let $\eta \in {}^{n-i-1}(|T|^+)$. The number of possible pairs $\langle \varphi, r \rangle$, φ a formula of $L(T)$, $r < \omega$ is $\leq |T| \aleph_0 = |T|$. Hence there are φ^α_η, r^α_η such that $|\{l < |T|^+ : \varphi^\alpha_{\eta\frown\langle l\rangle} = \varphi^\alpha_\eta, r^\alpha_{\eta\frown\langle l\rangle} = r^\alpha_\eta\}| = |T|^+$. By renaming we can assume that for every $l < |T|^+$, $\varphi^\alpha_{\eta\frown\langle l\rangle} = \varphi^\alpha_\eta$, $r^\alpha_{\eta\frown\langle l\rangle} = r^\alpha_\eta$.

Now for every $l < |T|^+$, $q^{\alpha+1}_{\eta\frown\langle l\rangle}$ satisfies $\langle \alpha; \varphi_{n-i+1}, r_{n-i+1}; \ldots; \varphi_n, r_n; \varphi^\alpha_{\eta\frown\langle l\rangle}, r^\alpha_{\eta\frown\langle l\rangle} \rangle$ by some $\bar{a}\,(l, \tau)$ $\tau \in {}^{i+1\geq}(|T|^+)$. Now clearly $q^{\alpha+1}_{\eta}$ satisfies $\langle \alpha; \varphi_{n-i}, r_{n-i}; \ldots; \varphi_n, r_n; \varphi^\alpha_\eta, r^\alpha_\eta \rangle$ by $\bar{a}_\tau$ $\tau \in {}^{i+2}(|T|^+)$, where $\bar{a}_\tau = \bar{a}\,[\tau(0), \langle \tau(1), \ldots, \tau(i+1)\rangle]$. So we prove (*).

Now the number of pairs $\langle \varphi, r \rangle$ is $\leq |T|$, so there are φ_{n+1}, r_{n+1} such that for $|T|^+$ α's $\varphi^\alpha_{\langle\,\rangle} = \varphi_{n+1}$, $r^\alpha_{\langle\,\rangle} = r_{n+1}$. As $\alpha < \beta$, implies that: if p satisfies $\langle \beta; \varphi_1, r_1; \ldots; \varphi_{n+1}, r_{n+1} \rangle$ then p satisfies $\langle \alpha; \varphi_1, r_1; \ldots; \varphi_{n+1}, r_{n+1} \rangle$; clearly we end the proof of the induction setp on n. So we prove A $\rightarrow$ B.

Proof of B $\rightarrow$ C. We shall define by induction on $k < \omega$ formulas φ^k_n and natural numbers $r^k_n \geq 2$ such that:

(1) for every k, they satisfy the conditions mentioned in B.

Let $f(k) = \min\{n : r^k_n > 2\}$.

(2) for every k, if $f(k) < \omega$ then $f(k+1) > f(k)$ or $l = f(k) = f(k+1)$, and $r^k_l > r^{k+1}_l$.

(3) if $n < f(k)$ then $\varphi^k_n = \varphi^{k+1}_n$.

Clearly if we succeed in defining them, and define φ'_n as φ^k_n for large enough k ($f(k) > n$) then clearly C is satisfied by $\varphi_n = \bigwedge_{i \leq n} \varphi'_i(\bar{x}; \bar{y}^i)$. It is also clear that for $k = 0$ there are such φ^0_n, r^0_n (by B). So it suffices to prove, that if φ^l_n, r^l_n are defined for $l \leq k$, $0 < n < \omega$, then we can define φ^{k+1}_n, r^{k+1}_n, $0 < n < \omega$.

Now we can assume there are $\bar{a}_\eta$ $\eta \in {}^{\omega>}(|T|^+)$ such that (B) is satisfied by φ^k_n, r^k_n $(0 < n < \omega)$ $\bar{a}_\eta$; and

(iii) if $\eta \in {}^{\omega>}(|T|^+)$, $i < \omega$, $l_1 \leq l_2 \leq \ldots \leq l_i < |T|^+$, $j_1 \leq j_2 \leq \ldots \leq j_i < |T|^+$ (where $l_\alpha = l_{\alpha+1} \Leftrightarrow j_\alpha = j_{\alpha+1}$ for $\alpha = 1, \ldots, i-1$) and $\tau_1, \ldots, \tau_i \in {}^{\omega>}(|T|^+)$ then the two sequences

$$\bar{a}_{\eta\frown\langle l_1\rangle\frown\tau_1} \quad \ldots \quad \bar{a}_{\eta\frown\langle l_i\rangle\frown\tau_i}$$

$$\bar{a}_{\eta\frown\langle j_1\rangle\frown\tau_1} \quad \ldots \quad \bar{a}_{\eta\frown\langle j_i\rangle\frown\tau_i}$$

realize the same type over

$$A_\eta = \bigcup\{\text{Rang}\, \bar a_\tau : l(\tau) \le l(\eta) \text{ or } \tau | l(\eta) \ne \eta; \tau \in {}^{\omega>}(|T|^+)\}$$

Remark. (1) We can choose the τ's as void sequences. Hence in particular, $\langle \bar a_{\eta^\frown\langle l\rangle} : l < |T|^+\rangle$ is an indiscernible sequence over A_η.

