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INTERPRETING GROUPS AND FIELDS IN SOME NONELEMENTARY CLASSES

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This paper is concerned with extensions of geometric stability theory to some nonelementary classes. We prove the following theorem:

Theorem. Let $\mathfrak{C}$ be a large homogeneous model of a stable diagram D. Let $p, q \in S_D(A)$, where p is quasiminimal and q unbounded. Let $P = p(\mathfrak{C})$ and $Q = q(\mathfrak{C})$. Suppose that there exists an integer $n < \omega$ such that

$$\dim(a_1 \cdots a_n / A \cup C) = n,$$

for any independent $a_1, \ldots, a_n \in P$ and finite subset $C \subseteq Q$, but

$$\dim(a_1 \cdots a_n a_{n+1} / A \cup C) \leq n,$$

for some independent $a_1, \ldots, a_n, a_{n+1} \in P$ and some finite subset $C \subseteq Q$.

Then $\mathfrak{C}$ interprets a group G which acts on the geometry P' obtained from P. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n = 1, 2, 3$ and

- If $n = 1$, then G is abelian and acts regularly on P'.
- If $n = 2$, the action of G on P' is isomorphic to the affine action of $K \rtimes K^*$ on the algebraically closed field K.
- If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.

We prove a similar result for excellent classes.

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0. Introduction

The fundamental theorem of projective geometry is a striking example of interplay between geometric and algebraic data: let k and ℓ distinct lines of, say, the complex projective plane $\mathbb{P}^2(\mathbb{C})$, with ∞ their point of intersection. Choose two distinct points 0 and 1 on $k\backslash\{\infty\}$. We have the *Desarguesian property*: for any 2 pairs of distinct points (P_1, P_2) and (Q_1, Q_2) on $k\backslash\{\infty\}$, there is an automorphism σ of $\mathbb{P}^2(\mathbb{C})$ fixing ℓ pointwise, preserving k, such that $\sigma(P_i) = Q_i$, for $i = 1, 2$. But for some triples (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) on $k\backslash\{\infty\}$, this property fails. From this, it is possible to endow k with the structure of a division ring, and another geometric property guarantees that it is a field. Model-theoretically, in the language of *points* (written $P, Q, \ldots$), *lines* (written $\ell, k, \ldots$), and an *incidence relation* $\in$, we have a saturated structure $\mathbb{P}^2(\mathbb{C})$, and two strongly minimal types $p(x) = \{x \in k\}$ and $q(x) = \{x \in \ell\}$. The Desarguesian property is equivalent to the following statement in *orthogonality calculus*, which is the area of model theory dealing with independence between types: p^2 is weakly orthogonal to q^ω, but p^3 is not almost orthogonal to q^ω (the abstract gives another equivalent condition in terms of dimension). From this, we can define a division ring on k. Model theory then gives us more: strong minimality guarantees that it is an algebraically closed field, and further conditions that it has characteristic 0; it follows that it must be $\mathbb{C}$.

A central theorem of geometric stability, due to Hrushovski [7] (extending Zilber [34]), is a generalization of this result to the context of stable first order theory: let $\mathfrak{C}$ be a large saturated model of a stable first order theory. Let $p, q \in S(A)$ be stationary and p regular such that for some $n < \omega$ the type p^n is weakly orthogonal to q^ω but p^{n+1} is not almost orthogonal to q^ω. Then $n = 1, 2, 3$ and if $n = 1$ then $\mathfrak{C}$ interprets an abelian group and if $n = 2, 3$ then $\mathfrak{C}$ interprets an algebraically closed field. He further obtains a description of the action for $n = 1, 2, 3$ (see the abstract). The first order notions of saturation, nonforking and orthogonality calculus, and canonical bases are some of the key consequences of the compactness theorem used in the proof.

Geometric stability theory is a branch of first order model theory that grew out of Shelah's classification theory [33]. It began with the discovery by Zilber and Hrushovski that certain model-theoretic problems (finite axiomatisability of totally categorical first order theories [34], existence of strictly stable unidimensional first order theories [8]) imposed abstract (geometric) model-theoretic conditions implying the existence of definable classical groups. The structure of these groups was then invoked to solve the problems (see also [35], where a special case of our result is fully analyzed). Geometric stability theory has now matured into a sophisticated body of techniques which have found remarkable applications both within model

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theory (see [25] and [3]) and in other areas of mathematics (see, for example, the surveys [9, 10]). However, its applicability is limited at present to mathematical contexts which are first order axiomatisable. To extend the scope of these techniques to deal with more general mathematical contexts, it is necessary to develop geometric stability theory without compactness. In this paper, we generalize Zilber's and Hrushovski's results to two contexts where the compactness theorem fails: homogeneous model theory and excellent classes.

Homogeneous model theory was initiated by Shelah [28]; it consists of studying the class of elementary submodels of a large homogeneous, rather than saturated, model. Homogeneous model theory is very well-behaved, with a good notion of stability [28, 30, 11, 6], superstability [17, 15], ω-stability [19, 20], and even simplicity [4]. Its scope of applicability is very broad, as many natural model-theoretic constructions fit within its framework: first order, Robinson theories, existentially closed models, Banach space model theory, many generic constructions, classes of models with set-amalgamation (L^n, infinitary), as well as many classical non-first order mathematical objects like free groups or Hilbert spaces. We will consider the stable case (but note that this context may be unstable from a first order standpoint), *without* assuming simplicity [4] i.e. without assuming that there is a dependence relation with all the properties of forking in the first order stable case. (This contrasts with the work of Berenstein [1], who carries out some Zilber group configuration theorems under the assumption of stability, simplicity, and the existence of canonical bases.)

Excellence is a property discovered by Shelah [31, 32] in his work on categoricity for nonelementary classes: for example, he proved that, under GCH, a sentence in $L_{\omega_1,\omega}$ which is categorical in all uncountable cardinals is excellent. On the other hand, excellence is central in the classification of almost-free algebras [24] and also arises naturally in Zilber's work around complex exponentiation [36, 37] (the structure $(\mathbb{C}, \exp)$ has intractable first order theory since it interprets the integers, but is manageable in an infinitary sense). Excellence is a condition on the existence of prime models over certain countable sets (under an ω-stability assumption). Classification theory for excellent classes is quite developed; we have a good understanding of categoricity ([31, 32, 21] for a Baldwin–Lachlan style proof involving pregeometries), and Grossberg and Hart proved the Main Gap [5]. Excellence follows from uncountable categoricity in the context of homogeneous model theory. However, excellence is at present restricted to ω-stability (see [31] for the definition), so excellent classes and stable homogeneous model theory, though related, are not comparable.

As we pointed out, the main technical difficulty is that the compactness theorem fails at this level of generality. In both our contexts, we lose saturation: we use various forms of homogeneity instead. We lose nonforking: we replace it with an appropriate independence relation with weaker properties, for example, extension or symmetry may fail (no better independence relation exists in general [15]).

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We lose canonical bases and the $\mathfrak{C}^{\mathrm{eq}}$-machinery: in this paper, we avoid their use entirely.

Each context comes with a notion of monster model $\mathfrak{C}$ (homogeneous or full), which functions as a universal domain; all relevant realizable types are realized in $\mathfrak{C}$, and models may be assumed to be submodels of $\mathfrak{C}$. We consider a *quasiminimal* type p, i.e. every definable subset of its set of realizations in $\mathfrak{C}$ is either bounded or has bounded complement. Quasiminimal types are a generalization of strongly minimal types in the first order case, and play a similar role, for example, in Baldwin–Lachlan theorems. We introduce the natural closure operator on the subsets of $\mathfrak{C}$; it induces a pregeometry and a notion of dimension $\dim(\cdot/C)$ on the set of realizations of p, for any $C \subseteq \mathfrak{C}$. We prove:

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Theorem 0.1. *Let $\mathfrak{C}$ be a large homogeneous stable model or a large full model in the excellent case. Let p, q be complete types over a finite set A, with p quasiminimal. Assume that there exists $n < \omega$ such that*

(1) *For any independent sequence $(a_0, \ldots, a_{n-1})$ of realizations of p and any countable set C of realizations of q we have*

$$\dim(a_0, \ldots, a_{n-1}/A \cup C) = n.$$

(2) *For some independent sequence $(a_0, \ldots, a_{n-1}, a_n)$ of realizations of p there is a countable set C of realizations of q such that*

$$\dim(a_0, \ldots, a_{n-1}, a_n/A \cup C) \leq n.$$

Then $\mathfrak{C}$ interprets a group G which acts on the geometry P' induced on the realizations of p. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n = 1, 2, 3$ and

- *If $n = 1$, then G is abelian and acts regularly on P'.*
- *If $n = 2$, the action of G on P' is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebraically closed field K.*
- *If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

As mentioned before, the phrasing in terms of dimension theory is equivalent to the statement in orthogonality calculus in Hrushovski's theorem. The main difference with the first order result is the appearance of the so-called *non-classical groups*, which are nonabelian ω-homogeneous groups carrying a pregeometry. In the first order case, it follows from Reineke's theorem [27] that such groups cannot exist. Another difference is that in the interpretation, we must use invariance rather than definability; since we have some homogeneity in our contexts, invariant sets are definable in infinitary logic (in the excellent case, for example, they are type-definable). Finally, we use quasiminimal types rather than regular types. This is only to obtain a field (for $n \geq 2$); in order to interpret a group, it is enough to start with p regular (see [17, 5] for the definitions of regular in these contexts).

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The paper is divided into four sections. The first two sections are group-theoretic and, although motivated by model theory, contain none. The first section is concerned with generalizing classical theorems on strongly minimal saturated groups and fields. We consider groups and fields whose universe carries an ω-homogeneous pregeometry. We introduce generic elements and ranks, but make no stability assumption. We obtain a lot of information on the structure of non-classical groups, for example, they are not solvable, their center is 0-dimensional, and the quotient with the center is divisible and torsion-free. Nonclassical groups are very complicated; in addition to the properties above, any two nonidentity elements of the quotient with the center are conjugate. Fields carrying an ω-homogeneous pregeometry are more amenable; as in the first order case, we can show that they are algebraically closed.

In the second section, we generalize the theory of groups acting on strongly minimal sets. We consider groups G *n-acting* on a pregeometry P, i.e. the action of the group G respects the pregeometry, and further (1) the integer n is maximal such that for each pair of independent n-tuples of the pregeometry P, there exists an element of the group G sending one n-tuple to the other, and (2) two elements of the group G whose actions agree on an $(n+1)$-dimensional set are identical. As a nontriviality condition, we require that this action must be ω-homogeneous (in [12] Hyttinen considered this context under a stronger assumption of homogeneity, but in order to apply the results to excellent classes we must weaken it). We are able to obtain a picture very similar to the classical first order case. We prove (see the section for precise definitions):

Theorem 0.2. *Suppose that G n-acts on a geometry P' and admits hereditarily unique generics with respect to the automorphism group Σ. Then either there is an A-invariant non-classical unbounded subgroup of G (for some finite $A \subseteq P'$), or $n = 1, 2, 3$ and*

- *If $n = 1$, then G is abelian and acts regularly on P'.*
- *If $n = 2$, the action of G on P' is isomorphic to the affine action of $K \rtimes K^*$ on the algebraically closed field K.*
- *If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

The last two sections are completely model-theoretic. In Sec. 3, we consider the case of stable homogeneous model theory, and in the fourth the excellent case. In each case, the group we interpret is based on the automorphism group of the monster model $\mathfrak{C}$: let p, q be unbounded types, say over a finite set A, and assume that p is quasiminimal. Let $P = p(\mathfrak{C})$ and $Q = q(\mathfrak{C})$. Bounded closure induces a pregeometry on P and we let P' be its associated geometry. In the stable homogeneous case, the group we interpret is the group of permutations of P' induced by automorphisms of $\mathfrak{C}$ fixing $A \cup Q$ pointwise. However, in the excellent case, we may not have enough homogeneity to carry this out. The new idea to remedy this, is to consider the

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group G of permutations of P' which agree *locally* with automorphisms of $\mathfrak{C}$, i.e. a permutation g of P' is in G if for any finite $X \subseteq P$ and countable $C \subseteq Q$, there is an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that the permutation of P' induced by σ agrees with g on X. In each case, we show that the group n-acts on the geometry P' in the sense of Sec. 2. The interpretation in $\mathfrak{C}$ follows from the n-action.

Although the construction we provide for excellent classes works for the stable homogeneous case also, for expositional reasons we present the construction with the obvious group in the homogeneous case first, and then present the modifications with the less obvious group in the excellent case.

To apply Theorem 0.2 to G and obtain Theorem 0.1, it remains to show that G admits hereditarily unique generics with respect to some group of automorphisms Σ. For this, we deal with an invariant (and interpretable) subgroup of G, the connected component, and deal with the group of automorphisms Σ induced by the *strong automorphisms* i.e. automorphisms preserving Lascar strong types. Hyttinen and Shelah introduced Lascar strong types for the stable homogeneous case in [17]; this is done without stability by Buechler and Lessmann in [4]. In the excellent case, this is done in detail in [16]; we only use the results over finite sets.

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1. Groups and Fields Carrying a Homogeneous Pregeometry

Recall that a pregeometry is a pair (P, cl), where cl is a finitary closure on the subsets of P, which is monotonic, transitive and satisfies exchange (see, for example [3]). In this section, we study algebraic structures carrying an ω-homogeneous pregeometry. The definition we give is similar to the definition from [12], except that the homogeneity requirement is weaker. We make no assumption on the size of the language.

Definition 1.1. An infinite model M *carries an ω-homogeneous pregeometry* if there exists a closure operator cl on the subsets of M satisfying the axioms of a pregeometry with $\dim(M) = \|M\|$, and such that whenever $A \subseteq M$ is finite and $a, b \notin \mathrm{cl}(A)$, then there is an automorphism of M preserving cl, fixing A pointwise, and sending a to b.

Remark 1.2. In model-theoretic applications, the model M is generally uncountable, and $|\mathrm{cl}(A)| < \|M\|$, when A is finite. Furthermore, if $a, b \notin \mathrm{cl}(A)$ and $|A| < \|M\|$ one can often find an automorphism of M fixing $\mathrm{cl}(A)$ pointwise, and not just A. However, we find this phrasing more natural and in non first order contexts like excellence, ω_1-homogeneity may fail.

Strongly minimal $\aleph_0$-saturated groups are the simplest example of groups carrying an ω-homogeneous pregeometry. In this case, Reineke's famous theorem [27] asserts that it must be abelian. Groups whose universe is a regular type are also of this form, and when the ambient theory is stable, Poizat [26] showed that they are also abelian. We are going to consider generalizations of these theorems, but first, we need to remind the reader of some terminology.

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Fix an infinite model M and assume that it carries an ω-homogeneous pregeometry. Following model-theoretic terminology, we will say that a set Z is A-*invariant*, where A and Z are subsets of the model M, if any automorphism of M fixing A pointwise, fixes Z setwise. In particular, if $f : M^m \to M^n$ is A-invariant and σ is an automorphism of M fixing A pointwise, then $f(\sigma(\bar{a})) = \sigma(f(\bar{a}))$, for any $\bar{a} \in M^m$. We use the term *bounded* to mean of size less than $\|M\|$.

The ω-homogeneity requirement has strong consequences. Obviously, any model carries the trivial pregeometry, but it is rarely ω-homogeneous; for example, no group can carry a trivial ω-homogeneous pregeometry.

We list a few consequences of ω-homogeneity which will be used repeatedly. First, if Z is A-invariant, for some finite A, then either Z or $G\backslash Z$ is contained in $\mathrm{cl}(A)$ and hence has finite dimension: if not, choose $x, y \notin \mathrm{cl}(A)$, such that $x \in Z$ and $y \notin Z$; then some automorphism of M fixing A sends x to y, contradicting the invariance of Z. Hence, if Z is an A-invariant set, for some finite A, and has bounded dimension, then $Z \subseteq \mathrm{cl}(A)$. It follows that if a has bounded orbit under the automorphisms of M fixing the finite set A, then $a \in \mathrm{cl}(A)$. This observation has the following consequence:

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Lemma 1.3. *Suppose that M carries an ω-homogeneous pregeometry. Let $A \subseteq M$ be finite. Let $f : M^n \to M^m$ be an A-invariant function. Then, for each $\bar{a} \in M^n$ we have* $\dim(f(\bar{a})/A) \leq \dim(\bar{a}/A)$.

Proof. Write $f = (f_0, \dots, f_m)$ with A-invariant $f_i : M^n \to M$, for $i < m$. Let $\bar{a} \in M^n$. If $\dim(f(\bar{a}/A) > \dim(\bar{a}/A)$, then there is $i < m$ such that $f_i(\bar{a}) \notin \mathrm{cl}(\bar{a}A)$. But this is impossible since any automorphism M fixing $A\bar{a}$ pointwise fixes $f_i(\bar{a})$. □

We now introduce generic tuples.

Definition 1.4. Suppose that M carries an ω-homogeneous pregeometry. A tuple $\bar{a} \in M^n$ is said to be *generic over* A, for $A \subseteq M$, if $\dim(\bar{a}/A) = n$.

Since M is infinite-dimensional, for any finite $A \subseteq M$ and any $n < \omega$, there exists a generic $\bar{a} \in M^n$ over A. Further, by ω-homogeneity, if $\bar{a}, \bar{b} \in M^n$ are both generic over the finite set A, then $\bar{a}$ and $\bar{b}$ are automorphic over A. This leads immediately to a proof of the following lemma.

Lemma 1.5. *Suppose that M carries an ω-homogeneous pregeometry. Let $A \subseteq M$ be finite and let Z be an A-invariant subset of M^n. If Z contains a generic tuple over A, then Z contains all generic tuples over A.*

We now establish a few more lemmas when M is a group $(G, \cdot)$. Generic elements are particularly useful here. For example, let $\bar{a} = (a_0, \dots, a_{n-1})$ and $\bar{b} = (b_0, \dots, b_{n-1})$ belong to G^n. If $\bar{a}$ is generic over $A \cup \{b_0, \dots, b_{n-1}\}$, then it follows immediately from Lemma 1.3. that $(a_0 \cdot b_0, \dots, a_{n-1}b_{n-1})$ is generic over A.

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When $n = 1$, the next lemma asserts that if H is a proper A-invariant subgroup of G (A finite), then $H \subseteq \mathrm{cl}(A)$.

Lemma 1.6. *Let G be a group carrying an ω-homogeneous pregeometry. Suppose that H is an A-invariant subgroup of G^n (with A finite and $n < \omega$). If H contains a generic tuple over A, then $H = G^n$.*

Proof. Let $(g_0, \dots, g_{n-1}) \in G^n$. By the previous lemma, H contains a generic tuple $(a_0, \dots, a_{n-1})$ over $A \cup \{g_0, \dots, g_{n-1}\}$. Then $(a_0 \cdot g_0, \dots, a_{n-1} \cdot g_{n-1})$ is also generic over A and therefore belongs to H by another application of the previous lemma. It follows that $(g_0, \dots, g_{n-1}) \in H$. $\square$

The previous lemma implies that groups carrying an ω-homogeneous pregeometry are *connected* (see the next definition).

