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ALL CREATURES GREAT AND SMALL

MARTIN GOLDSTERN AND SAHARON SHELAH

Abstract. Let λ be an uncountable regular cardinal. Assuming $2^\lambda = \lambda^+$, we show that the clone lattice on a set of size λ is not dually atomic.

0. Introduction

A clone $\mathscr{C}$ on a set X is a set of finitary operations $f : X^n \to X$ which contains all the projections and is closed under composition. (Alternatively, $\mathscr{C}$ is a clone if $\mathscr{C}$ is the set of term functions of some universal algebra over X.)

The family of all clones forms a complete algebraic lattice $Cl(X)$. (A lattice is complete if every subset has a greatest lower bound and a least upper bound; a complete lattice is algebraic if it is isomorphic to the lattice of subalgebras of some universal algebra.) The greatest element of this lattice is $\mathscr{O} = \bigcup_{n=1}^{\infty} X^{X^n}$, where X^{X^n} is the set of all n-ary operations on X. (In this paper, the underlying set X will be a fixed uncountable set.) The coatoms of this lattice $Cl(X)$ are called "precomplete clones" or "maximal clones" on X. The classical reference for older results about clones is [PK79].

For singleton sets X the lattice $Cl(X)$ is trivial; for $|X| = 2$ the lattice $Cl(X)$ is countable and well understood ("Post's lattice"). For $|X| \geq 3$, $Cl(X)$ has uncountably many elements. Many results for clones on finite sets can be found in [Sz86]. In particular, there is an explicit description of all (finitely many) precomplete clones on a given finite set ([R70]; see also [Q71] and [B96]); this description also includes a decision procedure for the membership problem for each of these clones. It is also known that every clone $\mathscr{C} \neq \mathscr{O}$ is contained in a precomplete clone, that is: the clone lattice $Cl(X)$ on any finite set X is *dually atomic*. (This gives an explicit criterion for deciding whether a given set of functions generates all of $\mathscr{O}$: just check if it is contained in one of the precomplete clones.)

Fewer results are known about the lattice of clones on an infinite set, and they are often negative or "nonstructure" results: [R76] showed that there are always 2^{2^κ} precomplete clones on a set of infinite cardinality κ (see also [GS02]). See the survey [GP07] for more background and references.

Rosenberg and Schweigert [RSch82] investigated "local" clones on infinite sets (clones that are closed sets in the product topology, where X is viewed as a discrete space). It is easy to see that the lattice of local clones is far from being dually atomic.

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Already Gavrilov [G59, pages 22–23] asked whether the lattice of all clones on a *countable* set is also dually atomic, since a positive answer would be an important component for a completeness criterion, as remarked above. The same question for *all* infinite sets is listed as problem P8 in [PK79, page 91].

In [GS05] we showed that (assuming the continuum hypothesis CH) the answer is negative for countable sets. We are now able to extend this construction to work on all regular uncountable cardinals as long as they satisfy the corresponding version of CH. The question whether such a theorem is provable in ZFC alone remains open.

We will write CH_λ for the statement $2^\lambda = \lambda^+$, or equivalently,

CH_λ: If $|X| = \lambda$, then every subset of $\mathscr{P}(X)$ (the power set of X) either has cardinality $\leq \lambda$ or is equinumerous with $\mathscr{P}(X)$.

We will show here the following for every uncountable regular cardinal λ:

Theorem 0.1. *Assume that X is a set of size λ and that CH_λ holds. Then the lattice of clones on the set X is* **not dually atomic***; i.e., there is a clone $\mathscr{C} \neq \mathscr{O}$ which is not contained in any precomplete clone.*

The clone $\mathscr{C}_U$ that we construct has the additional feature that we can give a good description of the interval $[\mathscr{C}_U, \mathscr{O}]$.

The method behind our proof is "forcing with large creatures", a new method which is rooted in "forcing with normed creatures" ([Sh84], [RSh99]). However, for the purposes of this paper the connection with forcing machinery is sufficiently shallow to allow us to be self-contained. In particular, no knowledge of set theory is required for our theorem, except for a basic understanding of ordinals, well-founded relations and transfinite induction.

Remark 0.2. The reader familiar with our previous paper [GS05] may appreciate the following list of differences/modifications:

(1) In our previous paper, the "largeness property" was connected with cardinalities of finite sets going to infinity, and we could show several partition theorems of the form: if the norms of a sequence of creatures (S_n) goes to infinity, we can find a subsequence (T_n) of "nice" creatures (e.g., homogeneous for some coloring function) such that their norm still goes to infinity.

This point has become easier now; rather than sets "large in cardinality", our large sets are now sets in certain ultrafilters.

(2) In our previous paper we had "unary" and "binary" partition theorems guaranteeing that we can thin out creatures to creatures that are homogeneous with respect to certain coloring functions. In the current paper we only have a unary partition theorem (see Lemma 2.25). This means that our notions of "f-weak" and "f-strong" are somewhat weaker than the notions in [GS05], which in turn means that we know somewhat less about the structure of the clone interval we construct. In particular, instead of showing that this interval is linearly ordered, we can only show that there is a linearly ordered cofinal set.