(2) We can assume that (iii) holds, by using Ramsay's theorem, or using the following theorem which is proved in Shelah [H]. The theory is combinatorical, in fact, generalizing Erdös, Hajnal and Rado [B].

Theorem. *For every $n, j < \omega$ there is $g = g(n, j) < \omega$ (an explicit expression can be obtained) such that: Let N be a model, with language L_1, $|L_1| \le \beth_\alpha < \beth_{\alpha+g} = \lambda$, and for every $\tau \in {}^{n>}\lambda$ there is a sequence $\bar b_\tau \in |N|$. Then there is a function $f : {}^{n>}(\beth_\alpha) \to {}^{n>}\lambda$ such that: (α) $f(\langle\rangle) = \langle\rangle$ (β) $l(\eta) = l[f(\eta)]$ (γ) $\tau = \eta | r$ iff $f(\tau) = f(\eta)|r$; (δ) if we define $\bar a_\tau = \bar b_{f(\tau)}$, then* (iii) *is satisfied, if we replaced $|T|^+$ by $\beth_\alpha$, and restrict ourselves to sequence (of ordinal) of length $< n$, and to $i \le j$.*

So we have φ_n^k, r_n^k $(0 < n < \omega)$ and $\bar a_\eta$, $\eta \in {}^{\omega>}(|T|^+)$ such that (i) and (ii) from (B) hold, and also (iii). We should define $\varphi_n^{k+1}, r_n^{k+1}$.

If $f(k) = \omega$, clearly C holds. So let $f(k) < \omega$ and $\eta \in {}^{f(k)-1}(|T|^+)$, and let 1_n be a sequence of n *ones.* Suppose first that

$$p^* = p \cup \{\varphi_n^k(\bar x; \bar a_{\eta|n}) : n < f(k)\} \cup \{\varphi_{f(k)+n}^k(\bar x, \bar a_{\eta^\frown 1_n}) : n < \omega\}$$

$$\cup \{\varphi_{f(k)}^k(\bar x, \bar a_{\eta^\frown\langle 0\rangle})\}$$

is consistent. Then define

$$\varphi_n^{k+1} = \varphi_n^k \quad \text{for} \quad n \ne f(k)$$

$$\varphi_{f(k)}^{k+1} = \varphi_{f(k)}^{k+1}(\bar x; \bar y^1, \bar y^2) = \varphi_{f(k)}^k(\bar x; \bar y^1) \wedge \varphi_{f(k)}^k(\bar x; \bar y^2)$$

$$r_n^{k+1} = r_n^k \quad \text{for} \quad n \ne f(k)$$

$$r_{f(k)}^{k+1} = r_{f(k)}^k - 1 \text{ (or, in fact, } (r_{f(k)}^k + 1)/2)$$

Clearly this definition satisfies our demands.

So suppose that p^* is inconsistent. Hence it has an inconsistent finite subtype, which we can assume is

$$p^1 = p \cup \{\varphi_n(\bar{x};\bar{a}_{\eta \restriction n}) : n < f(k)\} \cup \{\varphi_{f(k)+n}(\bar{x};\bar{a}_{\eta\ 1_n}) : n < n_0\}$$
$$\cup \{\varphi_{f(k)}(\bar{x};\bar{a}_{\eta\ \langle 0 \rangle})\}.$$

Let us define

$$\varphi_n^{k+1} = \varphi_n^k \quad \text{for } n < f(k)$$

$$\varphi_{f(k)}^{k+1} = \varphi_{f(k)}^{k+1}(\bar{x};\bar{y}^0, \ldots, \bar{y}^{n_0-1}) = \bigwedge_{i<f(k)+n_0} \varphi_i^k(\bar{x},\bar{y}^i) \wedge \bigwedge p$$

$$\varphi_n^{k+1} = \varphi_{n+n_0}^k \quad \text{for } n > f(k)$$

$$r_n^{k+1} = r_n^k \quad \text{for } n < f(k)$$

$$r_{f(k)}^{k+1} = 2$$

$$r_n^{k+1} = r_{n+n_0}^k \quad \text{for } n > f(k)$$

Clearly this definition satisfies our demands. So we end the definition, hence the proof of B → C.

Proof of C → D. Suppose $\operatorname{Deg} p < \infty$. Let $\varphi_n, r_n, \bar{a}_\eta$ be as in C, and $p_\eta = p \cup \{\varphi_n(\bar{x}, \bar{a}_{\eta \restriction n}) : 0 < n < l(\eta)\}$ for every $\eta \in {}^{\omega>}(|T|^+)$. Among the p_η there is one with minimal degree q_τ, $\operatorname{Deg} q_\tau = \alpha$. Hence for every $l < |T|^+$, $\operatorname{Deg}(q_{\tau\ \langle l \rangle}) \geq \operatorname{Deg} q_\tau = \alpha$. But as every subset of $\{\varphi_{l(\tau)+1}(\bar{x}, \bar{a}_{\tau\ \langle l \rangle}) : l < |T|^+\}$ with at least two elements is inconsistent; we get a contradiction by Lemma 6.2F.

