Journal of Applied Analysis
Vol. 10, No. 2 (2004), pp. 169–289

Special invited paper

PROPERNESS WITHOUT ELEMENTARICITY

S. SHELAH

Received December 23, 1997 and, in revised form, May 5, 2003

Abstract. We present reasons for developing a theory of forcing notions which satisfy the properness demand for countable models which are not necessarily elementary sub-models of some $(\mathcal{H}(\chi), \in)$. This leads to forcing notions which are "reasonably" definable. We present two specific properties materializing this intuition: nep (non-elementary properness) and snep (Souslin non-elementary properness) and also the older Souslin proper. For this we consider candidates (countable models to which the definition applies). A major theme here is "preservation by iteration", but we also show a dichotomy: if such forcing notions preserve the positiveness of the set of old reals for some naturally defined c.c.c. ideal, then they preserve the positiveness of any old positive set hence preservation by composition of two follows. Last but not least, we prove that (among such forcing notions) the only one commuting with Cohen is Cohen itself; in other words, any other such forcing notion make the set of old reals to a meager set. In the end we present some open problems in this area.

2000 *Mathematics Subject Classification.* Primary: 03E40; Secondary: 03E05, 03E47.
Key words and phrases. set theory, forcing, iterated forcing, preservation for iterated forcing, nep, snep, Cohen forcing, commuting forcing, Fubini theorem.
I thank Mr Ilmari Hyvönen and Einar Karttunen for making so nicely large revision in the text.
The research was partially supported by the Israeli Science Foundation.
Publication 630.

ISSN 1425-6908

ANNOTATED CONTENT

Section 0: Introduction We present reasons for developing the theory of forcing notions which satisfy the properness demand for countable models which are not necessarily elementary submodels of some $(\mathcal{H}(\chi), \in)$. This will lead us to forcing notions which are "reasonably" definable.

Section 1: Basic definitions We present two specific properties materializing this intuition: nep (non-elementary properness) and snep (Souslin non-elementary properness). For this we consider candidates (countable models to which the definition applies), and we also consider the older Souslin proper. We end by a criterion for being "explicitely nep".

Section 2: Connections between the basic definitions We point out various implications (snep implies nep, etc.). We also point out how much the properties are absolute.

Section 3: There are examples We point out that not just the reasonably definable forcing notions in use fit our framework, but that all the general theorems of Rosłanowski and Shelah [19], which prove properness, actually prove the stronger properties introduced earlier.

Section 4: Preservation under iteration: first round First we address a point we ignored earlier (it was not needed, but is certainly part of our expectations). In the definition of "q is $(N, \mathbb{Q})$-generic" predensity of each $\mathcal{I} \in \mathrm{pd}(N, \mathbb{Q})$ was originally designed to enable us to say things on $N[\underset{\sim}{G}_{\mathbb{Q}}]$, i.e. $N[G_{\mathbb{Q}}] \cap \mathcal{H}(\chi)^{\mathbf{V}} = N$, but we should be careful saying what we intend by $N[G_{\mathbb{Q}}]$ now, so we replace it by $N\langle \underset{\sim}{G}_{\mathbb{Q}} \rangle$. The preservation Theorem 4.8 says that CS iterations of nep forcing notions have the main property of nep. For this we define $p^{\langle\langle N \rangle\rangle}$ if $N \models$" $p \in \mathrm{Lim}(\bar{\mathbb{Q}})$ ". We also define and should consider (4.4) the "K-absolute nep".

Section 5: True preservation theorems We consider three closure operations of nep forcing notions $(\mathrm{cl}_1, \mathrm{cl}_2, \mathrm{cl}_3)$, investigate what is preserved and what is gained. The main result is a general preservation theorem for nep (5.18). This is done for the "straight" version of nep, which however is a further restriction on the definition of the forcing but not really on the forcing itself (as proved there). We then deal with restricting the iteration to a subsequence (of the sequence of forcing notions) and conclude that such iteration tend not to add an intersection to families of Borel sets from the ground model. In particular, considering iterating nice forcing notions, what a countable length iteration cannot do, iteration of any length cannot do.

Section 6: When a real is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over V We define the class $\mathcal{K}$ of pairs $(\mathbb{Q}, \underset{\sim}{\eta})$, in particular when $\underset{\sim}{\eta}$ is the generic real for $\mathbb{Q}$, and how nice is the subforcing $\mathbb{Q}'$ of $\mathbb{Q}$ generated by $\underset{\sim}{\eta}$. We then present FS iterations of c.c.c. forcing notions satisfying: each elements is ord-hc such that this holds also for the limit of the iteration.

Section 7: Preserving a little implies preserving much We are interested in the preservation of the property (of forcing notions) "retaining positiveness modulo the ideal derived from a c.c.c. nep forcing notion", e.g. being non-null (by forcing notions which are not necessarily c.c.c.). Concerning such preservations, [25, Chapter VI, §1, §2] dealt with cases such that "every new real belong to some perfect set of reals from the old universe which is small (according to a definition we choose for the specific application)", §1 there deal with the framework whereas §2 there deal mainly with several examples.; and [25, §3, Chapter XVIII, §3] replace perfect by F_σ with some price including dealing mainly with the limit case. Our main aim is to show that for "nice" enough forcing notions we have a dichotomy: retaining the positiveness of any $X \subseteq {}^{\omega}\omega$ is equivalent to retaining every positive Borel set. This implies preservation of the property above under, e.g., CS-iterations (of proper forcings).

Section 8: Non-symmetry We start to investigate for c.c.c. nep forcing: when do we have "if η_0 is $(\mathbb{Q}_0, \underset{\sim}{\eta}_0)$-generic over N and η_1 is $(\mathbb{Q}_1, \underset{\sim}{\eta}_1)$-generic over $N[\eta_0]$ then η_1 is $(\underset{\sim}{\mathbb{Q}}_0, \underset{\sim}{\eta}_0)$-generic over $N[\eta_1]$"? This property is known when both are Cohen reals and when both are random reals above.

Section 9: Poor Cohen commutes only with himself We prove that commuting with Cohen is quite rare. In fact, c.c.c. Souslin forcing which adds $\underset{\sim}{\eta}$, a name of a new real which is (absolutely) nowhere essentially Cohen does not commute with Cohen. So such forcing makes the set of old reals meager. This continues [24]. Papers continuing this are [21] and [22].

Section 10: Some absolute c.c.c. nicely defined forcing notions are not so nice We define such forcing notions which are not essentially Cohen as long as $\aleph_1$ is not too large in $\mathbf{L}$. This shows that "c.c.c. Souslin" cannot be outright replaced by "absolutely c.c.c. nep".

Section 11: Open problems We formulate several open questions. Continued in [21], [22], [28].

0. Introduction

The theme of [23], [25] is:

Thesis 0.1. It is good to have general theory of forcings, particularly for iterated forcing.

Some years ago, Haim Judah asked me some questions (on inequalities on cardinal invariants of the continuum). Looking for a forcing proof the following question arises:

Question 0.2. Will it not be nice to have a theory of forcing notions $\mathbb{Q}$ such that:

($\oplus$) *if* $\mathbb{Q} \in N \subseteq (\mathcal{H}(\chi), \in)$, N a countable model of ZFC$^-$ and $p \in N \cap \mathbb{Q}$, *then* there is $q \in \mathbb{Q}$ which is $(N, \mathbb{Q})$-generic?

Note the absence of $\prec$ (i.e. N is just a submodel of $(\mathcal{H}(\chi), \in)$), which is the difference between this property and "properness", and is alluded to in the name of this paper. This evolved to "Souslin proper forcing" (see 1.13) in Judah and Shelah [13], which was continued in Goldstern and Judah [12].

There are still some additional desirable properties absent there:

(a) not all "nicely defined" forcing notions satisfy "Souslin proper", in fact quite many and not so esoteric ones: the Sacks forcing, the Laver forcing (the "reason" being that incompatibility is not Σ^1_1); we like to have all of them;

(b) actual preservation by CS iteration was not proved, just the desired conclusion ($\oplus$) hold for $\mathbb{P}_\alpha$ when $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \alpha, j < \alpha \rangle$ is a countable support iteration and $i < \alpha \quad \Rightarrow \quad \Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is a Souslin proper forcing notion";

(c) to prove for such forcing notions better preservation theorems when we add properties in addition to properness.

Martin Goldstern asked me some years ago on the inadequacy of Souslin proper from clause (a). I suggested a version of the definition of nep, and this was preliminarily announced in Goldstern [11].

The intention here is to include forcing notions with "nice definition" (not ones constructed by diagonalization like e.g. Baumgartner's "every $\aleph_1$-dense sets of reals are isomorphic" [3] or the forcing notions constructed for the oracle c.c.c., see [23, Chapter IV], or forcing notions defined from an ultrafilter).

Our treatment (nep/snep) in a sense stands between [25] and Rosłanowski and Shelah [19]. In [25] we like to have theorems on iterations $\bar{\mathbb{Q}}$, mainly CS, getting results on the whole $\mathrm{Lim}(\bar{\mathbb{Q}})$ from assumptions on each $\mathbb{Q}_i$, but with no closer look at $\mathbb{Q}_i$ – by intention, as we would like to cover as much as we can. In Rosłanowski and Shelah [19] we deal with forcing notions which are quite concrete, usually built from countably many finite "creatures" (still relative to specific forcing this is quite general).

Here, our forcing notions are definable but not in so specific way as in [19], which still provides examples (all proper ones are included), and the theorems are quite parallel to [25]. So we are solving the "equations"

x/theory of proper forcing [23],[25]/
theory of forcing based on creatures [19]
=
"theory of manifolds" /general topology/
theory of manifolds in $\mathbb{R}^3$.

Thesis 0.3. "Nice" forcing notions which are proved to be proper, normally satisfy (even by same proof) the stronger demands defined in the next section.

We finish commenting on some subsequent works. Zapletal's memoir [29] looks at definable forcing notions from other point of view. Kellner and Shelah [15] continue Section 7, dealing with random reals. There has been a breakthrough relevant to the problems concerning random reals, see [24].

History: The paper is based on the author's lectures in Rutgers University in Fall 1996, which results probably in too many explanations. Answering Goldstern's question was mentioned above. A version of §8 (on non-symmetry) was done in Spring of '95 aiming at the symmetry question.

The rest was developed in the Summer and Fall of '96. The material was revised in Fall 2000 (in the Mittag-Leffler Institute) and again in fall 2001, spring 2002. I thank the audience of the lectures for their remarks and mainly Andrzej Rosłanowski for correcting the paper. Moreover, I thank the two referees and Jakob Kelner for detecting many unclarities and mistakes and asking for details, and I thank the logic group in Helsinki and particularly Jouko Vaananen for their generous help.

The reader may be helped by a list of defined notions at the end.

Notation: We try to keep our notation standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [2] or Jech [14]). However in forcing we keep the older/Cohen tradition that *a stronger condition is the larger one.*

For a regular cardinal χ, $\mathcal{H}(\chi)$ stands for the family of sets which are hereditarily of size less than χ. $\mathrm{Tc}^{\mathrm{ord}}(x)$, the hereditary closure relative to the ordinals, is defined by induction on $\mathrm{rk}(x) = \gamma$ as follows:

1. if $\gamma = 0$ or x is an ordinal then $\mathrm{Tc}^{\mathrm{ord}}(x) = \emptyset$
2. if $\gamma > 0$ and x is not an ordinal *then*

$$\mathrm{Tc}^{\mathrm{ord}}(x) = \bigcup\{\mathrm{Tc}^{\mathrm{ord}}(y) : y \in x\} \cup x.$$

The collection of all sets which are hereditarily countable relatively to κ, i.e., $\mathrm{Tc}^{\mathrm{ord}}(x)$ is countable and $\mathrm{Tc}^{\mathrm{ord}}(x) \cap \mathrm{Ord} \subseteq \kappa$, is denoted by $\mathcal{H}_{<\aleph_1}(\kappa)$, equivalently $\mathcal{H}_{<\aleph_1}(\kappa) = \{x \in \mathbf{V}_\kappa : \mathrm{Tc}^{\mathrm{ord}}(x)$ is countable and $\mathrm{Ord} \cap \mathrm{Tc}(x)$ is a bounded subset of $\kappa\}$. Let "x is an ord-hc-set" (or sometimes hc-set, abusing our notation) mean that x is a member of $\mathcal{H}_{<\aleph_1}(\kappa)$ for some κ. Let "x is a strict ord-hc set" mean that x is an ord-hc set but is not an ordinal.

We say that a set $M \subseteq \mathcal{H}(\chi)$ is ord-transitive if

$$x \in M \ \&\ x \text{ is not an ordinal} \quad \Rightarrow \quad x \subseteq M.$$

So every set of ordinals (e.g., any ordinal) is ord-transitive and for any non-ordinal x, we have: x is ord-transitive iff $\mathrm{Tc}^{\mathrm{ord}}(x) = x$. Clearly for every model $M \subseteq (\mathcal{H}(\chi), \in)$ satisfying exstensionality for non-ordinals there is a model $M' \subseteq (\mathcal{H}(\chi), \in)$ ord-transitive and isomorphic to M over $M \cap \mathrm{Ord}$.

We should consider the ordinals as urelements, i.e., $\omega \neq \{n : n < \omega\}$; the disadvantage is that this is not standard hence we have tried to avoid it, but it is closer to the spirit of the paper. Without this, a Borel operation giving $\{x_n :$ if y_n is true $\}$ may be the ordinal α instead of being the set $\{\beta : \beta < \alpha\}$ and if a candidate N "thinks" that ω_1 is countable, then $\mathbf{B}(x_1, \dots, y_1, \dots) = z$ is not absolute from N to $\mathbf{V}$.
WARNING: *So $\emptyset$ is not the ordinal 0 !*

Notation 0.4. We will keep the following rules for our notation:

1. $\alpha, \beta, \gamma, \delta, \epsilon, \xi, \zeta, i, j \dots$ will denote ordinals.

2. $\theta, \kappa, \lambda, \mu, \chi \ldots$ will stand for cardinal numbers, infinite if not said otherwise, sometimes we use them for ordinals.
3. a tilde indicates that we are dealing with a name for an object in a forcing extension (like $\underset{\sim}{x}$).
4. a bar above a name indicates that the object is a sequence, usually $\bar{X}$ will be $\langle X_i : i < \ell g(\bar{X}) \rangle$, where $\ell g(\bar{X})$ denotes the length of $\bar{X}$.
5. For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\ell g(\eta)$.
6. A *tree* is a non empty family of finite sequences closed under initial segments. For a tree T the family of all ω–branches through T is denoted by $\lim(T)$.
7. The Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$ are the spaces of all functions from ω to 2, and to ω, respectively, equipped with natural (Polish) topology.
8. The fixed "version" ZFC^-_* normally is such that the forcing theorem holds and for any large enough χ, the set of $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates (defined in 1.1) is cofinal in $\{N : N \subseteq (\mathcal{H}(\chi), \in)$ and N is countable$\}$ and in the scheme of ZFC^-_* we allow some extra relation (from $\mathfrak{B}$ etc) and whatever else we shall use (fully see 1.15). Usually, dealing with the simple case, we can restrict ourselves to $(\mathcal{H}_{<\aleph_1})$, which follows by collapsing N to an ord-transitive model.
9. $\mathfrak{C}$, $\mathfrak{B} \ldots$ will denote models (with some countable vocabulary). For a model $\mathfrak{C}$, its universe is denoted $|\mathfrak{C}|$ and its cardinality is $\|\mathfrak{C}\|$. Usually $\mathfrak{C}$'s universe is an ordinal $\alpha(\mathfrak{C})$ and $\kappa(\mathfrak{B}) \subseteq |\mathfrak{B}| \subseteq \mathcal{H}_{<\aleph_1}(\kappa(\mathfrak{B}))$, $\kappa(\mathfrak{B})$ a cardinal (or an ordinal).

   Let Δ denote a subset of $L_{\omega_1,\omega}(\tau_\Delta)$, (usually closed under subformulas but this is not required) and $\mathfrak{B}_1 \prec_\Delta \mathfrak{B}_2$ mean that for any $\varphi(\bar{x}) \in \Delta$, $\bar{a} \in {}^{\ell g(\bar{a})}\mathfrak{B}_1$ and $\mathfrak{B}_1 \models \varphi(\bar{a})$ implies that $\mathfrak{B}_2 \models \varphi(\bar{a})$; similarly for $\mathfrak{C}$.
10. K will denote a family of forcing notions including the trivial one (so a K–forcing extension of $\mathbf{V}$ is $\mathbf{V}[G]$ when $G \subseteq \mathbb{P} \in K$ is generic over $\mathbf{V}$) and we demand $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in K^{\mathbf{V}^{\mathbb{P}}}$" $\Rightarrow$ $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in K$. Usually K is the class of (set) forcing notions; as we need to say "$\underset{\sim}{\mathbb{Q}} \in K^N$" clearly K is a definition of such a family.

Definition 0.5. We define "the family of ord-hc Borel operations" to be the minimal family $\mathcal{F}$ of functions such that the following conditions are satisfied:

(a) Each $\mathbf{B} \in \mathcal{F}$ is a function with $\leq \omega$ places and with a designation of the possible results as ord-hc sets or strict ord-hc sets or as ordinals or as truth values.

(b) For every $\mathbf{B} \in \mathcal{F}$, each place of $\mathbf{B}$ is designated to an ord-hc-set or to an ordinal or to a truth value or to strictly ord-hc sets. We also allow a sequence of ordinals of length $\leq \omega$ or a sequence of truth values of given length $\leq \omega$.

(c) $\mathcal{F}$ contains the following atomic functions (with obvious interpretations):
 (α) $\neg x$ for truth value x;
 (β) $x_1 \vee x_2$ for truth values x_1 and x_2;
 (γ) $\bigwedge_{i<\alpha} x_i$ for $\alpha \leq \omega$ and truth values x_i;
 (δ) the constant values *true* and *false*;
 (ϵ_1) for all $\alpha \leq \omega$ and x_n varying on truth values and for all y_n varying on hc- sets (or on ordinals or on strict ord-hc sets) for $n < \omega$:
  - if x_n but not x_m for $m < n$ then y_n;
  - if $\neg x_n$ for every $n < \alpha$ then y_α;

 (ϵ_2) similarly for ordinals,
 (ζ) $\{y_i : i < \alpha, x_i \text{ is true}\}$, where $\alpha \leq \omega$ and each y_i varies on ord-hc-sets or on ordinals, x_n on truth values;
 note that by our conventions this is always a strict ord-hc set, never an ordinal.
 (η) the truth value of "x is an ordinal" where x vary on ord-hc-sets,

(d) $\mathcal{F}$ is closed under composition (preserving the designation to [strict] ord-hc-sets, ordinals and truth values).

Observation 0.6. *The family of ord-hc Borel operations is closed under the following perations:*

- *countable unions;*
- *set difference;*
- *definitions by cases + default;*
- *every such* $\mathbf{B}(....)$ *with values ord-hc-sets and variable ordinals can be represented as*

$$\{\mathbf{B}^1_n(y_0,\dots,y_k,\dots;x_0,\dots,x_\ell,\dots)_{k<\alpha,\ell<\beta} : n<\omega \textit{ and } \mathbf{B}^2_n(y_0,\dots,y_k,\dots;x_0,\dots,x_\ell,\dots)_{k<\alpha,\ell<\beta} \textit{ is true }\}.$$

 where $\alpha, \beta \leq \omega$ *and* y_ℓ *vary on ordinals and* x_n *vary on truth values;*
- *if* N *is an hc-ord model (i.e.,* $\{x : x \in N\} \in \mathcal{H}_{<\aleph_1}(\mathrm{Ord})$*), then* $|N|$ *as well as each relation (and function) first order definable in* N *possibly with parameters from* N *can be represented as an ord-hc Borel function with variable ordinals operating on any list of* $N \cap \mathrm{Ord}$*;*
- *for every such* $\mathbf{B}_1(\bar{x})$, $\mathbf{B}_2(\bar{x})$ *with values ord-hc-sets, there are* $\mathbf{B}_3(\bar{x})$, $\mathbf{B}_4(\bar{x})$ *with values truth values, such that:*
 $\mathbf{B}_3(\bar{x})$=*true iff* $\mathbf{B}_1(\bar{x}) = \mathbf{B}_2(\bar{x})$*; and*
 $\mathbf{B}_4(\bar{x})$=*true iff* $\mathbf{B}_1(\bar{x}) \in \mathbf{B}_2(\bar{x})$.
 Similarly if one or two of them has values ordinals.

Definition 0.7. We say that "$\mathcal{I}$ is a predense antichain above p in $\mathbb{Q}$" if

$$p \Vdash_{\mathbb{Q}} \text{" } \mathcal{I} \cap \underset{\sim}{G}_{\mathbb{Q}} \text{ has exactly one element ".}$$

So when we write $\{p_n : n < \omega\}$ instead $\mathcal{I}$ we mean that $p \Vdash_{\mathbb{Q}} \exists! n (p_n \in \underset{\sim}{G}_{\mathbb{Q}})$. In order to allow $\{p_n : n < \omega\}$ to enumerate also finite sets, we may use the constant value *false* or $\{p_n : n < \alpha\}$ for some $\alpha \leq \omega$. Note that we allow $q \in \mathcal{I}$ which are incompatible with p but they have little influence.

1. Basic definitions

Let us try to analyze the situation. Our intuition is that: looking at $\mathbb{Q}$ inside N we can construct a generic condition q for N, but if $N \not\prec (\mathcal{H}(\chi), \in)$, then $\mathbb{Q} \cap N$ might be arbitrary. So let $\mathbb{Q}$ be a definition. What is the meaning of, say, $N \models$"$r \in \mathbb{Q}$"? It is $N \models$"r satisfies $\varphi_0(-)$" for a suitable φ_0. It seems quite compelling to demand that inside N we can say in some sense "$r \in \mathbb{Q}$", and as we would like to have

$$q \Vdash \text{" } \underset{\sim}{G}_{\mathbb{Q}} \cap \mathbb{Q}^N \text{ is a subset of } \mathbb{Q}^N \text{ generic over } N \text{ ",}$$

we should demand

$(*)_1$ $N \models$" $r \in \mathbb{Q}$ " implies $\mathbf{V} \models$" $r \in \mathbb{Q}$ ".

So φ_0 (the definition of $\mathbb{Q}$) should have this amount of absoluteness. Similarly we would like to have:

$(*)_2$ if $N \models$"$p_1 \leq_{\mathbb{Q}} p_2$" and $p_2 \in G_{\mathbb{Q}}$, then $p_1 \in G_{\mathbb{Q}}$.

So we would like to have a φ_1 (or $<^{\varphi_1}$) (the definition of the partial order of $\mathbb{Q}$) and to have the upward absoluteness for φ_1.

But before we define this notion of properness without elementaricity, we should define the class of models N to which it applies.

We may have put in this section the "straight nep" (see 5.13) and/or "absolute nep" (see 4.4).

Advice: The reader may believe in the "nice" names, that is concentrate on the case of correct explicit simple and good nep forcing notions which are normal and local for K the class of set forcing notions (see Definitions 1.3(11), 1.3(2), 1.3(5), 1.15(1), 1.3(1), 1.15(4) 1.11, 1.3, respectively), remember that politeness is always assumed (see 1.1(4)).

The intention is that our forcing $\mathbb{Q}$ is a subset of $\mathcal{H}_{<\aleph_1}(\kappa)$. To find a witness for "$p \in \mathbb{Q}$", we consider general structures $\mathfrak{B}$ such that $\kappa \subseteq \mathfrak{B} \subseteq \mathcal{H}_{<\aleph_1}(\kappa)$.

The Easy Life / The Lazy Man Plan

$\boxdot_1$ (a) $\mathfrak{B}$ and $\mathfrak{C}$ denote models with vocabularies $\subseteq \mathcal{H}(\aleph_0)$ such that their universes are ordinals (usually cardinals) denoted by $\kappa(\mathfrak{B})$ and $\alpha_*(\mathfrak{C})$ respectively ($\mathfrak{C}$ may code very general information, $\mathfrak{B}$ may code an iteration).

(b) ZFC^-_* is a weak set theory extending ZC^- (which tells us what $(\mathcal{H}(\chi), \in)$ with $\chi > \aleph_0$ satisfies, i.e., ZFC without power set and without replacement but with comprehension) with individual constants for $\mathfrak{B}$, $\mathfrak{C}$, and for an ordinal θ saying:

(α) both $\beth_\omega(|\theta|)$ and $\beth_\omega(||\mathfrak{B}|| + ||\mathfrak{C}||)$ exist.

(β) If $\mathbb{P}$ is a forcing notion and $\beth_\omega(|\mathbb{P}|)$ exists then forcing with $\mathbb{P}$ preserves ZFC^-_*.

(γ) $\mathfrak{B}$ and $\mathfrak{C}$ are models as in (a) (ZFC^-_* does not require more on them).

The reader may add the power set axiom.

(c) A candidate is a countable model N of ZFC^-_* such that $\in^N = \in \restriction N$, $\mathrm{Ord}^N = \mathrm{Ord}^{\mathbf{V}} \cap N$, $\mathfrak{B}^N = \mathfrak{B} \restriction N, \mathfrak{C}^N = \mathfrak{C} \restriction N$ (or at least $\mathfrak{B}^N, \mathfrak{C}^N$ are submodels of $\mathfrak{B}, \mathfrak{C}$ resp.) and for every x, $N \models$ "x is countable" implies that $x \subseteq N$ and for every $x \in N$, N "think" that x is an ordinal iff it is.

We assume that ZFC^-_* is recursive or at least definable in $\mathfrak{C} \restriction \omega$ This is in order to make "$N \models [N'$ is a candidate]" well defined. Note that:

> if $N \models$ "N' is a candidate" and the candidate N is ord-transitive, then N' is a candidate.

Also

> if $N \models [N'$ is an ord-hc candidate] and N is a candidate, then N' is a candidate.

Without ord-transitivity, this only almost follows, mainly as the implication $N \models$ "x is countable" $\Rightarrow x \subseteq N$ may fail.

(d) $\mathbb{Q}$ is a forcing, or more exactly, a definition of a forcing using $\mathfrak{B}$ and θ (and possibly $\mathfrak{C}$), such that $\mathbb{Q} \subseteq \mathcal{H}_{<\aleph_1}(\theta)$ (i.e., simple), $\overline{\varphi}$ is the definition of it, and for all p, q we have $p \leq_{\mathbb{Q}} q$ holds iff some relevant candidate "thinks" so (i.e., correctness).

An Alternative Plan

$\boxdot_2$ (a) χ_0 is a strong limit cardinal[1]. Note that χ_0 serves also as an individual constant.

(b) $\chi_1 = \beth_{\omega+2}(\chi_0)^+$ and ZFC^-_* is ZC^- together with the following demand:

> "χ_0 is as in (a) and the class $\mathcal{H}(\beth_{\omega+1}(\chi_0))$ exists".

(c) $\mathfrak{C}$ just "codes κ".

[1]Why? Then forcings of cardinality $< \chi_0$ preserve the theory ZFC^-_*.

(d) $\theta < \chi$, $\kappa < \chi$.
(e) all the $\mathbb{Q}$, $\bar{\mathbb{Q}}$ we shall consider are from $\mathcal{H}(\chi')$, for some $\chi' < \chi_0$ (i.e., a definition of such an object inside $\mathcal{H}(\chi')$). Each $\mathbb{Q}$, $\mathbb{Q}_i$ is a creature forcing as in §3 from [19] or the limit of a CS-iteration of such forcing notions defined so that $\lg(\bar{\mathbb{Q}}) < \theta$ and $\mathbb{Q} \subseteq \mathcal{H}_{<\aleph_1}(\theta)$.
(e)$^+$ So all the forcings we use are provably (in ZFC^-_*) nep.
(f) $\mathfrak{B}$ codes CS iteration of such forcing notions with $\mathbb{Q}_i$ defined by $(\bar{\varphi}_i, \mathfrak{B}_i)$, objects and not names for simplicity. The universe of $\mathfrak{B}$ is θ and $\mathfrak{B}$ has a countable vocabulary. $\mathbf{p}$ is trivial.
(g) Candidates are countable $N \subseteq (\mathcal{H}(\chi_0), \in)$ such that they are models of $\mathrm{ZFC}^-_* = \mathrm{ZC}$, the relations of $\mathfrak{B}$ and $\mathfrak{C}$ together with "$x \in \theta$" and "$x \in \kappa = \mathfrak{B} \cap \mathrm{Ord}$" are allowed as predicates, and for every x, $N \models$ "x is countable" implies that $x \subseteq N$.
(h) We concentrate on ord-transitive ones, so when we use $N \prec (\mathcal{H}(\chi) \in)$ then we replace it by its ord-transitive collapse.

Hence we get all the good properties, and enough absoluteness.
NOW (assuming $\boxdot_1$ or $\boxdot_2$)

(α) Glance at 1.1, 1.3, 1.15, pipe at §3 (containing lots of examples and 1.18.1) to see that the plan fits.
(β) Read 7.18 (and 7.17) *if* you like to know how to quote preservation theorems on CS-iterations of nice forcings,
(γ) Read §6, §7 *if* you like to know why (β) holds using only $\underset{\sim}{\eta} \in {}^{\omega}\omega$.
(δ) Read §8, §9 *if* you like to know how special is Cohen forcing among nice c.c.c. forcings.
(ε) Read §4 *if* you like to know that CS-iterating nep forcing preserves its main properties.
(ζ) Read §5 *if* you like to know that CS-iterating nep gives you nep.
(η) Read §10 *if* you like to know the reason of some limitations.
(θ) Read 5.39 if you like to know when iteration of nice forcing, does not add a real in the intersection of some family of pregiven Borel sets, in particular if short iteration add no real to the intersection then also long ones does not; (and read 5.28–5.39 if you like to understand why).

If you like to do (α)—(θ) you may consider just reading.

When we consider "preservation by iteration", it is natural to define the following:

Definition 1.1. 1. Let $\mathfrak{C}$ denote a model with universe $\alpha_*(\mathfrak{C})$ (an infinite ordinal) and vocabulary $\tau_{\mathfrak{C}} \subseteq \mathcal{H}(\aleph_0)$, where the equality $=$ is one of the relation symbols (for the notational convenience).

Let Δ_1 denote a countable subset of $L_{\omega_1,\omega}(\tau_{\mathfrak{C}})$ which includes the atomic formulas and is closed under subformulas (e.g., first order or quantifier free).

2. Let $\mathfrak{B}$ denote a model whose vocabulary $\tau_{\mathfrak{B}}$ is a countable subset of $\mathcal{H}(\aleph_0)$ so that equality $=$ is one of the relation symbols. Moreover, suppose that for some ordinal $\kappa(\mathfrak{B})$ (possibly a cardinal) we have $\kappa(\mathfrak{B}) \subseteq |\mathfrak{B}| \subseteq \mathcal{H}_{<\aleph_1}(\mathfrak{B})$.

   Let $\Delta = \Delta_2$ be a countable subset of $L_{\omega_1,\omega}(\tau_{\mathfrak{B}})$, which is closed under subformulas.
3. A pre-semi-candidate or a pre-semi $(\mathfrak{C}, \Delta_1, \mathfrak{B}, \Delta_2)$-candidate is a model
$$N = (|N|, \in^N, \mathrm{Ord}^N, \mathfrak{B}^N, \mathfrak{C}^N),$$
where (note that we write N instead of $|N|$):
   (a) N is a set with $\omega + 1 \subseteq N$;
   (b) $\in^N = \in \restriction N$ is a two place relation;
   (c) $\mathrm{Ord}^N = \mathrm{Ord} \cap N$ is a unary relation;
   (d) $\mathfrak{C}^N$ is a unary relation on N defined in the following way:
   a tuple $(R, a_1, \ldots, a_n)$ is in $\mathfrak{C}^N$ iff R is an n–place predicate[2] in $\tau_{\mathfrak{C}}$, $(a_1, \ldots, a_n) \in R^{\mathfrak{C}}$, and $a_1, \ldots, a_n$ are in $|\mathfrak{C}^N|$, where $|\mathfrak{C}^N|$ is some set (not necessarily belonging to N though included in $|N|$) which is closed under the functions of $\mathfrak{C}$ and which satisfies that $|\mathfrak{C}^N| \subseteq \mathrm{Ord}^N$ is an initial segment of Ord^N. Additionaly, $\mathfrak{C}^N$ is a $\prec_{\Delta_1}$–submodel of $\mathfrak{C}$ whose universe is a subset of N, and $\Delta_1 \in N$ (recall that $\tau_{\mathfrak{C}} \subseteq \mathcal{H}(\aleph_0)$ hence $\Delta_1 \subseteq \mathcal{H}(\aleph_0)$ and $\tau_{\mathfrak{C}} \in N$).
   So essentially $\mathfrak{C} \restriction |\mathfrak{C}^N|$ is defined in the model N.
   We may omit $\mathfrak{C}$ if it is clear from the context.
   (e) Similarly for $\mathfrak{B}^N$ and Δ_2, except that $x \in \mathfrak{B}$ implies $x \in \mathcal{H}_{<\aleph_1}(\mathrm{Ord})^N$ and if they are not present, $\mathfrak{B}^N$ is the empty unary relation (in order to fix the vocabulary).
4. N is called polite if $\mathrm{Ord}^N = \mathrm{Ord} \cap N$, and we keep polite company here.
5. K denotes a definition of a family of forcing notions.
6. ZFC^-_* is an appropriate version of the set theory, *if*
   (a) it is in the vocabulary of a pre-semi-candidates;
   (b) it contains the axioms of ZC^-; so ω exists, this does not include the power set axiom, but in the subset schema we allow all formulas in the vocabulary, including $\tau_{\mathfrak{B}}, \tau_{\mathfrak{C}}$;
   (c) it cannot say more on $\mathfrak{B}$ (we may allow saying more on $\mathfrak{C}$).
7. For given $\mathfrak{C}$, $\mathfrak{B}$, ZFC^-_*, we say that N is a semi class candidate (or a $(\mathfrak{C}, \mathfrak{B}, \mathrm{ZFC}^-_*)$–candidate) if:
   (a) N is a pre-semi $(\mathfrak{C}, \mathfrak{B})$-candidate;
   (b) N is a model of ZFC^-_*;
   (c) for all x, $N \models$ " x is countable " implies $x \subseteq N$.
   We may say $\mathfrak{B}$-candidate if $\mathfrak{C}$ and ZFC^-_* are clear from the context.

[2]We treat "$F(x_1, \ldots, x_n) = x_{n+1}$" as an $n+1$–place predicate.

8. We omit "semi" if N is countable. We replace class by set if $N \models$ "$\mathfrak{B}^N$ is a set". If neither set nor class appear we mean set.
9. N_1 is a strong sub candidate of N_2, written $N_1 \leq N_2$, if $N_1 \subseteq N_2$ (see part (16) below), and $\mathfrak{C}^{N_1}$ and $\mathfrak{B}^{N_1}$ are definable in N_2 (with parameters).
10. We say that $(\mathfrak{B}, \mathbf{p}, \theta)$ is a frame *if*:
    (a) $\mathfrak{B}$ is as above;
    (b) θ is a cardinal (or ordinal);
    (c) $\mathbf{p}$ is a finite sequence (usually a sequence of formulas).
11. We say that N is a [semi] [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidate (with $\mathfrak{C}$ and ZFC^-_* understood from the context) *if*:
    (a) N is a [semi] [class] $\mathfrak{B}$-candidate;
    (b) $\theta \in N$, moreover $N \models$ " the set $\mathcal{H}_{<\aleph_1}(\theta)$ exists" but in the class case, we allow also $\theta = \mathrm{Ord}$ and $N \models x \in \theta$ means $x \in \mathrm{Ord}^N$;
    (c) $\mathbf{p} \in N$, ususally $\mathbf{p}$ is $\bar{\varphi}$, a finite sequece of formulas with parameters in $\mathcal{H}_{<\aleph_1}(\theta)$.
12. $\kappa' = \alpha_*(\mathfrak{C}) \cup \kappa(\mathfrak{B}) \cup \theta$ and $\kappa'' = \kappa' + ||\mathfrak{B}||$.
13. We say that a formula φ is upward absolute for (or from) [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidates when:

    if N_1 is a [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidate, $N_1 \models$ "$\varphi[\bar{x}]$", N_2 is a [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidate or it is $(\mathcal{H}(\chi), \in)$, for χ large enough, and $N_1 \leq N_2$ (see part (9)),
    then $N_2 \models \varphi[\bar{x}]$.

    We say above "through [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidates" if N_2 is demanded to be a [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidate (i.e., we omit the second possibility).

    If $\mathfrak{B}$, $\mathbf{p}$, and θ are clear from the context, we may forget to say "for [class] $(\mathfrak{B}, \mathbf{p}, \theta)$-candidates".
14. We say that "φ defines X absolutely through $(\mathfrak{B}, \mathbf{p}, \theta)$-candidates" if
    (α) $\varphi = \varphi(x)$ is upward absolute from $(\mathfrak{B}, \mathbf{p}, \theta)$-candidates,
    (β) $X = \bigcup\{X^N : N$ is a $(\mathfrak{B}, \mathbf{p}, \theta)$-candidate $\}$, where $X^N = \{x \in N : N \models \varphi(x)\}$.

    If only clause (α) holds then we add "weakly".
15. We say that N is ord-hereditary if $|N| = \mathrm{Tc}^{\mathrm{ord}}(|N|)$. Hence if in addition N is a candidate (not only a semi-candidate), *then* $|N| \subseteq \mathcal{H}_{<\aleph_1}(\mathrm{Ord})$, moreover $N \in \mathcal{H}_{<\aleph_1}(\mathrm{Ord})$.
16. For candidates N_1 and N_2 let $N_1 \subseteq N_2$ mean that
    (a) $|N_1| \subseteq |N_2|$;
    (b) $\mathfrak{C}^{N_1} \prec_{\Delta_1} \mathfrak{C}^{N_2}$;
    (c) $\mathfrak{B}^{N_1} \prec_{\Delta_2} \mathfrak{B}^{N_2}$.

    So we allow that $|\mathfrak{C}^{N_1}| \neq |\mathfrak{C}^{N_2}| \cap |N_1|$. (The reason is that when we collapse some N to an ord-hereditary N', maybe $N' \cap \mathfrak{B} \neq N \cap \mathfrak{B} \setminus \mathcal{H}_{\aleph_1}(\kappa(\mathfrak{B}))$.)

Discussion 1.2. 1. Should we prefer $|\mathfrak{B}| = \alpha$ an ordinal (here $\alpha = \theta$) or $|\mathfrak{B}| \subseteq \mathcal{H}_{<\aleph_1}(\alpha)$? The former is more convenient when we "collapse N over θ" (see 2.12). Also then we can fix the universe, whereas for $|\mathfrak{B}| = \mathcal{H}_{<\aleph_1}(\alpha)$ this is less reasonable as it is less absolute. On the other hand, when we would like to prove preservation by iteration the second is more useful (see §5). To have the best of both we use $\mathfrak{B}, \mathfrak{C}$.

2. Note that, for most of the properties listed, we know that we can usually assume all of them. *But* even though both standard and ord-hereditary are desirable, they are contradictory. A plus for ord-hereditary N is that if $N \models$ "$\mathbb{Q}$ is a small forcing" (e.g., set forcing is the nice case) and $G \subseteq (\mathbb{P}^{\mathbb{N}}, \leq_{\mathbb{P}^{\mathbb{N}}})$ is generic over N, then $N[G]$ is well defined and it is a candidate (see more later). A plus for standard is the normality (see Definition 1.16).

This motivates (nep abbreviates "**n**on–**e**lementary **p**roperness"):

Definition 1.3. 1. Let $\bar{\varphi} = \langle \varphi_0, \varphi_1 \rangle$ and $\mathfrak{B}$ be a model as in 1.1, $\kappa = \kappa(\mathfrak{B})$, of course of countable vocabulary $\subseteq \mathcal{H}(\aleph_0)$, the formulas φ_ℓ are first order in the vacabulary of pre-semi $(\mathfrak{C}, \Delta_1, \mathfrak{B}, \Delta_2)$-candidates + a predicate for θ. We say that $\bar{\varphi}$ or $(\bar{\varphi}, \mathfrak{B})$ is a temporary (κ, θ)–definition, or $(\mathfrak{B}, \theta)$–definition, of a nep-forcing notion[3] $\mathbb{Q}$ if, in $\mathbf{V}$:

(a) φ_0 defines the set of elements of $\mathbb{Q}$ and φ_0 is upward absolute from $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates,

(b) φ_1 defines the partial (or quasi) ordering of $\mathbb{Q}$, also in every $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate, and φ_1 is upward absolute from $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates,

(c) if N is a $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate and $p \in \mathbb{Q}^N$, *then* there is $q \in \mathbb{Q}$ such that $p \leq^{\mathbb{Q}} q$ and

$$q \Vdash \text{" } \underset{\sim}{G}_{\mathbb{Q}} \cap \mathbb{Q}^N \text{ is a subset of } \mathbb{Q}^N \text{ generic over } N \text{ "}$$

where, of course, $\mathbb{Q}^N = \{p : N \models \varphi_0(p)\}$. Of course, "$G$ is a subset of $\mathbb{Q}^N$ or $(\mathbb{Q}^N, <^N_{\mathbb{Q}})$ generic over N" means that: $G \subseteq \mathbb{Q}^N$ is $\leq^N_{\mathbb{Q}}$-directed and $N \models$ "$\mathcal{I} \subseteq \mathbb{Q}$ is dense in $(\mathbb{Q}^N, <_{\mathbb{Q}})$ implies $G \cap \mathcal{I}^N \neq \emptyset$".

We omit the "nep" when omitting clause (c).

2. We add the adjective "explicitly" if $\bar{\varphi} = \langle \varphi_0, \varphi_1, \varphi_2 \rangle$ and additionally

$(b)^+$ we add: φ_2 is an $(\omega+1)$-place relation, upward absolute from $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates and $\varphi_2(\langle p_i : i \leq \omega \rangle) \Rightarrow$ "$\{p_i : i \leq \omega\} \subseteq \mathbb{Q}$ and $\{p_i : i < \omega\}$ is predense antichain above p_ω", not just in $\mathbf{V}$ but in every $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate which satisfy $\varphi_2(\langle p_i : i \leq \omega \rangle)$; in this situation we say: $\{p_i : i < \omega\}$ is explicitly predense antichain above p_ω,

[3]So in the normal case (see 1.15(3), 1.17), $\bar{\varphi}$ defines $\mathbb{Q}$.

(c)$^+$ we add to clause (c): if $N \models$ "$\mathcal{I} \subseteq \mathbb{Q}$ is a predense antichain above p" (so $\mathcal{I} \in N$) *then* for some list $\langle p_i : i < \omega\rangle$ of $\mathcal{I} \cap N$ we have $\varphi_2(\langle p_i : i < \omega\rangle {}^\frown \langle q\rangle)$. We then say "$q$ is explicitly $\langle N, \mathbb{Q}\rangle$–generic above p".

(2A) We add the adjective "class" if we allow ourselves (in clauses (b), (c) of part (1) and (c)$^+$ of part (2)) class $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates N; so in clauses (c), (c)$^+$, $\mathcal{I}$ is a class of N; i.e. first order definable with parameters from N, and use the weak version of absoluteness.

3. For a class $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate N we let $\mathrm{pd}(N, \mathbb{Q}) = \mathrm{pd}_{\mathbb{Q}}(N) = \{\mathcal{I} :$ $\mathcal{I}$ is a class of N (i.e. defined in N by a first order formula with parameters from N) and is a predense subset of $\mathbb{Q}^N\}$. If N is a set candidate, it is $\{\mathcal{I} \in N : N \models$"$\mathcal{I}$ is predense subset of $\mathbb{Q}$ "$\}$, i.e., note that N thinks that $\mathrm{pd}_{\mathbb{Q}}(N)$ is a set if it is a set candidate and ZFC^-_* say that the power set of $\mathbb{Q}$ exists; we also let

$$\mathrm{pdac}(N, \mathbb{Q}) = \{\mathcal{I} : N \models \text{“ } \mathcal{I} \text{ is a maximal antichain in } \mathbb{Q} \text{ ”}\}.$$

In part (2) to allow $\{p_i : i < \omega\}$ to be finite we allow p_i to be false (i.e., 0 in the Boolean algebra terminology). Similarly,

$$\mathrm{pdac}(p, N, \mathbb{Q}) = \mathrm{pdac}(p, N, \mathbb{Q}) =$$
$$\{\mathcal{I} : N \models \text{“ } \mathcal{I} \subseteq \mathbb{Q} \text{ is a predense antichain above p ”}\}.$$

4. We replace "temporary" by K if the relevant proposition holds not only in $\mathbf{V}$ but in any forcing extension of $\mathbf{V}$ by a forcing notion $\mathbb{P} \in K$. If K is understood from the context (normally: all forcing notions we will use in that application) we may omit it.
5. We say that $(\bar{\varphi}, \mathfrak{B})$ is simple [explicitly] K–(κ, θ)–definition of a nep–forcing notion $\mathbb{Q}$, *if*:
   (α) $(\bar{\varphi}, \mathfrak{B})$ is [explicitly] K–definition of a nep-forcing notion $\mathbb{Q}$,
   (β) $\mathbb{Q} \subseteq \mathcal{H}_{<\aleph_1}(\theta)$; i.e., $\mathbb{P} \in K$ implies $\Vdash_{\mathbb{P}}$ "if $\varphi_0(x)$ then $x \in \mathcal{H}_{<\aleph_1}(\theta)$", moreover this holds for any $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate,
   (γ) $\mathfrak{B}, \kappa, \theta$ and possibly some elements of $\mathfrak{B}$ are the only parameters of $\bar{\varphi}$ (meaning there are no others, but even $\mathfrak{B}, \kappa, \theta$ do not necessarily appear).
   (δ) φ is absolute between $\mathcal{H}(\chi)$ and $\mathbf{V}$ for χ large enough, hence
   (ϵ) if N is a $\mathbb{Q}$-candidate and M is the ord-hereditary collapse of N, then: M is a candidate, $\mathbb{Q}^M <^M_{\mathbb{Q}}$ are the images of $\mathbb{Q}^N, <^N_{\mathbb{Q}}$ and q is $(N, \mathbb{Q})$-generic iff q is $(M, \mathbb{Q})$-generic, etc.
6. We add "very simple" *if* in addition:
   (δ) $\mathbb{Q} \subseteq {}^\omega\theta$.
7. We may say "$\mathbb{Q}$ is a nep-forcing notion", "N is a $\mathbb{Q}$-candidate" abusing notation. If not clear, we write $\mathbb{Q}^{\bar{\varphi}}$ or $(\mathbb{Q}^{\bar{\varphi}})^{\mathbf{V}}$. Conversely, we write $(\mathfrak{B}, \bar{\varphi}, \theta) = (\mathfrak{B}^{\mathbb{Q}}, \bar{\varphi}^{\mathbb{Q}}, \theta^{\mathbb{Q}})$ and $\mathrm{ZFC}^{\mathbb{Q}}$ for the relevant $(\mathfrak{B}, \bar{\varphi}, \mathbb{Q})$ and ZFC^-_*.

8. We say "$\mathcal{I} \subseteq \mathbb{Q}^N$ is explicitly predense antichain over p_ω" if $\varphi_2(\langle p_i : i \le \omega \rangle)$ for some list $\{p_i : i < \omega\}$ of a subset of $\mathcal{I}$.
9. If we use $(\mathfrak{B}, \mathbf{p}, \theta)$ we mean $\bar{\varphi}$ is an initial segment of $\mathbf{p}$.
10. We say $(\mathfrak{B}, \bar{\varphi}, \theta)$ (or abusing notation, $\mathbb{Q}$) is a class=set frame if every class $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate is a set $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate.
11. In 1.3(1) we add the adjective "correctly" (and we say that $(\mathfrak{B}, \bar{\varphi}, \theta)$ is *correct*) *if*, for a large enough regular cardinal χ:
    (a) the formula φ_0 defines the set of members of $\mathbb{Q}$ absolutely through $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates from $\mathcal{H}(\chi)$, that is
    $$\mathbb{Q} = \bigcup \{\mathbb{Q}^N : N \text{ is a } (\mathfrak{B}, \bar{\varphi}, \theta)\text{–candidate, and } N \subseteq \mathcal{H}(\chi)\},$$
    recalling $\mathbb{Q}^N = \{x : N \models \varphi_0(x)\}$,
    (b) the formula φ_1 defines the quasi order of $\mathbb{Q}$ absolutely through $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates, that is
    $$\le_{\mathbb{Q}} = \bigcup \{(\le_{\mathbb{Q}})^N : N \text{ is a } (\mathfrak{B}, \bar{\varphi}, \theta)\text{–candidate}\},$$
    where $(\le_{\mathbb{Q}})^N = \{(p, q) : N \models \varphi_1(p, q)\}$.
12. Similarly when we add "explicitly", (see 1.3(2)).

Convention 1.4. 1. So for correct frames, abusing our notation, we can ignore $(\mathcal{H}(\chi), \in) \models \varphi_\ell(x)$ and just ask for satisfaction in suitable candidates. So in particular in 1.3(1)(a),(b) it is equivalent if we replace "absolute from" by "absolute through". (Note: being correct is less relevant to snep.) Alternatively, use the "absolutely through" version rather then the "absolutely from", but this seem to just "move the dirt".
2. We may say "$\mathbb{Q}$ is …" when we mean "$(\mathfrak{B}, \bar{\varphi}, \theta)$ is …" or "$(\mathfrak{B}, \mathbf{p}, \theta)$ is …", and more fully adding ZFC^-_*.

Remark 1.5. The main case for us is candidates (not class ones), etc; still mostly we can use the class version of nep. Also we can play with various free choices.

Discussion 1.6. 1. Note: if $x \in \mathcal{I} \in N$, $N \models$" $\mathcal{I} \subseteq \mathbb{Q}$", possibly $x \notin \mathbb{Q}$, $x \notin \mathcal{I}^N$ and so those x's are not relevant (e.g., though $\alpha < \kappa(\mathfrak{B})$ have a special role).
2. We think of using CS iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \delta \rangle$, each $\underset{\sim}{\mathbb{Q}}_i$ has a definition $\bar{\varphi}^i$ and we would like to prove things on $\mathbb{P}_\alpha$ for $\alpha \le \delta$. So the relevant family K_i of forcing notions we really should consider for $\bar{\varphi}^i$ is $\{\mathbb{P}_\beta / \mathbb{P}_i : \beta \in [i, \delta)\}$, (in $\mathbf{V}^{\mathbb{P}_i}$), at least this holds almost always (maybe we can look as help in other extensions).

3. Note that a significant fraction of iterated forcing of proper forcing related to reals are forcing notions informally called "nice" in the introduction; the proof that they are proper usually gives more and we think that they will be included even by the same proof.
4. If K is trivial, (i.e., has only the trivial forcing notion as a member) this means we can replace it by "temporarily".
5. See also 4.4 for "K-absolutely".
6. Note a crucial point in Definition 1.3, the relation "$\{p_n : n < \omega\}$ is predense antichain above p" is not demanded to be absolute; only a "dense" family of cases of it is demanded (in our definition other important relations are not required to be upward absolute, e.g., $\neg\varphi_0(x)$, $\neg\varphi_1(x, y)$, that is, $x \notin \mathbb{Q}$, $x \not\leq^{\mathbb{Q}} y$). This change may seem technical, but is central being the difference between including not few natural examples and including all those we have in mind.
7. Note that the demand described in φ_2 is close to implying "incompatibility is upward absolute from N", but not quite.

Discussion 1.7. A variant of explicitly nep from Definition 1.3(2) is *explicitly' nep*: in (b)$^+$ replace

$$p_\omega \Vdash_{\mathbb{Q}} \text{“ } \underset{\sim}{G}_{\mathbb{Q}} \cap \{p_n : n < \omega\} \text{ has one and only one element ”}$$

by

$$p_\omega \Vdash_{\mathbb{Q}} \text{“ } \underset{\sim}{G}_{\mathbb{Q}} \cap \{p_n : n < \omega\} \neq \emptyset\text{”}.$$

We may ask ourselves: What is the difference?

Then in clause (c)$^+$ it does not matter if we use $\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q})$ or $\mathcal{I} \in \{\mathcal{I}^* : N \models \mathcal{I}^* \subseteq \mathbb{Q}$ is predense above $p\}$. Now in the definition as it stands we have the fact below, but if we use explicitly' nep this is not clear. The problem is that possibly p, q are incompatible in $\mathbb{Q}^N$ but not in $\mathbb{Q}$ (so if $q \Vdash$ "$\underset{\sim}{G} \cap \mathbb{Q}^N$ is $\leq^N_{\mathbb{Q}}$-directed", then all is OK).

Fact 1.8. *If $p \in \mathbb{Q}^N$, and N is a $\mathbb{Q}$-candidate, and $p <_{\mathbb{Q}} q \in \mathbb{Q}$ and $\varphi_2(\bar{p}, q)$ for some list $\bar{p}$ of $\mathcal{I}^N$ for every $\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q})$ then q is $(N, \mathbb{Q})$–generic.*

Proof. Easy (as if $p_1, p_2 \in \mathbb{Q}^N$, then there is $\mathcal{I} \in N$ such that $N \models$ "$\mathcal{I}$ is a predense antichain above p, and if $r \in \mathcal{I}$ then r is above p' or r is incompatible with p' for every $p' \in \{p, p_1, p_2\}$"). $\blacksquare_{1.8}$

Let us consider a more restrictive class, where the absoluteness holds because of more concrete reasons, the usual ones for upward absoluteness, the relevant relations are Σ^1_1, or more generally, κ–Souslin.

Definition 1.9. 1. We say that $\bar{T}$ is a temporarily (κ,θ)–definition of a snep–forcing notion $\mathbb{Q}$ *if* (the temporary mean just in the present universe):

(a) $\bar{T} = \langle T_0, T_1\rangle$ where $T_0 \subseteq {}^{\omega>}(\theta\times\kappa)$ and $T_1 \subseteq {}^{\omega>}(\theta\times\theta\times\kappa)$ are trees (i.e., closed under initial segments, non-empty)

(b) the set of elements of $\mathbb{Q}$ is

$$\mathrm{proj}_0(T_0) \stackrel{\mathrm{def}}{=} \{\nu\in{}^{\omega}\theta : \text{ for some } \eta\in{}^{\omega}\kappa \text{ we have } \nu*\eta \stackrel{\mathrm{def}}{=} \langle(\nu(n),\eta(n)) : n<\omega\rangle \in \lim(T_0)\},$$

(c) the partial order of $\mathbb{Q}$, $\{(p_0,p_1) : \mathbb{Q}\models p_0\le p_1\}$ is

$$\mathrm{proj}_1(T_1) \stackrel{\mathrm{def}}{=} \{(\nu_0,\nu_1) : \nu_0,\nu_1\in\mathbb{Q} \text{ and for some } \eta\in{}^{\omega}\kappa \text{ we have } \nu_0*\nu_1*\eta \stackrel{\mathrm{def}}{=} \langle(\nu_0(n),\nu_1(n),\eta(n)) : n<\omega\rangle\in\lim(T_1)\},$$

(d) for a large enough regular cardinal χ, if $N\subseteq(\mathcal{H}(\chi),\in)$ is a $(\mathfrak{B}_{\bar{T}},\theta)$–candidate (on $\mathfrak{B}_{\bar{T}}$ see below) and $p\in\mathbb{Q}^N$, *then* there is $q\in\mathbb{Q}$ such that $p\le_{\mathbb{Q}} q$ and

$$q \Vdash \text{“ } \underset{\sim}{G}_{\mathbb{Q}}\cap\mathbb{Q}^N \text{ is a generic subset of } \mathbb{Q}^N \text{ over } N \text{ ”},$$

where $\mathfrak{B}_{\bar{T}}$ is the model with universe $\mathcal{H}_{<\aleph_0}(\kappa)$ and the relations “$x\in T_\ell$”.

2. We add “explicitly” if $\bar{T} = \langle T_0,T_1,T_2\rangle$ (so $x\in T_\ell$ is a relation of $\mathfrak{B}$) and we add

$(a)^+$ also $T_2\subseteq{}^{\omega>}(\theta\times\theta\times\kappa)$ and we let

$$\mathrm{proj}_2(T_2) \stackrel{\mathrm{def}}{=} \big\{\langle\nu_i : i\le\omega\rangle : \text{ for some } \eta\in{}^{\omega}\kappa \text{ we have } \nu*\nu_\omega*\eta\in\lim(T_2) \text{ where } \nu=\mathrm{code}(\langle\nu_\ell:\ell<\omega\rangle) \text{ is the member of } {}^{\omega}\theta \text{ satisfying } \nu\big(\tbinom{\ell+k+1}{2}+\ell\big)=\nu_\ell(k)\big\}$$

and $\langle\nu_i : i\le\omega\rangle\in\mathrm{proj}_2(T_2)$ implies $\{\nu_i : i\le\omega\}\subseteq\mathbb{Q}$ (even in candidates; the natural case is that witnesses are coded).

$(d)^+$ we add: q is $\bar{T}$–explicitly $(N,\mathbb{Q})$–generic, which means that *if* $N\models$ “$\mathcal{I}$ is a predense antichain above p in $\mathbb{Q}$” *then* for some list $\langle p_n : n<\omega\rangle$ of $\mathcal{I}^N$ we have $\langle p_n : n<\omega\rangle^\frown\langle q\rangle\in\mathrm{proj}_2(T_2)$,

$(e)^+$ if $\nu_i\in\mathbb{Q}$ for $i\le\omega$ and for some $\eta\in{}^{\omega}\kappa$ we have $\mathrm{code}(\nu_0,\nu_1,\dots)*\nu_\omega*\eta\in\lim(T_2)$, *then* $\{\nu_0,\nu_1,\dots\}\subseteq\mathbb{Q}$ is predense antichain above ν_ω (and this holds in $(\mathfrak{B},\theta)$–candidates too).

3. We will also say “$\mathbb{Q}$ is a snep-forcing notion”, “N is a $\mathbb{Q}$–candidate”, etc.

4. We say η is a witness for $\nu\in\mathbb{Q}$ if $\nu*\eta\in\lim(T_0)$; similarly for T_1,T_2. We say that $\mathcal{I}$ is explicitly predense antichain over p_ω if $\mathrm{code}(\langle p_i : i\le\omega\rangle)\in\mathrm{proj}_2(T_2)$ for some list $\{p_i : i<\omega\}$ of $\mathcal{I}$.

Remark 1.10. In clause $(a)^+$ we would like the $\mathrm{proj}_2(T_2)$ to be an $(\omega+1)$–place relation on $\mathbb{Q}$, but we do not like the first coordinate to give too much information so we use the above coding, but it is in no way special. Note: we do not want to have one coordinate giving $\langle \nu_\ell(0) : \ell < \omega \rangle$.

Another possible coding is $\mathrm{code}(\nu_0, \nu_1, \dots) \cong \langle \langle \nu_\ell \restriction i : \ell \leq i \rangle : i < \omega \rangle$, so $T \subseteq {}^{\omega>}({}^{\omega>}({}^{\omega>}\theta) \times \theta \times \kappa)$.

Definition 1.11. 1. Let $\mathbb{Q}$ be explicitly snep. We add the adjective "local" *if* in the "properness clause, i.e., $1.9(2)(d)^{+}$" we add:

($\otimes$) the witnesses for "$q \in \mathbb{Q}$", "$\langle p_n^{\mathcal{I}} : n < \omega \rangle$ is $\mathbb{Q}$–explicitly predense antichain above q" are from ${}^{\omega}(N \cap \kappa)$.

2. Let $\mathbb{Q}$ be explicitly nep. We add the adjective "K-local" if in the "properness clause, i.e., $1.3(2)(c)^{+}$" we add:
for each candidate N which is ord–transitive we have

($\oplus$) for some K–extension N^+ of N (see below), we have: N^+ is a $\mathbb{Q}$–candidate (in particular a model of ZFC^-_*) and $N^+ \models$ "$\mathbb{Q}^N$ is countable" and for every $p \in \mathbb{Q}^N$ there is $q \in N^+$, $N^+ \models$ "$p \leq^{\mathbb{Q}} q$ and for each $\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q})$, $\mathcal{I}^N$ is explicitly predense over q".

(Note that $\mathfrak{B}^{N^+} = \mathfrak{B}^N$).

3. If K is the family of set forcing notions $\mathbb{Q}$ for which every $\beth_n(\mathbb{Q})$ exists, or constant understood from the context, we may omit K.

4. Assume N_1, N_2 are candidates and $N_1 \models$ "$\mathbb{Q}$ a forcing notion from K", and $G \subseteq \mathbb{Q}^{N_1}$ is generic over N_1. We say $N_2 = N_1[G]$ *if* this is true in N_2, so $|N_1| \subseteq |N_2|$, $\mathrm{Ord}^{N_1} = \mathrm{Ord}^{N_2}$, $N_2 \models$ "$x \in y$", $y \in N_1 \Rightarrow x \in N_1$, $\mathfrak{C}^{N_1} = \mathfrak{C}^{N_2}$, $\mathfrak{B}^{N_2} = \mathfrak{B}^{N_1}$ (so if $\mathbb{Q} \in N_1$ then $G \in N_2$) and if we use Mostowski collapse Mos_{N_2} then $\mathrm{Mos}(N_2) = (\mathrm{Mos}(N_1))[\mathrm{Mos}(G)]$. It is more natural here to use the transitive collpase over the ordinals. In this case we say "N_2 is a K-extension of N" (on the existence see §2).

Discussion 1.12. 1. Couldn't we fix $\theta = \omega$? Well, if we would like to have the result of "the limit of a CS iteration $\bar{\mathbb{Q}}$ of such forcing notions is such a forcing notion", we normally need $\theta \geq \ell g(\bar{\mathbb{Q}})$. Also $\kappa > \aleph_0$ is good for including Π^1_2–relations.

2. We may in Definition 1.11 have two versions of ZFC^-_*, one before the forcing and one after. Helpful mainly if we are interested in such theories not implying "$\beth_n(|\mathbb{Q}|)$ exists for each n".

3. In "Souslin proper" (starting with [13]) the demands were as in Definition 1.13 below.

Definition 1.13. A forcing notion $\mathbb{Q}$ is Souslin proper if ($\mathbb{Q} \subseteq {}^{\omega}\omega$ and) it is proper and: the relations "$x \in \mathbb{Q}$", "$x \leq^{\mathbb{Q}} y$" are Σ^1_1 and the notion of "incompatibility in $\mathbb{Q}$" is Σ^1_1 (noting that, of course, the compatibility relation is Σ^1_1). So really we have $(\varphi_0(x), \varphi_1(x,y), \varphi_2(x,y))$.