(3) (A crucial difference) In our previous paper, the construction took ω_1 steps, so in each intermediate step we only had to consider the countably many steps taken before. In particular, the σ-closure of our set of creatures was easily proved via a "diagonal" construction. In the current paper we again

have a simple diagonal construction (Lemma 4.8) to find a lower bound of a decreasing chain of creatures of length λ, but we also have to deal with shorter infinite sequences in Lemma 4.7, which necessitates a more complicated setup.

(4) For any $f : \lambda \to \lambda$ let $\bar{f} : \lambda \to \lambda$ be defined as $\bar{f}(x) = \sup\{f(x) : x \leq y\}$. If $\lambda = \omega$, then we have $f \in \mathscr{C}$ iff $\bar{f} \in \mathscr{C}$ for all (relevant) clones $\mathscr{C}$, so in our previous paper we could wlog assume that all unary functions that we considered were monotone. But for $\lambda > \omega$ we cannot assume that anymore.

(5) We introduce "coordinates" for elements of creatures. This will obviate the notational difficulties we had in [GS05, 3.10] (involving the possible "recycling" of deleted nodes).

(6) Another notational change: Rather than defining a linear order of equivalence classes of fronts as in [GS05, 5.2], we will work directly with the induced order on the functions in $\mathscr{O}$.

1. Preliminaries

Our base set will be a fixed uncountable regular cardinal λ, equipped with the usual order. We are interested in operations on λ, i.e., elements of $\mathscr{O} = \bigcup_{k=1,2,\dots} \lambda^{\lambda^k}$, and in subsets of $\mathscr{O}$.

Definition 1.1. We write $\mathscr{C}_{\max}$ for the set of all functions f which satisfy $f(x_1, \dots, x_k) \leq \max(x_1, \dots, x_k)$ for all $x_1, \dots, x_k \in \lambda$.

For each set $\mathscr{D} \subseteq \mathscr{O}$ we write $\langle \mathscr{D} \rangle$ for the clone generated by $\mathscr{D}$. We will write $\langle \mathscr{D} \rangle_{\max}$ for $\langle \mathscr{C}_{\max} \cup \mathscr{D} \rangle$.

Fact 1.2. (1) $\mathscr{C}_{\max}$ is a clone.

(2) Any clone containing $\mathscr{C}_{\max}$ is downward closed (in the sense of the pointwise partial order on each of the sets λ^{λ^n}).

(3) Assume that $\mathscr{C} \supseteq \mathscr{C}_{\max}$ is a clone, and assume that $f_1, \dots, f_k$ are functions of the same arity. Then $\langle \mathscr{C} \cup \{f_1, \dots, f_k\} \rangle = \langle \mathscr{C} \cup \{\max(f_1, \dots, f_k)\} \rangle$. (Here, max is the pointwise maximum function.)

Proof. (1) is trivial, and (2) is easy (see [GS05]): If $g \in \mathscr{C}$, and f is k-ary, $f(\vec{x}) \leq g(\vec{x})$ for all $\vec{x}$, then we can find a $(k+1)$-ary function $F \in \mathscr{C}_{\max}$ with $f(\vec{x}) = F(\vec{x}, g(\vec{x}))$ for all $\vec{x}$.

In (3), the inclusion $\subseteq$ follows from the downward closure of $\langle \mathscr{C} \cup \{f_1, \dots, f_k\} \rangle$ and (2), and the inclusion $\supseteq$ follows from the assumption that the k-ary maximum function is in $\mathscr{C}$. $\square$

1.1. Proof outline.

Fact 1.3. Let $(L, <)$ be a complete linear order, $\mathscr{C} \supseteq \mathscr{C}_{\max}$ a clone, and $\rho : \mathscr{O} \to L$ a map into L with properties (a), (b), (c) for all $f, g \in \mathscr{O}$, where we write $f <_\rho g$ for $\rho(f) < \rho(g)$, $f \sim_\rho g$ for $\rho(f) = \rho(g)$ and $f \leq_\rho g$ for $\rho(f) \leq \rho(g)$. Then (1), (2), (3) hold.

(a) $f <_\rho g \;\Rightarrow\; f \in \langle \mathscr{C} \cup \{g\} \rangle$.

(b) $f \in \langle \mathscr{C} \cup \{g\} \rangle \;\Rightarrow\; f \leq_\rho g$.

(c) $\rho(\max(f, g)) = \max(\rho(f), \rho(g))$.

(1) For every $d \in L$ the sets $\mathscr{D}_{<d} := \{f : \rho(f) < d\}$ and $\mathscr{D}_{\leq d} := \{f : \rho(f) \leq d\}$ are clones (unless they are empty).

(2) For every clone $\mathscr{D}$ in $[\mathscr{C},\mathscr{O})$ there is some $d \in L$ with $\mathscr{D}_{<d} \subseteq \mathscr{C} \subseteq \mathscr{D}_{\le d}$.
(3) If, moreover, $\rho[\mathscr{O}]$ has no last element, then the interval $[\mathscr{C},\mathscr{O}]$ has no coatom.