So $\operatorname{Deg} p = \infty$.

Proof of D → A. As $|T|^+ < \infty$, $\operatorname{Deg} p = \infty$ implies $\operatorname{Deg} p \geq |T|^+$.

Theorem 6.5. *If for some m-type p,* $\operatorname{Deg} p = \infty$, *then T is unstable in every* λ *for which* $\lambda^{\aleph_0} > \lambda$; *hence T is not superstable.*

Remark. T is superstable iff it is stable in every $\lambda \geq 2^{|T|}$.

Proof. By Theorem 6.4, D, C and the compactness theorem, there are $\varphi_n(\bar{x}, \bar{y}^n)$ $0 < n < \omega$, and $\bar{a}_\eta$, $\eta \in {}^{\omega>}\lambda$ such that

(i) for every $\eta \in {}^{\omega}\lambda$, $p_\eta = p \cup \{\varphi_n(\bar{x}, \bar{a}_{\eta|n}) : 0 < n < \omega\}$ is consistent.

(ii) for every $\eta \in {}^{r-1}\lambda$, $k \neq l < \lambda$, $\{\varphi_r(\bar{x}; \bar{a}_{\eta\frown\langle l\rangle}), \varphi_r(\bar{x}; \bar{a}_{\eta\frown\langle k\rangle})\}$ is inconsistent.

For every $\eta \in {}^{\omega}\lambda$ let $\bar{c}_\eta$ realize p_η, let $A = \bigcup\{\operatorname{Rang} \bar{a}_\eta : \eta \in {}^{\omega>}\lambda\}$ and q_η be the type $\bar{c}_\eta$ realize over A. Clearly

$$|S^m(A)| \geq |\{q_\eta : \eta \in {}^{\omega}\lambda\}| = \lambda^{\aleph_0},$$

$$|A| \leq \lambda \cdot \aleph_0 = \lambda .$$

By Lemma 2.10 the theorem is proved.

Theorem 6.6. *For any m-type p on A, there is an extension $q \in S^m(A)$ of the same degree.*

Proof. Let us first prove that
(*) for every $\varphi(\bar{x}; \bar{a})$, $\alpha_1 = \alpha$ or $\alpha_2 = \alpha$ where $\alpha = \operatorname{Deg} p$,
$\alpha_1 = \operatorname{Deg}[p \cup \{\varphi(\bar{x}; \bar{a})\}]$ $\alpha_2 = \operatorname{Deg}[p \cup \{\neg\varphi(\bar{x}, \bar{a})\}]$.

By 6.2 there is a finite $p_1 \subset p$ such that

$$\alpha = \operatorname{Deg} p_1 \quad \alpha_1 = \operatorname{Deg}[p_1 \cup \{\varphi(\bar{x}; \bar{a})\}] \quad \alpha_2 = \operatorname{Deg}[p_1 \cup \{\neg\varphi(\bar{x}, \bar{a})\}]$$

and denoting $\Psi(\bar{x}, \bar{b}) = \bigwedge_{\varphi \in p_1} \varphi$, we get

$$\alpha = \operatorname{Deg} \Psi(\bar{x}, \bar{b}),\ \alpha_1 = \operatorname{Deg}[\Psi(\bar{x}, \bar{b}) \wedge \varphi(\bar{x}; \bar{a})],$$

$$\alpha_2 = \operatorname{Deg}[\Psi(\bar{x}; \bar{b}) \wedge \neg\varphi(\bar{x}; \bar{a})]$$

So it suffices to prove by induction on α, that
(**) if $\operatorname{Deg} \Psi(\bar{x}, \bar{b}) \geq \alpha$, then $\operatorname{Deg}[\Psi(\bar{x}, \bar{b}) \wedge \varphi(\bar{x}; \bar{a})] \geq \alpha$ or $\operatorname{Deg}[\Psi(\bar{x}; \bar{b}) \wedge \neg\varphi(\bar{x}; \bar{a})] \geq \alpha$ [as always $\operatorname{Deg}[\Psi(\bar{x}, \bar{b}) \wedge \theta(\bar{x}, \bar{c})] \leq \operatorname{Deg} \Psi(\bar{x}, \bar{b})$ by 6.1A].