Remark 1.14. This makes "$\{p_n : n < \omega\}$ is predense antichain over p_ω" a Π^1_1–property[4], hence an $\aleph_1$–Souslin one. So we can get the "explicitly" cheaply, *however* possibly increasing κ. Note that for a Souslin proper forcing notion $\mathbb{Q}$, also $p \in \mathbb{Q}^N \Leftrightarrow p \in \mathbb{Q}$ & $p \in N$ and similarly for $p \leq_{\mathbb{Q}} q$.

* * *

If you like to be more pedantic on the ZFC^-_*, look at the following definition, if not go directly to 1.16–1.18. Normally there is no problem in having ZFC^-_* as required and we are assuming enough goodness.

Definition 1.15. 1. We say ZFC^-_* is a K–good version [with parameter $\mathfrak{C}$, possibly "for $(\mathfrak{B}, \mathbf{p}, \theta)$" for $\mathfrak{B}, \mathbf{p}, \theta$ as in 1.3 from the relevant family] *if* :

(a) ZFC^-_* contains ZC^-; i.e., Zermelo set theory without power set [as in 1.1(6) the axioms may speak on relations of $\mathfrak{C}$, only through the axiom schemes (of comprehension, also of cases of replacement if used), i.e, we allow to substitute formulas with relations of $\mathfrak{C}$ (and of $\mathfrak{B}$, we may also restrict the use of the relations of $\mathfrak{B}$, no lose as at present)]

(b) $\mathfrak{C}$ is a model with countable vocabulary ($\subseteq \mathcal{H}(\aleph_0)$) (given as a well ordered sequence) and its universe $|\mathfrak{C}|$ is an ordinal $\alpha_*(\mathfrak{C})$,

(c) for every χ large enough, if $X \subseteq \mathcal{H}(\chi)$ is countable then for some $\mathfrak{C}$–candidate (or $(\mathfrak{B}, \mathbf{p}, \theta)$–candidate) $N \subseteq (\mathcal{H}(\chi), \in)$, we have $X \subseteq N$. If we have the "for $\mathbb{Q}$" then for every countable set X there is candidate N extending it such that

(α) $X \subseteq N$

(β) $X \cap \mathbb{Q} \subseteq \mathbb{Q}^N$

(γ) $\leq^{\mathbb{Q}} \restriction X \subseteq \leq^N_{\mathbb{Q}}$

(δ) if $\models \varphi_2[\langle p_i : i < \omega\rangle^\frown\langle p\rangle]$ and $\langle p_i : i < \omega\rangle, \langle p\rangle \in X$, then $N \models \varphi_2[\langle p_i : i < \omega\rangle^\frown\langle p\rangle]$.

We may need a substitute for the "bare" nep with explicitly. In the simple case (which is the main one), we may restrict the assumption to the case $N \subseteq \mathcal{H}_{<\aleph_1}(\chi)$ and add to the conclusion $\subseteq \mathcal{H}_{<\aleph_1}(\chi)$.

(d) ZFC^-_* satisfies the forcing theorem[5] (see e.g. [25, Chapter I]) at least for forcing notions in K,

[4]We express it as: $\forall q$ [if q, p_ω are compatible then $\bigvee\limits_n (q, p_n$ are compatible)].

[5]That is: if $N \models \mathrm{ZFC}^-_*$ and $N \models$"$\mathbb{P}$ is a (set) forcing notion in K"and $G \subseteq \mathbb{P}^N$ is generic over N thus $N[G]$ is a model of ZFC^-_* if we do not need to prove the existence of a candidate $N_2 = N[G]$, we can ignore $\mathbf{V}$ and think on N only, recall that $\mathfrak{B}^{N[G]} = \mathfrak{B}^N$,

(e) those properties are preserved by forcing notions in K (if $\mathbb{P} \in K$, $G \subseteq \mathbb{P}$ generic over $\mathbf{V}[G]$ then $K^{\mathbf{V}[G]}$ will be interpreted as $\{\mathbb{Q}[G] : \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in K\}$), so after forcing with $\mathbb{Q} \in K$ we still have a model of ZFC^-_*, and we normally allow, e.g., $\mathrm{Levy}(\aleph_0, 2^{|\mathbb{Q}|})$,

(f) ZFC^-_* as well as Δ_1, Δ_2 are recursive or at least definable in say $\mathfrak{C}\restriction\omega$ (this[6] to enable us to say "N is a candidate").

2. If K is the class of all (set) forcing notions $\mathbb{P}$ such that ZFC^-_* says that $\beth_n(|\mathbb{P}|)$ exists for each n, we may omit it. If $K = \{\emptyset\}$, we replace K by "temporarily".
3. We say ZFC^-_* is normal[7] [for $(\mathfrak{B}, \mathbf{p}, \theta)$] *if* for χ large enough any countable $N \prec (\mathcal{H}(\chi), \in)$ to which $\mathfrak{C}$ [as well as $(\mathfrak{B}, \mathbf{p}, \theta)$] belongs is OK (for clause (1B)(c) above).
4. We say ZFC^-_* is semi-normal for $\mathbb{Q}$ [that is for $(\mathfrak{B}, \bar{\varphi}, \theta)$] *if* for χ large enough, for any countable $N \prec (\mathcal{H}(\chi), \in)$ to which $\mathfrak{C}$, $\overline{\varphi}^{\mathbb{Q}}, \mathfrak{B}^{\mathbb{Q}}, \theta$ $(\in \mathcal{H}(\chi))$ belong, for some $\mathbb{P} \in N$ such that $N \models$"$\mathbb{P}$ is a forcing notion" we have:

   (∗) *if* N' is countable $N \subseteq N' \subseteq (\mathcal{H}(\chi), \in)$, $N' \cap \chi = N \cap \chi$ and
   $$(\forall x)[N' \models \text{“}x \text{ is countable ”} \quad \Rightarrow \quad x \subseteq N'],$$
   and N' is a generic extension of N for $\mathbb{P}^N$,
   then $N'' = (\mathbf{N}, \in \restriction N', \mathrm{Ord}^N, \mathfrak{B}\restriction N)$ is $(\mathbb{Q}, \mathfrak{C}\restriction N)$–candidate and
   $$\mathbb{Q}^{N'}\restriction N = \mathbb{Q}\restriction N, \qquad \varphi_2^{N'}\restriction N = \varphi_2^N\restriction N.$$

   We say "K–semi-normal" if we demand $N \models$"$\mathbb{P} \in K$".
5. We say ZFC^-_* is weakly normal for $(\mathfrak{B}, \overline{\varphi}, \theta)$ if clause (c) of part (1) holds; similarly weakly K-normal is defined.
6. In parts (4), (5) we can replace $(\mathfrak{B}, \mathbf{p}, \theta)$ by a family of such triples meaning N is a candidate for all of them.
7. In parts (4), (5), (6) if $(\mathfrak{B}, \mathbf{p}, \theta) = (\mathfrak{B}^{\mathbb{Q}}, \bar{\varphi}^{\mathbb{Q}}, \theta^{\mathbb{Q}})$ we may replace $(\mathfrak{B}, \mathbf{p}, \theta)$ by $\mathbb{Q}$.

Discussion 1.16. 1. What are the points of parameters? E.g., we may have κ^* an Erdös cardinal, $\mathfrak{C}$ codes every $A \in \mathcal{H}(\chi)$ for each $\chi < \kappa^*$,

---

$\mathfrak{C}^{N[G]} = \mathfrak{C}^N$, see Definition 1.11(3)]. So this is an axiom scheme. We can weaken the demand if we use more than one set theory, and we say that $\mathrm{ZFC}^-_2 > \mathrm{ZFC}^-_1$ if the forcing theorem for ZFC^-_1 belongs to ZFC^-_2, but the gain seems meager. In addition for 7.10 we need: if $\mathbb{P}, \mathbb{Q}$ are forcing notions, $\underset{\sim}{G}$ is a $\mathbb{P}$–name for a subset of $\mathbb{Q}$ such that $\Vdash$" $\underset{\sim}{G}$ is a generic subset of $\mathbb{Q}$ ", and $q \in \mathbb{Q} \Rightarrow \nVdash_{\mathbb{P}}$"$q \notin \underset{\sim}{G}$" then for some $\mathbb{Q}$–name $\underset{\sim}{\mathbb{R}}$ of a forcing notion, $\mathbb{Q} * \underset{\sim}{\mathbb{R}}$, $\mathbb{P}$ are equivalent.

[6]We may say "ZFC^-_*, as a set of sentences belong to every candidate or even better put this demand in the definition of a $(\mathfrak{B}, \mathbb{P}, \theta)$-candidate, but we can just make it definable by having the set $(\subseteq \mathcal{H}(\aleph_0))$ appearing in $\mathbb{P}$ or as a relation of $\mathfrak{C}$ or $\mathfrak{B}$.

[7]If we use nice ZFC^-_* (see Definition 4.5), then it would be natural to restrict ourselves to strong limit uncountable χ.

$\mathrm{ZFC}^-_* = \mathrm{ZFC}^- +$ "κ^* is an Erdös cardinal, $\mathfrak{C}$ as above", $K =$ the class of forcing notions of cardinality $< \kappa^*$. Then we have stronger absoluteness results to play with.

2. On the other hand, we may use $\mathrm{ZFC}^-_* = \mathrm{ZFC}^- + (\forall r \in {}^\omega 2)(r^\#$ exists$)$ + "$\beth_7$ exists". This is a good version if $\mathbf{V} \models (\forall r \in {}^\omega 2)(r^\#$ exists$)$ so we can, e.g., weaken the definition snep (or Souslin-proper or Souslin-c.c.c.).
3. What is the point of semi-normal? E.g. if we would like $\mathrm{ZFC}^-_* \vdash \mathrm{CH}$, whereas in $\mathbf{V}$ the Continuum Hypothesis fails. But as we have said in the beginning, the normal case is usually enough.

Proposition 1.17. 1. *Assume* ZFC^-_* *is* $\{\emptyset\}$*–good.* Then *the clause (c)*$^+$ *of* 1.3(2) *follows from clause (c) + (*), where*

(*) if $p \in \mathbb{Q}$ *and* $\mathcal{I}_n$ *is predense antichain above* p *(for* $n < \omega$*) and each* $\mathcal{I}_n$ *is countable,*
then *for some* q, $p \le q \in \mathbb{Q}$, *and for some sequence* $\langle p^n_\ell : \ell < \omega\rangle$ *such that* $\mathcal{I}_n = \{p^n_\ell : \ell < \omega\}$ *we have* $\varphi_2(\langle p^n_\ell : \ell < \omega\rangle {}^\frown \langle q\rangle)$.

2. *If* ZFC^-_* *is normal for* $(\mathfrak{B}, \mathbf{p}, \theta)$ then *in Definition* 1.3(1), (2) *there is no difference between "absolutely through" and "weakly absolutely".*
3. *If* $\mathbb{Q}$ *is a definition of a forcing (so clauses (a), (b) of Definition* 1.3(1) *apply) and* M *is a* $\mathbb{Q}$*-candidate and* $M \models$ *"*$p, q \in \mathbb{Q}$ *are compatible ", then* $p, q \in \mathbb{Q}$ *are compatible (in* $\mathbb{Q}$*, that is in* $\mathbf{V}$*).*
4. *If* $\mathbb{Q}$ *is explicitly nep,* M *is a* $\mathbb{Q}$*-candidate and* $M \models$ *"*$p \in \mathbb{Q}$, $\mathcal{I} \subseteq \mathbb{Q}$ *is countable ", and in* $\mathbf{V}$, $\mathcal{I}$ *is a predense antichain above* p then *also in* M *this holds.*
5. *If* $\mathbb{Q}$ *is explicitly nep, and* $p \in \mathbb{Q}$, $\mathcal{I}_n$ *is a predense antichain in* $\mathbb{Q}$ *above* p *for each* $n < \omega$, then *for some* q *we have* $p \le_{\mathbb{Q}} q$ *and, for each* $n < \omega$*, some countable subset of* $\mathcal{I}_n$ *is an explicitly predense antichain above* q*.*

Proof. (1) So given a $\mathbb{Q}$-candidate M, and $p \in \mathbb{Q}^M$ we can find $q \in \mathbb{Q}$ which forces that $G_{\mathbb{Q}} \cap \mathbb{Q}^N$ is generic over N. Now, if $\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q})$, then necessarily $q \Vdash$ "$\underset{\sim}{G}_{\mathbb{Q}} \cap \mathcal{I}$ has one and only one element". Let $\langle \mathcal{I}_n : n < \omega\rangle$ list $\mathrm{pdac}(p, N, \mathbb{Q})$ so $q, \langle \mathcal{I}^N_n : n < \omega\rangle$ are as in the assumption of (*) for p and $\langle \mathcal{I}_n : n < \omega\rangle$ hence there is r such that $\mathbb{Q} \models q \le r$, and $\varphi^{\mathbb{Q}}_2(\langle p^n_i : i < \omega\rangle {}^\frown \langle r\rangle)$ where $\{p^n_i : i < \omega\} = \mathcal{I}^N_n$. So $r, \langle p^n_i : i < \omega\rangle$ are as required in (c)$^+$.
(2) Easy.
(3) As otherwise for some r we have $M \models$ "$p \le_{\mathbb{Q}} q \le_{\mathbb{Q}} r$", hence this holds also in $\mathbf{V}$, contradiction.
(4) If the set $\mathcal{I}$ is not a predense antichain above p in M, then we can find q such that

$$M \models \text{“}q \in \mathbb{Q},\ p \le_{\mathbb{Q}} q, \ \text{ and } q \text{ is incompatible with every member of } \mathcal{I}\text{”}.$$

Let $q_* \in \mathbb{Q}$ be such that $q \leq_{\mathbb{Q}} q_*$ and

$$q \Vdash \text{“}\underset{\sim}{G} \cap \mathbb{Q}^M \text{ is a generic subset of } (\mathbb{Q}^M, \leq^N_{\mathbb{Q}})\text{”}.$$

Now easy contradiction.
(5) By part (2), it suffices to prove $(*)$. So assume p and $\langle p^n_i : i < \omega\rangle$ for $n < \omega$ are as there. By the weak normality (see 1.15(6)) which we assume (see 1.15(1)), there is a $\mathbb{Q}$–candidate M satisfying clauses $(\beta), (\gamma), (\delta)$ of 1.15(1) and $\{p\} \cup \{p^n_i : i < \omega\} \subseteq \mathbb{Q}^M$. Now, by part (4) for each n the set $\{p^n_i : i < \omega\}$ is predense antichain above p in M. Using clause $(c)^+$ of 1.3(2) we get q as required. $\blacksquare_{1.17}$

Proposition 1.18. *Assume* ZFC^-_* *is normal for* $(\mathfrak{B}, \bar\varphi, \theta)$ *and* $\mathbb{Q}$ *(*$= \mathbb{Q}^{\bar\varphi}$*) is (temporarily) nep* then $\mathbb{Q}$ *is proper.* $\blacksquare$

Discussion 1.19. We may wonder: is normality necessary for 1.18? Yes. Consider:

(a) $\mathbf{V}$ satisfies CH,
(b) ZFC^-_* is ZC^-+“$2^{\aleph_0} > \aleph_1$”,
(c) $\mathbb{Q}$ is $\{p$: p is a function from some countable ordinal $\alpha = \mathrm{Dom}(p)$ into $\mathcal{H}_{\aleph_1}(\theta)$ such that for every limit $\delta \leq \alpha$, $\mathrm{Rang}(p \restriction \delta)$ is a model of ZFC^-_* + “$2^{\aleph_0}$ exists and is $> \aleph_1$”$\}$
ordered by inclusion.

Now

(d) ZFC^-_* is semi normal,
(e) $\mathbb{Q}$ is explicitly nep (in fact for any $\mathbb{Q}$-candidate N_1 any subset of $\mathbb{Q}^{N_1}$ generic over N_1 determine $\underset{\sim}{G}_{\mathbb{Q}} \cap N$ and has an upper bound),
(f) forcing with $\mathbb{Q}$ make $|({}^\omega 2)^{\mathbf{V}}| \leq |\omega_1^{\mathbf{V}}|$ and even $|\mathcal{H}_{<\aleph_1}(\theta(\aleph_3)^{\mathbf{V}}| \leq |\omega_1^{\mathbf{V}}|$
(g) forcing with $\mathbb{Q}$ collapse $\aleph_1$ (why? if $\bar\eta = \langle \eta_i : i < \omega_1\rangle$ list ${}^\omega 2$ in $\mathbf{V}$, $G \subseteq \mathbb{Q}$ generic, $f = \bigcup\{p : p \in G\}$, if $\aleph_1^{\mathbf{V}[G]} = \aleph_1$ then $\{\delta < \omega_1 : \mathrm{Rang}(f \restriction \delta) \cap {}^\omega 2 = \{\eta_i : i < \delta\}\}$ should be a club, but it is disjoint to some club of ω_1 from $\mathbf{V}$).

A sufficient condition for replacing $\mathrm{pdac}(p, N, \mathbb{Q})$ by $\mathrm{pd}(N, \mathbb{Q})$ is (this splits “$\mathcal{I}_n$ is an explicitly predense antichain” into two components):

Observation 1.20. *Assume that* $\mathbb{Q}$ *is nep. A sufficient condition for its being explicitly nep is that for some* φ_2', φ_2 *we have*

$(*)_1$ $\varphi_2'(x, y, z)$ *is an upward absolute formula for* $\mathbb{Q}$*-candidate and it defines incompatibility (i.e., the three have no common upper bound in* $\mathbf{V}$ *and in* $\mathbb{Q}$*-candidates).*

$(*)_2$ *$\varphi_2(x_0, x_1, \dots, x_\omega)$ is upward absolute for $\mathbb{Q}$-candidate and $\varphi_2(\langle p_i : i < \omega\rangle, p_\omega)$ implies $\{p_i : i < \omega\} \subseteq \mathbb{Q}$ is predense above p_ω in $\mathbb{Q}$; this holds in candidates too.*

$(*)_3$ *If N is a $\mathbb{Q}$-candidate, $p \in \mathbb{Q}^N$,* then *for some $q \in \mathbb{Q}$ we have $p \leq q$, q is $(N, \mathbb{Q})$-generic and for every $\mathcal{I} \in \mathrm{pd}(N_1, \mathbb{Q})$ for some $p_0, p_1, \dots \in \mathcal{I}^N$ (not necessarily listing it) we have $\varphi_2(p_0, p_1, \dots, p_\omega)$.*

Proof. Easy. ■

A sufficient condition for explicitly nep is given by the following observation (and used in Section 3).

Observation 1.21. 1. *Assume*

(a) *$\varphi_0(x)$ is a Borel definition of a set, say a subset of $\mathcal{P}(\mathcal{H}(\aleph_0))$, the set of elements of $\mathbb{Q}$.*

(b) *$\varphi_1(x, y)$ is a Borel quasi order on $\{x : \varphi_0(x)\}$ defining $\leq_{\mathbb{Q}}$.*

(c) *$\overline{\mathbf{B}} = \langle \mathbf{B}_n : n < \omega\rangle$, each $\mathbf{B}_n$ a Borel function from $\mathbb{Q}$ to $\mathbb{Q}$ such that $p \leq_{\mathbb{Q}} \mathbf{B}_n(p)$ for every $p \in \mathbb{Q}$.*

(d) *if N is a $\mathbb{Q}$-candidate, so $\varphi_0, \varphi_1 \in N$, N a model of appropriate ZFC^-_*, and $p \in \mathbb{Q}^N$,*
then *there is $q \in \mathbb{Q}^N$ such that:*

$$\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q}) \ \wedge \ q \leq_{\mathbb{Q}} r \in \mathbb{Q} \quad \Rightarrow \quad \bigvee_{n<\omega} \bigvee_{p' \in \mathcal{I}[N]} p' \leq_{\mathbb{Q}} \mathbf{B}_n(r).$$

Then *for some φ_2 (a relation which is the conjunction of a Π^1_1 and a Σ^1_1 formula), $\mathbb{Q}$ is explicitly nep (temporarily and absolutely when the conditions, mainly clause (d) are absolute)*

2. *We can*[8] *in (d) replace $\leq_{\mathbb{Q}}$ in $r \leq_{\mathbb{Q}} \mathbf{B}_n(r)$ by any Σ^1_1-relation guaranteeing compatibility.*

Proof. Straightforward using 1.20, note that incompatibility being a Π^1_1 relation, is upward absolute from candidates. $\blacksquare_{1.21}$

Observation 1.22. 1. *Assume*

(a) *$\varphi_0(x)$ is a Σ^1_1-formula defining a subset of $\mathcal{P}(\mathcal{H}(\aleph_0))$, this is membership in $\mathbb{Q}$.*

(b) *$\varphi_1(x, y)$ is a Σ^1_1-formula defining a quasi order on $\{x : \varphi_0(x)\}$, this is $x \leq_{\mathbb{Q}} y$.*

(c) *$\bar{\mathbf{B}} = \langle \mathbf{B}_n : n < \omega\rangle$ is a sequence of Borel functions satisfying $\varphi_0(x) \Rightarrow x \leq_{\mathbb{Q}} \mathbf{B}_n(x)$.*

[8] Useful, e.g., in preserving the measured creature forcing of [16].

(d) *If N is a $\mathbb{Q}$-candidate, $p \in \mathbb{Q}^N$,* then *there is q such that $p \leq_{\mathbb{Q}} q$ and: for every $\mathcal{I} \in \mathrm{pd}(N, \mathbb{Q})$ we can find $\langle q_n^{\mathcal{I}} : n < \omega \rangle$ such that: $q \leq_{\mathbb{Q}} q_n^{\mathcal{I}}$ for $n < \omega$, $(\forall n)(\exists r \in \mathcal{I}^N)(r \leq_{\mathbb{Q}} q_n^{\mathcal{I}})$ and for every r we have $q \leq_{\mathbb{Q}} r \;\Rightarrow\; (\exists n < \omega)(q_n^{\mathcal{I}} \leq_{\mathbb{Q}} \mathbf{B}_n(r))$.*

Then *the conclusion of 1.21 holds.*

2. *In the clause (d) we can replace $\leq_{\mathbb{Q}}$ in $q_n^{\mathcal{I}} \leq_{\mathbb{Q}} \mathbf{B}_n(r)$ by any Σ_1^1 (or just absolute for $\mathbb{Q}$-candidates) relation implying $\leq_{\mathbb{Q}}$–compatibility.*

Proof. Easy, really the same proof work. ■

2. Connections between the basic definitions

We first give the most transparent implications: we can omit "explicitly" and we can replace snep by nep (this is 2.1) and the model $\mathfrak{B}$ can be expanded, κ, θ increased, (see 2.2). Then we note that if $\kappa \geq \theta + \aleph_1$ and we are in the correct simple nep case, we can get from nep to snep because saying "there is a countable model $N \subseteq (\mathcal{H}(\chi), \in)$ such that ..." can be expressed as a κ–Souslin relation (see 2.3) and comment on the non-simple case. Then we discuss how the absoluteness lemmas help us to change the universe (in 2.4), to get the case with a class K from the case of temporarily (2.7) and to get explicit case from snep or from Souslin proper (in 2.9).

Proposition 2.1. 1. *If $(\bar{\varphi}, \mathfrak{B})$ is explicitly a K–definition of a nep-forcing notion $\mathbb{Q}$,* then *$\bar{\varphi}{\restriction}2$ is a K–definition of a nep-forcing notion $\mathbb{Q}$ (of course for the same $\mathfrak{C}, \mathrm{ZFC}_*^-$, so we normaly do not mention this).*

2. *If $\bar{T}$ is explicitly a K-definition of a snep-forcing notion $\mathbb{Q}$,* then *$(\bar{T}{\restriction}2)$ is a K–definition of an snep-forcing notion $\mathbb{Q}$,*
3. *If $\bar{T}$ is [explicitly] a K–(κ, θ)–definition of a snep-forcing notion $\mathbb{Q}$, and $\mathfrak{B}$ any model with universe κ coding the T_ℓ's and φ_ℓ is defined as $\mathrm{proj}_\ell(T_\ell)$ and ZFC_*^- is natural (e.g., $ZC+$"$\mathfrak{B}$ exists (as a set)"+ "$\beth_n(\kappa)$ exists for each n"),* then *$(\bar{\varphi}, \mathfrak{B})$ is correctly very simple [explicitly] K–(κ, θ)–definition of a nep forcing notion $\mathbb{Q}$ (and let $\mathfrak{B} = \mathfrak{B}_{\bar{T}}$, $\bar{\varphi} = \bar{\varphi}_{\bar{T}}$).*
4. *Very simply implies simply (see* 1.3(5), 1.3(6)*).*

Proof. Read the definitions. ■

Proposition 2.2. 0. *If we increase Δ_1 we essentially just have fewer candidates; fully, assume $\Delta_1 \subseteq \Delta_1' \subseteq L_{\omega_1,\omega}(\tau_{\mathfrak{C}})$ and $\mathfrak{C}$ codes the set Δ_1, e.g., by quantifier free formulas, then every candidate in the new sense is a candidate in the old sense, all relevant properties being preserved except* ZFC_*^- *being nice (see Definition* 4.5 *below) and semi-normal* *(but normal is included). Hence if $\mathbb{Q}$ is [explicitly] nep in the old sense it will be so in the new sense [and all relevant desirable properties are preserved].*

1. *Assume $\Delta' \subseteq L_{\omega_1,\omega}[\tau_{\mathfrak{B}'}]$ and $\mathfrak{B}$ is definable in $\mathfrak{B}'$ (that is, every $R \in \tau_{\mathfrak{B}}$ including the equality, has a definition $\vartheta_R \in L_{\omega,\omega}[\tau_{\mathfrak{B}'}]$) and $\Delta \subseteq \{\psi \in L_{\omega,\omega}[\tau_{\mathfrak{B}'}]$: making the substitution $R \mapsto \vartheta_R$ in ψ (and restrict ourselves to $\{x : \vartheta_=(x,x)\}$) we get $\psi' \in \Delta'\}$.*

   Assume further that $\bar\varphi$ is as in 1.3 and $\bar\varphi'$ is gotten from $\bar\varphi$ when we make the substitution $R \mapsto \vartheta_R$ (and restrict ourselves to $\{x : \vartheta_=(x,x)\}$). Lastly assume that Δ'_2, Δ_2 are naturally related (see 2.1(1) or just $\mathfrak{B}\restriction\mathcal{H}(\aleph_0)$ simply codes them).

   (a) *If N is a $(\mathfrak{B}', \bar\varphi', \theta')$–candidate,* then *$N$ is also a $(\mathfrak{B}, \bar\varphi, \theta)$–candidate; pedantically*

   (a') *Assume N' is a $(\mathfrak{B}', \bar\varphi', \theta')$–candidate, for Δ'_1 of course; so $N' = (|N'|, \in^{N'}, \mathrm{Ord}^{N'}, \mathfrak{B}^{N'}, \mathfrak{C}^{N'})$ and we let*

   $$N = (|N'|, \in^{N'}, \mathrm{Ord}^{N'}, (\mathfrak{B}')^{N'}, \mathfrak{C}^{N'}),$$

   where[9] $(\mathfrak{B}')^{\mathfrak{B}^{N'}} = \mathfrak{B}' \restriction |\mathfrak{B}^{N'}|$.

   Then *N is a $(\mathfrak{B}, \bar\varphi, \theta)$–candidate for Δ_1, of course.*

2. *Assume that $(\bar\varphi, \mathfrak{B})$ is [explicitly] a K–definition of a nep-forcing notion and $\mathfrak{B}$ is definable in $\mathfrak{B}'$ (and change Δ_2 accordingly to the interpretation as done in the previous part).*

   Then *$(\bar\varphi, \mathfrak{B}')$ is [explicitly] a K-definition of a nep-forcing notion; moreover, if $\mathfrak{B}$ is the only parameter of the φ_ℓ, we can replace it by $\mathfrak{B}'$ (changing trivially the φ_ℓ's).*

3. *Hence we can increase κ and θ and/or add "simply" (to the assumption and to the conclusion of the previous part); we may also add "very simply".*

Proof. Straightforward. ■

A converse to 2.1(1)+(2) is the following.

Proposition 2.3. 1. *Recall that $\kappa'' = \theta + \aleph_1 + \|\mathfrak{B}\| + \alpha_*(\mathfrak{C})$ and assume*

($\oplus$) *$(\bar\varphi, \mathfrak{B})$ is a correct very simple [explicit] K–(κ, θ)–definition of a nep forcing notion $\mathbb{Q}$.*

Then *some $\bar T$ codes the relevant relations ($\in \mathbb{Q}$, $\in_{\mathbb{Q}}$ and "explicit predense antichain"). If every $\mathfrak{B}_{\bar T}$–candidate is a $\mathbb{Q}$–candidate, then $\bar T$ is an [explicitly] K–(κ'', θ)–definition of a snep forcing-notion $\mathbb{Q}$ (the same $\mathbb{Q}$ as a forcing notion).*

[9] If we allow more complicated situations than $\mathfrak{B}^N = \mathfrak{B} \restriction |\mathfrak{B}^N|$ we have to say more here.

2. *If $\kappa = \theta = \kappa' = \aleph_0$ we get a similar result with the φ_ℓ being Π^1_2-sets.*
3. *If in clause ($\oplus$) of 2.3(1) we replace very simple by simple (so we weaken $\mathbb{Q} \subseteq {}^\omega\theta$ to $\mathbb{Q} \subseteq \mathcal{H}_{<\aleph_1}(\theta)$),* then *part* (1) *still holds for some $\mathbb{Q}'$ isomorphic to $\mathbb{Q}$.*

Proof. 1) This is, by now, totally straight; still we present the case of φ_0 for part (1) for completeness; for simplicity assume that $\mathfrak{C}$ is interpretable in $\mathfrak{B}$, say by quantifier free formulas and $\Delta_1 = \Delta_2$. If in Definition 1.1(2), clause (e) we use $\prec$, let $\langle \psi^1_n(y, x_0, \dots, x_{\ell_{n-1}}) : n < \omega\rangle$ list the first order formulas in the vocabulary of $\mathfrak{B}$ in the variables $\{y, x_\ell : \ell < \omega\}$, (so in ψ^1_n no x_ℓ, $\ell \geq n$ appears, but some x_ℓ, $\ell < n$ may not appear); if we use $\prec_\Delta$ let it list the ψ^1 such that $\exists y \psi^1$ is a subformula of member of Δ. Similarly $\langle \psi^2_n(y, x_3, \dots, x_{k_{n-1}}) : 4 \leq n < \omega\rangle$ for the vocabulary of set theory plus that of $\mathfrak{B}$. Let us define T_0 by defining a set of ω-sequences Y_0, and then we will let $T_0 = \{\rho \restriction n : \rho \in Y_0 \text{ and } n < \omega\}$. For $\alpha < \omega_1$ let $\{\beta_{\alpha,\ell} : \ell < \omega\}$ list $\{\beta : \beta \leq \alpha\}$.

Now let Y_0 be the set of ω-sequences $\rho \in {}^\omega(\theta \times \kappa'')$ such that for some $(\mathfrak{B}, \bar\varphi, \theta)$–candidate $N \subseteq (\mathcal{H}(\chi), \in)$ (so you can concentrate on the case $\mathfrak{B}, \theta, \kappa$ belong to N, e.g., in the normal case) and some list $\langle a_n : n < \omega\rangle$ and ω–sequences ν, η we have: $\rho = \nu * \eta$; i.e., $\rho(n) = (\nu(n), \eta(n))$ and

(i) $a_0 = \mathfrak{B}$, $a_1 = \theta$, $a_2 = \kappa$, $a_3 = \nu$, and $|N| = \{a_\ell : \ell \in [3, \omega)\}$
(ii) $\{n : n \geq 3 \text{ and } N \models a_n \in \kappa'\} = \{\eta(8n+1) : 0 < n < \omega\}$,
(iii) every $\eta(8n+2)$ is a countable ordinal such that:
$$N \models \text{“}\operatorname{rk}(a_n) < \operatorname{rk}(a_m)\text{”} \quad \text{iff} \quad \eta(8n+2) < \eta(8m+2) < \aleph_1 \leq \kappa',$$
(iv) if $\mathfrak{B} \models (\exists y)\psi^1_n(y, a_0, \dots, a_{n-1})$ then $\mathfrak{B} \models \psi^1_n[a_{\eta(8(n+1)+3)}, a_3, \dots, a_{\ell_n - 1}]$,
(v) $N \models \varphi_0[\nu]$; i.e. $N \models \varphi_0[a_3]$,
(vi) $N \models$“ $a_\ell \in a_m$ ” iff $\eta(8(\binom{\ell+m+1}{2} + \ell) + 4) = 0$,
(vii) if $n \geq 4$ and $N \models (\exists y)\psi^2_n(y, a_3, \dots, a_{k_n - 1})$, then $N \models \psi^2_n[a_{\eta(8n+5)}, a_3, \dots, a_{k_n-1}]$ and $\eta(8n+6) = 1$,
(viii) if $N \models$“a_n is a countable ordinal” and $a_k = \beta_{a_n,\ell}$, then $\eta(8(\binom{\ell+n+1}{2} + \ell) + 7) = k$.
(ix) $N \models \psi^2_n(a_3, a_3, \dots, a_{k_n-1})$ iff $\eta(8n+8) = 0$

Let $T_0 = \{\rho \restriction n : \rho \in Y_0, n < \omega\}$.

Claim 2.3.1. 1. *Y_0 is a closed subset of ${}^\omega(\theta \times \kappa)$.*
2. $\mathbb{Q} = \{\nu \in {}^\omega\theta : (\exists \eta)(\eta \in {}^\omega(\kappa') \;\&\; \nu * \eta \in Y_0 (= \lim(T_0)))\} = \operatorname{proj}_0(T_0)$.

Proof of the claim: 1) Given $\nu * \eta \in \lim(T_0)$ we can define a model N' with set of elements say $\{a'_n : 3 \leq n < \omega\}$ such that $N' \models$ “$a'_{n_\ell} \in a'_m$” iff $\eta(8(\binom{\ell+m+1}{2} + \ell) + 4) = 0$, that is according to clause (vi). We similarly define $\mathfrak{B}^{N'}$. Now $\eta(8n+8) = 0$ iff $N' \models \psi^2_n(a'_3, a'_3, \dots, a'_{k_n-1})$; we can prove this by induction on the formula ψ^2_n starting with atomic, the first a'_3 is just to save another list of ψ's; during the induction we use clause (vii) for the existential quantifier. So N' is a model of ZFC^-_* by clause (vii) (and the demand $N \models \mathrm{ZFC}^-_*$), it is well founded by clause (iii) (and the earlier information).

We start to define an embedding h of N' into $\mathcal{H}(\chi)$ and we put $h(a'_0) = \mathfrak{B}$, $h(a'_1) = \theta$, $h(a'_2) = \kappa$ (in the case we demand that $\mathfrak{B}, \theta, \kappa$ belong to $\mathbf{V}$ our candidates) and $h(a'_n) = \eta(8n+1)$ if $N' \models a'_n \in a'_2$, $n \geq 3$. Then let $h(a'_3) \in {}^\omega\theta$ be such that $h(a'_3)(\ell) = \gamma$ iff letting n be such that $\psi^2_n \equiv [y = x_3(\ell)]$, so necessarily $N' \models$"$a'_3(\ell) = a'_{\eta(8n+5)}$", we have $\eta(8(n+5)+1) = \gamma$ (see clause (vii)).

Lastly we define $h(a'_n)$ for the other a'_n by induction of $\mathrm{rk}^{N'}(a'_n)$, note that we should add to $h(a'_n)$ when $N' \models$ "a'_n is an ordinal" and $n \notin \eta(8m+1) : 0 < m < \omega\}$ dummy elements to retain $\mathrm{Rang}(h) \cap \kappa = \{\eta(8(n+1)+1) : n < \omega\}$.

The model $h[N']$ above should be built in such a way that it is ord–transitive. This (and clause (viii)) will ensure that the clause (g) of the demand 1.1(2) is satisfied.

Note that, actually, the coding (of candidates) which we use above does not change when passing to the ord–collapse.

2) Should be clear from the above noting: $p \in \mathbb{Q}$ iff for some N as above, $N \models \varphi_0(p)$ [as $\Leftarrow$ holds by the definition and $\Rightarrow$ holds as there are countable $N \prec (\mathcal{H}(\chi), \in)$ to which $p, \mathfrak{B}, \theta, \kappa$ belong]. $\blacksquare_{2.3.1}$

Continuation of the proof of 2.3.
This finishes the proof of the claim and so the proof that there is φ_0 as required; we can similarly define φ_1 and prove the other statement in the first part of the proposition.
(2), (3) Easy. $\blacksquare_{2.3}$

What if in 2.3 we omit "the only parameters of $\bar\varphi$ are $\mathfrak{B}, \theta, \kappa$"; so what do we do? Well, the role of $\mathfrak{B}$ is assumed by the transitive closure of $\langle \bar\varphi, \mathfrak{B}, \theta, \kappa\rangle$, which we can then map onto some $\kappa^* \geq \kappa$ we can use $\mathbf{p}$ instead.

Now we look at the connection in the situations in two universes.

Proposition 2.4. 1. *Assume* ZFC^-_* *is* $\{\emptyset\}$*-normal for* $(\mathfrak{B}, \bar\varphi, \theta)$ *(see Definition* 1.15(3)*), and, in* $\mathbf{V}$, $\bar\varphi$ *is a* $(\mathfrak{B}, \theta)$*–definition of an [explicit] nep forcing notion.* Then *we get "correctly" (see Definition* 1.3(11)*). For this, semi* $\{\emptyset\}$*-normal and even weakly* $\{\emptyset\}$*-normal (see Definition* 1.15*) suffice.*

2. *Assume $\bar\varphi$ is a K–$(\mathfrak{B},\theta)$–definition of a forcing notion $\mathbb{Q}$ ("nep" part is not needed). Let $\mathbf{V}'$ be a transitive class of $\mathbf{V}$ such that*
   (i) *$\bar\varphi$ and $\mathfrak{B}$ belong to $\mathbf{V}'$ (and of course $\mathfrak{C}$),*
   (ii) *$\mathbb{Q}$ is correct in $\mathbf{V}'$.*
   Then*:*
   (a) *if $\mathbf{V}' \models$ "$p \in \mathbb{Q}$" (i.e., $\varphi_0(p)$) then $\mathbf{V} \models$ "$p \in \mathbb{Q}$",*
   (b) *if $\mathbf{V}' \models$ "$p \leq_{\mathbb{Q}} q$" (i.e., $\varphi_1(p,q)$) then $\mathbf{V} \models$ "$p \leq_{\mathbb{Q}} q$",*
   (c) *if in $\mathbf{V}'$, N is a $(\mathfrak{B},\bar\varphi,\theta)$–candidate* then *also in $\mathbf{V}$, N is a $(\mathfrak{B},\bar\varphi,\theta)$–candidate; this does no require assumption (ii).*

2a. *Assume $\bar\varphi$ is a $K-(\mathfrak{B},\theta)$–definition of a forcing notion $\mathbb{Q}$; let $\mathbf{V}'$ be a transitive class of $\mathbf{V}$ such that*
   (i) *$\bar\varphi$ and $\mathfrak{B}$ belong to $\mathbf{V}'$ (as well as $\mathfrak{C}$, of course),*
   (ii) *$\mathbb{Q}$ is correct in $\mathbf{V}$.*
   Then
   (a) *if $p \in \mathcal{H}_{<\aleph_1}(\theta)^{\mathbf{V}'}$ and $\mathbf{V} \models$ "$p \in \mathbb{Q}$", then $\mathbf{V}' \models$ "$p \in \mathbb{Q}$",*
   (b) (i) *if $p,q \in \mathcal{H}_{<\aleph_1}(\theta)^{\mathbf{V}'}$ and $\mathbf{V} \models$ "$p \leq_{\mathbb{Q}} q$", then $\mathbf{V}' \models$ "$p \leq_{\mathbb{Q}} q$", and*
   (ii) *if $p,q \in \mathcal{H}_{<\aleph_1}(\theta)$ and $\mathbf{V} \models$ "p,q are compatible in $\mathbb{Q}$", then $\mathbf{V}' \models$ "p,q are compatible in $\mathbb{Q}$",*
   (c) *if $N \in \mathbf{V}$ and $N \in \mathcal{H}_{<\aleph_1}(\kappa)^{\mathbf{V}'}$ and $\mathbf{V} \models$ "N is a $(\bar\varphi,\mathfrak{B})$–candidate"* then *$\mathbf{V}' \models$ "N is a $(\bar\varphi,\mathfrak{B})$–candidate"; note that here assumption (ii) is not used. (it's natural to assume that $\mathbb{Q}$ is simple in $\mathbf{V}$)*

3. *If in* (2) *we add "explicitly" (including the "correct"),* then
   (d) *if $\langle p_n : n<\omega\rangle$ and p_ω belong to $\mathcal{H}_{<\aleph_1}(\kappa)^{\mathbf{V}}$ and $\mathbf{V}' \models \varphi_2(\langle p_i : i \leq \omega\rangle)$, then $\mathbf{V} \models \varphi_2(\langle p_i : i \leq \omega\rangle)$,*
   (e) *if in $\mathbf{V}'$, N is a $(\bar\varphi,\mathfrak{B})$–candidate and q is explicitly $(N,\mathbb{Q})$–generic,* then *this holds in $\mathbf{V}$.*

3a. *Similarly for* (2a).
   That is
   (d) *if $\langle p_n : n<\omega\rangle$ and p_ω belong to $\mathcal{H}_{<\aleph_1}(\kappa)^{\mathbf{V}'}$ and $\mathbf{V} \models$ "$\{p_n : n<\omega\}$ is an explicitly predense antichain over p_ω (in $\mathbb{Q}$)",* then *$\mathbf{V}' \models$ "$\{p_n : n<\omega\}$ is an explicitly predense antichain over p_ω (in $\mathbb{Q}$)".*

4. *If in* (2) *we add "$\bar\varphi$ is a temporary explicit correct $(\mathfrak{B},\theta)$–definition of a nep forcing notion" (in $\mathbf{V}$),* then *also in $\mathbf{V}'$, $\bar\varphi$ is a temporary explicit correct $(\mathfrak{B},\theta)$–definition of a nep-forcing notion.*

5. *If in* (4) *we add "local" to the assumption,* then *also in $\mathbf{V}'$, $\bar\varphi$ is a temporary explicit correct $(\mathfrak{B},\theta)$–definition of a local nep-forcing notion.*

Discussion 2.5. Note that in parts (2), (2a) there is no implication between the two versions of clause (ii), for $\mathbf{V}$ and for $\mathbf{V}'$. The reason is that possibly, e.g., $\mathbf{V}'$ satisfies CH while $\mathbf{V}$ satisfies its negation and ZFC^-_* decide it.

Proof. 1) Straightforward.

2) For clauses (a), (b), by the correctness in $\mathbf{V}'$ (i.e., assumption (ii)) there is a witness in $\mathbf{V}'$ which continues to be so in $\mathbf{V}$ (using upward absoluteness). Clause (c) is immediate.

2a) For clause (c) note that $N \in \mathcal{H}_{<\aleph_1}(\chi)^{\mathbf{V}'}$ is required as there may be $a \in N$, $a \nsubseteq N$, a is countable in $\mathbf{V}$ but not in $\mathbf{V}'$; anyhow clause (c) is immediate.

For clauses (a), (b) there is a candidate N in $\mathbf{V}$ witnessing the relevant fact hence (see 2.12 below), we can find $M \subseteq \mathcal{H}_{<\aleph_1}(\chi)^{\mathbf{V}}$ isomorphic to N over $N \cap \mathcal{H}_{<\aleph_1}(\chi)^{\mathbf{V}}$ (in particular, $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, $M \cap \mathfrak{B} = N \cap \mathfrak{B}$). Now use Shönfield-Levy absoluteness lemma.

3), 3a) Straightforward.

4) We concentrate on the main point: clause $(c)^+$ in 1.3(2). Suppose that

$$\mathbf{V}' \models \text{“ } N \text{ is a } (\bar{\varphi}, \mathfrak{B})\text{–candidate and } p \in \mathbb{Q}^N \text{ ”}.$$

In $\mathbf{V}'$, let $\langle \mathcal{I}_n : n < \omega \rangle$ list the $\mathcal{I}$s such that $N \models$“$\mathcal{I}$ is a predense antichain above p in $\mathbb{Q}$”. We know (by 2.4(2)(c)) that N is a candidate in $\mathbf{V}$. Hence, in $\mathbf{V}$, there are $q, \langle p^n_\ell : \ell < \omega, n < \omega \rangle$ such that:

(i) $\langle p^n_\ell : \ell < \omega \rangle$ lists $\mathcal{I}_n \cap N$,
(ii) $p \leq_{\mathbb{Q}} q \in \mathbb{Q}$,
(iii) $\varphi_2(\langle p^n_\ell : \ell < \omega \rangle ^\frown \langle q \rangle)$ for each $n < \omega$.

So in $\mathbf{V}$ there is a $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate N_1 such that $N \in N_1$, $\langle p^n_\ell : \ell < \omega \rangle : n < \omega \rangle$, q and $\langle \mathcal{I}_n : n < \omega \rangle$ belong to N_1, and $N_1 \models$“$p \leq_{\mathbb{Q}} q$”, and $N_1 \models \varphi_2(\langle p^n_\ell : \ell < \omega \rangle ^\frown \langle q \rangle)$ for $n < \omega$ (by “correct”). It is enough to find such $N_1 \in \mathbf{V}'$, (and $q_1, \langle \langle {}^1p^n_\ell : \ell < \omega \rangle : n < \omega \rangle$ which follows as in 2.7 below. (We use an amount of downward absoluteness which holds as $\mathbf{V}'$ is a transitive class including enough ordinals).

5) Similar proof. $\blacksquare_{2.4}$

Proposition 2.6. *Assume that $\bar{T}$ is in $\mathbf{V}$ a temporary (κ, θ)–definition of a snep forcing notion which we call $\mathbb{Q}$. Let $\mathbf{V}'$ be a transitive class of $\mathbf{V}$ containing $\bar{T}$ (and all ordinals or just $(\kappa')^+$) and satisfying* ZFC^-_*. Then*:*

(a) *if $\mathbf{V}' \models$ “$p \in \mathbb{Q}$” then $\mathbf{V} \models$ “$p \in \mathbb{Q}$”,*
(b) *if $\mathbf{V}' \models$ “$p \leq_{\mathbb{Q}} q$” then $\mathbf{V}' \models$ “$p \leq_{\mathbb{Q}} q$”,*
(c) *if in $\mathbf{V}'$, the model N is a $(\mathfrak{B}_{\bar{T}}, \bar{\varphi}_{\bar{T}}, \kappa_{\bar{T}})$–candidate* then *also in $\mathbf{V}$, N is a $(\mathfrak{B}_{\bar{T}}, \bar{\varphi}_{\bar{T}}, \kappa_{\bar{T}})$–candidate,*
(d) *also in $\mathbf{V}'$, $\mathbb{Q}$ is snep,*
(e) *if $\mathbf{V} \models$ “$p \in \mathbb{Q}$ and $p \in \mathcal{H}_{<\aleph_1}(\theta)^{\mathbf{V}}$”, then $\mathbf{V}' \models$ “$p \in \mathbb{Q}$”,*
(f) *if $\mathbf{V} \models$ “$p \leq_{\mathbb{Q}} q$ and $p, q \in \mathcal{H}_{<\aleph_1}(\theta)^{\mathbf{V}}$”, then $\mathbf{V}' \models$ “$p \leq_{\mathbb{Q}} q$”,*

(g) *if* $\mathbf{V} \models$ *"*$\varphi_2(\langle p_1 : i \leq \omega\rangle)$ *and* $\langle p_i : i \leq \omega\rangle \in \mathcal{H}_{<\aleph_1}(\theta)^{\mathbf{V}'}$*",*
then $\mathbf{V}' \models$ *"*$\varphi_2(\langle p_i : i \leq \omega\rangle)$*".*

Proof. By Shönfield-Levy absoluteness lemma; e.g., for clause (d) repeat the proof of 2.6(4). $\blacksquare_{2.6}$

Proposition 2.7. 1. *Assume that* $(\bar{\varphi}, \mathfrak{B})$ *is a simple correct local explicit temporary* (κ, θ)*–definition of a nep forcing notion* $\mathbb{Q}$*.* Then *for any extension* $\mathbf{V}_1$ *of* $\mathbf{V}$ *this still holds, provided that:*

$(*)_4$ $([\kappa'']^{\leq\aleph_0})^{\mathbf{V}}$ *is cofinal in* $([\kappa'']^{\leq\aleph_0}, \subseteq)^{\mathbf{V}_1}$ *recalling*

$$\kappa'' = \kappa + \theta + \alpha_*(\mathfrak{C}) + ||\mathfrak{B}||.$$

2. *Assume* $\bar{T}$ *is a local explicit temporary* (κ, θ)*–definition of a nep [snep]–forcing notion* $\mathbb{Q}$*. For any extension* $\mathbf{V}_1$ *of* $\mathbf{V}$*, this still holds if* $(*)_4$ *of above holds. So we can replace "temporary" by* $K =$ *class of all proper set forcing notions.*

Proof. 1) We concentrate on the main point (other ones are similar and certainly not harder). Let $a \in [\kappa'' \cup \mathfrak{B}]^{\aleph_0}$, and consider the statement

$\boxtimes_a$ if N is a $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate satisfying $N \cap (\kappa'' \cup \mathfrak{B}) \subseteq a$ and $p \in \mathbb{Q}^N$ (i.e., $N \models \varphi_0[p]$),
then there are N', a generic extension of N (so have the same ordinals and N is a class of N' and the same $\mathfrak{B}, \mathfrak{C}, \theta$) which is a $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate such that[10] $N' \models$"$\mathcal{P}(\theta)^N$ is countable" and

$$N' \models \text{" } (\exists q)[q \in \mathbb{Q} \ \& \ p \leq_{\mathbb{Q}} q \ \& \ q \text{ is explicitly } (N \cap \mathcal{P}(\mathbb{Q}), \mathbb{Q})\text{–generic] ".}$$

Note: in order to guarantee $[x \in N' \wedge N' \models$"$x$ is countable" $\Rightarrow x \subseteq N']$ just use a suitable collapse, see 2.12 below.

Now, only $(N, \alpha)_{\alpha \in a}/\cong$ and the choice (inside N) of the forcing notion are important and we can code N as a subset of a (as all three are countable). Also the issue of saying "N' is well founded" does not arise as N', N have the same ordinals. Thus the statement $\boxtimes_a$ is essentially

$$(\forall N)[(N \text{ is not well founded (or not } \mathfrak{B}{\restriction}(N \cap a) \prec_\Delta \mathfrak{B}, \text{ etc.)} \vee \vee (\exists N')(N' \text{ as above})].$$

As in $\mathbf{V}$, the set a is countable, it can be treated as ω so this is a Π^1_2 statement, hence it is absolute from $\mathbf{V}$ to $\mathbf{V}_1$. Now recalling in particular the definition of local (1.11(2)), both in $\mathbf{V}$ and in $\mathbf{V}_1$ the statement "$\mathbb{Q}$ is simply, locally, explicitly nep" is equivalent to $(\forall a \in [\kappa'' \cup \mathfrak{B}]^{\aleph_0})\boxtimes_a$, which is equivalent to $\mathcal{S} = \{a \in [\kappa \cup \mathfrak{B}]^{\aleph_0} : \boxtimes_a\}$ is cofinal in $[\kappa'' \cup \mathfrak{B}]^{\aleph_0}$. But by the previous paragraph it suffices to prove that $\mathcal{S}[\mathbf{V}]$ is cofinal in $\mathcal{S}[\mathbf{V}_1]$. Now $(*)_4$ gives the needed implication.

[10]Note that $\mathcal{P}(\theta)^N$ is countable in $\mathbf{V}$.

2) Similarly. $\blacksquare_{2.7}$

Remark 2.8. 1. So for the local version, we can replace "temporary" by "for the class of proper forcing (or just preserving "$([\kappa'']^{\leq\aleph_0})^{\mathbf{V}}$ is cofinal in $[\kappa'']^{\leq\aleph_0}$")".

2. We can replace this assumption (i.e., $(*)_4$ of 2.7) by enough absoluteness (so large enough cardinals). If those are strong enough, we can omit "local". The problem with local (even when $\kappa = \theta = \aleph_0$) is that we have to say "N' is well founded" arriving to Π^1_3.

Proposition 2.9. *If $\mathbb{Q}$ (i.e. $\langle\varphi_0, \varphi_1\rangle$) is a Souslin proper forcing notion (see* 1.13*) and $\mathfrak{B}$ codes the parameter (so has universe $\kappa = \aleph_0$ and let $\theta = \aleph_0$) and the parameter of a Σ^1_1–relation equivalent to incompatibility,* then $(\bar{\varphi}, \mathfrak{B})$ *is a simple explicit temporary (κ, θ)–definition of the nep-forcing notion $\mathbb{Q}$; in fact it is locally $(\aleph_0, \aleph_0)$–snep.*

Proof. Straightforward. $\blacksquare$

Definition 2.10. Assume that $(\bar{\varphi}, \mathfrak{B})$ is a temporary (κ, θ)–definition of a nep forcing notion $\mathbb{Q}$, and N is a $\mathbb{Q}$–candidate. We say that a condition $q' \in \mathbb{Q}$ is essentially explicitly $(N, \mathbb{Q})$–generic *if* for some candidate N', $N \subseteq N'$, $N \in N'$, q' is explicitly $(N', \mathbb{Q})$–generic and for some $q_0 \in \mathbb{Q}^{N'}$, $q_0 \leq_{\mathbb{Q}} q'$ and $N' \models$"q_0 is $(N, \mathbb{Q})$–generic". We say "over q" if we can choose $q_0 = q$.

Note: if $\mathbb{Q}$ is a snep-forcing for $\bar{T}$, this relation is $(\kappa + \theta + \aleph_1)$–Souslin, too.

Proposition 2.11. *Assume $\mathbb{Q}$ is an explicitly nep-forcing notion, say by $(\bar{\varphi}, \mathfrak{B})$ and* ZFC^-_* *is weakly normal for $\mathbb{Q}$.*

1. *If q is $(N, \mathbb{Q})$–generic,* then *for some q' we have*

$$q \leq q' \in \mathbb{Q} \quad \textit{and} \quad q' \textit{ is essentially explicitly } (N, \mathbb{Q})\textit{-generic above } q.$$

2. *If q' is essentially explicitly $(N, \mathbb{Q})$–generic (or just explicitly $(N, \mathbb{Q})$–generic),* then *q' is $(N, \mathbb{Q})$–generic.*
3. *In* 1.17 *we can add: and every essentially explicitly $(N, \mathbb{Q})$–generic is explicitly $(N, \mathbb{Q})$–generic (changing $\varphi_2^{\mathbb{Q}}$ slightly).*

Proof. 1) Let $N' \subseteq \mathcal{H}(\chi)$ be a countable $\mathbb{Q}$–candidate satisfying

$$\{N, q\} \in N'.$$

Such N exists by clause (c) of definition 1.15. By our assumptions there is q' such that: $q \leq q' \in \mathbb{Q}$ and q' is explicitly $(N', \mathbb{Q})$–generic.

2), 3) Easy. $\blacksquare_{2.11}$

The following claim tell us that asking about being nep looking for [explicitly] $(N, \mathbb{Q})$–generic, we may restrict ourselves to ord-hereditary $\mathbb{Q}$ candidates.

Proposition 2.12. 1. *If N is a $(\mathfrak{B}, \theta)$–candidate, so in particular*[11]

$$[N \models \text{“}\alpha < \kappa'\text{”}] \Rightarrow \alpha \in \theta \vee \alpha \in \alpha_*(\mathfrak{C}) \vee \alpha \in \kappa(\mathfrak{B})$$

then *there is one and only one hereditary over κ' model $N' \cong N$;* that *is: there are $N' = \mathrm{Mos}_{\kappa''}(N)$ and f such that (recalling $\kappa' = \kappa \cup \theta \cup \alpha_*(\mathfrak{C})$):*

(a) *f is an isomorphism from N onto N',*

(b) *$f(\alpha) = \alpha$ if $N \models$ “$\alpha < \kappa'$” and $f(y) = \beta$ if $N \models$ “(y is an ordinal) $\wedge$ $(y \geq \kappa')$” and $\beta = \kappa' + \mathrm{otp}(\{x : N \models$ “(x an ordinal $< y) \wedge \neg(x < \kappa')$”),*

(c) *if $x \in N \backslash \mathrm{Ord}^N$ then $f(x) = \{f(y) : N \models \text{“}y \in x\text{”}\}$,*

(d) *N' is a $\mathfrak{B}$–candidate.*

2. *Note that if $N \models$ “$x \in \mathcal{H}_{<\aleph_1}(\kappa')$” then $f(x) = x$ (but $\mathfrak{B} \cap N = \mathfrak{B}^N$ does not necessarily implies $\mathfrak{B} \cap N' = \mathfrak{B}^{N'}$).*
3. *For N, N' as above and $q \in \mathbb{Q}$ we have: q is [explicitly] $(N, \mathbb{Q})$–generic* iff *q is [explicitly] $(N, \mathbb{Q})$–generic.*

Proof. Easy but by a request we give details:
1) We try to define by induction on $i < \omega_1$ a function f_i with domain $\subseteq N$, increasing continuous with i.

Case 1: $i = 0$
Let $\mathrm{Dom}(f_0)$ be the set ordinals which belong[12] to N and $f_0(\alpha)$ is α if $\alpha \in N \cap \kappa'$ and is $N \cap \kappa' + \mathrm{otp}(N \cap \alpha \backslash \kappa')$ otherwise (so $f_0(\kappa') = \kappa'$ if $\kappa' \in N$); we are using clause (f) of Definition 1.1(4).

Case 2: i is a limit ordinal
$f_i = \bigcup \{f_j : j < i\}$

Case 3: $i = j + 1$
Let

$$\mathrm{Dom}(f_i) = \{x \in N : x \in \mathrm{Dom}(f_i) \text{ or } (\forall y)(y \in N \wedge y \in x \rightarrow y \in \mathrm{Dom}(f_j))\}.$$

For $x \in \mathrm{Dom}(f_i)$ let $f_i(x) = f_j(x)$ if $x \in \mathrm{Dom}(f_j)$ and $f_i = \{f_j(y) : y \in N, y \in x\}$ otherwise.

So we can carry out the induction. As $\langle \mathrm{Dom}(f_i) : i < \omega_1 \rangle$ is an increasing (not necessarily strictly) sequence of subsets of N that is, of $|N|$ which is

[11] As in some other place “ordinals are urelements” simplifies this.
[12] As we are assuming politeness; otherwise use $\mathrm{Ord}^N = \{x : N \models \text{“}x \text{ is an ordinal ”}\}$.

countable, it is necessarily eventually constant, say for $i \in [i(*), \omega_1))$. As $\in$ is well founded, by the definition of $f_{i(*)+1}$ necessarily $\mathrm{Dom}(f_{i(*)}) = N$. So let $f = f_{i(*)}$ and define N' by $|N'| = \{f(x) : x \in N\}$, and $\in^{N'} = \in \restriction |N'|$, $\mathfrak{B}^{N'} = f(\mathfrak{B}^N)$, $\mathfrak{C}^{N'} = f(\mathfrak{C}^N)$.

Now we prove by induction on i that f_i is one to one. For $i = 0$ check the definition, for i limit trivial, for $i = j + 1$, use "f_j is one to one and N satisfies the comprehension axiom". So $f = f_{i(*)}$ is one to one. Clearly f is the identity on $|\mathfrak{C}^N|$ and also on $\mathfrak{B}^N$, as we can prove this for $x \in \mathfrak{B}^N$ by induction on the rank: if x is an ordinal then $x < \kappa'$ hence $f(x) = x$, if x is not an ordinal, recall that $|\mathfrak{B}^N| \subseteq \mathcal{H}_{<\aleph_1}(\kappa)$ and by the definition of a candidate, $x \subseteq |N|$, so $f(x) = \{f(y) : y \in x\} = \{y : y \in x\} = x$. We can prove similarly by induction on i that $x, y \in \mathrm{Dom}(f_i) \Rightarrow [x \in y \equiv f_i(x) \in f_i(y)]$. So f is an isomorphism from N onto N'. This (and the definition) suffice for proving clause (a), (b), (c) in part (1) which we are proving. As $N' \cap \kappa' = N \cap \kappa'$ clearly $N' \restriction \kappa = N \restriction \kappa$, $N' \restriction \theta = N \restriction \theta$, $N' \restriction \alpha_*(\mathfrak{C}) = N \restriction \alpha_*(\mathfrak{C})$ and $x \in N \cap \mathcal{H}_{<\aleph_1}(\kappa')$ implies $f(x) = x$ recalling clause (c) of Definition 1.1(7). This implies that $N' \restriction \mathfrak{B} = N \restriction \mathfrak{B} \prec_{\Delta_2} \mathfrak{B}$ as $\kappa \subseteq |\mathfrak{B}| \subseteq \mathcal{H}_{<\aleph_1}(\kappa) \subseteq \mathcal{H}_{<\aleph_1}(\kappa')$, similarly for $\mathfrak{C}$. Now it should be clear that clause (d) holds too, i.e., N is a $\mathfrak{B}$–candidate.

This proves there is one and by the proof it is clear that there is only one. $\blacksquare_{2.12}$

Fact 2.13. 1. *In the definition of nep (or snep) in the "properness" clause, it is enough for each N to restrict ourselves to a family $\mathbf{I}$ of dense subsets $\mathcal{I}$ of $\mathbb{Q}$ (in the sense of N) such that:*
   (*) *if $\mathcal{I} \in \mathrm{pdac}(N, \mathbb{Q})$*
   then for some $\mathcal{J} \in \mathbf{I}$ we have $N \models (\forall p \in \mathcal{I} \cap N)(\exists q \in \mathcal{J})(p \leq_{\mathbb{Q}} q)$.
2. *For the explicit version we should speak of "predense antichains above p" (or use a variant as in* 1.17*).*
3. *We can in (*) use $\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q})$ for the $p \in \mathbb{Q}^N$ in clause (c) of* 1.3.

Proof. Straight. $\blacksquare$

Moving from nep to snep (and inversely) we may ask what occurs to "local". It is usually preserved.