Note that (b) is equivalent to (b'), and (a)+(b)+(c) is equivalent to (a)+(b')+(c'):

(b') $f <_\rho g \;\Rightarrow\; g \notin \langle \mathscr{C} \cup \{f\}\rangle$.
(c') Whenever $f <_\rho g$ or $f \sim_\rho g$, then $\max(f,g) \sim_\rho g$.

Proof. Writing 0 for $\inf \rho[\mathscr{O}]$, we conclude from (b):

$$f \in \mathscr{C} \Rightarrow \rho(f) = 0.$$

Property (c) implies that the sets $\mathscr{D}_{<e}$ and $\mathscr{D}_{\le e}$ are closed under the pointwise max function; if they are nonempty, they contain $\mathscr{C}$ (and hence also all projections). For $e \ge 0$, $k > 0$ we show that $\langle f_1,\dots,f_k\rangle \subseteq \mathscr{D}_{\le e}$ for any $f_1,\dots,f_k \in \mathscr{D}_{\le e}$:

Let $f := \max(f_1,\dots,f_k) \in \mathscr{D}_{\le e}$. So $\langle \mathscr{C}\cup\{f_1,\dots,f_k\}\rangle = \langle \mathscr{C}\cup\{f\}\rangle$.
If $h \in \langle \mathscr{C}\cup\{f\}\rangle$, then (by (b)) $\rho(h) \le \rho(f) \le e$. So $\langle \mathscr{C}\cup\{f\}\rangle \subseteq \mathscr{D}_{\le e}$.

Hence $\mathscr{D}_{\le e}$ is a clone. The argument for $\mathscr{D}_{<e}$ (with $e > 0$) is similar.

Now, given any clone $\mathscr{D} \supseteq \mathscr{C}$, let $d_0 := \sup\{\rho(f) : f \in \mathscr{D}\}$. We claim $\mathscr{D}_{<d_0} \subseteq \mathscr{D} \subseteq \mathscr{D}_{\le d_0}$:

Clearly $\mathscr{D} \subseteq \mathscr{D}_{\le d_0}$.
Let $h \in \mathscr{D}_{<d_0}$; then $\rho(h) < d_0$. Hence there is some $f \in \mathscr{D}$ with $\rho(h) < \rho(f)$. So $h \in \langle \mathscr{C} \cup \{f\}\rangle \subseteq \mathscr{D}$ by (a). Hence $\mathscr{D}_{<d_0} \subseteq \mathscr{D}$.

Finally, we see that the map $d \mapsto \mathscr{D}_{\le d}$ is 1-1 from $\rho[\mathscr{O}]$ into $[\mathscr{C},\mathscr{O})$, since $\rho(f) = d$ implies $f \in \mathscr{D}_{\le d} \setminus \mathscr{D}_{\le e}$ for $e < d$. Hence $[\mathscr{C},\mathscr{O})$ contains a cofinal copy of $\rho[\mathscr{O}]$, thus no maximal element. □

We will try to find a linear order L and a map ρ that will allow us to apply the lemma. But rather than finding L explicitly, we will first construct relations $<_\rho$ and $\sim_\rho$:

$$(**) \qquad f <_\rho g \Leftrightarrow \rho(f) < \rho(g), \qquad\qquad f \sim_\rho g \Leftrightarrow \rho(f) = \rho(g)$$

on $\mathscr{O}$. The order L will then appear as the Dedekind completion of the quotient order $\mathscr{O}/{\sim}$.

We will construct $<$ and $\sim$ in λ^+ many stages as unions $\bigcup_i <_i$ and $\bigcup_i \sim_i$. Each $<_i$ will be a partial order on $\mathscr{O}$, and each $\sim_i$ will be an equivalence relation, but only at the end will we guarantee that any two operations f and g are either $<$-comparable or $\sim$-equivalent.

The relation $f <_i g$ will say that on a "large" set, f grows faster than g. This i-th notion of "large" will come from a filter D_i on λ. Eventually, the clone $\mathscr{C}$ at the bottom of our interval will be determined by the filter $\bigcup_i D_i$.

1.2. **Filter clones.**

Definition 1.4. For any unbounded $A \subseteq \lambda$, let h_A be the function $h_A(x) = \min\{y \in A : y > x\}$. For any family U of unbounded subsets of λ let $\mathscr{C}_U$ be the clone $\langle h_A : A \in U\rangle_{\max}$.

(The function h_F will be defined below in Definition 3.9.)

Definition 1.5. For any unbounded $A \subseteq \lambda$ we write $f \le_A g$ iff $f \in \langle h_A, g\rangle_{\max}$.

Fact 1.6. The relation $\le_A$ is transitive.

Lemma 1.7. *Assume that U is a filter on λ containing no bounded sets. Then $\mathscr{C}_U = \{f : \exists A \in U\, \exists k\, \forall \vec{x}\ f(\vec{x}) \le h_A^{(k)}(\max \vec{x})\} = \bigcup_{A\in U} \langle h_A\rangle_{\max}$. (Here, $h_A^{(k)}$ is the k-fold iteration of the function h_A.)*

Proof. Write $\mathscr{C}'_U := \{f : \exists A \in U\, \exists k\, \forall \vec{x}\ f(\vec{x}) \le h_A^{(k)}(\max \vec{x})\}$, $\mathscr{C}''_U = \bigcup_{A\in U} \langle h_A\rangle_{\max}$. The inclusions $\mathscr{C}''_U \subseteq \mathscr{C}'_U \subseteq \mathscr{C}_U$ are trivial, and the inclusion $\mathscr{C}'_U \subseteq \mathscr{C}''_U$ follows from the downward closure of $\langle h_A\rangle_{\max}$.