Definition 1.7. A group G is *connected* if it has no proper subgroup of bounded index which is invariant over a finite set.

We now introduce the *rank* of an invariant set.

Definition 1.8. Suppose that M carries an ω-homogeneous pregeometry. Let $A \subseteq M$ be finite and let Z be an A-invariant subset of M^n. The *rank of Z over A*, written $rk(Z)$, is the largest $m \leq n$ such that there is $\bar{a} \in Z$ with $\dim(\bar{a}/A) = m$.

Notice that if Z is A-invariant and if B contains A is finite, then the rank of Z over A is equal to the rank of Z over B. We will therefore omit the parameters A. The next lemma is interesting also in the case where $n = 1$; it implies that any invariant homomorphism of G is either trivial or onto.

Lemma 1.9. *Let G be a group carrying an ω-homogeneous pregeometry. Let $f : G^n \to G^n$ be an A-invariant homomorphism, for $A \subseteq G$ finite. Then*

$$rk(\ker(f)) + rk(\mathrm{ran}(f)) = n.$$

Proof. Let $k \leq n$ such that $rk(\ker(f)) = k$. Fix $\bar{a} = (a_0, \dots, a_{n-1}) \in \ker(f)$ be of dimension k over A. By a permutation, we may assume that $(a_0, \dots, a_{k-1})$ is independent over A.

Notice that by ω-homogeneity and A-invariance of $\ker(f)$, for each generic $(a'_0, \dots, a'_{i-1})$ over A (for $i < k$), there exists $(b_0, \dots, b_{n-1}) \in \ker(f)$ such that $b_i = a'_i$ for $i < k$. We now claim that for any $i < k$ and any $b \notin \mathrm{cl}(A)$, there is $\bar{b} = (b_0, \dots, b_{n-1}) \in \ker(f)$ such that $b_j = 1$ for $j < i$ and $b_i = b$. To see this, notice that $\left(a_0^{-1}, \dots, a_{i-1}^{-1}\right)$ is generic over A (by Lemma 1.3). Choose $c \in G$ generic over $A\bar{a}$. Then there is $(d_0, \dots, d_{n-1}) \in \ker(f)$ such that $d_j = a_j^{-1}$ for $j < i$ and $d_i = c$. Let $(e_0, \dots, e_{n-1}) \in \ker(f)$ be the product of $\bar{a}$ with $(d_0, \dots, d_{n-1})$. Then $e_j = 1$ if $j < i$ and $e_i = a_i \cdot c^{-1} \notin \mathrm{cl}(A)$. By ω-homogeneity, there is an automorphism of G fixing A sending e_i to b. The image of $(e_0, \dots, e_{n-1})$ under this automorphism is the desired $\bar{b}$.

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We now show that $rk(\mathrm{ran}(f)) \leq n-k$. Let $\bar{d} = f(\bar{c})$. Observe that by multiplying $\bar{c} = (c_0, \ldots, c_{n-1})$ by appropriate elements in $\ker(f)$, we may assume that $c_i \in \mathrm{cl}(A)$ for each $i < k$. Hence $\dim(\bar{c}/A) \leq n-k$ so the conclusion follows from Lemma 1.3.

To see that $rk(\mathrm{ran}(f)) \geq n-k$, choose $\bar{c} \in G^n$ such that $c_i = 1$ for $i < k$ and $(c_k, \ldots, c_{n-1})$ is generic over A. It is enough to show that $\dim(f(\bar{c})/A) \geq \dim(\bar{c}/A)$. Suppose, for a contradiction, that $\dim(f(\bar{c})/A) < \dim(\bar{c}/A)$. Then there is $i < n$, with $k \leq i$ such that $c_i \notin \mathrm{cl}(f(\bar{c})A)$. Let $d \in G \backslash \mathrm{cl}(Af(\bar{c})\bar{c})$ and choose an automorphism σ fixing $Af(\bar{c})$ such that $\sigma(c_i) = d$. Let $\bar{d} = \sigma(\bar{c})$. Then

$$f(\bar{d}) = f(\sigma(\bar{c})) = \sigma(f(\bar{c})) = f(\bar{c}).$$

Let $\bar{e} = (e_0, \ldots, e_{n-1}) = \bar{c} \cdot \bar{d}^{-1}$. Then $\bar{e} \in \ker(f)$, $e_j = 1$ for $j < k$, and $e_i = c_i \cdot d^{-1} \notin \mathrm{cl}(A)$. By ω-homogeneity, we may assume that $e_i \notin \mathrm{cl}(A\bar{a})$. But $\bar{a} \cdot \bar{e} \in \ker(f)$, and $\dim(\bar{a} \cdot \bar{e}/A) \geq k+1$ (since the ith coordinate of $\bar{a} \cdot \bar{e}$ is not in $\mathrm{cl}(a_0, \ldots, a_{k-1}A)$). This contradicts the assumption that $rk(\ker(f)) = k$. $\square$

The next theorem is obtained by adapting Reineke's proof to our context. For expository purposes, we sketch some of the proof and refer to reader to [12] for details. We are unable to conclude that groups carrying an ω-homogeneous pregeometry are abelian, but we can still obtain a lot of information.

Theorem 1.10. *Let G be a nonabelian group which carries an ω-homogeneous pregeometry. Then the center $Z(G)$ has dimension* 0, *G is not solvable, any two nonidentity elements in the quotient group $G/Z(G)$ are conjugate, and $G/Z(G)$ is torsion-free and divisible. Also the first order theory of G is unstable.*

Proof. If G is not abelian, then the center of G, written $Z(G)$, is a proper subgroup of G. Since $Z(G)$ is invariant, Lemma 1.6 implies that $Z(G) \subseteq \mathrm{cl}(\emptyset)$.

We now claim that if H is an A-invariant proper normal subgroup of G then $H \subseteq Z(G)$.

By the previous lemma, H is finite-dimensional. For $g, h \in H$, define $X_{g,h} = \{x \in G : g^x = h\}$. Suppose, for a contradiction, that $H \not\subseteq Z(G)$ and choose $h_0 \in H \backslash Z(G)$. If for each $h \in H$, the set $X_{h_0,h}$ is finite-dimensional, then $X_{h_0,h_1} \subseteq \mathrm{cl}(h_0 h)$, and so $G \subseteq \bigcup_{h \in H} X_{h_0,h} \subseteq \mathrm{cl}(H)$, which is impossible since H has finite dimension. Hence, there is $h_1 \in H$, such X_{h_0,h_1} is infinite-dimensional and has finite-dimensional complement. Since h_1 is conjugate to h_0, then $h_1 \notin Z(G)$. Similarly, there is $h_2 \in H$ such that X_{h_1,h_2} has finite-dimensional complement. This allows us to choose $a, b \in G$ such that a, b, ab belong to both X_{h_0,h_1} and X_{h_1,h_2}. Then, $h_1 = h_0^{ab} = \left(h_0^b\right)^a = h_2$. This implies that the centralizer of h_1 has infinite dimension (since it is X_{h_1,h_2}) and must therefore be all of G by the first paragraph of this proof.

We now claim that $G^* = G/Z(G)$ is not abelian. Suppose, for a contradiction, that G^* is abelian. Let $a \in G \backslash Z(G)$. Then the sets $X_{a,k} = \{b \in G : a^b = ak\}$, where $k \in Z(G)$ form a partition of G, and so, as above, there is $k_a \in Z(G)$ such that

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X_{a,k_a} has infinite dimension, thus finite-dimensional complement. Thus, cX_{a,k_a} has finite-dimensional complement also, and hence nontrivial intersection with X_{a,k_a}. Arguing with an element in the intersection, we obtain that $k_a = 1$. But then, $X_{a,1}$ is a subgroup of G of infinite dimension and so is equal to G, which implies that $a \in Z(G)$, a contradiction.

Since $G/[G,G]$ is abelian, and $[G,G]$ is normal and invariant, then it cannot be proper (otherwise $[G,G] \leq Z(G)$). It follows that G is not solvable.

It follows easily from the previous claims that G^* is centerless. We now show that any two nonidentity elements in G^* are conjugate: let $a^* \in G^*$ be a nonidentity element. Since G^* is centerless, the centralizer of a^* in G^* is a proper subgroup of G^*. Hence, the inverse image of this centralizer under the canonical homomorphism induces a proper subgroup of G, which must therefore be of finite dimension. Hence, the set of conjugates of a^* in G^* is all of G^*, except for a set of finite dimension. It then follows that the set of elements of G^* which are not conjugates of a^* must have bounded dimension.

Since this holds for any nonidentity $b^* \in G^*$, this implies that any two nonidentity elements of G^* must be conjugates. The instability of $Th(G)$ now follows as in the proof of [26, Theorem 3.10]: since $G/Z(G)$ is not abelian and any two nonidentity elements of it are conjugate, we can construct an infinite strictly ascending chain of centralizers. This contradicts first order stability.

That G is torsion free and divisible is proved similarly (see [12] for details). □

Hyttinen called such groups *bad* in [12], but this conflicts with a standard notion, so we re-baptize them:

Definition 1.11. We say that a group G is *non-classical* if it is nonabelian and carries an ω-homogeneous pregeometry.

Question 1.12. Are there non-classical groups? And if there are, can they arise in the model-theoretic contexts we consider in this paper?

We now turn to fields. Here, we are able to adapt the proof of Macintyre's classical theorem [23] that ω-stable fields are algebraically closed.

Theorem 1.13. *A field carrying an ω-homogeneous pregeometry is algebraically closed.*

Proof. To show that F is algebraically closed, it is enough to show that any finite-dimensional field extension K of F is perfect, and has no Artin–Schreier or Kummer extension.

Let K be a field extension of F of finite degree $m < \omega$. Let $P \in F[X]$ be an irreducible polynomial of degree m such that $K = F(\xi)$, where $P(\xi) = 0$. Let A be the finite subset of F consisting of the coefficients of P. We can represent K in F as follows: K^+ is the vector space F^m, i.e. $a = a_0 + a_1\xi + \cdots + a_{m-1}\xi^{m-1}$ is represented as $(a_0, \ldots, a_{m-1})$. We can then easily represent addition in K and multiplication

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(the field product in K induces a bilinear form on $(F^+)^m$) as A-invariant operations. Notice that an automorphism σ of F fixing A pointwise induces an automorphism of K, via

$$(a_0, \ldots, a_{m-1}) \mapsto (\sigma(a_0), \ldots, \sigma(a_{m-1})).$$

We now consider generic elements of the field. For a finite subset $X \subseteq F$ containing A, we say that an $a \in K$ is *generic over* X if $\dim(a_0 \cdots a_{m-1}/X) = m$ (that is, $(a_0, \ldots, a_{m-1})$ is generic over X), where $a_i \in F$ and $a = a_0 + a_1\xi + \cdots + a_{m-1}\xi^{m-1}$. Notice that if $a, b \in K$ are generic over X (with $X \subseteq F$ finite containing A) then there exists an automorphism of K fixing X sending a to b. We prove two claims about generic elements.

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Claim 1.14. *Assume that $a \in K$ is generic over the finite set X, with $A \subseteq X \subseteq F$. Then a^n, $a^n - a$ for $n < \omega$, as well as $a + b$ and ab for $b = b_0 + b_1\xi + \cdots + b_{m-1}\xi^{m-1}$, $b_i \in X$ $(i < m)$ are also generic over X.*

Proof. We prove that a^n is generic over X. The other proofs are similar. Suppose, for a contradiction, that $a^n = c_0 + c_1\xi + \cdots + c_{m-1}\xi^{m-1}$ and $\dim(c_0 \cdots c_{m-1}/X) < m$. Then $\dim(a_0 \cdots a_{m-1}/Xc_0 \cdots c_{m-1}) \geq 1$, so there is $a_i \notin \mathrm{cl}(Xc_0 \cdots c_{m-1})$. Since F is infinite-dimensional, there are infinitely many $b \in F \setminus \mathrm{cl}(Xc_0 \cdots c_{m-1})$, and by ω-homogeneity there is an automorphism of F fixing $Xc_0 \cdots c_{m-1}$ sending a_i to b. It follows that there are infinitely many $x \in K$ such that $x^n = a^n$, a contradiction. □

Claim 1.15. *Let G be an A-invariant subgroup of K^+ (respectively, of K^*). If G contains an element generic over A then $G = K^+$ (respectively, $G = K^*$).*

Proof. We prove only one of the claims, as the other is similar. First, observe that if G contains an element of K generic over A, it contains all elements of K generic over A. Let $a \in K$ be arbitrary. Choose $b \in K$ generic over Aa. Then $b \in G$, and since $a + b$ is generic over Aa (and hence over A), we have also $a + b \in G$. It follows that $a \in G$, since G is a subgroup of K^+. Hence $G = K^+$. □

Consider the A-invariant subgroup $\{a^n : a \in K^*\}$ of K^*. Let $a \in K$ be generic over A. Since a^n is generic over A by the first claim, we have that $\{a^n : a \in K^*\} = K^*$ by the second claim. This shows that K is perfect (if the characteristics is a prime p, this follows from the existence of pth roots, and every field of characteristics 0 is perfect).

Suppose F has characteristics p. The A-invariant subgroup $\{a^p - a : a \in K^+\}$ of K^+ contains a generic element over A and hence $\{a^p - a : a \in K^+\} = K^+$.

The two previous paragraphs show that K is perfect and has no Kummer extensions (these are obtained by adjoining a solution to the equation $x^n = a$, for some $a \in K$) or Artin–Schreier extensions (these are obtained by adjoining a solution

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to the equation $x^p - x = a$, for some $a \in K$, where p is the characteristics). This finishes the proof. □

Question 1.16. If there are non-classical groups, are there also division rings carrying an ω-homogeneous pregeometry which are not fields?

2. Group Acting on Pregeometries

In this section, we generalize some classical results on groups acting on strongly minimal sets. We start by recalling some of the facts, terminology, and results from [12].

The main concept is that of *a group* $(G, \cdot)$ (Σ, n)*-acting on a pregeometry* (P, cl) This consists essentially of the following data: a group $(G, \cdot)$ acting on the universe P of a pregeometry (P, cl) in a way which respects the closure operator and which is homogeneous with respect to a specified group of automorphisms Σ of this action. We now make this more precise.

We consider an infinite-dimensional pregeometry (P, cl) with $\mathrm{cl}(\emptyset) = \emptyset$. We assume, in addition, that $\dim(P) > 2^{|\mathrm{cl}(A)|}$, for $A \subseteq P$ finite (see Remark 2.1 below).

We consider a group $(G, \cdot)$ which acts on the elements of P (we write $g(a)$ for the action of $g \in G$ on $a \in P$) and respects the pregeometry, i.e.

$$a \in \mathrm{cl}(A) \quad \text{if and only if} \quad g(a) \in \mathrm{cl}(\{g(b) : b \in A\}),$$

for $a \in P$, $A \subseteq P$ and $g \in G$.

For a tuple $\bar{x} = (x_1, \ldots, x_n) \in P^n$ and $g \in G$, we write $g(\bar{x})$ for $(g(x_1), \ldots, g(x_n))$. We assume that the action of G on P is an *n-action*, i.e. satisfies the following two properties:

- The action has *rank* n: Whenever $\bar{x}$ and $\bar{y}$ are two n-tuples of elements of P such that $\dim(\bar{x}\bar{y}) = 2n$, then there is $g \in G$ such that $g(\bar{x}) = \bar{y}$. However, for some $(n+1)$-tuples $\bar{x}, \bar{y}$ with $\dim(\bar{x}\bar{y}) = 2n+2$, there is no $g \in G$ is such that $g(\bar{x}) = \bar{y}$.
- The action is $(n+1)$*-determined*: Whenever the action of $g, h \in G$ agree on an $(n+1)$-dimensional subset X of P, then $g = h$.

By an *automorphism of the group action*, we mean a pair of automorphisms (σ_1, σ_2), where σ_1 is an automorphism of the group $(G, \cdot)$ and σ_2 is an automorphism of the pregeometry (P, cl), which preserve the group action, i.e.

$$\sigma_2(g(x)) = \sigma_1(g)\sigma_2(x).$$

Following model-theoretic practice, we will simply think of (σ_1, σ_2) as a single automorphism σ acting on two disjoint structures (the group and the pregeometry) and write $\sigma(g(x)) = \sigma(g)\sigma(x)$.

We consider a group of automorphisms Σ of this group action and we assume finally that the group action is ω*-homogeneous with respect to* Σ, i.e. whenever

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$X \subseteq P$ is *finite* and $x, y \in P \setminus \text{cl}(X)$, then there is an automorphism $\sigma \in \Sigma$ such that $\sigma(x) = y$ and $\sigma \upharpoonright X = \text{id}_X$.

Remark 2.1. The assumption that P has big dimension compared to the size of the closure of finite sets is very natural and can easily be seen to hold in the model-theoretic contexts of Secs. 3 and 4, as we always work inside sufficiently large universal domains. This assumption will be used in Proposition 2.29, in the form of the pigeonhole principle.

This is essentially the set-up that Hyttinen isolated in [12, Definition 1.1]. There are two slight differences: (1) We specify the automorphism group Σ, whereas [12] works with *all* automorphisms of the action (but there he allows extra structure on P, thus changing the automorphism group, so the settings are equivalent). (2) We require the existence of $\sigma \in \Sigma$ such that $\sigma(x) = y$ and $\sigma \upharpoonright X = \text{id}_X$, when $x, y \notin \text{cl}(X)$ only for *finite* $X \subseteq P$. All the statements and proofs from [12] can be easily modified. The results of the beginning of this section (up to Hypothesis 2) are easy adaptations from the proofs in [12]. To avoid unnecessary repetitions, we sometimes list some of these results as facts and refer the reader to [12].

We sometimes simply say homogeneous group action, when the identity of Σ or n is clear from the context.

From now, until the end of this section we make the following hypothesis.

Hypothesis 2.2. The group $(G, \cdot)$ (Σ, n)-acts on the pregeometry (P, cl).

This includes all the assumptions on P above, in particular that $\text{cl}(\emptyset) = \emptyset$ and the dimension of P is large compared the the size of the closure of finite sets.

Recall that a set X is *invariant over* A if X is fixed setwise by each automorphism $\sigma \in \Sigma$ fixing A pointwise. We simply say *invariant* for $\emptyset$-invariant. We isolate a few observations in the fact below, which we prove as a warm-up exercise.

Fact 2.3.