Let $\alpha = 0$, and so by the definition there is $\bar{c}$ such that $\models \Psi[\bar{c};\bar{b}]$. So either $\models \varphi[\bar{c};\bar{a}]$ or $\models \neg\varphi[\bar{c},\bar{a}]$. In the first case $\models (\exists\bar{x})[\Psi(\bar{x};\bar{b}) \wedge \varphi(\bar{x};\bar{a})]$, hence $\alpha_1 = \mathrm{Deg}[\Psi(\bar{x};\bar{b}) \wedge \varphi(\bar{x};\bar{a})] \geq 0 = \alpha$ (by Definition 6.1), and similarly if $\models \neg\varphi[\bar{c},\bar{a}]$ then $\alpha_2 \geq \alpha$.

So suppose we have proved (**) for α, and we shall prove it for $\alpha + 1$. As $\mathrm{Deg}\,\Psi(\bar{x},\bar{b}) \geq \alpha + 1 > \alpha$, there are $n < \omega$, $\theta(\bar{x},\bar{c}^{\alpha})\ \alpha < |T|^+$, such that:

(i) for every $\alpha < |T|^+$, $\mathrm{Deg}[\Psi(\bar{x},\bar{b}) \wedge \theta(\bar{x},\bar{c}^{\alpha})] \geq \alpha$

(ii) for every $w \subset |T|^+$, $|w| \geq n$, $\{\theta(\bar{x};\bar{c}^{\alpha}) : \alpha \in w\}$ is inconsistent.

By the industion hypothesis, for every $\beta < |T|^+$ there is $\delta(\beta) \in \{0, 1\}$ such that $\mathrm{Deg}[\Psi(\bar{x},\bar{b}) \wedge \theta(\bar{x},\bar{c}^{\beta}) \wedge \varphi(\bar{x},\bar{a})^{\delta(\beta)}] \geq \alpha$. Hence for $|T|^+$ β's $\delta(\beta) = 0$, or for $|T|^+$ β's $\delta(\beta) = 1$. So by renaming we can assume for every β $\delta(\beta) = \delta(0)$. So by Definition 6.1

$$\mathrm{Deg}[\Psi(\bar{x};\bar{b}) \wedge \varphi(\bar{x};\bar{a})^{\delta(0)}] \geq \alpha + 1.$$

So remains the case α is a limit ordinal. But if (**) fails, $\alpha_3 = \max(\alpha_1,\alpha_2) < \alpha$, hence

$$\mathrm{Deg}[\Psi(\bar{x},\bar{b})] \geq \alpha_3 + 1,\ \mathrm{Deg}[\Psi(\bar{x};\bar{b}) \wedge \varphi(\bar{x};\bar{a})] =$$

$$\alpha_1 < \alpha_3 + 1,\ \mathrm{Deg}[\Psi(\bar{x};\bar{b}) \wedge \neg\varphi(\bar{x};\bar{a})] \leq \alpha_2 < \alpha_3 + 1.$$

Contradiction to the induction hypothesis.

So we prove (**) hence (*). Now let $\{\varphi_i(\bar{x},\bar{a}^i) : i < i_0\}$ be the list of all formulas with parameters from A, and $\bar{x}$ as variables. We can define by induction $\delta(i) \in \{0, 1\}$ for $i < i_0$, such that $\mathrm{Deg}\,p_{i+1} = \mathrm{Deg}\,p_i$ where

$$p_i = p \cup \{\varphi_j(\bar{x},\bar{a}^j)^{\delta(j)} : j < i\}.$$

By 6.2B $\mathrm{Deg}\,p = \mathrm{Deg}\,p_{i_0}$, and clearly $p \subset p_{i_0} \in S^m(A)$.

Theorem 6.7. *Suppose T is stable, p an m-type and $\alpha_0 = \mathrm{Deg}\,p < \infty$. Then there are no φ, and φ-m-types p_k $k < |T|^+$, contradictory in pairs, such that $\alpha_0 = \mathrm{Deg}(p \cup p_k)$ for every $k < |T|^+$.*

Proof. Suppose Theorem 6.7 fails, and $\{p_k : k < |T|^+\}$ is such a set of types. Let A be a set such that p, p_k are types on A. By Theorem 6.6 we can assume $p_k \in S^m_\varphi(A)$ (by replacing p_k by a suitable extension). By Theorem 2.5 for every k there is a finite $q^k \subset p_k$, such that $\mathrm{Rank}^m_\varphi(q^k) = \mathrm{Rank}^m_\varphi(p_k)$; and as $p \subset p \cup q^k \subset p \cup p_k$, clearly $\alpha_0 = \mathrm{Deg}(p \cup q_k)$. As there are $|T|^+ > \aleph_0$ q^k's, we can, by dropping and remaining assume that for every $k < |T|^+$, $|q^k| = n_0$, $q^k = \{\varphi(\bar{x}, \bar{a}^{k,i})^{\eta(i)} : i < n_0\}$, and $\mathrm{Rank}^m_\varphi(q^k) = n_1$. So let $\Psi(\bar{x}; \bar{c}^k) = \bigwedge\{\Psi : \Psi \in q^k\}$. So clearly $\mathrm{Deg}[p \cup \{\Psi(\bar{x}; \bar{c}^k)\}] = \alpha_0$. Now let $\bar{b}^k$ realize $p \cup p_k$. Then clearly it does not realize q^l for $l \neq k$; for otherwise

$$q^l \subset p_k, q^l \subset p_l, n_1 = \mathrm{Rank}^m_\varphi q^l = \mathrm{Rank}^m_\varphi p_k = \mathrm{Rank}^m_\varphi p_l$$

contradiction by 2.5. So for every k, $p \cup \{\Psi(\bar{x}; \bar{c}^l)^{\mathrm{if}\,(l=k)} : l < |T|^+\}$ is consistent, and of degree α_0, by 6.2D.