To check $\mathscr{C}_U \subseteq \mathscr{C}'_U$, it is enough to see that $\mathscr{C}'_U$ is a clone. So let $f, g_1, \dots, g_n \in \mathscr{C}'_U$, witnessed by $A, A_1, \dots, A_n, k, k_1, \dots, k_n$. Let $k^* = \max(k_1, \dots, k_n)$, $A^* = A_1 \cap \dots \cap A_n$. Then

$$\begin{aligned} f(g_1(\vec{x}), \dots, g_n(\vec{x})) &\le h_A^{(k)}(\max(g_1(\vec{x}), \dots, g_n(\vec{x}))) \\ &\le h_A^{(k)}(h_{A^*}^{(k^*)}(\max \vec{x})) \le h_{A\cap A^*}^{(k+k^*)}(\max \vec{x}). \end{aligned}$$

□

All clones constructed in this paper will be of the form $\mathscr{C}_U$ for some filter U.

2. Creatures

2.1. Definitions.

Definition 2.1. A *planar tree* is a tuple $(T, \trianglelefteq, <)$ where:

(A) T is a nonempty set. (Elements of trees are often called "nodes".)
(B) $\trianglelefteq$ is a partial order on T in which every set $\{\eta : \eta \trianglelefteq \nu\}$ is well-ordered by $\trianglelefteq$.
(We take $\trianglelefteq$ to be reflexive, and write $\triangleleft$ for the corresponding irreflexive relation.)
(C) $<$ is an irreflexive partial order on T such that any two $\eta \ne \nu$ in T are $<$-comparable iff they are $\trianglelefteq$-incomparable. ($x \le y$ means $x < y \vee x = y$.)
(D) Whenever $\eta \trianglelefteq \eta'$ and $\nu \trianglelefteq \nu'$, then $\eta < \nu$ implies $\eta' < \nu'$.

Example 2.2. Let T be a downward closed set of nonempty (possibly transfinite) sequences of ordinals. Then T admits a natural tree order $\trianglelefteq$: $\eta \trianglelefteq \nu$ iff η is an initial segment of ν. We also have a natural partial order $<$, namely, the usual lexicographic order of sequences of ordinals (where sequences $\eta \triangleleft \nu$ are $<$-incomparable). Thus $(T, \trianglelefteq, <)$ is a planar tree.

It is easy to see that every planar tree in which the relation $<$ is well-founded is isomorphic to a planar tree as described in this example. None of our trees will contain infinite $\triangleleft$-chains, so they could be represented using sets of finite (or even strictly decreasing) sequences of ordinals.

For notational reasons, however, we will use a completely different way to represent trees. The problem with the particular implementation described above is that we will have to "glue" old trees together to obtain new trees (see Definition 2.24); this means that the roots of the old trees will no longer be roots in the new tree. Since we want to view the old trees as subtrees of the new trees, it is not reasonable to demand that roots are always sequences of length 1.

Notation 2.3. Let $(T, \trianglelefteq, <)$ be a planar tree.

* We call $\trianglelefteq = \trianglelefteq^T$ the "tree order" and $< = <^T$ the "lexicographic order" of T.

* For $\eta \in T$ we write $\mathrm{Succ}_T(\eta)$ or sometimes $\mathrm{Succ}(\eta)$ for the set of all direct successors of η:
$$\mathrm{Succ}_T(\eta) := \{\, \nu \in T : \eta = \max\{\nu' : \nu' \lhd \nu\} \,\}.$$
* $\mathbf{ext}(T)$, the set of *external nodes* or *leaves* of T, is the set of all η with $\mathrm{Succ}_T(\eta) = \emptyset$.
$\mathbf{int}(T) := T \setminus \mathbf{ext}(T)$ is the set of *internal nodes*.
* We let $\mathbf{Root}(T)$ be the set of minimal elements of T (in the tree order $\unlhd$). If $\mathbf{Root}(T)$ is a singleton, we call its unique element $\mathbf{root}(T)$.
* A *branch* is a maximal linearly ordered subset of T (in the sense of $\unlhd$).
The tree T is called "well-founded" iff T has no infinite branches, or equivalently, no infinite linearly ordered subsets, equivalently, if $(T, \unrhd)$ is well-founded in the usual sense.
If T is well-founded, then there is a natural bijection between external nodes and branches, given by $\nu \mapsto \{\eta \in T : \eta \unlhd \nu\}$.
* For any $\eta \in T$ we let $T^{[\eta]} := \{\nu : \eta \unlhd \nu\}$; this is again a planar tree (with the inherited relations $\unlhd$ and $<$).
More generally, if H is a set of pairwise $\unlhd$-incomparable nodes of S (often $H \subseteq \mathbf{Root}(S)$), then we define
$$S^{[H]} := \{\eta \in S : \exists \gamma \in H \ \gamma \unlhd \eta\} = \bigcup_{\gamma \in H} S^{[\gamma]}.$$
This is again a planar tree, and $\mathbf{Root}(S^{[H]}) = H$.
If $H = \{\gamma \in \mathbf{Root}(S) : \gamma_0 < \gamma\}$ for some $\gamma_0 \in \mathbf{Root}(S)$, then we write $S^{[\mathrm{root} > \gamma_0]}$ for $S^{[H]}$.
* A *front* is a subset of T which meets each branch exactly once. (Equivalently, a front is a maximal subset of T that is linearly ordered by $<$.)
For example, $\mathbf{ext}(T)$ is a front, and $\mathbf{Root}(T)$ is also a front. If $F \subseteq \mathbf{int}(T)$ is a front, then also $\bigcup_{\eta \in F} \mathrm{Succ}_T(\eta)$ is a front.
Let $\eta \in \mathbf{int}(T)$, $F \subseteq T^{[\eta]}$. We say that F is a "front above η" iff F is linearly ordered by $<$ and meets every branch of T containing η. Equivalently, F is a front above η if F is a front in $T^{[\eta]}$. (For example, $\mathrm{Succ}_T(\eta)$ is a front above η.)
* All trees S that we consider will satisfy $\mathbf{ext}(S) \subseteq \lambda$, so it makes sense to define the following notation:
  - Let S be a tree with $\mathbf{ext}(S) \subseteq \lambda$, and let $\eta \in S$. Then $\min_S[\eta] := \min(\mathbf{ext}(S^{[\eta]}))$.
  - Similarly $\sup_S[\eta] := \sup(\mathbf{ext}(S^{[\eta]}))$.