(1) Let $x \in P$ and $A \subseteq P$ be finite. If x is fixed under any automorphism in Σ fixing A pointwise, then $x \in \text{cl}(A)$.
(2) Let $X, A \subseteq P$ with A finite. If X is A-invariant and $\dim(X) < \dim(P)$, then $X \subseteq \text{cl}(A)$ and so X is finite-dimensional.
(3) Let $\bar{x}$ and $\bar{y}$ be n-tuples each of dimension n, then there is $g \in G$ such that $g(\bar{x}) = \bar{y}$.
(4) For no pair of $(n+1)$-tuples $\bar{x}$, $\bar{y}$ with $\dim(\bar{x}\bar{y}) = 2n+2$, is there a $g \in G$ sending $\bar{x}$ to $\bar{y}$.
(5) If $\dim(\bar{x}g(\bar{x})) = 2n$ and $y \notin \text{cl}(\bar{x}g(\bar{x}))$, with $g \in G$, then

$$g(y) \in \text{cl}(\bar{x}yg(\bar{x})).$$

Proof. (1) and (2) use Σ and the ω-homogeneity in a similar way to the corresponding facts proved in the first section.

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(3) Choose an independent n-tuple $\bar{z}$ such that $\dim(\bar{x}\bar{z}) = \dim(\bar{y}\bar{z}) = 2n$, which is possible since $\dim(P)$ is infinite. Since the action has rank n, there are $g_0, g_1 \in G$ such that $g(\bar{x}) = \bar{z}$ and $g(\bar{z}) = \bar{y}$. Then $g_0 \cdot g_1 \in G$ sends $\bar{x}$ to $\bar{y}$.

(4) Since the action has rank n, there are $(n+1)$-tuples $\bar{x}'$ and $\bar{y}'$ with $\dim(\bar{x}'\bar{y}') = 2n+2$ for which there does not exist $g \in G$ with $g(\bar{x}') = \bar{y}'$. Let $\bar{x}, \bar{y}$ be any other $(n+1)$-tuples with $\dim(\bar{x}\bar{y}) = 2n+2$. Suppose, for a contradiction, that there is $h \in G$ such that $h(\bar{x}) = \bar{y}$. By homogeneity, there is $\sigma \in \Sigma$ such that $\sigma(\bar{x}) = \bar{x}'$ and $\sigma(\bar{y}) = \bar{y}'$ since $\bar{x}\bar{y}$ and $\bar{x}'\bar{y}'$ are two independent sequences of the same length. Then $\sigma(h) \in G$ and

$$\sigma(h)(\bar{x}') = \sigma(h)\sigma(\bar{x}) = \sigma(h(\bar{x})) = \sigma(\bar{y}) = \bar{y}',$$

a contradiction to the choice of $\bar{x}'$ and $\bar{y}'$.

(5) Assume, for a contradiction, that $g(y) \notin \mathrm{cl}(\bar{x}g(\bar{x})y)$. By homogeneity, for any $z \notin \mathrm{cl}(\bar{x}g(\bar{x})y)$, there is $\sigma \in \Sigma$ fixing $\bar{x}g(\bar{x})y$ such that $\sigma(g(y)) = z$. Then $\sigma(g) \in G$ agrees with g on $\bar{x}$ and sends y to z. By homogeneity again, this implies that whenever $\bar{x}$ is an n-tuple and $y, z \in P$ are such $\dim(\bar{x}yz) = 2n+2$, there is $h \in G$ such that $h(\bar{x}) = \bar{x}$ and $h(y) = z$. Furthermore, by choosing an extra element as in (3), this implies that whenever $\bar{x}$ is an n-tuple and $y, z \in P$ such that $\dim(\bar{x}y) = \dim(\bar{x}z) = n+1$, there is $h \in G$ fixing $\bar{x}$ and sending y to z.

We use this to contradict (4): let $\bar{x}, \bar{x}'$ be n-tuples and $y, y' \in P$ such that $\dim(\bar{x}y\bar{x}'y') = 2n+2$. Choose $g \in G$ such that $g(\bar{x}) = \bar{x}'$, which is possible since the action has rank n. Then $\dim(g(\bar{x})g(y)g(\bar{x}')g(y')) = 2n+2$. But $g(\bar{x}) = \bar{x}'$, so $\dim(\bar{x}'g(y)) = n+1$. Since also $\dim(\bar{x}'y') = n+1$, there exists $h \in G$ such that $h(\bar{x}') = \bar{x}'$ and $h(g(y)) = y'$ by the previous paragraph. In all, we have found an element $h \cdot g \in G$ sending $\bar{x}y$ to $\bar{x}'y'$, which contradicts (4). □

As we pointed out, the classical example of homogeneous group actions on a pregeometry are definable groups acting on a strongly minimal sets inside a saturated model. Model theory provides important tools to deal with this situation; we now give generalizations of these tools and define *types*, *stationarity*, *generic elements*, *connected component*, and so forth in this general context. Notice also that the forthcoming definitions will depend on all this data, including the automorphism group Σ (for example, the notion of invariance, the definition of types, stationarity, etc.).

Let A be a k-element independent subset of P with $k < n$. We can form a new homogeneous group action by *localizing at* A: the group $G_A \leq G$ is the pointwise stabilizer of A; the pregeometry P_A is obtained from P by considering the new closure operator $\mathrm{cl}_A(X) = \mathrm{cl}(A \cup X) \backslash \mathrm{cl}(A)$ on the set $P \backslash \mathrm{cl}(A)$; then G_A acts on P_A by restriction; and let Σ_A be the group of automorphisms in Σ fixing A pointwise. Then the group G_A $(\Sigma_A, n-k)$-acts on the pregeometry P_A. In the rest of this section, when we consider localizations of the group n-action at some set A, it is always assumed to be given with this particular Σ_A (this will be used in Definition 2.9 for

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example). Generally, for $A \subseteq G \cup P$, we denote by Σ_A the group of automorphisms in Σ which fix A pointwise.

Since we have a Σ-homogeneous group action of G on P, we can use Σ to talk about *types* of elements of G: these are the orbits of elements of G under Σ. Similarly, the *type of an element* $g \in G$ *over* $X \subseteq P$ is the orbit of g under Σ_X. We write $\mathrm{tp}(g/X)$ for the type of g over X. In the next definition, the integer n is the one given by the (Σ, n)-action.

Definition 2.4. We say that $g \in G$ is *generic* over $X \subseteq P$, if there exists an independent n-tuple $\bar{x}$ of P such that

$$\dim(\bar{x}g(\bar{x})/X) = 2n.$$

It is immediate that if g is generic over X then so is its inverse. It follows immediately from the fact that P is infinite-dimensional and that the action has rank n, that given a finite set $X \subseteq P$, there is a $g \in G$ generic over X. Observe that if g is generic over X, so is $\sigma(g)$, where $\sigma \in \Sigma_X$, i.e. the genericity of g over X is a property of its type over X; we can therefore talk about *generic types over* X, which are simply types of elements generic over X. Finally, if $\mathrm{tp}(g/X)$ is generic over X, $X \subseteq Y$ are finite, then there is $h \in G$ generic over Y such that $\mathrm{tp}(h/X) = \mathrm{tp}(g/X)$.

We can now define stationarity in the natural way (notice the extra condition on the number of types; this condition holds trivially in model-theoretic contexts).

Definition 2.5. We say that G is *stationary* if whenever $g, h \in G$ with $\mathrm{tp}(g/\emptyset) = \mathrm{tp}(h/\emptyset)$ and $X \subseteq P$ is finite and both g and h are generic over X, then $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$. Furthermore, we assume that the number of types over each finite set is bounded.

The following is a strengthening of stationarity.

Definition 2.6. We say that G has *unique generics* if for all finite $X \subseteq P$ and $g, h \in G$ generic over X we have $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$.

If the (Σ, n)-action of G on P is n-determined (rather than simply $(n+1)$-determined), then G has unique generics: if $g, h \in G$ are generic over the finite set X and $\bar{x}, \bar{y} \in P$ are two n-tuples such that

$$\dim(\bar{x}g(\bar{x})/X) = 2n = \dim(\bar{y}h(\bar{y})/X),$$

then there is $\sigma \in \Sigma$ fixing X such that $\sigma(\bar{x}) = \bar{y}$ and $\sigma(g(\bar{x})) = h(\bar{y})$. From this, it follows that $\sigma(g) = h$, since $\sigma(g)$ and h agree on the n-dimensional set $\bar{y}$.

We now introduce the *connected component* G^0: we let G^0 be the intersection of all $\emptyset$-invariant, normal subgroups of G with bounded index.

The proof of the next fact is left to the reader; it is [12, Lemma 3.2].

Fact 2.7. If G is stationary then G^0 is a normal invariant subgroup of G of bounded index. Then G^0 (Σ^0, n)-acts on the pregeometry (P, cl) by restriction, where Σ^0 is obtained from Σ by restriction to G^0.

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We provide the proof of the next proposition to convey the flavour of these arguments. Note that because of the previous fact, if two elements of G^0 are automorphic with respect to an element $\sigma \in \Sigma$, then they are also automorphic with respect to Σ^0 above (since G^0 is invariant under Σ). Hence, in the next proposition, proving uniqueness of generics with respect to Σ or Σ^0 is the same.

Proposition 2.8. *If G is stationary then G^0 has unique generics.*

Proof. Let Q be the set of generic types over the empty set. For $q \in Q$ and $g \in G$, we define gq as follows: let $X \subseteq P$ with the property that $\sigma \restriction X = \mathrm{id}_X$ implies $\sigma(g) = g$ for any $\sigma \in \Sigma$. Choose $h \models q$ which is generic over X. Define $gq = \mathrm{tp}(gh/\emptyset)$.

Notice that by stationarity of G, the definition of gq does not depend on the choice of X or the choice of h. Similarly, the value of gq depends on $\mathrm{tp}(g/\emptyset)$ only. We claim that

$$q \mapsto gq$$

is a group action of G on Q. Since $1q = q$, in order to prove that this is indeed an action on Q, we need to show that gq is generic and $(gh)(q) = g(hq)$.

This is implied by the following claim: if $X \subseteq P$ is finite containing $\bar{x}$ and $g(\bar{x})$, where $\bar{x}$ is an independent $(n+1)$-tuple of elements in P, and $h \models q$ is generic over X, then gh is generic over X.

To see the claim, choose $\bar{z}$ an n-tuple of elements of P such that

$$\dim(\bar{z}h(\bar{z})/X) = 2n.$$

Notice that $h(\bar{z}) \subseteq \mathrm{cl}(Xgh(\bar{z}))$, since any $\sigma \in \Sigma$ fixing $Xgh(\bar{z})$ pointwise fixes $h(\bar{z})$ (for any such σ, we have $\sigma(h(\bar{z})) = \sigma(g^{-1}gh(\bar{z})) = \sigma(g^{-1})\sigma(gh(\bar{z})) = g^{-1}gh(\bar{z}) = h(\bar{z})$). Thus, $\dim(\bar{z}gh(\bar{z})/X) \geq \dim(\bar{z}h(\bar{z})/X) = 2n$, so $\bar{z}$ demonstrates that gh is generic over X.

Now consider the kernel H of the action, namely the set of $h \in G$ such that $hq = q$ for each $q \in Q$. This is clearly an invariant subgroup, and since the action depends only on $\mathrm{tp}(h/\emptyset)$, H must have bounded index (this condition is part of the definition of stationarity). Hence, by definition, the connected component G^0 is a subgroup of H.

By stationarity of G, if G^0 does not have unique generics, there are $g, h \in G^0$ generic over the empty set such that $\mathrm{tp}(g/\emptyset) \neq \mathrm{tp}(h/\emptyset)$. Without loss of generality, we may assume that h is generic over $\bar{x}g(\bar{x})$, where $\bar{x}$ is an independent $(n+1)$-tuple of P. Now it is easy to check that $hg^{-1}(\mathrm{tp}(g/\emptyset)) = \mathrm{tp}(h/\emptyset))$, so that $hg^{-1} \notin H$. But $hg^{-1}h \in G^0 \subseteq H$, since $g, h \in G^0$, a contradiction. □

We consider a further strengthening of stationarity which will arise naturally in model-theoretic contexts. Note again that this definition depends on Σ.

Definition 2.9. We say that G *admits hereditarily unique generics* if G has unique generics and for any independent k-set $A \subseteq P$ with $k < n$, there is a normal

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subgroup G' of G_A such that G' $(\Sigma', n-k)$-acts on P_A (for some subgroup $\Sigma' \leq \Sigma$), which has unique generics with respect to Σ.

Notice that whether an element of $g \in G$ is generic does not depend on whether or not g belongs to some subgroup $G' \leq G$. Uniqueness of generics depends on the automorphism group and in the previous definition, we only require the easiest condition: that the generics be automorphic under Σ (and not any particular $\Sigma' \leq \Sigma$).

Admitting hereditarily unique generics is connected to n-determinacy and non-classical groups in the following way. The next fact is [12, Theorem 2.7] (note that "nonclassical" groups are called "bad" groups there).

Fact 2.10. Suppose that G admits hereditarily unique generics. Then either $(G_A)^0$ is non-classical, for some independent $(n-1)$-subset $A \subseteq P$ or the action of G on P is n-determined.

A key idea in the proof is the next fact, which we prove, as it will be used several times in this section.

Fact 2.11. Let $n = 1$. If G has unique generics, then G carries an ω-homogeneous pregeometry.

Proof. We define a closure operator cl on the subsets of G as follows: for $g \in G$ and $g_0, \ldots, g_k \in G$ we let

$$g \in \mathrm{cl}(g_0, \ldots, g_k),$$

if for some independent 2-tuple $\bar{y} \in P$ and some $x \in P \backslash \mathrm{cl}(\bar{y} g(\bar{y}) g_0(\bar{y}) \cdots g_k(\bar{y}))$ then

$$g(x) \in \mathrm{cl}(x g_0(x), \ldots, g_k(x)).$$

Notice first that this definition does not depend on the choice of x and $\bar{y}$: let $x' \notin \mathrm{cl}(\bar{y}' g(\bar{y}') g_0(\bar{y}') \cdots g_k(\bar{y}'))$ for another independent 2-tuple $\bar{y}'$. Let z be such that

$$z \notin \mathrm{cl}(\bar{y} g(\bar{y}) g_0(\bar{y}) \cdots g_k(\bar{y}) \bar{y}' g_0(\bar{y}') \cdots g_k(\bar{y}')).$$

Then by homogeneity, there exists $\sigma \in \Sigma_{\bar{y} g_0(\bar{y}) \cdots g_k(\bar{y})}$ such that $\sigma(x) = z$, and $\tau \in \Sigma_{\bar{y}' g_0(\bar{y}') \cdots g_k(\bar{y}')}$ such that $\tau(z) = x'$. Notice that $\sigma(g) = \tau(g) = g$ and $\sigma(g_i) = \tau(g_i) = g_i$ for $i \leq k$ by 2-determinacy. Hence $g(x) \in \mathrm{cl}(x g_0(x), \ldots, g_k(x))$ if and only if $g(x') \in \mathrm{cl}(x' g_0(x), \ldots, g_k(x))$ by applying $\sigma \circ \tau$.

We define $g \in \mathrm{cl}(A)$ for g, A in G, where A may be infinite, if there are $g_0, \ldots, g_k \in G$ such that $g \in \mathrm{cl}(g_0, \ldots, g_k)$. It is not difficult to check that this induces a pregeometry on G with same infinite dimension as P. Notice however, that even though the closure of the empty set is empty in P by assumption, the induced closure on G contains the identity element of G.

The unicity of generics implies that the pregeometry is ω-homogeneous: suppose $g, h \notin \mathrm{cl}(A)$, where $A \subseteq G$ is finite. We need to find $\sigma \in \Sigma$ fixing A sending g to h.

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For a tuple $\bar{z}$, write $A(\bar{z}) = \{f(\bar{z}) : f \in A\}$. We will find an independent pair $z_1, z_2 \in P$ such that both g and h are generic over $z_1 z_2 A(z_1) A(z_2)$. This is enough: by uniqueness of generics there is $\sigma \in \Sigma$ sending g to h fixing $z_1 z_2 A(z_1) A(z_2)$, and since $n = 1$ (and so the action is 2-determined) σ must fix A pointwise. Here is how we find z_1 and z_2: first, choose $\bar{y} \in P^2$ an independent pair and choose $z_1 \in P$ such that

$$z_1 \notin \mathrm{cl}(\bar{y} g(\bar{y}) h(\bar{y}) A(\bar{y})).$$

Since $g \notin \mathrm{cl}(A)$, we have $g(z_1) \notin \mathrm{cl}(z_1 A(z_1))$ by definition of the closure on G. Let $x \in P$ with

$$x \notin \mathrm{cl}(z_1 g(z_1) h(z_1) A(z_1) \bar{y} g(\bar{y}) h(\bar{y}) A(\bar{y})).$$

Since $n = 1$, we must have $g(z_1) \in \mathrm{cl}(x z_1 g(x))$. Hence $g(x) \notin \mathrm{cl}(z_1 x A(x))$, so $\dim(x g(x)/z_1 A(z_1)) = 2$. Let $z_2 \in P$ with

$$z_2 \notin \mathrm{cl}(z_1 x g(z_1) g(x) h(z_1) h(x) A(z_1) A(x)).$$

Then since $n = 1$, we have that $f(z_2) \in \mathrm{cl}(z_1 z_2 A(z_1))$, for each $f \in A$, and so

$$\dim(x g(x)/z_1 z_2 A(z_1) A(z_2)) = 2.$$

Since the situation is entirely symmetric, we can apply the same argument to h and show that

$$\dim(x h(x)/z_1 z_2 A(z_1) A(z_2)) = 2.$$

Hence, since $n = 1$, we have that g, h are generic over $z_1 z_2 A(z_1) A(z_2)$. This finishes the argument. □

So in the case of $n = 1$, either the connected component is non-classical, or it is abelian and the action of G on P is 1-determined. Hence, the action of G^0 on P is regular.

The next fact is [12, Lemma 2.8].

Fact 2.12 If the action of G on P is n-determined then $n = 1, 2, 3$.

Definition 2.13. We say that the (Σ, n)-action of G on the pregeometry (P, cl) is *sharp* if it is n-determined.

We will also say that G (Σ, n)-acts sharply on the pregeometry (P, cl). If G (Σ, n)-acts sharply on P, then the element of G sending a given independent n-tuple of P to another is unique. However, this does not mean that the action is sharp on the set of elements of P.

This finishes the preliminaries. The results in the rest of this section are new. We are interested in producing fields, so we assume $n \geq 2$. We assume further that G (Σ, n)-acts sharply on the pregeometry P. Then $n = 2, 3$. If $n = 3$ and $a \in P$, then G_a $(\Sigma_a, 2)$-acts sharply on the pregeometry P_a, obtained by localizing

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at a. So, without loss of generality, we can assume that $n = 2$. From now until Proposition 2.31 we make the following additional hypothesis.

Hypothesis 2.14. The group $(G, \cdot)$ $(\Sigma, 2)$-acts sharply on the pregeometry (P, cl).