Proposition 2.14. 1. *Assume $\bar{T}$ defines an explicit (κ, θ)–snep forcing notion. Let $\bar{\varphi} = \bar{\varphi}_T$, $\mathfrak{B} = \mathfrak{B}_{\bar{T}}$ (see* 2.1(3) *and* ZFC^-_* *as there). If $\mathbb{Q}^{\bar{T}}$ is local,* then $\mathbb{Q}^{\bar{\varphi}}$ *is local, in fact as in Definition* 1.11(2)*.*

2. *If* (ZFC^-_* *is* K*–good and)* ZFC^-_* *says that it is preserved by collapsing* $2^{|\mathbb{Q}|}$ *and that* $(\mathfrak{B}, \bar{\varphi}, \theta)$ *is explicitly nep, and* $\bar{\varphi}$ *is correct,* then $(\mathfrak{B}^{\bar{\varphi}}, \bar{\varphi}, \theta)$ *is explicitly nep and local.*

Proof. 1) The point is that if $N' \subseteq (\mathcal{H}(\chi, \in)$ "thinks" a tree T is countable and has no ω–branch, this is true as N' "thinks" it has ω_1.
2) Straight. $\blacksquare_{2.14}$

3. There are examples

In this section we show that a large family of natural forcing notions satisfies our definition. Later we will deal with preservation theorems but to get nicer results we better "doctor" the forcing notions[13] but this is delayed to the next section.

In fact all the theorems of Rosłanowski and Shelah [19], which were designed to prove properness, actually give one notion or another from §1 here (confirming the thesis 0.3 of §0). We will prove them without giving the definitions from [19] and give a proof of (hopefully) well known specific cases, indicating why it works.

Lemma 3.1. 1. *Suppose that* $\mathbb{Q}$ *is a forcing notion of one of the following types:*

(a) $\mathbb{Q}^{\mathrm{tree}}_e(K, \Sigma)$ *for some finitary tree-creating pair* (K, Σ)*, where* $e = 1$ *and* (K, Σ) *is* 2*-big* or $e = 0$ *and* (K, Σ) *is t-omittory (see* [19, §2.3]*; so, e.g., this covers the Sacks forcing notion),*

(b) $\mathbb{Q}^*_{\mathrm{s}\infty}(K, \Sigma)$ *for some finitary creating pair* (K, Σ) *which is growing, condensed and of the AB–type or omittory, of the* AB^+*–type and satisfies* $\oplus_0, \oplus_3$ *of* [19, 4.3.8] *(see* [19, §3.4]*; this captures the forcing notion of Blass–Shelah* [4]*),*

(c) $\mathbb{Q}^*_{\mathrm{w}\infty}(K, \Sigma)$ *for some finitary creating pair which captures singletons (see* [19, §2.1]*)*

(d) $\mathbb{Q}^*_f(K, \Sigma)$ *for some finitary, 2-big creating pair* (K, Σ) *with the Halving Property which is either simple or gluing and an* H*-fast function* $f : \omega \times \omega \longrightarrow \omega$ *(see* [19, §2.2]*).*

Then $\mathbb{Q}$ *is an explicit* $\aleph_0$*–snep forcing notion, moreover, it is local.*

2. *Assume that* $\mathbb{Q}$ *is a forcing notion of one of the following types:*

(a) $\mathbb{Q}^{\mathrm{tree}}_e(K, \Sigma)$ *for* $e < 3$ *and a tree-creating pair* (K, Σ)*, which is bounded if* $e = 2$ *(see* [19, §2.3]*; this includes the Laver forcing notion),*

[13] Alternatively, we can directly prove that they are very straight, see Definition 5.13 below.

(b) $\mathbb{Q}^*_\infty(K,\Sigma)$ *for a finitary growing creating pair* (K,Σ) *(see* [19, §2.1]*; this covers the Mathias forcing notion).*

Then $\mathbb{Q}$ *is an explicit* $\aleph_0$*–nep forcing notion, moreover, it is local.*

Proof. Actually it should be immediate if you know [19] (and 1.20+ our present definition) particularly if using 1.21; that is let $\langle \mathbf{B}_n : n < \omega\rangle$ list $\{\mathbf{B}_t : t$ is the root of some $p \in \mathbb{Q}\}$ and we let $\mathbf{B}_t(p)$ be $p^{[t]}$, that is p when we restrict ourselves to increasing the root to t (and no more) and is p if this is impossible. Without this you can use 1.20 to justify ignoring the "antichain" part in what follows.

As usual we concentrate on the main point, the properness clause. Let N be a $\mathbb{Q}$-candidate and $p \in \mathbb{Q}^N$. Let $\langle \mathcal{J}_n : n < \omega\rangle$ list $\mathrm{pdac}(p, N, \mathbb{Q})$. Then there is a sequence $\langle (p_n, \mathcal{I}_n) : n < \omega\rangle$ such that $p_n, \mathcal{I}_n \in N$, $N \models p_n \le p_{n+1}$, $\mathcal{I}_n$ is a countable set such that for some function f_n from $\mathcal{I}_n$ into $\mathcal{J}_n$ we have $r \in \mathcal{I}_n \Rightarrow f_n(r) \le_{\mathbb{Q}} r$, $\langle p_n : n < \omega\rangle$ has an upper bound in $\mathbb{Q}$ and $\mathcal{I}_n$ is predense antichain above p_{n+1}, moreover, in an explicit way as described below (see the respective subsections in [19]). Moreover,

$(*)_1$ if $r_1 \ne r_2$ are in $\mathcal{I}_n$ then for some k and $\mathcal{I}_{n,r_1,r_2}$ predense in an explicit way above p_n (as above), for each $r \in \mathcal{I}_{n,r_1,r_2}$ in an explicit way, $\{r_1, r_2, p_n\}$ has no common upper bound, (this happen to hold, but is not so needed),

$(*)_2$ in part (1) of 3.1 cases (a)+(c), $\mathcal{I}_n$ is finite and moreover, we can say "$\mathcal{I}_n$ is predense above p_{n+1}" in a Borel way.

For example for the Sacks forcing notion: for some $k < \omega$, $\mathcal{I}_n = \{p^{[\eta]}_{n+1} : \eta \in p_{n+1}, \ell g(\eta) = k\}$, so $\mathcal{I}_n$ corresponds to a front of p_{n+1}, which necessarily is finite. This property serves as φ_2 (compare with more detailed description for the Laver forcing below).

In part (1) case (b) (e.g., the forcing notion from Blass–Shelah [4]) $\mathcal{I}_n$ is countable. We do not know which level will be activated, but if in the generic, we use the n–th creature in $\lim\langle p_\ell : \ell < \omega\rangle$, then we get into $\mathcal{I}_n$, so $\mathcal{I}_n$ is countable but the property (i.e., the φ_2) is Borel not just Π^1_1.

Similarly in part (1) cases (c), (d).

Now, in part (2), $\mathcal{I}_n$ is countable and again it corresponds to some front A of p_{n+1} in an appropriate sense. So $\mathcal{I}_n = \{p^{[\eta]}_{n+1} : \eta \in A\}$, but to say "$A$ is a front" is Π^1_1 (in some instances of 2(a) we have e-thick antichains instead of fronts, but the complexity is the same), but we may have more "explicite" front A.

Recall that for a subset $T \subseteq {}^{\omega>}\omega$, $A \subseteq T$ is a front of T if

$$(\forall \eta \in \lim(T))(\exists n)(\eta \restriction n \in A)$$

(usually members of A are pairwise incomparable).

Specifically, for the Laver forcing notion, we can guarantee $\mathcal{I}_n = \{p^{[\eta]}_{n+1} : \eta \in A\}$, where A is a front of p_{n+1}. Now being a front is a Π^1_1–sentence (see the definition above) which is upward absolute and this is our choice for φ_2. Let us write this formula in a more explicit way (for the case of the Laver forcing notion):

$$\begin{aligned}
&\varphi_2(\langle p_i : i \le \omega\rangle) \equiv \text{ each } p_i \text{ is a Laver condition and}\\
&\qquad \bigwedge_{i\in\omega} (\exists! \eta)(\eta \in p_\omega \ \&\ p_{2i} = p^{[\eta]}_\omega)\\
&[\text{call this unique } \eta \text{ by } \eta_i] \quad \text{and}\\
&\qquad \bigwedge_{i\neq j} \eta_i \ntrianglelefteq \eta_j \text{ (incomparable)} \ \ \& \ (\forall \rho \in \lim(p_\omega))(\bigvee_n \bigvee_m \rho{\restriction}n = \eta_{2m})\\
&[\text{this is: } \{p_i : i \in \omega\} \text{ is explicitly predense above } p_\omega].
\end{aligned}$$

So it is Π^1_1 (of course, Σ^1_2 is okay, too.)

Note that even for the Sacks forcing notion, "p, q are incompatible" is complete Π^1_1. So "$\{p_n : n \in \omega\}$ is predense above p" will be Π^1_2. For Laver forcing we cannot do better. Now, generally Π^1_2 is not upward absolute from countable submodels, whereas Π^1_1 is. $\blacksquare_{3.1}$

Proposition 3.2. *All the forcing notions $\mathbb{Q}$ defined in* [19], [18], *are simple correct, very straight (see Definition* 5.13*) and we can use* $\mathrm{ZFC}^-_* = \mathrm{ZC}^-$ *which is good and normal (see* 1.15*). Also the relation "p, q are incompatible members of $\mathbb{Q}$" is upward absolute from $\mathbb{Q}$–candidates (as well as $p \in \mathbb{Q}$, $p \notin \mathbb{Q}$, $p \le q$, and "p, q are compatible").*

Proof. Just check. $\blacksquare$

4. Preservation under iteration: first round

We give here one variant of the preservation theorem, but for it we need some preliminary clarification. We have said "there is q which is $(N, \mathbb{Q})$–generic"; i.e. $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{G}_{\mathbb{Q}} \cap \mathbb{Q}^N$ is a generic subset of $\mathbb{Q}^N$ over N". Note that we have said $\mathbb{Q}^N$ and not $\mathbb{Q} \cap N$ as we intended to demand $\mathbb{Q}^N \subseteq \mathbb{Q} \cap N$ rather than $\mathbb{Q}^N = \mathbb{Q} \cap N$, in other words

$$N \models \text{“ } r \in \mathbb{Q} \text{ ”} \quad \Rightarrow \quad \mathbf{V} \models \text{“ } r \in \mathbb{Q} \text{ ”}$$

rather than

$$r \in N \quad \Rightarrow \quad [N \models \text{“} r \in \mathbb{Q} \text{ ”} \ \Leftrightarrow \ \mathbf{V} \models \text{“ } r \in \mathbb{Q} \text{ ”}]$$

(the version we use is, of course, weaker and so better). A trivial example of the possible non equality is the following:

Example 4.1. let $\varphi_0(x)$ say $x = \langle \eta_i : i < \alpha \rangle$, $\alpha < \omega_1$, $\eta_i \in {}^\omega 2$ and $i < j \Rightarrow \eta_i \in \mathbf{L}_\alpha[\eta_j]$ for some countable α; $\varphi_1(x, y)$ is $x \trianglelefteq y$. So if $\eta_0 \in \mathbf{L}[\eta_1]$, but $N \models$" $\eta_0, \eta_1 \in {}^\omega 2$, $\eta_0 \notin \mathbf{L}[\eta_1]$ ", then $\langle \eta_0, \eta_1 \rangle \in \mathbb{Q} \cap N \setminus \mathbb{Q}^N$; clearly we can find such N, η_0, η_1 (using generic extensions of countable models inside $\mathbf{V}$).

Now, using properness we usually use $N[G]$ (e.g., when iterating). But what is $N[G]$ here? In fact, what is the connection between $N \models$"$\underset{\sim}{\tau}$ is a $\mathbb{Q}$–name" and $\mathbf{V} \models$"$\underset{\sim}{\tau}$ is a $\mathbb{Q}$–name"? Because $[x \in Y \in N \not\Rightarrow x \in N]$, there is (in general) no implication between the two statements.

For our purpose, the usual $N[G] = \{\underset{\sim}{\tau}[G] : \underset{\sim}{\tau} \in N \text{ is a } \mathbb{Q}\text{-name}\}$ is not appropriate as it is not clear where being a $\mathbb{Q}$-name is defined. We use $N\langle G \rangle$ which is $N[G \cap \mathbb{Q}^N]$ when we disregard objects in $\mathbf{V} \setminus N$. Of course, if the models are $\subseteq \mathcal{H}_{<\aleph_1}(\mathrm{Ord})$ life is easier; but we may lose the case $N \prec (\mathcal{H}(\chi), \in)$ (see Definition 1.1(15)), which is not so bad by 2.12.

We then prove (in 4.8) the first version of preservation by CS iteration. We aim at proving only that $\mathbb{P}_\alpha = \mathrm{Lim}(\bar{\mathbb{Q}})$ satisfies the main clause, i.e. clause (c) of Definition 1.3(1) (but did not say that $\mathbb{P}_\alpha$ is nep itself). For this we need again to define what is $N\langle G \rangle$. The second treatment (in §5) depends just on Definition 4.4 from this section.

A reader who is not happy with Definition 4.4, may restrict himself to ord-transitive candidates consoling ourselves with Proposition 2.12. Also, instead of changing the $\mathbb{Q}$'s we may use free iterations instead of CS ones, see [25, Chapter IX], and $\mathrm{cl}_3(\mathbb{Q})$ in §5; the two are quite related.

Definition 4.2.
1. We define $\mathbb{P}$-names and their ranks for a forcing notion $\mathbb{P}$ slightly different than usual. We define by induction on the ordinal γ what it means for $\underset{\sim}{\tau}$ to be a $\mathbb{P}$-name of rank $\leq \gamma$, and for $G \subseteq \mathbb{P}$ generic over $\mathbf{V}$ what $\underset{\sim}{\tau}[G]$ mean
   (a) $\gamma = 0, \underset{\sim}{\tau} = \check{x}$ and $\underset{\sim}{\tau}[G] = x$,
   (b) $\gamma > 0$, $\underset{\sim}{\tau}$ is $\{(p_i, \underset{\sim}{\tau}_i) : i < i^*\}$, and $\underset{\sim}{\tau}[G] = \{\underset{\sim}{\tau}_i[G] : i < i^* \,\&\, p_i \in G\}$ where $p_i \in \mathbb{P}$, $\underset{\sim}{\tau}_i$ are $\mathbb{P}$-names of rank $< \gamma$. (Pedantically, we may use $\check{x} = \langle 0, x \rangle$ in (a) and $\underset{\sim}{\tau} = \langle \gamma, \{(p_i, \underset{\sim}{\tau}_i) : i < i^*\} \rangle$ in clause (b) but we normaly forget to use this).
2. For a regular cardinal κ we define when $\underset{\sim}{\tau}$ is $(< \kappa)$-hereditary[14] by induction on the rank γ: if $\gamma = 0$ always, if $\gamma > 0, \underset{\sim}{\tau} = \{(p_i, \underset{\sim}{\tau}_i)\} : i < i^*\}$ then each $\underset{\sim}{\tau}_i$ is and $i^* < \kappa$.

Definition 4.3.
1. Assume $N \models$"$\mathbb{Q}$ is a nep-forcing notion" and $G \subseteq \mathbb{Q}^N$ is generic over N. We define $N\langle G \rangle = N\langle G \cap \mathbb{Q}^N \rangle$ "ignoring $\mathbf{V}$"

[14] In §1.2 hereditary countable mean over the ordinals but the notions are different.

and[15] letting $\mathfrak{B}^{N\langle G\rangle} = \mathfrak{B}^N$ (and of course $\mathfrak{C}^{N\langle G\rangle} = \mathfrak{C}^N$ and $\mathrm{Ord}^N\langle G\rangle = \mathrm{Ord}^N$) for the relevant $\mathfrak{B}$ (and $\mathfrak{C}$). In details,

$$N\langle G\rangle \stackrel{\mathrm{def}}{=} \{\underset{\sim}{\tau}^N\langle G\rangle : N \models \text{“ } \underset{\sim}{\tau} \text{ is a } \mathbb{Q}\text{–name ”}\},$$

where $\underset{\sim}{\tau}^N\langle G\rangle$ is defined by induction on $\mathrm{rk}^N(\underset{\sim}{\tau})$ (see above in Definition 4.2):

(a) if for some $p \in G \cap \mathbb{Q}^N$ and $x \in N$ we have $N \models [p \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{\tau} = x\text{”}]$ then $\underset{\sim}{\tau}^N\langle G\rangle = x$,

(b) if not (a) then necessarily $N \models$“$\underset{\sim}{\tau}$ has the form $\{(p_i, \underset{\sim}{\tau}_i) : i < i^*\}$, $p_i \in \mathbb{Q}$, $\underset{\sim}{\tau}_i$ a $\mathbb{Q}$–name of rank $< \mathrm{rk}(\underset{\sim}{\tau})$”; now we let $\underset{\sim}{\tau}\langle G\rangle$ be $\underset{\sim}{\tau}\langle G, N\rangle$ (see below) if $\underset{\sim}{\tau}\langle G, N\rangle \notin \mathrm{Tc}(|N|)$ and be $(\underset{\sim}{\tau}\langle G, N\rangle) \cup \{|N|\}$ if $\underset{\sim}{\tau}\langle G, N\rangle \in \mathrm{Tc}(|N|)$, *where*

$$\underset{\sim}{\tau}\langle G, N\rangle = \{(\underset{\sim}{\tau}')^N\langle G\rangle : \underset{\sim}{\tau}' \in N \text{ and for some } p \in G \cap \mathbb{Q}^N \text{ we have } (p, \underset{\sim}{\tau}') \in \underset{\sim}{\tau} \cap N\}.$$

2. If $N \models$“$\underset{\sim}{\tau}$ is a $\mathbb{Q}$–name” we define a $\mathbb{Q}$–name $\underset{\sim}{\tau}^{\langle N\rangle}$ as follows: it is implicitly defined in part(2)), assuming $G \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$ and $G \cap \mathbb{Q}^N$ is generic over N.
3. We say “q is $\langle N, \mathbb{Q}\rangle$–generic” if $q \Vdash_{\mathbb{Q}}$“ $\underset{\sim}{G}_{\mathbb{Q}} \cap \mathbb{Q}^N$ is a subset of $(\mathbb{Q}^N, <^{\mathbb{Q}^N})$ generic over N”; see Definition 1.3(1).

A relative of 1.3(4) is

Definition 4.4. 1. Replacing “temporarily” by “K–absolutely” in Definition 1.3(1) means:

(a) if $\mathbf{V}_1$ is a K–extension of $\mathbf{V}$ (i.e., a generic extension of $\mathbf{V}$ by a forcing notion from $K^{\mathbf{V}}$), *then*
  (i) $\mathbf{V} \models$“$x \in \mathbb{Q}^{\bar{\varphi}}$” $\Rightarrow$ $\mathbf{V}_1 \models$“$x \in \mathbb{Q}^{\bar{\varphi}}$”,
  (ii) $\mathbf{V} \models$“$x <^{\mathbb{Q}^{\bar{\varphi}}} y$” $\Rightarrow$ $\mathbf{V}_1 \models$“$x <^{\mathbb{Q}^{\bar{\varphi}}} y$”,
  (iii) in the explicit case we have a similar demand for φ_2; otherwise, if N is a $\mathbb{Q}^{\bar{\varphi}}$–candidate in $\mathbf{V}$, $q \in \mathbb{Q}^{\bar{\varphi}}$ is $\langle N, \mathbb{Q}\rangle$–generic (see 4.3(3)) in $\mathbf{V}$, *then*[16] q is $\langle N, \mathbb{Q}\rangle$–generic in $\mathbf{V}_1$,

(b) if $\mathbf{V}_1$ is a K–extension of $\mathbf{V}$, *then* the relevant part of Definition 1.3 and clause (a) here holds in $\mathbf{V}_1$,

(c) if $\mathbf{V}_{\ell+1}$ is a K–extension of $\mathbf{V}_\ell$ for $\ell \in \{0, 1\}$, $\mathbf{V}_0 = \mathbf{V}$ *then* $\mathbf{V}_2$ is a K–extension of $\mathbf{V}_1$.

2. We omit K when we mean: any set forcing.

[15] If N is ord-hereditarily countable, this is the usual $N[G]$, as forcing add no ordinals, recalling that the ordinals are ureelements.

[16] So the explicit nep case seems not to imply the nep case.

Note that (a)(i) + (ii) of 4.4 is automatic for explicitly snep, also (a)(iii). One can make "absolutely nep" to the main case.

The following is natural to assume.

Definition 4.5. 1. We say ZFC_*^- is nice *if* ZFC_*^- says ZC (so that the power set axiom holds) and ZFC_*^- is preserved by forcing by set forcing notions.

2. We say $\mathbb{Q}$ is nice (or ZFC_*^- nice to $\mathbb{Q}$) *if* (it is a set definition and) for every axiom φ of ZFC_*^-, $\mathrm{ZFC}_*^- \vdash$ "$\Vdash_{\mathrm{Levy}(\aleph_0,\mathcal{P}(\mathbb{Q}))} \varphi$ ".

3. We say $\mathbb{Q}$ is weakly nice (for ZFC_*^-) if:
   (a) $\beth_n(\kappa')$ exists for each $n<\omega$,
   (b) ZFC_*^- is K–good for for $K=\{\mathbb{R} : \mathbb{R}$ is a forcing notion of cardinality at most $\beth_n(\kappa')$ for some $n<\omega\}$.

Remark 4.6. An alternative is: We say[17] ZFC_*^- is χ-nice *if* χ is a strong limit uncountable cardinal, which is $>\theta,\kappa(\mathfrak{B}),\alpha_*(\mathfrak{C})$, and χ serves as individual constant of ZFC_*^-, which say that it is strong limit, ZFC_*^- says that it is preserved by forcing by forcing notions of cardinality $<\chi$.

Claim 4.6.1. 1. *If* ZFC_*^- *is nice,* $\mathbb{Q}$ *a (definition of a) set forcing,* then ZFC_*^- *is nice to* $\mathbb{Q}$.

2. *Assume* ZFC_*^- *is nice to* $\mathbb{Q}$ *and*
   (a) $\mathbb{Q}$ *is explicitly nep,*
   (b) $\mathrm{ZFC}_*^- \vdash$ "$\mathbb{Q}$ *is explicitly nep".*

   Then
   (α) $\mathbb{Q}$ *is local for* ZFC_*^-
   (β) $\mathbb{Q}$ *is absolutely local explicit* (θ,κ)*–definition nep forcing for* ZFC_*^-.

Proof. Straight. ■

Proposition 4.7. *If N is a $\mathbb{Q}$–candidate, $\mathbb{Q}$ is a nep-forcing notion, $G\subseteq\mathbb{Q}$ is a subset of $(\mathbb{Q}^{\mathbf{V}},<_{\mathbb{Q}}^{\mathbf{V}})$ generic over $\mathbf{V}$ and $G_{\mathbb{Q}}\cap\mathbb{Q}^N$ is generic over N* then*:*

(a) *If $N\models$ "$\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name" and N is ord-hereditary (see Definition* 1.1(15)*) or just $N\models$ "$\underset{\sim}{\tau}$ is ord-hc ", then $\underset{\sim}{\tau}^N\langle G\rangle=\underset{\sim}{\tau}^{\langle N\rangle}[G]=\underset{\sim}{\tau}[G]$.*

(b) *$N\langle G\rangle$ is a model of ZFC_*^- and moreover it is a $\mathbb{Q}$–candidate and is a forcing extension of N for $\mathbb{Q}^N$, provided that the forcing theorem applies, i.e., ZFC_*^- is K–good, $\mathbb{Q}\in K$ (see Definition* 1.15*).*

(c) *$N\langle G\rangle\cap\kappa'=N\cap\kappa'$, so $N\langle G\rangle\cap\theta=N\cap\theta$, moreover $N\langle G\rangle\cap\mathrm{Ord}=N\cap\mathrm{Ord}$.*

[17]For natural $\mathfrak{C}$'s we have in mind it is natural to omit $\alpha_*(\mathfrak{C})<\chi$, but then we have to prove a suitable absoluteness lemma; so we avoid the issue here.

(d) *If N' is the ord-collapse of N (see 2.12) and* Mos_N *is the isomorphism from N onto N' and $G' = \mathrm{Mos}''_N(G \cap \mathbb{Q}^N)$, i.e., $G' = \{\mathrm{Mos}_N(p) : p \in \mathbb{Q}^N\}$*, then
  (α) *G' is a subset of $(\mathbb{Q}^{N'}, <^{N'}_{\mathbb{Q}})$ generic over N',*
  (β) $\mathrm{Mos}_{N\langle G\rangle}$ *extends* Mos_N *and is an isomorphism from $N\langle G\rangle$ onto $N'\langle G\rangle$.*

(e) *Assume that $\mathbb{Q}$ is simple (the main case anyhow).* Then *in clause (d) we can add*
  (γ) *$\mathbb{Q}^N = \mathbb{Q}^{N'}$ and $\mathrm{Mos}_N \restriction \mathbb{Q}^N$ is the identity,*
  (δ) *$N'\langle G \cap N\rangle = N'[G]$,*
  (ϵ) *$q \in \mathbb{Q}$ is $(N, \mathbb{Q})$–generic iff q is $(N', \mathbb{Q})$–generic.*

Proposition 4.8. *Assume*

(a) *$\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \le \alpha, j < \alpha\rangle$ is a CS iteration,*

(b) *for each $i < \alpha$*

$$\Vdash_{\mathbb{P}_i} \text{“ } (\bar{\varphi}_i, \mathfrak{B}_i) \text{ is a temporary } (\kappa_i, \theta_i)\text{–definition of a simple nep-forcing notion } \underset{\sim}{\mathbb{Q}}_i \text{ ”},$$

so $\mathfrak{C}$ is a constant and for simplicity $\Delta_2^{\mathfrak{B}_i}$ is quantifier free and the only parameter of $\bar{\varphi}_i$ is $\mathfrak{B}_i$, so we are demanding $\langle(\bar{\varphi}_i, \mathfrak{B}_i) : i < \alpha\rangle \in \mathbf{V}$, and for simplicity $\mathfrak{B}_i$ has universe κ_i (rather than $\mathcal{H}_{<\aleph_1}(\kappa_i)$),

(c) *$\mathfrak{B}$ is a model with the universe*[18] *κ^*, or including κ^* and included in $\mathcal{H}_{<\aleph_1}(\kappa^*)$, where $\kappa^* \ge \alpha$, $\kappa^* \ge \kappa_i = \kappa(\mathfrak{B}_i)$, $\kappa^* \ge \theta_i$ (for $i < \alpha$) and $\kappa^* \ge \alpha_*(\mathfrak{C})$ and $\mathfrak{B}$ codes $\langle(\mathfrak{B}_i, \bar{\varphi}_i, \theta_i) : i < \alpha\rangle$ (see below) and the functions $\alpha - 1, \alpha + 1$,*

(d) ZFC^*_- *is nice.*

We can use a vocabulary $\subseteq \{P_{n,m} : n, m < k^*(\le \omega)\}$, where $P_{n,m}$ is an n-place predicate to code $\langle \mathfrak{B}^i : i < \alpha\rangle$: let

$$P^{\mathfrak{B}}_{n+1,2m} = \{\langle i, x_1, \dots, x_n\rangle : \langle x_1, \dots, x_n\rangle \in P^{\mathfrak{B}_i}_{n,m}\},$$
$$P^{\mathfrak{B}}_{2,1} = \{(\alpha, \alpha+1) : \alpha + 1 < \alpha^*\}$$

(and $\Delta_2^{\mathfrak{B}}$ is the set of first order formulas).
Then*: if $N \subseteq (\mathcal{H}(\chi), \in)$ is an ord-hereditary $(\mathfrak{B}, \kappa)$–candidate, $p \in \mathbb{P}^N_\alpha$,* then *some condition $q \in \mathbb{P}_\alpha$ is $\langle N, \mathbb{P}_\alpha\rangle$–generic (in particular $\mathbb{P}_\alpha$ is defined from $\mathfrak{B}$) which is defined below, and it satisfies*

$$q \Vdash_{\mathbb{P}_\alpha} \text{“ if } \beta \in \mathrm{Dom}(p) \text{ then } p(\beta)\langle G_{\mathbb{P}_\beta}\rangle \in G_{\underset{\sim}{\mathbb{Q}}_\beta} \text{ ”}.$$

Definition 4.9. Under the assumptions of 4.8, inside N we have a definition of the countable support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \le \alpha, j < \alpha\rangle$. We define by induction on $j \in N \cap (\alpha + 1)$ when $q \in \mathbb{P}_j$ is $\langle N, \mathbb{P}_j\rangle$–generic:

[18] No real need for κ^* to be a cardinal.

($\circledast$) if $q \in G_j \subseteq \mathbb{P}_j$ and G_j is generic over $\mathbf{V}$, *then* $G_j^{\langle N \rangle}$ is a generic subset of $\mathbb{P}_j^N$ over N, where

(a) $G_j^{\langle N \rangle} \stackrel{\text{def}}{=} \{p : N \models \text{“}p \in \mathbb{P}_j\text{ ” and } p^{\langle\langle N \rangle\rangle} \in G_j\}$,

(b) $p^{\langle\langle N \rangle\rangle}$ is a function with domain $\mathrm{Dom}(p)^N$, and $p(\gamma)$ is the following $\mathbb{P}_\gamma$–name: *if* $p(\gamma)^{\langle N \langle \underset{\sim}{G}_\gamma \cap N \rangle\rangle} \in \underset{\sim}{\mathbb{Q}}_\gamma$, *then* it is $p(\gamma)$; *if* not, *then* it is $\emptyset_{\underset{\sim}{\mathbb{Q}}_\gamma}$.

Remark 4.10. The major weakness of 4.8 is that $\mathbb{P}_\alpha$ is not proved to be in some of our classes (nep or snep). We get the “main original property” without the “support team”, i.e., the $\underset{\sim}{\mathbb{Q}}_i$ are nep, but on $\mathbb{P}_\alpha$ we just say it satisfies the main part of nep. A minor one is that $\mathfrak{B}_i$ is not allowed to be a $\mathbb{P}_i$–name in any way, both are dealt with in Section 5. In later theorems (in the next section), we use $\mathbb{P}'_\alpha \subseteq \mathbb{P}_\alpha$ consisting of “hereditarily countable” names.

Note: inside N, if “$N \models p \in \mathbb{P}_\alpha$” then $\mathrm{Dom}(p_\alpha) \in [\alpha]^{\leq\aleph_0}$ in N's sense hence (see Definition 1.1(7)(c)) $\mathrm{Dom}(p_\alpha) \subseteq N$ and similarly the names are actually from N, members outside N do not count, they may not be in $\mathbb{P}_\alpha$ at all.

Proof of 4.8. We imitate the proof of the preservation of properness. So we prove by induction on $j \in (\alpha+1) \cap N$ that:

$(*)_j$ if $p \in \mathbb{P}_\alpha^N, i \in j \cap N$, $q \in \mathbb{P}$ is $\langle N, \mathbb{P}_i \rangle$–generic, and $q \Vdash_{\mathbb{P}_i}$ “$(p \restriction i)^{\langle N \rangle} \in \underset{\sim}{G}_{\mathbb{P}_i}$”,
then we can find a condition $r \in \mathbb{P}_j$ such that r is $\langle N, \mathbb{P}_j \rangle$–generic, $\mathrm{Dom}(r) \setminus i \subseteq N$, and $r \restriction i = q$, and $r \Vdash_{\mathbb{P}_j}$ “ $(p \restriction j)^{\langle N \rangle} \in \underset{\sim}{G}_{\mathbb{P}_j}$”.

Case 0: $j = 0$.
Left to the reader.

Case 1: $j = j_1 + 1$.
So $j_1 \in N$ (why? use $P_{2,1}$ and 1.3(2)(e)), and by the inductive hypothesis and the form of the conclusion without loss of generality $i = j_1$. Let $q \in G_i \subseteq \mathbb{P}_i$, G_i generic over $\mathbf{V}$. So $N\langle G_i^{\langle N \rangle}\rangle \cap \kappa^* = N \cap \kappa$ (by 4.7), and hence $\mathfrak{B} \restriction N\langle G_i^{\langle N \rangle}\rangle = \mathfrak{B} \restriction N \prec_{\Delta_1} \mathfrak{B}$. But $i \in N$, so this applies to $\mathfrak{B}_i$, too. So $\mathbf{V}[G_i] \models$“ $N\langle G_i^{\langle N \rangle}\rangle$ is a $\mathfrak{B}_i$–candidate”. Also $N\langle G_i^{\langle N \rangle}\rangle \models$“ $p(i)^{\langle N \langle G_i^{\langle N \rangle}\rangle\rangle} \in \mathbb{Q}_i$” because $G_i^{\langle N \rangle}$ is a generic subset of $\mathbb{P}_i^N = \{x : x \in N,\ N \models\text{“ } x \in \mathbb{P}_i \text{ ”}\}$ over N and use the property of $\mathbb{Q}_i$.

Case 2: j is a limit ordinal.
As in the proof for properness (see [25, Chapter III, 3.2]). $\blacksquare_{4.8}$

Remark 4.11. Note that if $N \models$"w is a subset of α" then we can deal with $\mathbb{P}_w$, see later in §5.

5. True preservation theorems

Let us recall that $\mathbb{Q}$ is nep if "$p \in \mathbb{Q}$", "$p \leq_{\mathbb{Q}} q$" are defined by upward absolute formulas for models N which are $\mathbb{Q}$-candidates so $N = (|N'|, \in\restriction |N|, \mathfrak{B}^N, \mathfrak{C}^N)$ where $(|N|, \in\restriction |N|) \subseteq (\mathcal{H}(\chi), \in)$ countable, $\mathfrak{B}^{\mathbb{Q}} \in N$ a model on some κ (or $\mathcal{H}_{<\aleph_1}(\kappa)$), $\mathfrak{B}^{\mathbb{Q}}\restriction N \prec_{\Delta_2} \mathfrak{B}^{\mathbb{Q}}$, N model of ZFC$^-_*$ and for each such model we have the properness condition. Usually $\mathbb{Q} \subseteq {}^\omega\theta$, or $\mathbb{Q} \subseteq \mathcal{H}_{<\aleph_1}(\theta)$ or so. We would like to prove that CS iteration preserves "being nep", but CS (countable support) iteration may give "too large" names of conditions (of $\underset{\sim}{\mathbb{Q}}_i$, $i > 0$) depending say on large maximal antichains (of $\mathbb{P}_i$). Note: if $\mathbb{Q}_0$ is not c.c.c., typically it has maximal antichains which are not absolutely maximal antichains: start with a perfect set of pairwise incompatible elements and extend it to a maximal antichain. Then whenever a real is added, the maximality is lost. Finally, c.c.c. is normally lost in $\mathbb{P}_\omega$. So we will revise our iteration so that we consider only hereditarily countable names (as in 4.7 above).

But in the iteration, trying to prove a case of properness for a candidate N and $p \in \mathbb{P}^N_{\alpha+1}$, considering $q \in \mathbb{P}_\alpha$ which is $\langle N, \mathbb{P}^N_\alpha\rangle$–generic, we know that in $\mathbf{V}[G_{\mathbb{P}_\alpha}]$ (if $q \in G_{\mathbb{P}_\alpha}$), there is $q' \in \underset{\sim}{\mathbb{Q}}_\alpha[G_{\mathbb{P}_\alpha}]$ which is $\langle N[G_{\mathbb{P}_\alpha}], \mathbb{Q}_\alpha[G_\alpha]\rangle$–generic. But under present circumstances, we have no idea where to look for q', so no way to make a name of it, $\underset{\sim}{q}'$, which is hereditarily countable, without increasing $q \in \mathbb{P}_\alpha$. Except when $\underset{\sim}{\mathbb{Q}}$ is local (see 1.11), of course; it is not unreasonable to assume it but we prefer not to and even then, we just have to look for it in, essentially, a copy of the set of reals. The solution is to increase $\mathbb{Q}_i$ insubstantially so that we will exactly have the right element q', essentially it is:

$$p(\alpha) \ \& \bigwedge_{\mathcal{I} \in \mathrm{pd}_{\mathbb{Q}}(N)} \ \bigvee_{p \in \mathcal{I}[N]} p,$$

as explained below. We give two variants. That is, toward the iteration Theorem 5.18 the derivation of the first variant of the forcing is done in 5.2, 5.3, 5.4, and the second is done in 5.7, 5.8. But Definition 5.10 + Proposition 5.12 dealing with hc-names are essential, as well as 5.13, 5.14 being straight and very straight in particular. Those are stronger properties for the $\bar{\varphi}$ though not for $\mathbb{Q}$, as we deduce to be the case in 5.14(3) for the first variant and in 5.14(4) if you use the second variant. Being straight will be clear if you recall Observation 1.22.

Notation 5.1. Let

$$\mathrm{pd}_{\mathbb{Q}}(N) = \mathrm{pd}(N,\mathbb{Q}) = \{\mathcal{I} : N \models \text{“ } \mathcal{I} \text{ is a predense subset of } \mathbb{Q} \text{ ”} \},$$

and $\mathcal{I}[N] = \mathcal{I}^N = \mathcal{I} \cap N$ and

$$\begin{aligned}&\mathrm{pdac}_{\mathbb{Q}}(p,N) = \mathrm{pdac}(p,\mathbb{Q},N) = \\ &\quad \{\mathcal{I} : N \models \text{“ is a predense antichain of } \mathbb{Q}^N \text{ above } p \text{ ”} \},\end{aligned}$$

and similalry without p. Let

$$\mathrm{do}_{\mathbb{Q}}(N) = \mathrm{do}(N,\mathbb{Q}) = \{\mathcal{I} : N \models \text{ “ } \mathcal{I} \text{ is dense open subset of } \mathbb{Q} \text{ ”} \};$$

note that $\mathcal{I}$ is here a set and not a definition, as we are dealing with set candidates.

Definition 5.2. Let $\mathbb{Q}$ be an explicitly nep-forcing notion. *Then* we define $\mathbb{Q}' = \mathrm{cl}(\mathbb{Q})$; this depend on our choice of the family of the candidates, e.g., on $(\mathrm{ZFC}^-_*, \mathfrak{C}, \mathfrak{B})$, as follows:

(a) the set of elements [19] is

$$\mathbb{Q} \cup \Big\{p \;\&\; \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q})} \;\bigvee_{r\in\mathcal{I}[N]} r : \quad p \in \mathbb{Q}^N \text{ and } N \text{ is a } \mathbb{Q}\text{–candidate}\Big\}$$

[we are assuming no incidental identification] but

(α) code them in any reasonable way, if $\mathbb{Q}$ is simple, as members of $\mathcal{H}_{<\aleph_1}(\theta)$,

(β) for snep (or very simple) $\mathbb{Q}$ we work slightly more to code them as members of ${}^{\omega}\theta$, pedantically easier in ${}^{\omega}(\theta+\omega)$,

(b) the order $\leq_{\mathbb{Q}'}$ is given by $q_1 \leq_{\mathbb{Q}'} q_2$ if and only if one of the following occurs:

(α) $q_1, q_2 \in \mathbb{Q}$, and $q_1 \leq_{\mathbb{Q}} q_2$,

(β) $q_1 \in \mathbb{Q}$, $q_2 = p \;\&\; \Big(\bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q})} \;\bigvee_{r\in\mathcal{I}[N]} r\Big)$ and $q_1 \leq_{\mathbb{Q}} p$,

(γ) $q_1 = p \;\&\; \Big(\bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q})} \;\bigvee_{r\in\mathcal{I}[N]} r\Big)$ and $q_2 \in \mathbb{Q}$, $p \leq_{\mathbb{Q}} q_2$ and if $\mathcal{I} \in \mathrm{pdac}(N,\mathbb{Q})$, then

$$\begin{aligned}(\exists q' \in \mathbb{Q})(\exists \langle p_n : n \in \omega\rangle)(q' \leq_{\mathbb{Q}} q_2 \;\&\; \varphi_2^{\mathbb{Q}}(\dots, p_n, \dots, q') \;\&\\ \{p_n : n < \omega\} \subseteq \mathcal{I}^N),\end{aligned}$$

[19] The sentence $p \;\&\; \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q})} \;\bigvee_{r\in\mathcal{I}[N]} r$ is not accurate, our intension would be better served by adding $p \;\&\; \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q})} \;\bigwedge_{r_1\neq r_2\in\mathcal{I}[N]} \psi_{\neg(r_1\wedge r_2\wedge p)}$ but anyhow we do not really use the meaning. We could have used $can(p,N')$, $N \cong N'$, N' is hc-ord (“*can*” for canonical explicitly generic) and, e.g., $\mathrm{Ord}^N \subseteq \kappa' + \omega_1$.

(δ) $q_\ell = p_\ell \ \& \left(\bigwedge_{\mathcal{I} \in \mathrm{pdac}(p, N_\ell, \mathbb{Q})} \ \bigvee_{r \in \mathcal{I}[N_\ell]} r \right)$ (for $\ell = 1, 2$) and: $q_1 = q_2$ *or* $q_1 \le p_2$ by clause (γ).

Remark 5.3. 1. In [25, Chapter III], for a hereditarily countable name, instead of

$$p \ \& \bigwedge_{\mathcal{I} \in \mathrm{pdac}_p(N, \mathbb{Q})} \ \bigvee_{r \in \mathcal{I}[N]} r$$

we use the first member of $\mathbb{Q}_{\underset{\sim}{i}}$ which forces this. Simpler, but when we ask whether this guy is $\le q$ (for some $q \in \mathbb{Q}$) we run into uncountable antichains.

2. We may weaken the demand on N being a candidate, by demanding less than ZFC^-_*. Anyhow at present we are not sensitive to the exact choice of ZFC^-_* so using two such versions of set theory as done in 5.4 suffices. We can consider there $\mathrm{ZFC}^-_* = \mathrm{ZFC}^-_{**}$ but it will be somewhat artificial: in some $\mathbb{Q}$ candidates, N we will have $(\mathbb{Q}')^N = \mathbb{Q}^N$.
3. We could allow q is $p \ \& \bigwedge_{\mathcal{I} \in \mathrm{pdac}(N, \mathbb{Q}')} \ \bigvee_{r \in \mathcal{I}[N]} r$ but the gain is small — if there are canonically $\langle N, \mathbb{Q} \rangle$–generic we do not have to increase p.

Proposition 5.4. 1. *Assume $\mathbb{Q}$ is explicitly nep.* Then*:*

(a) *in Definition* 5.2, $\mathbb{Q}'$ *is a (quasi) order,*
(b) $\le_{\mathbb{Q}'} \restriction \mathbb{Q} = \le_{\mathbb{Q}}$,
(c) $\mathbb{Q}$ *is a dense subset of* $\mathbb{Q}'$.

2. *Assume in addition:*

($\boxtimes_2$) *for some suitable versions of set theory* ZFC^-_{**} *we have*
  (α) $\mathbb{Q}$ *is explicitly nep for* ZFC^-_{**},
  (β) $\mathrm{ZFC}^-_* \vdash \mathrm{ZFC}^-_{**}$,
  (γ) $\mathrm{ZFC}^-_* \vdash$ "ZFC^-_{**} *is weakly normal for* $\mathbb{Q}'$, $\mathbb{Q}$ *is explicitly nep for* ZFC^-_{**}",
($\boxtimes_3$) $\mathbb{Q}' = \mathrm{cl}(\mathbb{Q})$ *is defined*[20] *as in Definition* 5.2 *for* ZFC^-_{**},
($\boxtimes_4$) $\mathrm{ZFC}^-_{\mathbb{Q}'}$ *is* ZFC^-_* *and* $\mathfrak{B}^{\mathbb{Q}'}$ *are defined inside the proof.*

Then*:*

(d) *if N is a $\mathbb{Q}'$–candidate, and* $N \models$ "$p \in \mathbb{Q}'$",then *for some $q \in N$ we have* $N \models$ "$p \le_{\mathbb{Q}'} q \ \& \ q \in \mathbb{Q}$",
(e) $\mathbb{Q}'$ *is explicitly nep (with the same* $\mathfrak{B}^{\mathbb{Q}}$ *and parameters),*
(f) *if N is a $\mathbb{Q}'$-candidate, $p \in \mathbb{Q}^N$ and* $q = p \ \& \bigwedge_{\mathcal{I} \in \mathrm{pd}(N, \mathbb{Q})} \bigvee \{r : r \in \mathcal{I}^N\}$, then *$q$ is $(N, \mathbb{Q}')$–generic.*
(g) $\mathbb{Q}'$ *is correct if* $\mathbb{Q}$ *is correct.*

[20] We could have alternatively change $\mathfrak{B}$, but this seems more straightforward.

3. *We can replace above (in the assumption and conclusion) nep by snep, or nep by simple nep; similarly for K–absolutely.*

Remark 5.5. The definition of "local" (in 1.11(1)) can be handled a little differently. We can (in 1.11(2)) demand less on N' (it is not a $\mathbb{Q}$–candidate), just have some of its main properties and in $\boxtimes_3$ of 5.4(3), ZFC^-_* says that $\mathcal{P}(\theta)$ is a set (so has a cardinality) and is a $\mathbb{Q}$–candidate. So we may consider having ZFC^-_ℓ for several ℓ's, ZFC^*_ℓ speaks on $\chi_0 > \ldots > \chi_{\ell-1}$ and the generic extensions of a model of $\mathrm{ZFC}^*_{\ell+1}$ for the forcing notion $\mathrm{Levy}(\aleph_0, \chi_\ell)$ is a model of ZFC^-_ℓ. Similar remarks hold for §7. But, as we can deal with the nice case (see Definition 4.5 above), we may start with a countable $N \prec (\mathcal{H}(\beth_\omega), \in)$ (or even better $(\mathcal{H}(\beth_{\omega_1}), \in)$ so that "countable depth can be absorbed"), we ignore this in our main presentation.

Proof of 5.4. 1) **Clause (a)**: Assume $q_1 \le q_2 \le q_3$; we have $2^3 = 8$ cases according to truth values of $q_\ell \in \mathbb{Q}$:

Case (A): $q_1, q_2, q_3 \in \mathbb{Q}$.
Trivial (as $\le_{\mathbb{Q}}$ is transitive).

Case (B): $q_1, q_2 \in \mathbb{Q}$, $q_3 \notin \mathbb{Q}$.
Check (i.e., $q_1 \le_{\mathbb{Q}} q_2$ and clause (β) of Definition 5.2(b) apply to $q_2 \le q_3$, so letting

$$q_3 = p_3 \quad \& \quad (\bigwedge_{\mathcal{I} \in \mathrm{pdac}(p_3, N, \mathbb{Q})} \ \bigvee_{r \in \mathcal{I}[N]} r)$$

we have $q_2 \le_{\mathbb{Q}} p_3$; but $q_1 \le_{\mathbb{Q}} q_2$ so $q_1 \le_{\mathbb{Q}} p_3$ so $q_1 \le q_3$ by clause (b)(β) of Definition 5.2.)

Case (C): $q_1 \notin \mathbb{Q}$, $q_2, q_3 \in \mathbb{Q}$.
Check (similar to case (B), using clause (b)(γ) of Def 5.2, using the same witness q' for any $\mathcal{I} \in \mathrm{pdac}(p_1, N, \mathbb{Q})$.

Case (D): $q_1 \in \mathbb{Q}$, $q_2 \notin \mathbb{Q}$, $q_3 \in \mathbb{Q}$.
Let $q_2 = p_2 \ \& \bigwedge_{\mathcal{I} \in \mathrm{pdac}(p_2, N, \mathbb{Q})} \ \bigvee_{r \in \mathcal{I}[N]} r$, hence $q_1 \le_{\mathbb{Q}} p_2$ (because $q_1 \le q_2$ by 5.2(b)(β)) and $p_2 \le_{\mathbb{Q}} q_3$ (because $q_2 \le q_3$ by 5.2(b)(γ)). Hence $q_1 \le_{\mathbb{Q}} q_3$ follows.

Case (E): $q_1 \in \mathbb{Q}$, $q_2 \notin \mathbb{Q}$, $q_3 \notin \mathbb{Q}$.
Let $q_\ell = p_\ell \ \& \bigwedge_{\mathcal{I} \in \mathrm{pdac}(p_\ell, N_\ell, Q)} \ \bigvee_{r \in \mathcal{I}[N_\ell]} r$ for $\ell = 2, 3$. So $q_1 \le_{\mathbb{Q}} p_2$ (because $q_1 \le_{\mathbb{Q}} q_2$ by 5.2(b)(β)) and $p_2 \le_{\mathbb{Q}} p_3$ (because $q_2 \le_{\mathbb{Q}} q_3$ by 5.2(b)$(\gamma), (\delta)$). Hence $q_1 \le_{\mathbb{Q}} p_2$ (as $\le_{\mathbb{Q}}$ is transitive) and so $q_1 \le q_3$ (by 5.2(b)(β)).

Case (F): $q_1 \notin \mathbb{Q}$, $q_2 \notin \mathbb{Q}$, $q_3 \in \mathbb{Q}$.
Let $q_\ell = p_\ell \ \& \bigwedge_{\mathcal{I} \in \mathrm{pdac}(p_\ell, N_\ell, \mathbb{Q})} \ \bigvee_{r \in \mathcal{I}[N_\ell]} r$ for $\ell = 1, 2$ and suppose that $q_1 \ne q_2$

(otherwise trivial). Then, by 5.2(b)(δ), $q_1 \le p_2$ and by 5.2(b)(γ), $p_2 \le q_3$ so by the previous case (C), $q_1 \le q_3$ as required.

Case (G): $q_1 \notin \mathbb{Q}$, $q_2 \in \mathbb{Q}$, $q_3 \notin \mathbb{Q}$.
Let $q_\ell = p_\ell \,\&\, \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p_\ell,N,\mathbb{Q})} \bigvee_{r\in\mathcal{I}[N]} r$ for $\ell = 1,3$. Now, by 5.2(b)(β), $q_2 \le p_3$ and by the previous case (C), $q_1 \le p_3$ and hence, by 5.2(b)(β), $q_1 \le q_3$ as required.

Case (H): $\bigwedge_\ell q_\ell \notin \mathbb{Q}$.
Let $q_\ell = p_\ell \,\&\, \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p_\ell,N,\mathbb{Q})} \bigvee_{r\in\mathcal{I}[N]} r$ for $\ell = 1,2,3$. If $q_1 = q_2$ or $q_2 = q_3$ then the conclusion is totally trivial. So assume not. Thus

$$\begin{array}{ll} q_1 \le p_2 & \text{(by clause 5.2}(\delta)\text{ a case defined in 5.2}(\gamma)) \\ q_2 \le p_3 & \text{(by clause 5.2}(\delta)). \end{array}$$

Hence $p_2 \le p_3 \le q_3$ (see clause 5.2 (γ)), so the previous case (G) applies.
This finishes the proof of the clause (a) of 5.4(1).

Clause (b): Totally trivial.

Clause (c): Let $q \in \mathbb{Q}'$; if $q \in \mathbb{Q}$, then there is nothing to do; otherwise for some $\mathbb{Q}$–candidate N and p ($\in \mathbb{Q}^N$) we have $q = p \,\&\, \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q})} \bigvee_{r\in\mathcal{I}[N]} r$ and use nep (i.e., clause (c) of 1.3(1)) on the $\mathbb{Q}$–candidate N and p ($\in \mathbb{Q}^M$) and 5.2(b)(γ) from the definition of the order on $\mathbb{Q}'$.

2) Assume ($\boxtimes_2$).
Clause (d): Proved inside the proof of clause (e).

Clauses (e), (f), (g): We have to define

$$\varphi_0^{\mathbb{Q}'}, \varphi_1^{\mathbb{Q}'}, \varphi_2^{\mathbb{Q}'}, \mathfrak{B}^{\mathbb{Q}'}, \theta^{\mathbb{Q}'}$$

and then prove the required demands for a $\mathbb{Q}'$–candidates. We let $\mathfrak{B}^{\mathbb{Q}'} = \mathfrak{B}^{\mathbb{Q}}$, $\theta^{\mathbb{Q}'} = \theta^{\mathbb{Q}}$, the formulas will be different, but with the same parameters. But recall that the versions of set theory are different, so we know only that every $\mathbb{Q}'$–candidate is a $\mathbb{Q}$–candidate, but not inversely. We say that M is an str $\mathbb{Q}$–candidate *if* M is a $\mathbb{Q}$-candidate and

$$M \models \text{“ } \mathbb{Q} \text{ is nep and weakly normal ”};$$

clearly ZFC_*^- implies that every countable set is included in a str $\mathbb{Q}$-candidate. What is $\varphi_0^{\mathbb{Q}'}$? It is

$$\begin{aligned} \varphi_0^{\mathbb{Q}}(x) \vee \text{“ } x \text{ has the form } \quad & p \,\&\, \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,M,\mathbb{Q})} \bigvee_{r\in\mathcal{I}[M]} r, \qquad \text{where} \\ & M \text{ is a str } \mathbb{Q}\text{-candidate (so countable) and } M \models \varphi_0^{\mathbb{Q}}(p)\text{”}. \end{aligned}$$

Why is "M is a str $\mathbb{Q}$–candidate" expressible in the relevant language? As we can define ZFC^-_{**} (see definition 1.15(1) clause (f)), and the way we have phrased Definition 1.1.

Clearly if $\mathbb{Q}$ is correct then $\varphi_0^{\mathbb{Q}'}$ defines $\mathbb{Q}'$ through $\mathbb{Q}'$–candidates so correctness is preserved, i.e., clause (g). Note that if N is a $\mathbb{Q}'$–candidate (or even just $\mathbb{Q}$-candidate) and $N \models$"M is a $\mathbb{Q}$–candidate", then we have $M \subseteq N$, because $N \models$ "$\{x : x \in M\}$ is countable", also $M \models$"x is countable", implies $x \subseteq M \subseteq N$; hence M is really a $\mathbb{Q}$–candidate, similarly for str $\mathbb{Q}$-candidate. Consequently, $\varphi_0^{\mathbb{Q}'}$ is upward absolute for $\mathbb{Q}'$-candidates and it defines the set of elements of $\mathbb{Q}'$. So clause (a) of Definition 1.3(1) holds.

Now we pay our debt proving clause (d). Let N be a $\mathbb{Q}'$–candidate and assume $N \models$"$p \in \mathbb{Q}'$", i.e., $N \models \varphi_0^{\mathbb{Q}'}(p)$. By the definition of $\mathbb{Q}'$, either $N \models$"$p \in \mathbb{Q}$" and we are done, or for some $p', M \in N$ we have

$$\begin{aligned} &N \models \text{“}M \text{ is a } \mathbb{Q}\text{–candidate, } p' \in \mathbb{Q}^M, \text{ and} \\ &p = \big(p' \;\&\; \bigwedge_{\mathcal{I} \in \mathrm{pdac}(p'_a, M, \mathbb{Q})} \; \bigvee_{r \in \mathcal{I}[M]} r\big)\text{”}. \end{aligned}$$

By clause (γ) of the assumption $(\boxtimes_2)$, for some $q \in \mathbb{Q}^N$ we have

$$N \models \text{“ } q \text{ is explicitly } \langle M, \mathbb{Q}\rangle\text{–generic ”}$$

and $N \models$"$p' \leq_{\mathbb{Q}} q$". By Definition 1.3(2) for some

$$\langle \langle r_{\mathcal{I},\ell} : \ell < \omega\rangle : \mathcal{I} \in \mathrm{pdac}(p, M, \mathbb{Q})\rangle \in N$$

we have:

- $N \models$" $\{r_{\mathcal{I},\ell} : \ell < \omega\}$ enumerates $\mathcal{I}[M]$ ", and
- $N \models$" $\varphi_2^{\mathbb{Q}}(r_{\mathcal{I}_\ell,0}, r_{\mathcal{I},1}, \dots, q)$ ".

It follows from the definition of $\mathbb{Q}'$ that $N \models$"$p \leq_{\mathbb{Q}'} q$", so q is as required.

Now clause (f) follows easily.

What is $\varphi_1^{\mathbb{Q}'}$? Just write the definition of $p \leq_{\mathbb{Q}'} q$ from clause (b) of 5.2. Clearly also $\varphi_1^{\mathbb{Q}'}$ is upward absolute for $\mathbb{Q}'$–candidates and it defines the partial order of $\mathbb{Q}'$ (even in $\mathbb{Q}'$–candidates and even in $\mathbb{Q}$–candidate). So clause (b) of Definition 1.3(1) holds.

What is $\varphi_2^{\mathbb{Q}'}$? Let it be:

$\varphi_2^{\mathbb{Q}'}(p_0, p_1, \dots, p_\omega) \overset{\mathrm{def}}{=} [\varphi_2^{\mathbb{Q}}(p_0, \dots, p_\omega)$ *or* "there are M, p, q such that: M is a str $\mathbb{Q}$–candidate and $p \in \mathbb{Q}^M$ and $q = \big(p \;\&\; \bigwedge_{\mathcal{I} \in \mathrm{pdac}(p,M,\mathbb{Q})} \; \bigvee_{r \in \mathcal{I}[M]} r\big)$ satisfying $q \leq_{\mathbb{Q}'} p_\omega$ and $p_n \in (\mathbb{Q}')^M$ and for $m < n < \omega$, in $M \models$ "p, p_n, p_m has no common upper bound" and for some

$\mathcal{J} \in \mathrm{pdac}(p, M, \mathbb{Q})$, if $r \in \mathcal{J}[M]$ then there is ℓ such that $M \models$ "$p_\ell \leq_{\mathbb{Q}'} r$]".

To show that $\varphi_2^{\mathbb{Q}'}$ is upward absolute for $\mathbb{Q}'$–candidates suppose that N is a $\mathbb{Q}'$–candidate and $N \models \varphi_2^{\mathbb{Q}'}(p_0, p_1, \dots, p_\omega)$. Then, in N, if $\varphi_2^{\mathbb{Q}}(p_0, \dots, p_\omega)$ holds we are done easily so assume the second clause in the definition of $\varphi_2^{\mathbb{Q}'}$ holds, say for M, p, q. So in particular, in N, M is a str $\mathbb{Q}$—candidate, $p \in \mathbb{Q}$, $q \in \mathbb{Q}'$ and for some $\mathcal{J} \in \mathrm{pdac}(p, M, \mathbb{Q})$ we have:

if $r \in \mathcal{J}[M]$, then there is ℓ such that $p_\ell \leq_{\mathbb{Q}'} r$.

By the known upward absoluteness all those statements hold in $\mathbf{V}$ too, and in any $\mathbb{Q}'$-candidate N' satisfying $N \subseteq N'$. Next we prove that

($\boxtimes$) $\varphi_2^{\mathbb{Q}'}(p_0, p_1, \dots, p_\omega)$ implies $\{p_0, \dots\}$ is a predense antichain in $\mathbb{Q}'$ above p_ω, inside N', where N' is a $\mathbb{Q}'$–candidate or N' is $\mathbf{V}$.

So assume (ignoring the trivial case) that $\varphi_2^{\mathbb{Q}'}(p_0, p_1, \dots, p_\omega)$ holds as witnessed by N and $p, q \in \mathbb{Q}^N$ and $\mathcal{J} \in \mathrm{pdac}(p, N, \mathbb{Q})$.

Let $G' \subseteq \mathbb{Q}'$ be generic over $\mathbf{V}$ or just $G' \subseteq (\mathbb{Q}')^{N'}$ generic over N', such that $p_\omega \in G'$. Let G be $G' \cap \mathbb{Q}^{N'}$. Now clearly G' is a subset of $\mathbb{Q}^{N'}$, and by part (1) (which holds also in every ZFC^-_*–candidate) we know that G is $\leq_{\mathbb{Q}^{N'}}$-directed, and moreover is generic over N'. Similarly, as $p_\omega \in G'$ hence $p_\omega \in G$, and clearly there is $q' \in G$ such that $p_\omega \leq_{\mathbb{Q}'} q'$. But $q \leq_{\mathbb{Q}'} p_\omega$ (by the choice of $N, p, q, \mathcal{J}$). As $\leq_{\mathbb{Q}'}$ is transitive, clearly $q \leq q'$ for $\leq_{\mathbb{Q}'}$, and so (by clause (b) (γ) of Definition 5.2 of the quasi order $\leq_{\mathbb{Q}'}$) there is $q'' \in \mathbb{Q}$ such that $q'' \leq_{\mathbb{Q}} q'$ and q'' is explicitly $\langle N, \mathbb{Q}\rangle$–generic. As $q'' \leq_{\mathbb{Q}} q'$, $q' \in G$ and G is a subset of $\mathbb{Q}^N$ generic over N, clearly $q'' \in G$. So $G \cap \mathbb{Q}^N$ is generic over N and of course $p \in G \cap \mathbb{Q}^N$.

Now

$(*)$ if $q'_* \in G' \cap (\mathbb{Q}')^N$, *then* for some $q_* \in G \cap \mathbb{Q}^N$ we have $N \models q'_* \leq_{\mathbb{Q}'} q_*$.

[Why? If $q'_* \in \mathbb{Q}^N$ we let $q_* = q'_*$, and we are done. So we may assume that for some M, p_* we have

$$N \models \text{" } q'_* = p_* \& \bigwedge_{\mathcal{I} \in \mathrm{pdac}(p,M,\mathbb{Q})} \bigvee_{r \in \mathcal{I}[M]} r \text{ and } M \text{ is a } \mathbb{Q}\text{–candidate and } p_* \in \mathbb{Q}^M\text{"} .$$

In N we can define

$$\begin{aligned}\mathcal{J} = \{r \in \mathbb{Q} : \ & r \text{ satisfies one of the following:}\\ & \text{(a) } r, p_* \text{ are incompatible (by } \leq_{\mathbb{Q}}),\\ & \text{(b) } p_* \leq r \text{ and for some } \mathcal{I} \in \mathrm{pdac}(p_*, M, \mathbb{Q}) \text{ we have}\\ & \qquad r \Vdash_{\mathbb{Q}} |\underset{\sim}{G}_{\mathbb{Q}} \cap \mathcal{I}^M| \neq 1\\ & \text{(c) for some } r', p_* \leq_{\mathbb{Q}} r' \leq_{\mathbb{Q}} r'' \leq_{\mathbb{Q}} r \text{ and for some}\\ & \ \mathbb{Q}\text{–candidate } M'_* \text{ we have } r', M \in M'_* \text{ and}\\ & M'_* \models \text{“ } p_* \leq r' \text{ and } r' \text{ is } \langle M, \mathbb{Q}\rangle\text{–generic and is above } p_* \text{ ”}\\ & \text{and } r'' \text{ is explicitly } \langle M'_*, \mathbb{Q}\rangle\text{–generic}\}\end{aligned}$$

Clearly $N \models$ “$\mathcal{J}$ is a dense open subset of $\mathbb{Q}$” (recalling that N is a str $\mathbb{Q}$–candidate). But $G \cap \mathbb{Q}^N$ is generic over N and hence there is $r^* \in \mathcal{I}^N \cap G$, so r^* satisfies (a) or (b) or (c). If it satisfies clause (a), then $p, r^* \in G \cap \mathbb{Q}^N$ but $G \cap \mathbb{Q}^N$ is generic over N, so they are compatible in $\mathbb{Q}^N$, a contradiction.