When $<$ and $\unlhd$ are clear from the context we may just call the tree "S"; we may later write $\unlhd^S$, $<^S$ for the respective relations.

We visualize such trees as being embedded in the real plane $\mathbb{R}^2$, with the order $\unlhd$ pointing from the bottom to the top, whereas the order $<$ can be viewed as pointing from left to right. (See Figure 1, where we have $\eta_1 \unlhd \eta_2 \unlhd \eta_3$, $\nu_1 \unlhd \nu_2$, $\nu_1 \unlhd \nu_3$, $\nu_2 < \nu_3$, and $\eta_i < \nu_j$ for all $i, j \in \{1, 2, 3\}$.)

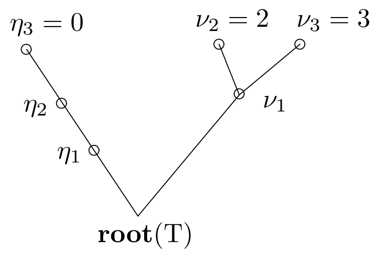


FIGURE 1. $\trianglelefteq$ and $\leq$

Definition 2.4. Let $(L, <)$ be a linear order, D a filter on L. We say that "D converges to $\sup L$" iff for all $x_0 \in L$ the set $\{y \in L : x_0 < y\}$ is in D.

Fact 2.5. If $(L, <)$ is a linear order, D a filter on L converging to $\sup L$, then L has no last element, and, moreover, each $A \in D$ has no last element.

Proof. If $A \in D$, $x_0 \in A$, then the set $\{x \in A : x > x_0\}$ is in D and hence cannot be empty. $\square$

Definition 2.6. An abstract creature is a tuple $(S, \trianglelefteq, <, D)$, where:

(A-D) $(S, \trianglelefteq, <)$ is a planar well-founded tree (see Definition 2.1).
(E) $D = (D_\eta : \eta \in \mathbf{int}(S))$ is a family of ultrafilters.
(F) For all $\eta \in \mathbf{int}(S)$, the linear order $(\mathrm{Succ}_S(\eta), <)$ has no last element.
(G) For all $\eta \in \mathbf{int}(S)$, D_η is an ultrafilter on $\mathrm{Succ}_S(\eta)$ which "converges to $\sup(\mathrm{Succ}_S(\eta))$".

We sometimes write (S, D) or just S for creatures if the other parameters are clear from the context. When an argument involves several creatures $S, T, \ldots$, we may write D^S, D^T, etc., for the respective families of ultrafilters. (The notation D_S will be reserved for a quite different notion; see Definition 5.3.)

Remark 2.7. Since a creature S is really a well-founded tree $(S, \trianglelefteq)$, we have that both $(S, \trianglelefteq)$ and $(S, \trianglerighteq)$ are well-founded. So when we prove theorems about the nodes of a creature S or when we define a function on a creature, we can use one of two kinds of induction/recursion:

- "Upward induction". Every nonempty $X \subseteq S$ has a minimal element with respect to $\triangleleft$. So if we want to define a function f "by recursion" on S, we may use the values of $f \restriction \{\eta : \eta \triangleleft \nu\}$ when we define $f(\nu)$. Similarly, we can prove properties of all $\eta \in T$ indirectly by considering a minimal counterexample and deriving a contradiction.
- "Downward induction". Every nonempty $X \subseteq S$ has a maximal element with respect to $\triangleleft$. So we can define a function f on S by downward recursion; to define $f(\eta)$ we may use the function $f \restriction \{\nu : \eta \triangleleft \nu\}$ or more often the function $f \restriction \mathrm{Succ}(\eta)$. Similarly, we may use "maximal counterexamples" in proofs of properties of all $\eta \in S$.