Remark 2.15. The application (Theorem 2.32) only deals with groups acting on *geometries*. However, when we study the case $n = 3$ where G $(\Sigma, 3)$-acts sharply on the geometry P, and localize at a point $a \in P$, then the group G_a $(\Sigma_a, 2)$-acts sharply on the localized *pregeometry* P_a, but it is not clear *a priori* why the action should be sharp on the *geometry* $(P_a)'$ obtained from P_a. In fact, we will prove that when G $(\Sigma, 2)$-acts sharply on the pregeometry (P, cl), then (P, cl) is a geometry and the action is sharply 2-transitive on the set of elements of P (Proposition 2.29).

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We start by proving a few useful lemmas on generic elements.

Lemma 2.16. *Let $a, b \in P$ and $g \in G$ such that $g(a) \in \mathrm{cl}(a)$ and $a, b, g(b)$ are independent. Then g is generic.*

Proof. Choose $x_1, x_2 \in P$ such that $\dim(abg(b)x_1x_2) = 5$. We will show that $\dim(x_1x_2g(x_1)g(x_2)) = 4$, so g is generic. Suppose, for a contradiction, that $\dim(x_1x_2g(x_1)g(x_2)) \leq 3$. Notice that $g(b) \in \mathrm{cl}(bx_1x_2g(x_1)g(x_2))$, since any $\sigma \in \Sigma$ fixing $bx_1x_2g(x_1)g(x_2)$ fixes g (since the action is 2-determined), and hence fixes $g(b)$. So $\dim(bg(b)x_1x_2g(x_1)g(x_2)) \leq 4$. Notice further that

$$b \notin \mathrm{cl}(x_1x_2g(x_1)g(x_2)),$$

for otherwise $\dim(bg(b)x_1x_2g(x_1)g(x_2)) \leq 3$, contradicting the choice of x_1, x_2.

We now claim that $a \in \mathrm{cl}(x_1x_2g(x_1)g(x_2))$. If not, by ω-homogeneity with respect to Σ, we can find $\sigma \in \Sigma$ fixing $x_1x_2g(x_1)g(x_2)$ such that $\sigma(a) = b$. But σ fixes g, so applying σ to $g(a) \in \mathrm{cl}(a)$, gives $g(\sigma(a)) \in \mathrm{cl}(\sigma(a))$, so $g(b) \in \mathrm{cl}(b)$, which is impossible since b and $g(b)$ are independent.

So, $a \in \mathrm{cl}(x_1x_2g(x_1)g(x_2))$, but then $\dim(abg(b)x_1x_2g(x_1)g(x_2)) \leq 4$, which contradicts the choice of x_1, x_2. $\square$

We say that $a \in P$ is a *fixed point of* $g \in G$ if $g(a) = a$. The previous lemma implies immediately that each generic element of G has a fixed point: let a, b, c be independent. Any element $g \in G$ fixing a sending b to c satisfies the assumptions of the previous lemma, and hence must be generic. But G has unique generics (since the action is 2-determined) so all generics have a fixed point. It is obvious that generics cannot have two independent fixed points, since the action is 2-determined. The next lemma shows that generics fix all the elements in the closure of a fixed point. Recall that $2^{|\mathrm{cl}(A)|} < \dim(P)$, for $A \subseteq P$ finite.

Lemma 2.17. *Let $g \in G$ be generic and $a \in P$ such that $g(a) = a$. Then g fixes $\mathrm{cl}(a)$ pointwise.*

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Proof. Since G has unique generics, it is enough to find *some* generic $g \in G$ such that $g \restriction \mathrm{cl}(a) = \mathrm{id}$.

Let $b \notin \mathrm{cl}(a)$. For each $c \notin \mathrm{cl}(ab)$, there is $g_c \in G$ such that $g_c(a) = a$ and $g_c(b) = c$, since the action has rank 2. By the previous lemma, g_c is generic for each $c \notin \mathrm{cl}(ab)$. Observe that there are only $2^{|\mathrm{cl}(A)|} < \dim(P)$ distinct functions on $\mathrm{cl}(A)$, hence, by the pigeonhole principle, we can find c, d independent over a such that $g_c \restriction \mathrm{cl}(a) = g_d \restriction \mathrm{cl}(a)$. Let $g = g_d^{-1} g_c$. Then $g \restriction \mathrm{cl}(a) = \mathrm{id}$. Also, $g(b) = g_d^{-1} g_c(b) = g_d^{-1}(c)$. Now $c \notin \mathrm{cl}(ad)$, so $g_d^{-1}(c) \notin \mathrm{cl}(g_d^{-1}(a) g_d^{-1}(d)) = \mathrm{cl}(ab)$. So $a, b, g(b)$ are independent, and therefore g is generic by Lemma 2.16. □

Lemma 2.18. *Let $b \notin \mathrm{cl}(a)$. Let $g \in G$ such that $g(a) = a$ and $b \neq g(b) \in \mathrm{cl}(b)$. Then g is generic.*

Proof. Suppose, for a contradiction, that g is not generic. Choose $x_1, x_2 \in P$ such that $\dim(abx_1x_2) = 4$. Since g is not generic, have $\dim(x_1x_2g(x_1)g(x_2)) \leq 3$. Hence, either $a \notin \mathrm{cl}(x_1x_2g(x_1)g(x_2))$ or $b \notin \mathrm{cl}(x_1x_2g(x_1)g(x_2))$.

If $a \notin \mathrm{cl}(x_1x_2g(x_1)g(x_2))$, then for each $c \notin \mathrm{cl}(x_1x_2g(x_1)g(x_2))$ we have $g(c) = c$ (by choosing $\sigma \in \Sigma$ fixing $x_1x_2g(x_1)g(x_2)$, and hence g, and sending a to c). This implies immediately that $g = 1$ by 2-determinacy, which is a contradiction since $g(b) \neq b$.

So $a \in \mathrm{cl}(x_1x_2g(x_1)g(x_2))$. Hence, $b \notin \mathrm{cl}(x_1x_2g(x_1)g(x_2))$. Choose $y_1, y_2 \in P$ independent over $ax_1x_2g(x_1)g(x_2)$. Then, $g(y_\ell) \in \mathrm{cl}(y_\ell)$, for $\ell = 1, 2$ (by choosing $\sigma_\ell \in \Sigma$ fixing $x_1x_2g(x_1)g(x_2)$ sending b to y_ℓ, for $\ell = 1, 2$). So $a \notin \mathrm{cl}(y_1y_2g(y_1)g(y_2))$ since $\mathrm{cl}(y_1y_2g(y_1)g(y_2)) = \mathrm{cl}(y_1y_2)$. But this implies that $g = 1$ as in the previous paragraph, which is a contradiction. □

Following Hrushovski [7], we now consider involutions.

Definition 2.19. Let $I = \{g \in G : g^2 = 1\}$.

The set I is invariant under Σ but may not be a group. We will use I to define a group (Definition 2.22). The next couple of lemmas will be handy.

Lemma 2.20. *Let $a, b \in P$ be independent. Then there is a unique $g \in I$ such that $g(a) = b$.*

Proof. Since G $(\Sigma, 2)$-acts on P sharply, there is a unique $g \in G$ such that $g(a) = b$ and $g(b) = a$. Now $g \in I$, since $g^2(a) = a$ and $g^2(b) = b$, so that g^2 agrees with $1 \in G$ on a two-dimensional set $\{a, b\}$ and so $g^2 = 1$. But any $h \in I$ with $h(a) = b$ must be such that $h(b) = a$, so we are done. □

The next lemma is a consequence of this.

Lemma 2.21. *No $g \in I$ is generic. Moreover, if $g, h \in I$ then gh is not generic.*

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Proof. Suppose first, for a contradiction, that $g \in I$ is generic. Let a, b such that $\dim(abg(a)g(b)) = 4$. Choose $\sigma \in \Sigma$ fixing $abg(a)$ such that $\sigma(g(b)) \neq g(b)$. Now, $\sigma(g) \in I$ and $\sigma(g) \neq g$, as they disagree on b. However, $\sigma(g)(a) = \sigma(g(a)) = g(a)$, and $g(a)$ is independent from a, so $\sigma(g) = g$, by the previous lemma, a contradiction.

Now suppose, for a contradiction, that $g, h \in I$ and gh is generic. Then gh has a fixed point by the paragraph following Lemma 2.16, so choose $a \in P$ such that $gh(a) = a$. Then $g(a) = h(a)$ since $g, h \in I$.

If $h(a) \notin \mathrm{cl}(a)$, then $g = h$ by the previous lemma, and so $gh = g^2 = 1$, so gh is not generic, a contradiction.

Hence, $h(a) \in \mathrm{cl}(a)$ (so $g(a) \in \mathrm{cl}(a)$). Choose any $x_1, x_2 \in P$ independent. Since h is not generic, then $\dim(ax_1h(x_1)) \leq 2$, and also $\dim(ax_2h(x_2)) \leq 2$ by Lemma 2.16. It follows that

$$\dim(ax_1x_2h(x_1)h(x_2)) \leq 3.$$

Similarly, using the fact that g is not generic, we have that $\dim(ah(x_1)g(h(x_1))) \leq 2$ and $\dim(ah(x_2)g(h(x_2))) \leq 2$, from which we derive that

$$\dim(ah(x_1)h(x_2)g(h(x_1))g(h(x_2))) \leq 3.$$

Together, these inequalities imply that $\dim(ax_1x_2gh(x_1)gh(x_2)) \leq 3$. In particular, gh is not generic, a contradiction. □

We now define N_a, which, intuitively, is the group generically generated by I. The short-term goal will be to show that N_a is a normal, invariant subgroup of G, which does not depend on a, and acts regularly on P. N_a is actually abelian, but we will not need to prove this to prove Theorem 2.32, as this follows immediately from the additional assumption that there are no nonclassical groups.

Definition 2.22. Let $a \in P$. We let $N_a \subseteq G$ consists of those elements $g \in G$ for which the set

$$\{h(a) : h \in I, gh \notin I\}$$

has bounded dimension in P.

Let us make a few simplifying observations on the definition of N_a.

Choose b independent from a. We first show that $\{h(a) : h \in I, gh \notin I\}$ is invariant over $abg(a)g(b)$: let $h \in I$, $gh \notin I$. Let $\sigma \in \Sigma$ fixing $abg(a)g(b)$. We must show that $\sigma(h(a)) \in \{h(a) : h \in I, gh \notin I\}$. But, $\sigma(h) \in I$ and $\sigma(g) = g$ since σ fixes $abg(a)g(b)$. Thus $\sigma(h)g \notin I$. Finally, $\sigma(h)(a) = \sigma(h(a))$, since σ fixes a, and hence $\sigma(h(a)) \in \{h(a) : h \in I, gh \notin I\}$. This shows invariance over $abg(a)g(b)$. Hence, since the action is ω-homogeneous with respect to Σ, the set $\{h(a) : h \in I, gh \notin I\}$ has bounded dimension if and only if it is finite-dimensional if and only if it is contained in $\mathrm{cl}(abg(a)g(b))$.

Thus, with b independent from a, we have $g \in N_a$ if and only if

$$\{h(a) : h \in I, gh \notin I\} \subseteq \mathrm{cl}(abg(a)g(b)).$$

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Since $\{h(a) : h \in I, gh \notin I\}$ is similarly invariant, we also have $g \notin N_a$ if and only if

$$\{h(a) : h \in I, gh \in I\} \subseteq \mathrm{cl}(abg(a)g(b)).$$

In other words, for $g \in G$, a, b independent, and $h \in I$ such that

$$h(a) \notin \mathrm{cl}(abg(a)g(b)),$$

then $g \in N_a$ if and only if $gh \in I$. Notice finally that none of these considerations depend on the particular b, provided b is independent from a.

We now show that N_a does not depend on a.

Lemma 2.23. *Let $a, b \in P$. Then $N_a = N_b$.*

Proof. By using a third element c independent from a and b, we may assume that a and b are independent. By symmetry, it is enough to show that if $g \in N_a$ then $g \in N_b$. Let $g \in N_a$.

Choose $d \notin \mathrm{cl}(abg(a)g(b))$ and $h \in I$ such that $h(a) = d$, which exists by Lemma 2.20. It is enough to show that $h(b) \notin \mathrm{cl}(abg(a)g(b))$: since $g \in N_a$ and $h(a) \notin \mathrm{cl}(abg(a)g(b))$, we must have $gh \in I$, and if $h(b) \notin \mathrm{cl}(abg(a)g(b))$, then necessarily $g \in N_b$.

Suppose, for a contradiction, that $h(b) \in \mathrm{cl}(abg(a)g(b))$. By Lemma 2.20, any $\sigma \in \Sigma$ fixing $abh(a)$ fixes $h(b)$, so $h(b) \in \mathrm{cl}(abh(a))$. If $h(b) \notin \mathrm{cl}(ab)$, then by exchange we have $h(a) \in \mathrm{cl}(abh(b)) \subseteq \mathrm{cl}(abg(a)g(b))$, a contradiction. Hence $h(b) \in \mathrm{cl}(ab)$. But h is not generic by Lemma 2.21, and so $h(b) \notin \mathrm{cl}(b)$ by Lemma 2.16. So $a \in \mathrm{cl}(bh(b))$ by exchange, so that $h(a) \in \mathrm{cl}(h(b)h^2(b)) = \mathrm{cl}(h(b)b)$, by applying h (which preserves cl). But then $h(a) \in \mathrm{cl}(bh(b)) \subseteq \mathrm{cl}(abg(a)g(b))$, another contradiction. □

It follows immediately that N_a is $\emptyset$-invariant. The next lemma shows that elements of N_a are determined by their action on one element, in particular there are no generics in N_a. Observe that, by the pigeonhole principle, for each $f \in G$ and finite $A \subseteq P$, there is $c \notin \mathrm{cl}(A)$ such that $f(c) \notin \mathrm{cl}(A)$. This observation will be used repeatedly.

Lemma 2.24. *Let $g, f \in N_a$ and $c \in P$. If $g(c) = f(c)$, then $g = f$.*

Proof. By the previous observation, there is $d \in P$ such that

$$d, g(d) \notin \mathrm{cl}(abg(a)g(b)f(a)f(b)).$$

Hence, by Lemma 2.23, using an automorphism sending d to c and moving a, b if necessary, we may assume that

$$c, g(c) \notin \mathrm{cl}(abg(a)g(b)f(a)f(b))).$$

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Let $h \in I$ such that $h(a) = c$. Then $gh, fh \in I$, since $g, f \in N_a$. But also $gh(a) = g(c) = f(c) = fh(a) \notin \mathrm{cl}(a)$. Hence, by Lemma 2.20, we have that $gh = fh$, so $g = f$. □

We now show that N_a is a group.

Lemma 2.25. *N_a is a group.*

Proof. Fix $b \in P$ independent from a. First, the identity element 1 of G is clearly in N_a, since $\{h(a) : h \in I, 1h \in I\}$ has unbounded dimension by Lemma 2.20.

N_a is closed under products: let $g, f \in N_a$. By the pigeonhole principle, we can choose $c \notin \mathrm{cl}(abg(a)g(b)f(a)f(b))$ such that

$$f(c) \notin \mathrm{cl}(abg(a)g(b)f(a)f(b)).$$

Let $h \in I$ such that $h(a) = c$. Then $fh \in I$, since $f \in N_a$. In order to show that $gf \in N_a$, it suffices to show that $(gf)h \in I$. But $(gf)h = g(fh)$, and to show $g(fh) \in I$, it suffices to show that $fh(a) \notin \mathrm{cl}(abg(a)g(b))$, since $fh \in I$ and $g \in N_a$. But $fh(a) = f(c)$ so we are done by the choice of c.

Finally, N_a is closed under inverses: let $g \in N_a$. Choose $h \in I$ such that $h(a) \notin \mathrm{cl}(abg(a)g(b))$. Observe that $\mathrm{cl}(abg(a)g(b)) = \mathrm{cl}(abg^{-1}(a)g^{-1}(b))$ (since any $\sigma \in \Sigma$ fixing $abg(a)g(b)$ fixes g and hence g^{-1}). Thus, since $h(a) \notin \mathrm{cl}(abg^{-1}(a)g^{-1}(b))$, to show that $g^{-1} \in N_a$, it is enough to show that $g^{-1}h \in I$. But $gh \in I$ since $g \in N_a$, so $(gh)^2 = 1$, so $g^{-1} = hgh$. It follows that $g^{-1}h = hg$. Now $hg \in I$, since $(hg)^2 = hghg = g^{-1}g = 1$. This shows that $g^{-1}h \in I$ and finishes the proof. □

Recall that the action of a group on a set is called *regular* if it is transitive and sharp.

Lemma 2.26. *The group N_a $(\Sigma', 1)$-acts sharply on (P, cl) by restriction, where Σ' is obtained from Σ by restriction. In fact, N_a acts regularly on the set of elements of P.*

Proof. Since N_a is invariant, let Σ' be obtained from Σ by restriction. By Σ-homogeneity N_a acts regularly on the elements of P if and only if N_a $(\Sigma', 1)$-acts sharply on the pregeometry (P, cl) (since $\mathrm{cl}(\emptyset) = \emptyset$). Lemma 2.24 shows that the action is 1-determined so it is enough to show that it has rank 1. Since N_a is invariant, it is enough to show that there is $g \in N_a$ such that c and $g(c)$ are independent (by Σ'-homogeneity, which is immediate from Σ-homogeneity).

Now to show that g as above exists, it is enough to find $f, h \in I$ such that $fh(c)$ is independent from c and $h(a) \notin \mathrm{cl}(abfh(a)fh(b))$: for then, $fh \in N_a$ since $h \in I$ and $(fh)h = f \in I$, and g can be taken to be fh.

For this, choose $h \in I$ such that $h(a) \notin \mathrm{cl}(ab)$. Now choose $f \in I$ such that $f(h(a)) \notin \mathrm{cl}(abh(a))$ (in particular $fh(a) \notin \mathrm{cl}(a)$). This implies that

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$h(a) \notin \text{cl}(abfh(a))$ by exchange. Notice further that $fh(b) \in \text{cl}(abfh(a))$, otherwise $\dim(abfh(a)fh(b)) = 4$, so fh is generic which contradicts Lemma 2.21. Hence $h(a) \notin \text{cl}(abfh(a)fh(b))$, which is what we wanted to show. □

Recall that for $a \in P$, we denote by G_a the elements of G fixing a. G_a $(\Sigma_a, 1)$-acts sharply on P_a. Notice also that since $\text{cl}(\emptyset) = \emptyset$, G_a is isomorphic to G_b for $a \neq b$.

Proposition 2.27. *The invariant group N_a is normal in G and $G = N_a \rtimes G_a$.*

Proof. We first show that N_a is normal in G: let $g \in N_a$ and $f \in G$. Then $\{h(a) : h \in I, gh \in I\}$ has unbounded dimension. Since conjugation by f permutes the elements of I, we must have that the set $\{fhf^{-1}(a) : fhf^{-1} \in I, fghf^{-1} \in I\}$ has unbounded dimension. But $fghf^{-1} = fgf^{-1}fhf^{-1}$, so $\{fhf^{-1}(a) : fhf^{-1} \in I, fgf^{-1}fhf^{-1} \in I\}$ has unbounded dimension. Hence, $\{h(a) : h \in I, fgf^{-1}h \in I\}$ has unbounded dimension, so $fgf^{-1} \in N_a$.