If r^* satisfies clause (b) say for $\mathcal{I}$, then

$$N \models r^* \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{G} \cap \mathcal{I}^M \text{ is not a singleton ”},$$

hence $N\langle G \cap \mathbb{Q}^N\rangle \models$ “$(G \cap \mathbb{Q}^N) \cap \mathcal{I}^M$ is not a singleton”. Therefore (by absoluteness) $N'[G] \models$ “$G \cap \mathcal{I}^M = (G \cap \mathbb{Q}^N) \cap \mathcal{I}^N$ is not a singleton”. But $q'_* \in G \cap \mathbb{Q}^N$, so we may easily show that “$G \cap \mathbb{Q}^M \subseteq \mathbb{Q}^M$ is generic over M”, a contradiction.

So necessarily r^* satisfies clause (c), say for r', r'', M'_*. Now as $r^* \in G$ and

$$N \models \text{“ } r'' \text{ is explicitly } \langle \mathbb{Q}, M'_*\rangle\text{–generic above } r' \text{ ”}$$

and $G \cap \mathbb{Q}^N$ is generic over N, and $r' \leq_{\mathbb{Q}} r'' \leq_{\mathbb{Q}} r^* \in G$, clearly $G \cap \mathbb{Q}^M$ is generic over M. Moreover, $N \models q'_* \leq_{\mathbb{Q}'} r^*$ because $\mathrm{pdac}(p_*, M, \mathbb{Q}) \subseteq \mathrm{pdac}(r', M'_*, \mathbb{Q})$ (and read clause (b)(γ) of Definition 5.2), so r^* is as required and $(*)$ holds.]

$(**)$ If $\mathcal{I} \in \mathrm{pdac}(p, \mathbb{Q}', N)$, *then* $G' \cap \mathcal{I}^N$ is a singleton.

[Why? In N define

$$\begin{aligned}\mathcal{I}_0 = \{r \in \mathbb{Q} : \ & r \text{ is incompatible with } p, \text{ or } r \text{ is above } p \text{ and}\\ & \text{above some member of } \mathcal{I}\},\end{aligned}$$

so $N \models$“ $\mathcal{I}_0 \subseteq \mathbb{Q}$ is dense open ”, so there is a maximal antichain $\mathcal{I}_1 \subseteq \mathcal{I}_0$ of $\mathbb{Q}'$ in N. Choose such $\mathcal{I}_1$ so $G \cap \mathcal{I}_1^N$ is a singleton say $\{r^*\}$. As “$G \cap \mathbb{Q}^N$ is a subset of $\mathbb{Q}^N$ generic over N”, clearly r^* cannot be incompatible with p (in N sense), hence r^* (by the definition of $\mathcal{I}_0$) is above p_* and r^* is, in N, $\leq_{\mathbb{Q}'}$–above some member r of $\mathcal{I}^N$, so $G' \cap \mathcal{I}^N \supseteq \{r\} \neq \emptyset$. If also $r' \in G' \cap \mathcal{I}^N \setminus \{r\}$ we can find r^2, $r' \leq^N_{\mathbb{Q}} r^2 \in \mathbb{Q}^N \cap G' = \mathbb{Q}^N \cap G$. Similarly we can find r^3 such that $p \leq^N_{\mathbb{Q}} r^3 \in \mathbb{Q}^N \cap G' = \mathbb{Q}^N \cap G$. So $p, r^3, r^* \in \mathbb{Q}^N \cap G' = \mathbb{Q}^N \cap G$, but there is no common upper bound in $\mathbb{Q}^N$ (as in $(\mathbb{Q}')^N, r^3, r^*$ are above

$r', r \in \mathcal{I}^N$ and $N \models \mathcal{I}$ is a predense antichain above p). But, as said above, $\mathbb{Q}^N \cap G$ is generic over N, giving a contradiction. Thus $(**)$ holds.]

It follows from $(**)$ that $G' \cap (\mathbb{Q}')^N$ is generic over N.

$(***)$ $\{p_n : n < \omega\} \cap G'$ is a singleton.

[Why? The intersection has at most one member as for any $n < m < \omega$,

$$N \models \text{“}\{p, p_n, p_m\} \text{ has no common upper bound in } \mathbb{Q}' \text{”},$$

so as $G' \cap (\mathbb{Q}')^N$ is generic over N we are done. The intersection has at least one member as by the definition of $\varphi_2^{\mathbb{Q}'}$ there is $\mathcal{J} \in \mathrm{pdac}(p, N, \mathbb{Q})$ witnessing this so in particular each member of $\mathcal{J}^N$ is above (by $\leq^N_{\mathbb{Q}'}$) some p_n, but

$$\mathcal{J}^N \cap G' \neq \emptyset \quad \& \quad G' \cap (\mathbb{Q}')^N \text{ is } \leq^N_{\mathbb{Q}'}\text{–downward closed (and } \leq^N_{\mathbb{Q}'}\text{-directed).}$$

Together we are done.]

We are left with clause $(c)^+$ but its proof is actually included in the proof of $(b)^+$.

3) Similar proof. $\square_{5.4}$

Discussion 5.6. The closure from 5.2 is somewhat artificial and we cannot express in it Borel compositions of conditions, i.e., we may like to be closer to the free limit of [25, IX, §1]; a natural closure is $\hat{\mathbb{Q}}$ from part (3) of Definition 5.7 below. Note that there: $\mathrm{cl}_1(\mathbb{Q})$ cannot serve as a forcing notion as it contains “false” (e.g., the empty disjunction), $\mathrm{cl}_2(\mathbb{Q})$ is the reasonable restriction, and $\mathrm{cl}_3(\mathbb{Q})$ has the same elements but more “explicit” quasi order. We do not define a quasi order on $\mathrm{cl}_1(\mathbb{Q})$, but it is natural to use the one of $\mathrm{cl}_2(\mathbb{Q})$ adding: $\psi \leq \varphi$ if $\varphi \in \mathrm{cl}_1(\mathbb{Q}) \setminus \mathrm{cl}_2(\mathbb{Q})$. No harm in allowing in the definition of $\mathrm{cl}_1(\mathbb{Q})$ also $\neg$ (the negation). The previous $\mathrm{cl}(\mathbb{Q})$ is close to $\mathrm{cl}_3(\mathbb{Q})$.

Definition 5.7. Let $\mathbb{Q}$ be a forcing notion.

1. Let $\mathrm{cl}_1(\mathbb{Q})$ be the closure of the set $\mathbb{Q}$ by conjunctions and disjunctions over sequences of members of length $\leq \omega$ [we may add: and $\neg$ (the negation)]; wlog there are no incidental identification and $\mathbb{Q} \subseteq \mathrm{cl}_1(\mathbb{Q})$.
2. For a $G \subseteq \mathbb{Q}$ (possibly outside $\mathbf{V}$) and $\psi \in \mathrm{cl}_1(\mathbb{Q})$ let $\psi[G]$ be the truth value of ψ under G where for $\psi = p \in \mathbb{Q}$, $\psi[G]$ is the truth value of $p \in G$. (We will use $\mathfrak{t}$ for “true”.)
3. $\hat{\mathbb{Q}} = \mathrm{cl}_2(\mathbb{Q}) = \{\psi \in \mathrm{cl}_1(\mathbb{Q}) :$ for some $p \in \mathbb{Q}$ we have $p \Vdash$“$\psi[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}$”$\}$ is ordered by $\leq_{\hat{\mathbb{Q}}} = \leq_2^{\mathbb{Q}}$ defined by:

$$\psi_1 \leq_{\hat{\mathbb{Q}}} \psi_2 \;\Leftrightarrow\; (\forall p \in \mathbb{Q})[p \Vdash_{\mathbb{Q}} \text{“}\psi_2[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}\text{”} \quad\Rightarrow\quad p \Vdash_{\mathbb{Q}} \text{“}\psi_1[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}\text{”}].$$

4. Let $\mathbb{Q}$ be explicitly nep. We let $\mathrm{cl}_3(\mathbb{Q})$ be the following forcing notion:

(a) the set of elements is $\mathrm{cl}_2(\mathbb{Q})$ but now the formula $\varphi(x)$ says that "there are $q, \bar{r}$ such that $r = \langle r_{n,\ell} : n < \omega, \ell < \omega\rangle$, for each $n < \omega$ the sequence $\langle r_{n,\ell} : \ell < \omega\rangle$ is explicitely a predense maximal antichain above q and they give explicit witnesses for the truth value of ψ being truth[21],"

(b) the order $\leq_3^{\hat{\mathbb{Q}}} = \leq_3^{\mathbb{Q}} = \leq_3^{\mathrm{cl}_3(\mathbb{Q})} = \leq_{\mathrm{cl}_3(\mathbb{Q})}$ is the transitive closure of $\leq_0^{\hat{\mathbb{Q}}}$ which is defined by

$\psi_1 \leq_0^{\hat{\mathbb{Q}}} \psi_2$ *iff* one of the following occurs

(i) $\psi_1, \psi_2 \in \mathbb{Q}$ and $\psi_1 \leq_{\mathbb{Q}} \psi_2$,

(ii) ψ_1 is a conjunct of ψ_2 (meaning: $\psi_1 = \psi_2$ or $\psi_2 = \bigwedge\limits_{n<\alpha} \psi_{2,n}$, and $\psi_1 \in \{\psi_{2,n} : n < \alpha\}$),

(iii) $\psi_2 \in \mathbb{Q}$ and there is a sequence $\bar{r} = \langle r_{n,\ell} : n < \omega, \ell < \omega\rangle$ such that together with ψ_2 witness that $\psi_1 \in \mathbb{Q}$[22].

Proposition 5.8. 1. *$\mathbb{Q} \subseteq \hat{\mathbb{Q}}$ as sets, $\leq^{\hat{\mathbb{Q}}}$ is a quasi order, and $\leq_{\hat{\mathbb{Q}}} \restriction \mathbb{Q} = \{(p,q) : q \Vdash_{\mathbb{Q}}$ "$p \in \underset{\sim}{G}_{\mathbb{Q}}$"$\}$ and $\mathbb{Q}$ is a dense subset of $\hat{\mathbb{Q}}$, and $\Vdash_{\hat{\mathbb{Q}}}$ "$\underset{\sim}{G}_{\hat{\mathbb{Q}}} \cap \mathbb{Q}$ is a generic subset", so if $\mathbb{Q}$ is separative,* then: $\leq_{\hat{\mathbb{Q}}} \restriction \mathbb{Q} = \leq_{\mathbb{Q}}$.

2. *Assume $\mathbb{Q}$ is temporarily explicitly nep.* Then *the two definitions of the set of elements of $\mathrm{cl}_3(\mathbb{Q})$ given in clause (a) of Definition* 5.7 *are equivalent, and:*

(a) *$\mathbb{Q} \subseteq \mathrm{cl}_3(\mathbb{Q})$ as sets and $\leq_3^{\mathbb{Q}} \restriction \mathbb{Q} \supseteq \leq_{\mathbb{Q}}$ and $\leq_3^{\mathbb{Q}} \subseteq \leq_{\hat{\mathbb{Q}}}$,*

(b) *$\mathbb{Q}$ is a dense subset of $\mathrm{cl}_3(\mathbb{Q})$ and $p \in \mathrm{cl}_3(\mathbb{Q}) \Rightarrow p \in \mathrm{cl}_2(\mathbb{Q})$.*

3. *Assume in addition*

($\circledast_3$) *$\mathbb{Q}$ is correctly explicitly nep in* **V** *and moreover this holds in every $\mathbb{Q}$–candidate.*

Then

(d) *if N is a $\mathbb{Q}$–candidate and $N \models$ "$p \in \mathrm{cl}_3(\mathbb{Q})$",* then *for some $q \in N$ we have $N \models$ "$p \leq_{\mathrm{cl}_3(\mathbb{Q})} q$ & $q \in \mathbb{Q}$",*

(e) *$cl_3(\mathbb{Q})$ is explicitly nep and correct.*

Proof. Straight, e.g.,

(2) The equivalence: The implication "$\Leftarrow$" is trivial. Assume $\psi \in \mathrm{cl}_2(\mathbb{Q})$, so for some $p \in \mathbb{Q}$ we have $p \Vdash_{\mathbb{Q}}$"$\psi[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}$". There is a $\mathbb{Q}$–candidate M to which p and ψ belong (as ZFC_*^- is $\emptyset$–good). Let p_1 be such

[21] That is, for each subformula ψ' of ψ for some $n(*) < \omega$, for every $\ell(*) < \omega$, for some n we have $r_{n(*),\ell(*)}, \langle r_{n,\ell} : n = k \bmod m, \ell < \omega\rangle$ explicitely witness the truth value of ψ' is, say, $\mathfrak{t}_{\psi',n(*),\ell(*)}$, which is naturally defined by induction on formulas. This is done for variety, as we could have acted as in Definition 5.2.

[22] If in clause (a) above we have choosen to immitate Definition 5.2, then we would have said here: $\psi_2 \in \mathbb{Q}$ and there are a $\mathbb{Q}$–candidate M and p such that $p, \psi_1 \in M$, $p \in \mathbb{Q}^M$, $p \leq_{\mathbb{Q}} \psi_2$, ψ_2 is explicitly $\langle M, \mathbb{Q}\rangle$–generic and $M \models$"$p \Vdash \psi_1[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}$".

that

$$N \models \text{“ } p_1 \in \mathbb{Q} \text{ and } [p_1 \Vdash_{\mathbb{Q}} \psi[\underset{\sim}{G}] = \text{true or } p_1 \Vdash_{\mathbb{Q}} \psi[\underset{\sim}{G}] = \text{false}] \text{ ”}.$$

Let q be explicitly $\langle M, \mathbb{Q}\rangle$–generic satisfying $\mathbb{Q} \models p_1 \leq q$. If $N \models$ “ $p_1 \Vdash_{\mathbb{Q}} \psi[\underset{\sim}{G}] = \text{false}$ ”, let $G \subseteq \mathbb{Q}$ be generic over N such that $q \in G$. Then $G \cap \mathbb{Q}^N$ is generic over N, so $N\langle G\rangle \models$ “ $\psi[G] = \text{false}$ ”. But $N[G] \models$ “ $\psi[G] = \text{true}$ ”, contradicting easy absoluteness. Thus $N \models$ “ $p_1 \Vdash_{\mathbb{Q}} \psi[\underset{\sim}{G}] = \text{true}$ ”.

Clause (b): Let q be as said in φ_0 (and $\mathcal{I} \in \text{pdac}(p, N, \mathbb{Q})$ be as required). By clause (iii) of 5.7(4)(b), we have $\text{cl}_3(\mathbb{Q}) \models$“$\psi \leq q$”, as required.

(3) Clause (d): By the formula defining $\text{cl}_3(\mathbb{Q})$.

Clause (e): Let $\varphi^3_0(x)$ be as in the definition[23]. Let $\varphi^3_1(x, y)$ say the definition of $\leq^{\mathbb{Q}}_3$. Lastly, $\varphi^3_2(\langle x_i : i \leq \omega\rangle)$ says that for some $\langle y_i : i \leq \omega\rangle$ we have

$$\varphi^{\mathbb{Q}}_2(\langle y_i : i \leq \omega\rangle), \quad y_\omega \leq^{\mathbb{Q}}_3 x_\omega \quad (\text{i.e. } \varphi^3_1(y_\omega, x_\omega)) \text{ and } \bigwedge_{i<\omega} \bigvee_{j<\omega} x_j \leq^{\mathbb{Q}}_3 y_i.$$

$\blacksquare_{5.8}$

Remark 5.9. Instead of using $\text{cl}(\mathbb{Q})$ from 5.2 (or the ones from 5.7) we can have in $\bar{\varphi}$, a function which from an ω–list of the elements of N and from p computes an element of $\mathbb{Q}$ having the role of $p \,\&\, \bigwedge_{\mathcal{I} \in \text{pdac}(p,N,\mathbb{Q})} \bigvee_{r \in \mathcal{I}[N]} r$. The choice does not seem to matter. Similarly to 4.2 we define:

Definition 5.10. For a forcing notion $\mathbb{P}$ and a cardinal (or ordinal) κ, we define what is an hc-κ-$\mathbb{P}$–name (here hc stands for hereditarily countable), and for this we define by induction on $\zeta < \omega_1$ what is such a name of depth $\leq \zeta$.

$\zeta = 0$: It is α, that is $\check{\alpha} = (0, \alpha)$, for some $\alpha < \kappa$ and $\check{\alpha}[G] = \alpha$.

$\zeta > 0$:

(α) It has the form $\underset{\sim}{\tau} = \{\langle p_i, \underset{\sim}{\tau}_i\rangle : i < i^*\}$, where $i^* < \omega_1$, $p_i \in \text{cl}_1(\mathbb{P})$ from Definition 5.7(1) and $\underset{\sim}{\tau}_i$ an hc-κ-$\mathbb{P}$–name of some depth $< \zeta$; that is for $G \subseteq \mathbb{P}$ generic over $\mathbf{V}$, we let

$$\underset{\sim}{\tau}[G] = \{\underset{\sim}{\tau}_i[G] : p_i[G] = \mathfrak{t} \text{ and } \underset{\sim}{\tau}_i[G] \text{ is defined }\}.$$

(β) it has the form $\underset{\sim}{\tau} = \{(p_i, \underset{\sim}{\tau}_i) : i < i^*\}$, where i^*, p_i, τ_i are as above and $\underset{\sim}{\tau}[G] = \underset{\sim}{\tau}_i[G]$ if $p_i[G]$ is truth and $j < i \Rightarrow p_i[G] =$false, and is $\emptyset$ otherwise, i.e., if $j < i \Rightarrow \underset{\sim}{\tau}[G] =$false.

An hc-κ-$\mathbb{P}$–name is an hc-κ-$\mathbb{P}$–name of some depth $< \omega_1$. An hc-κ-$\mathbb{P}$–name $\underset{\sim}{\tau}$ has depth ζ if it has depth $\leq \zeta$, but not $\leq \xi$ for $\xi < \zeta$.

[23]If we have immitated Definition 5.2 we should say that there is a $\mathbb{Q}$–candidate M such that $M \models$“$x \in \text{cl}_3(\mathbb{Q})$”.

Remark 5.11. 1. Why did we use above $p \in \mathrm{cl}_1(\mathbb{Q})$ and not $p \in \mathrm{cl}_3(\mathbb{Q})$? As the membership in $\mathrm{cl}_1(\mathbb{Q})$ is easier to define and we do not need to worry if p_i is "false".

2. The undefined case is not necessary as we can add $\psi_{i^*} = \bigwedge_{i<i^*} \neg p_i$.

Proposition 5.12. 1. *If $\underset{\sim}{\tau}$ is an hc-κ-$\mathbb{P}$–name and $G \subseteq \mathbb{P}$ is generic over* **V** then $\underset{\sim}{\tau}[G] \in \mathcal{H}_{<\aleph_1}(\kappa)$. *If in addition* $\mathbb{P} \subseteq \mathcal{H}_{<\aleph_1}(\kappa)$, then $\underset{\sim}{\tau} \in \mathcal{H}_{<\aleph_1}(\kappa)$.

2. *Let $\varphi(x_0, \dots, x_{n-1})$ be a first order formula and $\underset{\sim}{\tau}_0, \dots, \underset{\sim}{\tau}_{n-1}$ be* hc-κ-$\mathbb{P}$*–names.* Then *there is $p \in \mathrm{cl}_1(\mathbb{P})$ such that for every $G \subseteq \mathbb{P}$ generic over* **V**:

$$(\bigcup_{\ell<n} \mathrm{Tc}^{\mathrm{ord}}(\underset{\sim}{\tau}_\ell[G]), \in) \models \varphi(\underset{\sim}{\tau}_0[G], \dots, \underset{\sim}{\tau}_{n-1}[G]) \quad \text{iff} \quad p[G] = \mathfrak{t}.$$

(So if $p \notin \mathrm{cl}_2(\mathbb{P})$, then we get always "false".)

3. *The set of* hc-κ-$\mathbb{P}$*–names is closed under the following operations as hc $-$ κ-objects, i.e., recalling ordinals are urelements (see* 0.6, 2.12*)*
   (a) *difference,*
   (b) *union and intersection of two, finitely many and even countably many,*
   (c) *definition by cases: for $p_n \in \mathrm{cl}_1(\mathbb{P})$ and* hc-κ-$\mathbb{P}$*–names $\underset{\sim}{\tau}_n$ (for $n < \omega$) there is a* hc-κ-$\mathbb{P}$*–name $\underset{\sim}{\tau}$ such that for a generic $G \subseteq \mathbb{Q}$ over* **V** *we have*

$$\underset{\sim}{\tau}[G] \textit{ is}: \quad \begin{cases} \underset{\sim}{\tau}_n[G] & \textit{if} \quad p_n[G] = \mathfrak{t} \ \& \ \bigwedge_{\ell<n} \neg p_\ell[G] = \mathfrak{t} \\ \emptyset & \textit{if} \quad \bigwedge_{\ell<\omega} \neg p_\ell[G] = \mathfrak{t}. \end{cases}$$

Proof. Straight. $\blacksquare_{5.12}$

Definition 5.13. 1. A forcing notion $\mathbb{Q}$ (or $\bar{\varphi}$) is temporarily, explicitly *straight* (κ, θ)–nep for $\mathfrak{B}$ if: all the conditions from Definition 1.3(1), (2) (for explicitly (κ, θ)–nep) hold but possibly $\kappa \subset \mathfrak{B} \subseteq \mathcal{H}_{<\aleph_1}(\kappa)$; and
   (d) $\mathbb{Q} \subseteq \mathcal{H}_{<\aleph_1}(\theta)$ (i.e., $\mathbb{Q}$ is simple) and $\aleph_1 + \theta \leq \kappa$,
   (e) for $\ell < 3$ the formula $\varphi_\ell^{\mathbb{Q}}(\bar{x})$ is of the form

$$(\exists t)[t \in \mathcal{H}_{<\aleph_1}(\kappa) \ \& \ t = \mathrm{Tc}^{\mathrm{ord}}(t) \ \& \ (\exists s)((s \in t \vee s = t) \wedge \psi_\ell^{\mathbb{Q}}(\bar{x}, s))],$$

where in the formula $\psi_\ell^{\mathbb{Q}}$ the quantifiers are of the form $(\exists s' \in s)$ and the atomic formulas are "$x \in y$","x is an ordinal", "$x < y$ are ordinals" and those of $\mathfrak{B}^N, \mathfrak{C}^N$, i.e., for $\mathfrak{B}^N \models R(x)$.

2. In clause (e) of part (1), we call such t an explicit witness for $\varphi^{\mathbb{Q}}_\ell(\bar{x})$. We call t a weak witness, if for every $\mathbb{Q}$–candidate N satisfying $\bar{x} \in N$, if $t \in N$ then $N \models$ "$\varphi^{\mathbb{Q}}_\ell(\bar{x})$ & t is hc" and $\mathfrak{B}^t \subseteq \mathfrak{B}^N$. We call such t an almost witness if:
   (i) $\ell = 0$ and it is an explicit witness, or
   (ii) $\ell = 1$ (so $\bar{x} = \langle x_0, x_1 \rangle$) and t gives k, $y_0, \dots, y_k$, $t_0, \dots, t_{k-1}$, $s_0, \dots, s_k$ such that: s_ℓ explicitly witnesses $\varphi_0(y_\ell)$, t_ℓ explicitly witnesses $y_\ell \leq_{\mathbb{Q}} y_{\ell+1}$ and $y_0 = x_0$, $y_k = x_1$ (so $y_\ell \in t$, $s_\ell \in t$, $x_\ell \in t$),
   (iii) $\ell = 2$ (so $\bar{x} = \langle x_i : i \leq \omega \rangle$) and t gives $\langle y_i : i \leq \omega \rangle$, $\langle k_i : i < \omega \rangle$, $\langle m_i : i < \omega \rangle$, $\langle s_i : i \leq \omega + 1 \rangle$ such that s_ω is a witness as in (i) or (ii) to $y_\omega \leq x_\omega$, $s_{\omega+1}$ is an explicit witness to $\varphi^{\mathbb{Q}}_2(\langle y_i : i \leq \omega \rangle)$, $m_i \neq m_j \ \Rightarrow k_i \neq k_j$ for $i \neq j < \omega$, s_i is an almost witness to $x_{m_i} \leq^{\mathbb{Q}} y_{k_i}$ for some $k_i < \omega$ (so also they all belong to t, as well as witnesses to $x_i, y_j \in \mathbb{Q}$).
3. We say that $\mathbb{Q}$ is very straight if it is straight and in addition
   (f) for some ord-hc Borel[24] functions $\mathbf{B}_1, \mathbf{B}_2$, if N is a $\mathbb{Q}$-candidate, $\bar{a} = \langle a_i : i < \omega \rangle$ list N, and $p \in \mathbb{Q}^N$, *then* $q = \mathbf{B}_1(p, N, \bar{a})$ is explicitly $\langle N, \mathbb{Q} \rangle$–generic, and $\mathbf{B}_2(p, N, \bar{a})$ is a witness.[25] For such q we say it is canonically generic [the main case is as in Definition 5.2, Proposition 5.4].

   We can make $y\mathfrak{B}$ depend on $N \cap \mathrm{Ord}$, i.e., for each such intersection we give a different ord–hc Borel function.
4. In part (3) we say that the $\mathbf{B}_1, \mathbf{B}_2$ witness $\mathbb{Q}$ is very straight. For notational simplicity, if not said otherwise they are coded uniformly by $\mathfrak{B} \restriction \omega$.

Proposition 5.14.

1. *Assume $\mathbb{Q}$ is temporarily explicitly straight (κ, θ)–nep for $\mathfrak{B}$.* Then $\mathbb{Q}$ *is temporarily simple explicitly (κ, θ)–nep for $\mathfrak{B}$.*
2. *Assume $\mathbb{Q}$ is temporarily correctly simple explicitly (κ, θ)–nep for $\mathfrak{B}$ and $\theta + \aleph_1 \leq \kappa$.* Then *we can find $\bar{\varphi}'$ such that*
   (a) $\mathbb{Q}^{\bar{\varphi}'} = \mathbb{Q}$ *(i.e., same members, same quasi order)*
   (b) $\mathbb{Q}^{\bar{\varphi}'}$ *is temporarily straight explicitly (κ, θ)–nep for $\mathfrak{B}$ and is correct.*

   [Nevertheless, "simple" and "straight" are distinct as properties of $(\mathfrak{B}, \bar{\varphi}, \theta)$, i.e., the point is changing $\bar{\varphi}$.]
3. *In Proposition* 5.4, $\mathrm{cl}(\mathbb{Q})$ *is very straight for* ZFC^-_* *(not* ZFC^-_{**}*) with witnesses as in the proof there.*

[24]See Definition 0.5, $\mathbf{B}_1(p, N, \bar{v})$ means that $\mathbf{B}_1(\bar{x})$, where $\bar{x}$ is as in 5.17 below.

[25]That is, it witnesses $p \leq q$ and $\varphi_2(\langle p_{\mathcal{I},n} : n < \omega \rangle, q)$ for some sequence $\langle p_{\mathcal{I},n} : n < \omega \rangle$ of members of $\mathcal{I}^N$ for every $\mathcal{I} \in \mathrm{pdac}(p, N, \mathbb{Q})$.

4. *Assume that* $\mathbb{Q}$ *is correct , simple and explicitely nep.* Then $\mathrm{cl}_3(\mathbb{Q})$ *is simple, correct and very straight.*

Proof. (1) We have to check the demands (α), (β), (γ) in Definition 1.3(5).
Clause (α) is part of preliminary demands in Definition 5.13(1).
Clause (β) holds by (d) of Definition 5.13(1).
Clause (γ) holds by clause (e) of Definition 5.13(1).
(2) Let $\varphi'_0(x)$ say that "there is N, a $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate, with $|N| \in \mathcal{H}_{<\aleph_1}(\kappa)$, $|N| = \mathrm{Tc}^{\mathrm{ord}}(|\mathrm{N}|)$, $N \cap \omega_1 \in \omega_1$ and $N \models \varphi_0(x)$". We define $\varphi'_1(x,y)$, φ'_2 similarly. Now check.
(3), (4) Straightforward. $\blacksquare_{5.14}$

Definition 5.15. Let $\mathrm{cl}'(\mathbb{Q})$ be defined as in Definition 5.2 but the candidates mentioned are ord-transitive and from[26] $\mathcal{H}_{\aleph_1}(\kappa'(\mathbb{Q}) + \omega_1)$.

Proposition 5.16. 1. *Assume*

(a) ZFC^-_* *is normal,*
(b) $\mathbb{Q} = (\mathfrak{B}, \bar{\varphi}, \theta)$ *is correct,*
(c) $\mathbb{Q}$ *is simple nep in every* K*-extension.*

Then *we can define* $\mathbb{Q}' = (\mathfrak{B}', \bar{\varphi}', \theta)$ *and normal,* ZFC^-_{**} *such that*

(α) $\mathbb{Q}', \mathbb{Q}$ *are equivalent as forcing notions in every* K*-extension of* $\mathbf{V}$*,*
(β) $\mathrm{ZFC}^-_{**} \vdash$ *"*ZFC^-_* *and* ZFC^-_* *is weakly normal for* $\mathbb{Q}'$*",*
(γ) $\mathbb{Q}'$ *as a forcing notion is* $\mathrm{cl}'(\mathbb{Q})$ *from Definition* 5.15 *above,*
(δ) $\mathbb{Q}'$ *is correct simple very straight explicitly nep,*
(ϵ) ZFC^-_{**} *is normal for* $\mathbb{Q}'$*,*
(ζ) $\kappa(\mathfrak{B}) = \kappa'(\mathbb{Q}') + \omega_1$*, with little more work* $\kappa(\mathfrak{B}) = \kappa'(\mathbb{Q})$ *suffices.*

2. *Similarly for semi-normal.*

Proof. Let $\mathfrak{B}'$ has universe $|\mathfrak{B}| \cup (\kappa'(\mathbb{Q}) + \omega_1)$ so $\kappa(\mathbb{Q}') = \kappa'(\mathbb{Q}') = \kappa(\mathfrak{B}')$ and it expand $\mathfrak{B}, \mathfrak{C}$ and $(\kappa'(\mathbb{Q}) + \omega_1, <)$. Let $\varphi'_0(x)$ say that $x \in \mathrm{cl}'(\mathbb{Q})$ where for $p \in \mathrm{cl}'(\mathbb{Q})$ a witness is naturally defined. Let $\varphi_1(x,y)$ be defined similarly as in Definition 5.2, i.e., it contains the finite chain of the "easy" cases which witness $x \leq_{\mathbb{Q}} y$. Let $\varphi'_2(\langle x_i : i < \omega \rangle, x)$ say that for some ord-transitive $\mathbb{Q}$-candidate M, $p \in \mathbb{Q}^M$, $x = p \ \& \bigwedge\limits_{\mathcal{I} \in \mathrm{pdac}(M,\mathbb{Q})} \bigvee\limits_{r \in \mathcal{I}[M]} r$ and $\{x_i : i < \omega\}$ is a list enumerating $\mathcal{I}^M$ for some $\mathcal{I} \in \mathrm{pd}(N, \mathbb{Q})$. The proof as in 5.4. $\blacksquare_{5.16}$

A variant of "straight" (for which a parallel of 5.18 works) is:

Definition 5.17. 1. $\mathbb{Q}$, i.e., $(\mathfrak{B}, \bar{\varphi}, \theta) = (\mathfrak{B}^{\mathbb{Q}}, \bar{\varphi}^{\mathbb{Q}}, \theta^{\mathbb{Q}})$ is *real nep* whenever $\bar{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \mathfrak{B})$, and ($\mathbb{Q}$-candidate means $(\mathfrak{B}, \bar{\varphi}, \theta)$-candidate):

[26] may use coding to show $\kappa'(\mathbb{Q})$ suffices.

(a) φ_0 defines the set of elements of $\mathbb{Q}$ which is upward absolute for $\mathbb{Q}$-candidate and for simplicity $x \in \mathbb{Q} \Rightarrow x \in \mathcal{H}_{<\aleph_1}(\theta)$,
(b) φ_1 defines the quasi order $\leq_{\mathbb{Q}}$ and is upward absolute for $\mathbb{Q}$-candidates,
(c) there is an ord-hc Borel function $\mathbf{B}$ such that if N is a $\mathbb{Q}$-candidate and $\bar{a} = \langle a_n : n < \omega \rangle$ list of the members of N and $\bar{x}$ is an ω-list as below then $\mathbf{B}(\bar{x})$ is $(N, \mathbb{Q})$–generic where $\bar{x}$ lists:
  (α) $a'_n = \begin{cases} a_n & \text{if } N \models a_n \in \mathcal{H}_{<\aleph_1}(\kappa') \\ \emptyset & \text{if otherwise} \end{cases}$,
  (β) truth value$(a_n \in a_m)$,
  (γ) $(R, \bar{a})$ such that $\mathfrak{B}^N \models R(\bar{a})$,
  (δ) $(R, \bar{a})$ such that $\mathfrak{C}^N \models R(\bar{a})$,

2. We add "explicitly" if we have φ_2 as usual, $\mathbf{B}(\bar{x})$ is explicitely $(N, \mathbb{Q})$–generic and $\mathbf{B}(\bar{x})$ is explicitly given, i.e., we have $(\mathfrak{B}^{\mathbb{Q}}, \bar{\varphi}^{\mathbb{Q}}, \theta^{\mathbb{Q}}, \mathbf{B}^{\mathbb{Q}})$.

Claim 5.17.1. 1. *The forcing notions from §3 are real nep.*
2. *Real nep implies very straight correct (and more).*

Proof. Should be clear. ∎

Definition/Theorem 5.18. We assume (not really necessary but in the cases we have in mind)

($*$) ZFC^-_* is normal and nice to forcing notion of cardinality $\leq \lambda^{\mathfrak{C}}$ ($\lambda^{\mathfrak{C}}$ an individual constant of $\mathfrak{C}$) and Δ_1, Δ_2 are the set of quantifier free formulas,

or just

$(*)^-$ ZFC^-_* is nice[27] to forcing notion cardinality $\leq \lambda^{\mathfrak{C}}$ (an individual constant of $\mathfrak{C}$) and Δ_1, Δ_2 are the set of quantifier free formulas.

By induction on the ordinal α we define and prove the following:

(A) *[Definition]* $\bar{\mathbb{Q}} = \langle (\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \mathbf{B}_i, \kappa_i, \theta_i) : i < \alpha \rangle$ is nep–CS–iteration for ZFC^-_* and $\mathfrak{C}$.
(B) *[Definition]* $\kappa^{\bar{\mathbb{Q}}} = \kappa[\bar{\mathbb{Q}}]$, in short κ^α abusing notation when $\alpha = lg(\bar{\mathbb{Q}})$.
(C) *[Definition]* We define $\mathfrak{B}^{\bar{\mathbb{Q}}}$ and call it in short also $\mathfrak{B}^\alpha$ when $\alpha = \ell g(\bar{\mathbb{Q}})$ and $\bar{\mathbb{Q}}$ is as in clause (A).
(D) *[Definition]* $\mathrm{Lim}_{\mathrm{nep}}(\bar{\mathbb{Q}}) = \mathbb{P}_\alpha$ for $\bar{\mathbb{Q}}$ as in clause (A).
So $\bar{\varphi}^{\mathbb{P}_\alpha}$ is a temporary $(\mathfrak{B}^\alpha, \kappa^\alpha)$-definition of a forcing notion (so $\mathfrak{B}^{\mathbb{P}_\alpha} = \mathfrak{B}^\alpha, \theta^{\mathbb{P}_\alpha} = \kappa^\alpha$, $\bar{\varphi}^{\mathbb{P}_\alpha} = (\varphi_0^{\mathbb{P}_\alpha}, \varphi_1^{\mathbb{P}_\alpha})$ but $\varphi_2^{\mathbb{P}_\alpha}$ is defined later and

[27]We can replace this by demanding below that for each β, if N is $\mathfrak{B}^\beta$–candidate, $G_\beta \subseteq \mathbb{P}_\beta$ generic over $\mathbf{V}$, $N \models$"$\mathbb{R}$ a forcing of cardinality $\leq (\kappa^\beta)^{\aleph_0}$" (or more restricted, like $\mathbb{P}_\beta$), and $G \cap \mathbb{R}$ is generic over N then $N[G]$ is a $\mathbb{Q}_\beta[G_\beta]$–candidate.

also for each $\beta \leq \alpha$ we define $\mathbb{P}^\alpha_\beta$, a temporary $(\mathfrak{B}^\alpha, \kappa^\beta)$-definition of a forcing notion (so $\mathfrak{B}^{\mathbb{P}^\alpha_\beta} = \mathfrak{B}^\alpha, \theta^{\mathbb{P}_\alpha} = \kappa^\beta$) (this is a variant of $\mathbb{P}_\beta$, the same as forcing notion; naturally $\bar{\varphi}^{\mathbb{P}^\alpha_\beta}$ has β as a parameter).

(E) *[Claim]* If $\bar{\mathbb{Q}}$ is a nep–CS–iteration, and $\beta \leq \alpha = \ell g(\bar{\mathbb{Q}})$, *then* $\bar{\mathbb{Q}} \restriction \beta$ is a nep–CS–iteration, $\mathrm{Lim}_{\mathrm{nep}}(\bar{\mathbb{Q}} \restriction \beta) = \mathbb{P}_\beta$ if $\beta < \alpha$; and $\mathbb{P}_\beta \subseteq \mathcal{H}_{<\aleph_1}(\kappa^\beta)$. For $\beta < \alpha$, $(\mathfrak{B}^\alpha_\beta, \bar{\varphi}^{\mathbb{P}^\alpha_\beta}, \kappa^\beta)$ is another definition of $\mathbb{P}_\beta$ as in claim 1.1 for quantifier free formulas, moreover the derivation of $\bar{\varphi}^{\mathbb{P}^\alpha_\beta}$ from $(\mathfrak{B}_\alpha, \bar{\varphi}^{\mathbb{P}_\beta}, \kappa^\beta)$ is uniform. So $\mathbb{P}^\alpha_\beta$ is explicitly straight correct κ^α–nep.

(F) *[Claim]* For $\bar{\mathbb{Q}}$ as in (A), a $\mathfrak{B}^\alpha$–candidate N, $\gamma \leq \beta \leq \alpha$ and $p, q \in \mathbb{P}_\beta$ we have:
  (a) p is a function with domain a countable subset of β (more, see in clause (D) below),
  (b) $\mathbb{P}_\beta$ is a forcing notion (i.e. a quasi order) satisfying (d) + (e) of 5.13 and (a), (b), $(b)^+$ of 1.3(1),(2),
  (c) $p \restriction \gamma \in \mathbb{P}_\gamma$ and $\mathbb{P}_\beta \models$"$p \restriction \gamma \leq p$",
  (d) $\mathbb{P}_\gamma \models$"$p \restriction \gamma \leq q$" implies $\mathbb{P}_\beta \models$"$p \leq (q \cup p \restriction [\gamma, \beta))$",
  (e) $\mathbb{P}_\gamma \subseteq \mathbb{P}_\beta$ and even $\mathbb{P}_\gamma \lessdot \mathbb{P}_\beta$,
  (f) $p \in \mathbb{P}_\beta$ iff p a function with domain $\in [\beta]^{\leq \aleph_0}$ and

$$\zeta \in \mathrm{Dom}(p) \quad \Rightarrow \quad p \restriction (\zeta + 1) \in \mathbb{P}_{\zeta+1},$$

  (g) $|\mathbb{P}_\beta| \leq (\kappa^\beta)^{\aleph_0}$.

(G) *[Definition]* For a $\mathfrak{B}^\alpha$–candidate N and β, γ such that $\gamma < \beta \leq \alpha$, $\gamma \in N$, $\beta \in N$, and $q \in \mathbb{P}_\beta$, $p \in N$ such that $N \models$"$p \in \mathbb{P}_\beta$" and $q \restriction \gamma$ is $\langle N, \mathbb{P}_\gamma \rangle$–generic we define when q is $[\gamma, \beta)$–canonically $\langle N, P_\beta \rangle$–generic above p. We also define $\varphi_2^{\mathbb{P}_\alpha}$.

(H) *[Theorem]*
  (a) If $q \in N$ is a $[\gamma, \beta)$–canonically $\langle N, \mathbb{P}_\beta \rangle$–generic above p, *then* q is $\langle N, \mathbb{P}_\beta \rangle$–generic and $\mathbb{P}_\beta \models p \leq q$.
  (b) "q is $[0, \beta)$-canonically $\langle N, \mathbb{P}_\beta \rangle$–generic above p" can be defined as in 5.13(3), i.e., by an ord-hc-κ-Borel function.
  (c) If $G_\beta \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$ and $\beta < \alpha$, *then* $\mathbb{Q}_\beta[G_\beta]$, i.e., defining it by its defining formulas $(\bar{\varphi}_\beta)$, and $\mathbb{P}_{\beta+1}/\underset{\sim}{G}_\beta$ are essentially the same modulo renaming.

(I) *[Theorem]* $\mathbb{P}_\alpha$ is explicitly straight correct κ^α–nep for $(\mathfrak{B}^\alpha, \bar{\varphi}^{\mathbb{P}_\alpha}, \kappa^\alpha)$ with ZFC^-_*; in fact very straight.

(J) *[Theorem]* For any $\kappa \geq \kappa^\alpha$,

$$\Vdash_{\mathbb{P}_\alpha} \text{“ } (\mathcal{H}_{<\aleph_1}(\kappa))^{\mathbf{V}[\mathbb{P}_\alpha]} = \{\underset{\sim}{\tau}[G_{\mathbb{P}_\alpha}] : \underset{\sim}{\tau} \text{ is an hc–}\kappa\text{–}\mathbb{P}_\alpha\text{–name }\})\text{”}.$$

(K) *[Definition]* For a nep-CS-iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}'_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i : i < \alpha \rangle$ we define $F[\bar{\mathbb{Q}}] = \langle \mathbb{P}'_i, \underset{\sim}{\mathbb{Q}}'_i : i < \alpha \rangle$.

(L) *[Claim]* $F[\bar{\mathbb{Q}}]$ is CS iteration such that $\mathbb{P}_i \subseteq \mathbb{P}'_i$ even $\mathbb{P}_i$ is a dense subset of $\mathbb{P}'_i$ for $i \le \alpha$.

Let us carry out the clauses one by one.

Clause (A), Definition: $\bar{\mathbb{Q}} = \langle (\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i) : i < \alpha \rangle$ is a nep–CS–iteration if:

(α) $\beta < \alpha \quad \Rightarrow \quad \bar{\mathbb{Q}} \restriction \beta$ is a nep–CS–iteration,

(β) if $\alpha = \beta + 1$ then

(i) $\mathbb{P}_\beta = \mathrm{Lim}_{\mathrm{nep}}(\bar{\mathbb{Q}} \restriction \beta)$ (use clause (D))

(ii) $\bar{\varphi}_\beta = \langle \varphi_{\beta,\ell} : \ell < 3 \rangle$ is formally as in the definition of explicitly nep (the substantial demand is (v) below, but the parameter $\mathfrak{B}_\beta$ is a name!)

(iii) $\kappa_\beta, \theta_\beta$ are infinite cardinals (or ordinals)

(iv) $\underset{\sim}{\mathfrak{B}}_\beta$ is a $\mathbb{P}_\beta$–name of a model with universe[28] κ_β whose vocabulary is a fixed countable set $\tau_0 \subseteq \mathcal{H}(\aleph_0)$ (fixed means "does not depend on β") definable in $\mathfrak{C} \restriction \omega$, but for each atomic formula $\psi(x_0, \ldots, x_{n-1})$ and $\alpha_0, \ldots, \alpha_{n-1} < \kappa_\beta$ the name of the truth value $\underset{\sim}{\mathfrak{B}}_\beta \models$"$\psi(\alpha_0, \ldots, \alpha_{n-1})$" is an hc–$\kappa$-$\mathbb{P}_\beta$–name (i.e., is defined by one $p = p^\beta_{\psi^*} \in \mathrm{cl}_1(\mathbb{P}_\beta)$) where $\psi^* = \psi(\alpha_0, \ldots, \alpha_{n-1})$ so actually $\underset{\sim}{\mathfrak{B}}_\beta$ is the function

$$(\beta, \psi^*(\alpha_0, \ldots, \alpha_{n-1})) \mapsto p^\beta_{\psi^*(\alpha_0, \ldots, \alpha_{n-1})};$$

this function is from $\mathbf{V}$.

(v) $\Vdash_{\mathbb{P}_{\beta s}}$ " the $\underset{\sim}{\mathbb{Q}}_\beta$ defined by $\bar{\varphi}_\beta$, is temporarily very straight[29] correct explicitly $(\kappa_\beta, \theta_\beta)$–nep as witnessed by $\underset{\sim}{\mathbf{B}}_\alpha$ ", so we should add it to $\bar{\mathbb{Q}}$ or to $\bar{\varphi}$. But

(α) $\underset{\sim}{\mathbf{B}}_\alpha$ is a $\mathbb{P}_\beta$–name such that we can write it as an ord–hc Borel function having extra ω variables in which we put $p_n \in \mathrm{cl}_1(\mathbb{P}_\beta)$,

(β) $\underset{\sim}{\mathbf{B}}_\alpha$ may depend on a countable set of ordinals (the $N \cap \mathrm{Ord}$).

(vi) $\mathbb{Q}_\beta$ has a minimal element $\emptyset_{\mathbb{Q}_\beta}$, e.g., just $\emptyset$ for notaional simplicity,

(γ) ZFC^-_* is good for $\{\mathbb{P}_\beta : \beta < \alpha\}$ or even for forcing notions of cardinality $\le \Sigma\{\kappa_\beta^{\aleph_0} : \beta < \alpha\}$.

Clause (B), Definition: We define $\kappa^\alpha = \sup[\{\kappa_i + 1, \theta_i + 1 : i < \alpha\} \cup \{\alpha\}]$ (ordinal is here more natural; of course, if the result is an ordinal we can replace it by $\min\{\kappa : \kappa$ a cardinal $\kappa > \alpha$ and $\kappa > \kappa_i, \theta_i$ for $i < \alpha\}$. We use $\kappa_i + 1$, $\theta_i + 1$ just to clarify, "$\mathfrak{B}^\alpha$ codes them". We could here allowed $\underset{\sim}{\kappa}_i$, $\underset{\sim}{\theta}_i$ to be hc–$\kappa_i^{\alpha+1} - \mathbb{P}_i$–names but then let $\kappa^\alpha = \min\{\zeta : |\zeta| \ge \alpha$ and for any

[28] The reader can concentrate on this case, we shall just remerk on the changes needed to assume just that the model has universe just a subset of $\mathcal{H}_{<\aleph_1}(\kappa_\beta)^{\mathbf{V}[G_{\mathbb{P}_\beta}]}$ which includes κ_β.

[29] There are many such by 5.14(3).

$\beta < \alpha, \Vdash_{\mathbb{P}_\beta}$ " $\underset{\sim}{\kappa}_i, \underset{\sim}{\theta}_i$ are $< \zeta$ "}. Why the "$< \zeta$" instead of "$\leq \zeta$"? Formally to allow such names).

Clause (C), Definition: We define $\mathfrak{B}^\alpha = \mathfrak{B}^{\bar{\mathbb{Q}}}$, a model with universe included in $\mathcal{H}_{<\aleph_1}(\kappa^\alpha)$ or write κ^α and the usual vocabulary such that:

(*) $\mathfrak{B}^\alpha$ codes (by its relations uniformly) $\alpha, \{(\beta, \bar{\varphi}_\beta, \kappa_\beta, \theta_\beta) : \beta < \alpha\}$ and $\langle \mathfrak{B}_\beta : \beta < \alpha \rangle$; i.e., for every atomic formula $\psi = \psi(x_0, \dots, x_{n-1})$ in the vocabulary τ_0 (so is of $\mathfrak{B}_\beta$), for some function symbol F_ψ we have:
if $\alpha_\ell < \kappa_\beta$ for $\ell < n$ then $F_{\psi(\bar{x})}(\beta; \alpha_0, \dots, \alpha_{n-1})$ is $p^\beta_{\psi(\alpha_0, \dots, \alpha_{n-1})}$ (see clause (A)(iv)) and
if the $\mathfrak{B}$'s are on κ, we have also $F_{\psi,\ell}$, functions of $\mathfrak{B}^\alpha$ such that:
if $\alpha_\ell < \kappa_\beta$ for $\ell < n$ then $\{F_\psi(\beta, \ell, \alpha_0, \dots, \alpha_{n-1}) : \ell < \omega\}$ lists the ordinals in $\mathrm{Tc}^{\mathrm{ord}}(p^\beta_{\psi(\alpha_0, \dots, \alpha_{n-1})})$ (the condition in $\mathbb{P}_\beta$ saying...) and F'_ψ codes how p was gotten from them (so we need $\kappa^\alpha \geq \omega_1$).

So in any case

(**) if N is a $\mathfrak{B}^\alpha$–candidate, and $\beta \in \alpha \cap N$ then N essentially is a $\mathfrak{B}^\beta$–candidate. Better, note that as $\mathfrak{B}^\beta$ is so easily definable in $\mathfrak{B}^\alpha$ by some parameter, we can for $\mathbb{P}_\beta$ use $\mathfrak{B}_\alpha$-candidates to which β belongs. Formally done in clause (D) below.

Similarly with the $\langle \underset{\sim}{\mathbf{B}}_\beta : \beta < \alpha \rangle$.

Clause (D), Definition:

Case 1: If $\alpha = 0$ then $\mathbb{P}_\alpha = \{\emptyset\}$.
Case 2: If $\alpha = \beta + 1$ then

$\mathbb{P}_\alpha = \big\{p :$ p is a function, $\mathrm{Dom}(p) \subseteq \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ and if $\beta \in \mathrm{Dom}(p)$ then we have $r = r_{p,\beta} \in \mathrm{cl}_1(\mathbb{P}_\beta)$ determined by p such that
(α) if q appears in r then $\mathrm{Dom}(q) \subseteq \mathrm{Dom}(p \restriction \beta)$
(β) $p(\beta)$ is defined by cases:
if $r[\underset{\sim}{G}_{P_\beta}] = \mathfrak{t}$, then $p(\beta)$ is a hc-θ_β-$\mathbb{P}_\beta$-name in $\underset{\sim}{\mathbb{Q}}_\beta$, and an explicit witness is provided (say $p(\beta)$ codes it and having $r[\underset{\sim}{G}_{\mathbb{P}_\beta}] = \mathfrak{t}$ says so),
if not, $p(\beta)$ is $\emptyset = \emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} = \min(\underset{\sim}{\mathbb{Q}}_\beta)\big\}$.

In details, $p \in \mathbb{P}_\alpha$ if and only if p has the form $p' \cup \{\langle \beta, \{(\ell, \underset{\sim}{x}_\ell) : \ell < 3\} \rangle\}$ where $p' \in \mathbb{P}_\beta$, $\underset{\sim}{x}_0 \in \mathrm{cl}_1(\mathbb{P}_\beta)$, $\underset{\sim}{x}_1, \underset{\sim}{x}_2$ are hc–κ_α–$\mathbb{P}_\beta$–names of members of $\mathcal{H}_{<\aleph_1}(\theta_\beta)$ and $\underset{\sim}{x}_0[G_\beta]$ is the truth value of "$\underset{\sim}{x}_2[G_\beta]$ is a witness to $\underset{\sim}{x}_1[G_\beta] \in \mathbb{Q}_\beta$", see Proposition 5.12(2) and Definition 5.13(1) clause (e).
Case 3: If α is limit, then

$$\mathbb{P}_\alpha = \{p : p \text{ is a function, } \mathrm{Dom}(p) \in [\alpha]^{\leq \aleph_0} \text{ and } \beta \leq \alpha \quad \Rightarrow \quad p \restriction \beta \in \mathbb{P}_\beta\}.$$

The order:
For $\alpha = 0$ nothing to do.
For α limit: $p \le q$ if and only if $\bigwedge_{\beta<\alpha} \mathbb{P}_\beta \models$ "$p \restriction \beta \le q \restriction \beta$" (equivalently:

$$\bigwedge_{\beta \in \mathrm{Dom}(p)} \mathbb{P}_{\beta+1} \models \text{“ } p \restriction (\beta+1) \le q \restriction (\beta+1) \text{ ”}).$$

For $\alpha = \beta + 1$: the order is the transitive closure of the following cases:

(α) $p \in \mathbb{P}_\beta$, $q \in \mathbb{P}_\alpha$ and $\mathbb{P}_\beta \models$ "$p \le q \restriction \beta$",

(β) $p(\beta) = q(\beta)$ and $\mathbb{P}_\beta \models$ "$p \restriction \beta \le q \restriction \beta$",

(γ) $p \restriction \beta = q \restriction \beta$ and there is a $\mathfrak{B}^\alpha$–candidate N and $p' \in \mathbb{P}^N_\alpha$ such that $q \restriction \beta$ is a $[0, \beta)$–canonical $(N, \mathbb{P}_\beta)$–generic above $p' \restriction \beta$, hence $\mathbb{P}_\beta \models$ "$p' \restriction \beta \le q \restriction \beta$", and (i) or (ii) where
  (i) $p' \in \mathbb{P}_\beta$ and $N \models$ "$[p' \Vdash_{\mathbb{P}_\beta} p(\beta) \le_{\mathbb{Q}_\beta} q(\beta)]$",
  (ii) $N \models$ "$p' \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p(\beta) \le_{\mathbb{Q}_\beta} p'(\beta)$" and $q(\beta)$ is canonically generic for $(\mathbb{Q}_\beta[G_\beta], N[\underset{\sim}{G}_\beta])$ over $p(\beta)$, i.e., is $\underset{\sim}{\mathbf{B}}^{\mathbb{Q}_\beta}_1(p'(\beta),\ N[\underset{\sim}{G}_\beta], \bar{a}[\underset{\sim}{G}_\beta])$ where for an ω-list of N we produce on ω–list $\bar{a}$ of $N[\underset{\sim}{G}_\beta]$ by

$$a'_n = \begin{cases} a_n & N \models \text{“}a_n \text{ not a } \mathbb{P}_\beta\text{-name”} \\ a_n \langle \underset{\sim}{G}_\beta \rangle & N \models a_n \text{ is a } \mathbb{P}_\beta\text{-name”}. \end{cases}$$

We have so far defined $\mathbb{P}_\alpha$ just as a forcing notion. But $\varphi_0^{\mathbb{P}_\alpha}$ is implicit in the definition of the set of the elements, and $\varphi_1^{\mathbb{P}_\alpha}$ is explicit in the definition of the orders. As for $\varphi_2^{\mathbb{P}_\alpha}$ we do define it, yet this will be done in clause (G), so pedantically we should write $\mathbb{P}^-_\alpha$. We also have to define $\mathbb{P}^\alpha_\beta$ for $\beta < \alpha$, this is obvious.

Clause (E), Claim: Trivial.

Clause (F), Claim: Subclauses (a) and (c)–(f) are trivial.
Subclause (b): Here we should be careful concerning transitivity of $\le_{\mathbb{P}_\beta}$ as we did not ask just that the order is forced but there is a hc witness. However we define the order as the transitive closure of cases with a witness so we can combine them, i.e., use almost witness in the sense of Definition 5.13. See more in the proof of clause (I).

Clause (G), Definition:
Case 1: For $\beta < \alpha$ note that N is also a $\mathfrak{B}^\beta$–candidate as $\beta \in N$ and use the definition for $\bar{\mathbb{Q}} \restriction \beta$, i.e., the induction hypothesis.
Case 2: If $\gamma = \beta = \alpha$ – trivial.
Case 3: For $\beta = \alpha$, $\alpha = 0$ – trivial.
Case 4: Suppose $\gamma < \beta = \alpha$, $\alpha = \beta' + 1$.
Then: $q \restriction \beta'$ is $[\gamma, \beta')$–canonically $\langle N, \mathbb{P}_\beta \rangle$–generic and for some $\underset{\sim}{\tau}$, and $\underset{\sim}{r} = \langle (p_i, \underset{\sim}{r}_i) : i < \omega \rangle$ we have

(α) $\{p_i : i < \omega\}$ list a subset of $\mathbb{P}^N_{\beta'}$ which is predense in it for $\leq^N_{\mathbb{P}_{\beta'}}$,

(β) $\underset{\sim}{r}_i \in N$ and $N \models$ "$\underset{\sim}{r}_i$ is an hc-κ_β-$\mathbb{P}_\beta$-name of a member of $\underset{\sim}{\mathbb{Q}}_\beta$ which is above $p(\beta)$,

(γ) $\underset{\sim}{r}$ is $\underset{\sim}{r}_i$ for the first i such that $p_i \in \underset{\sim}{G}_{\mathbb{P}_{\beta''}}$ and $\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta}$ if there is no such i and $q(\beta) = \underset{\sim}{\mathbf{B}}_1^{\mathbb{Q}_\beta}(\underset{\sim}{\tau}, N, \bar{a}[\underset{\sim}{G}_\beta])$ for some ω-list $\bar{a}$ of N.

(δ) $\mathrm{Dom}(q) \setminus \gamma = N \cap \gamma \setminus \beta$

Case 5: $\gamma < \beta = \alpha$, β a limit.
Say that diagonalization was used (and for $\beta' \in N \cap \beta \setminus \gamma$ use the induction); see more in clause (H) below.

Lastly we define $\varphi_2^{\mathbb{P}_\alpha}(\langle p_i : i < \omega\rangle^\frown\langle q\rangle)$ it say that for some $\mathfrak{B}^\alpha$-candidate N, which is hereditarily countable, q is $[0,\alpha]$-canonically $\langle N, \mathbb{P}_\alpha\rangle$–generic (above some $p \in \mathbb{P}^N_\alpha$) and $\{p_i : i < \alpha\}$ list some subset of $\mathbb{P}^N_\alpha$ predense under $\leq^N_{\mathbb{P}_\alpha}$ (or dense open if you prefer). Note that it does not matter if we use N or its ord–hereditary collapse (see 4.7; this applies to the proof of (H)).

Clause (H), Theorem: First about the $[0,\beta]$–canonical $\langle N, \mathbb{P}\rangle$–generic being defined by a witness the zero case is trivial. In the limit case this holds as all the construction is carried in N, and any ω-list of the members of N (and $N \cap \mathrm{Ord}$) suffices to make all the free choices. So we first compute from it a set $\langle \beta_n : n < \omega\rangle$ of ordinals in $N \cap \beta$ such that $\beta_0 = \gamma, \beta_n < \beta_{n+1}$ and $(\forall \beta' \in N \cap \beta)(\exists n)[\beta' < \beta_n]$, recall that by the definition of $\mathfrak{B}^\alpha$, $N \cap \beta$ is closed under the successor function hence $N \cap \beta$ has no last element. Second choose $\langle \mathcal{I}_n : n < \omega\rangle$ listing $\mathrm{pdac}(N, \mathbb{P}_\beta)$. Third we choose $\langle \underset{\sim}{p}_n : n < \omega\rangle$ by induction in n, such that $\underset{\sim}{p}_n$ is a hc-κ^{β_n}-$\mathbb{P}_{\beta_n}$-name of a member[30] of $\mathbb{P}^N_\beta$, $\underset{\sim}{p}_0 = p, \underset{\sim}{p}_{n+1}$ is the first member of $\mathbb{P}^N_\beta$ (by the list above) such that if $\underset{\sim}{G}_\beta \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$, then $\underset{\sim}{p}_{n+1}\restriction\beta_n = \underset{\sim}{p}_n\restriction\beta_n$, and for some $\mathcal{I}$, $N \models$ "$\mathcal{I}$ is a maximal antichain of $\mathbb{P}_{\beta_n}$ and if $r \in \underset{\sim}{\mathcal{I}} \cap \underset{\sim}{G}_{\beta_n}$ then $\underset{\sim}{p}_{n+1}$ is the first $p'' \in \mathbb{P}^N_\beta / \underset{\sim}{G}_{\beta_n}$ satisfying $\underset{\sim}{p}_n[G_{\beta_n}] \leq_{\mathbb{P}_\beta} p' \in \mathcal{I}_n$, $p'\restriction\beta_n \leq_{\mathbb{P}_{\beta_n}} r$. The successor case is easy using $\mathbf{B}_1^{\mathbb{Q}_{\beta'}}$.

We turn to proving that canonically generic is generic, that is we have to prove

$\otimes_0$ if N is a $\mathfrak{B}^\beta$-candidate $\gamma \leq \beta \leq \alpha$, $\{\gamma, \beta\} \subseteq N \cap (\alpha + 1)$, and $q \in \mathbb{P}_\beta$ is canonically $[\gamma, \beta]$–generic above p, $p \in \mathbb{P}^N_\beta$, *then* q is $\langle N, \mathbb{P}_\beta\rangle$–generic, i.e., $q \Vdash$ "$\underset{\sim}{G}_{\mathbb{P}_\beta} \cap \mathbb{P}_\beta$ is not disjoint to $\mathcal{I}^N$ for $\mathcal{I}^N \in \mathrm{pd}(N, \mathbb{P}_\beta)$ and is directed by $\leq_{\mathbb{P}_\beta}$ and p belongs to it".

We do it by the same cases as in clause (G).

[30] but in general, itself not a member of N

case 1: $\beta < \alpha$: By the induction hypothesis (pedantically we define N' which is the same as N except $\mathfrak{B}^{N'} = (\mathfrak{B}^\alpha_\beta$ as defined inside $\mathfrak{B}^\alpha)$, etc or just use "essentially a candidate".
case 2: $\gamma = \beta = \alpha$: In the Definition (G) we assume q is $\langle N, \mathbb{P}_\gamma\rangle$–generic.
case 3: $\alpha = 0$: Trivial
case 4: $\gamma < \beta = \alpha = \beta' + 1$:
So assume that $G_\beta \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$ and $q \in \mathbb{P}_\beta$. Let $G_{\beta'} = G_\beta \cap \mathbb{P}_{\beta'}$, clearly it is a generic subset of $\mathbb{P}_{\beta'}$ over $\mathbf{V}$ and $q \restriction \beta' \in G_{\beta'}$. By the definition in stage G, the condition $q' = q \restriction \beta'$ is canonically $\langle N, \mathbb{P}_{\beta'}\rangle$–generic as N is essentially a $\mathbb{P}_{\beta'}$-candidate. So by the induction hypotheses:

$\otimes_1$ $G_{\beta'} \cap \mathbb{P}^N_{\beta'}$ is not disjoint to any $\mathcal{I}^N, \mathcal{I} \in \mathrm{pd}(N, \mathbb{P}_{\beta'})$ and is directed by $\leq^N_{\mathbb{P}_{\beta'}}$.