Motivation 2.8. Mainly for notational reasons it will be convenient to be able to read off information about the relations $\eta \trianglelefteq \nu$ and $\eta < \nu$ directly from η and ν. So we will restrict our attention to a subclass of the class of all creatures:

First we will require all external nodes of our creatures to come from a fixed linearly ordered set, the set of ordinals $< \lambda$. We also require that the "lexicographic" order (see Notation 2.3) agrees with the usual order of ordinals.

We then want to encode information about the location of any internal node $\eta \in T$ within T into the node T itself. It turns out that we can use the pair $(\min \mathbf{ext} T^{[\eta]}, \sup \mathbf{ext} T^{[\eta]})$ as "coordinates" for η. Thus, all our creatures will be subsets of $\lambda \cup (\lambda \times \lambda)$.

Definition 2.11 below is motivated by the following fact:

Fact 2.9. Let S be a creature with $\mathbf{ext}(S) \subseteq \lambda$ such that the lexicographic order on $\mathbf{ext}(S)$ agrees with the usual order on λ, which (in this paragraph only) we will denote by $\leq_{\mathrm{Ord}}$. Then for all $\eta, \nu \in S$:

- $\eta \lhd^S \nu$ iff $\min_S[\eta] \leq_{\mathrm{Ord}} \min_S[\nu]$ and $\sup_S[\nu] <_{\mathrm{Ord}} \sup_S[\eta]$.
- If η, ν are $\trianglelefteq$-incomparable, then: $\eta <^S \nu$ iff $\sup_S[\eta] \leq_{\mathrm{Ord}} \min_S[\nu]$.

Proof. $\eta \lhd \nu$ implies that $\mathbf{ext} T^{[\nu]} \subsetneq \mathbf{ext} T^{[\eta]}$, so $\min \mathbf{ext} T^{[\eta]} \leq \min \mathbf{ext} T^{[\nu]}$ and $\sup \mathbf{ext} T^{[\eta]} \geq \sup \mathbf{ext} T^{[\nu]}$. In fact, using Definition 2.6(F) it is easy to see that $\eta \lhd \nu$ even implies $\sup \mathbf{ext} T^{[\eta]} > \sup \mathbf{ext} T^{[\nu]}$, so the map $\eta \mapsto \sup_S[\eta]$ is 1-1. ☐

Remark 2.10. Let S be a creature with $\mathbf{ext}(S) \subseteq \lambda$ such that the lexicographic order on $\mathbf{ext}(S)$ agrees with the usual order on λ. Then for every $\eta \in S$ (except possibly $\eta = \mathbf{root}(S)$) the set $\mathrm{Succ}_S(\eta)$ has cardinality $< \lambda$.

Proof. If $|\mathrm{Succ}_S(\eta)| \geq \lambda$, then we must have $\sup_S[\eta] \geq \lambda$. This can only happen if η is the unique root of S. ☐

Definition 2.11. Let $\Lambda := \lambda \cup \{(i,j) \in \lambda \times \lambda : i < j\}$. We define two functions α and β from Λ into λ: $\alpha(i,j) = i$, $\beta(i,j) = j$, $\alpha(i) = \beta(i) = i$ for all $i, j \in \lambda$.

We define two partial orders $\trianglelefteq$ and $<$ on Λ. For all $\eta \neq \nu$ in Λ:

- $\eta \lhd \nu \;\Leftrightarrow\; \alpha(\eta) \leq \alpha(\nu)$ and $\beta(\eta) > \beta(\nu)$.
- If η, ν are $\trianglelefteq$-incomparable, then $\eta < \nu \;\Leftrightarrow\; \beta(\eta) \leq_{\mathrm{Ord}} \alpha(\nu)$.

Definition 2.12. A concrete creature (in the following, just "creature") is a tuple $(S, \trianglelefteq, <, D)$, where:

(A-G) $(S, \trianglelefteq^S, <^S, D^S)$ is an abstract creature (see Definition 2.6).

(H) $S \subseteq \Lambda$, $<^S$ and $\lhd^S$ agree with the relations $<$ and $\lhd$ defined in Definition 2.11.

(I) Each $\eta \in \mathbf{int}(S)$ is a pair $\eta = (\alpha(\eta), \beta(\eta))$, and $\mathbf{ext}(S) \subseteq \lambda$.

(J) For all $\eta \in \mathbf{int}(T)$, $\alpha(\eta) \leq \min \mathbf{ext}(T^{[\eta]})$ and $\sup \mathbf{ext} T^{[\eta]} = \beta(\eta)$.

Fact 2.13. Every creature (whose external nodes are a subset of λ with the natural order) is isomorphic to a concrete creature (replacing each internal node η by the pair $(\min[\eta], \sup[\eta])$).

Fact 2.14. If S and T are concrete creatures and $\eta, \nu \in S \cap T$, then $\eta \trianglelefteq^S \nu$ iff $\eta \trianglelefteq^T \nu$, and similarly $\eta <^S \nu$ iff $\eta <^T \nu$.