To show that $G = N_a \rtimes G_a$, it is enough to show that $N_a \cap G_a = \{1\}$ and $G = N_aG_a$. But $N_a \cap G_a = \{1\}$ since 1 is the only element of N_a fixing a by Lemma 2.24. To see that $G = N_aG_a$, let $g \in G$. Choose $c \in P$ such that $g(c) = a$. Let $h' \in N_a$ be such that $h'(c) = a$, which exists by Lemma 2.26. By the pigeonhole principle, there is $d \in P$ such that $g(d), h'(d) \notin \text{cl}(ac)$. Choose $g' \in G_a$ sending $h'(d)$ to $g(d)$, which exists since G_a acts regularly on P_a. Then $g'h'(c) = g'(a) = a = g(c)$, and $g'h'(d) = g(d)$, so $g'h'$ and g agree on a 2-dimensional set, so $g = g'h'$. □

The next lemma will be used to show that P is a geometry.

Lemma 2.28. *Let b, c be distinct from a such that $c \in \text{cl}(a)$ and $b \notin \text{cl}(a)$. Let $f \in N_a$ such that $f(a) = c$. Let $h \in I$ such that $h(a) = b$. Then $f \in I$, $f(b) \in \text{cl}(b)$, and $fh = hf$.*

Proof. Observe first that $fh \in I$: let b' be independent from a and b. Notice that $f(b') \in \text{cl}(acb') = \text{cl}(ab')$, since any automorphism fixing acb' fixes f (by Lemma 2.24) and hence fixes $f(b')$. So $h(a) \notin \text{cl}(ab'f(a)f(b'))$, by definition of f and h. Hence, $fh \in I$, since $f \in N_a$.

Since $fh \in I$, we have $fhfh = 1$, so $f(b) = hf^{-1}h(b) = hf^{-1}(a)$. But $a \in \text{cl}(f(a))$, so $f^{-1}(a) \in \text{cl}(a)$, so $hf^{-1}(a) \in \text{cl}(h(a))$. In all $f(b) \in \text{cl}(b)$.

Now to show that f commutes with h, we claim that whenever $g \in G$ is such that $\dim(abg(a)g(b)) = 4$, then g commutes with f.

This is enough: choose $g \in G$ such that $\dim(abg(a)g(b)) = 4$. Then $\dim(abgh(a)gh(b)) = 4$, since h simply permutes a and b. Hence, by applying the claim to gh and then g, we have $ghf = fgh = gfh$, so $hf = fh$.

We now prove the claim. By 2-determinacy, it is enough to prove the claim for *some* $g \in G$ with $\dim(abg(a)g(b)) = 4$. Observe that since $f(a) \in \text{cl}(a)$ and $f(b) \in \text{cl}(b)$ and a and b are independent, then $f(d) \in \text{cl}(d)$ for any $d \in P$: for $d \in P$, either $d \notin \text{cl}(a)$ or $d \notin \text{cl}(b)$; assume the latter and choose $\sigma \in \Sigma$ fixing $b, f(b)$

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such that $\sigma(a) = d$; then σ fixes f by Lemma 2.24, so $f(d) \in \mathrm{cl}(d)$ by applying σ to $f(a) \in \mathrm{cl}(a)$. The case when $d \notin \mathrm{cl}(a)$ is similar.

Hence, for each $k \in G$ we have $f(k^{-1}(a)) \in \mathrm{cl}(k^{-1}(a))$, so $kfk^{-1}(a) \in \mathrm{cl}(a)$, and similarly $kfk^{-1}(b) \in \mathrm{cl}(b)$. Therefore, by the pigeonhole principle (since $\mathrm{cl}(a)$ and $\mathrm{cl}(b)$ are bounded), there are $k, \ell \in G$ such that

$$\dim(abk(a)\ell(a)k(b)\ell(b)) = 6$$

satisfying $kfk^{-1}(a) = \ell f \ell^{-1}(a)$ and $kfk^{-1}(b) = \ell f \ell^{-1}(b)$. By 2-determinacy, we have $kfk^{-1} = \ell f \ell^{-1}$, so $k^{-1}\ell$ commutes with f. Since

$$\dim(k^{-1}(a)k^{-1}(b)abk^{-1}\ell(a)k^{-1}\ell(b)) = 6,$$

so $\dim(abk^{-1}\ell(a)k^{-1}\ell(b)) = 4$. This proves the claim by taking $g = k^{-1}f$.

Finally, observe that $f^2 = 1$: first $1 = fhfh$ since $fh \in I$, but $fhfh = f^2h^2$ since f and h commute, and lastly $f^2 = 1$ since $h^2 = 1$. □

Recall that a pregeometry (P, cl) is a *geometry* if $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(a) = \{a\}$, for each $a \in P$.

Proposition 2.29. *G acts sharply 2-transitively on the set of elements of P and P is a geometry.*

Proof. It is enough to prove that P is a geometry, since then, any two distinct points are independent.

Suppose, for a contradiction, that P is not a geometry. So there is some element $b \in P$, such that $\mathrm{cl}(b) \neq \{b\}$. By using an automorphism sending b to a, we must have $\mathrm{cl}(a) \neq \{a\}$. Fix $c \in \mathrm{cl}(a)$ with $c \neq a$.

Since N_a acts transitively on P by Lemma 2.26, there is $f \in N_a$ such that $f(a) = c$. Let $b \notin \mathrm{cl}(a)$. Choose $h \in I$ such that $h(a) = b$. Then f, h are as in the previous lemma, so f and h commute, $f \in I$, and $f(b) \in \mathrm{cl}(b)$.

Since in particular $f(b) \notin \mathrm{cl}(a)$ and G_a acts transitively on P_a, there exists $g \in G_a$ such that $g(b) = f(b)$. By Lemma 2.18 g is generic and so g fixes $\mathrm{cl}(a)$ pointwise by Lemma 2.17.

But $gfg^{-1} \in N_a$ by Proposition 2.27, and $gfg^{-1}(a) = gf(a) = f(a)$, since $f(a) \in \mathrm{cl}(a)$ and g is the identity on $\mathrm{cl}(a)$. By Lemma 2.24, this implies that $gfg^{-1} = f$. It follows that f and g commute. All together, we have that $g^2(b) = gg(b) = gf(b) = fg(b) = ff(b) = b$, since $g(b) = f(b)$, f and g commute, and $f^2 = 1$. But $g^2(a) = a$ since $g \in G_a$, so $g^2 = 1$ by 2-determinacy. Hence $g \in I$, which contradicts Lemma 2.21 since g is generic. □

We now construct a field when N_a and G_a are abelian.

Lemma 2.30. *If G_a and N_a are abelian, then the action of G_a on N_a by conjugation induces the structure of an algebraically closed field on N_a.*

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Proof. By the previous proposition, G_a acts regularly on $N_a \setminus \{1\}$: let $g, h \in N_a \setminus \{1\}$. Then, since $g, h \neq 1$, we must have $g(a), h(a) \neq a$ by Lemma 2.24. Hence $g(a), h(a) \in P_a$, since P is a geometry. But G_a acts regularly on P_a so there is a unique $f \in G_a$ such that $f(g(a)) = h(a)$. Then $fgf^{-1}(a) = h(a)$, furthermore $fgf^{-1} \in N_a$, hence $fgf^{-1} = h$ by Lemma 2.24.

We use this to define the structure of a field $(K, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ on the elements of N_a: the additive group $(K, \oplus, \mathbf{0})$ is simply the group $(N_a, \cdot, 1)$. Now fix an arbitrary nonidentity element in N_a, which we denote by $\mathbf{1}$ (not to confuse with the identity element of N_a, previously denoted by 1, which we now call $\mathbf{0}$). We put the multiplicative structure $(K^*, \otimes, \mathbf{1})$ on $N_a \setminus \{\mathbf{0}\}$, as follows: for each $g \in N_a \setminus \{\mathbf{0}\}$, let $f_g \in G_a$ be the unique element such that $\mathbf{1}^{f_g} = g$, whose existence is proved in the first paragraph. We define the multiplication $\otimes$ of elements $g, h \in N_a$ as follows: $g \otimes h = h^{f_g}$. It is a routine exercise to see that this makes N_a into a field K e.g. for $g, h \in N_a$ we have

$$h \otimes g = f_h g f_h^{-1} = f_h \big(f_g \mathbf{1} f_g^{-1} \big) f_h^{-1} = f_g f_h \mathbf{1} f_h^{-1} f_g^{-1} = g \otimes h.$$

Since N_a $(\Sigma', 1)$-acts sharply on (P, cl) by Lemma 2.26, the group $(K, \oplus, \mathbf{0})$ carries an ω-homogeneous pregeometry. But the field structure on N_a is preserved by the automorphism in Σ' which fix $\mathbf{1}$ (recall that N_a is invariant Lemma 2.23). Hence, the field $(K, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ carries an ω-homogeneous pregeometry. It follows that K is algebraically closed by Theorem 1.13. □

Proposition 2.31. *Assume that G_a and N_a are abelian. Then P can be given the structure of an algebraically closed field $(K, +, \cdot, 0, 1)$, and the action of G on P is isomorphic to the affine action of $K \rtimes K^*$, $x \mapsto \ell + kx$, on K. Moreover, the field structure on P and the isomorphism of the group action are invariant once the identities of the field $0, 1$ are chosen.*

Proof. Assume that G_a and N_a are abelian. In the previous proposition, we defined the structure of an algebraically closed field $(K, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ on N_a, where the additive group $(K, \oplus, \mathbf{0})$ is the group $(N_a, \cdot, 1)$ and the multiplicative group $(K^*, \otimes, \mathbf{1})$ is isomorphic to the group $(G_a, \cdot, 1)$, via the bijection $g \mapsto f_g$, where $f_g \in G_a$ is the unique element such that $\mathbf{1}^{f_g} = g$.

Observe that N_a acts on K by translation: if $g, x \in N_a$ then $gx = g \oplus x$ by definition. And conjugation by an element $g \in G_a$ corresponds to dilation on K, via the above bijection: for $g \in G$ and $x \in N_a$, we have $f_g x f_g^{-1} = g \otimes x$. Hence, the action of $G = N_a \rtimes G_a$ on K is isomorphic to the affine action $x \mapsto \ell \oplus (k \otimes x)$, of $K \rtimes K^*$ on the algebraically closed field K.

Now, since N_a acts regularly on P, via $(g, a) \mapsto g(a)$, we can transfer the algebraically closed field structure of N_a onto P by the natural bijection $\phi : N_a \to P$ given by $\phi(g) = g(a)$. Define the addition $+$ and multiplication $\cdot$ using ϕ: $x + y := \phi^{-1}(x) \oplus \phi^{-1}(y)$ and $x \cdot y := \phi^{-1}(x) \otimes \phi^{-1}(y)$. This makes P into an algebraically closed field $(P, +, \cdot, 0, 1)$, isomorphic to $(K, \oplus, \otimes, \mathbf{0}, \mathbf{1})$, where $0 = a$ and $1 = \mathbf{1}(a)$.

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By unravelling the definitions, one checks easily that the action of G on P is isomorphic to the affine action $x \mapsto \ell + k \cdot x$ of $K \rtimes K^*$ on K. Moreover, this isomorphism and the field structure are invariant over $0, 1 \in P$. □

We finish this section with the full picture when groups act on *geometries*. The structure follows roughly [3], where this is done for groups acting on the universe of a strongly minimal set.

Theorem 2.32. *Let G be a group (Σ, n)-acting on a geometry P. Assume that G admits hereditarily unique generics. Then, either there is an unbounded nonclassical A-invariant subgroup of G (for some finite $A \subseteq P$), or $n = 1, 2, 3$ and*

(1) *If $n = 1$, then G is abelian and acts regularly on P.*
(2) *If $n = 2$, then P can be given the A-invariant structure of an algebraically closed field K (for $A \subseteq P$ finite), and the action of G on P is isomorphic to the affine action of $K \rtimes K^*$ on K.*
(3) *If $n = 3$, then $P \backslash \{\infty\}$ can be given the A-invariant structure of an algebraically closed field K (for some $\infty \in P$ and $A \subseteq P$ finite), and the action of G on P is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$.*

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Proof. Suppose that there are no A-invariant unbounded non-classical subgroup of P, for any finite A. Then the (Σ, n)-action of G on P is n-determined, by Fact 2.10, since groups of the form $(G_A)^0$ are A-invariant and unbounded. Thus $n = 1, 2, 3$ by Fact 2.12.

Let $n = 1$: since G $(\Sigma, 1)$-act sharply on the geometry (P, cl), then G acts regularly on the elements of P. Moreover, G carries an ω-homogeneous pregeometry by Fact 2.11. Hence, G must be abelian, otherwise it is nonclassical.

Let $n = 2$: since G_a $(\Sigma_a, 1)$-acts sharply on P_a and N_a $(\Sigma', 1)$-acts sharply on P, then G_a and N_a carry ω-homogeneous pregeometries by Fact 2.11. But G_a is a-invariant and N_a is invariant, and both are unbounded, so they must be abelian since G has no nonclassical invariant, unbounded subgroups. The result now follows from Proposition 2.31.

Let $n = 3$: choose a point $b \in P$ and call it ∞. Then G_∞ $(\Sigma_\infty, 2)$-acts sharply on the pregeometry P_∞. Choose $a \in P_\infty$ and call it 0. Call $N_{\infty,0}$ the group defined in Definition 2.22 for the pregeometry P_∞. Let $G_{\infty,0}$ consist of those elements of G_∞ fixing also 0. Then, $G_{\infty,0}$ $(\Sigma, 1)$-acts sharply on $P_{\infty,0}$ and $N_{\infty,0}$ $(\Sigma'_\infty, 1)$-acts sharply on P_∞. Hence $G_{\infty,0}$ and $N_{\infty,0}$ carry ω-homogeneous pregeometries, so they must be abelian as they are both unbounded and invariant over the finite set $\infty, 0$. By Proposition 2.31, the action of $G_\infty = N_{\infty,0} \rtimes G_{\infty,0}$ on P_∞ is isomorphic to the affine action of $K \rtimes K^*$ on the algebraically closed field K (notice that $0 \in P$ chosen above is the 0 of the field). Let $1 \in P_\infty$ the identity element for the multiplicative structure of the field K. Notice that P_∞ is a geometry by Proposition 2.29, so in particular, the set $\{0, 1, \infty\} \subseteq P$ is 3-dimensional.

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Since G $(\Sigma, 3)$-acts sharply on P, there is a unique $\alpha \in G$ such that $\alpha(0) = \infty$, $\alpha(\infty) = 0$, and $\alpha(1) = 1$. Notice that $\alpha^2 = 1$, by 3-determinacy.

We leave it to the reader to check that conjugation by α induces an idempotent automorphism σ of $G_{\infty,0}$, which is not the identity. Furthermore, $\sigma(g) = g^{-1}$ for each $g \in G_{\infty,0}$: to see this, consider the proper invariant subgroup $B = \{c \in G_{\infty,0} : \sigma(c) = c\}$ of $G_{\infty,0}$. Then B is 0-dimensional in the pregeometry cl' of $G_{\infty,0}$. Consider also $C = \{c \in G_{\infty,0} : \sigma(c) = c^{-1}\}$. Let $\tau : G_{\infty,0} \to G_{\infty,0}$ be the homomorphism defined by $\tau(x) = \sigma(x)x^{-1}$. Then for $x \in G_{\infty,0}$ we have

$$\sigma(\tau(x)) = \sigma^2(x)\sigma(x^{-1}) = x\sigma(x)^{-1} = \tau(x)^{-1},$$

so τ maps $G_{\infty,0}$ into C. If $\tau(x) = \tau(y)$, then $x \in yB$, so $x \in \mathrm{cl}'(y)$ (in the pregeometry of $G_{\infty,0}$). It follows that the kernel of τ is finite-dimensional, and therefore $C = G_{\infty,0}$ (using essentially Lemma 1.9).

We can now complete the proof: the geometry P is now isomorphic to the projective line $\mathbb{P}^1(K)$, with ∞ being the point at infinity. Given $x \in K^*$, choose $h \in G_{\infty,0}$ such that $h1 = x$. Then $\alpha x = \alpha h1 = h^{-1}\alpha 1 = h^{-1}1 = x^{-1}$. Further, α permutes 0 and ∞ by choice, so α acts like an inversion on $\mathbb{P}^1(K)$. It follows that G contains the group of automorphisms of $\mathbb{P}^1(K)$ generated by the affine transformations and inversion. Hence $\mathrm{PGL}_2(K)$ embeds in G. Since the action of $\mathrm{PGL}_2(K)$ and G are both sharply 3-transitive, the embedding is all of G.

The projective line structure and the isomorphism of the group action are invariant over the points $0, 1, \infty \in P$. □

3. The Stable Homogeneous Case

We remind the reader of a few basic facts in homogeneous model theory, which can be found in [28, 17], or [6]. Let L be a language and let $\bar{\kappa}$ be a suitably big cardinal. Let $\mathfrak{C}$ be a *strongly $\bar{\kappa}$-homogeneous* model, i.e. any elementary map $f : \mathfrak{C} \to \mathfrak{C}$ of size less than $\bar{\kappa}$ extends to an automorphism of $\mathfrak{C}$. We denote by $\mathrm{Aut}_{\mathrm{A}}(\mathfrak{C})$ or $\mathrm{Aut}(\mathfrak{C}/A)$ the group of automorphisms of $\mathfrak{C}$ fixing A pointwise. A set Z will be called *A-invariant* if Z is fixed setwise by any automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$. This will be our substitute for definability; by homogeneity of $\mathfrak{C}$ an A-invariant set is the disjunction of complete types over A.

Let D be the *diagram* of $\mathfrak{C}$, i.e. the set of complete L-types over the empty set realized by finite sequences from $\mathfrak{C}$. For $A \subseteq \mathfrak{C}$ we denote by

$$S_D(A) = \{p \in S(A) : \text{ for any } c \models p \text{ and } a \in A \text{ the type } \mathrm{tp}(ac/\emptyset) \in D\}.$$

The homogeneity of $\mathfrak{C}$ has the following important consequence. Let $p \in S(A)$ for $A \subseteq \mathfrak{C}$ with $|A| < \bar{\kappa}$. The following conditions are equivalent:

- $p \in S_D(A)$;
- p is realized in $\mathfrak{C}$;
- $p \restriction B$ is realized in $\mathfrak{C}$ for each finite $B \subseteq \mathfrak{C}$.

The equivalence of the second and third item is sometimes called *weak compactness*, it is the chief reason why homogeneous model theory is so well-behaved.