As T is stable, it has not the independence p (by 4.1) so there is $n^2 < \omega$ such that: there are no $\bar{b}^0, \ldots, b^{n^2-1}$ such that for every $w \subset n^2$, $\{\Psi(\bar{x}; \bar{b}^n)^{\mathrm{if}\,(n \in w)} : n < n^2\}$ is consistent.

Now by Theorem 5.8 we can assume $\{\bar{c}^k : k < |T|^+\}$ is Δ-$2n^2$-indiscernible set where

$$\Delta = \left\{(\exists \bar{x}) \bigwedge_{n < l(\eta)} \Psi(\bar{x}, \bar{y}^n)^{\eta(n)} : \eta \in {}^{2n^2}2\right\}.$$

Define for $k < |T|^+$

$$\Psi^*(\bar{x}; \bar{a}^k) = \Psi(\bar{x}; \bar{c}^{k+n^2}) \wedge \bigwedge_{i<n^2} \neg\Psi(\bar{x}; \bar{c}^i).$$

Clearly by 6.2D $\alpha_0 = \mathrm{Deg}[p \cup \{\Psi^*(x; \bar{a}^k)\}]$ On the other hand by the definition of n^2 and the indiscernibility of $\{\bar{c}^k : k < |T|^+\}$, every set of n^2 $\Psi^*(\bar{x}; \bar{a}^k)$'s is inconsistent. This is a contradiction by 6.2F. So we prove the theorem.

Definition 6.5. A m-type p is fixed, if for every formula $\varphi(\bar{x}, \bar{a})$, either

$$\mathrm{Deg}[p \cup \{\varphi(\bar{x}, \bar{a})\}] = \mathrm{Deg}\, p \text{ or } \mathrm{Deg}[p \cup \{\neg\varphi(\bar{x}, \bar{a})\}] = \mathrm{Deg}\, p.$$

but not both. Hence if p is a fixed type on A, it has a unique extension in $S^m(A)$ of the same degree.

Corollary 6.8. *Let T be superstable.* (A) *For every m-type p there is a set A, $|A| \leq |T|$ such that if $q \in S^m(A)$,* $\operatorname{Deg} p = \operatorname{Deg} p \cup q$, *then q is fixed.*

(B) *every m-type over A has $\leq 2^{|T|}$ extensions in $S^m(A)$ of the same degree.*

(C) *If p is an m-type over C, $A = \{a_i : i < |T|\}$ is as in* (A), *$A_1 = \{a^i : i < |T|\}$, and for every $i_1 < \ldots < i_n < |T|$, $\langle a_{i_1}, \ldots, a_{i_n} \rangle$, $\langle a^{i_1}, \ldots, a^{i_n} \rangle$ realizes the same type over C then we can replace A by A_1.*

(D) *If in* (A) *p is a type over M, M is $|T|$-saturated, then we can choose $A \subset |M|$.*

Proof. Immediate.

Theorem 6.9. *If* $\operatorname{Deg}^1(x = x) < \infty$, *$T$ is stable, then T is superstable, i.e. for every A, $|S(A)| \leq |A| + 2^{|T|}$.*

Proof. By Lemma 6.1, every $\varphi(x; \bar{a})$ has degree $< \infty$, hence every 1-type has degree $< \infty$. Let A be a set. By Lemma 6.2 every type $p \in S(A)$ has a finite subtype $q[p]$ of the same degree. Clearly the number of finite types on A is $\leq |A| + |T|$ and by Corollary 6.8B, for every finite q $|\{p : p \in S(A), q[p] = q\}| \leq 2^{|T|}$. Hence

$$|S(A)| = \sum_q |\{p : p \in S(A), q[p] = q\}| \leq (|A| + |T|) 2^{|T|} =$$

$$|A| + 2^{|T|}.$$

Corollary 6.10. *The following conditions are equivalent; if T is stable:*

(A) $\operatorname{Deg}^1(x = x) < \infty$

(B) *every m-type has degree $< |T|^+$ (hence $< \alpha_0 < |T|^+$)*

(C) *T is superstable, i.e. stable in every $\lambda \geq 2^{|T|}$*

(D) *T is stable in one λ, $\lambda^{\aleph_0} > \lambda \geq 2^{|T|}$*

(E) *there are no $\varphi_n(x; \bar{y}^n)$ $0 < n < \omega$; $\bar{a}_\tau$ $\tau \in {}^{\omega>}\omega$ such that:*