We will often "thin out" creatures to get better behaved subcreatures. It will be easy to check that starting from a concrete creature, each of these thinning-out processes will again yield a concrete creature.

2.2. Small is beautiful.

Definition 2.15. Let $(S, D) = (S, \trianglelefteq, <, D)$ be a creature. We say that (S, D) is

small, if $\mathbf{Root}(S)$ has a unique element: $\mathbf{Root}(S) = \{\mathbf{root}(S)\}$.
(This is a creature in the usual sense. We usually require $|\mathrm{Succ}(\mathbf{root}(S)| < \lambda$.)

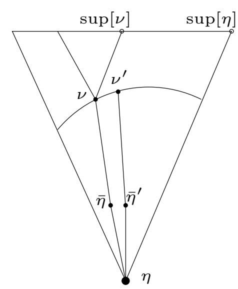
medium, if $\mathbf{Root}(S)$ is infinite without a last element but of cardinality $< \lambda$.
(Such a creature is often identified with the set (or naturally ordered sequence) $\{S^{[\gamma]} : \gamma \in \mathbf{Root}(S)\}$ of small creatures.)

large, if $\mathbf{Root}(S) \subseteq \lambda$ has size λ.
(These creatures are usually called "conditions" in forcing arguments. They correspond to "zoos" in [GS05]. Again, it may be convenient to identify such a large creature with a λ-sequence of small creatures.)

(We will not consider creatures S with $1 < |\mathbf{Root}(S)|$ where $\mathbf{Root}(S)$ has a last element.)

Fact 2.16. Let F be a front above η (see 2.3). Assume $F \neq \{\eta\}$. Then:

(1) F is linearly ordered by $<$ and has no last element.
(2) For all $\nu \in F$: $\sup[\nu] < \sup[\eta]$.
(3) $\sup[\eta] = \sup\{\sup[\nu] : \nu \in F\}$.
(4) $\sup[\eta] = \sup\{\min[\nu] : \nu \in F\}$.



Proof. We only show (1); the rest is clear. Let $\nu \in F$. We will find $\nu' \in F$, $\nu < \nu'$.
Let $\eta \trianglelefteq \bar\eta \trianglelefteq \nu$, with $\bar\eta \in \mathrm{Succ}_T(\eta)$. As $\mathrm{Succ}_T(\eta)$ has no last element, we can find $\bar\eta' \in \mathrm{Succ}(\eta)$, $\bar\eta < \bar\eta'$. So $\sup[\nu] \leq \min[\bar\eta']$.
There is $\nu' \in F$ with $\bar\eta' \trianglelefteq \nu'$. By the definition of a planar tree, $\nu < \nu'$. $\square$

2.3. Thinner creatures.

Fact and Definition 2.17 (THIN). If (S, D) is a small (or large) creature, $S' \subseteq S$, then we write $S' \leq_{\text{thin}} S$ iff

- $\mathbf{Root}(S) = \mathbf{Root}(S')$.
- $\forall \eta \in S' \cap \mathbf{int}(S) : \mathrm{Succ}_{S'}(\eta) \in D_\eta$.

In this case, S' naturally defines again a small (or large, respectively) creature (S', D') by letting $D'_\eta := \{X \cap \mathrm{Succ}_{S'}(\eta) : X \in D_\eta\}$ for all $\eta \in S'$ and by restricting $\trianglelefteq$ and $<$.

Fact 2.18. If S is a concrete creature and $S' \leq_{\text{thin}} S$, then also S' is a concrete creature.

Proof. Let $\eta = (\alpha, \beta) \in S'$. We have to show that $\alpha \leq \min_{S'}[\eta]$ and $\beta = \sup_{S'}[\eta]$. The first property follows from $\alpha \leq \min_S[\eta] \leq \min_{S'}[\eta]$.

For the second property we use downward induction. Arriving at η, we may assume $\sup_{S'}[\nu] = \sup_S[\nu]$ for all $\nu \in \mathrm{Succ}_{S'}(\eta)$. Now $\mathrm{Succ}_{S'}(\eta)$ is cofinal in $\mathrm{Succ}_S(\eta)$; hence also $\{\sup_{S'}[\nu] : \nu \in \mathrm{Succ}_{S'}(\eta)\} = \{\sup_S[\nu] : \nu \in \mathrm{Succ}_{S'}(\eta)\}$ is cofinal in $\{\sup_S[\nu] : \nu \in \mathrm{Succ}_S(\eta)\}$. $\square$

The following facts are easy:

Fact 2.19. If T and S are small or large creatures, $T \leq_{\text{thin}} S$, then for any $\eta \in T$ we also have $T^{[\eta]} \leq_{\text{thin}} S^{[\eta]}$.

Fact 2.20. $\leq_{\text{thin}}$ is transitive.

2.4. **Drop, short, sum, glue.**

Fact and Definition 2.21 (DROP). Let S and T be large creatures. We write $T \leq_{\text{drop}} S$ iff $\mathbf{Root}(T) \subseteq \mathbf{Root}(S)$ (with the same order $<$) and $T = S^{[\mathbf{Root}(T)]}$. (See Notation 2.3.)