We will use $\mathfrak{C}$ as a universal domain; each set and model will be assumed to be inside $\mathfrak{C}$ of size less than $\bar{\kappa}$, satisfaction is taken with respect to $\mathfrak{C}$. We will use the term *bounded* to mean "of size less than $\bar{\kappa}$" and *unbounded* otherwise. By abuse of language, a type is bounded if its set of realizations is bounded.

We will work in the *stable* context. We say that $\mathfrak{C}$ (or D) is *stable* if one of the following equivalent conditions are satisfied:

Fact 3.1 (Shelah). The following conditions are equivalent:

(1) For some cardinal λ, D is λ-stable, i.e. $|S_D(A)| \leq \lambda$ for each $A \subseteq \mathfrak{C}$ of size λ.
(2) D does not have the order property, i.e. there does not exist a formula $\phi(x, y)$ such that for arbitrarily large λ we have $\{a_i : i < \lambda\} \subseteq \mathfrak{C}$ such that

$$\mathfrak{C} \models \phi(a_i, a_j) \quad \text{if and only if } i < j < \lambda.$$

Note that a diagram D may be stable while the first order theory of $\mathfrak{C}$ is unstable.

Recall that a type $p \in S_D(A)$ is *quasiminimal* (also called *strongly minimal*) if it is unbounded but has a unique unbounded (hence quasiminimal) extension to any $S_D(B)$, for $A \subseteq B$. Quasiminimal types carry a pregeometry:

Fact 3.2. Let $p \in S_D(A)$ be quasiminimal and let $P = p(\mathfrak{C})$. Then (P, bcl_A), where for $a, B \subseteq P$

$$a \in \mathrm{bcl}_A(B) \quad \text{if} \quad \mathrm{tp}(a/A \cup B) \text{ is bounded},$$

satisfies the axioms of a pregeometry.

We can therefore define $\dim(X/B)$ for $X \subseteq P = p(\mathfrak{C})$ and $B \subseteq \mathfrak{C}$ with respect to bcl_A. Observe that if $X \subseteq P$ is finite, then $\mathrm{cl}(X)$ has size at most $\beth_{(2^{|L|+|A|})^+}$.

This induces a dependence relation $\mathop{\smile\!\!\!\!\!\!\;|}\;$ as follows:

$$a \underset{B}{\perp} C,$$

for $a \in P$ a finite sequence, and $B, C \subseteq \mathfrak{C}$ if and only if

$$\dim(a/B) = \dim(a/B \cup C).$$

We write $\not\perp$ for the negation of $\perp$. The next fact follows easily from the definitions:

Fact 3.3. Let $a, b \in P$ be finite sequences, and $B \subseteq C \subseteq D \subseteq E \subseteq \mathfrak{C}$.

(1) (Finite character) If $a \underset{B}{\not\perp} C$, then there exists a finite $C' \subseteq C$ such that $a \underset{B}{\not\perp} C'$.
(2) (Monotonicity) If $a \underset{B}{\perp} E$ then $a \underset{C}{\perp} D$.

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(3) (Transitivity) $a \underset{B}{\downarrow} D$ and $a \underset{D}{\downarrow} E$ if and only if $a \underset{B}{\downarrow} E$.
(4) (Symmetry) $a \underset{B}{\downarrow} b$ if and only if $b \underset{B}{\downarrow} a$.

This dependence relation defined when the left-hand side is a subset of P allows us to extend much of the theory of forking. Notice however, that the left-hand side will always be finite and always a subset of P, whereas the basis and the right-hand side can be any subset in $\mathfrak{C}$.

From now until Theorem 3.19, we make the following hypothesis.

Hypothesis 3.4. Let $\mathfrak{C}$ be stable. Let p, $q \in S_D(A)$ be unbounded, with p quasi-minimal. Let $n < \omega$ be such that:

(1) For any independent sequence $(a_1, \ldots, a_n)$ of realizations of p and any (finite) set C of realizations of q we have

$$\dim(a_1, \ldots, a_n/A) = \dim(a_1, \ldots, a_n/A \cup C).$$

(2) For some independent sequence $(a_1, \ldots, a_{n+1})$ of realizations of p there is a finite set C of realizations of q such that

$$\dim(a_1, \ldots, a_{n+1}/A) > \dim(a_1, \ldots, a_{n+1}/A \cup C).$$

Notice that by uniqueness of unbounded types, condition (2) is equivalent to *for all* independent sequence $(a_1, \ldots, a_{n+1})$ of realizations of p there is a finite set C of realizations of q such that

$$\dim(a_1, \ldots, a_{n+1}/A) > \dim(a_1, \ldots, a_{n+1}/A \cup C).$$

Remark 3.5. In case we are in the ω-stable [19] or even the superstable [15] case, there is a dependence relation on all the subsets, induced by a rank, which satisfies many of the properties of forking (symmetry and extension only over certain sets, however). This dependence relation, which coincides with the one defined when both make sense, allows us to develop orthogonality calculus in much the same way as the first order setting, and would have enabled us to phrase the conditions (1) and (2) in the same way as the one we phrased for Hrushovski's theorem. Without canonical bases, however, it is not clear that the, apparently weaker, condition that p^n is weakly orthogonal to q^ω implies (1).

We now make the pregeometry P into a *geometry* P/E by considering the equivalence relation E on elements of $P \setminus \mathrm{bcl}_A(\emptyset)$ given by

$$E(x, y) \quad \text{if and only if } \mathrm{bcl}_A(x) = \mathrm{bcl}_A(y).$$

We now proceed with the construction. Before we start, recall that the notion of interpretation we use in this context is like the first order notion, except that we replace definable sets by invariant sets (see Definition 3.16).

Let $Q = q(\mathfrak{C})$. The group we are going to interpret is the following:

$$\mathrm{Aut}_{Q \cup A}(P/E).$$

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The group $\mathrm{Aut}_{Q\cup A}(P/E)$ is the group of permutations of the geometry obtained from P, which are induced by automorphisms of $\mathfrak{C}$ fixing $Q \cup A$ pointwise. There is a natural action of this group on the geometry P/E. We will show in this section that the action has rank n, is $(n+1)$-determined. Furthermore, considering the automorphisms induced from $\mathrm{Aut}_A(\mathfrak{C})$, we have a group acting on a geometry in the sense of the previous section. By restricting the group of automorphism to those induced by the group of strong automorphisms $\mathrm{Saut}_A(\mathfrak{C})$, we will show in addition that this group is stationary and admits hereditarily unique generics. The conclusion will then follow easily from the last theorem of the previous section.

We now give the construction more precisely.

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Notation 3.6. We denote by $\mathrm{Aut}(P/A \cup Q)$ the group of permutations of P which extend to an automorphism of $\mathfrak{C}$ fixing $A \cup Q$.

Then $\mathrm{Aut}(P/A\cup Q)$ acts on P in the natural way. Moreover, each $\sigma \in \mathrm{Aut}(P/A\cup Q)$ induces a unique permutation on P/E which we denote by σ/E, i.e. $\sigma/E(c/E) = \sigma(c)/E$, for $c/E \in P/E$. We now define the group that we will interpret:

Definition 3.7. Let G be the group consisting of the permutations σ/E of P/E induced by elements $\sigma \in \mathrm{Aut}(P/A \cup Q)$.

Since Q is unbounded, $\mathrm{Aut}_{A\cup Q}(\mathfrak{C})$ could be trivial (this is the case even in the first order case if the theory is not stable). The next lemma shows that this is not the case under stability of $\mathfrak{C}$. By abuse of notation, we write

$$\mathrm{tp}(a/A \cup Q) = \mathrm{tp}(b/A \cup Q),$$

if $\mathrm{tp}(a/AC) = \mathrm{tp}(b/AC)$ for any bounded $C \subseteq Q$.

The proof of the next lemma involves splitting. Recall that a complete type $r \in S_D(A)$ *splits over* B if there is $\phi(x,y) \in L$ and $c, d \in A$ with $\mathrm{tp}(c/B) = \mathrm{tp}(d/B)$ such that $\phi(x,c) \in r$ but $\neg\phi(x,d) \in r$. Shelah proved that stability is equivalent to the existence of a cardinal κ such that for each $r \in S_D(A)$ the type r does not split over a subset $B \subseteq A$ of size less than κ. Moreover, he showed that κ is at most the first stability cardinal, hence less that $\beth_{(2^{|L|})^+}$ (see [28] or [6] for an exposition). Notice also that splitting coincides with our dependence relation inside P.

Lemma 3.8. *Let a, b be bounded sequences in $\mathfrak{C}$ such that*

$$\mathrm{tp}(a/A \cup Q) = \mathrm{tp}(b/A \cup Q).$$

Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{C})$ sending a to b which is the identity on $A \cup Q$.

Proof. By induction, it is enough to prove that for all $a' \in \mathfrak{C}$, there is $b' \in \mathfrak{C}$ such that $\mathrm{tp}(aa'/A \cup Q) = \mathrm{tp}(bb'/A \cup Q)$.

Let $a' \in \mathfrak{C}$. We claim that there exists a bounded $B \subseteq Q$ such that for all $C \subseteq Q$ bounded, we have $\mathrm{tp}(aa'/ABC)$ does not split over AB.

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Otherwise, for any λ, we can inductively construct an increasing sequence of bounded sets $(C_i : i < \lambda)$ such that $\operatorname{tp}(aa'/C_{i+1})$ does not split over C_i. This contradicts stability (such a chain must stop at the first stability cardinal).

Now let $\sigma \in \operatorname{Aut}_{A\cup B}(\mathfrak{C})$ sending a to b and let $b' = \sigma(a')$. We claim that $\operatorname{tp}(aa'/A\cup Q) = \operatorname{tp}(bb'/A\cup Q)$. If not, let $\phi(x,y,c) \in \operatorname{tp}(aa'/A\cup Q)$ and $\neg\phi(x,y,c) \in \operatorname{tp}(bb'/A \cup Q)$. Then, $\phi(x,y,c), \neg\phi(x,y,\sigma(c)) \in \operatorname{tp}(aa'/A \cup Bc\sigma(c))$, and therefore $\operatorname{tp}(aa'/ABc\sigma(c))$ splits over AB, a contradiction. □

It follows that the action of $\operatorname{Aut}(P/A \cup Q)$ on P, and *a fortiori* the action of G on P/E, has some transitivity properties. The next corollary implies that the action of G on P has rank n (condition (2) in Hypothesis 3 prevents two distinct independent $(n+1)$-tuples of realizations of p from being automorphic over $A\cup Q$).

Corollary 3.9. *For any $a_1,\dots,a_n$ and $b_1,\dots,b_n \in P^n$ with both $a_1,\dots,a_n$ and $b_1,\dots,b_n$ independent, there is $g \in G$ such that $g(a_i/E) = b_i/E$, for $i = 1,\dots,n$.*

Proof. By Hypothesis 3(1) $\dim(a_1\cdots a_n/A \cup C) = \dim(b_1\cdots b_n/A \cup C) = n$, for each finite $C \subseteq Q$. By uniqueness of unbounded extensions, we have that $\operatorname{tp}(a_1,\dots,a_n/A \cup C) = \operatorname{tp}(b_1,\dots,b_n/A \cup C)$ for each finite $C \subseteq Q$. It follows that $\operatorname{tp}(a_1,\dots,a_n/A \cup Q) = \operatorname{tp}(b_1,\dots,b_n/A \cup Q)$ so by the previous lemma, there is $\sigma \in \operatorname{Aut}(\mathfrak{C}/A \cup Q)$ such that $\sigma(a_i) = b_i$, for $i = 1,\dots,n$. Then $g = \sigma/E$. □

The next two lemmas are in preparation to show that the action is $(n+1)$-determined. We first give a condition ensuring that two elements of G coincide.

Lemma 3.10. *Let $a_1/E,\dots,a_{n+1}/E \in P/E$ be independent with $\sigma(a_i)/E = a_i/E$ for $i = 1,\dots,n+1$ and $\sigma \in \operatorname{Aut}(P/A \cup Q)$. Let $c \in P\setminus \operatorname{bcl}_A(a_1,\dots,a_{n+1})$. Then $\sigma(c)/E = c/E$.*

Proof. We first claim that

$$\sigma(c) \underset{\{a_1,\dots,a_{n+1}\}\setminus\{a_i\}}{\not\downarrow} c, \quad \text{for each } i = 1,\dots,n+1.$$

By a re-enumerating if necessary, it is enough to prove that

$$\sigma(c) \underset{a_1,\dots,a_n}{\not\downarrow} c.$$

Assume, for a contradiction, that this fails. Then, $\sigma(c) \notin \operatorname{bcl}_A(ca_1\cdots a_n)$ (note that $\sigma(c) \notin \operatorname{bcl}_A(a_1\cdots a_{n+1})$, since $\sigma(c) \notin \operatorname{bcl}_A(\sigma(a_1)\cdots\sigma(a_{n+1}))$ and also $\operatorname{bcl}_A(a_i) = \operatorname{bcl}_A(\sigma(a_i))$ for $i = 1,\dots,n+1$ by assumption on σ). By Hypothesis 3(2), there exists a finite $C \subseteq Q$ such that

$$\dim(ca_1\cdots a_n/A \cup C) = n. \qquad (*)$$

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Let $d \in P \backslash \mathrm{bcl}_A(Cca_1 \cdots a_n)$, and choose an automorphism f of $\mathfrak{C}$ such that $f(d) = \sigma(c)$ fixing $Aca_1 \cdots a_n$. Applying f on (*), we obtain

$$\dim(ca_1 \cdots a_n / A \cup f(C)) = n. \tag{**}$$

On the other hand, $\sigma(c) \notin \mathrm{bcl}_A(f(C)a_1 \cdots a_n)$, by choice of d, and so also $\sigma(c) \notin \mathrm{bcl}_A(f(C)\sigma(a_1) \cdots \sigma(a_n))$ (since $\sigma(a_i)/E = a_i/E$, for $i = 1, \ldots, n$). But by Hypothesis 3(1) we have $\dim(\sigma(a_1) \cdots \sigma(a_n)/A \cup f(C)) = n$, so together we have

$$\dim(\sigma(c)\sigma(a_1) \cdots \sigma(a_n)/A \cup f(C)) = n + 1.$$

But this contradicts (**) since σ fixes $f(C)$. This proves the claim.

We now prove that $\sigma(c)/E = c/E$. Suppose not, then $c \notin \mathrm{bcl}_A(\sigma(c))$. But $c \in \mathrm{bcl}_A(\sigma(c)a_1, \ldots, a_n)$ by the previous claim. Hence, by exchange, we can find i, with $1 \leq i \leq n$, such that $a_i \in \mathrm{bcl}_A(c\sigma(c) \cup \{a_1 \cdots a_n\} \backslash \{a_i\})$. Using the previous claim again, we have that $\sigma(c) \in \mathrm{bcl}_A(c \cup \{a_1, \ldots, a_n, a_{n+1}\} \backslash \{a_i\})$, so that

$$\dim(c, \sigma(c), a_1, \ldots, a_n, a_{n+1}/A) = n + 1.$$

But this implies that $\mathrm{bcl}_A(c\sigma(c)a_1 \cdots a_n a_{n+1}) = \mathrm{bcl}_A(a_1 \cdots a_n a_{n+1})$, which contradicts the assumption that $c, \sigma(c) \notin \mathrm{bcl}_A(a_1 \cdots a_{n+1})$. □

We now eliminate the assumption that $c \notin \mathrm{bcl}(Aa)$.

Lemma 3.11. *Let $\sigma \in \mathrm{Aut}(P/A \cup Q)$, and $a_1/E, \ldots, a_{n+1}/E \in P/E$ be independent with $\sigma(a_i)/E = a_i/E$ for $i = 1, \ldots, n+1$. Let $c \in P \setminus \mathrm{bcl}_A(\emptyset)$. Then $\sigma(c)/E = c/E$.*

Proof. We reduce this case to the previous lemma: choose $\{b_i : i < \omega\}$ in P independent. Then $\{\sigma(b_i) : i < \omega\}$ is in P and independent. Since $c\sigma(c)a_1 \cdots a_{n+1}$ is finite-dimensional, there are $i_1 < \cdots < i_{n+1} < \omega$ such that $b_{i_1}, \ldots, b_{i_{n+1}}$ is independent over $c\sigma(c)a_1 \cdots a_{n+1}$. By the previous lemma, we therefore have that $\sigma(b_{i_\ell})/E = b_{i_\ell}/E$ for each $\ell = 1, \ldots, n+1$. Hence, since $c \notin \mathrm{bcl}_A(b_{i_1}, \ldots, b_{i_{n+1}})$ (by exchange, since $c \notin \mathrm{bcl}_A(\emptyset)$) we have that $\sigma(c)/E = c/E$ by another application of the previous lemma. □

The next corollary follows immediately by applying the lemma to $\tau^{-1} \circ \sigma$. Together with Corollary 3.9, it shows that the action of G on P/E is an n-action.

Corollary 3.12. *Let $\sigma, \tau \in \mathrm{Aut}(P/A \cup Q)$ and assume there is an $(n+1)$-dimensional subset X of P/E on which σ/E and τ/E agree. Then $\sigma/E = \tau/E$.*

We now consider automorphisms of this group action. Let $\sigma \in \mathrm{Aut}_A(\mathfrak{C})$. Then, σ induces an automorphism σ' of the group action as follows: σ' is σ/E on P/E, and for $g \in G$ we let $\sigma'(g)(a/E) = \sigma(\tau(\sigma^{-1}(a)))/E$, where τ is such that $\tau/E = g$. It is easy to verify that

$$\sigma' : G \to G$$

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is an automorphism of G (as $\sigma \circ \tau \circ \sigma^{-1} \in \mathrm{Aut}_{\mathrm{Q} \cup \mathrm{A}}(\mathfrak{C})$ if $\tau \in \mathrm{Aut}_{\mathrm{Q} \cup \mathrm{A}}(\mathfrak{C})$, and both P and Q are A-invariant). Finally, one checks directly that σ' preserves the action.

For stationarity, it is more convenient to consider strong automorphisms. Recall that two sequences $a, b \in \mathfrak{C}$ have the same *Lascar strong types* over C, written $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, if $E(a,b)$ holds for any C-invariant equivalence relation E with a bounded number of classes. An automorphism $f \in \mathrm{Aut}(\mathfrak{C}/C)$ is called *strong* if $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(f(a)/C)$ for any $a \in \mathfrak{C}$. We denote by $\mathrm{Saut}(\mathfrak{C}/C)$ or $\mathrm{Saut}_C(\mathfrak{C})$ the group of strong automorphisms fixing C pointwise. We let $\Sigma = \{\sigma' : \sigma \in \mathrm{Saut}_A(\mathfrak{C})\}$. The reader is referred to [16] or [4] for more details.

First, we show that the action is ω-homogeneous with respect to Σ.

Lemma 3.13. *If $X \subseteq P/E$ is finite and $x, y \in P/E$ are outside $\mathrm{bcl}_A(X)$, then there is an automorphism $\sigma \in \Sigma$ of the group action sending x to y which is the identity on X.*

Proof. By uniqueness of unbounded extensions, there is an automorphism $\sigma \in \mathrm{Saut}(\mathfrak{C})$ fixing $A \cup X$ pointwise and sending x to y. The automorphism σ' is as desired. □

We are now able to show the stationarity of G.