(i) *for every $\eta \in {}^{\omega}\omega$, $\{\varphi_n(x; \bar{a}_{\eta|n}) : 0 < n < \omega\}$ is consistent.*

(ii) *for every $\tau \in {}^{\omega>}\omega$, $k < l < \omega$; $\varphi_n(x, \bar{a}_{\tau\frown\langle k\rangle})$, $\varphi_n(x, \bar{a}_{\tau\frown\langle l\rangle})$ are contradictory where $n = l(\tau) + 1$.*

Proof. A $\to$ C by 6.9, C $\to$ D trivially; by 6.5 D implies B (noticing that by 6.4, $\mathrm{Deg}^m p \geq |T|^+$ implies $\mathrm{Deg}^m p = \infty$) and trivially B $\to$ A. A $\leftrightarrow$ E by 6.4.

Remark. It can be easily proven that if T has the strict order property, then $\mathrm{Deg}^1 (x = x) = \infty$ (like the proof of C $\to$ D in 6.4).

Lemma 6.11. *Suppose T is superstable, p is an m-type over A and p is fixed. If $\bar{a}$, $\bar{b}$ realize the same type over A, $\bar{a}$, $\bar{b} \in B$, $A \subset B$, $p \subset q \in S^m(B)$,* $\mathrm{Deg}\, p = \mathrm{Deg}\, q$ *and φ is a formula then $\varphi(\bar{x}; \bar{a}) \in q \iff \varphi(\bar{x}, \bar{b}) \in q$.*

Proof. By Lemma 6.1B, 6.2A.

Theorem 6.12. *Suppose p_i is the m-type $\bar{a}^i$ realizes over $A_i = \cup\{$*Rang *$\bar{a}^j : j < i\} \cup A$ for every $i < \alpha \geq |T|^+$ and $i < j < \alpha$ implies $p_i \subset p_j$. Then, assuming T is superstable:*

(A) *If p_0 is fixed, and* $\mathrm{Deg}\, p_i = \mathrm{Deg}\, p_0$ *for $i < \alpha$, then $\{\bar{a}_i : i < \alpha\}$ is an indiscernible set over A.*

(B) *if for every $i < \alpha$,* $\mathrm{Deg}\, p_i = \mathrm{Deg}\, p_0$, *then there is $\beta < |T|^+$ such that $\{\bar{a}_i : \beta \leq i < \alpha\}$ is an indiscernible set over $A \cup \{$*Rang *$\bar{a}_i : i < \beta\}$.*

Proof. (A) The proof is as the proof of 5.7, using 6.11.

(B) By 5.17 there is $\beta < |T|^+$, such that $\{\bar{a}_i : \beta \leq i < |T|^+\}$ is an indiscernible set over A_γ. If γ is the first such that $\{\bar{a}_i : \beta \leq i \leq \gamma\}$ is not an indiscernible set over A_β, then p_γ splits strongly over A_β (see Definition 6.4) contradiction by 6.3. [In fact in (B) we can take $\beta \leq \omega$.]

Theorem 6.13. (A) *Suppose T is superstable, $\{\bar{a}^i : i \in I\}$ an indiscernible set over A, $\bar{c}$ a sequence. Then there is a finite set $I_1 \subset I$ such that $\{\bar{a}^i : i \in I - I_1\}$ is an indiscernible set over $A \cup \mathbf{U}\{$*Rang *$\bar{a}^i : i \in I_1\} \cup$* Rang *$\bar{c}$.*

(B) *In* (A) *we can replace superstability by stability, and $|I_1| < \aleph_0$ by $|I_1| < \kappa(T)$.*

Remark. See Remark to 5.9.

Proof. For any $J \subset I$ let $B(J) = A \cup \bigcup\{\operatorname{Rang} \bar{a}^i : i \in J\}$. By Lemma 6.2 there is finite $I_1 \subset I$, such that the type p that $\bar{c}$ realizes over $B(I)$ satisfies $\operatorname{Deg}[p \upharpoonright B(I_1)] = \operatorname{Deg} p$. By Lemma 6.3 p does not split strongly over $B(I_1)$. If $\{\bar{a}^i : i \in I - I_1\}$ is not an indiscernible set, over $B(I_1) \cup \operatorname{Rang} \bar{c}$ there are different $s_1, ..., s_n \in I - I_1$, and different $t_1, ..., t_n \in I - I_1$, such that for some φ

$$(*) \qquad \models \varphi[\bar{a}^{s_1}, ..., \bar{a}^{s_n}; \bar{c}] \qquad \models \neg\varphi[\bar{a}^{t_1}, ..., \bar{a}^{t_n}; \bar{c}].$$

Without loss of generality $\{s_1, ..., s_n\} \cap \{t_1, ..., t_n\} = 0$ (otherwise we take a third set, disjoint to both of them, and replace one of them with it, so that (*) still holds). Now it is easy to see that p splits strongly over $B(I_1)$, contradiction.

Remark. More refined theorems about degrees will appear, including some remarks which were not proven here. See Shelah [N*].