Let $N^+ = N\langle G_{\beta'} \cap \mathbb{P}^N_{\beta'}\rangle$, so

$\otimes_2$ N^+ is a generic extension of N for $\mathbb{P}_{\beta'}$.

Now as ZFC^-_* is $\{\mathbb{P}_\beta\}$–good

$\otimes_3$ $\mathbf{V}[G_{\beta'}] \models$ "N^+ is a $\underset{\sim}{\mathbb{Q}}_{\beta'}[G_{\beta'}]$-candidate".

So by Definition 5.13(3) clause (f) and our definition of "q is $[0, \beta]$-canonically $\langle N, \mathbb{P}_\beta\rangle$–generic" we have

$\otimes_4$ $\mathbf{V}[G_{\beta'}] \models$ "$q(\beta')[G_{\beta'}]$ is $(N^+, \underset{\sim}{\mathbb{Q}}_{\beta'}[G_{\beta'}])$–generic".

In $\mathbf{V}[G_{\beta'}]$ we have $\mathbb{Q}_{\beta'} = \underset{\sim}{\mathbb{Q}}_{\beta'}[G_{\beta'}]$, i.e., $(\mathfrak{B}_\beta[G_{\beta'}], \bar{\varphi}_\beta, \theta_\beta)$ defines it, and we have $\underset{\sim}{\mathbb{Q}}'_{\beta'} = \mathbb{P}_\beta / G_{\beta'}$. Now every member of $\mathbb{Q}_{\beta'}$ belongs to $\mathcal{H}_{<\aleph_1}(\theta_\beta)$ and has a witness in $\mathcal{H}_{<\aleph_1}(\kappa^\alpha)$, a $\mathbb{P}_{\beta'}$-names say $\underset{\sim}{x}_1[G_{\beta'}], \underset{\sim}{x}_2[G_{\beta'}]$ hence by Proposition 5.12(2), there is $\underset{\sim}{x}_0$, a hc-κ^α-$\mathbb{P}_{\beta'}$-name of the truth value of this, So $p \in G_{\beta'}$ which forces $\underset{\sim}{x}, \underset{\sim}{x}_1, \underset{\sim}{x}_2$ to have those properties, that is $\underset{\sim}{x}_0$ to be truth. So $p \cup \{\langle \beta', \{(\ell, \underset{\sim}{x}_\ell) : \ell < 3\}\rangle\} \in \mathbb{P}_\alpha$ and so $q = \underset{\sim}{x}_1[G_{\beta'}] \in \underset{\sim}{\mathbb{Q}}_{\beta'}$ is actually $p(\beta')$. The inverse inclusition is also easy. Similarly for the order; so

$\otimes_5$ $\mathbb{Q}_{\beta'}[G_{\beta'}]$ is (essentially) equal to $\mathbb{P}_\alpha / G_{\beta'}$.
$\otimes_6$ Similarly inside $N[G_{\beta'} \cap \mathbb{P}^N_{\beta'}]$.
$\otimes_7$ $G_\beta \cap \mathcal{I}^N \neq \emptyset$ for $\mathcal{I} \in \mathrm{pdac}(N, \theta, \mathbb{P}_\beta)$.

We leave the checking to the reader as it is the same as the usual CS iteration and prove that

$\otimes_8$ $\mathbb{G}_\beta \cap \mathbb{P}^{\mathbb{N}}_\beta$ is directed by $\leq^N_{\mathbb{P}_\beta}$

Why? Let $p_1, p_2 \in \mathbb{G}_\beta \cap \mathbb{P}^{\mathbb{N}}_\beta$, hence $p_1 \restriction \beta, p_2 \restriction \beta \in G_{\beta'} \cap \mathbb{P}^N_{\beta'}$ hence by $\otimes_1$ above, there is $r_* \in G_{\beta'} \cap \mathbb{P}^N_{\beta'}$ such that $N \models$ "$p_1 \restriction \beta' \leq_{\mathbb{P}_{\beta'}} r$ and $p_2 \restriction \beta' \leq_{\mathbb{P}_{\beta'}} r$".

In $\mathbf{V}[G_{\beta'}]$, the condition $q(\beta')[G_{\beta'}]$ is $\langle N^+, \mathbb{Q}_\beta\rangle$–generic, by the def of canonical generic so $\mathbf{V}[G_{\beta'}] \models [q(\beta')[G_{\beta'}] \Vdash_{\mathbb{Q}_\beta}$ "$\underset{\sim}{G}_{\mathbb{Q}_\beta} \cap \mathbb{Q}^{N^+}_\beta$ is directed by $\leq^{N^+}_{\mathbb{Q}_\beta}$"]. As $q \in G_\beta$ we can finish.

$\otimes_9$ $p\restriction\beta' \in G_{\beta'}$.

Why? By the inductive hypothesis.

$\otimes_{10}$ $p(\beta')[G_{\beta'}] \in G_\beta/G_{\beta'}$.

Why? By the assumption on $\mathbf{B}_{\beta'}$. Now, it follows from $\otimes_9 + \otimes_{10}$ that

$\otimes_{11}$ $p \in G_\beta$.

case 5 $\beta = \alpha$ is a limit ordinal $> \gamma$.
Should also be clear.

Clause (I), Theorem: We have defined $\mathfrak{B}^\alpha$ and κ^α (so $\theta^\alpha = \kappa^\alpha$). The formulas $\varphi_\ell^{\mathbb{P}_\alpha}$ ($\ell < 3$) are implicitly defined (in the induction).

Why is $\varphi_0^{\mathbb{P}_\alpha}$ absolute enough? As the demand on $p(\beta)$ above says that $r_{p\restriction(\beta+1),\beta}$, the witness for $p(\beta) \in \mathrm{cl}(\mathbb{Q})$, is such that $r[\underset{\sim}{G}_{\mathbb{P}_\beta}] = \mathfrak{t}$ gives all the required information.

Why is $\varphi_1^{\mathbb{P}_\alpha}$ absolute enough? Because the canonical genericity is about φ_2 and the properness requirement, see clause (G), are designed such that they fit.

Now one proves by induction on $\beta \leq \alpha$:

($\otimes$) *if* N is a $\mathfrak{B}^\alpha$–candidate, $\gamma_0 \leq \gamma_1 \leq \beta$, $\{\gamma_0, \gamma_1, \beta\} \subseteq (\alpha+1)\cap N$, $p \in \mathbb{P}_\beta^N$, $q \in \mathbb{P}_\gamma$, $p\restriction\gamma_1 \leq q$, q is $[\gamma_0, \gamma_1)$–canonically $(N, \mathbb{P}_{\gamma_1})$–generic, *then* we can find q^+ such that:
- (α) $q^+ \in \mathbb{P}_\beta$, $q^+\restriction\gamma = q$,
- (β) $p \leq q^+$,
- (γ) q^+ is $[\gamma, \beta)$–canonically $(N, \mathbb{P}_\beta)$–generic.

Clause (J), Theorem: Straight.

Clauses (K),(L): Done elaborately in 5.19 below. $\blacksquare_{5.18}$

Proposition 5.19. *Assume $(*)$ of* 5.18 *(i.e., that* ZFC_*^- *is normal and nice to $\mathbb{P}_i$'s for $i \leq \alpha$ below). The iteration in* 5.18 *is equivalent to a CS iteration. More formally, assume*

$$\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i) : i < \alpha\rangle \textit{ is a CS–nep iteration.}$$

We can define $\bar{\mathbb{Q}}' = \langle\mathbb{P}'_i, \underset{\sim}{\mathbb{Q}}'_i : i < \alpha\rangle$ and $\langle F_i : i < \alpha\rangle$ such that for $i \leq \alpha$

- (a) *$\bar{\mathbb{Q}}'\restriction i$ is a CS iteration (and $\mathbb{P}'_\alpha$ is the limit),*
- (b) *F_i is a mapping from $\mathbb{P}_i$ into $\mathbb{P}'_i$, such that* $\mathrm{Dom}(F_i(p)) = \mathrm{Dom}(p)$ *for $p \in \mathbb{P}_i$, mapping $\emptyset_{\mathbb{P}_i}$ to $\emptyset_{\mathbb{P}'_i}$ (the minimal elements),*
- (c) *if $j < i$* then *$F_j = F_i\restriction\mathbb{P}_j$, and $p \in \mathbb{P}_i \quad\Rightarrow\quad F_i(p)\restriction j = F_j(p\restriction i)$,*
- (d) *F_i is an embedding of $\mathbb{P}_i$ into $\mathbb{P}'_i$ with dense range, more exactly:*
  - (i) $\mathbb{P}_i \models p \leq q \quad\Rightarrow\quad \mathbb{P}'_i \models F_i(p) \leq F_i(q)$,

(ii) *if* $p \in \mathbb{P}_i$ *and* $\mathbb{P}'_i \models F_i(p) \le p'$, then *for some* q *we have* $\mathbb{P}_i \models p \le q$ *and* $\mathbb{P}'_i \models p' \le F_i(q)$,

(iii) *if* $p(\gamma) = \emptyset_{\mathbb{Q}_\gamma}$ *and* $p \in \mathbb{P}_\beta$, *then* $(F_\beta(p))(\gamma) = \emptyset_{\mathbb{P}_\gamma}$,
hence

(iv) *if* G_i *is a generic subset of* $\mathbb{P}_i$ *over* $\mathbf{V}$, then

$$\{p' \in \mathbb{P}'_i : \text{ for some } p \in G_i,\ \mathbb{P}'_i \models p' \le F_i(p)\}$$

is a subset of $\mathbb{P}'_i$ *generic over* $\mathbf{V}$,

(v) *if* G'_i *is a generic subset of* $\mathbb{P}'_i$ *over* $\mathbf{V}$, then $F_i^{-1}(G'_i)$ *is a subset of* $\mathbb{P}_i$ *generic over* $\mathbf{V}$,

(e) $\underset{\sim}{\mathbb{Q}}_i$ *is mapped by* F_i *to* $\underset{\sim}{\mathbb{Q}}'_i$,

(f) $\Vdash_{\mathbb{P}'_i}$ " $\mathbb{Q}'_i$ *is proper* ".

Proof. Straight. Still, by (simultaneous) induction on i (of course, $F_i, \bar{\mathbb{Q}}' \restriction i$ depend only on $\bar{\mathbb{Q}} \restriction i$).

Case 1: $i = 0$ Trivial.

Case 2: $i = \varepsilon + 1$

By clause (d) for ε, F_i maps $\mathbb{P}_\varepsilon$–names to $\mathbb{P}'_\varepsilon$–names naturally. Let $\underset{\sim}{\mathbb{Q}}'_\varepsilon = F_\varepsilon(\underset{\sim}{\mathbb{Q}}_\varepsilon)$, it is a $\mathbb{P}'_\varepsilon$–name of a forcing notion, so $\bar{\mathbb{Q}}' \restriction i$ is defined naturally. So for every $p \in \mathbb{P}_i$ we define $F_i(p)$ by (recall clause (b)):

(i) for $j \in \mathrm{Dom}(p) \cap \varepsilon$, $(F_i(p))(j) = (F_\varepsilon(p \restriction \varepsilon))(j)$,

(ii) if $j = \varepsilon \in \mathrm{Dom}(p)$, then $(F_i(p))(j)$ is the F_ε–image of the $\mathbb{P}_\varepsilon$–name $p(\varepsilon)$ (see 5.18 clause (D)). As $\Vdash_{\mathbb{P}_\varepsilon}$ "$p(\varepsilon) \in \underset{\sim}{\mathbb{Q}}_\varepsilon$" clearly $\Vdash_{\mathbb{P}'_\varepsilon}$ "$(F_i(p))(\varepsilon) \in \underset{\sim}{\mathbb{Q}}'{}_\varepsilon$", so $F_i(p)$ actually belongs to $\mathbb{P}'_i$.

Now clauses (a), (b), (c), (d)(i)+(iii) should be clear and we shall now prove clause (d)(ii), then clause (e) will follow. So we are given $p \in \mathbb{P}_i$ and $p' \in \mathbb{P}'_i$ such that $\mathbb{P}'_i \models F_i(p) \le p'$, and we should find q such that $\mathbb{P}_i \models p \le q$ and $\mathbb{P}'_i \models p' \le F_i(q)$. Applying the induction hypothesis (clause (d)(ii)) to $p \restriction \varepsilon$, $p' \restriction \varepsilon$ we can find $q_0 \in \mathbb{P}_\varepsilon$ satisfying $\mathbb{P}'_\varepsilon \models p' \restriction \varepsilon \le F_\varepsilon(q_0)$. Note that $p'(i)$ is a $\mathbb{P}'_\varepsilon$–name of a member of $\underset{\sim}{\mathbb{Q}}'_\varepsilon$. Let χ be large enough, N a countable elementary submodel of $(\mathcal{H}(\chi), \in)$ to which $\{\mathfrak{C}, \mathfrak{B}, \mathbb{Q}, \varepsilon, F_\varepsilon, p, p', q, q_0\}$ belongs, and hence is a $\mathbb{P}_\varepsilon$–candidate (we are assuming normality).

We can find $q_1 \in \mathbb{P}_\varepsilon$ which is $(N, \mathbb{P}_\varepsilon)$–generic and $\mathbb{P}\varepsilon \models$"$q_0 \le q_1$". Let $\underset{\sim}{r} = p'(\varepsilon)$; by the inductive hypothesis $F(q)$ is $(N, \mathbb{P}_\epsilon)$-generic, hence above q_1, $\underset{\sim}{r}$ is equivalent to some r' which is as in 5.18(D) case(2). By the induction hypothesis there is a $\mathbb{P}_\varepsilon$–name $\underset{\sim}{r}$ such that $q_1 \Vdash_{\mathbb{P}_\varepsilon}$ "$F(\underset{\sim}{r}) = r'$".

Now $q_1 \cup \{(\varepsilon, \underset{\sim}{r}')\}$ is as required.

Case 3: i a limit ordinal of countable cofinality

Define $F_i(p) = q$ iff $p \in \mathbb{P}_i$, $\mathrm{Dom}(q) \subseteq i$ and $(\forall j < i)(F_i(p) \restriction j = F_j(q))$.

Now $F_i(p)$ is well defined by clause (c). As $\mathbb{P}_i = \{p : \mathrm{Dom}(p) \subseteq i$ and $(\forall j < i)(p \restriction j \in \mathbb{P}_j)\}$ and we let $\mathbb{P}'_i = \{p : \mathrm{Dom}(p) \subseteq i$ and $(\forall j < i)(p \restriction j \in \mathbb{P}'_j)\}$ clearly clauses (a), (b), (c), (d)(i) hold. Also clause (d)(iii) holds by the inductive definition of the order in both cases, and clause (e) will follow (if $i < \alpha$) once we prove clause (d)(ii); for proving it we prove that for any $i_0 \leq i_1 \leq i$ from N the statement $\boxtimes_{i_0,i_1}$, see below (for $(i_0, i_1) = (0, i)$ and appropriate N we get the desired conclusion): for χ large enough:

$\boxtimes_{i_0,i_1}$ Assume $N \prec (\mathcal{H}(\chi), \in)$ is countable, and $\{\mathfrak{C}, \mathfrak{B}, \bar{\mathbb{Q}}, \bar{\mathbb{Q}}' \restriction i_1, F_{i_1}, i_0, i_1\} \in N$, and $p \in N \cap \mathbb{P}_{i_1}$, $q \in \mathbb{P}_{i_0}$, $\mathbb{P}_{i_0} \models p \restriction i_0 \leq q$, q is $(N, \mathbb{P}_{i_0})$–generic (may add canonically) $\mathbb{P}'_{i_1} \models F_{i_1}(p) \leq p'$ and $p' \in N$ and $\mathbb{P}'_{i_0} \models p' \restriction i_0 \leq q$. *Then* we can find q^+ such that $q^+ \in \mathbb{P}_{i_1}$, $q^+ \restriction i_0 = q$, $\mathbb{P}_{i_1} \models p \leq q^+$, $\mathbb{P}'_{i_1} \models p' \leq F_{i_1}(q^+)$ and q is $(N, \mathbb{P}_{i_1})$–generic (may add canonically or at least) $\mathrm{Dom}(q^+) \setminus i_0 = N \cap [i_0, i_1)$.

This is done as usually by induction on i_1.

Case 4: $\mathrm{cf}(i) > \aleph_0$ Easier. $\blacksquare_{5.19}$

Discussion 5.20. If we have a CS iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ in mind, and each $\mathbb{Q}_i$ is a correct explicit nep for $(\mathfrak{B}_i, \underset{\sim}{\bar{\varphi}}_i, \underset{\sim}{\theta}_i)$, still we may not translate it to a CS–nep iteration when those names are too complicated. By whatever we have in mind we can first define it as a CS–nep iteration and then we are assured by 5.19 that actually we have the CS iteration we have in mind to begin with, using of course:

Proposition 5.21. *For any function F and an ordinal α we can find a CS–nep iteration $\bar{\mathbb{Q}} = \langle (\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i) : i < \beta \rangle$ such that*

(a) *for $i < \beta$, $(\bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i) = F(\bar{\mathbb{Q}} \restriction i)$,*
(b) *$\beta \subseteq \alpha$,*
(c) *if $\beta < \alpha$ then $F(\bar{\mathbb{Q}})$ is not as required in* 5.18 *clause (A)(β) with* $\mathrm{Lim}_{\mathrm{nep}}(\bar{\mathbb{Q}})$, *$F(\bar{\mathbb{Q}})$ here standing for $\mathbb{P}_\beta$, $(\underset{\sim}{\mathfrak{B}}_\beta, \underset{\sim}{\bar{\varphi}}_{\beta'} \kappa_\beta)$ there.*

Proof. Obvious. $\blacksquare_{5.21}$

Proposition 5.22. *In the context of* 5.18*:*

Assume that each $\underset{\sim}{\mathfrak{B}}_\beta$ is essentially a real; i.e., $\kappa_\beta = \omega$ and so for R in the vocabulary of $\underset{\sim}{\mathfrak{B}}_\beta$ we have $R^{\mathfrak{B}_\beta} \subseteq {}^{n(R)}\omega$. If $\alpha < \omega_1$ then *so is the $\mathfrak{B}_\alpha$. (If $\alpha \geq \omega_1$ we get weaker results).*

Proof. Left to the reader. $\blacksquare$

Remark 5.23. 1. Note that 5.18, 5.19 (and 5.22) say something even for $\alpha = 1$ so it speaks on $\mathrm{cl}_3(\mathbb{Q}_0) = \mathbb{P}_1$ (or $\mathrm{cl}(\mathbb{Q}_0) = \mathbb{P}_1$).

2. Concerning 5.22 note that if $\kappa(\mathfrak{B}) \geq \omega_1$, the difference between nep and snep is not large, however the case $\alpha < \omega_1$ has special interest.
3. In 5.18, 5.19, we can replace the use of $\mathrm{cl}_3(\mathbb{Q})$ from Definition 5.7 (using 5.8) by $\mathbb{Q}' = \mathrm{cl}(\mathbb{Q})$ from 5.2.
4. We can derive a theorem on local in 5.19, but for strong enough ZFC^-_*, then it anyhow follows.

Of course, we can get forcing axioms.

Proposition 5.24. 1. *Assume for simplicity that* $\mathbf{V} \models 2^{\aleph_0} = \aleph_1$ & $2^{\aleph_1} = \aleph_2$. Then *for some proper* $\aleph_2$*–c.c. forcing notion* $\mathbb{P}$ *of cardinality* $\aleph_2$ *and we have*

($\otimes$) $\mathbb{P}$ *is the limit of a CS iteration of* $(\aleph_1, \aleph_1)$*-nep forcing notions and we have in* $\mathbf{V}^{\mathbb{P}}$*:*

($\oplus$) $\mathrm{Ax}_{\omega_1}[(\aleph_1, \aleph_1)$*–nep*]*:* if $\mathbb{Q}$ *is a* (κ, θ)*–nep forcing notion,* $\kappa, \theta \leq \aleph_1$ *and* $\mathcal{I}_i$ *is a dense subset of* $\mathbb{Q}$ *for* $i < \omega_1$ *and* $\underset{\sim}{S}_i$ *as a* $\mathbb{Q}$*–name of a stationary subset of* ω_1 *for* $i < i(*) \leq \omega_1$, then *for some directed* $G \subseteq \mathbb{Q}$ *we have: for any* $i < \omega_1$ *we have* $G \cap \mathcal{I}_i \neq \emptyset$ *and*

$$\underset{\sim}{S}_i[G] \stackrel{\text{def}}{=} \{\zeta < \omega_1 : \textit{for some } q \in G \textit{ we have } q \Vdash_{\mathbb{Q}} \text{“} \zeta \in \underset{\sim}{S}_i \text{”}\}$$

is a stationary subset of ω_1.

2. *We can demand that* $\mathbb{P}$ *is explicitly* $(\aleph_2, \aleph_2)$*–nep provided that in* ($\oplus$) *we add "explicitly simply" to the requirements on* $\mathbb{Q}$.
3. *Assume* κ *is supercompact with the Laver indestructibility, and we replace* $\aleph_2$ *by* κ. Then *in parts* 1) *and* 2)*, we can strengthen* ($\oplus$) *to* $\mathrm{Ax}_{\omega_1}[$*nep*$]$.

Proof. Straight (as failure of "$\mathbb{Q}$, i.e., $\bar{\varphi}$ is nep" is preserved when extending the universe by a proper forcing). $\blacksquare_{5.24}$

Proposition 5.25. *In* 5.14–5.19 *we can replace CS countable support (CS) by free limit (see* [25, Chapter IX, §1.8, pp. 436–443]*) this influences just how the elements of* $\mathrm{Lim}_{\mathrm{nep}}(\bar{\mathbb{Q}})$ *look like, essentially what we have done to each* $\mathbb{Q}_i$ *(replace it say by* $\mathrm{cl}_3(\mathbb{Q}_i)$*) is done also in limit of countable cofinality.*

Proposition 5.26. *We can generalize the definitions and claims so far by:*

(a) *a forcing notion* $\mathbb{Q}$ *is* $(\mathbb{Q}, \leq, \leq_{\mathrm{pr}}, \emptyset_{\mathbb{Q}})$*, where* $\leq_{\mathrm{pr}}$ *is a quasi order,* $p \leq_{\mathrm{pr}} q \Rightarrow p \leq q$ *and* $\emptyset_{\mathbb{Q}}$ *the minimal element;*
(b) *in the definition of nep in addition to* φ_1 *we have* $\varphi_{1,\mathrm{pr}}$ *defining* $\leq_{\mathrm{pr}}$*, which is upward absolute from* $\mathbb{Q}$*–candidates, and in Definition* 1.3(2)(*c*) *we strengthen* $p \leq q$ *to* $p \leq_{\mathrm{pr}} q$*;*

(c) *the definition of CS iteration* $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ *is modified in the following way:*

$$\mathbb{P}_i = \{p: \ p \text{ is a function, } \mathrm{Dom}(p) \text{ is a countable subset of } i, \\ j \in \mathrm{Dom}(p) \ \Rightarrow \Vdash_{\mathbb{P}_j} \text{ “} p(j) \in \underset{\sim}{\mathbb{Q}}_j \text{ ”}\},$$

with the order given by: $p \leq q$ *if and only if*

- $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$,
- $\beta \in \mathrm{Dom}(p) \ \Rightarrow \ q \restriction \beta \Vdash p(\beta) \leq_{\mathbb{Q}_\beta} q(\beta)$, *and*
- $\{\beta \in \mathrm{Dom}(p) : \neg(q \restriction \beta \Vdash p(\beta) \leq_{pr, \mathbb{Q}_\beta} q(\beta)\}$ *is finite.*

(d) *Similarly for the CS-nep iteration.*

Proof. Left to the reader. $\blacksquare_{5.26}$

Discussion 5.27. 1. Why not $\{\beta \in \mathrm{Dom}(p) : \nVdash_{\mathbb{P}_p} \text{“}\emptyset_{\mathbb{Q}_\beta} \leq_{\mathrm{pr}} p(\beta)\text{”}\}$ finite? Let $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \omega_1 \rangle$ be such iteration (just CS or CS–nep but with purity condition as above), and assume that $\underset{\sim}{\tau}_i$ is a $\underset{\sim}{\mathbb{Q}}_i$–name, such that

$$\Vdash_{\mathbb{P}_i} \text{ “ there is } p \in \underset{\sim}{\mathbb{Q}}_i,\ p \Vdash_{\underset{\sim}{\mathbb{Q}}_i} \text{“}\underset{\sim}{\tau}_i = \emptyset\text{”, but } \underset{\sim}{\mathbb{Q}}_i \models \emptyset_{\underset{\sim}{\mathbb{Q}}_i} \leq_{\mathrm{pr}} p \text{ implies} \\ \text{that there is } q \text{ satisfying } \underset{\sim}{\mathbb{Q}}_i \models p \leq_{\mathrm{pr}} q \text{ and } q \Vdash_{\underset{\sim}{\mathbb{Q}}_j} \underset{\sim}{\tau}_i \neq \emptyset \text{ ”.}$$

This is a very reasonable demand. Now let $\underset{\sim}{w}$ be the $\mathbb{P}_{\omega_1}$–name $\underset{\sim}{w} = \{i : \underset{\sim}{\tau}_i[G] = 0\}$, and $\underset{\sim}{i}_n$ the n–th member of $\underset{\sim}{w}$. So $\underset{\sim}{w} \subseteq \omega_1$, and for every $p \in \mathbb{P}_{\omega_1}$ and $j < \omega_1$ we can find q, such that $p \leq q \in \mathbb{P}_{\omega_1}$, $j \subseteq \mathrm{Dom}(q)$ and for some finite $u \subseteq j$, $q \Vdash_{\mathbb{P}_{\omega_1}} \text{“}\underset{\sim}{w} \cap j \subseteq u\text{”}$, in fact $u = \{i < j : q \restriction i \nVdash_{\mathbb{P}_j} \text{“}\phi_{\mathbb{Q}_i} \leq^{\mathbb{Q}_i}_{\underset{\sim}{\tau}_i > 0} q(i)\text{”}\}$. Also obviously $\Vdash_{\mathbb{P}_{\omega_1}} \text{“}\underset{\sim}{w}$ is infinite”. So $\Vdash_{\mathbb{P}_{\omega_1}} \text{“}\sup\{\underset{\sim}{i}_n : n < \omega\} = \omega_1\text{”}$ (so $\mathbb{P}_{\omega_1}$ is not proper). See more in [25, XIV, §2].

2. In fact being able to define for $\bar{\mathbb{Q}}$ a nep-CS iteration (see in Proposition 5.18) the forcing notion $\mathbb{P}_w$ for any subset w of $\alpha = \ell g(\mathbb{Q})$ is one of the advantages of using nep forcing. As we like to deal with the connection between iteration and “subiteration”, it is convenient to have the following.

Definition 5.28. 1. For any set w of ordinals we can define when $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i \in w \rangle$ is CS iteration of forcing notions and its limit $\mathrm{Lim}(\bar{\mathbb{Q}})$, by induction on $\mathrm{otp}(w)$:

(a) $\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}_{i \cap w}$-name of a forcing notion,
(b) $\mathbb{P}_i = \mathrm{Lim}(\bar{\mathbb{Q}} \restriction (w \cap i))$,
(c) $\mathrm{Lim}(\bar{\mathbb{Q}}) = \{p : p$ a function such that $\mathrm{Dom}(p) \in [w]^{\leq \aleph_0}$ and $\Vdash_{\mathbb{P}_i} p(i) \in \underset{\sim}{\mathbb{Q}}_i$ for $i \in \mathrm{Dom}(p)\}$,
(d) with the usual order.

2. We can define similarly CS-nep forcing.

Observation 5.29. *Definition* 5.28(2) *gives nothing new (see* 5.35*).*

Definition 5.30. Assume that $\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i : i < \alpha\rangle$ is a nep-CS-iteration, and $\kappa^\alpha, \mathfrak{B}^\alpha$ are as in 5.18. For $\beta \le \alpha$ we define what is $\mathrm{supp}(p)$ for $p \in \mathbb{P}_\beta$ and $\mathrm{supp}(\underset{\sim}{\tau})$ for $\underset{\sim}{\tau}$ a hc-κ-$\mathbb{P}_\beta$-name, by induction on β as follows:
case 1 if $\beta = 0$ they are both the empty set;
case 2 if β *is limit*
For $p \in \mathbb{P}_\beta$ let $\mathrm{supp}(p) = \bigcup\{\mathrm{supp}(p \restriction \gamma) : \gamma < \beta\}$
For $\underset{\sim}{\tau}$ a hc-κ-$\mathbb{P}_\beta$-name let $\mathrm{supp}(\underset{\sim}{\tau}) = \bigcup\{\mathrm{supp}(q) :\ q$ is a condition from $\mathbb{P}_\beta$ which appears in $\underset{\sim}{\tau}\}$.
case 3 if $\beta = \gamma + 1$
For $p \in \mathbb{P}_\beta$, if $p \in \mathbb{P}_\gamma$ we are done otherwise $\mathrm{supp}(p) = \mathrm{supp}(p \restriction \gamma) \cup \{\gamma\} \cup \bigcup\{\mathrm{supp}(\underset{\sim}{x}_\ell) : \underset{\sim}{x}$ is one of the hc-$\kappa^{\beta+1}$-names appearing in $p(\gamma)\}$. For $\underset{\sim}{\tau}$ a hc-κ-$\mathbb{P}_\beta$-name let

$$\mathrm{supp}(\underset{\sim}{\tau}) = \bigcup\{\mathrm{supp}(q) :\ q \text{ is a condition from } \mathbb{P}_\beta \text{ which appears in } \underset{\sim}{\tau}\}.$$

Definition 5.31. Let $\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i) : i < \alpha\rangle$ be a nep-CS-iteration. By induction on $\beta \le \alpha$ we define for $w \subseteq \alpha$ the meaning of "w is $\bar{\mathbb{Q}}$-closed" and $\mathbb{P}_w = \mathbb{P}_w[\mathbb{Q}]$

(A) We say $w \subseteq \beta$ is $\bar{\mathbb{Q}}$-closed if
  (a) $i \in w \Rightarrow w \cap i$ is $\bar{\mathbb{Q}}$-closed,
  (b) if w has a last member, say i, then the name $\mathfrak{B}_i$ involves conditions from $\mathbb{P}_{w \cap i}$ only, that is for each of the hc-$\kappa(\mathfrak{B}_i)$-$\mathbb{P}_i$-names $\underset{\sim}{x}$ in the definition, $\mathrm{supp}(\underset{\sim}{x}) \subseteq w \cap i$.

(B) If $w \subseteq \beta$ is $\bar{\mathbb{Q}}$-closed, $\mathbb{P}_w = \{p \in \mathbb{P}_\beta : \mathrm{supp}(p) \subseteq w\}$ and let the order $\le_{\mathbb{P}_w}$ be $\le_{\mathbb{P}_\alpha} \restriction \mathbb{P}_w$.

Discussion 5.32. 1. A natural case is $\mathfrak{B}_i = \mathfrak{B}_i$, then every $w \subseteq \lg(\bar{\mathbb{Q}})$ is $\bar{\mathbb{Q}}$-closed.
2. A more general natural case is when for each $\mathfrak{B}_i$ there are hc-κ-$\mathbb{P}_i$-names $\underset{\sim}{x}_{i,n}$ for $n < \omega$ such that all $\mathfrak{B}_i$ is computed from then. So w is $\bar{\mathbb{Q}}$-closed if $\beta \in w \Rightarrow \bigcup(\mathrm{supp}(\underset{\sim}{x}_{\beta,n}) : n < \omega\} \subseteq w$.
3. In the more general case for $w \subseteq \alpha$, we can allow to "compute" $\mathfrak{B}_i$ only as far as we can by $\mathbb{P}_{i \cap w}$, seems reasonable, but no urgent need now, so not included in Definition 5.31.

Observation 5.33. *Let* $\bar{\mathbb{Q}}$ *be as in* 5.18
1. *If* $p \in \mathbb{P}_\gamma$, $\gamma \le \beta \le \alpha$, then $\mathrm{supp}(p)$ *does not change if we use the definition for* $\bar{\mathbb{Q}}$ *or for* $\bar{\mathbb{Q}} \restriction \beta$.

2. *If* $w \subseteq \beta < \alpha$ then w *is* $\bar{\mathbb{Q}}$*-closed iff* w *is* $(\bar{\mathbb{Q}}\restriction\beta)$*-closed, and similarly for* $\mathbb{P}_w$*.*
3. *If* $w_1 \subseteq w_2$ then $p \in \mathbb{P}_w \;\Rightarrow\; p \in \mathbb{P}_{w_2}$*.*

Proof. Trivial. $\blacksquare_{5.33}$

We know that $\mathbb{P}_{w_1} \lessdot \mathbb{P}_{w_2}$ in general fails for $\bar{\mathbb{Q}}$-closed $w_1 \subseteq w_2$. Still we intuitively feel that there is a connection of this sort and we shall have a substitute for it. The first step is

Observation 5.34. *Let* $\bar{\mathbb{Q}}$*-be a nep-CS-iteration (as in* 5.18*). If* w *and* $w_1 \subseteq w_2$ *are* $\bar{\mathbb{Q}}$*-closed,* then

(a) $\mathbb{P}_{w_1} \subseteq \mathbb{P}_{w_2}$ *as quasi orders,*
(b) *if* $p \leq_{\mathbb{P}_w} q$ *then* $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$*,*
(c) *if* $p \leq_{\mathbb{P}_w} q$ *then* $\mathrm{supp}(p) \subseteq \mathrm{supp}(q) \subseteq w$*.*

Proof. First note that clause (b) is a property of the CS-nep iteration which we can prove by induction on $\beta \leq \alpha$ for $p, q \in \mathbb{P}_\beta$. For $\beta = 0$, β limit trivial. For $\beta = \beta' + 1$ we define the order on $\mathbb{P}_\beta$ as the transitive closure of some "atomic cases". So check each using the induction hypothesis on $\mathbb{P}_{\beta'}$.
The proof of clauses (a),(c) is similar. $\blacksquare_{5.34}$

Observation 5.35. 1. *Assume* $\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \bar{\varphi}_i, \mathfrak{B}_i, \kappa_i, \theta_i) : i < \alpha\rangle$ *is a nep-CS-iteration, and each* $\mathfrak{B}_i$ *is an object in* $\mathbf{V}$ *not just a* $\mathbb{P}_i$*-name.*

- (A) *Every* $w \subseteq \alpha$ *is* $\bar{\mathbb{Q}}$*-closed.*
- (B) *If* $w \subseteq \alpha$ *and* $\bar{\mathbb{Q}}' = \langle \mathbb{P}'_i, \underset{\sim}{\mathbb{Q}}' : i < \mathrm{otp}(w)\rangle$ *is the CS iteration satisfying* $(*)$ *below,* then $\mathbb{P}'_{\mathrm{otp}(w)} \cong \mathbb{P}_w$*, where* $\bar{\mathbb{P}}'_{\mathrm{otp}(w)} = \mathrm{Lim}_{\mathrm{nep}}(\bar{\mathbb{Q}}')$*:*
  - $(*)$ *if* $j = \mathrm{otp}(\gamma \cap w)$ & $\gamma \in w$, then $\underset{\sim}{\mathbb{Q}}'_j$ *is the forcing notion defined by* $(\mathfrak{B}_\gamma, \bar{\varphi}_\gamma, \theta_\gamma)$ *in* $\mathbf{V}^{\mathbb{P}'_i}$*.*
- (C) *If* N *is a* $(\mathfrak{B}^\alpha, \bar{\varphi}^\alpha, \theta^\alpha)$*–candidate with* $(\mathfrak{B}^\alpha, \bar{\varphi}^\alpha, \theta^\alpha)$ *as in* 5.28*, and* $N \models$ *"*$U \subseteq \alpha$*" and* $w = \{i : N \models i \in U\} (\subseteq \alpha)$ *and* $p \in \mathbb{P}^N_U$, then *for some* q *we have*
  - (a) $\mathbb{P} \models p \leq q$*,*
  - (b) $q \in \mathbb{P}_w$*,*
  - (c) $q \Vdash_{\mathbb{P}_U}$ *"*$\underset{\sim}{G}_{\mathbb{P}_U} \cap \mathbb{P}^N_{\mathbb{P}_U}$ *is a* $\langle\mathbb{P}_U, N\rangle$*–generic (i.e., directed subsets a* $(\mathbb{P}^N_U, \leq^N_{\mathbb{P}_U})$ *generic over* N*)", and moreover*
  - (d) *clause (c) also holds for* $\Vdash_{\mathbb{P}_\alpha}$*.*

2. *Similar to* (1) *above, but each* $\mathfrak{B}_i$ *is "almost" an object in* $\mathbf{V}$*: it is computed by an ord-hc Borel function from some hc name* $\underset{\sim}{x}_i$ *of a real.*

Proof. Should be clear.

Observation 5.36. *In 5.35 we can deal with $\bar{\mathbb{Q}} \restriction w$ instead of mapping to* $\mathrm{otp}(w)$.

Observation 5.37. *Assume $\bar{\mathbb{Q}}$ is a nep-CS-iteration as in 5.18 so $\alpha = \ell g(\bar{\mathbb{Q}})$.*

1. *If $N \cap \alpha \subseteq w \subseteq \alpha$ and w is $\bar{\mathbb{Q}}$–closed, and N is a $(\bar{\mathbb{Q}} \restriction w)$–candidate (if you assume normality, as normally done, then any countable $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ to which $\bar{\mathbb{Q}}$, w belong is fine) and q is canonically $\langle N, \mathbb{P}_w\rangle$–generic, then q is $\langle N, \mathbb{P}_\alpha\rangle$–generic.*
2. *If N is a $\mathbb{P}_\alpha$-candidate and q is $[0, \alpha]$–canonically $\langle N, \mathbb{P}_\alpha\rangle$–generic, and w is $\bar{\mathbb{Q}}$-closed containing $N \cap \alpha$,* then *$q \in \mathbb{P}_w$ is also $\langle N, \mathbb{P}_w\rangle$–generic.*

Proof. Repeat the proof that canonically generic implies generic. $\blacksquare_{5.37}$

Conclusion 5.38. 1. *Assume that:*

(a) *$\mathbb{Q}^\epsilon = (\mathbb{Q}_\epsilon, \bar{\varphi}_\epsilon, \theta_\epsilon)$ for $\epsilon < \epsilon^*$ is a definition of a very straight correct simple explicit forcing notion even for*

$$K = \{\mathbb{P} : \mathbb{P} \text{ the limit of CS-iteration of forcing among } \{\mathbb{Q}^\epsilon : \epsilon < \epsilon^*\}\ \}$$

for ZFC^-_*, *which is normal,*

(b) *$\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ is a CS iteration, each $\mathbb{Q}_i$ is $(\mathbb{Q}^{\epsilon(i)})^{\mathbf{V}^{\mathbb{P}_i}}$ for some $\epsilon < \epsilon^*$,*

(c) *$N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is countable, $p \in \mathbb{P}$.*

Then *for some q we have*

(α) *$p \le q \in \mathbb{P}_\alpha$,*

(β) *q is $(N, \mathbb{P}_\alpha)$–generic,*

(γ) *if we let $w = \alpha \cap N$ and $\bar{\mathbb{Q}}^w = \langle \mathbb{P}^w_i, \underset{\sim}{\mathbb{Q}}^w_j : i \in w \cup \{\alpha\}, j \in w\rangle$ be the CS iteration with $\underset{\sim}{\mathbb{Q}}^w_j = (\mathbb{Q}^{\epsilon(j)})^{\mathbf{V}^{\mathbb{P}^w_j}}$, then we have $\mathbb{P}^w_\alpha \subseteq \mathbb{P}_\alpha$ (but not in general $\mathbb{P}^w_\alpha \lessdot \mathbb{P}_\alpha$), and there is $q \in \mathbb{P}^w_\alpha$ such that*

$$q \Vdash_{\mathbb{P}^w_\alpha} \text{“ } \underset{\sim}{G}_{\mathbb{P}^w_\alpha} \cap N \text{ is a generic subset of } (\mathbb{P}_\alpha)^N \text{ over } N \text{ ”},$$

and

$$q \Vdash_{\mathbb{P}_\alpha} \text{“} \underset{\sim}{G}_{\mathbb{P}_\alpha} \cap \mathbb{P}^N_\alpha \text{ is a generic subset of } (\mathbb{P}^N_\alpha, \le^N_{\mathbb{P}_\alpha}) \text{ ”}.$$

(δ) *Note that $\mathbb{P}^w_i$ is a CS-iteration of countable length of cases of $\mathbb{Q}^\epsilon$.*

2. *As in* 5.35(2).

Proof. Should be clear.

Conclusion 5.39. 1. *Assume that:*

(a) *$\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ is a CS iteration.*

(b) *each* $\mathbb{Q}_i$ *is one of the creature forcing proved in* [19] *to be proper, or more generally is a very straight nep forcing with definition in* $\mathbf{V}$, *being nep in* $\mathbf{V}^{\mathbb{P}_i}$,

(c) $\{\mathbf{B}_i : i < i^*\}$ *is a family of Borel sets (i.e., definitions of Borel subsets say of* $\mathbb{R}$ *or* ${}^{\omega}2$*), or just* Π^1_2 *sets,*

(d) *for every countable* $w \subseteq \alpha$, $\Vdash_{\mathbb{P}_w}$ "$\bigcap_{i<i^*} \mathbf{B}_i = \emptyset$".

Then $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}})} \models$ "$\bigcap\limits_{i<i^*} \mathbf{B}_i = \emptyset$".

2. *In part* (1), *we can allow* $\mathbb{Q}_i$ *to be as in* 5.37.
3. *In parts* (1) *and* (2), *in the definition of* $\mathbb{Q}_i$, *we can allow a* $\mathbb{P}_i$*-name* $\underset{\sim}{\eta}_i$ *of a real provided that* $\underset{\sim}{\eta}_i = \mathbf{B}^*_i(\langle \underset{\sim}{\nu}_{j(i,n)} : n < \omega\rangle)$ *with* $j(i,n) < \omega$, $\underset{\sim}{\nu}_i$ *is hc-*$\mathbb{Q}_j$*-name.*

Proof. (Recall that a Π^1_2 set is the intersection of $\aleph_1$ Borel sets.)

By 5.18 we can deal with CS-nep iteration. Assume toward contradiction that

$$p_1 \Vdash_{\mathbb{P}_\alpha} \text{“ } \underset{\sim}{\eta} \in \bigcap_{i<i^*} \mathbf{B}_i \text{ ”}.$$

Let χ be large enough and let $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be countable and such that $\bar{\mathbb{Q}} \in N$. Let q_1 be as in 5.38, so $w = \alpha \cap N$ is countable and $q \in \mathbb{P}_w \subseteq \mathbb{P}_\alpha$, and $q \Vdash \underset{\sim}{\eta} = \underset{\sim}{\eta}'$ where η' is computed from the truth values of $q \in G$ for $q \in \mathbb{P}_\alpha \cap N \subseteq \mathbb{P}_w$. By hypothesis (d), $q \Vdash \mathbb{P}_w \underset{\sim}{\eta}' \notin \bigcap\limits_{i<i^*} \mathbf{B}_i$. Hence there are q_1, i such that $\mathbb{P}_w \models q \le q_1$, $i < i^*$ and $q_1 \Vdash_{\mathbb{P}_w}$ "$\underset{\sim}{\eta}' \notin \mathbf{B}_i$". Let $N_1 \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be a countable model such that $N, \bar{\mathbb{Q}}, q_1, \underset{\sim}{\eta}' \in N_1$. Let q_2 be $\langle N_1, \mathbb{P}_w\rangle$–generic above q_1. Now, $q_2 \in \mathbb{P}_\alpha$ and

$$q_2 \Vdash \text{“ } \underset{\sim}{G}_{\mathbb{P}_\alpha} \cap N_1 \cap \mathbb{P}_w \text{ is a generic subset of } \mathbb{P}^{N_1}_w \text{ over } N_1 \text{ ”},$$

so we get an easy contradiction. $\blacksquare_{5.39}$

Discussion 5.40. Inside a forcing $\mathbb{Q}$ we may be able to find $q \in \mathbb{Q}$ and a forcing notion $\mathbb{Q}_*$ definable in N such that

$$q \Vdash \text{“}\underset{\sim}{G}_{\mathbb{Q}} \cap N \text{ is a directed subset of } (\mathbb{Q}^N, \le^N_{\mathbb{Q}}) \text{ generic over } N\text{”}.$$

This looks like a cheating so we call it faking; one example occurs in 5.35(3). Another example for faking is gotten by considering the Miller rational perfect forcing $\mathbb{Q} =$, which is $\{T : T \subseteq {}^{\omega>}\omega$ is a subtree such that for every $\eta \in T$ for some ν we have $\eta \lhd \nu \in T$ and $\mathrm{suc}_T(\eta)$ infinite $\}$. With the order $\le = \le_1 =$ inverse inclusion. Let

$p \le_0 q$ iff $q \subseteq p$ & $\mathrm{sp}(q) = \mathrm{sp}(p) \cap q$,

where $\mathrm{sp}(p) = \{\eta \in p : (\exists^{\mathrm{inf}} n)(\eta^\frown\langle n\rangle \in p\}$, and let

$p \le_2 q$ iff $p \le_0 q$ & $(\forall \eta)(\mathrm{tr}(q) \trianglelefteq \eta \in q \;\&\; \eta \in \mathrm{sp}(p) \to \eta \in \mathrm{sp}(q))$.

Now for every $r \in \mathbb{Q}$ the quasi order $(\{p \in \mathbb{Q} : r \leq_2 p\}, \leq_2)$ is isomorphic to Laver forcing. So if $\bar{\mathbb{Q}}$ is nep-CS-iteration of rational perfect forcing, we can define also $\leq^2_{\mathbb{P}_i}$. Then

(a) $\leq^2_{\mathbb{P}_i}$ is a quasi order on $\mathbb{P}_i$, $p \leq^2_{\mathbb{P}_i} q \Rightarrow p \leq_{\mathbb{P}_i} q$,

(b) if N is a $\bar{\mathbb{Q}}$-candidate, $p \in (\mathrm{Lim}(\bar{\mathbb{Q}}))^N$, then there is q such that: $p \leq^2_{\mathbb{P}\mathrm{Lim}(\bar{\mathbb{Q}})} q$ and $q \Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{G}_{\mathbb{P}_i} \cap N$ is a subset of $\mathbb{P}_i^N$ directed under $\leq^{2,N}_{\mathrm{Lim}(\bar{\mathbb{Q}})}$ and not disjoint to $\mathcal{I}^N$ to which p belongs" whenever $N \models$ "$\mathcal{I}$ is a subset of $\mathrm{Lim}(\bar{\mathbb{Q}})$ dense under $\leq^2_{\mathrm{Lim}(\bar{\mathbb{Q}})}$". Now $\bar{\mathbb{Q}}$ is nw–nep (see [26], [22]) whereas Laver forcing is just nep. Here the "nw-nep" dominates, i.e., we succeed even for non-well-founded models by thinning each $q(i)$ naturally (think on the case $\alpha = 1$).

6. When a real is $(\mathbb{Q}, \eta)$–generic over V

A first time reader may be advised that the case $\tau \in \{2, \omega\}$, $\sigma = \omega$ is typical, interesting and may suffice for you hence we write "$n < \sigma$".

Definition 6.1. 1. We say that $(\mathbb{Q}, \bar{W})$ is a temporary $(\mathfrak{B}, \theta, \sigma, \tau)$–pair *if* for some $\mathbb{Q}$–name $\underset{\sim}{\eta}$ the following conditions are satisfied:

(a) $\mathbb{Q}$ is a nep-forcing notion for $(\mathfrak{B}, \bar{\varphi}, \theta)$; possibly $\mathfrak{B}$ expands $\mathfrak{B}^{\mathbb{Q}}$,

(b) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in {}^{\sigma}\tau$",

(c) $\bar{W} = \langle W_n : n < \sigma \rangle$,

(d) for each $n < \sigma$, $W_n \subseteq \{(p, \alpha) : p \in \mathbb{Q} \text{ and } \alpha < \tau\}$,

(e) if $(p_\ell, \alpha_\ell) \in W_n$ for $\ell = 1, 2$ and α_1, α_2 are not equal, *then* p_1, p_2 are incompatible in $\mathbb{Q}$,

(f) for each $n < \sigma$ the set $\mathcal{I}_n = \mathcal{I}_n[\bar{W}] \stackrel{\text{def}}{=} \{p : (\exists \alpha)[(p, \alpha) \in W_n]\}$ is a predense subset of $\mathbb{Q}$,

(g) so $\sigma = \sigma[\bar{W}] = \sigma[\mathbb{Q}, \bar{W}]$ and (abusing notation) let $\tau = \tau[\bar{W}] = \tau[\mathbb{Q}, \bar{W}]$.

2. For $(\mathbb{Q}, \bar{W})$ as above, $\underset{\sim}{\eta} = \underset{\sim}{\eta}[\bar{W}] = \underset{\sim}{\eta}[\mathbb{Q}, \bar{W}]$ is the $\mathbb{Q}$–name

$$\{(n, \alpha) : (\exists p \in \underset{\sim}{G}_{\mathbb{Q}})[(p, \alpha)) \in W_n], \text{ so } n < \sigma\}.$$

3. We replace the temporary by K if this (specifically the demand (f)) holds in any K–extension of $\mathbf{V}$; similarly below.

4. We may write $(\mathbb{Q}, \underset{\sim}{\eta}), \bar{W} = \bar{W}^{\underset{\sim}{\eta}}$ abusing notation. If we omit $\mathfrak{B}$ we mean $\mathfrak{B} = \mathfrak{B}^{\mathbb{Q}}$. If $\tau = \aleph_0$ we may omit it; if $\tau = \sigma = \aleph_0$ we may omit them, if $\kappa(\mathfrak{B}) = \theta = \sigma = \tau = \aleph_0$, we may write κ.

5. We say that $\underset{\sim}{\eta}[\mathbb{Q}, \bar{W}]$ is a temporarily generic real (or function) for $\mathbb{Q}$ *if* for no distinct $G_1, G_2 \subseteq \mathbb{Q}$ generic over $\mathbf{V}$ do we have $\underset{\sim}{\eta}[G_1] = \underset{\sim}{\eta}[G_2]$.

6. In part (5) we add *directly* if for every $p \in \mathbb{Q}$ there is an ord-hc Borel function which from $\underset{\sim}{\eta}[\mathbb{Q}, \bar{W}]$ computes the truth value of $p \in \underset{\sim}{G}$.
7. Instead $(\mathbb{Q}, \bar{W})$ we may write $((\mathfrak{B}^{\mathbb{Q}}, \bar{\varphi}^{\mathbb{Q}}, \theta^{\mathbb{Q}}), \bar{W})$ (or with $\underset{\sim}{\eta}$ instead $\bar{W}$).

Definition 6.2. 1. Let $\mathcal{K}_{\kappa,\theta,\sigma,\tau}$ be the class of all $(\mathbb{Q}, \underset{\sim}{\eta})$ which are temporary $(\mathfrak{B}, \theta, \sigma, \tau)$–pairs for some $\mathfrak{B}$ with $\kappa(\mathfrak{B}) \leq \kappa$, $\|\mathfrak{B}\| \leq \kappa$.

2. Let $(\mathbb{Q}, \underset{\sim}{\eta})$ be a temporary (κ, θ)–pair (actually more accurately write $((\mathfrak{B}, \bar{\varphi}, \tilde{\theta}), \bar{W}))$; and $\sigma = \aleph_0 \geq \tau \geq 2$.

   Let N be a $\mathbb{Q}$–candidate and $\eta \in {}^{\omega}\omega$. We say that η is a $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over N *if* for some $G \subseteq \mathbb{Q}^N$ which is generic over N we have $\eta = \underset{\sim}{\eta}[G]$.

   We add satisfying q if above $q \in G$ (note that "satisfying" has more direct meaning if $\mathbb{Q}$ is generated by $\underset{\sim}{\eta}$ as in the proof of 6.5).

3. We say that $\underset{\sim}{\eta} = \underset{\sim}{\eta}[\bar{W}]$ is hereditarily countable if each W_n is countable (note: the generic reals of the forcing notions from [19] are like that, but for our purpose just "absolute enough" suffices).

Definition 6.3. 1. $(\mathbb{Q}, \bar{W})$ is a temporary explicitly $(\mathfrak{B}, \theta, \sigma, \tau)$–pair (or nep pair) *if* for some $\mathbb{Q}$–name $\underset{\sim}{\eta}$ we have:

   (a) $\mathbb{Q}$ is an explicit nep forcing notion for $(\mathfrak{B}, \bar{\varphi}, \theta)$,
   (b) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in {}^{\sigma}\tau$",
   (c) $\bar{W} = \langle \psi_{\alpha,\zeta} : \alpha < \sigma, \zeta < \tau \rangle$,
   (d) $\psi_{\alpha,\zeta} \in \mathrm{cl}_1(\mathbb{Q})$ for $\alpha < \sigma$, $\zeta < \tau$,
   (e) $\Vdash_{\mathbb{Q}}$ " $\underset{\sim}{\eta}(\alpha) = \zeta$ iff $\psi_{\alpha,\zeta}[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}$ ".

2. In this case $\underset{\sim}{\eta} = \underset{\sim}{\eta}[\bar{W}] = \underset{\sim}{\eta}[\mathbb{Q}, \bar{W}]$ is the $\mathbb{Q}$–name above (it is unique). Abusing notation we may write $(\mathbb{Q}, \underset{\sim}{\eta})$ instead $(\mathbb{Q}, \bar{W})$ and then let $\bar{W} = \bar{W}[\underset{\sim}{\eta}] = \bar{W}[\mathbb{Q}, \underset{\sim}{\eta}]$.

3. We introduce the notions from 6.1(3)–(6) for the current case with almost no changes.

Definition 6.4. $\mathcal{K}^{\mathrm{ex}}_{\kappa,\theta,\sigma,\tau} = \{(\mathbb{Q}, \underset{\sim}{\eta}) \in K_{\kappa,\theta,\sigma,\tau} : (\mathbb{Q}, \underset{\sim}{\eta})$ is temporarily explicitly $(\mathfrak{B}, \theta, \sigma, \tau)$–pair for some model $\mathfrak{B}$ with $\kappa(\mathfrak{B}) \leq \kappa, \|\mathfrak{B}\| \leq \kappa\}$.

Proposition 6.5. *Assume that:*

(a) $\mathbb{Q}$ *is an explicitly nep forcing notion which satisfies the c.c.c.*
(b) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in {}^{\sigma}\omega$" *and (for* $\alpha < \sigma$ *and* $m < \omega$*)* $\psi_{\alpha,m} \in \mathrm{cl}_1(\mathbb{Q})$ *are such that*

$$\Vdash_{\mathbb{Q}} \text{ `` } \underset{\sim}{\eta}(\alpha) = m \quad \textit{iff} \quad \psi_{\alpha,m}[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t} \text{ ''}.$$

(c) $\mathbb{Q}' \stackrel{\text{def}}{=} B_2(\mathbb{Q}, \underset{\sim}{\eta})$ *is the following suborder of* $\text{cl}_2(\mathbb{Q})$*: the set of elements is* $B'(\mathbb{Q}, \underset{\sim}{\eta})$*, where*

$$B'(\mathbb{Q}, \underset{\sim}{\eta}) =: \{p \in \text{cl}_2(\mathbb{Q}) : \begin{array}{l} p \text{ is generated by the} \\ \psi_{\alpha,m}\text{'s i.e. it belongs to the closure of} \\ \{\psi_{\alpha,m} : \alpha < \sigma, m < \omega\} \\ \text{under } \neg, \bigwedge_{i<\gamma} \text{ for } \gamma < \omega_1 \text{ in } \text{cl}_1(\mathbb{Q})\} \end{array}$$

(i.e., it is the quasi order $\leq_2^{\mathbb{Q}}$ *restricted to this set).*

Then*:*

1. $\mathbb{Q}' \lessdot \text{cl}_2(\mathbb{Q})$ *and* $\underset{\sim}{\eta} \in {}^\sigma\omega$ *is a generic function for* $\mathbb{Q}'$*.*
2. *Assume additionally that*
   (∗) *if* M *is a* $\mathbb{Q}$*–candidate,* $M \models$ *"*$\mathcal{I}$ *is a maximal antichain of* $\mathbb{Q}$*",* then $\mathcal{I}^M$ *is a maximal antichain of* $\mathbb{Q}$*.*

   Then *we also have*
   (α) $\mathbb{Q}'$ *is* (κ, θ)*-nep and strong c.c.c. forcing notion (see 6.12),*
   (β) *if* $\mathbb{Q}$ *is simple,* then $\mathbb{Q}'$ *is simple, (really (*) is not needed),*
   (γ) *if* $\mathbb{Q}$ *is* K*–local,* then $\mathbb{Q}'$ *is* K*–local,*
   (δ) *if* $\mathbb{Q}$ *is Souslin, then so is* $\mathbb{Q}'$*.*

Proof. Straight. $\blacksquare_{6.5}$

Now the hypothesis (∗) in 6.5(2) is undesirable, so we use $B_3(\mathbb{Q}, \underset{\sim}{\eta})$ (see 6.6(c) below), which has a suitable quasi order.

Proposition 6.6. *Assume that:*

(a) $\mathbb{Q}$ *is a correct explicitly nep forcing notion which satisfies the c.c.c.*
(b) $\Vdash_{\mathbb{Q}}$ *"*$\underset{\sim}{\eta} \in {}^\sigma\omega$*" and* $\psi_{\alpha,m} \in \text{cl}_2(\mathbb{Q})$ *for* $\alpha < \sigma, m < \omega$ *are such that*

$$\Vdash_{\mathbb{Q}} \text{“} \underset{\sim}{\eta}(\alpha) = m \quad \textit{iff} \quad \psi_{\alpha,m}[\underset{\sim}{G}_{\mathbb{Q}}] = \mathfrak{t}\text{”},$$

(c) $\mathbb{Q}' \stackrel{\text{def}}{=} B_3(\mathbb{Q}, \underset{\sim}{\eta})$ *is a forcing notion defined as follows:*
the set of elements is like $B_2(\mathbb{Q}, \underset{\sim}{\eta})$*; i.e. it is the closure of* $\{\psi_{\alpha,m} : \alpha < \sigma, m < \omega\}$ *under* $\neg, \bigwedge_{i<\gamma}$ *for* $\gamma < \omega_1$ *inside* $\text{cl}_2(\mathbb{Q})$*;*
the quasi order $\leq_3 = \leq_3^{B_3(\mathbb{Q},\underset{\sim}{\eta})}$ *is* $\leq^{\text{cl}_3(\mathbb{Q})}$ *restricted to* $B_3(\mathbb{Q}, \underset{\sim}{\eta})$*,*
(d) $\mathbb{Q}$ *is correctly explicitly nep forcing and c.c.c. in* $\mathbf{V}$ *and in every* $\mathbb{Q}$*–candidate. (This strengthens clause (a))*

Then*:*

(α) $\mathbb{Q}'$ *is a complete suborder of* $\text{cl}_3(\mathbb{Q})$*; including* $\psi \in \mathbb{Q}' \Rightarrow \psi \in \text{cl}_3(\mathbb{Q})$*, and for* $\psi_1, \psi_2 \in \mathbb{Q}'$ *we have:* $\psi_1 \leq_3 \psi_2 \Leftrightarrow \psi_1 \leq^{\text{cl}_3(\mathbb{Q})} \psi_2$*,*
(β) $\underset{\sim}{\eta}$ *is a* $\mathbb{Q}'$*–name,* $\Vdash_{\mathbb{Q}'}$ *"*$\underset{\sim}{\eta} \in {}^\sigma\omega$*" and* $\underset{\sim}{\eta}$ *is a generic function for* $\mathbb{Q}'$*,*

(γ) *$\mathbb{Q}'$ is explicitly nep c.c.c. forcing notion with $\mathfrak{B}^{\mathbb{Q}'} = \mathfrak{B}^{\mathbb{Q}}$, $\bar{\varphi}^{\mathbb{Q}'} = \bar{\varphi}^{B_3(\mathbb{Q},\underset{\sim}{\eta})}$, $\theta^{\mathbb{Q}'} = \theta^{\mathbb{Q}}$,*

(γ)$^+$ *each forcing extension of* $\mathbf{V}$ *which preserves the assumption (a) (hence also (b)) preserves (γ),*

(δ) *if $\mathbb{Q}$ is simple (or straight)* then *$\mathbb{Q}'$ is simple (or straight).*

Proof. Straightforward, still we elaborate [note that in the proof of clauses (α), (β) we do not use (d)].

Clause (α) By the choice of $\mathbb{Q}'$ we have $\leq^{\mathbb{Q}'} = <^{\mathrm{cl}_3(\mathbb{Q})} \restriction \mathbb{Q}'$. Also if $\mathbb{Q}' \models$"ψ_1, ψ_2 incompatible" then they are incompatible in $\mathrm{cl}_3(\mathbb{Q})$, otherwise $\psi_1 \wedge \psi_2 \in \mathbb{Q}'$ is a counterexample. So the only problem is for $\mathbb{Q}' \lessdot \mathrm{cl}_3(\mathbb{Q})$. So assume $\mathcal{I} = \{\psi_j : j < i^*\}$ is a maximal antichain of $\mathbb{Q}'$ and we shall prove that it is a maximal antichain of $\mathrm{cl}_3(\mathbb{Q})$, this suffices as we are assuming the $\mathbb{Q}$ satisfies the c.c.c., also $\mathrm{cl}_3(\mathbb{Q})$ satisfies c.c.c..

Hence necessarily i^* is countable, so if $\mathcal{I}$ is not a maximal antichain of $\mathrm{cl}_3(\mathbb{Q})$, let $\psi^* \in \mathrm{cl}_3(\mathbb{Q})$ be incompatible with every ψ_i, let $p \in \mathbb{Q}$, $p \Vdash$ "$\psi^*[\underset{\sim}{G}_{\mathbb{Q}}]$ =truth". Consider $\psi = \bigwedge\limits_{i<i^*} \neg\psi_i$;

(i) ψ belongs to $\mathrm{cl}_1(\mathbb{Q})$,
(ii) ψ is in $\mathrm{cl}_3(\mathbb{Q})$ as $p \Vdash$" $\psi[\underset{\sim}{G}_{\mathbb{Q}}]$ =truth", by the choice of p and ψ^*,
(iii) ψ is incompatible with each ψ_i (see its definition),
(iv) $\psi \in B_2'(\mathbb{Q}, \underset{\sim}{\eta})$ (as $\psi_i \in \mathbb{Q}'$ for $i < i^*$),
(v) $\psi \in \mathbb{Q}'$ (by (ii)+(iv)).