Proposition 3.14. *G is stationary with respect to Σ.*

Proof. First, notice that the number of Lascar strong types is bounded by stability. Now, let $g \in G$ be generic over the bounded set X and let $\bar{x} \in P^n$ be an independent sequence witnessing this, i.e.

$$\dim(\bar{x}g(\bar{x})/X) = 2n.$$

If $x' \in P$ is such that $\dim(\bar{x}x'/X) = n+1$, then $\dim(\bar{x}x'g(\bar{x})g(x)'/X) = 2n+1$. By quasiminimality of p, this implies that

$$\bar{x}x'g(\bar{x})g(x') \underset{A}{\perp} X.$$

Now let $h \in G$ be also generic over X and such that $\sigma(g) = h$ with $\sigma \in \Sigma$. For $\bar{y}, y'$ witnessing the genericity of h as above, we have

$$\bar{y}y'h(\bar{y})h(y') \underset{A}{\perp} X.$$

Hence, by stationarity of Lascar strong types we have $\mathrm{Lstp}(\bar{x}x'g(\bar{x})g(x')/AX) = \mathrm{Lstp}(\bar{y}y'h(\bar{y})h(y')/AX)$. Thus, there is τ, a strong automorphism of $\mathfrak{C}$ fixing $A \cup X$ pointwise, such that $\tau(\bar{x}x'g(\bar{x}g(x')) = \bar{y}y'h(\bar{y})h(y')$. Then, $\tau'(g) = h$ ($\tau' \in \Sigma$) since the action is $(n+1)$-determined. □

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The previous proposition implies that G^0 has unique generics, but we can prove more:

Proposition 3.15. *G^0 admits hereditarily unique generics with respect to Σ.*

Proof. For any independent k-tuple $a \in P/E$ with $k < n$, consider the Σ_a-homogeneous $(n-k)$-action G_a on P/E. Instead of Σ_a, consider the smaller group Σ'_a consisting of σ' for strong automorphisms of $\mathfrak{C}$ fixing Aa and preserving strong types over Aa. Then, as in the proof of the previous proposition, G_a is stationary with respect to Σ'_a, which implies that the connected component G'_a of G_a (defined with Σ'_a) has unique generics with respect to restriction of automorphisms in Σ'_a by Theorem 2.8. But, there are even more automorphisms in Σ_a so G'_a has unique generics with respect to restriction of automorphisms in Σ_a. By definition, this means that G admits hereditarily unique generics. □

We now show that G is interpretable in $\mathfrak{C}$. We recall the definition of interpretable group in this context.

Definition 3.16. A group $(G, \cdot)$ *interpretable in* $\mathfrak{C}$ if there is a (bounded) subset $B \subseteq \mathfrak{C}$ and an unbounded set $U \subseteq \mathfrak{C}^k$ (for some $k < \omega$), an equivalence relation E on U, and a binary function $*$ on U/E which are B-invariant and such that $(G, \cdot)$ is isomorphic to $(U/E, *)$.

We can now prove:

Proposition 3.17. *The group G is interpretable in $\mathfrak{C}$.*

Proof. This follows from the $(n+1)$-determinacy of the group action. Fix a an independent $(n+1)$-tuple of elements of P/E. Let $B = Aa$.

We let $U/E \subseteq P^{n+1}/E$ consist of those $b \in P^{n+1}/E$ such that $ga = b$ for some G. Then, this set is B-invariant since if $b \in P^{n+1}/E$ and $\sigma \in \mathrm{Aut}_B(\mathfrak{C})$, then $\sigma'(g) \in G$ and sends a to $\sigma(b)$ (recall that σ' is the automorphism of the group action induced by σ).

We now define $b_1 * b_2 = b_3$ on U/E, if whenever $g_\ell \in G$ such that $g_\ell(a) = b_\ell$, then $g_1 \circ g_2 = g_3$. This is well-defined by $(n+1)$-determinacy and the definition of U/E. Furthermore, the binary function $*$ is B-invariant. It is clear that $(U/E, *)$ is isomorphic to G. □

Remark 3.18. As we pointed out, by homogeneity of $\mathfrak{C}$, any B-invariant set is equivalent to a disjunction of complete types over A. So, for example, if B is finite, E and U is expressible by formulas in $L_{\lambda^+,\omega}$, where $\lambda = |S^D(B)|$.

It follows from the same proof that G^0 is interpretable in $\mathfrak{C}$, and similarly G_a and $(G_a)^0$ are interpretable for any independent k-tuple a in P/E with $k < n$.

We can now prove the main theorem. We restate the hypotheses for completeness.

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Theorem 3.19. *Let $\mathfrak{C}$ be a large, homogeneous model of a stable diagram D. Let $p, q \in S_D(A)$ be unbounded with p quasiminimal. Assume that there is $n \in \omega$ such that*

(1) *For any independent n-tuple $(a_0, \ldots, a_{n-1})$ of realizations of p and any finite set C of realizations of q we have*

$$\dim(a_0, \ldots, a_{n-1}/A \cup C) = n.$$

(2) *For some independent sequence $(a_0, \ldots, a_n)$ of realizations of p there is a finite set C of realizations of q such that*

$$\dim(a_0, \ldots, a_n/A \cup C) < n + 1.$$

Then $\mathfrak{C}$ interprets a group G which acts on the geometry P' obtained from P. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n \leq 3$ and

- *If $n = 1$, then G is abelian and acts regularly on P'.*
- *If $n = 2$, the action of G on P' is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebraically closed field K.*
- *If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

Proof. First, by assuming that A is small compared to the size of $\mathfrak{C}$, we may assume that $2^{|\mathrm{cl}(X)|} < \|\mathfrak{C}\|$, for each finite set $X \subseteq P$. The group G is interpretable in $\mathfrak{C}$ by Proposition 3.17. This group acts on the geometry P/E; the action has rank n and is $(n+1)$-determined. Furthermore, G^0 admits hereditarily unique generics with respect to set of automorphisms Σ induced by strong automorphisms of $\mathfrak{C}$. Working now with the connected group G^0. G^0 is an invariant subgroup of G and therefore interpretable. But, G^0 (Σ^0, n)-acts on the geometry P/E (Σ^0 is simply obtained from Σ by restriction) and has hereditarily unique generics. Hence, the conclusion follows from Theorem 2.32. □

Remark 3.20. If we choose p to be regular (see [17] for a definition), instead of quasiminimal, and work with dim the natural dimension induced inside a regular type, then, under the assumptions (1) and (2) above, we can still interpret a group G (the proof is identical). We have used the fact that the dependence relation is given by bounded closure only to ensure the stationarity of G, and to obtain a field. Is it possible to do this for regular types also?

4. The Excellent Case

Here we consider a class $\mathcal{K}$ of atomic models of a countable first order theory. In the language of the previous section, this means that D is the set of isolated types over the empty set. We assume that $\mathcal{K}$ is *excellent* (see [31, 32, 5, 21], or the expository article [22] for the basics of excellence; in particular all the facts that we use in this section can be found there). We will use the notation $S_D(A)$ and splitting, which have been defined in the previous section.

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Excellence lives in the ω-*stable* context, i.e. $S_D(M)$ is countable, for any countable $M \in \mathcal{K}$. This notion of ω-stability is strictly weaker than the corresponding notion given in the previous section; in the excellent, non-homogeneous case, there are countable atomic sets A such that $S_D(A)$ is uncountable.

The ω-stability implies that *splitting* provides a dependence relation between sets, which satisfies all the usual axioms of forking, provided we only work over models in $\mathcal{K}$. For example, for each $p \in S_D(M)$, for $M \in \mathcal{K}$, there is a finite $B \subseteq M$ such that p does not split over B. Moreover, if $N \in \mathcal{K}$ extends M then p has a unique extension in $S_D(N)$ which does not split over B. We have symmetry, transitivity, and extension, and types over models have a unique nonsplitting extension. Generally, we say that a type with a unique nonsplitting extension is *stationary*.

Using splitting, we can define what we mean by a model being independent from another model over a third, and more generally, we can define *independent systems of models* in a natural way, similar to what is done in the first order case in Main Gap constructions.

The last ingredient of excellence is primary models. Recall that a model $M \in \mathcal{K}$ is *primary over* A, if $M = A \cup \{a_i : i < \lambda\}$ and for each $i < \lambda$ the type $\mathrm{tp}(a_i/A \cup \{a_j : j < i\})$ is isolated. Primary models are prime in $\mathcal{K}$.

Excellence is a condition on the existence of primary models over certain kinds of sets, namely those obtained as unions of independent systems of models. The definition is quite technical and will not be needed here. Rather, we isolate a few less technical consequences which are the only ones used for the rest of the paper.

The following facts are due to Shelah [31, 32] (see also [22]). The first concerns primary models.

Fact 4.1 (Shelah). Assume that $\mathcal{K}$ is excellent.

(1) If A is a finite atomic set, then there is a primary model $M \in \mathcal{K}$ over A.
(2) If $M \in \mathcal{K}$ and $p \in S_D(M)$, then for each $a \models p$, there is a primary model over $M \cup a$.

The second concerns *full models*. Full models are a substitute of homogeneous models in this context (uncountable homogeneous models do not necessarily exist) and can be used as universal domains: as in the first order case, or the homogeneous case, they are unique up to isomorphism, they are universal, and realize all the types realized in the models of the class. The existence of arbitrarily large full models follows from excellence.

Fact 4.2 (Shelah). Let M be a full model of uncountable size $\bar{\kappa}$.

(1) M is ω-homogeneous.
(2) M is model-homogeneous, i.e. if $a, b \in M$ have the same type over $N \prec M$ with $\|N\| < \bar{\kappa}$, then there is an automorphism of M fixing N sending a to b.
(3) M realizes any $p \in S_D(N)$ with $N \prec M$ of size less than $\bar{\kappa}$.

Remark 4.3. Any ω-stable class of atomic models $\mathcal{K}$ (in a countable language) which contains an uncountable homogeneous model is excellent. This is because

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in this case, Shelah showed [28] that there are prime models over any atomic set. The converse holds also: if $\mathcal{K}$ is an excellent class which has prime models over any countable atomic set, then it has arbitrarily large models [20]. So in the excellent case nonhomogeneous case, where there are few types over models, there are countable sets over which there are many types. This is the central difficulty.

Moreover, every ω-stable class $\mathcal{K}$ can be assumed to be in a countable language, and if it contains an uncountable homogeneous model can be thinned into an ω-stable atomic class by expanding the language (and discarding the models which are not $\aleph_0$-homogeneous). So in this sense, this section is a generalization of the previous section (except for the fact that there we can prove the result under stability, rather than ω-stability). More generally, any uncountably categorical sentence of $L_{\omega_1,\omega}$ can be made into an atomic class by expanding the language (see [30], but more details are in Baldwin's book [2]). However, expanding the language in these two cases may not be natural in specific contexts when one is interested in definability issues. As the reader will see in the sequel the atomicity is not the crucial issue, but rather the ω-homogeneity of the models. Hence, all our results can be obtained without expanding the language provided that we work with ω-homogeneous models (see [21], for example).

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We work inside a full $\mathfrak{C}$ of size $\bar{\kappa}$, for some suitably big cardinal $\bar{\kappa}$. All sets and models will be assumed to be inside $\mathfrak{C}$ of size less than $\bar{\kappa}$, unless otherwise specified. The previous fact shows that all types over finite sets, and all stationary types of size less than $\bar{\kappa}$ are realized in $\mathfrak{C}$. The paper is written in such a way that only the listed consequences of excellence in the two facts above will be used in the rest of the paper.

Since the automorphism group of $\mathfrak{C}$ is not as rich as in the homogeneous case, it will be necessary to consider another closure operator: for all $X \subseteq \mathfrak{C}$ and $a \in M$, we define the *essential closure* of X, written $\mathrm{ecl}(X)$, by

$$a \in \mathrm{ecl}(X), \quad \text{if } a \in M \quad \text{for each } M \prec \mathfrak{C} \text{ containing } X.$$

Observe that $\mathrm{bcl}(X) \subseteq \mathrm{ecl}(X)$ for any X: if $a \notin \mathrm{ecl}(X)$, then $a \notin N$ for some model N containing X, hence $\mathrm{tp}(a/N)$ is unbounded so $a \notin \mathrm{bcl}(X)$. The converse is more delicate, and may not hold for all sets. However, using homogeneity, we can show $\mathrm{bcl}(X) = \mathrm{ecl}(X)$ when X is finite or a model or of the form Ma, where $M \in \mathcal{K}$ and a is a finite sequence: for example, assume that X is finite and $a \notin \mathrm{bcl}(X)$. Let M be a model containing X. Since $\mathrm{tp}(a/X)$ is unbounded, there is $b \models \mathrm{tp}(a/X)$ outside of M. Then by ω-homogeneity, there is an automorphism σ fixing X sending b to a. Thus $\sigma(M)$ is a model containing X and avoiding a, so $a \notin \mathrm{ecl}(X)$. And using this, we can show $\mathrm{ecl}(X) = \mathrm{bcl}(X)$, when there is a prime model over X, for example, sets of the form Ma, where a is a finite sequence.

As usual, for $B \subseteq \mathfrak{C}$, we write $\mathrm{ecl}_B(X)$ for the closure operator on subsets X of $\mathfrak{C}$ given by $\mathrm{ecl}(X \cup B)$. Also, it is easy to check that $X \subseteq \mathrm{ecl}_B(X) = \mathrm{ecl}_B(\mathrm{ecl}_B(X))$, for each $X, B \subseteq \mathfrak{C}$. Furthermore, $X \subseteq Y$ implies that $\mathrm{ecl}_B(X) \subseteq \mathrm{ecl}_B(Y)$. So $\mathrm{ecl}_B(X)$ is a closure operator, but it is not necessarily finitary.

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Again we consider a *quasiminimal* type $p \in S_D(A)$, i.e. $p(\mathfrak{C})$ is unbounded and there is a unique unbounded extension of p over each subset of $\mathfrak{C}$. Since the language is countable in this case, and we have ω-stability, the bounded closure of a countable set is countable. Bounded closure satisfies exchange on the set of realizations of p (see [21]). This holds also for essential closure.

Lemma 4.4. *Let $p \in S_D(A)$ be quasiminimal. Let $B \subseteq \mathfrak{C}$ contain A. Suppose that $a, b \models p$ are such that $a \in \mathrm{ecl}_B(Xb) \setminus \mathrm{ecl}_B(X)$. Then $b \in \mathrm{ecl}_B(Xa)$.*

Proof. Suppose not, and let $M \prec \mathfrak{C}$ containing $B \cup X \cup a$ such that $b \notin M$. Let N containing $B \cup X$ such that $a \notin N$. In particular $a \notin \mathrm{bcl}_B(N)$ and $a \in \mathrm{ecl}_B(Nb)$. Let $b' \in \mathfrak{C}$ realize the unique free extension of p over $M \cup N$. Then $\mathrm{tp}(b/M) = \mathrm{tp}(b'/M)$ since there is a unique big extension of p over M. It follows that there exists $f \in \mathrm{Aut}(\mathfrak{C}/M)$ such that $f(b) = b'$. Let $N' = f(N)$. Then $b' \notin \mathrm{bcl}_B(N'a)$. On the other hand, we have $a \in \mathrm{ecl}_B(Nb) \setminus \mathrm{ecl}_B(N)$ by monotonicity and choice of N, so $a \in \mathrm{ecl}_B(N'b') \setminus \mathrm{ecl}_B(N')$. But, then $a \in \mathrm{bcl}_B(N'b') \setminus \mathrm{bcl}(N')$ (if $a \notin \mathrm{bcl}_B(N'b')$, then $a \notin N'(b')$, for some (all) primary models over $N' \cup b'$). But this is a contradiction. □

It follows from the previous lemma that the closure relation $\mathrm{ecl}_{A \cup B}$ satisfies the axioms of a pregeometry on the *finite* subsets of $P = p(\mathfrak{C})$, when $p \in S_D(A)$ is quasiminimal for any set B.

Thus, for finite subsets $X \subseteq P$, and any set $B \subseteq \mathfrak{C}$, we can define $\dim(X/A \cup B)$ using the closure operator $\mathrm{ecl}_{A \cup B}$. We will now use the independence relation $\mathop{\smile\!\!\!\!\!\!|}$ as follows:

$$a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} C,$$

for $a \in P$ a finite sequence, and $B, C \subseteq \mathfrak{C}$ if and only if

$$\dim(a/A \cup B) = \dim(a/A \cup B \cup C).$$

The following lemma follows easily.

Lemma 4.5. *Let $a, b \in P$ be finite sequences, and $B \subseteq C \subseteq D \subseteq E \subseteq \mathfrak{C}$.*

(1) (*Monotonicity*) *If $a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} E$ then $a \underset{C}{\mathop{\smile\!\!\!\!\!\!|}} D$.*
(2) (*Transitivity*) *$a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} D$ and $a \underset{D}{\mathop{\smile\!\!\!\!\!\!|}} E$ if and only if $a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} E$.*
(3) (*Symmetry*) *$a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} b$ if and only if $b \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} a$.*

From now until Theorem 4.19, we make a hypothesis similar to Hypothesis 3, except that A is chosen finite and the witness C is allowed to be countable (the reason is that we do not have finite character in the right-hand side argument of $\mathop{\smile\!\!\!\!\!\!|}$). Since we work over finite sets, notice that p and q below are actually equivalent to formulas over A.

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Hypothesis 4.6. Let $\mathfrak{C}$ be a large full model of an excellent class $\mathcal{K}$. Let $A \subseteq \mathfrak{C}$ be finite. Let p, $q \in S_D(A)$ be unbounded with p quasiminimal. Let $n < \omega$. Assume that

(1) For any independent sequence $(a_1, \ldots, a_n)$ of realizations of p and any countable set C of realizations of q we have

$$\dim(a_1, \ldots, a_n/A) = \dim(a_1, \ldots, a_n/A \cup C).$$

(2) For some independent sequence $(a_1, \ldots, a_{n+1})$ of realizations of p there is a countable set C of realizations of q such that

$$\dim(a_1, \ldots, a_{n+1}/A) > \dim(a_1, \ldots, a_{n+1}/A \cup C).$$

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Write $P = p(\mathfrak{C})$ and $Q = q(\mathfrak{C})$, as in the previous section. Then, P carries a pregeometry with respect to bounded closure, which coincides with essential closure over finite sets. Thus, when we speak about finite sets or sequences in P, the term independent is unambiguous. We make P into a geometry P/E by considering the A-invariant equivalence relation

$$E(x, y), \quad \text{defined by } \mathrm{bcl}_A(x) = \mathrm{bcl}_A(y).$$

The group we will interpret in this section is defined slightly differently, because of the lack of homogeneity (in the homogeneous case, they coincide). We will consider the group G of all permutations g of P/E with the property that for each countable $C \subseteq Q$ and for each finite $X \subseteq P$, there exists $\sigma \in \mathrm{Aut}_{A\cup C}(\mathfrak{C})$ such that $\sigma(a)/E = g(a/E)$ for each $a \in X$. This is defined unambiguously since if $x, y \in P$ such that $x/E = y/E$ then $\sigma(x)/E = \sigma(y)/E$ for any automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$.