Question. What is the exact relation between the degree here, the rank in Shelah [C] (def. 2.1, p. 75), the strong splitting of Shelah [D] def. 4.1 p. 87 (here 6.3), and the notion suggested naturally from the remark to Lemma 6.3? (See 6.8B, 6.3, and Shelah [N*1).

References

[A] J.T. Baldwin, $\aleph_1$-categorical theories whose models are the closure of strongly minimal sets, Notices of A.M.S. 17 (1970) 834.
[B] J.T. Baldwin and A.H. Lachlan, On strongly minimal sets, J. of Symb. Logic 36 (1971) 79–96.
[C] J.T. Baldwin, α_T is finite for $\aleph_1$-categorical T, A.M.S. Notices 17 (1970) 967.
[D] J.T. Baldwin, Countable theories categorical in uncountable power, Ph. D. thesis, Simon Fraser Univ. 1970.
[E] J.T. Baldwin, A note on definability in totally transcendental theories, A.M.S. Notices 17 (1970) 1087.
[A] J.E. Baumgartner, Results and independence proofs in combinatorial set theory, Ph. D. thesis, Univ. of Calif., Berkeley, 1970.
[B] J.E. Baumgartner, On the cardinality of dense subsets of linear ordering I, A.M.S. Notices 15 (1968) 953.
[A] M. Benda, Reduced filters and Boolean ultrapowers, Ph. D. thesis, Univ. of Wisconsin, 1970, appearing in this Journal.
[A] L. Blum, Generalized algebraic structures: model theoretic approach, Ph. D. thesis, M.I.T. 1968.
[A] C.C. Chang, Some remarks on the model theory of infinitary languages. Lecture notes in Mathematics No. 72, The Syntax and Semantics of Infinitary languages (Springer, Berlin, 1968) 36–64.
[B] C.C. Chang and H.J. Keisler, Theory of Models, to appear, North-Holland Publ. Co.
[A] A. Ehrenfeucht, On theories categorical in power, Fundamenta Mathematica, 44 (1957) 241–248.
[B] E. Ehrenfeucht, Theories having at least continuum many non-isomorphic models in each infinite power, A.M.S. Notices 5 (1968) 680.
[C] A. Ehrenfeucht and A. Mostowski, Models of axiomatic theories admitting automorphism, Fundamenta Mathematica, 43 (1956) 50–68.
[A] E. Engeler, A characterization of theories with isomorphic denumerable models, Notices of A.M.S. 6 (1959) 161.
[A] P. Erdös and M. Makkai, Some remarks on set theory X, Studia Scientiarum Mathematirum Hungarica 1 (1966) 157–159.
[B] P. Erdös, A. Hajnal, and R. Rado, Partition relations for cardinal numbers, Acta Math. 16 (1965) 93–196.
[A] G. Gaifman, Results concerning models of Peano's arithmetic, Notices of A.M.S. 12 (1965) 377.
[A] W. Glassmire, A problem in categoricity, A.M.S. Notices 17 (1970) 295.
[A] V. Harnik, Masters thesis. The Hebrew Univ. Jerusalem, 1966.
[B] V. Harnik and J.P. Ressayre, Prime extensions and categoricity in power, Israel J. Math. 10 (1971)
[C] V. Harnik, Stable theories and related concepts, Ph. D. Thesis, The Hebrew Univ., Jerusalem, 1971.
[B] V. Harnik and J.P. Ressayre, Prime extensions and categoricity in power, Israel J. Math.
[C] V. Harnik, Ph. D. Thesis, The Hebrew Univ., Jerusalem, 1971.
[A] H.J. Keisler, Ultraproducts which are not saturated, J. of Symb. Logic 32 (1967) 23–46.
[B] H.J. Keisler, Some model theoretic results for ω-logic, Israel J. of Math. 4 (1966) 249–261.
[C] H.J. Keisler, Logic with the quantifier "there exist uncountably many", Annals of Math. Logic 1 (1970) 1–94.
[D] H.J. Keisler, Ultraproducts of finite sets, J. of Symb. Logic 32 (1967) 47–57.