So $\mathbb{Q}' \lessdot \mathrm{cl}_3(\mathbb{Q})$, i.e., clause ($\alpha$) holds.

Clause (β) The first two statements are obvious and the third one follows by

$(*)_\psi$ if G_1, G_2 are subsets of $\mathbb{Q}'$ generic over $\mathbf{V}$ (in some generic extension of $\mathbf{V}$) and $\underset{\sim}{\eta}[G_1] = \underset{\sim}{\eta}[G_2]$ and $\psi \in B_2(\mathbb{Q}, \eta)$, then $\psi[G_1] = \psi[G_2]$

We prove $(*)$ by induction on the depth of ψ. For depth zero we use $\underset{\sim}{\eta}[G_1] = \underset{\sim}{\eta}[G_2]$ and in the other cases the inductive definition of $\psi[G_\ell]$.

Clause (γ) As $\mathbb{Q}' \lessdot \mathrm{cl}_3(\mathbb{Q})$ by the c.c.c. most clauses of Definition 1.3(1),(2) follow by 5.8(3)(e) except clause (c)$^+$. So let N be a $\mathbb{Q}'$–candidate so a $\mathbb{Q}$–candidate, and $p \in (\mathbb{Q}')^N$ hence $p \in \mathrm{cl}_3(\mathbb{Q})^N$ by the definition of $\mathbb{Q}'$. Let

$$\psi_1 = p \;\&\; \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathbb{Q}')} \; \bigvee_{r\in\mathcal{I}^N} r \quad \text{and} \quad \psi_2 = p \;\&\; \bigwedge_{\mathcal{I}\in \mathrm{pdac}(p,N,\mathrm{cl}_3(\mathbb{Q}))} \; \bigvee_{r\in\mathcal{I}^N} r.$$

By 5.8 ψ_2 is explicitly $(N, \mathrm{cl}_3(\mathbb{Q}))$–generic. Now also $N \models$"$\mathbb{Q}' \lessdot \mathrm{cl}_3(\mathbb{Q})$", hence $\mathrm{pdac}(p, N, \mathbb{Q}) \subseteq \mathrm{pdac}(p, N, \mathrm{cl}_3(\mathbb{Q}))$, because by clause (d) as we have not used it in the proof of clauses (α)+(β) above so they are satisfied by N too.

So clearly $\mathrm{cl}_1(\mathbb{Q}) \models \psi_1 \leq \psi_2$ hence $\psi_1 \in \mathrm{cl}_3(\mathbb{Q})$, and its definition (and the definition of $\mathbb{Q}'$) we have $\psi_1 \in \mathbb{Q}'$. Obviously ψ_1 is $(N, \mathbb{Q}')$–generic by its definition, but what about explicitly? Just let ψ_2' say that there is ψ_2 as above (and use $\mathrm{cl}_3(\mathbb{Q})$ is correct explicitly nep by 5.8(3), as $(\circledast)$ of clause (d) here implies $((\circledast_3))$ there.)
Clauses $(\gamma)^+$**,** (δ) Left to the reader. $\blacksquare_{6.6}$

Proposition 6.7. *In* 6.1–6.6 *above, we can replace* $\mathbb{Q}$ *by* $\mathbb{Q}\restriction(\geq q) =: \mathbb{Q}\restriction\{p \in \mathbb{Q} : p \geq q\}$ *preserving the properties of* $(\mathbb{Q}, \underset{\sim}{\eta})$. $\blacksquare$

Fact 6.8. *If* $\mathbb{Q}$ *is simple correct nep for* K, $\mathbb{Q}$ *is in* $\mathbf{V}$, *and* $\mathbf{V}_1$ *is a* K–*extension of* $\mathbf{V}$, then

(i) *in* $\mathbf{V}_1$, $\mathbb{Q}^{\mathbf{V}} \leq_{ic} \mathbb{Q}^{\mathbf{V}_1}$ *which means: for* $p, q \in \mathbf{V}_0$, *"*$p \in \mathbb{Q}$*", "*$p \leq q$*", "*$\neg(p \leq q)$*", "*p, q *compatible", "*p, q *incompatible" all in the sense of* $\mathbb{Q}$ *are preserved from* $\mathbf{V}$ *to* $\mathbf{V}_1$,

(ii) *for* $p, p_n \in \mathbf{V}$ *the statements "*$p \notin \mathbb{Q}$*" and "*$\mathcal{I} = \{p_n : n < \omega\}$ *is predense above* p *in* $\mathbb{Q}$*" are preserved from* $\mathbf{V}$ *to* $\mathbf{V}_1$,

(iii) *if* $\mathbb{Q}$ *satisfies the c.c.c.,* then *in clause (ii) above we can omit the countability of* $\mathcal{I}$.

Proof. (i) Straight, for example:
"p, q are incompatible" *iff* there is no $\mathbb{Q}$–candidate M such that

$$M \models \text{“ } p, q \text{ have a common } \leq_{\mathbb{Q}}\text{–upper bound ”}.$$

So by Shöenfield–Levy absoluteness, if this holds in $\mathbf{V}$, it holds in $\mathbf{V}_1$.
(ii) Similarly.
(iii) Follows (and repeated in 7.14 below). $\blacksquare_{6.8}$

Proposition 6.9. *Let* $(\mathbb{Q}, \underset{\sim}{\eta})$ *be temporarily explicitly nep pair. Assume* N *is a* $\mathbb{Q}$–*candidate. If* $N \models$ *"*η^* *is* $(\mathbb{Q}, \underset{\sim}{\eta})$–*generic over the* $\mathbb{Q}$–*candidate* M*",* then η^* *is a* $(\mathbb{Q}, \underset{\sim}{\eta})$–*generic over* M.

Proof. Straight. $\blacksquare_{6.9}$

Proposition 6.10. *Assume that:*

(a) $\mathbb{Q}$ *is explicitly nep,*

(b) $\mathbb{Q}$ *is c.c.c. moreover it satisfies the c.c.c. in every* $\mathbb{Q}$–*candidate,*

(c) *incompatibity in* $\mathbb{Q}$ *is upward absolute from* $\mathbb{Q}$–*candidates (but see* 6.8*),*

(d) $\underset{\sim}{\eta}$ *is a* hc–$\kappa(\mathfrak{B}^{\mathbb{Q}})$–$\mathbb{Q}$–*name of a member of* ${}^{\omega}\omega$ *defined from* $\mathfrak{B}^{\mathbb{Q}}$ *(so we demand this in every* $\mathbb{Q}$–*candidate).*

Furthermore, suppose that

(A) *N_1, N_2 are $\mathbb{Q}$–candidates, N_2 is a generic extension of N_1 for a forcing notion $\mathbb{R}$, (so $\mathfrak{B}^{N_2} = \mathfrak{B}^{N_1}$ and $N_1 \models$ "$\mathbb{R}$ is a forcing notion"),*

(B) *$N_1 \models$ " for every countable $X \subseteq \mathbb{Q}$ there is a $\mathbb{Q}$-candidate $N_0 \prec_{\Sigma_m} N_1$ to which X and $\mathbb{R}$ belong, moreover N_0-s being a $\mathbb{Q}$-candidate is preserved by forcing for $\mathbb{R}$" for each $m < \omega$; recall N_0 is in particular countable,*

(C) *$\eta^* \in {}^{\omega}\omega$ is a $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over N_2.*

Then *$\eta^* \in {}^{\omega}\omega$ is a $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over N_1.*

Remark 6.11. 1. In clause (B), we can replace X by "a maximal antichain of $\mathbb{Q}$ or just of $B_3(\mathbb{Q}, \underset{\sim}{\eta})$".

2. Clearly we can replace "maximal antichain" by "predense set" or "predense set over p" (note $\mathcal{I}^{N_2} = \mathcal{I}^{N_1}$ as $N_2 = N_1^{\mathbb{R}}$).
3. We can weaken "$N_0 \prec_{\Sigma_m} N_1$" in clause (B).

Proof of 6.10. Clearly, it suffices to prove that (assuming (a)–(d), (A), (B) and (C)):

(∗) *if* $N_1 \models$"$\mathcal{I}$ is a maximal antichain of $\mathbb{Q}$",
then $\mathcal{I}^{N_1} = \mathcal{I}^{N_2}$ and $N_2 \models$"$\mathcal{I}$ is a maximal antichain of $\mathbb{Q}$".

Assume that this fails for $\mathcal{I}$. Then some $r \in \mathbb{R}^N$ forces this failure (in N_1). By assumption (b), in N_1 the set $\mathcal{I}^{N_1}$ is countable so let $N_1 \models$"$\mathcal{I} = \{p_n : n < \alpha\}$", where $\alpha \leq \omega$. Let $m < \omega$ be large enough. By clause (B) in N_1 there is a $\mathbb{Q}$–candidate N_0 to which $\mathcal{I}$ and r and $\mathbb{R}$ belong and $N_0 \prec_{\Sigma_m} N_1$. Since

$$N_1 \models \text{“}(\exists r \in \mathbb{R})[r \Vdash_{\mathbb{R}} \text{“}\mathcal{I} \text{ is not a maximal antichain of } \mathbb{Q} \\ \text{(and } N_1[\underset{\sim}{G}_{\mathbb{R}}] \text{ is a } \mathbb{Q}\text{–candidate)”]”},$$

there is $r_0 \in \mathbb{R}^{N_0}$ such that

$$N_0 \models \text{“}[r_0 \Vdash_{\mathbb{R}} \text{“}\mathcal{I} \text{ is not a maximal antichain of } \mathbb{Q} \\ \text{(and } N_0[\underset{\sim}{G}_{\mathbb{R}}] \text{ is a } \mathbb{Q}\text{–candidate)]”}.$$

Now, as N_1 satisfies enough set theory and N_1 "thinks" that N_0 is countable and $\mathbb{R}^{N_0}$ is a forcing notion in N_0, there is in N_1 a subset $G'_{\mathbb{R}}$ of $\mathbb{R}^{N_0}$ generic over N_0 to which r_0 belongs. So in $N_0[G'_{\mathbb{R}}]$ there is $p \in \mathbb{Q}^{N_0[G'_{\mathbb{R}}]}$ incompatible (in $\mathbb{Q}^{N_0[G'_{\mathbb{R}}]}$) with each p_n. By the assumption (c) this holds in N_1, contradiction to the choice of $\mathcal{I}$ (see (∗)). $\blacksquare_{6.10}$

Definition 6.12. 1. We say that $\bar{\varphi}$ or $(\bar{\varphi}, \mathfrak{B})$ is a temporarily (κ, θ)-definition of a strong c.c.c.–nep forcing notion $\mathbb{Q}$ for $(\mathfrak{B}, \bar{\varphi})$ if:

(a) φ_0 defines the set of elements of $\mathbb{Q}$ and φ_0 is upward absolute from $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates,

(b) φ_1 defines the partial ordering of $\mathbb{Q}$ (even in $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates) and φ_1 is upward absolute from $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidates,
(c) for any $(\mathfrak{B}, \bar{\varphi}, \theta)$–candidate N, if $N \models$"$\mathcal{I} \subseteq \mathbb{Q}$ is predense", *then* also in $\mathbf{V}$, $\mathcal{I}^N$ is a predense subset of $\mathbb{Q}$,
(d) incompatibility is upward absolute from $\mathbb{Q}$-candidates.

2. We say that $\bar{\varphi}$ or $(\bar{\varphi}, \mathfrak{B})$ is a temporarily [explicitly] (κ, θ)–definition of a c.c.c.–nep forcing notion $\mathbb{Q}$ if
   (α) it is a temporary [explicitly] (κ, θ)-definition of a nep forcing notion,
   (β) for every $\mathbb{Q}$–candidate N we have $N \models$"$\mathbb{Q}$ satisfies the c.c.c.".
3. The variants are defined as usual.

Proposition 6.13. 1. *If* $\mathbb{Q}$ *is temporarily strong explicit c.c.c.–nep forcing notion and* $N_1 \subseteq N_2$ *are* $\mathbb{Q}$*–candidates,* then *every* η *which is* $(\mathbb{Q}, \underset{\sim}{\eta})$*–generic over* N_2 *is also* $(\mathbb{Q}, \underset{\sim}{\eta})$*–generic over* N_1*.*

2. *If* ZFC^-_* *is normal and* $\mathbb{Q}$ *is temporarily c.c.c.–nep* then $\mathbb{Q}$ *satisfies the c.c.c.*

Proof. 1) As in 6.10 (by Definition 6.12).
2) Easy too. $\blacksquare_{6.13}$

Comment 6.14. We can spell out various absoluteness, e.g.

1. If $\mathbb{Q}$ is simple nep, c.c.c. and "$\langle p_n : n < \omega\rangle$ is predense" has the form $(\exists t \in \mathcal{H}_{<\aleph_1}((\kappa+\theta)))[t \models \dots]$ (e.g. $\kappa^{\mathbb{Q}} = \omega$ and it is Π^1_2) then predensity of countable sets is preserved in any forcing extension.
2. Note that strong c.c.c.–nep (from 6.12(1)) does not imply c.c.c.–nep (from 6.12(2)). But if $\mathrm{ZFC}^-_{**} \vdash \mathrm{ZFC}^-_*$ and ZFC^-_{**} says that ZFC^-_* is normal and $\mathbb{Q}$ is strong c.c.c.–nep for ZFC^-_*, then $\mathbb{Q}$ is c.c.c.–nep for ZFC^-_*.

The following is similar to 6.10, but for a simpler case so we make it self contained: it is really from [25, III] (in earlier version this was said offhand in §8).

Definition/Theorem 6.15. By induction on the ordinal α we define and prove the following

(A) *[Definition]* $\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\theta}_i, \theta_i : i < \alpha\rangle$ is a c.c.c-simple-FS-iteration, in full $\mathbb{P}_i = \mathbb{P}^{\bar{\mathbb{Q}}}_i = \mathbb{P}_i[\bar{\mathbb{Q}}]$ etc and $\alpha = \ell g(\bar{\mathbb{Q}})$,
(B) *[Definition]* $\theta^{\bar{\mathbb{Q}}} = \theta[\bar{\mathbb{Q}}]$ for $\bar{\mathbb{Q}}$ as in (A),
(C) *[Definition]* $\underset{\sim}{\tau}$ is a hereditarily countable $\mathbb{P}_\beta$-name of a member of $\mathcal{H}_{<\aleph_1}(\zeta)$ or hc-ζ-$\mathbb{P}_\beta$-name,

(D) *[Definition]* $\mathrm{Lim}(\mathbb{Q})$ denoted also by $\mathbb{P}_\alpha = \mathbb{P}^{\bar{\mathbb{Q}}}_\alpha$,

(E) *[Claim]* Assume $\bar{\mathbb{Q}}$ is a c.c.c-simple-FS-iteration as in (A)

(a) $\mathbb{P}_\alpha$ is a c.c.c. forcing notion with set of elements $\subseteq \mathcal{H}_{<\aleph_1}(\theta_\alpha)$,

(b) if $\beta \le \alpha(= lg(\bar{\mathbb{Q}}))$, *then* $\bar{\mathbb{Q}} \restriction \beta$ is a c.c.c.–simple–FS–iteration and $\mathrm{Lim}(\bar{\mathbb{Q}} \restriction \beta) = \mathbb{P}_\alpha$, $\theta[\bar{\mathbb{Q}}] = \theta_\alpha$ (hence $\mathbb{P}_\beta \subseteq \mathcal{H}_{<\aleph_1}(\theta_\beta)$) and $\mathbb{P}_\beta \subseteq \mathbb{P}_\alpha$,

(c) $p \in \mathbb{P}_\alpha$ iff p is a function with domain a finite subset of α such that $\gamma \in \mathrm{Dom}(p)$ implies that $p(\gamma)$ is an hc-$\mathbb{P}_\gamma$-name of a member of $\mathcal{H}_{<\aleph_1}(\theta_{\gamma+1})$ which is forced to be a member of $\underset{\sim}{\mathbb{Q}}_\gamma$ and of $\mathcal{H}_{<\aleph_1}(\theta_\gamma)$,

(d) the order on $\mathbb{P}_\alpha$ is as usual,

(e) if $\gamma \le \beta \le \alpha$ and $p \in \mathbb{P}_\beta$, then $p \restriction \gamma \in \mathbb{P}_\gamma$ and $\mathbb{P} \models (p \restriction \gamma) \le p$,

(f) if $\gamma \le \beta \le \alpha$, $p \in \mathbb{P}_\beta$ and $\mathbb{P}_\gamma \models (p \restriction \gamma) \le q$,
then $\mathbb{P}_\beta \models$ "$p \le (q \cup (p \restriction [\gamma, \beta])$",

(g) if $\gamma \le \beta \le \alpha$, then $\mathbb{P}_\gamma \lessdot \mathbb{P}_\beta$,

(h) if $\beta \le \alpha$, then $||\mathbb{P}_\beta|| \le (\theta_\beta)^{\aleph_0}$,

(i) $\mathbb{P}_\alpha$ is c.c.c.

(F) For any $\theta \ge \theta^\alpha$,

$$\Vdash_{\mathbb{P}_\alpha} \text{“}(\mathcal{H}_{<\aleph_1}(\theta))^{\mathbf{V}[\mathbb{P}_\alpha]} = \{\underset{\sim}{\tau}[\underset{\sim}{G}_{\mathbb{P}_\alpha}] : \underset{\sim}{\tau} \text{ is a hc-}\kappa\text{-}\mathbb{P}_\alpha\text{-name }\}\text{”}$$

Let us carry out the obvious induction.

Clause (A)

$\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\theta}_i, \theta_i) : i < \alpha\rangle$ being a c.c.c.-simple-FS-iteration is defined by cases

case 1 $\alpha = 0$ trivial

case 2 α limit

$\bar{\mathbb{Q}}$ is c.c.c.-simple-FS-iteration iff
$\bar{\mathbb{Q}} \restriction \beta$ is a c.c.c.-simple-FS-iteration for every $\beta < \alpha$.

case 3 $\alpha = \beta + 1$

$\bar{\mathbb{Q}}$ is a c.c.c.-simple-FS-iteration *iff* $\bar{\mathbb{Q}} \restriction \beta$ is such iteration, $\mathbb{P}_\beta = \mathrm{Lim}(\bar{\mathbb{Q}} \restriction \beta)$, $\theta_\beta \ge \theta[\bar{\mathbb{Q}} \restriction \beta]$, if equal we say standard; $\underset{\sim}{\theta}_\beta$ a $\mathbb{P}_\beta$-name of cardinal or ordinal and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is a c.c.c. forcing notion with minimal element $\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta}$ being for simplicity $\emptyset$ and set of elements $\subseteq \mathcal{H}_{<\aleph_1}(\underset{\sim}{\theta}_\beta)$".

Clause (B)

$\theta^{\bar{\mathbb{Q}}}$ is the minimal cardinal (or ordinal) θ such that $\beta < \alpha \Rightarrow \theta \ge \theta_\beta$ and $\alpha = \beta + 1$ implies $\theta \ge \beta + 1$ and $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\theta}_\beta \le \theta$".

Clause (C)

As in §5

Clause (D)

We define $\mathbb{P}_\alpha = \mathrm{Lim}(\bar{\mathbb{Q}})$ by clauses (c) + (d) of clause (E).

Clause (E)

As in the classical proofs on FS iterations of c.c.c forcings.

Clause (F)
Easy too. $\blacksquare_{6.15}$

Definition 6.16. We say that $\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_i, \mathfrak{B}^i, \bar{\underset{\sim}{\varphi}}_i, \underset{\sim}{\theta}_i, \kappa^i, \mathfrak{B}^i, \bar{\varphi}^i) : i < \alpha\rangle$ is a [strong] c.c.c.-nep-FS iteration if

(a) $\bar{\mathbb{Q}}' = \langle(\mathbb{P}_i, \mathbb{Q}_i, \theta_i, \kappa^i) : i < \alpha\rangle$ is a c.c.c.-simple-FS-iteration;
(b) $(\underset{\sim}{\kappa}_i, \mathfrak{B}_i, \underset{\sim}{\varphi}_i, \tilde{\theta}_i, \tilde{\mathbb{Q}}_i)$ are $\mathbb{P}_i$-names and it is forced ($\Vdash_{\mathbb{P}_i}$) that $\underset{\sim}{\mathbb{Q}}_i$ is a $(\underset{\sim}{\kappa}_i, \mathbb{Q}_i)$-definition of a [strong] c.c.c.-nep forcing for $(\mathfrak{B}_i, \bar{\underset{\sim}{\varphi}}_i)$, see Definition 6.12(1); it follows that this is done through $(\mathfrak{B}_i, \bar{\underset{\sim}{\varphi}}_i, \tilde{\theta}_i)$-candidates;
(c) $\mathfrak{B}^i$ is a model with universe $\mathcal{H}_{<\aleph_1}(\kappa^i)$ and the relation $\in$ and the (finitely many) relations implicit in $\bar{\mathbb{Q}}\restriction i, \mathbb{P}_i, \underset{\sim}{\kappa}_i, \mathfrak{B}_i, \bar{\varphi}_i, \underset{\sim}{\theta}_i$ so in particular $\Vdash$ "$\underset{\sim}{\kappa}_i, \underset{\sim}{\theta}_i < \kappa^i$";
(d) $\bar{\varphi}^i$ define $\mathbb{P}_i$ in $\mathfrak{B}^i$ naturally.

Proposition 6.17. 1. *In Definition* 6.16*, for every $\beta < \alpha$, $(\mathfrak{B}^i, \bar{\varphi}^i)$ is a [strong] (κ_1, κ_1)-definition of the [strong] c.c.c.-nep forcing notion $\mathbb{P}_i$.*
2. *If N is a $(\mathfrak{B}^i, \bar{\varphi}^i, \kappa^i)$-candidate and $j < i$ and $j \in N$,* then *N is a $(\mathfrak{B}^j, \bar{\varphi}^j, \kappa^j)$-candidate.*
3. *If N is a $(\mathfrak{B}^{j+1}, \bar{\varphi}^{j+1}, \kappa^{j+1})$-candidate, $j \in N$ and $G_j \subseteq \mathbb{P}_j$ is generic over* $\mathbf{V}$, then *$G = \mathbb{P}_j^N \cap G_j$ is $\langle N, \mathbb{P}_j^N\rangle$–generic and $N\langle G\rangle$ is a $(\mathfrak{B}_j[G], \bar{\underset{\sim}{\varphi}}_j[G], \underset{\sim}{\theta}_j[G])$–candidate with $(N\langle G\rangle)^{\mathfrak{B}_j[G]} = \mathfrak{B}_G\langle G\rangle$ defined naturally.*

Proof. 1) We prove by (1) by induction on i. Let N be a $(\mathfrak{B}, \bar{\varphi}, \kappa^i)$-candidate.

Case 1 $N \cap i$ has no last element (i.e., i is a limit ordinal).

So assume $N \models$ "$\mathcal{I} \subseteq \mathbb{Q}$ is predense", and let $q \in \mathbb{P}_i$. For some $j \in N \cap i$, $\mathrm{Dom}(q) \cap (N \cap i) \subseteq j$, and in N define $\mathcal{J} = \{p\restriction j : p \in \mathcal{I}\}$, so clearly $N \models$ "$\mathcal{I}$ is a predense subset of $\mathbb{P}_j$". Now N is a $(\mathfrak{B}^j, \bar{\varphi}^j, \theta^j)$-candidate by part (2) below. Hence by the induction hypothesis $\mathcal{I}^N$ is predense in $\mathbb{P}_j$ so there is $p' \in \mathcal{J}^N$, compatible with q in $\mathbb{P}_j$, so there is $r \in \mathbb{P}_j$ above both. As $p' \in \mathcal{J}^N$, for some $p \in \mathcal{I}^N$ we have $N \models p\restriction j = p'$, hence $p' = p\restriction j$ and $r \cap (p\restriction[i, j]) \in \mathbb{P}_j$ is a common upper bound of q and $p \in \mathcal{I}^N$ in $\mathbb{P}_i$ as required.

Case 2 $i = j + 1$

Similarly using part (3).

2), 3) Left to the reader. $\blacksquare_{6.17}$

Proposition 6.18. 1. *For every function F and ordinal α there is a unique c.c.c.-nep-FS-iteration $\bar{\mathbb{Q}} = \langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_i, \mathfrak{B}_i, \bar{\underset{\sim}{\varphi}}_i, \underset{\sim}{\theta}_i, \kappa^i, \varphi\mathfrak{B}_i^i, \varphi^*) : i < \beta\rangle$ such that*

(a) *$\beta \le \alpha$,*
(b) *if $i < \beta$, then $(\mathbb{Q}_i, \underset{\sim}{\kappa}_i, \mathfrak{B}_i, \bar{\underset{\sim}{\varphi}}_i, \underset{\sim}{\theta}_i) = F(\bar{\mathbb{Q}} \restriction i)$,*
(c) *if $\beta < \alpha$, then $(\underset{\sim}{\kappa}_i, \mathfrak{B}_i, \bar{\underset{\sim}{\varphi}}_i, \underset{\sim}{\theta}_i)$ is not a $\mathbb{P}_i$-name, or it is but is not forced (i.e. $\Vdash_{\mathbb{P}_\beta}$) to be as in Definition 6.10(b).*

2. *The parallel of* 5.19 *holds. We can add "strong" (c.c.c.).*

Proof. Straight. ∎

Proposition 6.19. *We can do the parallel of* 6.15, 6.18 *replacing finite support and c.c.c. by countable support and proper.*

7. Preserving a little implies preserving much

Our main intention is to show that, for example if a "nice" forcing notion $\mathbb{P}$ satisfies $\Vdash_{\mathbb{P}}$"$({}^\omega 2)^{\mathbf{V}}$ is not null", *then* it preserves "$X \subseteq {}^\omega 2 \quad (X \in \mathbf{V})$ is not null".

By Goldstern and Shelah ([25, Chapter XVIII, 3.11]) if a Souslin proper forcing preserves "$({}^\omega\omega)^{\mathbf{V}}$ is non-meagre" then it preserves "$X \subseteq {}^\omega\omega$ is non-meagre" and more (in a way suitable for the preservation theorems there).

The main question not resolved there was: is this preservation special for Cohen forcing (which is a way to speak on non-meagre), or does it hold for nice c.c.c. forcing notions in general, in particular does a similar theorem hold for "non-null" instead of "non-meagre". Though there have been doubts about it, we succeed to do it here. In fact, even for a wider family of forcing notions but we have to work more in the proof.

The reader may concentrate on the case that $\mathbb{Q}$ is strong c.c.c. nep and $\mathbb{P}, \mathbb{Q}$ are explicitly $\aleph_0$–nep and simple. It is natural to assume that $\underset{\sim}{\eta}$ is a generic real for $\mathbb{Q}$ but we do not ask for it when not used.

Convention 7.1. 1. $\mathbb{Q}$ is an explicitly nep forcing notion.

2. $\underset{\sim}{\eta} \in {}^\omega\omega$ is a hereditarily countable $\mathbb{Q}$-name which is $\mathfrak{B}$–definable, so as in 6.2(1).

We would like to preserve something like: "x is $\mathbb{Q}$–generic over N".

Definition 7.2. 1. $I_{(\mathbb{Q},\underset{\sim}{\eta})} \stackrel{\text{def}}{=} \{A \in \text{Borel}({}^\omega\omega) : \Vdash_{\mathbb{Q}}$"$\underset{\sim}{\eta} \notin A$"$\}$ (this is an ideal on the Boolean algebra of Borel subsets of ${}^\omega\omega$).

2. $I^{\text{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})}$ is the ideal generated by $I_{(\mathbb{Q},\underset{\sim}{\eta})}$ on $\mathcal{P}({}^\omega\omega)$. (So for $A \in \text{Borel}({}^\omega\omega)$ we have: $A \in I_{(\mathbb{Q},\underset{\sim}{\eta})} \Leftrightarrow A \in I^{\text{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})}$).

3. Let

$$I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})} \stackrel{\rm def}{=} \{X \subseteq {}^\omega\omega : \begin{array}{l}\text{for a dense set of } q \in \mathbb{Q}, \text{ for some Borel set } B \subseteq {}^\omega\omega,\\ \text{we have } X \subseteq B \text{ and } q \Vdash \text{“}\underset{\sim}{\eta} \notin B\text{”}\}.\end{array}$$

4. For an ideal I [an ideal on Borel sets], the family of I–positive [Borel, respectively] sets is denoted by I^+.
(Thus, for a Borel subset A of ${}^\omega\omega$, $A \in I^+_{(\mathbb{Q},\underset{\sim}{\eta})}$ *iff* there is $q \in \mathbb{Q}$ such that $q \Vdash_{\mathbb{Q}}$ “$\underset{\sim}{\eta} \in A$”).

Definition 7.3. 1. A forcing notion $\mathbb{P}$ is $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving *if* for every Borel set A

$$A \in (I_{(\mathbb{Q},\underset{\sim}{\eta})})^+ \quad \Rightarrow \quad \Vdash_{\mathbb{P}} \text{“ } A^{\mathbf{V}} \in (I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})})^{+}\text{”}$$

($A^{\mathbf{V}}$ means: the same set, which is $A^{\mathbf{V}[\mathbb{P}]} \cap \mathbf{V}$).

2. A forcing notion $\mathbb{P}$ is strongly $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving *if* for all $X \subseteq {}^\omega\omega$ (i.e. not only Borel sets)

$$X \in (I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})})^+ \quad \Rightarrow \quad \Vdash_{\mathbb{P}} \text{“}X \in (I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})})^{+}\text{”}.$$

[See 7.4(7) below for $\mathbb{Q}$ which is c.c.c.]

3. We say that a forcing notion $\mathbb{P}$ is weakly $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving *if* $\Vdash_{\mathbb{P}}$“ $({}^\omega\omega)^{\mathbf{V}} \in (I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})})^+$ ”.

4. $\mathbb{P}$ is super–$I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving *if* for all $X \subseteq {}^\omega\omega$ we have:

$$X \in (I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})})^+ \quad \Rightarrow \quad \Vdash_{\mathbb{P}} X \in (I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})})^+.$$

Proposition 7.4. 1. *$I_{(\mathbb{Q},\underset{\sim}{\eta})}$ is an $\aleph_1$–complete ideal (in fact, if $\langle A_i : i \leq \alpha\rangle \in \mathbf{V}$, each $A_i \in$* Borel$({}^\omega\omega)$ *and $\Vdash_{\mathbb{Q}}$ “$A_\alpha^{\mathbf{V}[G]} \subseteq \bigcup\limits_{i<\alpha} A_i^{\mathbf{V}[G]}$” (if α is a countable ordinal this is equivalent to $A_\alpha \subseteq \bigcup_{i<\alpha} A_i$) and $A_i \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$ for $i < \alpha$,* then *$A_\alpha \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$).*

2. *If $(\mathbb{Q},\underset{\sim}{\eta})$ is not trivial (i.e., $\Vdash_{\mathbb{Q}}$ “$\underset{\sim}{\eta} \notin ({}^\omega\omega)^{\mathbf{V}}$),* then *singletons belong to $I_{(\mathbb{Q},\underset{\sim}{\eta})}$.*

3. ${}^\omega\omega \notin I_{(\mathbb{Q},\underset{\sim}{\eta})}$.

4. *Assume (*ZFC$^-_*$ *is K–good and) $\mathbb{Q}$ is correct. If in $\mathbf{V}$, $X \in I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$ and* $\mathbb{P} \in K$, then *in $\mathbf{V}^{\mathbb{P}}$ still $X \in I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$.*

5. *Assume (*ZFC$^-_*$ *is K–good, particularly (c) of* 1.15 *and) $\mathbb{Q}$ is correct. If, in $\mathbf{V}$, B is a Borel subset of ${}^\omega\omega$ from $I_{(\mathbb{Q},\underset{\sim}{\eta})}$ and $\mathbb{P} \in K$ and $\mathbf{V}_1 =$* $\mathbf{V}^{\mathbb{P}}$, then *also $\mathbf{V}_1 \models$ “$B \in I_{\mathbb{Q},\underset{\sim}{\eta}}$” , of course here B mean $B^{\mathbf{V}_1}$.*

6. $I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$, $I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})}$ *are ideals of* $\mathcal{P}({}^{\omega}\omega)$ *and* $I^{\rm ex}_{\mathbb{Q},\underset{\sim}{\eta}} \subseteq I^{\rm dx}_{\mathbb{Q},\underset{\sim}{\eta}}$,

$I_{(\mathbb{Q},\underset{\sim}{\eta})} = I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}\restriction$*(the family of Borel sets)* $= I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})}\restriction$*(the family of Borel sets).*

7. *If* $\mathbb{Q}$ *satisfies the c.c.c.* then $I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})}$ *is generated by* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$, *so equal to* $I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$.
8. $I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$ *is* $\aleph_1$*–complete.*
9. *If for some stationary* $S \subseteq [\chi]^{\aleph_0}$ *the forcing notion* $\mathbb{Q}$ *is* S*–proper* then $I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})}$ *is* $\aleph_1$*–complete.*
10. *If* $\mathbb{Q}$ *is c.c.c.,* $\mathbf{V}_1$ *an extension of* $\mathbf{V}$ *(normally generic) and* $\eta \in ({}^{\omega}\omega)^{\mathbf{V}_1}$, then *: there is* $G \subseteq \mathbb{Q}$ *generic over* $\mathbf{V}$ *such that* $\eta = \underset{\sim}{\eta}[G]$ iff $(\forall B \in I^{\mathbf{V}}_{(\mathbb{Q},\underset{\sim}{\eta})})(\eta \notin B^{\mathbf{V}_1})$*; of course this applies to suitable candidates.*

Proof. We will prove parts 5) and 4), 6) only, the rest is left to the reader.
5) First work in $\mathbf{V}^{\mathbb{P}}$. If the conclusion fails then for some $q \in \mathbb{Q}$ we have $q \Vdash$"$\underset{\sim}{\eta} \in B$". So there is a $\mathbb{Q}$–candidate M to which q, B (i.e. the code of B) belong. There is q' such that $q \leq_{\mathbb{Q}} q'$ and q' is $(M,\mathbb{Q})$–generic. Now for every $G \subseteq \mathbb{Q}^{\mathbf{V}[\mathbb{P}]}$ generic over $\mathbf{V}^{\mathbb{P}}$ to which q' belong, $\underset{\sim}{\eta}[G] \in B^{\mathbf{V}^{\mathbb{P}}[G]}$. By absoluteness, also $M\langle G\cap\mathbb{Q}^M\rangle \models \underset{\sim}{\eta}\langle G\cap\mathbb{Q}^M\rangle \in B^{M\langle G\rangle}$ and hence (by the forcing theorem) for some $p \in G\cap\mathbb{Q}^M$ we have $M \models [p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in B$"$]$. Now, returning to $\mathbf{V}$, by Shöenfield–Levy absoluteness there are such M', p' in $\mathbf{V}$. Let p'' be $\langle M',\mathbb{Q}\rangle$–generic, $p' \leq_{\mathbb{Q}} p''$. So similarly to the above, $p'' \Vdash_{\mathbb{Q}}$"$\underset{\sim}{\eta} \in B$".
4) As $X \in I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$, clearly for some Borel set $B \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$ we have $X \subseteq B$. By part (5), also in $\mathbf{V}^{\mathbb{P}}$ we have $B \in I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$ and trivially $X \subseteq B^{\mathbf{V}} \subseteq B^{\mathbf{V}^{\mathbb{P}}}$.

6) E.g. assume B is a Borel set and it belong to $I^{\rm dx}_{(\mathbb{Q},\underset{\sim}{\eta})}$, *then* the set $\mathcal{I} = \{p \in \mathbb{Q}$: for some Borel set B_1 we have $B \subseteq B_1$ and $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin B_1$", is a dense subset of $\mathbb{Q}$. But for $p \in \mathcal{I}$ let B^p_1 witness it, now $B \subseteq B^p_1$, $p \Vdash$ "$\underset{\sim}{\eta} \notin B^p_1$, i.e., $\underset{\sim}{\eta} \notin (B^p_1)^{\mathbf{V}^{\mathbb{Q}}}$". But the inclusion of Borel sets is absolute, so $p \Vdash_{\mathbb{Q}}$ "$B^{\mathbf{V}^{\mathbb{Q}}} \subseteq (B^p_1)^{\mathbf{V}^{\mathbb{Q}}}$ and $\underset{\sim}{\eta} \notin (B^p_1)^{\mathbf{V}^{\mathbb{Q}}}$". Hence $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin B^{\mathbf{V}^{\mathbb{Q}}}$". As this holds for a dense set of $p \in \mathbb{Q}$ (i.e. $p \in \mathcal{I}$) clearly $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin B^{\mathbf{V}^{\mathbb{Q}}}$", so $B \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$. So we have shown $I^{\rm dx}_{\mathbb{Q},\underset{\sim}{\eta}}\restriction\{B : B \text{ Borel set}\} \subseteq I^{\rm ex}_{\mathbb{Q},\underset{\sim}{\eta}}$. The other parts are even easier. $\blacksquare_{7.4}$

Proposition 7.5. 1. *If a forcing notion* $\mathbb{P}$ *is* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving,* then $\mathbb{P}$ *is weakly* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving.*

2. *If* $\mathbb{P}$ *is super* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving,* then $\mathbb{P}$ *is strongly* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving. If* $\mathbb{P}$ *is strongly* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving,* then $\mathbb{P}$ *is* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving.*
3. *Assume that* $\mathbb{Q}$ *satisfies the c.c.c. and* $(\mathbb{Q},\underset{\sim}{\eta})$ *is homogeneous (see* $(\circledast)$ *below).* Then*:* $\mathbb{P}$ *is* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving iff* $\mathbb{P}$ *is weakly* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving, where*
   $(\circledast)$ $(\mathbb{Q},\underset{\sim}{\eta})$ *is specially homogeneous if: for any (Borel) sets* $B_1, B_2 \in (I_{(\mathbb{Q},\underset{\sim}{\eta})})^+$ *we can find a Borel set* B'_1 *satisfying* $B'_1 \subseteq B_1$, $B'_1 \in (I_{(\mathbb{Q},\underset{\sim}{\eta})})^+$ *and a Borel function* F *from* B'_1 *into* B_2 *such that*
   (α) *for every Borel set* $A \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$, $F^{-1}[A \cap B_2] \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$ *and this holds even in* $\mathbf{V}^{\mathbb{P}}$*, we say this* F *is* $(I_{(\mathbb{Q},\underset{\sim}{\eta})})^+$*–preserving in* $\mathbf{V}^{\mathbb{P}}$*,*
   (β) *this is absolute (or at least it holds also in* $\mathbf{V}^{\mathbb{P}}$*).*
4. *If* $\mathbb{Q}$ *is strong explicit c.c.c.–nep forcing and* N *is a* $\mathbb{Q}$*–candidate in generic for* $\mathbb{Q}$*,* then $B^-_N = \{\nu \in {}^{\omega}\omega : \nu$ *is not* $(\mathbb{Q},\eta)$*–generic over* $N\}$ *is Borel and belongs to* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$.

Remark 7.6. We say that the forcing notion $\mathbb{Q}$ [or a pair $(\mathbb{Q},\underset{\sim}{\eta})$] is homogeneous when for every $p, q \in \mathbb{Q}$ we can find p_1, q_1 such that $p \le_{\mathbb{Q}} p_1$, $q \le_{\mathbb{Q}} q_1$ and an isomorphism F from $\mathbb{Q} \restriction \{r : p_1 \le_{\mathbb{Q}} r\}$ onto $\mathbb{Q} \restriction \{r : p_1 \le_{\mathbb{Q}} r\}$ [mapping the appropriate restriction of $\underset{\sim}{\eta}$ to the one of $\underset{\sim}{\eta}$].

Proof. 1), 2) Easy.

3) By part (1) it suffices to show "non–preserving" assuming "not weakly preserving", and toward contradiction assume that this fails. So there are $p, B^*, \underset{\sim}{A}$ such that $B^* \in (I_{(\mathbb{Q},\underset{\sim}{\eta})})^+$ is a Borel subset of ${}^{\omega}\omega$ and

$$p \Vdash_{\mathbb{P}} \text{“}\begin{array}{l} (a)\ \underset{\sim}{A} \text{ is a Borel set} \\ (b)\ \underset{\sim}{A} \in I_{(\mathbb{Q},\underset{\sim}{\eta})},\ \text{that is } \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{\eta} \notin \underset{\sim}{A}\text{”} \\ (c)\ \nu \in \underset{\sim}{A} \text{ for every } \nu \in (B^*)^{\mathbf{V}}.\ \text{”} \end{array}$$

Let

$$\mathcal{J} = \{B : \begin{array}{l} B \in (I_{(\mathbb{Q},\underset{\sim}{\eta})})^+, \text{ so a Borel subset of } {}^{\omega}\omega, \text{ and} \\ \text{for some Borel function } F \text{ from } B \text{ into } B^* \text{ we have} \\ F \text{ is } (I_{(\mathbb{Q},\underset{\sim}{\eta})})^+\text{–preserving even in } \mathbf{V}^{\mathbb{P}}\}. \end{array}$$

Choose (in $\mathbf{V}$) a maximal family $\{B_i : i < i^*\} \subseteq \mathcal{J}$ such that $i \ne j \Rightarrow B_i \cap B_j \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$. As $\mathbb{Q}$ satisfies the c.c.c. necessarily $i^* < \omega_1$, so without loss of generality $i^* \le \omega$. By the assumption $(\circledast)$, clearly, ${}^{\omega}\omega \setminus \bigcup\limits_{i<i^*} B_i \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$. Let F_i witness that $B_i \in \mathcal{J}$. Let

$$\underset{\sim}{A}_i = \{\eta \in {}^{\omega}\omega : \eta \in B_i \ \text{ and } \ F_i(\eta) \in \underset{\sim}{A}, \text{ recalling } F_i \text{ is Borel}\}.$$

Then $\underset{\sim}{A}_i$ is a ($\mathbb{Q}$-name of a) Borel subset of ${}^{\omega}\omega$ and $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_i \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$" as $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A} \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$" and the choice of F_i. Hence

$$p \Vdash \text{“} \bigcup_{i<i^*} \underset{\sim}{A}_i \cup ({}^{\omega}\omega \setminus \bigcup_{i<i^*} B_i) \in I_{(\mathbb{Q},\underset{\sim}{\eta})} \text{”}$$

(call this set $\underset{\sim}{A}^*$). Now for every $\nu \in B_i^{\mathbf{V}}$, $F_i(\nu) \in (B^*)^{\mathbf{V}}$ hence (by clause (c) in the choice of p, B^*, $\underset{\sim}{A}$) we have $p \Vdash$ "$F_i(\nu) \in \underset{\sim}{A}$" hence by the definition of $\underset{\sim}{A}_i$ we have $p \Vdash$ "$\nu \in \underset{\sim}{A}_i$". So

$$p \Vdash_{\mathbb{P}} \text{“} ({}^{\omega}\omega)^{\mathbf{V}} = ({}^{\omega}\omega \setminus \bigcup_{i<i^*} B_i)^{\mathbf{V}} \cup \bigcup_{i<i^*} B_i^{\mathbf{V}} \subseteq ({}^{\omega}\omega \setminus \bigcup_{i<i^*} B_i) \cup \bigcup_{i<i^*} \underset{\sim}{A}_i \in I_{(\mathbb{Q},\underset{\sim}{\eta})} \text{”}$$

so we are done.

4) By 6.13(1). $\blacksquare_{7.5}$

Example 7.7. 1) It is easy to find a forcing notion $\mathbb{P}$ which is $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving, but not strongly $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving, e.g. for $\mathbb{Q}$ = Cohen (see Exaple 7.12 below). However, for sufficiently nice forcing notion $\mathbb{P}$, "$I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving" and "strongly $I_{(\mathbb{Q},\underset{\sim}{\eta})}$-preserving" coincide, as we will see in 7.10. (Parallel to the phenomenon that for "nice" sets, CH holds).
2) It is even easier to find a weakly $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving forcing notion $\mathbb{P}$ which is not $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving.

Assume that for $\ell < 2$ we have $(\mathbb{Q}_\ell, \underset{\sim}{\eta}_\ell)$ as in 7.1, e.g. $\mathbb{Q}_0$ is Cohen forcing, $\mathbb{Q}_1$ is random real forcing. Let $\mathbb{Q} = \{\emptyset\} \cup \bigcup_{\ell<2} \{\ell\} \times \mathbb{Q}_\ell$, $\emptyset$ minimal, $(\ell_1, q_1) \leq (\ell_2, q_2)$ iff $\ell_1 = \ell_2$ and $\mathbb{Q}_{\ell_1} \models q_1 \leq q_2$. We define a $\mathbb{Q}$–name $\underset{\sim}{\eta}$ by defining for a generic $G \subseteq \mathbb{Q}$ over $\mathbf{V}$:

$$\underset{\sim}{\eta}[G] \text{ is } \begin{array}{ll} \langle 0 \rangle^\frown(\underset{\sim}{\eta}_0[G_0]) & \text{if } \{0\} \times \mathbb{Q}_0 \cap G \neq \emptyset, \text{ and } G_0 = \{q \in \mathbb{Q}_0 : (0,q) \in G\} \\ \langle 1 \rangle^\frown(\underset{\sim}{\eta}_1[G_1]) & \text{if } \{1\} \times \mathbb{Q}_1 \cap G \neq \emptyset, \text{ and } G_1 = \{q \in \mathbb{Q}_1 : (1,q) \in G\}. \end{array}$$

Then usually (and certainly for our choice) we get a counterexample.

Proposition 7.8. *Assume that A is a Borel subset (better: a definition of a Borel subset) of ${}^{\omega}\omega$, M is a $\mathbb{Q}$–candidate (so $\underset{\sim}{\eta} \in M$, i.e. $\langle \psi_{\alpha,m} : \alpha < \omega, m < \omega \rangle \in M$ as in Definition 6.2) and $A \in M$ (i.e. the definition). Further, suppose that $q \in \mathbb{Q}^M$ is such that $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in A$". Then*

(α) *$M \models$ "$q \Vdash_{\mathbb{Q}} \underset{\sim}{\eta} \in A$",*
(β) *there is $\eta \in A$ which is a $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over M satisfying[31] q.*

[31] i.e. $\eta = \underset{\sim}{\eta}[G]$ where G is $\langle N, \mathbb{Q} \rangle$–generic and $q \in G$

Proof. As for (α), if it fails then for some $q' \in \mathbb{Q}^M$, we have

$$M \models \text{“ } q \leq^{\mathbb{Q}} q' \text{ and } q' \Vdash_{\mathbb{Q}} \underset{\sim}{\eta} \notin A \text{ ”},$$

and let $r \in \mathbb{Q}$ be $\langle M, \mathbb{Q}\rangle$–generic above q'. So if G is a subset of $\mathbb{Q}$ generic over $\mathbf{V}$ to which r belongs then $q' \in G$ and $G \cap \mathbb{Q}^M$ is a subset of $\mathbb{Q}^M$ generic over M to which q' belongs. Hence $M\langle G\rangle \models$“$\underset{\sim}{\eta}\langle G \cap \mathbb{Q}^M\rangle \notin A$” and $\underset{\sim}{\eta}\langle G \cap \mathbb{Q}^M\rangle \in {}^{\omega}\omega$. By absoluteness also $\mathbf{V}[G] \models \underset{\sim}{\eta}[G \cap \mathbb{Q}^M] \notin A$ and $\underset{\sim}{\eta}[G \cap \mathbb{Q}^M] \in {}^{\omega}\omega$. But as $\underset{\sim}{\eta} \in M$ clearly $\underset{\sim}{\eta}\langle G \cap \mathbb{Q}^M\rangle = \underset{\sim}{\eta}[G]$ and as $q' \in G$ also $q \in G$, so we get contradiction to $q \Vdash_{\mathbb{Q}}$“$\underset{\sim}{\eta} \in A$”.

By clause (α) clause (β) is easy: we can find a subset $G \in \mathbf{V}$ of $\mathbb{Q}^N$ to which q belongs which is generic over M. So $\underset{\sim}{\eta}[G] \in {}^{\omega}\omega$ and it belongs to A as $M \models$“$q \Vdash_{\mathbb{Q}} \underset{\sim}{\eta} \in A$”. $\blacksquare_{7.8}$

Proposition 7.9. *Assume $\mathbb{Q}$ is correct and satisfies the c.c.c., $\aleph_0 = \theta^{\mathbb{Q}} + \kappa^{\mathbb{Q}} + ||\mathfrak{B}^{\mathbb{Q}}|| + ||\underset{\sim}{\tau}|| + \|\mathfrak{C}^{\mathbb{Q}}\|$ and $\underset{\sim}{\eta}$, an ord-hc-$\mathbb{Q}$-name, is generic for $\mathbb{Q}$.* Then *the following conditions are equivalent for a set $X \subseteq {}^{\omega}\omega$:*

(A) $X \in I^{\mathrm{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})}$,

(B) *for some $\rho \in {}^{\omega}2$, for every $\mathbb{Q}$–candidate N to which ρ belongs there is no $\eta \in X$ which is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over N,*

(C) *for every $p \in \mathbb{Q}$ for some $\mathbb{Q}$–candidate N such that $p \in \mathbb{Q}^N$, there is no $\eta \in X$ which is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over N satisfying q.*

Proof. (A) $\Rightarrow$ (B): So assume (A), i.e. $X \in I^{\mathrm{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})}$. Then for some Borel set $A \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$ we have $X \subseteq A$. Let $\rho \in {}^{\omega}2$ code A. Since $\Vdash_{\mathbb{Q}}$“$\underset{\sim}{\eta} \notin A^{\mathbf{V}[\underset{\sim}{G}_{\mathbb{Q}}]}$”, it follows from 7.8 that

(*) for any $\mathbb{Q}$–candidate N to which ρ belongs, $N \models$“$q \Vdash_{\mathbb{Q}} \eta \notin A^{\mathbf{V}[\underset{\sim}{G}_{\mathbb{Q}}]}$”, hence there is no $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real η over N which belongs to X (or even just to A).

(B) $\Rightarrow$ (C): Easy as $\mathbb{Q}$ is correct. That is given $p \in \mathbb{Q}$, by correctness there is a $\mathbb{Q}$–candidate M such that $p \in \mathbb{Q}^M$. Also there is a $\mathbb{Q}$–candidate N such that $|M| \cup \{\rho\} \subseteq N$ where ρ witness clause (B); there is such N as ZFC^-_* is weakly normal (see 1.17). Now N is as required.

(C) $\Rightarrow$ (A): Assume (C). Let

$$\begin{aligned}\mathcal{I} = \{p \in \mathbb{Q} : \ & \text{for some Borel subset } A = A_p \text{ of } {}^{\omega}\omega \\ & \text{we have } p \Vdash \text{“}\underset{\sim}{\eta} \notin A_p\text{” and } X \subseteq A_p\}.\end{aligned}$$

Suppose first that $\mathcal{I}$ is predense in $\mathbb{Q}$. Clearly it is open, and we can find a maximal antichain $\mathcal{J}$ of $\mathbb{Q}$ such that $\mathcal{J} \subseteq \mathcal{I}$. As $\mathbb{Q}$ satisfies the c.c.c., necessarily $\mathcal{J}$ is countable. So $A \overset{\mathrm{def}}{=} \bigcap\limits_{p \in \mathcal{J}} A_p$ is a Borel subset of ${}^{\omega}\omega$ (as $\mathcal{J}$

is countable) and it includes X (as each A_p does). Moreover, since $\mathcal{J}$ is a maximal antichain of $\mathbb{Q}$ (and $p \in \mathcal{J} \Rightarrow p \in \mathcal{I} \Rightarrow p \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{\eta} \notin A_p\text{”} \Rightarrow p \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{\eta} \notin A\text{”}$) we have $\Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{\eta} \notin A\text{”}$. Consequently (A) holds.

Suppose now that $\mathcal{I}$ is not predense in $\mathbb{Q}$ and let $p^* \in \mathbb{Q}$ exemplify it, i.e. it is incompatible with every member of $\mathcal{I}$. Let N be a $\mathbb{Q}$–candidate such that $p^* \in \mathbb{Q}^N$ as is guaranteeded by clause (C) which we are assuming. Thus $p^* \in \mathbb{Q}^N$ and no $\eta \in X$ is $(\mathbb{Q}, \eta)$–generic over N. Let q be a member of $\mathbb{Q}$ which is above p^* and is $\langle N, \mathbb{Q}^{\tilde{N}}\rangle$–generic (i.e. $q \Vdash \text{“}G^P \cap \mathbb{Q}^N$ is generic over N”). Let $A \stackrel{\text{def}}{=} \{\eta \in {}^{\omega}\omega : \eta$ is not $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N\}$. Now

(a) A is a Borel subset of ${}^{\omega}\omega$.

(Why? As N is countable and $\underset{\sim}{\eta}$ being generic and hereditarily countable name, of course without $\underset{\sim}{\eta}$ being generic for $\mathbb{Q}$, we get only Σ^1_1, but see next proof.)

(b) $X \subseteq A$.

(Why? By the choice of N according to clause (C).)

(c) $q \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{\eta} \notin A^{\mathbf{V}[G_{\mathbb{Q}}]}\text{”}$.

(Why? By the definition of A.)

(d) $q \in \mathcal{I}$.

(Why? By (a)+(b)+(c)).

Thus $p^* \le q \in \mathcal{I}$ and we get contradiction to the choice of p^*. $\blacksquare_{7.9}$

Theorem 7.10. *Assume that:*

(a) *$\mathbb{Q}$, $\underset{\sim}{\eta}$ are as above (see 7.1), and $\mathbb{Q}$ is correct,*
(b) *$\mathbb{P}$ is nep-forcing notion with respect to* our fixed *version* ZFC^-_*,
(c) *$\mathbb{P}$ is $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving, moreover*[32],
(c)$^+$ *if M is a $\mathbb{P}$–candidate and a $\mathbb{Q}$–candidate, $p \in \mathbb{P}^M$, and $q \in \mathbb{Q}^M$,* then *there are $p_1 \in \mathbb{P}$ and $\eta \in {}^{\omega}\omega$ such that $p \le_{\mathbb{P}} p_1$, p_1 is $\langle M, \mathbb{P}\rangle$–generic and $p_1 \Vdash_{\mathbb{P}}$ “η is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic both for M and for $M\langle \underset{\sim}{G}_{\mathbb{P}}\rangle$ satisfying q”.*
(d) ZFC^-_{**} *is a stronger version of set theory including clauses (i)–(v) below for some*[33] χ
  (i) *$(\mathcal{H}(\chi), \in)$ is a (well defined) model of* ZFC^-_*,
  (ii) *(a), (b) and (c) and (c)$^+$ hold (with $\mathfrak{B}^{\mathbb{P}}$, $\mathfrak{B}^{\mathbb{Q}}$, $\underset{\sim}{\eta}$ as individual constants),*
  (iii) *$\mathbb{Q}, \mathbb{P} \in \mathcal{H}(\chi)$ and $(\mathcal{H}(\chi), \in)$ is a semi $\mathbb{P}$–candidate and a semi $\mathbb{Q}$-candidate with $(\mathfrak{B}^{\mathbb{P}})$ interpreted as $(\mathfrak{B}^{\mathbb{P}})^N \restriction \mathcal{H}(\chi)$ and similarly for*

[32]This follows from clause (c) if $\mathbb{Q}$ satisfies the c.c.c., by 7.5(4).
[33]but we can with more care, using several χ-s weaken the demands

$\mathbb{Q}$, so (natural to assume) $\mathfrak{B}^{\mathbb{P}}$, $\mathfrak{B}^{\mathbb{Q}} \in \mathcal{H}(\chi)$ (remember: "semi" means omitting the countability demand, see 1.1(7), (8)*)*

(iv) *forcing of cardinality $< \chi$ preserves the properties (i), (ii), (iii), and χ is a strong limit cardinal.*

Then*:*

(α) if*, additionally,*

(e) ZFC$^-_{**}$ *is normal (see Definition* 1.15(3)*)*

then $\mathbb{P}$ *is strongly $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving,*

(β) if N *is a $\mathbb{P}$–candidate and $\mathbb{Q}$–candidate and moreover it is a model of* ZFC$^-_{**}$ *and $N \models$ "$p \in \mathbb{P}$" and η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over N,*

then *for some p' we have:*

(i) *$p \leq_{\mathbb{P}} p'$ and $p' \in \mathbb{P}$,*

(ii) *p' is $\langle N, \mathbb{P}\rangle$–generic; i.e. $p' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N$ is generic over N" (see* 4.3*),*

(iii) *$p' \Vdash_{\mathbb{P}}$ "η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N[\mathbb{P}^N \cap \underset{\sim}{G}_{\mathbb{P}}]$".*

$(\alpha)^+$ *We can strengthen the conclusion of* (α) *to "$\mathbb{P}$ is super–$I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving".*

Remark 7.11. 1) We consider, for a nep forcing notion $\mathbb{Q}$

$(*)_1$ $\mathbb{Q}$ satisfies the c.c.c.

We also consider

$(*)_2$ being a predense subset (or just a maximal antichain) of $\mathbb{Q}$ is K–absolute.

By results of the previous section, $(*)_1 \Rightarrow (*)_2$ under reasonable conditions. You may wonder whether $(*)_2 \Rightarrow (*)_1$, but by the examples in Section 10 the answer is not.

2) Note that in $(\alpha), (\alpha)^+$ we can use only weak normality if $\mathbb{Q}$ satisfies the c.c.c., see 7.14. We do not use "$\mathbb{P}$ is explicitly nep" so we do not demand it (though would not mind it).

Before we prove the theorem, let us give an example for a forcing notion failing the conclusion and see why many times we can simplify assumptions.

Example 7.12. Start with $\mathbf{V}_0$. Let $\bar{s} = \langle s_i : i < \omega_1\rangle$ be a sequence of random reals over $\mathbf{V}_0$, i.e. generic sequence for the measure algebra on ${}^{\omega_1}({}^{\omega}2)$. Let $\mathbf{V}_1 = \mathbf{V}_0[\bar{s}]$, $\mathbf{V}_2 = \mathbf{V}_1[r]$, r a Cohen over $\mathbf{V}_1$ and

$$\mathbf{V}_3 = \mathbf{V}_2[\bar{t}] \quad \text{where } \bar{t} = \langle t_i : i < \omega_1\rangle \text{ is a sequence of random reals, i.e. generic over } \mathbf{V}_2 \text{ for the measure algebra } {}^{\omega_1}({}^{\omega}2).$$

Then in $\mathbf{V}_3$ (in fact, already in $\mathbf{V}_2$), $\{s_i : i < \omega_1\}$ is a null set, whereas $\{t_i : i < \omega_1\}$ is not null. But $\bar{t}$ is also generic for the measure algebra over

$\mathbf{V}_1$. So $\mathbf{V}_2' = \mathbf{V}_1[\bar{t}]$ is a generic extension of $\mathbf{V}_1$. We have $\mathbf{V}_3 = \mathbf{V}_2'[r]$, where r is generic for some forcing notion from $\mathbf{V}_2'$, more specifically for

$$\mathbb{R} \stackrel{\text{def}}{=} (\text{Cohen} * \text{measure algebra adding } \bar{t})/\bar{t}.$$

So in $\mathbf{V}_2'$ the sets $\bar{t}$ and $\bar{s}$ are not null and $\mathbb{R}$ makes $\bar{s}$ null, but not $\bar{t}$.

How can $\mathbb{R}$ do that? $\mathbb{R}$ uses $\langle t_i : i < \omega_1\rangle$ in its definition, so it is not "nice" enough. $\blacksquare_{7.12}$

Remark 7.13. 1) In the proof of 7.10, of course, we may assume $N \prec (\mathcal{H}(\chi), \in)$ if $(\mathcal{H}(\chi), \in) \models \mathrm{ZFC}^-_{**}$, as this normally holds. In (α) the use of such N does not matter. In (β) it slightly weakens the conclusion. Now, (α) is our original aim. But (β) both is needed for (α) and is a step towards preserving them (as in [25]). So typically N is an elementary submodel of appropriate $\mathcal{H}(\chi)$.
2) Below we use 7.5(4) to get $(c)^+$, may consider 7.9 too.

Proof of 7.10 **Clause** (α): To prove (α) we will use (β). So let $X \subseteq {}^\omega\omega$, $X \in (I^{\mathrm{dx}}_{(\mathbb{Q},\underset{\sim}{\eta})})^+$. Then there is a condition $q^* \in \mathbb{Q}$ such that

$(*)_1$ for no Borel subset B of ${}^\omega\omega$ and q satisfying $q^* \leq q \in \mathbb{Q}$ do we have: $X \subseteq B$ and $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin B$".

Let χ be large enough. We can find $N \subseteq (\mathcal{H}(\chi), \in)$ as in (β), moreover $N \prec (\mathcal{H}(\chi), \in)$ a model of ZFC^-_{**} (and so a $\mathbb{P}$–candidate and a $\mathbb{Q}$-candidate) [it exists because by clause (e) of the assumptions for (α), ZFC^-_{**} is normal so for χ large enough any countable $N \prec (\mathcal{H}(\chi), \in)$ to which $\mathfrak{C}, \mathfrak{B}^{\mathbb{Q}}, \mathfrak{B}^{\mathbb{P}}$ belong is a model of ZFC^-_{**} and is a $\mathbb{P}$–candidate and a $\mathbb{Q}$–candidate, so is as required].

Towards a contradiction, assume $p^* \in \mathbb{P}$ and $p^* \Vdash_{\mathbb{P}}$ "$X \in I^{\mathrm{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})}$". So for some $\mathbb{P}$–name $\underset{\sim}{A}$ we have

$p^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}$ is a Borel subset of ${}^\omega\omega$, $X \subseteq \underset{\sim}{A}$ and $\underset{\sim}{A} \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$, i.e. $\Vdash_{\mathbb{Q}} \underset{\sim}{\eta} \notin \underset{\sim}{A}$".