We will show first that for any pair of independent n-tuples in P and $C \subseteq Q$ there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ sending a to b. Next, we will show essentially that the action of G on P/E is $(n+1)$-determined, which we will then use to show that the action has rank n. It will follow immediately that G is interpretable in $\mathfrak{C}$, as in the previous section. Finally, we will consider Lascar strong types and strong automorphisms (over finite sets) to show that G admits hereditarily unique generics, again, exactly like in the previous section.

We now construct the group more formally.

Definition 4.7. Let G be the group of permutations g of P/E such that for each countable $C \subseteq Q$ and finite $X \subseteq P$ there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that $\sigma(a)/E = g(a/E)$ for each $a \in X$.

G is clearly a group. We now prove a couple of key lemmas that explain why we chose ecl rather than bcl; these will be used to show that G is not trivial.

Lemma 4.8. *Let $(a_i)_{i<k}$ be a finite sequence in P with $\dim(a_1, \ldots, a_{k-1}/C) = k$, for some $C \subseteq \mathfrak{C}$. Then there exists $M \prec \mathfrak{C}$ containing C such that*

$$a_i \notin \mathrm{bcl}(Ma_0 \cdots a_{i-1}), \quad \textit{for each } i < k.$$

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Proof. We find models M_i^j, for $i \leq j < k$, and automorphisms $f_j \in \mathrm{Aut}(\mathfrak{C}/M_j^j)$ for each $j < k$ such that:

(1) $A \cup C \cup a_0 \cdots a_{i-1} \subseteq M_i^j$ for each $i \leq j < k$.
(2) For each $i < j < n$, $M_i^{j-1} = f_j(M_i^j)$.
(3) $a_j \underset{M_j^j}{\perp} M_0^j \cup \cdots \cup M_{j-1}^j$.

This is possible: let $M_0^0 \prec \mathfrak{C}$ containing $A \cup C$ be such that $a_0 \notin M_0^0$, which exists by definition, and let f_0 be the identity on $\mathfrak{C}$. Having constructed M_i^j for $i \leq j$, and f_j, we let $M_{j+1}^{j+1} \prec \mathfrak{C}$ contain $A \cup C \cup a_0 \cdots a_j$ such that $a_{j+1} \notin M_{j+1}^{j+1}$, which exists by definition. Let $b_{j+1} \in \mathfrak{C}$ realize $\mathrm{tp}\big(a_{j+1}/M_{j+1}^{j+1}\big)$ such that

$$b_{j+1} \underset{M_{j+1}^{j+1}}{\perp} M_0^j \cup \cdots \cup M_j^j.$$

Such b_{j+1} exists by stationarity of $\mathrm{tp}\big(a_{j+1}/M_{j+1}^{j+1}\big)$. Let f_{j+1} be an automorphism of $\mathfrak{C}$ fixing $M_{j+1}^{j+1})$ sending b_{j+1} to a_{j+1}. Let $M_i^{j+1} = f_{j+1}(M_i^j)$, for $i \leq j$. These are easily seen to be as required.

This is enough: let $M = M_0^{k-1}$. To see that M is as needed, we show by induction on $i \leq j < k$, that $a_i \notin \mathrm{bcl}\big(M_0^j a_0 \cdots a_{i-1}\big)$. For $i = j$, this is clear since $a_i \notin \mathrm{bcl}\big(M_0^i \cup \cdots \cup M_i^i\big)$. Now if $j = \ell + 1$, $a_i \notin \mathrm{bcl}\big(M_0^\ell a_0 \cdots a_{i-1}\big)$ by induction hypothesis. Since $M_0^{\ell+1} = f_{\ell+1}\big(M_0^\ell\big)$ and $f_{\ell+1}$ is the identity on $a_0 \cdots a_i$, the conclusion follows. □

It follows from the previous lemma that the sequence $(a_i : i < k)$ is a Morley sequence of the quasiminimal type p_M, and hence that (1) it can be extended to any length, and (2) that any permutation of it extends to an automorphism of $\mathfrak{C}$ over M (hence over C).

Lemma 4.9. *Let $a_1, \ldots, a_n \in P$ and $b_1, \ldots, b_n \in P$, both independent, and let $C \subseteq Q$ be countable. Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/\mathrm{C})$ such that $\sigma(a_i) = b_i$, for $i = 1, \ldots, n$.*

Proof. By assumption, we have $\dim(a_1 \cdots a_n/A \cup C) = \dim(b_1 \cdots b_n/A \cup C)$. By using a third sequence if necessary, we may also assume that

$$\dim(a_1 \cdots a_n b_1 \cdots b_n/A \cup C) = 2n.$$

Then, by the previous lemma, there exists $M \prec \mathfrak{C}$ containing $A \cup C$ such that $a_1, \ldots, a_n, b_1, \ldots, b_n$ is a Morley sequence of M. Thus, the permutation sending a_i to b_i extends to an automorphism σ of $\mathfrak{C}$ fixing M (hence C). □

The fact that the previous lemma fails for independent sequences of length $n+1$ follows from item (2) of Hypothesis 4. Notice also, that *for some* independent $(a_1, \ldots, a_{n+1})$ in (2) is equivalent to *for all.*

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We now concentrate on the n-action. We first prove a lemma which is essentially like Lemmas 3.10 and 3.11. However, since we cannot consider automorphisms fixing all of Q, we need to introduce good pairs.

Definition 4.10. A pair (X, C) is a *good pair* if $X \subseteq P$ is countable and infinite-dimensional, $C \subseteq Q$ is countable, and $X = \mathrm{ecl}_A(X \cup C) \cap P$. Furthermore, for each $\bar{a} \in X^{n+1}$ independent over A, there is $C_{\bar{a}} \subseteq C$ such that

- $\dim(\bar{a}/A \cup C_{\bar{a}}) \leq n$.
- For all $c \in X \setminus \mathrm{ecl}_A(\bar{a})$ there is $d \in P \setminus \mathrm{ecl}_A(C_{\bar{a}}\bar{a})$ and f an automorphism of $\mathfrak{C}$ fixing $A \cup \bar{a}$ such that $f(d) = c$ and $f(C_{\bar{a}}) \subseteq C$.

Notice that, by Hypothesis 4.6, for each countable $X' \subseteq P$, there exists a good pair (X, C), with $X' \subseteq X$.

Lemma 4.11. *Let (X, C) be a good pair. Suppose that $a_1, \dots, a_{n+1} \in X$ are independent and $\sigma(a_i)/E = a_i$, for $i = 1, \dots, n+1$, for some $\sigma \in \mathrm{Aut}(P/A \cup C)$. Then $\sigma(c)/E = c/E$ for any $c \in X \setminus \mathrm{ecl}_A(\emptyset)$.*

Proof. We first show this for $c \notin \mathrm{ecl}_A(a_1 \cdots a_{n+1})$. As in the proof of Lemma 3.10, we claim that

$$\sigma(c) \underset{\{a_1,\dots,a_{n+1}\}\setminus\{a_i\}}{\not\downarrow} c, \quad \text{for each } i = 1, \dots, n+1.$$

Again, we only prove that

$$\sigma(c) \underset{a_1,\dots,a_n}{\not\downarrow} c.$$

Assume, for a contradiction, that this fails. Then $\sigma(c) \notin \mathrm{ecl}_A(ca_1 \cdots a_n)$ (since $\sigma(c) \notin \mathrm{ecl}_A(a_1 \cdots a_n)$ by choice of c and assumption on σ). Now $ca_1 \cdots a_n$ is an independent $n+1$ tuple in X, so by definition of good pair, there exists a countable $C' = C_{ca_1 \cdots a_n} \subseteq C$ such that

$$\dim(ca_1 \cdots a_n / A \cup C') = n. \tag{*}$$

So $c \in \mathrm{ecl}_A(C' a_1 \cdots a_n)$ and hence $\sigma(c) \in \mathrm{ecl}_A(C' a_1 \cdots a_n)$ by assumption on σ. This implies that $\sigma(c) \in X$ by definition of good pair. Now since $\sigma(c) \in X \setminus \mathrm{ecl}_A(ca_1 \cdots a_n)$, there is $d \in P \setminus \mathrm{ecl}_A(C' ca_1 \cdots a_n)$, and an automorphism f of $\mathfrak{C}$ fixing $Aca_1 \cdots a_n$, such that $f(d) = \sigma(c)$ and $f(C') \subseteq C$. Applying f on (*), we obtain

$$\dim(ca_1 \cdots a_n / A \cup f(C')) = n. \tag{**}$$

On the other hand, $\sigma(c) \notin \mathrm{ecl}_A(f(C') a_1 \cdots a_n)$, by choice of d, and so also $\sigma(c) \notin \mathrm{ecl}_A(f(C') \sigma(a_1) \cdots \sigma(a_n))$ (since $\sigma(a_i)/E = a_i/E$, for $i = 1, \dots, n$). But we have $\dim(\sigma(a_1) \cdots \sigma(a_n) / A \cup f(C')) = n$, so together we have

$$\dim(\sigma(c)\sigma(a_1) \cdots \sigma(a_n) / A \cup f(C')) = n + 1.$$

But this contradicts (**) since σ fixes $f(C') \subseteq C$.

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The rest of the proof is identical to Lemma 3.10, which allows us to deduce that $\sigma(c)/E = c/E$, when $c \in X \setminus \mathrm{ecl}_A(a_1 \cdots a_{n+1})$.

Now since X is infinite-dimensional, we can find as in Lemma 3.11 elements $b_1, \ldots, b_{n+1} \in X$ independent from $ca_1, \ldots, a_{n+1}$. The first part of the proof applies $n+1$ times to ensure that $\sigma(b_i)/E = b_i/E$, for $i = 1, \ldots, n+1$, and so $\sigma(c)/E = c/E$ since $c \notin \mathrm{ecl}_A(b_1, \ldots, b_{n+1})$. □

We now deduce easily the next proposition.

Proposition 4.12. *Let $a_1, \ldots, a_{n+1} \in P$ be independent. Let $c \in P \setminus \mathrm{ecl}_A(\emptyset)$. There exists a countable $C_c \subseteq Q$ such that if $\sigma, \tau \in \mathrm{Aut}(\mathfrak{C}/A \cup C_c)$ and*

$$\sigma(a_i)/E = \tau(a_i)/E, \quad \textit{for each } i = 1, \ldots, n+1$$

then $\sigma(c)/E = \tau(c)/E$.

Proof. Let (X, C) be a good pair with X containing $a_1, \ldots, a_{n+1}, c$. We let C_c be C. Then, for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ with $\sigma(a_i)/E = \tau(a_i)/E$, for $i = 1, \ldots, n+1$, then $\tau^{-1} \circ \sigma(a_i)/E = a_i/E$, for $i = 1, \ldots, n+1$. Hence, by the previous lemma, we have that $\tau^{-1} \circ \sigma(c)/E = c/E$. This implies that $\sigma(c)/E = \tau(c)/E$. □

The value of $\sigma(c)$ in the previous proposition is independent of C_c. It follows that the action of G on P/E is $(n+1)$-determined. We will now show that the action has rank n (so G is automatically nontrivial).

Proposition 4.13. *The action of G on P/E is an n-action.*

Proof. The $(n+1)$-determinacy of the action of G on P follows from the previous proposition. We now have to show that the action has rank n.

For this, we first prove the following claim: if $\bar{a} = (a_i)_{i<n}$ and $\bar{b} = (b_i)_{i<n}$ are in P such that $\dim(\bar{a}\bar{b}/A) = 2n$ and $c \notin \mathrm{ecl}_A(\bar{a}\bar{b})$, then there is $d \in P$ such that for each countable $C \subseteq Q$ there is $\sigma \in \mathrm{Aut}(\mathfrak{C}/AC)$ satisfying $\sigma(a_i) = b_i$ (for $i < n$) and $\sigma(c) = d$.

To see this, choose $D \subseteq Q$ such that $\dim(\bar{a}c/D) = n$ (this is possible by Hypothesis 4.6). Suppose, for a contradiction, that no such d exists. Any automorphism fixing D and sending $\bar{a}$ to $\bar{b}$ must send $c \in \mathrm{ecl}_A(D\bar{b}) \cap P$. Thus, for each $d \in \mathrm{ecl}_A(D\bar{b})$, there is a countable set $C_d \subseteq Q$ containing D with the property that no automorphism fixing C_d sending $\bar{a}$ to $\bar{b}$ also sends c to d. Since $\mathrm{ecl}_A(D\bar{b})$ is countable, we can therefore find a countable $C \subseteq Q$ containing D such that any $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ sending $\bar{a}$ to $\bar{b}$ is such that $\sigma(c) \notin \mathrm{ecl}_A(D\bar{b})$. By Lemma 4.9, there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that $\sigma(\bar{a}) = \bar{b}$, and by choice of D we have $\sigma(c) \in \mathrm{ecl}_A(D\bar{b})$. This contradicts the choice of C.

We can now show that the action of G on P/E has rank n. Assume that $\bar{a}, \bar{b}$ are independent n-tuples of realizations of p. We must find $g \in G$ such that $g(\bar{a}/E) = \bar{b}/E$. Let $c \in P \setminus \mathrm{ecl}_A(\bar{a}\bar{b})$ and choose $d \in P$ as in the previous claim. We now

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define the following function $g : P/E \to P/E$. For each $e \in P$, choose C_e as in the Proposition 4.12, i.e. for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{C}/A \cup C_e)$, such that $\sigma(\bar{a})/E = \bar{b}/E = \tau(\bar{a})/E$ and $\sigma(c)/E = d/E = \tau(c)/E$, we have $\sigma(e)/E = \tau(e)/E$. By the previous claim there is $\sigma \in \mathrm{Aut}(\mathfrak{C}/C_e)$ sending $\bar{a}c$ to $\bar{b}d$. Let $g(e/E) = \sigma(e)/E$. The choice of C_e guarantees that this is well-defined. It is easily seen to induce a permutation of P/E. Further, suppose a countable $C \subseteq Q$ is given and a finite $X \subseteq P$. Choose C_e as in the previous proposition for each $e \in X$. There is $\sigma \in \mathrm{Aut}(\mathfrak{C})$ sending $\bar{a}c$ to $\bar{b}d$ fixing each C_e pointwise. By definition of g, we have $\sigma(e)/E = g(e/E)$. This implies that $g \in G$. Since this fails for independent $(n+1)$-tuples, by Hypothesis 4.6, the action of G on P has rank n. □

The next proposition is now proved exactly like Proposition 3.17.

Proposition 4.14. *The group G is interpretable in $\mathfrak{C}$ (over a finite set).*

Remark 4.15. Recall that in this case, any complete type over a finite set is equivalent to a formula (as $\mathcal{K}$ is the class of atomic models of a countable first order theory). By ω-homogeneity of $\mathfrak{C}$, for any finite B, any B-invariant is subset of $\mathfrak{C}$ is a countable disjunction of formulas over A. Since the complement of a B-invariant set is B-invariant, any B-invariant set over a finite set is actually type-definable over B. Hence, the various invariant sets in the above interpretation are all type-definable over a finite set. Although this is a stronger result, it is due to the atomicity, and may not be obtainable in the nonatomic case, in general, we will have an $L_{\omega_1,\omega}$ definable set quotiented by an $L_{\omega_1,\omega}$-definable equivalence relation.

It remains to deal with the stationarity of G. As in the previous section, this is done by considering *strong automorphisms* and *Lascar strong types*. We only need to consider the group of strong automorphisms over finite sets C, which makes the theory easier.

In the excellent case, indiscernibles do not behave as well as in the homogeneous case: on the one hand, some indiscernibles cannot be extended, and on the other hand, it is not clear that a permutation of the elements induce an automorphism. However, Morley sequences over models have both of these properties. Recall that $(a_i : i < \alpha)$ is the *Morley sequence* of $\mathrm{tp}(a_0/M)$ if $\mathrm{tp}(a_i/M\{a_j : j < i\})$ does not split over M. (In the application, we will be interested in Morley sequences inside P, these just coincide with independent sequences.)

We first define Lascar strong types.

Definition 4.16. Let C be a finite subset of $\mathfrak{C}$. We say that a and b have the same *Lascar strong type over* C, written $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, if $E(a, b)$ holds for any C-invariant equivalence relation E with a bounded number of classes.

Equality between Lascar strong types over C is clearly a C-invariant equivalence relation; it is the finest C-invariant equivalence relation with a bounded number of

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classes. With this definition, one can prove the same properties for Lascar strong types as one has in the homogeneous case. The details are in [16]; the use of excellence to extract good indiscernible sequences from large enough sequences is a bit different from the homogeneous case, but once one has the fact below, the details are similar.

Fact 4.17. Let $I \cup C \subseteq \mathfrak{C}$ be such that $|I|$ is uncountable and C countable. Then there is a countable $M_0 \prec \mathfrak{C}$ containing C and $J \subseteq I$ uncountable such that J is a Morley sequence of some stationary type $p \in S_D(M_0)$.

The key consequences are that (1) The Lascar strong types are the orbits of the group Σ of strong automorphisms, and (2) Lascar strong types are stationary. We can then show a proposition similar to Propositions 3.14 and 3.15.

Proposition 4.18. *G is stationary and admits hereditarily unique generics with respect to Σ.*

We have therefore proved:

Theorem 4.19. *Let $\mathcal{K}$ be excellent. Let $\mathfrak{C}$ be a large full model containing the finite set A. Let $p, q \in S_D(A)$ be unbounded with p quasiminimal. Assume that there exists an integer $n < \omega$ such that*

(1) *For each independent n-tuple $a_0, \ldots, a_{n-1}$ of realizations of p and countable $C \subseteq Q$ we have*

$$\dim(a_0 \cdots a_{n-1}/AC) = n.$$

(2) *For some independent $(n+1)$-tuple $a_0, \ldots, a_n$ of realizations of p and some countable $C \subseteq Q$ we have*

$$\dim(a_0 \cdots a_n/AC) \leq n.$$

Then $\mathfrak{C}$ interprets a group G acting on the geometry P' induced on the realizations of p. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n \leq 3$ and

- *If $n = 1$, then G is abelian and acts regularly on P'.*
- *If $n = 2$, the action of G on P' is isomorphic to the affine action of $K \rtimes K^*$ on the algebraically closed field K.*
- *If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

Question 4.20. Again, as in the stable case, we can produce a group starting from a regular type only (see [5] for the definition). Is it possible to get the field (i.e. hereditarily unique generics) starting from a regular, rather than quasiminimal type?

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