[E] H.J. Keisler, Model theory for infinitary Logic (North-Holland, Amsterdam, 1971).
[F] H.J. Keisler and M.D. Morley, On the number of homogeneous models of a given power, Israel J. of Math. 5 (1967) 73–78.
[G] H.J. Keisler, Categorical theories in infinitary logic, Notices of A.M.S. 16 (1969) 437.
[H] H.J. Keisler, On theories categorical in their own power, J. of Symb. Logic.
[A] K. Kunen, Ultrafilter and independent sets, to appear, Trans. of A.M.S. (1971).
[A] A.H. Lachlan and J.T. Baldwin, On countable theories categorical in power $\aleph_1$ but not in power $\aleph_0$, A.M.S. Notices 16 (1969) 1087.
[B] A.H. Lachlan, The transcendental rank of a theory, Pacific J. Math.(1971) 119–122.
[C] A.H. Lachlan, The transcendental rank of a theory, Notices of A.M.S. 17 (1970) 673.
[A] J. Łoś, On the categoricity in power of elementary deductive systems and related problems, Colloq. Math. 3 (1954) 58–62.
[A] W.E. Marsh, On ω_1-categorical but not ω-categorical theories, Ph. D., Dartmouth College, 1966.
[A] W. Mitchell, On the cardinality of dense subsets of linear ordering II, A.M.S. Notices 15 (1968) 935.
[A] M.D. Morley, Categoricity in power, Trans. of A.M.S. 114 (1965) 514–538.
[B] M.D. Morley, Countable models of $\aleph_1$-categorical theories, Israel J. of Math. 5 (1967) 65–72.
[C] M.D. Morley, The number of countable models, J. of Symb. Logic 35 (1970) 14–18.
[D] M.D. Morley and R.L. Vaught, Homogeneous universal models, Mathematica Scandinavica II (1962) 37–57.
[A] F.D. Ramsey, On a problem of formal logic, Proceedings of the London Math. Society 30 (1929) 338–384.
[A] J.P. Ressayre, Sur les theories du premier ordre categorique en un cardinal, Trans. of A.M.S. 142 (1969) 481–505.
[A] F. Rowbottom, The Łoś conjecture for uncountable theories, Notices of A.M.S. 11 (1964) 248.
[A] C. Ryll-Nardzewski, On categoricity in power $\leq \aleph_0$, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 7 (1959) 545–548.
[A] S. Shelah, Categoricity of classes of models, Ph. D. Thesis, The Hebrew Univ., Jerusalem, 1969.
[B] S. Shelah, Stable theories, Israel J. of Math. 7 (1969) 187–202.
[C] S. Shelah, On theories T categorical in $|T|$, J. of Symb. Logic 35 (1970) 73–82.
[D] S. Shelah, Finite diagrams stable in power, Annals of Math. Logic 2 (1970) 69–118.
[E] S. Shelah, The number of non-isomorphic models of an unstable first-order theory, Israel J. of Math. 9 (1971) 473–487.
[F] S. Shelah, A combinatorial problem; stability and order for models and theories in infinitary languages, Pacific J. of Math., (submitted).
[G] S. Shelah, Saturation of ultrapower and Keisler's order, to appear in this Journal.
[H] S. Shelah, The number of non-isomorphic models of an unsuperstable first order theory, in preparation.
[I] S. Shelah, Classes with homogeneous models only, Notices of A.M.S. 15 (1968) 803.
[J] S. Shelah, On generalization of categoricity, Notices of A.M.S. 16 (1969) 683.
[K] S. Shelah, Proof of Łoś conjecture for uncountable theories, Lecture notes by M. Brown, Fall, 1970. To appear in the Proc. of the 1971 Symp. in Berkeley, in Honour of A. Tarski, ed. L. Henkin, published by A.S.L. (1972).
[L] S. Shelah, On the number of non almost-isomorphic models of a theory in a power, Pacific J. of Math. 36 (1971) 813–820.
[M] S. Shelah, Generalizations of saturativity, Notices of A.M.S. 18 (1971) January.
[A] L. Svenonius, $\aleph_0$-categoricity in first-order calculus, Theoria (Lund) 25, 82–94.

[A] R.L. Vaught, Denumerable models of complete theories; Infinitistic methods, Proceedings of the Symposium on foundation of Math., Warsaw 1959 (Pergamon Press, London and PWN, Warsaw, 1961) 303–321.

References added in proof

[A*] C.J. Ash, Undecidable $\aleph_0$-categorical theories, Notices of A.M.S. 19 (1971) 423.
[F*] J.T. Baldwin, α_T is finite for $\aleph_1$-categorical T, Pacific J. of Math. 1971, to appear.
[C*] C.C. Chang, Omitting types of prenex formulas, J. of Symb. Logic, 32 (1971) 61–74.
[D*] A.H. Lachlan, A property of stable theories, Fund. Math., to appear.
[E*] A.H. Lachlan, The number of countable models of countable superstable theory, Proc. of the 1971 Symp. in Bucharest.
[F*] A.H. Lachlan, A note on the strict order property, Pacific J. of Math., to appear.
[I*] H.J. Keisler, Proc. of the Summer School in Logic, Cambridge 1971.
[A*] Macintyre, On ω_1-categorical theories of abelian groups, Fund. Math., to appear.
[A*] Rosental, A new proof of a theorem of Shelah, Proc. of the 1971 Symp. in Berkeley in honour of Tarski, ed. Henkin, published [illegible] A.S.L. (1972).
[N*] S. Shelah, Lecture notes, spring 1971 [illegible]LA, written by R. Gail.
[O*] S. Shelah, Uniqueness and characterization of prime models over sets for totally transcendental first-order theories, J. of Symb. Logic 36 (1971) no. 3.
[P*] S. Shelah, Categoricity in Power, Notices of A.M.S. 15 (1968) 930.