Without loss of generality the name $\underset{\sim}{A}$ is hereditarily countable and $\underset{\sim}{A}, p^*, q^*$ belong to N. In $\mathbf{V}$, let

$$B = \{\eta \in {}^\omega\omega : \ \eta \text{ is a } (\mathbb{Q}^{\geq q^*}, \underset{\sim}{\eta})\text{–generic real over } N, \text{ which means that } \eta = \underset{\sim}{\eta}[G] \text{ for some } G \subseteq \mathbb{Q}^N \text{ generic over } N \text{ such that } q^* \in \underset{\sim}{G}\}.$$

Clearly, it is an analytic set (if $\underset{\sim}{\eta}$ was generic real then it is actually Borel; both holds as the relevant statement follows from ZFC^-_{**}). So for some sequence $\bar{B} = \langle B_i : i < \omega_1\rangle$ we have $B = \bigcup\limits_{i<\omega_1} B_i$, each B_i is Borel (absolutely as long $\aleph_1$ is not collapsed). Let $q \in \mathbb{Q}$ be $\langle N, \mathbb{Q}\rangle$–generic and $q^* \leq q$. Then

$q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in B$" by the definition of B and hence possibly increasing q, for some $i < \omega_1$ we have $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in B_i$". Since $q^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin ({}^{\omega}\omega \setminus B_i)$" (as $q \Vdash$ "$\underset{\sim}{\eta} \in B_i$"), we may apply $(*)_1$ to the set ${}^{\omega}\omega \setminus B_i$ to conclude that $X \not\subseteq {}^{\omega}\omega \setminus B_i$. Choose $\eta^* \in X \cap B_i$ (so it is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over N by the choice of B and $\bar{B}$). So by clause (β) (proved below), there is a condition $p \in \mathbb{P}$, $p \geq p^*$ which is $\langle N, \mathbb{P} \rangle$–generic (i.e. it forces that $\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N$ is generic over N, generally not necessarily $\underset{\sim}{G}_{\mathbb{P}} \cap N$, but in our case $N \prec (\mathcal{H}(\chi), \in)$ hence they are equal) and such that

$$p \Vdash_{\mathbb{P}} \text{“}\eta^* \text{ is } (\mathbb{Q}, \underset{\sim}{\eta})\text{–generic over } N[\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N]\text{”}.$$

Choose $G_{\mathbb{P}} \subseteq \mathbb{P}$ generic over $\mathbf{V}$, such that $p \in G_{\mathbb{P}}$. In $\mathbf{V}[G_{\mathbb{P}}]$, $N[G_{\mathbb{P}} \cap \mathbb{P}^N]$ is a generic extension of N (for $\mathbb{P}^N$!), a $\mathbb{Q}$–candidate (see (d) of the assumptions), and η^* is $(\mathbb{Q}, \eta)$–generic over $N[G_{\mathbb{P}} \cap N]$ (and over N). As $p^* \leq p \in G_{\mathbb{P}}$, clearly if $G_{\mathbb{Q}} \subseteq \mathbb{Q}^{\mathbf{V}[G_{\mathbb{P}}]}$ is generic over $\mathbf{V}[G_{\mathbb{P}}]$, then $\underset{\sim}{\eta}[G_{\mathbb{Q}}] \notin \underset{\sim}{A}[G_{\mathbb{P}}]$ in $\mathbf{V}[G_{\mathbb{P}}, G_{\mathbb{Q}}]$, simply by the choice of $\underset{\sim}{A}$. But $N[G_{\mathbb{P}} \cap \mathbb{P}^N] \prec (\mathcal{H}(\chi)^{\mathbf{V}[G_{\mathbb{P}}]}, \in)$ by the choice of N, so $N[G_{\mathbb{P}} \cap \mathbb{P}^N]$ satisfies the parallel statement.

Since η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N[G_{\mathbb{P}} \cap \mathbb{P}^N]$, it cannot belong to $\underset{\sim}{A}[G_{\mathbb{P}} \cap \mathbb{P}^N]$, all in $\mathbf{V}[G_{\mathbb{P}}]$. But easily $\underset{\sim}{A}[G_{\mathbb{P}} \cap \mathbb{P}^N] = \underset{\sim}{A}[G_{\mathbb{P}}]$ as definitions of Borel sets and $X \subseteq \underset{\sim}{A}[G_{\mathbb{P}}]$ by the choice of $\underset{\sim}{A}$ as $p^* \in \underset{\sim}{G}_{\mathbb{P}}$. But $\eta^* \in X$ by the choice of η^*, hence, $\eta^* \in X \subseteq \underset{\sim}{A}[G_{\mathbb{P}}]$, a contradiction to the previous paragraph. This ends the proof of clause (α) of 7.10.

Clause $(\alpha)^+$: Similar to the proof of clause (α). We start as there so toward contradiction we assume $p^* \Vdash_{\mathbb{P}}$ "$X \in I^{\mathrm{dx}}_{(\mathbb{Q}, \underset{\sim}{\eta})}$" but now we choose functions $r^*, A^*, \mathcal{I}$ such that

$(*)_2$ $\mathrm{Dom}(r^*) = \mathrm{Dom}(\mathcal{I})$ is the set of all hereditarily countable canonical $\mathbb{P}$–names for elements of $\mathbb{Q}$ (so each is a member of $\mathcal{H}_{<\aleph_1}(\kappa(\mathbb{P}) + \kappa(\mathbb{Q}))$), and[34]

$$\mathrm{Dom}(A^*) = \{(p, \underset{\sim}{q}) : p \in \mathcal{I}(\underset{\sim}{q}),\ \underset{\sim}{q} \in \mathrm{Dom}(r^*)\},$$

each $A^*(p, \underset{\sim}{q})$ is a $\mathbb{P}$–name,

$(*)_3$ for each $\underset{\sim}{q} \in \mathrm{Dom}(r^*) = \mathrm{Dom}(\mathcal{I})$, $\mathcal{I}(\underset{\sim}{q})$ is a predense subset of $\mathbb{P}$ above p^* such that for each $p \in \mathcal{I}(\underset{\sim}{q})$ we have:

$$\begin{aligned} p \Vdash_{\mathbb{P}} &\text{“ } A^*(p, \underset{\sim}{q}) \text{ is a Borel subset of } {}^{\omega}\omega\text{”}, \\ p \Vdash_{\mathbb{P}} &\ [r^*(\underset{\sim}{q}) \Vdash_{\mathbb{Q}} \text{“ } \underset{\sim}{\eta} \notin A^*(p, \underset{\sim}{q}) \text{ ”}], \\ p \Vdash_{\mathbb{P}} &\text{“ } X \subseteq A^*(p, \underset{\sim}{q}) \text{ ”}, \\ p \Vdash_{\mathbb{P}} &\text{“ } \mathbb{Q} \models \underset{\sim}{q} \leq r^*(\underset{\sim}{q}) \text{ ”}. \end{aligned}$$

Those functions exist by the definition of $I^{\mathrm{dx}}_{(\mathbb{Q}, \underset{\sim}{\eta})}$. Without loss of generality, the set X, and the functions $r^*, A^*, \mathcal{I}$ belong to N choosen as above. We

[34] If we change $\mathrm{Dom}(r^*)$ to be $\mathrm{Dom}(A^*)$ we can ask also $r^*(p, \underset{\sim}{q})$ to be a hereditary countable name.

choose also conditions $q \in \mathbb{Q}$, $p \in \mathbb{P}$ and a real $\eta^* \in X$ and a generic filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ over $\mathbf{V}$ in a similar manner as in the proof of clause (α). We note that by $(*)_3$

$$\begin{aligned} \underset{\sim}{q} \in \mathrm{Dom}(r^*) \cap N \quad & \Rightarrow \quad \text{“}\mathcal{I}(\underset{\sim}{q}) \text{ a predense subset of } \mathbb{P}\text{”} \\ & \Rightarrow \quad \mathbb{P}^N \cap \mathcal{I}(\underset{\sim}{q}) \cap G_{\mathbb{P}} \neq \emptyset, \end{aligned}$$

so we can choose $p[\underset{\sim}{q}] \in \mathbb{P}^N \cap \mathcal{I}(\underset{\sim}{q}) \cap G_{\mathbb{P}}$ for $\underset{\sim}{q} \in \mathrm{Dom}(\mathcal{I} \cap N$. Since η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N[G_{\mathbb{P}} \cap \mathbb{P}^N]$, there is $G^* \subseteq \mathbb{Q}^{N[G_{\mathbb{P}} \cap \mathbb{P}^N]}$ generic over $N[G_{\mathbb{P}} \cap \mathbb{P}^N]$ such that $\eta^* = \underset{\sim}{\eta}[G^*]$. By the choice of the function r^* the set $\{r^*[\underset{\sim}{q}][G_{\mathbb{P}}] : \underset{\sim}{q} \in \mathrm{Dom}(r^*)\}$ is a dense subset of $\mathbb{Q}^{\mathbf{V}[G_{\mathbb{P}}]}$, hence also $N[G_{\mathbb{P}} \cap \mathbb{P}^N]$ satisfies this hence there is $\underset{\sim}{q} \in N \cap \mathrm{Dom}(r^*)$ such that $r \stackrel{\mathrm{def}}{=} r^*[\underset{\sim}{q}][G_{\mathbb{P}} \cap \mathbb{P}^N] \in G^*$. Now, $A = A^*(p[\underset{\sim}{q}], \underset{\sim}{q})[G_{\mathbb{P}}] \in N[G_{\mathbb{P}} \cap \mathbb{P}^N]$ is a Borel subset of ${}^{\omega}\omega$ and $N[G_{\mathbb{P}} \cap \mathbb{P}^N] \models$“$r \Vdash_{\mathbb{Q}} \underset{\sim}{\eta} \notin A$”, hence $N[G_{\mathbb{P}} \cap \mathbb{P}^N][G^*] \models$“$\underset{\sim}{\eta}[G^*] \notin A$”. But $N[G_{\mathbb{P}} \cap \mathbb{P}^N] \models$ “$X \subseteq A$”, so $(\mathcal{H}(\chi), \in)^{\mathbf{V}[G_{\mathbb{P}}]} \models X \subseteq A$, hence $\mathbf{V} \models X \subseteq A$. Therefore, recalling $A \subseteq \mathbf{V}$, we have $N[G_{\mathbb{P}} \cap \mathbb{P}][G^*] \models$“$X \subseteq A$ that is $X \setminus A = \emptyset$”. But this contradicts $\underset{\sim}{\eta}[G^*] = \eta^* \in X \setminus A$ (see the choice of G^* and η^*).

Clause (β): So $N, \eta^*, \mathbb{Q}, \mathbb{P}, p$ are given; if below the use of $N\langle G_1\rangle\langle G_2\rangle$ seen suspicious, we may without lost of generality assume N is ordinal transitive, so if $G \subseteq \mathbb{P} * \mathbb{R}^N_*$ is generic over $\mathbf{V}$ then $N\langle G\rangle = N[G]$ and $N[G]$ is ord-transitive. Let $N_1 = N[G^*]$ be a generic extension of N by a subset $G^* \in \mathbf{V}$ of $\mathbb{Q}^N$ generic over N such that $\eta^* = \underset{\sim}{\eta}[G^*]$, note that $G^* \in \mathbf{V}$ exist by an assumption and N being countable (see 6.2). As $\mathrm{ZFC}^-_{**} \vdash$ “χ is strong limit”, clearly the power sets of $\mathbb{P}$ and $\mathbb{Q}$ belong to $\mathcal{H}(\chi)$. Now choose (in N) for $\ell = 1, 2$ a model $M_\ell \prec (\mathcal{H}(\chi), \in)^N$ such that

$\boxdot_1$ (i) $\mathbb{P}, \mathbb{Q}, \underset{\sim}{\eta}, p \in M_\ell$,
(ii) $\mathbb{Q}^N \subseteq M_\ell$ and $\mathbb{P}^N \subseteq M_\ell$,
(iii) the family of maximal antichains of $\mathbb{P}$ and of $\mathbb{Q}$ from N are included in M_ℓ,
(iv) $M_\ell \in N$, moreover $M_\ell \in \mathcal{H}(\chi)^N$,
(v) $M_1 \cup \mathcal{P}^N(M_1) \subseteq M_2$.

Hence, by assumption (d), clause (iii)

$\boxdot_2$ M_ℓ is a $\mathbb{P}$-candidate and a $\mathbb{Q}$-candidate and

$$N \models \text{“}M_\ell \text{ is a semi } \mathbb{P}\text{-candidate and semi } \mathbb{Q}\text{-candidate”}.$$

Let $\mathbb{R}_\ell = \mathrm{Levy}(\aleph_0, |M_\ell|)^N$ i.e. $\{f$: a function from some $n < \omega$ into M_ℓ in N-'s sense$\}$ ordered by inclusion; of course we can replace $|M_\ell|$ by $||M_\ell||^N$; lastly let $\mathbb{R} = \mathbb{R}_2, M = M_2$. In $\mathbf{V}$ let $G_{\mathbb{R}} \subseteq \mathbb{R}$ be generic over $N_1 = N\langle G^*\rangle$ (note that as N_1 is countable, clearly $G_{\mathbb{R}}$ exists) and let $N_2 = N_1[G_{\mathbb{R}}] =$

$N\langle G^*\rangle\langle G_{\mathbb{R}}\rangle$ (note that it too is a $\mathbb{P}$-candidate and a $\mathbb{Q}$-candidate), by clause (iv) of clause (d) of the assumption of 7.10. Note that

$\boxdot_3$ $\eta^* \in N_1 \subseteq N_2$.

Note: η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over M too and G^* is a subset of $\mathbb{Q}^M$ generic over M (by clauses (ii) + (iii) of $\boxdot_1$) and $\mathbb{Q}^N = \mathbb{Q}^M$, $\mathbb{P}^N = \mathbb{P}^M$ (note that in N_2 the model M is countable).

Now we ask the following question:

$\circledast$ Is there $p' \in \mathbb{P}^{N_2}$ such that $N_2 \models$" $p \leq_{\mathbb{P}} p'$, p' is $(M, \mathbb{P})$–generic and $p' \Vdash_{\mathbb{P}}$"η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $M\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^M\rangle$" "?

Depending on the answer, we consider two cases.

Case 1: The answer is "yes".
Choose p' as in $\circledast$ and choose $p'' \in \mathbb{P}$, such that $p' \leq_{\mathbb{P}} p''$ and p'' is $(N_2, \mathbb{P}^{N_2})$–generic. Then we have

$$p'' \Vdash_{\mathbb{P}} \text{“} \begin{array}{l} \text{in } \mathbf{V}[\underset{\sim}{G}_{\mathbb{P}}],\ \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^{N_2} \text{ is generic over } N_2,\ \{p, p'\} \in \underset{\sim}{G}_{\mathbb{P}}, \text{ and} \\ \text{in } N_2\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^{N_2}\rangle,\ \eta^* \text{ is } (\mathbb{Q}, \underset{\sim}{\eta})\text{–generic over } M\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^M\rangle, \\ \text{hence also over } N\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N\rangle\text{”.} \end{array}$$

[Why does p'' force this? As:

(A) "$\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^{N_2}$ is generic over N_2" holds because p'' is $(N_2, \mathbb{P}^{N_2})$–generic;
(B) "$p, p' \in \underset{\sim}{G}_{\mathbb{P}}$" holds as $p \leq p' \leq p'' \in G_{\mathbb{P}}$
(C) "in $N_2\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^{N_2}\rangle$, η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $M\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^M\rangle$" holds because of clause (A) and the choice of p' (i.e. the assumption of the case and as $p' \in G_{\mathbb{P}}$);
(D) "η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N\rangle$" holds by clause (C) above and clause (iii) of the choice of M that is $\boxdot_1$.]

So, $p'' \Vdash_{\mathbb{P}}$"η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N[\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N]$", i.e., p'' is as required.

Case 2: The answer is "no".
Let $\psi(x)$ be the following statement:

$\boxdot^4_\psi$ x is $(\mathbb{Q}, \underset{\sim}{\eta})$-generic real over M and there is no p' satisfying: $p' \in \mathbb{P}$, $\mathbb{P} \models$"$p \leq p'$", p' is $\langle M, \mathbb{P}^M\rangle$–generic and $p' \Vdash_{\mathbb{P}}$"x is a $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over $M\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^M\rangle$".

So ψ is a first order formula in set theory, all parameters are in $N \subseteq N_1 = N\langle G^*\rangle \subseteq N_2 = N\langle G^*\rangle\langle G_{\mathbb{R}_2}\rangle$, and by the assumption of the case

$$N\langle G^*\rangle\langle G_{\mathbb{R}_2}\rangle \models \psi[\eta^*].$$

As $G_{\mathbb{R}} \subseteq \mathbb{R}$ is generic over $N_1 = N\langle G^*\rangle$ for $\mathbb{R}$, necessarily (by the forcing theorem), for some $r = r_2 \in G_{\mathbb{R}}$ we have

$$N\langle G^*\rangle \models \text{“ } r \Vdash_{\mathbb{R}} \psi[\eta^*] \text{ ”}.$$

Since $\mathbb{R}$ is homogeneous we may assume that $r = \emptyset$. So necessarily, for some $q \in G^* \subseteq \mathbb{Q}^N = \mathbb{Q}^M$ we have

$$N \models \big(q \Vdash_{\mathbb{Q}} [r \Vdash_{\mathbb{R}} \psi(\underset{\sim}{\eta}[G_{\mathbb{Q}}])]\big).$$

Now $\mathbb{R} = \mathbb{R}_2 \in N$ (by its definition, as $M \in N$) so

$\boxdot_5$ $N \models \big((q,r) \Vdash_{\mathbb{Q}\times\mathbb{R}} \psi(\underset{\sim}{\eta}[G_{\mathbb{Q}}])\big)$.

For the rest of the proof we can forget η^*, and derive, eventually, a contradiction thus finishing. Next, we deal with $\mathbb{R}_1$, let $G_{\mathbb{R}_1} \subseteq {\mathbb{R}_1}^N$ be generic over N hence over M_2. For a time everything said on N holds for M_2 as well, so $N\langle G_{\mathbb{R}_1}\rangle$ is a generic extension by a "small" forcing of N which is a model of ZFC^-_{**}, so $N\langle G_{\mathbb{R}_1}\rangle$ satisfies (i), (ii) and (iii) of the clause (d) of the assumptions. Note that $N \models$"M_1 is a semi $\mathbb{Q}$-candidate and a semi $\mathbb{P}$–candidate", see clause (d)(iii) of the assumptions and the choice of M_1, so also $N\langle G_{\mathbb{R}_1}\rangle$ satisfies this. Moreover, $N\langle G_{\mathbb{R}_1}\rangle \models$" M_1 is countable", so $N\langle G_{\mathbb{R}_1}\rangle \models$"$M_1$ is a $\mathbb{Q}$-candidate and a $\mathbb{P}$–candidate". Hence by assumption (d)(ii), that is in $(c)^+$ there are $p^1, \eta^\otimes, G^\otimes_{\mathbb{Q}} \in N\langle G_{\mathbb{R}_1}\rangle$ such that:

$\boxdot_6$ $N\langle G_{\mathbb{R}_1}\rangle \models$ " $p^1 \in \mathbb{P}$, $p \le_{\mathbb{P}} p_1$, p^1 is $\langle M_1, \mathbb{P}^M\rangle$–generic and
$p^1 \Vdash_{\mathbb{P}} [\eta^\otimes$ is a $(\mathbb{Q}, \underset{\sim}{\eta})$–real over $M_1[\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^M]$
and over M_1 satisfying $q]$
more explicitly, $p_1 \Vdash_{\mathbb{P}}$ "$\eta^\otimes = \underset{\sim}{\eta}\langle G^\otimes_{\mathbb{Q}}\rangle$", where
$G^\otimes_{\mathbb{Q}} \subseteq \mathbb{Q}^{M_1}$ is a generic set over M_1 such that
$q \in G^\otimes_{\mathbb{Q}}$".

[Why such p^1, $\eta^\otimes$, $G^\otimes_{\mathbb{Q}}$ exist? As $N\langle G_{\mathbb{R}_1}\rangle$ satisfies the assumption $(c)^+$, (apply clause (iv) of the assumption (d) to $\mathbb{R}_1$).]

As said above, wlog in $\boxdot_6$ we can replace N by M_2 and in particular $p^1, \eta^\otimes, G^\otimes_{\mathbb{Q}}$ belong to M_2. Let $\underset{\sim}{p}^1, \underset{\sim}{\eta}^\otimes, \underset{\sim}{G}^\otimes_{\mathbb{Q}} \in N$, moreover $\in M_2$, be $\mathbb{R}_1$–names such that $\underset{\sim}{\eta}^\otimes\langle G_{\mathbb{R}_1}\rangle = \eta^\otimes$, $G^\otimes_{\mathbb{Q}} = \underset{\sim}{G}^\otimes_{\mathbb{Q}}\langle G_{\mathbb{R}_1}\rangle$ and $\underset{\sim}{p}^1\langle G_{\mathbb{R}_1}\rangle = p^1$, and some $r_1 \in G_{\mathbb{R}_1}$ forces all this. Now, without loss of generality, $r_1 = \emptyset_{\mathbb{Q}}$ (again by homogenity of $\mathbb{R}$) so in N we have

$\boxdot_7$ $r_1 \Vdash_{\mathbb{R}_1}$ " $\underset{\sim}{\eta}^\otimes$ is a $\#(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over M_1 satisfying q and $\underset{\sim}{p}^1 \in \mathbb{P}$
and $\underset{\sim}{p}^1$ forces ($\Vdash_{\mathbb{P}}$) that
$\underset{\sim}{\eta}^\otimes$ is also $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $M_1\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^{M_1}\rangle$ satisfying q,
and $\underset{\sim}{G}^\otimes_{\mathbb{Q}}$ is a subset of $\mathbb{Q}^{M_1}$ generic overM_1, $\underset{\sim}{\eta}^\otimes = \eta[\underset{\sim}{G}^\otimes_{\mathbb{Q}}]$".

Let $G'_{\mathbb{R}_2} \subseteq \mathbb{R}_2$ be a generic over $N\langle G_{\mathbb{R}_2}\rangle$, to which r_2 belongs, $N[G_{\mathbb{R}_1}][G'_{\mathbb{R}_2}]$ is a forcing extension of $N[G_{\mathbb{R}_1}]$. So both are generic extensions of N by a small forcing and $G_{\mathbb{R}_1} \times G'_{\mathbb{R}_2}$ is generic for $\mathbb{R}_1 \times \mathbb{R}_2$ over N.

Now $\underset{\sim}{G}^\otimes_{\mathbb{Q}}$ is essentially a complete embedding of $\mathbb{Q}\restriction(\ge q)$ in N into $\mathbb{R}_1$ (by basic forcing theory, see the footnote to 1.15(1)(d)); and we can use the value for 0 of the function $\bigcup\{f : f \in G_{\mathbb{R}_1}\}$ to choose q', $q \le q' \in \mathbb{Q}^N = \mathbb{Q}^{M_2}$

recalling $r_1 = \emptyset$). Hence, in N, possibly increasing q, for some $\mathbb{Q}$–name $\underset{\sim}{\mathbb{R}}^* = \underset{\sim}{\mathbb{R}}_1^*$ we have $(\mathbb{Q}\restriction(\geq q)) * \underset{\sim}{\mathbb{R}}^*$ is $\mathbb{R}_1$, more exactly $\mathbb{R}_1\restriction\{r : r$ is above some member of $\mathcal{I}\}$ for some non empty subset $\mathcal{I}$ of $\mathbb{R}_1$. So $G_{\mathbb{R}_1} = G^{\otimes}_{\mathbb{Q}} * G_{\mathbb{R}^*}$ for some $G_{\mathbb{R}^*} \in N\langle G_{\mathbb{R}_1}\rangle$, where $\mathbb{R}^* = \underset{\sim}{\mathbb{R}}^*[G^{\otimes}_{\mathbb{Q}}]$, $\underset{\sim}{\mathbb{R}}^*$ a $\mathbb{Q}^N$–name. So we can represent $N\langle G_{\mathbb{R}_1}\rangle\langle G'_{\mathbb{R}_2}\rangle$ also as $N^3 \overset{\text{def}}{=} N\langle G^{\otimes}_{\mathbb{Q}}\rangle\langle G'_{\mathbb{R}_2}\rangle\langle G_{\mathbb{R}_1^*}\rangle$; i.e., forcing first with $\mathbb{Q}^N\restriction(\geq q)$, then with $\mathbb{R}_2$, lastly with $\underset{\sim}{\mathbb{R}}^*\langle G^{\otimes}_{\mathbb{Q}}\rangle$. Now let $N^2 \overset{\text{def}}{=} N\langle G^{\otimes}_{\mathbb{Q}}\rangle\langle G'_{\mathbb{R}_2}\rangle$, so N^2 is a generic extension of N and N^3 is a generic extension of N^2 (both by "small" forcing), and in N^3 we have p^1 and $\eta^{\otimes} = \underset{\sim}{\eta}\langle G^{\otimes}_{\mathbb{Q}}\rangle$ and $G^{\otimes}_{\mathbb{Q}}$. But $G^{\otimes}_{\mathbb{Q}} \times G'_{\mathbb{R}_2}$ is a generic subset of $(\mathbb{Q}^N\restriction\geq q) \times \mathbb{R}_2$ over N, so essentially a generic (over N) subset of $\mathbb{Q}^N \times \mathbb{R}_2$ to which (q, r_2) belongs, hence (by $\boxdot_5$ above) $N^2 \models \psi(\underset{\sim}{\eta}\langle G^{\otimes}_{\mathbb{Q}}\rangle)$. Therefore by the choice of ψ above (see $\boxdot^4_{\psi}$) we have

$\boxtimes_1$ there is no $p' \in N^2$ such that[35]:

($\boxtimes_2$) $N^2 \models [p' \in \mathbb{P}$ is $\langle M, \mathbb{P}^M\rangle$-generic, $p \leq p'$ and $p' \Vdash_{\mathbb{P}}$ "$\eta^{\otimes}$ is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $M\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^M\rangle$"].

Recall that $\underset{\sim}{p}', \underset{\sim}{G}^{\mathbb{Q}}_{\otimes}, \underset{\sim}{\eta}^{\otimes}, \underset{\sim}{R}^*$ belong to M_2. Now in $N\langle G^{\otimes}_{\mathbb{Q}}\rangle$, the forcing notion $\underset{\sim}{\mathbb{R}}^*\langle G^{\otimes}_{\mathbb{Q}}\rangle$ has cardinality at most $||M_1||^N$.

So $N\langle G^{\otimes}_{\mathbb{Q}}\rangle \models$ "the cardinality of $\mathcal{P}(\underset{\sim}{\mathbb{R}}^*\langle G^{\otimes}_{\mathbb{Q}}\rangle)$ is $\leq 2^{\|M_1\|}$, see $\boxdot_1$(iii) hence if we force with $\mathbb{R}_2$, we can find a generic for $\underset{\sim}{\mathbb{R}}^*\langle G^{\otimes}_{\mathbb{Q}}\rangle$".

Hence there is $G' \in N\langle G^{\otimes}_{\mathbb{Q}}\rangle\langle G_{\mathbb{R}_2}\rangle = N^2$ such that $N^2 \models$ "G' is generic of $\underset{\sim}{R}^*\langle G^{\otimes}_{\mathbb{Q}}\rangle$ over $N\langle G^{\otimes}_{\mathbb{Q}}\rangle$, i.e., it is a directed and meet all relevant dense subsets".

So in N^2, $G^{\otimes}_{\mathbb{Q}}$ is a generic subset of $\mathbb{Q}^{M_2}$ over M_2, to which q belongs and G' is a generic subset of $\underset{\sim}{\mathbb{R}}^*\langle G^{\otimes}_{\mathbb{Q}}\rangle$ over $M_2[G^{\otimes}_{\mathbb{Q}}]$, hence $G_{\mathbb{R}_1} = G^{\otimes}_{\mathbb{Q}} * G^*$ is a generic subset of $\mathbb{R}_1$ over M_2, and so $M_2^* \overset{\text{def}}{=} M_2[G_{\mathbb{R}_1}] \in N^2$ satisfies

- M_2^* is countable and a $\mathbb{P}$-candidate and a $\mathbb{Q}$-candidate and a generic extension of M_2 for $\mathbb{R}_1$.

$\boxdot_7$
- $G^{\otimes}_{\mathbb{Q}} \in M_2^*$ is a subset of $\mathbb{Q}^{M_2}$ generic over M_2,
- $q \in G^{\otimes}_{\mathbb{Q}}$,
- $\underset{\sim}{\eta}^{\otimes}\langle G_{\mathbb{R}_1}\rangle = \underset{\sim}{\eta}\langle G^{\otimes}_{\mathbb{Q}}\rangle$, call it η',
- $p_1^* = \underset{\sim}{p}^1\langle G_{\mathbb{R}_1}\rangle$ belong to $\mathbb{P}^{M_2^*}$ and is $\langle M_1, \mathbb{P}^M\rangle$-generic,
- $M_2^* \models [p_1^* \Vdash$ "η' is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $M_1\langle \underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^{M_1}\rangle$"].

[Why? See $\boxdot_6$.]

Now in N^2, by $\boxdot_6$, $M_2\langle G_{\mathbb{R}_1}\rangle$ is countable and a $\mathbb{P}$–candidate (see the assumptions) so there is p_2^* which is $\langle M_2\langle G_{\mathbb{R}_1}\rangle, \mathbb{P}^{M_2\langle G_{\mathbb{R}_1}\rangle}\rangle$–generic and $p_1^* \leq_{\mathbb{P}}$

[35] Of course, we can use a weaker demand on $G^{\otimes}_{\mathbb{Q}}$.

p_2^*. Now p_2^* contradicts $\boxdot_6$, that is $G^{\otimes}_{\mathbb{Q}} \times G_{\mathbb{R}_2} \subseteq \mathbb{Q} \times \mathbb{R}_2$ is generic over N and (q, r_2) belongs to it, but by $\boxdot_7$ above $N\langle G^{\otimes}_{\mathbb{Q}}\rangle\langle G^{\otimes}_{\mathbb{R}_2}\rangle \models \psi[\eta^{\otimes}]$ where $\eta^{\otimes} = \underset{\sim}{\eta}^{\otimes}\langle G^{\otimes}_{\mathbb{Q}}\rangle$ hence p_2^* can serve as the "no p'" in the definition of $\psi(x)$, see $\boxdot^4_{\psi}$. So we are done. $\blacksquare_{7.10}$

Proposition 7.14. *Assume (a),(b),(c) and (d) of* 7.10 *and*

(e) ZFC^-_{**} *is weakly normal,*
(f) $\mathbb{Q}$ *is c.c.c. and simple (for simplicity) and correct,*
(g) $\aleph_0 = \theta^{\mathbb{Q}} \vdash \kappa^{\mathbb{Q}} + ||\mathfrak{B}^{\mathbb{Q}}|| + |\alpha_*(\mathfrak{C})|$,
(h) $\underset{\sim}{\eta} \in {}^{\omega}\omega$ *is a generic for* $\mathbb{Q}$.

Then $\mathbb{P}$ *is super* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving.*

Proof. First note by 7.4(7) that $I^{\mathrm{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})} = I^{\mathrm{dx}}_{(\mathbb{Q},\underset{\sim}{\eta})}$. Now, if the conclusion fails in $\mathbf{V}$ as witnessed by a set X, then, by 7.9, the statements (A), (B), (C) of 7.9 fail in $\mathbf{V}$. Hence, by $\neg$(A), $X \in (I^{\mathrm{ex}}_{(\mathbb{Q},\underset{\sim}{\eta})})^+$ and by $\neg$(C) there are $p \in \mathbb{P}$ and a hereditarily countable canonical $\mathbb{P}$–name $\underset{\sim}{y}$ such that

$$p \Vdash_{\mathbb{P}} \text{“}\ \underset{\sim}{y} \in \mathcal{H}_{<\aleph_1}(\mathbb{Q}) \text{ and for no } \mathbb{Q}\text{-candidate } M \text{ to which } \underset{\sim}{y}, p \text{ belong there is } \nu \in X \text{ which is } (\mathbb{Q},\underset{\sim}{\eta})\text{–generic over } N[\underset{\sim}{G}_{\mathbb{P}}]\text{”.}$$

As ZFC^-_{**} is weakly normal we can find a model N of ZFC^-_{**} which is a $\mathbb{P}$–candidate and a $\mathbb{Q}$–candidate and to which $\underset{\sim}{y}$ and p belong. Let $\eta^* \in X \subseteq {}^{\omega}\omega$ (in $\mathbf{V}$) be $(\mathbb{Q},\underset{\sim}{\eta})$–generic over N (exists by the negation of (B) of 7.9). By (β) of 7.10 there is $q \in \mathbb{P}$ such that $p \le q$, q is $\langle N, \mathbb{P}\rangle$–generic and $q \Vdash_{\mathbb{P}}$"η^* is $(\mathbb{Q},\underset{\sim}{\eta})$–generic over $N[\underset{\sim}{G}_{\mathbb{P}}]$", a contradiction. $\blacksquare_{7.14}$

Conclusion 7.15. *Assume (a), (b), (c) of* 7.10. *Let* $\mathbb{P},\mathbb{Q}$ *be normal* K*–good for* $K = \{\mathbb{R} : \mathbb{R}$ *a forcing notion of cardinality* $\le \beth_{n(*)}(\kappa^*)\}$ *where* $\kappa^* = \theta^{\mathbb{P}} + \theta^{\mathbb{Q}} + ||\mathfrak{B}^{\mathbb{P}}|| + ||\mathfrak{B}^{\mathbb{Q}}|| + |\alpha_x(\mathfrak{C})|$ *for* $n(*)$ *large enough (3 suffices) and*

(c)* *if* χ *is large enough and* $M \prec (\mathcal{H}(\chi),\in)$ *is countable and* $\mathbb{P},\mathbb{Q},\underset{\sim}{\eta} \in M$ *and* $p \in \mathbb{P} \cap M$, $q \in \mathbb{Q} \cap M$ then *there are* p_1, η *such that* $p_1 \in \mathbb{P}$ *is* $(M,\mathbb{P})$*–generic,* η *is* $(\mathbb{Q},\underset{\sim}{\eta})$*–generic over* M *satisfying* q *and* $p_1 \Vdash$ "η *is* $(\mathbb{Q},\underset{\sim}{\eta})$*–generic over* $M[\underset{\sim}{G}_{\mathbb{P}}]$ *satisfying* q*".*

Then

$(\alpha)'$ $\mathbb{P}$ *is super* $I_{(\mathbb{Q},\underset{\sim}{\eta})}$*–preserving,*
(β) *for* χ *large enough, if* $N \prec (\mathcal{H}(\chi),\in)$ *is countable (and* $\mathfrak{C}, \mathfrak{B}^{\mathbb{Q}}, \mathfrak{B}^{\mathbb{P}}, \underset{\sim}{\eta} \in N$*) and* $N \models$ "$p \in \mathbb{P}$" *and* η^* *is* $(\mathbb{Q},\underset{\sim}{\eta})$*–generic over* N *then for some* q *we have*
  (i) $p \le q$, $q \in \mathbb{P}$,

(ii) *q is $\langle N, \mathbb{P}\rangle$–generic; i.e., $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{G}_{\mathbb{P}} \cap \mathbb{P}^N$ is generic over N",*
(iii) *$q \Vdash_{\mathbb{P}}$ "η^* is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $N[\mathbb{P}^N \cap G_{\mathbb{P}}]$".*

Proof. Let χ_1 be a large enough strong limit, and $\chi_2 = \beth_\omega(\chi_1)$, $\chi_3 = \beth_\omega(\chi_2)$, and repeat the proof of 7.10 (that is of clause $(\alpha)^+$) using $N \prec (\mathcal{H}(\chi_3), \in)$ to which $\mathfrak{C}, \mathfrak{B}^{\mathbb{Q}}, \mathfrak{B}^{\mathbb{P}}$ and $\mathbb{P}, \mathbb{Q}, \theta, \chi_1, \chi_2$ belong. $\blacksquare_{7.15}$

Observation 7.16. *Assume $\mathbb{Q}$ is strong c.c.c.–nep. and $\underset{\sim}{\eta}$ generic for $\mathbb{Q}$.* Then *in* 7.10 *we can omit assumption (c)$^+$ as it follows.*

Proof. Let M, p, q be given. Choose $p_1 \in \mathbb{P}$ which is $\langle M, \mathbb{P}\rangle$–generic and is above p, and let $G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$ such that $p_1 \in G$. In $\mathbf{V}[G]$, $M\langle G\rangle$ is a $\mathbb{P}$–candidate, hence by 7.5(4) the Borel set $B_1 = \{\nu \in {}^\omega\omega : \nu$ is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over $M\langle G_{\mathbb{P}} \cap \mathbb{P}^M\rangle\}$ satisfies ${}^\omega\omega \setminus B_1 \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$. Let q', $q \leq q' \in \mathbb{Q}^{\mathbf{V}}$ be $\langle M, \mathbb{Q}\rangle$–generic in $\mathbf{V}$, the Borel set $B_0 = \{\nu \in {}^\omega\omega : \nu$ is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic over M and is above $q'\}$ belongs to $I^+_{(\mathbb{Q},\underset{\sim}{\eta})}$. But as (by clause (c)) the forcing notion $\mathbb{P}$ is $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving, so $B_0^{\mathbf{V}} \notin I^{\rm ex}_{(\mathbb{Q},\underset{\sim}{\eta})}$ in $\mathbf{V}_1$. As ${}^\omega\omega \setminus B_1 \in I_{(\mathbb{Q},\underset{\sim}{\eta})}$ necessarily $B_0^{\mathbf{V}} \cap B_1 \neq \emptyset$, and choose $\eta^* \in B_0^{\mathbf{V}} \cap B_1$, lastly, some $p_1' \in G_{\mathbb{P}}$ which is above p_1 (hence p) forces this so we are done. $\blacksquare_{7.16}$

We can conclude (phrased for simplicity for strong-c.c.c. nep).

Conclusion 7.17. *Assume that*

(a) *$\mathbb{Q}$ is strong c.c.c. explicitly nep (see Definition* 6.12*) and simple and correct,*
(b) *$\underset{\sim}{\eta} \in {}^\omega\omega$ generic for $\mathbb{Q}$, a hereditarily countable $\mathbb{Q}$–name.*

If $\mathbb{P}_0$ is nep, $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving and $\Vdash_{\mathbb{P}_0}$ "$\underset{\sim}{\mathbb{P}}_1$ is nep, $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving", then $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}_1$ is (nep and) $I_{(\mathbb{Q},\underset{\sim}{\eta})}$–preserving. $\blacksquare$

The reader may ask: what about ω-limits (etc.)? We shall address these problems in the continuation [21] and [22].

The point is to combine the results here with, e.g., the proof say in [25, XVIII, §3]. The following example is characteristic for many cases, and is central in itself

Conclusion 7.18. *Assume $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ is a nep–CS–iteration, and forcing with $\underset{\sim}{\mathbb{Q}}_i$ does not make ${}^\omega 2$ null (forcing over $\mathbf{V}^{\mathbb{P}_i}$, but normally this is absolute). Then forcing with $\mathbb{P}_\alpha = \mathrm{Lim}(\bar{\mathbb{Q}})$ preserve non–nullity of any $X \subseteq {}^\omega 2$ (from $\mathbf{V}$).*

Proof. See [15]. $\blacksquare_{7.18}$

8. Non-symmetry

The following Hypothesis 8.1 will be assumed in this and the next section. Sometimes (including the main Theorems 9.11–9.15) we assume snep (i.e., 8.2). The FS iteration we use is from the end of Section 6.

Hypothesis 8.1. $\mathbb{Q}$ is correct c.c.c. simple, strongly c.c.c. nep, $\underset{\sim}{\eta}$ is a hereditarily countable name of a generic real for $\mathbb{Q}$, so $(\mathbb{Q}, \underset{\sim}{\eta}) \in \mathcal{K}$ (see Definition 6.2; by 6.5 the assumption that $\underset{\sim}{\eta}$ is generic real for $\mathbb{Q}$ is not a great loss) and ZFC^-_* and ZFC^- (and the properties above) are preserved by a forcing of cardinality $< \chi^*$, $|\mathbb{Q}|^{\aleph_0} < \chi^*$, for all the $\mathbb{Q}$–candidates we shall consider, e.g., χ^* is an individual constant.

Hypothesis 8.2. Like 8.1 with snep.

Definition 8.3. Let $\mathbb{Q}, \underset{\sim}{\eta}$ be as in 8.1 and let α be an ordinal.

1. Let $\mathbb{Q}^{[\alpha]}$ be $\mathbb{P}_\alpha$, where $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \alpha, j < \alpha \rangle$ is a FS iteration and $\underset{\sim}{\mathbb{Q}}_j = \mathbb{Q}^{\mathbf{V}[\mathbb{P}_j]}$.
2. We let $\underset{\sim}{\eta}^{[\alpha]}$ be $\langle \underset{\sim}{\eta}_i : i < \alpha \rangle$, where $\underset{\sim}{\eta}_i$ is $\underset{\sim}{\eta}$ "copied to $\underset{\sim}{\mathbb{Q}}_i$" (see 8.4(1) below).
3. $(\mathbb{Q}^{\langle \alpha \rangle}, \underset{\sim}{\eta}^{\langle \alpha \rangle})$ is defined similarly as a FS product.
4. For a finite set $u \subseteq \alpha$ with $n < \omega$ elements we define the function $F = F^{\alpha,u}_{\mathbb{Q}}$ from $\mathbb{Q}^{[n]}$ into $\mathbb{Q}^{\langle \alpha \rangle}$ by induction on n naturally.
5. The FS iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j, \underset{\sim}{\eta}_j : i \leq \alpha, j < i \rangle$ of nep means $(\underset{\sim}{\mathbb{Q}}_j, \underset{\sim}{\eta}_j) \in \mathcal{K}$.

Proposition 8.4. 1. *In Definition* 8.3(4)*, for finite* $u \subseteq \alpha$*,* $F = F^{\alpha,u}_{\mathbb{Q}}$ *is a complete* ($\lessdot$*–) embedding, as "*$p \leq q$*", "*p, q *compatible", "*p, q *incompatible", "*$\langle p_n : n < \omega \rangle$ *is predense set above* q*" are upward absolute from* $\mathbb{Q}$*–candidates (holds as* $\mathbb{Q}$ *is strongly c.c.c. by* 8.1*). So* $\underset{\sim}{\eta}_\alpha$ *is* $F^{\alpha,\{\alpha\}}_{\mathbb{Q}}(\underset{\sim}{\eta})$ *if* $\alpha \in u$*.*

2. $\mathbb{Q}^{[\alpha]}$ *satisfies the c.c.c.*
3. *The same holds for* $\mathbb{Q}^{\langle \alpha \rangle}$*.*
4. $(\mathbb{Q}^{[\alpha]}, \underset{\sim}{\eta}^{[\alpha]})$ *for* $\alpha < \omega_1$ *are as in* 8.1*, too.*
5. *If "*$\underset{\sim}{\eta}$ *is generic for* $\mathbb{Q}$*" is absolute,* then $\underset{\sim}{\eta}^{[\alpha]}$ *is generic for* $\mathbb{Q}^{[\alpha]}$ *(and* $\underset{\sim}{\eta}^{\langle \alpha \rangle}$ *is generic for* $\mathbb{Q}^{\langle \alpha \rangle}$*).*

6. *Similarly for non constant sequence of such (definitions) of forcings (so we have $F^{\alpha,u}_{\bar{\mathbb{Q}}}$).*

Proof. For example:

3) It is enough to prove it for finite α, and this we prove by induction on α for $\alpha = n+1$. For the c.c.c. use "incompatibility is upward absolute" for forcing by $\mathbb{Q}^{\langle n\rangle}$, so we can use the last phrase in 8.1.

4) The main point here is the strong c.c.c.-nep, so let N be a $\mathbb{Q}$-candidate (and $\alpha+1 \subseteq N$) and

$$N \models \text{“ } \mathcal{I} \subseteq \mathbb{Q}^{[\alpha]} \text{ is predense ”}.$$

Let $G^{[\alpha]} \subseteq \mathbb{Q}^{[\alpha]}$ be generic over $\mathbf{V}$ and for $\beta \leq \alpha$, $G^{[\beta]} = G^{[\alpha]} \cap \mathbb{Q}^{[\beta]}$. Show by induction on β that $G^{[\beta]} \cap (\mathbb{Q}^{[\beta]})^N$ is a generic subset of $(\mathbb{Q}^{[\beta]})^N$ over $N\langle G^{[\beta]}\rangle$. This is clearly enough. $\blacksquare_{8.4}$

Definition 8.5. 1. We say that $\mathbb{Q}$ is $[n]$–symmetric whenever:
if $\langle \eta^*_\ell : \ell < n\rangle$ is generic for $\mathbb{Q}^{[n]} = \langle \mathbb{P}_\ell, \underset{\sim}{\mathbb{Q}}_\ell, \underset{\sim}{\eta}_\ell : \ell < n\rangle$ and π is a permutation of $\{0,\dots,n-1\}$,
then $\langle \underset{\sim}{\eta}_{\pi(\ell)} : \ell < n\rangle$ is generic for $\langle \mathbb{P}_\ell, \underset{\sim}{\mathbb{Q}}_\ell, \underset{\sim}{\eta}_\ell : \ell < n\rangle$.

2. If $(\mathbb{Q}', \underset{\sim}{\eta}')$, $(\mathbb{Q}'', \underset{\sim}{\eta}'')$ are as in 8.1, we say that they commute whenever: *if* r' is $(\mathbb{Q}', \underset{\sim}{\eta}')$–generic over $\mathbf{V}$ and r'' is $(\mathbb{Q}'', \underset{\sim}{\eta}'')$–generic over $\mathbf{V}[r']$, *then* r' is $(\mathbb{Q}', \underset{\sim}{\eta}')$–generic over $\mathbf{V}[r'']$.
(Note that η'' is $(\mathbb{Q}'', \underset{\sim}{\eta}'')$–generic over $\mathbf{V}$ is always true by 6.6.)
3. For $(\mathbb{Q}', \underset{\sim}{\eta}')$, $(\mathbb{Q}'', \underset{\sim}{\eta}'')$ we say that they weakly commute if $(\mathbb{Q}' \restriction (\geq q'), \underset{\sim}{\eta}')$, $(\mathbb{Q}'' \restriction (\geq q''), \underset{\sim}{\eta}'')$ commute for some $q' \in \mathbb{Q}'$ and $q'' \in \mathbb{Q}''$.
4. Similarly for a set or sequence of such pairs.

Proposition 8.6. 1. *"Commute" is a commutative relation.*
2. *For $n \geq 2$ we have:*
$\mathbb{Q}$ *is* $[n]$*–symmetric* *iff*
$\mathbb{Q}, \mathbb{Q}^{[n-1]}$ *commute and* $\mathbb{Q}$ *is* $[n-1]$*–symmetric* *iff*
$\mathbb{Q}$ *is* $[2]$*-symmetric.*
3. *If* $\mathbb{P}, \mathbb{Q}^{[n]}$ *commute,* $m \leq n$, then $\mathbb{P}, \mathbb{Q}^{[m]}$ *commute. Similarly, if* $\mathbb{P}, \mathbb{Q}$ *commute and* $\mathbb{Q}' \lessdot \mathbb{Q}$, then $\mathbb{P}, \mathbb{Q}'$ *commute.*
4. *In part* 3) *we can replace* $[-]$ *by* $\langle - \rangle$.
5. *If* $(\mathbb{Q}', \underset{\sim}{\eta}')$, $(\mathbb{Q}'', \underset{\sim}{\eta}'')$ *weakly commute and* $\mathbb{Q}', \mathbb{Q}''$ *are strongly homogeneous, then they commute.*
6. $\langle (\mathbb{Q}_i, \underset{\sim}{\eta}_i) : i < i^*\rangle$ *commute* iff *for every* $i < j < i^*$ *the pair* $(\mathbb{Q}_i, \eta_i)$, $(\mathbb{Q}_j, \underset{\sim}{\eta}_j)$ *commutes.*
7. *Similarly for a sequence* $\langle \mathbb{Q}_\ell : \ell < n\rangle$, $\langle (\mathbb{Q}_\ell, \underset{\sim}{\eta}_\ell) : \ell < \omega\rangle$.

Proof. 1) Let $(\mathbb{Q}', \underset{\sim}{\eta}')$, $(\mathbb{Q}'', \underset{\sim}{\eta}'')$ be as in 8.1. Then "$(\mathbb{Q}', \underset{\sim}{\eta}')$, $(\mathbb{Q}'', \underset{\sim}{\eta}'')$ commute" says $\mathbb{Q}' * \underset{\sim}{\mathbb{Q}}'' = \mathbb{Q}'' * \underset{\sim}{\mathbb{Q}}'$, which is symmetric.

2) For the second "iff", use "the permutations $\pi_\ell = (\ell, \ell+1)$ for $\ell < n$ generate the group of permutations of $\{0, \dots, n-1\}$".

3)–7) Left to the reader. $\blacksquare_{8.6}$

Proposition 8.7. 1. *If* $\mathbb{Q}^{[\omega]}$ *and* Cohen *do not commute,* then *for some* $n < \omega$, $\mathbb{Q}^{[n]}$ *and* Cohen *do not commute.*
(The inverse holds by 8.6(3)*, second phrase.)*

2. *If* $\mathbb{Q}^{\langle\omega\rangle}$ *and* Cohen *do not commute,* then *for some* $n < \omega$, $\mathbb{Q}^{\langle n\rangle}$ *and* Cohen *do not commute.*

Proof. 1) Since Cohen and $\mathbb{Q}^{[\omega]}$ do not commute, there is a $\mathbb{Q}^{[\omega]}$–name $\underset{\sim}{\mathcal{I}}$ of a dense open subset of Cohen (i.e. of $({}^{\omega>}2, \triangleleft)$) such that for some condition $(p, \underset{\sim}{q}) \in \text{Cohen} * \underset{\sim}{\mathbb{Q}}^{[\omega]}$ we have

$$(p, \underset{\sim}{q}) \Vdash \text{“ } \underset{\sim}{\eta}^{\text{Cohen}} \text{ has no initial segment in } \underset{\sim}{\mathcal{I}} \text{ ”}.$$

Without loss of generality (possibly increasing p) for some $n^* < \omega$ we have

$$p \Vdash_{\text{Cohen}} \text{“ Dom}(\underset{\sim}{q}) \subseteq \{0, \dots, n^*-1\} \text{ ”}.$$

Let $\underset{\sim}{\mathcal{I}}'$ be the $\mathbb{Q}^{[n^*]}$–name for the following set:

$$\{\eta \in {}^{\omega>}2 : \text{for some } p' \in \mathbb{Q}^{[\omega]},\ p' \restriction n^* \in \underset{\sim}{G}_{\mathbb{Q}^{[n^*]}} \text{ and } p' \Vdash_{\mathbb{Q}^{[\omega]}} \text{“}\eta \in \underset{\sim}{\mathcal{I}}\text{”}\}.$$

It should be clear that $\Vdash_{\mathbb{Q}^{[n^*]}}$ "$\underset{\sim}{\mathcal{I}}'$ is a dense open subset of $({}^{\omega>}2, \triangleleft)$". We interprate $\mathcal{I}$ also as a (Cohen $*\underset{\sim}{\mathbb{Q}}^{[n]}$)–name naturally. Now we ask the following question.

$$\text{Does } (p, \underset{\sim}{q}) \Vdash_{\text{Cohen}*\underset{\sim}{\mathbb{Q}}^{[n^*]}} \text{“ } \underset{\sim}{\eta}^{\text{Cohen}} \restriction n \notin \underset{\sim}{\mathcal{I}}' \text{ for each } n < \omega \text{ ”?}$$

If yes, we have gotten the desired conclusion (i.e. Cohen and $\underset{\sim}{\mathbb{Q}}^{[n^*]}$ do not commute). If not, for some $(p', \underset{\sim}{q}')$ such that $(p, \underset{\sim}{q}) \leq (p', \underset{\sim}{q}') \in \text{Cohen} * \underset{\sim}{\mathbb{Q}}^{[n^*]}$ and for some $n < \omega$ and $\nu \in {}^n\omega$ we have:

$$(p', \underset{\sim}{q}') \Vdash_{\text{Cohen}*\underset{\sim}{\mathbb{Q}}^{[n^*]}} \text{“ } \underset{\sim}{\eta}^{\text{Cohen}} \restriction n = \nu \in \underset{\sim}{\mathcal{I}}' \text{ ”}.$$

So (by the definition of $\mathcal{I}'$) for some $(p'', \underset{\sim}{q}'') \in \text{Cohen} * \underset{\sim}{\mathbb{Q}}^{[n]}$ above $(p', \underset{\sim}{q}')$ for some $r \in (\mathbb{Q}^{[\omega]})^{\mathbf{V}}$ we have $(p'', \underset{\sim}{q}'') \Vdash$"$(\emptyset, r \restriction n^*) \in \underset{\sim}{G}_{\text{Cohen}*\underset{\sim}{\mathbb{Q}}^{[n^*]}}$" and $r \Vdash_{\mathbb{Q}^{[\omega]}}$ "$\nu \in \underset{\sim}{\mathcal{I}}$". So wlog $p'' \Vdash \mathbb{Q}^{[\omega]} \models [r \restriction n^* \leq \underset{\sim}{q}'']$. Then $(p'', \underset{\sim}{q}'' \cup r \restriction [n^*, \omega))$ forces (in Cohen $* \underset{\sim}{\mathbb{Q}}^{[\omega]}$) that $\underset{\sim}{\eta}^{\text{Cohen}} \restriction n = \nu \in \underset{\sim}{\mathcal{I}}$, a contradiction.

2) Similar to part (1), just easier (replacing $\mathbb{Q}^{[n]}, \mathbb{Q}^{[\omega]}$ by $\mathbb{Q}^{\langle n\rangle}, \mathbb{Q}^{\langle\omega\rangle]}$ and $\underset{\sim}{q}$ by q).

$\blacksquare_{8.7}$

Proposition 8.8. 1. *If* $\mathbb{Q}^{[n]}$ *and* Cohen *do not commute* ($\mathbb{Q}$ *as before*), *then* $\mathbb{Q}$ *and* Cohen *do not commute in* $\mathbf{V}^{\mathbb{Q}^{[m]}}$ *for some* $m \le n$ *(we get "do not commute in* $\mathbf{V}$*" if both "absolutely").*

2. *The following conditions (for nep forcing notion* $\mathbb{Q}$ *as in* 7.10*) are equivalent:*
   (i) $\mathbb{Q}$ *commutes with Cohen,*
   (ii) $\Vdash_{\mathbb{Q}}$ "$({}^{\omega}2)^{\mathbf{V}}$ *is not meagre",*
   (iii) $(\forall A)[\mathbf{V} \models$ "$A \subseteq {}^{\omega}2$ *non-meagre"* $\Rightarrow \Vdash_{\mathbb{Q}}$ "A *is non-meagre"*$]$
   (all three clauses are interprated "absolutely", i.e., not only in the present universe but in its generic extensions too, for all set forcing or just for large enough K*).*
3. *In part* (2) *we can replace* Cohen *by others to which* 7.10 *applies and are homogeneous (see* 7.5*).*

Proof. 1) Assume toward contradiction that $\mathbb{Q}$ and Cohen commute when forcing over $\mathbf{V}^{\mathbb{Q}^{[m]}}$ for every $m < n$ (i.e., the conclusion fails). Let $\eta \in {}^{\omega}2$ be a Cohen real over $\mathbf{V}$. Let $G_\ell \subseteq \mathbb{Q}^{\mathbf{V}[G_0,\dots,G_{\ell-1},\eta]}$ be generic over $\mathbf{V}[G_0,\dots,G_{\ell-1},\eta]$ for $\ell < n$, and let $\eta_\ell = \underset{\sim}{\eta}[G_\ell]$. We now prove by induction on ℓ, that η is a Cohen real over $\mathbf{V}[G_0,\dots,G_{\ell-1}]$. The induction step is by the assumption "$\mathbb{Q}$ and Cohen commute in $\mathbf{V}^{\mathbb{Q}^{[m]}}$". The net result is that η is a Cohen real over $\mathbf{V}[\eta_0,\dots,\eta_{n-1}]$, contradicting the assumption.

2) The second clause implies the third by 7.10. The third clause implies the second trivially.

Let us argue that the implication $(i) \Rightarrow (ii)$ holds.
Add $\aleph_1$ Cohen reals $\{\eta_i : i < \omega_1\}$ and then force by $\mathbb{Q}$, i.e., let $G_{\mathbb{Q}} \subseteq \mathbb{Q}^{\mathbf{V}[\langle \eta_i : i<\omega_1\rangle]}$ be generic over $\mathbf{V}[\langle \eta_i : i < \omega_1\rangle]$, and $\eta = \underset{\sim}{\eta}_{\mathbb{Q}}[G_{\mathbb{Q}}]$. Then (i) implies that for every $j < \omega_1$ we have: η_j is Cohen over $\mathbf{V}[\langle \eta_i : i < \omega_1, i \neq j\rangle, \eta]$. Hence in $\mathbf{V}[\langle \eta_i : i < \omega_1\rangle, \eta] = \mathbf{V}[\langle \eta_i : i < \omega_1\rangle][G_{\mathbb{Q}}]$, the set $\{\eta_i : i < \omega_1\}$ is not meagre and consequently (ii) holds.

Lastly, assume (ii) and let $\nu \in {}^{\omega}||\mathbb{Q}||$ be generic for Levy$(\aleph_0, ||\mathbb{Q}||)$. Let η be $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over $\mathbf{V}[\nu]$. By (ii), we can find in $\mathbf{V}[\nu]$ a real $\rho \in {}^{\omega}2$ which is in no meagre Borel set from $\mathbf{V}[\eta]$ (note that there are countably many such meagre sets from the point of view of $\mathbf{V}[\nu]$). Now we easily finish.

3) Same proof. $\blacksquare_{8.8}$

9. Poor Cohen commutes only with himself

Definition 9.1. 1. We say a $\mathbb{Q}$–name $\underset{\sim}{x}$ of a subset of some countable $a^* \in \mathbf{V}$ is [somewhere] essentially Cohen if $B_2(\mathbb{Q}, \underset{\sim}{x})$ is [somewhere] essentially countable; i.e., [above some p] has countable density; on $B_2(\mathbb{Q}, \underset{\sim}{x})$ see 6.5.

2. We say $(\mathbb{Q}, \underset{\sim}{\eta}) \in \mathcal{K}^{\neg c}$ (a non-Cohen pair) if:
   (a) $(\mathbb{Q}, \underset{\sim}{\eta})$ is as in 8.2,
   (b) $(\mathbb{Q}, \underset{\sim}{\eta})$ (see Definition 6.5) is nowhere essentially Cohen (i.e. above every condition).

Hypothesis 9.2. In addition to Hypothesis 8.2 χ is regular large enough cardinal, and $(\mathbb{Q}, \underset{\sim}{\eta}) \in \mathcal{K}^{\neg c}$ will be fixed as in 9.1, and ZFC^-_* is normal (see Definition 1.15).

Definition 9.3. 1. $\mathcal{D} = \mathcal{D}_{\leq\aleph_0}(\mathcal{H}(\chi))$ is the filter of clubs on $[\mathcal{H}(\chi)]^{\leq\aleph_0}$.
2. $\mathcal{C}_0 = \{a : a \prec (\mathcal{H}(\chi), \in)$ is countable, and $(\mathbb{Q}, \underset{\sim}{\eta}) \in a$ (i.e. their definitions) so a, i.e., $(a, \in)$ is a $\mathbb{Q}$–candidate$\}$.

Definition 9.4. We say that $q \in \mathbb{Q}$ is influential on $a \in \mathcal{C}_0$ if:

$(\circledast)_{a,q}$ the set $\{p \in a \cap \mathbb{Q} : p, q$ are incompatible in $\mathbb{Q}\}$ is dense in the (quasi) order $\mathbb{Q} \restriction a$.

Proposition 9.5. 1. *For every $a \in \mathcal{C}_0$ there is $q \in \mathbb{Q}$ which is influential on a.*
2. *Moreover, for every $p \in \mathbb{Q}$ and $a \in \mathcal{C}_0$ there is q influential on a such that $p \leq_{\mathbb{Q}} q$.*

Proof. 1) Follows by 2).

2) Let p, a be given. Clearly $\underset{\sim}{G}_{\mathbb{Q}} \cap a$ is a $\mathbb{Q}$–name of a countable subset of an old set $\mathbb{Q} \cap a$, so it can be considered as a real. We restrict ourselves to be above our fix $p \in \mathbb{Q}$. Note that

$(*)_1$ $\underset{\sim}{G}_{\mathbb{Q}} \cap a$ is not somewhere essentially Cohen.

[Why? Toward contradiction assume that this fail, say above q, $q \geq p$. From $\underset{\sim}{G}_{\mathbb{Q}} \cap a$ we can compute $\underset{\sim}{\eta}$ (as $\underset{\sim}{\eta} \in a$, i.e., the relevant (countably many, countable) maximal antichains belong to a), so $\underset{\sim}{\eta}$ can be considered a $B_2[\mathbb{Q}, \underset{\sim}{G}_{\mathbb{Q}} \cap a]$–name. But "any (name of) a real in an essentially Cohen forcing notion is essentially Cohen itself", so $\underset{\sim}{\eta}$ is essentially Cohen $\mathbb{Q}$–name, contradicting Hypothesis 9.2.]

Consequently, $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{G}_{\mathbb{Q}} \cap a$ is not a generic subset of $\mathbb{Q} \restriction a$ (over $\mathbf{V}$)". Thus there are q and $\mathcal{I}$ such that:

(i) $p \leq q \in \mathbb{Q}$,
(ii) $\mathcal{I} \subseteq \mathbb{Q} \cap a$ is a dense open subset of $\mathbb{Q} \restriction a$,
(iii) $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{G}_{\mathbb{Q}}$ is disjoint to $\mathcal{I}$".

But this means that

$(*)_2$ q is incompatible with every $r \in \mathcal{I}$.
[Why? Otherwise $q \nVdash_{\mathbb{Q}}$ "$r \notin \underset{\sim}{G}_{\mathbb{Q}}$".]
So $\{r \in a \cap \mathbb{Q} : q, r \text{ incompatible (in } \mathbb{Q})\}$ is a subset of $\mathbb{Q} \cap a$ including $\mathcal{I}$ hence it is dense in $\mathbb{Q} \restriction a$. $\blacksquare_{9.5}$

Choice 9.6. We choose $\bar{p} = \langle p_a : a \in \mathcal{C}_0 \rangle$ such that $p_a \in \mathbb{Q}$ is influential on a (possible by 9.5).

Definition 9.7.
1. For $R \subseteq \mathbb{Q}$ let $A[R] \stackrel{\text{def}}{=} \{a \in \mathcal{C}_0 : p_a \in R\}$.
2. $\mathcal{D}_{\bar{p}} = \mathcal{D}_{\mathbb{Q},\bar{p}} \stackrel{\text{def}}{=} \{R \subseteq \mathbb{Q} : A[R] \in \mathcal{D}\}$.
   The family of $\mathcal{D}_{\bar{p}}$–positive sets will be denoted $\mathcal{D}^+_{\bar{p}}$ (so for a set $S \subseteq \mathbb{Q}$, we have $S \in \mathcal{D}^+_{\bar{p}}$ iff $R \cap S \neq \emptyset$ for each $R \in \mathcal{D}_{\bar{p}}$).
3. For $R \subseteq \mathbb{Q}$ and $q \in \mathbb{Q}$ let $R[q] \stackrel{\text{def}}{=} \{p \in R : p, q \text{ are incompatible in } \mathbb{Q}\}$ (so $R[q]$ is in a sense the orthogonal complement of q inside R).

Fact 9.8.
1. *$\mathcal{D}_{\bar{p}}$ is an $\aleph_1$–complete filter on $\mathbb{Q}$.*
2. *For $R \subseteq \mathbb{Q}$ we have $R \in \mathcal{D}^+_{\bar{p}} \iff A[R] \in \mathcal{D}^+$.*

Proof. Immediate as $a \in \mathcal{C}_0 \Rightarrow p_a \notin a$ and for some $\mathcal{C}_1 \in \mathcal{D}$, $\mathcal{C}_1 \subseteq \mathcal{C}_0$ and $\langle p_a : a \in \mathcal{C}_1 \rangle$ is without repetitions. $\blacksquare$

Proposition 9.9. *If* $R \in \mathcal{D}^+_{\bar{p}}$ then *the set*

$$R^{\otimes} \stackrel{\text{def}}{=} \{q \in \mathbb{Q} : R[q] \in \mathcal{D}^+_{\bar{p}}\}$$

is dense in $\mathbb{Q}$.

Proof. Assume not, so for some $q^* \in \mathbb{Q}$ we have
$(*)_1$ there is no $q \in \mathbb{Q}$ such that $q^* \leq q \in \mathbb{Q}$ & $R[q] \in \mathcal{D}^+_{\bar{p}}$.
Thus

$$\begin{aligned} q^* \leq q \in \mathbb{Q} \quad &\Rightarrow \quad R[q] = \emptyset \bmod \mathcal{D}_{\bar{p}} \quad \Rightarrow \quad A[R[q]] = \emptyset \bmod \mathcal{D} \\ &\Rightarrow \quad \text{for some club } \mathcal{C}_q \subseteq \mathcal{C}_0 \text{ of } [\mathcal{H}(\chi)]^{\leq\aleph_0} \text{ we have } (\forall a \in \mathcal{C}_q)(q \in a) \\ &\qquad \text{and } (\forall a \in \mathcal{C}_q)[p_a \notin R[q], \text{ i.e., } p_a, q \text{ are compatible}]. \end{aligned}$$

Let $\mathcal{C}^* = \{a \in \mathcal{C}_0 : q^* \in a \text{ and } (\forall q)[q^* \leq q \in a \cap \mathbb{Q} \;\Rightarrow\; a \in \mathcal{C}_q]\}$. As each $\mathcal{C}_q$ is a club of $[\mathcal{H}(\chi)]^{\leq\aleph_0}$ clearly $\mathcal{C}^*$ (as a diagonal intersection) is a club of $[\mathcal{H}(\chi)]^{\leq\aleph_0}$, i.e., $\mathcal{C}^* \in \mathcal{D}$. Since $R \in \mathcal{D}^+_{\bar{p}}$ by 9.8 we have $A[R] \in \mathcal{D}^+$, so together with the previous sentence we know that there is $a^* \in A[R] \cap \mathcal{C}^*$. By the choice of $\bar{p}$ (see 9.6, and Definition 9.4) as $q^* \in a^* \cap \mathbb{Q}$ (see the choice of $\mathcal{C}^*$) for some q we have:

$$q^* \leq q \in a^* \quad \text{and} \quad p_{a^*}, q \text{ are incompatible.}$$

Now this contradicts "$a^* \in \mathcal{C}_q$". $\blacksquare_{9.9}$

Definition 9.10. Assume $\chi_1 = (2^\chi)^+$ (so $\mathcal{H}(\chi) \in \mathcal{H}(\chi_1)$) and N is a countable elementary submodel of $(\mathcal{H}(\chi_1), \in)$ to which $\{\chi, \mathbb{Q}, \bar{p}\}$ belong (so $\mathcal{D}_{\bar{p}} \in N$).

1. We let $\mathrm{Cohen}_N = \mathrm{Cohen}_{N,\mathbb{Q}}$ be $(\mathcal{D}^+_{\mathbb{Q},\bar{p}}, \supseteq) \restriction N$ (so this is a countable atomless forcing notion and hence equivalent to Cohen forcing).
2. If $G_N \in \mathrm{Gen}(N, \mathcal{D}^+_{\mathbb{Q},\bar{p}}) \stackrel{\text{def}}{=} \{G : G \subseteq \mathrm{Cohen}_N$ is generic over N for the partial order $((\mathcal{D}^+_{\mathbb{Q},\bar{p}}, \supseteq) \restriction N)\}$ (possibly in a universe $\mathbf{V}'$ extending $\mathbf{V}$), *then* let $\underset{\sim}{p}_N[G]$ be the sequence (i.e., in ${}^\omega\omega$ or just member of ${}^\omega\theta(\mathbb{Q})$) such that for each $\ell < \omega$ and γ
$$(\underset{\sim}{p}_N[G_N])(\ell) = \gamma \quad \Leftrightarrow \quad (\exists R \in G)(\forall p \in R)[p(\ell) = \gamma].$$

Proposition 9.11. *Assume (9.2 and, additionally), $\mathbb{Q}$ is Souslin c.c.c. (i.e., the incompatibility relation is $\mathbf{\Sigma}^1_1$)*[36]*. If χ_1, N and $G \in \mathrm{Gen}(N, \mathcal{D}^+_{\mathbb{Q}})$ are as in 9.10 (so G is possibly in some generic extension $\mathbf{V}_1$ of $\mathbf{V}$ but* Cohen_N *is from $\mathbf{V}$),* then

(a) $\underset{\sim}{p}_N[G]$ *is an ω–sequence (i.e. for each ℓ there is one and only one γ),*
(b) $\underset{\sim}{p}_N[G] \in \mathbb{Q}$*,*
(c) $\underset{\sim}{p}_N[G]$ *is influential on N (which belongs to $\mathcal{C}_0$).*

Proof. For every $p \in \mathbb{Q}$ there is $\nu_p \in {}^\omega\omega$ which witnesses $p \in \mathbb{Q}$, i.e., $p * \nu_p \in \lim(T_0^{\mathbb{Q}})$. So choose such a function $p \mapsto \nu_p$. Now in $\mathbf{V}$, for $n < \omega$ the function $p_a \mapsto (p_a \restriction n, \nu_{p_a} \restriction n)$ is a mapping from $\{p_a : a \in \mathcal{C}_0\} \in \mathcal{D}_{\bar{p}}$ with countable range. Since $\mathcal{D}_{\bar{p}}$ is $\aleph_1$–complete

$(*)_1$ in $\mathbf{V}$, if $R \in \mathcal{D}^+_{\bar{p}}$ and $n < \omega$ then for some $R' \subseteq R$ and (η^n, ν^n) we have
$$R' \in \mathcal{D}^+_{\bar{p}} \quad \text{and} \quad (\forall p \in R')[(p \restriction n, \nu_p \restriction n) = (\eta^n, \nu^n)].$$

This is inherited by N, so wlog the function $p \mapsto \nu_p$ belongs to N, hence $\underset{\sim}{p}_N[G]$ satisfies clauses (a), (b) (in fact
$$\underset{\sim}{\nu}[G] = \bigcup\{\nu^* : \text{ for some } n < \omega \text{ and } R \in G \text{ we have } (\forall p \in R)[\nu_p \restriction n = \nu^*]\}$$
is a witness for $\underset{\sim}{p}_N[G] \in \mathbb{Q}$). Also for each $q \in \mathbb{Q} \cap N$ the set
$$\mathcal{J}_q = \big\{R \in \mathcal{D}^+_{\bar{p}} : \begin{array}{l} \text{for some } q' \in \mathbb{Q} \text{ stronger than } q \text{ we have:} \\ (\forall p \in R)[p, q' \text{ are incompatible (in } \mathbb{Q})]\big\} \end{array}$$

[36] If we have more absolutness, we can omit this.

is a dense subset of $(\mathcal{D}^+_{\bar{p}}, \supseteq)$ (remember p_a is influential on a; use normality of the filter $\mathcal{D}$). Clearly $\mathcal{J}_q$ belongs to N, so by the demand on G we know that $G \cap \mathcal{J}_q \neq \emptyset$. Choose $R_q \in G \cap \mathcal{J}_q$ and let $q' \in \mathbb{Q} \cap N$ witness it, so

$$R_q \in \mathcal{D}^+_{\bar{p}} \cap N \quad \text{ and } \quad (\forall p \in R_q)[p, q' \text{ are incompatible}].$$

Now "incompatible in $\mathbb{Q}$" is a $\mathbf{\Sigma}^1_1$–relation (belonging to N) hence as above, $\underset{\sim}{p}_N[G], q'$ are incompatible. As q was any member of $\mathbb{Q} \cap N$ we have finished proving clause (c). $\blacksquare_{9.11}$

Proposition 9.12. *Assume* 9.2 *and let* $\mathbb{Q}$ *be Souslin c.c.c.* Then $\mathbb{Q}^{[\omega]}$ *(see* 8.3*) and* Cohen *do not commute.*

Proof. Assume that $\mathbb{Q}^{[\omega]}$ and Cohen do commute. Let χ be large enough, $N \prec (\mathcal{H}(\chi), \in)$ be countable such that $(\mathbb{Q}, \underset{\sim}{\eta}) \in N$ (as in 9.10 and we whall use freely 9.6, 9.7, 9.10). Now we can interpret a Cohen real ν (over $\mathbf{V}$) as a subset of $\mathcal{D}^+_{\bar{p}} \cap N$ called g_ν. Thus it is $\text{Cohen}_{N,\mathbb{Q}}$–generic over $\mathbf{V}$ so $\underset{\sim}{p}_N[g_\nu]$ is well defined, and it belongs to $\mathbb{Q}^{\mathbf{V}[\nu]}$ (by 9.11). Moreover, in $\mathbf{V}[\nu]$ we have:

$$\{q \in \mathbb{Q}^N : q \text{ and } \underset{\sim}{p}_N(g_\eta) \text{ are incompatible in } \mathbb{Q}\ \} \text{ is dense in } \mathbb{Q}^N.$$

Let $\langle \eta_\ell : \ell < \omega \rangle$ be generic for $(\mathbb{Q}^{[\omega]}, \underset{\sim}{\eta}^{[\omega]})$ over $\mathbf{V}$ and let ν be Cohen generic over $\mathbf{V}[\langle \eta_\ell : \ell < \omega \rangle]$. For each ℓ, clearly η_ℓ is $(\mathbb{Q}, \underset{\sim}{\eta})$–generic also over $\mathbf{V}$, so let $\eta_\ell = \underset{\sim}{\eta}[G_\ell]$, where $G_\ell \subseteq \mathbb{Q}$ is the unique such generic set over $\mathbf{V}$. Clearly $G_\ell \cap N$ is a subset of $\mathbb{Q}^N$ generic over $\mathbf{V}$ (by "$\mathbb{Q}$ is strongly c.c.c."). So $\langle G_\ell \cap N, g_\nu \rangle$ is a subset of $\mathbb{Q}^N \times (\mathcal{D}^+_{\bar{p}} \cap N, \supseteq)$ generic over N. By 9.11, for any $q \in \mathbb{Q}^N$ and $R \in (\mathcal{D}^+_{\bar{p}} \cap N)$, for some $R' \subseteq R$ and q' we have $R' \in (\mathcal{D}^+_{\bar{p}} \cap N)$, $N \models$"$q \leq q' \in \mathbb{Q}$" and

$$N \models (\forall a \in R')(p_a, q' \text{ are incompatible}).$$

So look at the set

$$\mathcal{I} \stackrel{\text{def}}{=} \{(q, R) \in \mathbb{Q}^N \times (\mathcal{D}^+_{\bar{p}} \cap N) : (\forall a \in R)(p_a, q \text{ are incompatible})\}.$$

By the previous sentence, this is a dense subset of $\mathbb{Q}^N \times (\mathcal{D}^+_{\bar{p}} \cap N, \supseteq)$. Hence there is $(q, R) \in (G_\ell \cap N) \times g_\nu$ which belongs to it. Hence, as in 9.11, for each ℓ, $\underset{\sim}{p}_N[g_\nu]$ is incompatible with some $q \in G_\ell$.

By the assumption that the forcing notions commute we know that $\langle \eta_\ell : \ell < \omega \rangle$ is generic for $(\mathbb{Q}^{[\omega]}, \underset{\sim}{\eta}^{[\omega]})$ over $\mathbf{V}[\nu]$. Necessarily (by FS + genericity) for some ℓ we have $F^{\omega,\{\ell\}}_{\mathbb{Q}}(\underset{\sim}{p}_N[g_\eta]) \in G_{\mathbb{Q}}[\langle \eta_\ell : \ell < \omega \rangle]$; a contradiction to the previous paragraph.

$\blacksquare_{9.12}$

Conclusion 9.13. *Assume* 9.2 *and let* $\mathbb{Q}$ *be Souslin c.c.c.* Then $(\mathbb{Q}, \underset{\sim}{\eta})$ *does not commute with* Cohen *but possibly only in some generic extensions of* $\mathbf{V}$ *(by* $\mathbb{Q}^{[n]}$*) (even above any* $q \in \mathbb{Q}$*).*

Proof. If we restrict ourselves above $q_0 \in \mathbb{Q}$, the Hypothesis 9.2 still holds so we can ignore this. By 9.12 we have $(\mathbb{Q}^{[\omega]}, \underset{\sim}{\eta}^{[\omega]})$ does not commute with Cohen. So by 8.7 we have that, for some n, $(\mathbb{Q}^{[n]}, \underset{\sim}{\eta}^{[n]})$ does not commute with Cohen and by 8.8 we finish. $\blacksquare_{9.13}$

Proposition 9.14. *If* $\mathbb{Q}$ *is Souslin c.c.c.,* $(\mathbb{Q}, \underset{\sim}{\eta}) \in \mathcal{K}^{\neg c}$*,* then $\mathbb{Q}$ *satisfies* 9.2 *for suitable* ZFC^-_**.*

Proof. Let $\rho \in {}^{\omega}2$ be the real parameter in the definition of $\mathbb{Q}$. Let ZFC^-_* say:

(a) ZC (i.e. the axioms of Zermelo satisfied by $(\mathcal{H}(\beth_\omega), \in)$),
(b) $\mathbb{Q}$ (defined from ρ which is an individual constant) satisfies the c.c.c.,
(c) for each $n < \omega$, generic extensions for forcing notions of cardinality $< \beth_\omega$ preserve (b) (and, of course (a)).

The "strong" comes for "being a maximal antichain is absolute from $\mathbb{Q}$–candidates". Now the desired properties are easy. $\blacksquare_{9.14}$

Conclusion 9.15. *If* $\mathbb{Q}$ *is a Souslin c.c.c. forcing notion which is not* ${}^{\omega}\omega$*–bounding (say* $\Vdash$ *" there is an unbounded* $\underset{\sim}{\eta} \in {}^{\omega}\omega$ *"), but adds an essentially non-Cohen real,* then $\mathbb{Q}$ *does not commute with itself.*

Proof. By 6.5(2)(δ), wlog $\underset{\sim}{\eta}$ is the generic of $\mathbb{Q}$ and $\mathbb{Q}$ is as in 9.5 if $\mathbb{Q}$ is.

By [24], $\mathbb{Q}$ adds a Cohen real; now by the assumptions, for some $\mathbb{Q}$–name $\underset{\sim}{\eta}$, $(\mathbb{Q}, \underset{\sim}{\eta}) \in \mathcal{K}^{\neg c}$. By 9.13 we know that $\mathbb{Q}$ and Cohen do not commute, so by 8.6(3) we are done. $\blacksquare_{9.15}$

Conclusion 9.16. *If* $\mathbb{Q}$ *is a Souslin c.c.c. forcing notion adding a non-Cohen real,* then *the forcing by* $\mathbb{Q}$ *makes the old reals meagre.*

10. Some absolute c.c.c. nicely defined forcing notions are not nice enough

We may wonder can we replace the assumption "$\mathbb{Q}$ is Souslin c.c.c." by the weaker one in §8 and in [24]. We review limitations and then see how much we can weaken it.

Proposition 10.1. *Assume that $\eta^* \in {}^{\omega}2$ and $\aleph_1 = \aleph_1^{\mathbf{L}[\eta^*]}$.* Then *there is a definition of a forcing notion $\mathbb{Q}$ (i.e. $\bar{\varphi}$) such that*

(a) *the definition is $\mathbf{\Sigma}^1_1$ (with parameter η^*), so $p \in \mathbb{Q}$, $p \leq^{\mathbb{Q}} q$, "p, q incompatible", "$\{p_n : n < \omega\} \subseteq a$ is a maximal antichain of $\mathbb{Q}$" are preserved by forcing extensions,*
(b) *$\mathbb{Q}$ is c.c.c. (even in a forcing extension; even σ–centered),*
(c) *there is hc–$\mathbb{Q}$–name $\underset{\sim}{\eta}$ of a generic for $\mathbb{Q}$,*
(d) *$\underset{\sim}{\eta}$ is everywhere not essentially Cohen (preserved by extensions not collapsing $\aleph_1$), in fact has cardinality $\aleph_1$,*
(e) *$\mathbb{Q}$ commutes with* Cohen.

Proof. For notational simplicity we ignore η^*.

A condition p in $\mathbb{Q}$ is a quadruple $\langle E_p, X_p, u_p, w_p\rangle$ consisting of: a 2-place relation E_p on ω, a subset X_p of ω, a finite subset u_p of X_p, and a finite subset w_p of ω such that:

$N_p \overset{\text{def}}{=} (\omega, E_p)$ is a model of $\mathrm{ZC}^- + \mathbf{V} = \mathbf{L}$ (let $<^{N_p}_*$ be the canonical ordering of N_p, we do not require well foundedness) such that:

$(N_p, X_p) \models$ " (α) every $x \in X_p$ is an infinite subset of ω,
(β) if $x \neq y$ are from X_p, then $x \cap y$ is finite,
(γ) if $x \in X_p$, then there is no y satisfying

$$y <^{N_p}_* x \quad \& \quad (\forall z \in X_p)(z <^{N_p}_* x \Rightarrow z \cap y \text{ finite}) \quad \&$$
$$(y \text{ an infinite subset of } \omega) \quad \&$$
$$\bigwedge_{n<\omega} (\forall z_1 \dots z_n \in X_p)\Big(\bigwedge_{\ell=1}^{n} z_\ell <^{N_p}_* x \Rightarrow (\exists^\infty m < \omega)(m \notin y \cup \bigcup_{\ell=1}^{n} z_\ell)\Big)\text{''}.$$

The order is defined by:

$p \leq q$ if and only if ($p, q \in \mathbb{Q}$ and) one of the following occurs:

(A) $p = q$,
(B) there are $Y \subseteq \omega$ and $a \in N_q$ and $f \in {}^{Y}\omega$ such that:
  (i) $[x \in Y \;\&\; N_p \models \text{“}y \in x\text{”}] \quad \Rightarrow \quad y \in Y$,
  (ii) $[N_p \models \text{“rk}(x) = y\text{”} \;\&\; y \in Y] \quad \Rightarrow \quad x \in Y$,
  (iii) $N_p \models \text{“rk}(x) = y\text{”} \;\&\; x \in Y \quad \Rightarrow \quad y \in Y$,
  (iv) the set $\{x : N_p \models \text{“}x \text{ an ordinal”}, x \notin Y\}$ has no first element by E_p, i.e., if $N_p \models$ "y is an ordinal , $y \notin Y$", then for some $x \in \omega \setminus Y$ we have $N_p \models$ "$x < y$, x is an ordinal".
  (v) $N_q \models$"a is a transitive set",
  (vi) f is an isomorphism from $N_p \restriction Y$ onto $N_q \restriction \{b : N_q \models b \in a\}$,
  (vii) f maps $X_p \cap Y$ onto $X_q \cap \mathrm{Rang}(f)$,
  (viii) f maps $u_p \cap Y$ into $u_q \cap \mathrm{Rang}(f)$,
  (ix) $w_p \subseteq w_q$,

(x) if $n \in w_q \backslash w_p$ and $x \in u_q \setminus f''(u_p)$, then $N_q \models$"the n-th natural number does not belong to x".

The reader can now check (note that $\underset{\sim}{w} = \bigcup\{w^p : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$ is forced to be an infinite subset of ω almost disjoint to every $A \in \mathcal{A}$, $\mathcal{A}$ a reasonably defined MAD family in $\mathbf{L}$); see more details in the proof of 10.4. $\blacksquare_{10.1}$

Remark 10.2. Is $\mathbb{Q}$ nep? Not; let N be a $\mathbb{Q}$-candidate, $A \in \mathcal{A} \setminus N$, and let $w^* \subseteq \omega$ be such that for some $G \subseteq \mathbb{Q}^N$ generic over N, $w^* = \underset{\sim}{w}[G]$, and $w^* \cap A = \emptyset$. Clearly there is no q which is which is $(N\langle G\rangle, \mathbb{Q})$–generic. Is there such G'? If $N \in \mathbf{L}[\eta^*]$ and $G \subseteq \mathbb{Q}^N$ is generic over N, they will do.

The following show that we cannot improve too much the results of [22] (compare with the conclusion in the end of Section 7).

Proposition 10.3. *Assume* $\mathbf{V} = \mathbf{L}$. *There is* $\mathbb{Q} = \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ *such that:*

(a) $\mathbb{Q}_0$ *is as nep c.c.c. not adding a dominating real,*
(b) $\Vdash_{\mathbb{Q}_0}$ "$\underset{\sim}{\mathbb{Q}}_1$ *is as nice as in* 10.1, *in particular, c.c.c. not adding a dominating real",*
(c) $\mathbb{Q}$ *adds a dominating real,*
(d) *in fact,* $\mathbb{Q}_0$ *is the Cohen forcing (so in any* $\mathbf{V}_1$ *it is c.c.c. and strongly c.c.c., correct, very simple nep (and snep), and it is really absolute, i.e., it is the same in* $\mathbf{V}_1$ *and* $\mathbf{V}$, *and its definition uses no parameters),*
(e) *moreover,* $\underset{\sim}{\mathbb{Q}}_1 = \mathbb{Q}_1$ *is defined in* $\mathbf{L}$, *really absolute, and in any* $\mathbf{V}_1$ *it is c.c.c. (and even snep). In* $\mathbf{V}_1$, $\mathbb{Q}_1$ *adds a dominating real iff* $({}^{\omega}\omega)^{\mathbf{L}}$ *is a dominating family in* $\mathbf{V}_1$.

Proof. Let $\mathbb{Q}_0$ be Cohen. We shall define $\mathbb{Q}_1$ in a similar manner as $\mathbb{Q}$ in the proof of 10.1.

A condition in $\underset{\sim}{\mathbb{Q}}_1$ is a triple $\langle E_p, u_p, w_p\rangle$ such that E_p is a 2-place relation on ω, u_p is a finite subset of ω and w_p is a finite function from a subset of ω to ω and:

> $N_p \stackrel{\text{def}}{=} (\omega, E_p)$ is a model of $\text{ZFC}^- + \mathbf{V} = \mathbf{L}$ (let $<_*^{N_p}$ be the canonical ordering of N_p, we do not require well foundedness); so in formulas we use $\in$.

[What is the intended meaning of a condition p? Let

$$M_p = N_p \restriction \{x : (\text{Tc}(x)^{N_p}, E_p \restriction \text{Tc}(x)^{N_p}) \text{ is well founded}\},$$

where $\text{Tc}(x)$ is the transitive closure of x. Let M'_p be the Mostowski collapse of M_p and $h_p : M_p \longrightarrow M'_p$ be the isomorphism. Now, p gives us information on the function $\underset{\sim}{w} = \bigcup\{w_p : p \in \underset{\sim}{G}\}$ from ω to ω, it says: $\underset{\sim}{w}$ extends the function w_p and if $x \in M_p \cap u_p$ is a function from ω to ω then for every natural number satisfying $n \notin \text{Dom}(w_p)$ we have $x(n) \leq \underset{\sim}{w}(n)$. Note that

$h_p(x)$ is a function from ω to ω iff $M_p \models$" x is a function from ω to ω " iff $N_p \models$" x is a function from ω to ω ".]

The order is defined by: $p \le q$ if and only if one of the following occurs:

(A) $p = q$,

(B) there are $Y \subseteq \omega$ and $a \in N_q$ and $f \in {}^{Y}\omega$ such that
- (i) $[x \in Y \ \& \ N_p \models y \in x] \Rightarrow y \in Y$,
- (ii) $[N_p \models$"$\mathrm{rk}(x) = y$" $\& \ y \in Y] \Rightarrow x \in Y$,
- (iii) $[N_p \models$"$\mathrm{rk}(x) = y$" $\& \ x \in Y] \Rightarrow y \in Y$,
- (iv) the set $\{x : N_p \models$ "x an ordinal", $x \notin Y\}$ has no first element (by E_p),
- (v) $N_q \models$"a is a transitive set",
- (vi) f is an isomorphism from $N_p \restriction Y$ onto $N_q \restriction \{b : N_q \models b \in a\}$,
- (vii) f maps $u_p \cap Y$ into $u_q \cap \mathrm{Rang}(f)$,
- (viii) $w_p \subseteq w_q$,
- (ix) if $n \in \mathrm{Dom}(w^q) \setminus \mathrm{Dom}(w^p)$ and $x \in u_p$, $N_p \models$"x is a function from the natural numbers to the natural numbers" and $x^* = f(x)$, so in particular $x \in \mathrm{Dom}(f) = Y$, *then* $N_q \models$"if y is the n-th natural number then $w^q(y) > x(y)$".

Clearly $\mathbb{Q}$ is equivalent to $\mathbb{Q}' =$ (the Hechler forcing)$^{\mathbf{L}}$, just let us define, for $p \in \mathbb{Q}_1$, $g(p) = (w^p, F^p)$ where $F^p = \{h_p(x) : x \in w^p \subseteq M_p\}$. Now, g is onto $\mathbb{Q}'$ and

$$\begin{aligned} \mathbb{Q}_1 \models p \le q \quad \Rightarrow \quad & \mathbb{Q}' \models g(p) \le g(q) \quad \Rightarrow \\ & \neg(\exists p')(p \le^{\mathbb{Q}} p' \ \& \ p', q \text{ are incompatible in } \mathbb{Q}). \end{aligned}$$

The rest is left to the reader. $\blacksquare_{10.3}$

Proposition 10.4. 1. *Assume that:*

- (a) $\bar\varphi = (\varphi_0(x), \varphi_1(x,y))$ *defines, in any model of* ZFC^-_*, *a forcing notion* $\mathbb{Q}_{\bar\varphi}$ *with parameters from* $\mathbf{L}_{\omega_1}$, *but we may write* $x <_{\varphi_1} y$ *instead of* $\varphi_1(x,y)$ *(or, say,* $x <_{\varphi_1} y$ *in* N*),*
- (b) *for every* $\beta < \omega_1$ *such that* $\mathbf{L}_\beta \models \mathrm{ZFC}^-_*$, *for every* $x, y \in \mathbf{L}_\beta$ *we have:*

$$[x \in \mathbb{Q}^{\mathbf{L}_\beta}_{\bar\varphi} \Leftrightarrow x \in \mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar\varphi}] \quad \textit{and} \quad [x < y \textit{ in } \mathbb{Q}^{\mathbf{L}_\beta}_{\bar\varphi} \Leftrightarrow x < y \textit{ in } \mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar\varphi}],$$

- (c) *for unboundedly many* $\alpha < \omega_1$ *we have* $\mathbf{L}_\alpha \models \mathrm{ZFC}^-_*$,
- (d) *any two compatible members of* $\mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar\varphi}$ *have a lub,*
- (e) *like (b) for compatibility and for existence of lub.*

Then *there is a* $\mathbf{\Sigma}^1_1$ *forcing notion* $\mathbb{Q}$ *equivalent to* $\mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar\varphi}$*; the relations have just the real parameters of* $\bar\varphi$ *and are actually Borel.*

2. *We can use a real parameter* ρ *and replace* $\mathbf{L}_\alpha$ *by* $\mathbf{L}_\alpha[\rho]$.

Proof. It is similar to the proof of 10.1. Let $\mathbb{Q}$ be the set of quadruples $p = (E_p, n_p, \bar{\alpha}_p, \bar{a}_p)$ such that:

(α) E_p is a two-place relation on ω,

(β) $N_p \stackrel{\text{def}}{=} (\omega, E_p)$ is a model of $\text{ZFC}^-_* + \mathbf{V} = \mathbf{L}$,

(γ) for some $n = n_p$ we have

$$\bar{\alpha}_p = \langle \alpha_{p,\ell} : \ell < n \rangle, \qquad \bar{a}_p = \langle a_{p,\ell} : \ell < n \rangle,$$

(δ) $N_p \models$"$\alpha_{p,\ell}$ is an ordinal, $a_{p,\ell} \in \mathbf{L}_{\alpha_{p,\ell}}$, $\mathbf{L}_{\alpha_{p,\ell}} \models \text{ZFC}^-_*$, and for $k \leq \ell < n$ we have $\mathbf{L}_{\alpha_{p,\ell}} \models \varphi_0(a_{p,k}), \alpha_{p,\ell} <_{\varphi_1} \alpha_{p,\ell+1}$", and

(ε) if $m \leq k \leq \ell < n$ then $N_p \models$" $\mathbf{L}_{\alpha_{p,\ell}} \models \varphi_1(a_{p,\alpha_m}, a_{p,\alpha_k})$ ".

The order is given by:

$p_0 \leq_{\mathbb{Q}} p_1$ if and only if ($p_0, p_1 \in \mathbb{Q}$ and) for some $Y_0, Y_1 \subseteq \omega$ and f we have:

(i) for $\ell = 0, 1$: Y_ℓ is an E_p–transitive subset of N_{p_ℓ}, and

$$(\forall x \in N_{p_\ell})(x \in Y_\ell \equiv \text{rk}^{N_{p_\ell}}(x) \in Y_\ell),$$

(ii) f is an isomorphism from $N_{p_0} \restriction Y_0$ onto $N_{p_1} \restriction Y_1$,

(iii) in $\{x \in N_{p_0} : x \notin Y_0,\ N_{p_0} \models$"$x$ is an ordinal"$\}$ there is no E_p–minimal element,

(iv) f maps $\{\alpha_{p_0,\ell} : \ell < n^*\} \cap Y_0$ into $\{\alpha_{p_{1},\ell} : \ell < n_{p_1}\} \cap Y_1$,

(v) if $f(\alpha_{p_0,k}) = \alpha_{p_1,m}$ then $N_{p_1} \models$" $\mathbf{L}_{\alpha_{p_1,m}} \models \varphi_1(f(a_{p_0,k}), a_{p_1,m})$ ".

Claim 10.4.1. *$\mathbb{Q}$ is a quasi order.*

Proof of the claim: Check.

Now for every $p \in \mathbb{Q}$ define N_p, M_p, h_p, M'_p as in the proof of 10.3.

Claim 10.4.2. *The set*

$$\mathbb{Q}' \stackrel{\text{def}}{=} \{p \in \mathbb{Q} : N_p \textit{ is well founded, } n_p > 0\}$$

is dense in $\mathbb{Q}$.

Proof of the claim: Check.

Define $g : \mathbb{Q}' \longrightarrow \mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar{\varphi}}$ by $g(p) = h_p(a_{p,n_p-1})$.

Claim 10.4.3. *g is really a function from $\mathbb{Q}'$ onto $\mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar{\varphi}}$ and and for $p_0, p_1 \in \mathbb{Q}'$ we have*

$$\begin{aligned} p_0 \leq_{\mathbb{Q}} p_1 \quad &\Rightarrow \quad \mathbb{Q}^{\mathbf{L}_{\omega_1}} \models g(p_0) \leq_{\varphi_1} g(p_1), \\ p_0, p_1 \textit{ are incompatible in } \mathbb{Q}' \quad &\Rightarrow \quad g(p_0), g(p_1) \textit{ are incompatible in } \mathbb{Q}^{\mathbf{L}_{\omega_1}}. \end{aligned}$$

Proof of the claim: The first statement is trivial, the second is immediate by clause (v) in the definition of $\leq_{\mathbb{Q}}$. For the last clause recall clause (e) of the assumptions.

Now 10.4 should be clear. $\blacksquare_{10.4}$

We may consider possible "inputs" to 10.4. Instead of looking at the case $\aleph_1^{L[\eta]} = \aleph_1$ we may look at other cases in which $\aleph_1$ is not large in inner models. For example:

Proposition 10.5. *Assume that $\varphi = \varphi(x, y)$ is such that*

(i) $\mathrm{ZFC}_*^- \vdash$ *"for every infinite ordinal $x \in X \overset{\mathrm{def}}{=} \{\alpha : \alpha = \omega$ or $\omega^\alpha = \alpha$ (ordinal exponentiation)$\}$, there is a unique A_x, an unbounded subset of x of order type x such that $\varphi(x, A_x)$, and $\psi(\cdot)$ defines a set $S \subseteq X$ not reflecting",*

(ii) $\mathrm{ZFC}_*^- \vdash$ *if $\mu_1 < \mu_2$ are from X then $A_{\mu_1} \nsubseteq A_{\mu_2}$,*

(iii) $\omega_1 = \sup\{\alpha : \mathbf{L}_\alpha \models \mathrm{ZFC}_*^-\}$, *and the truth value of "$\beta \in A_\gamma,\ \beta \in S$" is the same in $\mathbf{L}_\alpha$ for every $\alpha < \omega_1$ for which $\mathbf{L}_\alpha \models \mathrm{ZFC}_*^-$,*

(iv) *the set S, i.e. $\{\beta < \omega_1 : (\exists \alpha)(\mathbf{L}_\alpha \models \mathrm{ZFC}_*^- \ \&\ \psi(\beta))\}$, is a stationary subset of ω_1*
[this is close to saying "$\aleph_1^{\mathbf{V}}$ is below first ineffable and is not weakly compact in $\mathbf{L}$"].

Then *for some $\bar{\varphi}$ as in the assumptions*[37] *of* 10.4, *and $\underset{\sim}{\eta}$ we have:*

(a) $\mathbb{Q}_{\bar{\varphi}}^{\mathbf{L}_{\omega_1}}$ *is a c.c.c. forcing notion,*

(b) $\underset{\sim}{\eta} \in {}^{\omega}2$ *is a generic real of $\mathbb{Q}_{\bar{\varphi}}^{\mathbf{L}_{\omega_1}}$, and is nowhere essentially Cohen,*

(c) $\mathbb{Q}_{\bar{\varphi}}^{\mathbf{L}_{\omega_1}}$ *commute with Cohen.*

Proof. Let $pr(\alpha, \beta) = (\alpha + \beta)(\alpha + \beta) + \alpha$, it is a pairing function. By coding, without loss of generality

(ii)$'$ if $x, x_1, \ldots, x_n$ are distinct cardinals in $\mathbf{L}_{\omega_1}$, then $A_x \nsubseteq \bigcup_{\ell=1}^{n} A_{x_\ell}$.

[Why? E.g., letting

$$A'_\alpha = \{pr^+(n, pr_n(\beta_1, \ldots, \beta_n)) : n < \omega, \{\beta_1, \ldots, \beta_n\} \subseteq A_\alpha\},$$

where $pr_1(\beta) = \beta$, $pr_{n+1}(\beta_1, \ldots, \beta_{n+1}) = pr(pr_n(\beta_1, \ldots, \beta_n), \beta_{n+1})$). So replacing A_α by A'_α, we have (ii)$'$.]

For $\delta \in X$ let $f_0(\delta) = \min(X \setminus (\delta + 1))$ and let f_δ^1 be the first (in the canonical well ordering of $\mathbf{L}$) one-to-one function from $f_0(\delta)$ onto δ; exist by the definition of X. Let C_δ be the first club of δ disjoint to S. For $\alpha \in [\omega, \omega_1)$, let $\delta_\alpha = \sup(X \cap \alpha)$ and let

$$B^*_\alpha = \{pr_3(\varepsilon, \zeta, \xi) : \varepsilon \in C_{\delta_\alpha},\ \zeta = f^1_{\delta_\alpha}(\alpha),\ \xi \in A_{\delta_\alpha} \text{ and } \varepsilon > \zeta,\ \varepsilon > \xi\}.$$

Note that

$(*)$ B^*_α is an unbounded subset of δ_α such that
(a) $\beta \in S \cap \alpha \quad \Rightarrow \quad \beta > \sup(B^*_\alpha \cap \beta)$,

[37] so we can "translate" it to be Borel

(b) if $\alpha_1, \dots, \alpha_n \in [\omega, \omega_1) \setminus \{\alpha\}$ then $B^*_\alpha \setminus \bigcup_{\ell=1}^{n} B^*_{\alpha_\ell}$ is unbounded in δ_α.

[Why? For (a), suppose that $\beta \in S \cap \alpha$ is a counter example. Trivially, $\min(B^*_\alpha) > \min(C_\alpha)$, so $\gamma = \sup(C_{\delta_\alpha} \cap \beta)$ is well defined. But $C_{\delta_\alpha} \cap S = \emptyset$ whereas $\beta \in S$, hence $\beta \notin C_{\delta_\alpha}$. Also $\beta < \delta_\alpha = \sup(C_{\delta_\alpha})$ and hence necessarily $\beta > \sup(C_{\delta_\alpha} \cap \beta) = \gamma$. Now,

$$B^*_\alpha \cap \beta \subseteq \{\mathrm{pr}_3(\varepsilon, \zeta, \xi) : \varepsilon < \gamma \text{ and } \zeta, \xi < \varepsilon\} \subseteq (\gamma + \gamma + \gamma)^4 < \beta$$

(see the definition of B^*_α and note that $\varepsilon \leq pr_3(\varepsilon, \zeta, \xi)$; the last inequality follows from the fact that $\beta \in X$). To show (b) suppose that $\gamma_0 < \delta_\alpha$ and choose $\xi \in A_\alpha \setminus \bigcup_{\ell=1}^{n} A_{\alpha_\ell}$. Let $\zeta = f^1_{\delta_\alpha}(\alpha)$ and let $\varepsilon \in C_{\delta_\alpha}$ be large enough. So $\mathrm{pr}_3(\varepsilon, \zeta, \xi) \in B^*_\alpha$ (by definition) and $\mathrm{pr}_3(\varepsilon, \zeta, \xi) \notin B^*_{\alpha_\ell}$ (use the third coordinate) and $\mathrm{pr}_3(\varepsilon, \zeta, \xi) > \varepsilon > \gamma_0$.]

Let I_α be the ideal of subsets of B^*_α generated by

$$\{B^*_\alpha \cap B^*_\beta : \omega \leq \beta < \omega_1,\ \beta \neq \alpha\} \cup \{B^*_\alpha \cap \beta : \beta < \delta_\alpha\}.$$

Let $\mathbb{Q}$ be the set of finite functions p from $\omega_1 \setminus \omega$ to $\{0, 1, 2\}$ ordered by: $p \leq q$ if and only if $p \subseteq q$ and:

if $\omega \leq \alpha \in \mathrm{Dom}(p)$, $\beta \in \mathrm{Dom}(q) \cap B^*_\alpha \setminus \mathrm{Dom}(p)$
then $q(\beta) = p(\alpha) \ \vee\ q(\beta) = 2$.

Claim 10.5.1. *$\mathbb{Q}$ is a partial order.*

Claim 10.5.2. *For each $\alpha \in [\omega, \omega_1)$ the set $\mathcal{I}_\alpha = \{p : \alpha \in \mathrm{Dom}(p)\}$ is dense in $\mathbb{Q}$.*

Proof of the claim: Let $p \in \mathbb{Q}$ and suppose that $\alpha \notin \mathrm{Dom}(p)$. Let $q = p \cup \{\langle \alpha, 2 \rangle\}$.

Let $\underset{\sim}{f}$ be the $\mathbb{Q}$–name defined by $\Vdash \underset{\sim}{f} = \bigcup \underset{\sim}{G}_{\mathbb{Q}}$.

Claim 10.5.3. *For $\alpha \in [\omega, \omega_1)$.*

$$\Vdash_{\mathbb{Q}} \text{“ for some } \ell < 3, \text{ for any } m < 3 \text{ we have } \{\beta \in B^*_\alpha : \underset{\sim}{f}(\beta) = m\} \neq \emptyset \mod \mathcal{I}_\alpha \text{ iff } m \in \{2, \ell\} \text{ ”.}$$

Proof of the claim: Take $p \in \underset{\sim}{G}_{\mathbb{Q}}$ such that $\alpha \in \mathrm{Dom}(p)$ and let

$$B = B^*_\alpha \setminus \mathrm{Dom}(p)$$

so $B \in \mathcal{I}_\alpha$. Clearly, $p \Vdash$“ if $\beta \in B^*_\alpha \setminus B$, then $\underset{\sim}{f}(\beta) \in \{2, p(\alpha)\}$ ”, hence

$$p \Vdash_{\mathbb{Q}} \text{“ if } m \in \{0, 1, 2\} \setminus \{2, p(\alpha)\} \text{ then } m \notin \mathrm{Rang}(\underset{\sim}{f} \restriction (B^*_\alpha \setminus B)) \text{ ”.}$$

Now, if $B' \in \mathcal{I}_\alpha$, and $p \leq_{\mathbb{Q}} q$ then there is $\gamma \in A_\alpha \setminus B' \setminus \bigcup\{B^*_\gamma : \gamma \in \mathrm{Dom}(q) \setminus \{\alpha\}\}$ and hence $q \cup \{\langle \gamma, p(\alpha) \rangle\}$ as well as $q \cup \{\langle \gamma, 2 \rangle\}$ belong to $\mathbb{Q}$ and are above q. Reflecting we are done.

Claim 10.5.4. *One can define $\underset{\sim}{f}$ from $\underset{\sim}{f}\restriction\omega \in {}^{\omega}3$.*

Proof of the claim: Define $\underset{\sim}{f}\restriction\alpha$ by induction on $\alpha \in X$ using 10.5.3.

Claim 10.5.5. *The forcing notion $\mathbb{Q}$ is nowhere essentially Cohen.*

Proof of the claim: For every $\alpha^* < \omega_1$ and for every large enough $\gamma < \omega_1$, the condition $q^2_\gamma = \{\langle\gamma, 2\rangle\}$ is compatible with every $q \in \mathbb{Q}$ such that $\mathrm{Dom}(p) \subseteq \alpha^*$, but $q^1_\gamma = \{\langle\gamma, 1\rangle\}$ is incompatible with it. Together with 10.5.4 we are done.

Claim 10.5.6. *The $\mathbb{Q}$–name $\underset{\sim}{f}$ (for a real), hence $\underset{\sim}{f}\restriction u$, is nowhere essentially Cohen.*

Proof of the claim: By 10.5.4, 10.5.5, as obviously $\underset{\sim}{f}$ is generic, i.e., $\underset{\sim}{G} = \{p : p \subseteq \underset{\sim}{f}\}$.

Claim 10.5.7. *The forcing notion $\mathbb{Q}$ satisfies the demands in* 10.5.

Proof of the claim: Check.

Claim 10.5.8. *The forcing notion $\mathbb{Q}$ satisfies the c.c.c.*

Proof of the claim: Use "$S \subseteq \omega_1$ is stationary" and clause (a) of (*), (that is, if $p_\alpha \in \mathbb{Q}$ for $\alpha < \omega_1$, for each limit $\alpha \in S$ we let

$$\gamma_\alpha = \sup\big(\{B^*_\beta \cap \alpha : \beta \in \mathrm{Dom}(p) \setminus \{\alpha\} \text{ and } \beta \geq \omega\} \cup \{\mathrm{Dom}(p_\alpha) \cap \alpha\}\big),$$

so by (*)(a) we have $\gamma_\alpha < \alpha$, hence for some stationary $S' \subseteq S$ we have

$$\alpha \in S' \quad\Rightarrow\quad \gamma_\alpha = \gamma^* \ \&\ p_\alpha\restriction\alpha = p^*,$$

and wlog $\beta < \alpha \in S' \;\Rightarrow\; \mathrm{Dom}(p_\beta) \subseteq \alpha$. Now if $\alpha_1, \alpha_2 \in S'$, then $p = p_{\alpha_1} \cup p_{\alpha_2} \in \mathbb{Q}$ is a common upper bound of $p_{\alpha_1}, p_{\alpha_2}$, as required. $\blacksquare_{10.5}$

Remark 10.6.
1. Of course, such forcing can make $\aleph_1$ to be $\aleph_1^{\mathbf{L}[\eta]}$. But it seems that we can have such forcing which preserves the $\mathbf{L}_{\omega_1}$–cardinals (and even their being "large" in suitable senses). For this it should be like "coding the universe by a real" of Jensen, Beller and Welch [1], and see Shelah and Stanley [27].
2. Instead of coding $\aleph_1$–Cohen we can iterate adding dominating reals or whatever.

Definition 10.7.
1. We say that forcing notions $\mathbb{Q}_0, \mathbb{Q}_1$ are equivalent if their completions to Boolean algebras $(\mathrm{BA}(\mathbb{Q}_0), \mathrm{BA}(\mathbb{Q}_1))$ are isomorphic.
2. Forcing notions $\mathbb{Q}_0, \mathbb{Q}_1$ are locally equivalent if

(i) for each $p_0 \in \mathbb{Q}_0$ there are q_0, q_1 such that

$$p_0 \leq q_0 \in \mathbb{Q}_0 \ \& \ q_1 \in \mathbb{Q}_1 \ \& \ \mathrm{BA}(\mathbb{Q}_0 \restriction (\geq q_0)) \cong \mathrm{BA}(\mathbb{Q}_1 \restriction (\geq q_1)),$$

(ii) for every $p_1 \in \mathbb{Q}_1$ there are q_0, q_1 such that

$$q_0 \in \mathbb{Q}_0 \ \& \ p_1 \leq q_1 \in \mathbb{Q}_1 \ \& \ \mathrm{BA}(\mathbb{Q}_0 \restriction (\geq q_0)) \cong \mathrm{BA}(\mathbb{Q}_1 \restriction (\geq q_1)).$$

Now we may phrase the conclusions of 10.4, 10.5.

Proposition 10.8. 1. *Assume $\bar{\varphi}_1 = \langle \varphi^1_0, \varphi^1_1 \rangle$ and $\bar{\varphi}^2 = \langle \varphi^2_0, \varphi^2_1 \rangle$ are as in* 10.4. Then *we can find $\bar{\varphi}^3$ as there, only with the parameters of $\bar{\varphi}_1, \bar{\varphi}_2$ and such that:*

(a) *if in $\mathbf{L}_{\omega_1}$ there is a last cardinal μ (i.e., $\aleph_1^{\mathbf{V}}$ is a successor cardinal in $\mathbf{L}$),* then *$\mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar{\varphi}_3}$ is locally equivalent to*

$$\bigcup \{ \mathbb{Q}^{\mathbf{L}_\alpha}_{\bar{\varphi}_1} : \mu < \alpha, \ \mathbf{L}_\alpha \models \mu \textit{ is the last cardinal} \},$$

(b) *if in $\mathbf{L}_{\omega_1}$ there is no last cardinal (i.e. $\aleph_1^{\mathbf{V}}$ is a limit cardinal in $\mathbf{L}$),* then *$\mathbb{Q}^{\mathbf{L}_{\omega_1}}_{\bar{\varphi}_3}$ is locally equivalent to*

$$\bigcup \{ \mathbb{Q}^{\mathbf{L}_{\omega,\alpha}}_{\bar{\varphi}^1_2} : \mathbf{L}_{\omega_1} \models \alpha \textit{ a cardinal} \}.$$

2. *In* 10.4, 10.5(1) *we can replace $\mathbf{L}_{\omega_1}$ by $\mathbf{L}_{\omega_1}[\eta^*]$, $\eta^* \in {}^\omega\omega$.*
3. *In* 10.4, 10.5(1) *we can replace $\mathbf{L}_{\omega_1}$ by $\mathbf{L}_{\omega_1}[A]$ where $A \subseteq \omega_1$ but have $\aleph_1$-snep instead of $\aleph_0$-snep.*

Proof. Let $\varphi_{3,0}(x)$ say

(i) $x = \langle \bar{\alpha}^x, \bar{\beta}^x, \bar{a}^x, \bar{b}^x \rangle$, $\bar{\alpha}^x = \langle \alpha^x_\ell : \ell \leq n^x \rangle$, $\langle \langle \beta^x_{\ell,k} : k \leq k^x_\ell \rangle : \ell \leq n^x \rangle$, $\bar{a}^x = \langle \langle a^x_{\ell,k} : k \leq k^x_\ell \rangle : \ell < n^x \rangle$, $\bar{a}^x = \langle a^x_\ell : \ell \leq n^x \rangle$, $\bar{b}^x = \langle b^x_\ell : \ell < n^x \rangle$,

(ii) $\alpha^x_\ell < \beta^x_{\ell,0} < \beta^x_{\ell,1} \ldots$, and $\mathbf{L}_{\beta^x_\ell} \models$"$\alpha^\ell_x$ the last cardinal",

(iii) $\mathbf{L}_{\beta^x_\ell} \models$"$\varphi_{1,0}(b^x_\ell)$", $\mathbf{L}_{\alpha^x_{\ell+1}} \models$"$\varphi_{2,0}(a^x_\ell)$" for $\ell < n^x$,

(iv) $\mathbf{L}_{\beta^x_{n^x}} \models$"$\alpha^x_\ell$ is a cardinal",

(v) $\mathbf{L}_{\alpha^x_{\ell+1}} \models$"$\varphi^2_1(a^x_\ell, a^x_{\ell+1})$".

Let $\beta(x) = x$. Let $\varphi_{3,1}(x,y)$ say:

(α) $\beta^x_{n^x} \leq \beta^y_{n_y}$,

(β) $\{\alpha^x_\ell : \ell \leq n^x$ and $\mathbf{L}_{\beta^y_y} \models$"$\alpha^x_\ell$ is a cardinal"$\}$ is a subset of $\{\alpha^y_\ell : \ell \leq n^y\}$,

(γ) if $\alpha^x_{\ell(*)}$ is maximal in $\{\alpha^x_\ell : \ell < n^x, \mathbf{L}_{\beta^y_{n^y}} \models$"$\alpha^x_\ell$ is a cardinal"$\}$ then $\bar{\alpha}^x \restriction \ell(*) = \bar{\alpha}^y \restriction \ell(x)$, $\bar{\beta}^x \restriction \ell(*) = \bar{\beta}^y \restriction \ell(*)$, $\bar{a}^x \restriction \ell(*) = \bar{a}^y \restriction \ell(*)$, $\bar{b}^x \restriction \ell(*) = \bar{b}^y \restriction \ell(*)$,

(δ) $\alpha^x_{\ell(*)} = \alpha^y_{\ell(*)}$,

(ε) $\beta^x_{\ell(*)} \leq \beta^y_{\ell(*)}$ and $\mathbf{L}_{\beta^y_{\ell(*)}} \models \varphi^1_{0,1}(b^x_{\ell(*)}, b^y_{\ell(*)})$.

Now check. $\blacksquare_{10.8}$

Remark 10.9. Compare with 6.14.

Are such forcing notions nep? Generally not. E.g., (for $\mathbb{Q}$ from 10.3), if in $\mathbf{V}$, $({}^{\omega}\omega)^{\mathbf{L}}$ is not bounded (by any function from $\mathbf{V}$) then for any candidate N, there is an increasing sequence $\eta \in ({}^{\omega}\omega)^{\mathbf{L}}$ not dominated by any $f \in ({}^{\omega}\omega)^{N}$; let $\eta^* \in {}^{\omega}\omega$ be defined by $\eta^*(0) = 0$, $\eta^*(n+1) = \eta^*(\eta(n)+1)$. Then there is $\nu \leq \eta^*$, $\nu \in \prod\limits_{n<\omega} (\eta(n)+1)$ which is Cohen over N. So $N[\nu]$ is a candidate, $N[\nu] \models \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}$ does not dominate ν". But

$$\mathbf{V} \models \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\eta} \text{ dominates } \eta^* \text{ hence } \nu\text{”}.$$

Probably this is a general phenomena.

11. Open problems

Problem 11.1.
1. Can we in [24] weaken the assumptions (from Souslin c.c.c.) to "$\mathbb{Q}$ is nep and c.c.c."? (See [22, §2] for partial answer.)
2. Similarly in the symmetry theorem.
3. Similarly other problems here have such versions too.

Problem 11.2 (von Neumann)**.** Is every c.c.c. ${}^{\omega}\omega$–bounding atomless forcing notion a measure algebra? We may now rephrase: is the non-existence consistent?

A relative of the von Neumann problem is a problem which Fremlin [7] stresses and has many equivalent versions (see [7] on its history). Half way between them and our context is the following.

Problem 11.3. Assume $\mathbb{Q}$ is a Souslin c.c.c. ${}^{\omega}\omega$–bounding forcing notion. Is every $\mathbb{Q}$–name of a new real essentially a random real?

Problem 11.4.
1. Is it consistent that every c.c.c. forcing notion adding an unbounded real adds a Cohen real? (See Błaszczyk and Shelah [5] for a proof of the σ-centered version).
2. If $\mathbb{P}$ satisfies [24, 1.5], does it imply $\mathbb{P}$ adds a Cohen real?

Problem 11.5. Are there any symmetric (or $(<\omega)$-symmetric) c.c.c. Souslin forcing notions in addition to Cohen forcing and random forcing?

["Yes" here implies "no" to 11.3 so not of present interest.]

Problem 11.6 (Gitik and Shelah [8], [9]). 1. Assume I is an $\aleph_1$–complete ideal on κ such that $\mathcal{P}/I$ is atomless. Can I^+ (as a forcing notion) be a c.c.c. Souslin forcing generated by a real?
2. Replace Souslin by "definable in an $(\mathcal{H}_{<\sigma}(\theta), \in, \mathfrak{B})$", $\mathfrak{B}$ has universe κ or $\mathcal{H}_{<\sigma}(\kappa)$, and I is $(\theta+\kappa)^+$–complete (see [8]).
3. Generalize the results of the form "if $\mathcal{P}(\kappa)/I$ is the measure algebra with Maharam dimension μ (or is the adding of μ Cohen reals) then λ is large enough", see [9], [10] for those results.
4. Combine (2) and (3).

Problem 11.7. When do iterations (CS,FS) of Souslin c.c.c. forcing notions not adding a dominating real have this property? Is each almost ${}^{\omega}\omega$–bounding? (See [28]; generally try to continue 7.10, replace "η^* is generic real for $(N, \mathbb{Q}, \underset{\sim}{\eta})$" by less.)

Problem 11.8. 1. Is there a pair $(\mathbb{Q}, \underset{\sim}{r})$ such that:
(a) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{r} \in {}^{\omega}2$ is new no-where Cohen",
(b) if $\mathbb{P}$ is a Souslin c.c.c. forcing notion with no $\mathbb{P}$–name $\underset{\sim}{r}'$ of a real such that the forcing notion $\mathcal{B}_{\mathbb{P}}(\underset{\sim}{r}')$ is ${}^{\omega}\omega$–bounding but $\mathbb{P}$ adds a nowhere essentially Cohen real,
then forcing with $\mathbb{P}$ adds a $(\mathbb{Q}, \underset{\sim}{r})$ real, i.e. for some $\mathbb{P}$–name $\underset{\sim}{r}''$ for a real we have $\Vdash_{\mathbb{P}}$"for some $G'' \subseteq \mathbb{Q}^{\mathbf{V}}$ generic over $\mathbf{V}$, $\underset{\sim}{r}''[G_{\mathbb{P}}] = \underset{\sim}{r}[G'']$".
2. As above, $\mathbb{P}$ is σ–centered.
3. If $\mathbb{P}$ is a Souslin c.c.c. forcing notion adding new reals but not adding a real $\underset{\sim}{r}'$ with $\mathcal{B}_{\mathbb{P}}(\underset{\sim}{r}')$ being ${}^{\omega}\omega$–bounding,
then forcing with $\mathbb{P}$ adds a new real $\underset{\sim}{r}''$ such that $\mathcal{B}_{\mathbb{P}}(r'')$ is σ–centered.

Problem 11.9. Develop[38] the theory of "definable forcing notions" when we allow an ultrafilter on ω as a parameter.

Problem 11.10. Does nep$\neq$snep? (The case $\theta = \kappa = \aleph_0$, of course.)

Problem 11.11. Try to generalize our present context to λ–complete forcing notions (see Rosłanowski and Shelah [20], [17]). Is λ^+-c.c. preserved by $(<\lambda)$–support iterations of such forcing notions?

Problem 11.12. When does $\mathbb{Q}^{\mathbf{V}} \lessdot \mathbb{Q}^{\mathbf{V}^{\mathbb{P}}}$?

---

[38] But see [27].

Problem 11.13. Does $\mathrm{Ax}_{\omega_1}[(\aleph_1, \aleph_1)\text{–nep}]$ imply $2^{\aleph_0} = \aleph_2$?

Or does $\mathrm{Ax}_{\omega_1}[\mathrm{nep}]$ imply $2^{\aleph_0} = \aleph_2$?

[The parallel question for Souslin proper was formulated in Goldstern and Judah [12]]

Two other relatives of 11.3 are

Problem 11.14. Assume $\mathbb{Q}$ is a Souslin c.c.c. forcing notion which is snep and even "$x \in \mathbb{Q}$", "$x \leq^{\mathbb{Q}} y$", "$\{p_n : n < \omega\}$ predense above q'" are Σ^1_1 relations. Does $\mathbb{Q}$ add Cohen or random real?

Problem 11.15 (Judah)**.** Can a Souslin c.c.c. forcing notion add a minimal real? (Note: this is of interest only if the answer in 11.3 is NO and/or the answer to 11.16 is NO.)

We may also ask:

Problem 11.16.

1. Let $\mathbb{Q}$ be a Souslin c.c.c. forcing notion and $\Vdash_{\mathbb{Q}}$"$\underset{\sim}{r} \in {}^{\omega}2$". Is $B_2(\mathbb{Q}, \underset{\sim}{r})$ also a Souslin c.c.c. forcing notion?
2. Similarly for nep c.c.c.

List of defined concepts

References

[1] Beller, A., Jensen, R. B., Welch, P., *Coding the Universe*, London Math. Soc. Lecture Notes Ser. **47** Cambridge Univ. Press, Cambridge, 1982.

[2] Bartoszyński, T., Judah, H., *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, Massachusetts, 1995.

[3] Baumgartner, J. E., *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. **79** (1973), 101–106.

[4] Blass, A., Shelah, S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), 213–243.

[5] Błaszczyk, A., Shelah, S., *Complete σ-centered Boolean Algebra not adding Cohen reals*, J. Symbolic Logic (accepted), math. LO/9712285.

[6] Farah, I., Zapletal, J., *Between Maharam's and von Neumann's problems*, arxive math. LO/0401134.

[7] Fremlin, D., *Problem list.* Circulated notes; available from `http://www.essex.ac.uk/maths/staff/fremlin/measur.htm`.

[8] Gitik, M., Shelah, S,. *Forcings with ideals and simple forcing notions*, Israel J. Math. **68** (1989), 129–160.

[9] Gitik, M., Shelah, S., *More on simple forcing notions and forcings with ideals*, Ann. Pure Appl. Logic **59** (1993), 219–238.

[10] Gitik, M., Shelah, S., *More on real-valued measurable cardinals and forcing with ideals*, Israel J. Math. **124** (2001), 221–242, math. LO/9507208.

[11] Goldstern, M., *Tools for your forcing construction*, in: Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc. **6** (1993), 305–360.

[12] Goldstern, M, Judah, H., *Iteration of Souslin forcing, projective measurability and the Borel conjecture*, Israel J. Math. **78** (1992), 335–362.

[13] Ihoda, J. (Judah, H.), Shelah, S., *Souslin forcing*, J. Symbolic Logic, **53** (1988), 1188–1207.

[14] Jech, T., *Set Theory*, Academic Press, New York, 1978.

[15] Kellner, J., Shelah, S., *Preserving preservation*, preprint, arxive math. LO/040581

[16] Rosłanowski, A., Shelah, S., *Measured creatures*, Israel J. Math (submitted), math. LO/0010070.

[17] Rosłanowski, A., Shelah, S., *Sheva-Sheva-Sheva: large creatures*, Israel J. Math. (submitted), math. LO/0210205.

[18] Rosłanowski, A., Shelah, S., *Norms on possibilities II: more ccc ideals on 2^ω*, J. Appl. Anal. **3** (1997), 103–127, math. LO/9703222.

[19] Rosłanowski, A., Shelah, S., *Norms on Possibilities I: Forcing with Trees and Creatures*, Mem. Amer. Math. Soc. **141**(671), Amer. Math. Soc., Providence, RI, 1999. math. LO/9807172.

[20] Rosłanowski, A., Shelah, S., *Iteration of λ-complete forcing notions not collapsing λ^+*, Internat. J. Math. Math. Sci. **28** (2001),63–82, math. LO/9906024.

[21] Shelah, S., *Non Cohen oracle c.c.c.*, J. Appl. Anal. (submitted), math. LO/0303294.

[22] Shelah, S., *On nicely definable forcing notions*, J. Appl. Anal. **11**(1) (2005), (to appear), math. LO/0303293.

[23] Shelah, S., *Proper Forcing*, Lecture Notes in Math. **940**, Springer-Verlag, Berlin-New York, 1982.

[24] Shelah, S., *How special are Cohen and random forcings i.e. Boolean algebras of the family of subsets of reals modulo meagre or null*, Israel J. Math. **88** (1994), 159–174, math. LO/9303208.

[25] Shelah, S., *Proper and Improper Forcing*, Perspect. Math. Logic, Springer, New York, 1998.
[26] Shelah, S., *On what I do not understand (and have something to say)*, Fund. Math. **166** (2000), 1–82, math. LO/9906113.
[27] Shelah, S., Stanley, L., *A combinatorial forcing for coding the universe by a real when there are no sharps*, J. Symbolic Logic **60** (1995), 1–35, math. LO/9311204.
[28] Shelah, S., *More on non-elementary properness*, (in preparation).
[29] Zapletal, J., *Descriptive Set Theory and Definable Forcing*, Mem. Amer. Math. Soc. **167**(793), Providence, RI, 2004.

Institute of Mathematics
The Hebrew University
Jerusalem, 91904, Israel
and Mathematics Department
Rutgers University
New Brunswick, NJ 08854, USA
email: shelah@math.huji.ac.il