Sometimes we drop only an initial part of the creature. This relation deserves a special name:

Definition 2.22 (SHORT)**.** Let S and T be large creatures. We write $T \leq_{\text{short}} S$ iff there is some $\gamma \in \mathbf{Root}(S)$ such that $S^{[\text{root}>\gamma]} = T$.

We write $T \leq_{\text{thin/short}} S$ iff there is some T' with $T \leq_{\text{short}} T' \leq_{\text{thin}} S'$. (Equivalently, if there is some T' with $T \leq_{\text{thin}} T' \leq_{\text{short}} S'$.)

Definition 2.23 (SUM)**.** Let (S, D) be a medium concrete creature. (See Figure 2.)

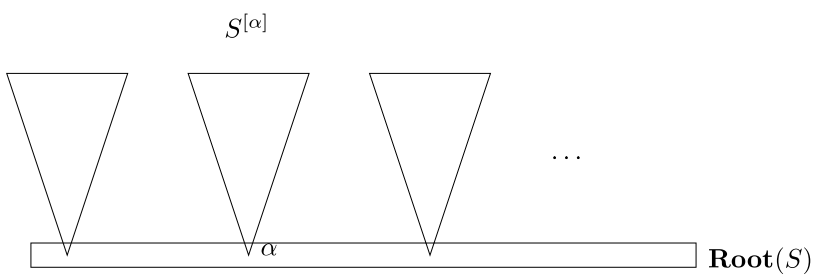


FIGURE 2. A medium creature S...

Let U be an ultrafilter on $\mathbf{Root}(S)$ converging to $\sup \mathbf{Root}(S)$ (see Definition 2.4). Let

$$\alpha := \min\{\alpha(\eta) : \eta \in S\}, \quad \beta := \sup\{\beta(\eta) : \eta \in S\} = \sup \mathbf{ext}(S), \quad \gamma := (\alpha, \beta).$$

(Note that $\gamma \lhd \eta$ for all $\eta \in S$.) Then $\sum(S, D) = \sum_U (S, D) = \sum_U S$ is defined as the following small concrete creature (T, E) (see Figure 3):

- $T := \{\gamma\} \cup S$, $\mathbf{root}(T) = \gamma$, $D_\gamma = U$.
- For all $\eta \in \mathbf{Root}(S) = \mathrm{Succ}_T(\eta)$: $T^{[\eta]} = S^{[\eta]}$.

Definition 2.24 (GLUE)**.** Let S and T be large concrete creatures. We write $T \leq_{\text{glue}} S$ iff for each $\gamma \in \mathbf{Root}(T)$ the set $H_\gamma := \mathrm{Succ}_T(\gamma)$ is an interval in $\mathbf{Root}(S)$ with no last element and each $T^{[\gamma]}$ can be written as $\sum_{U_\gamma, \gamma} S^{[H_\gamma]}$ for some ultrafilters U_γ (see Figures 4 and 5).

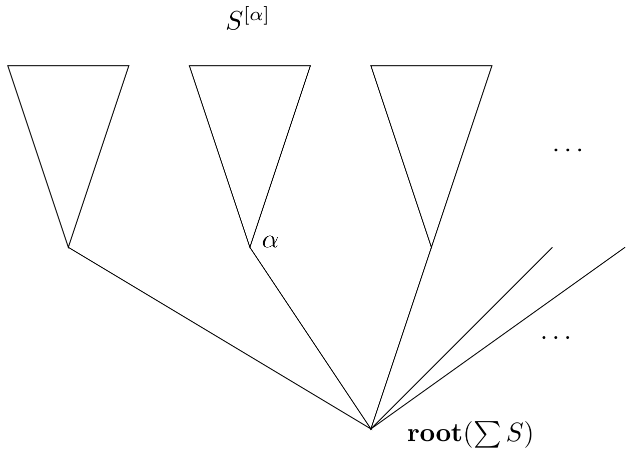


FIGURE 3. …whose sum is a small creature $\sum S$

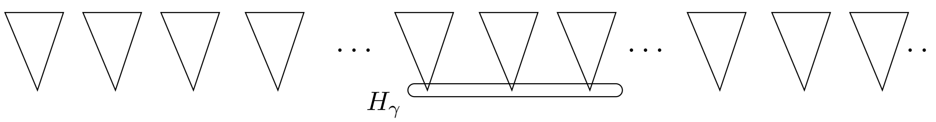


FIGURE 4. A large creature S

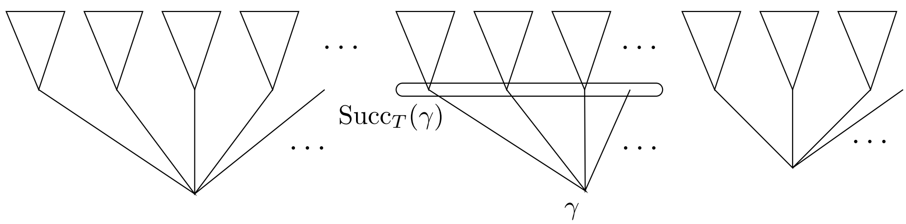


FIGURE 5. A large creature $T \leq_{\mathrm{glue}} S$

2.5. Partition theorems.

Lemma 2.25. *Let E be a finite set.*

(1) *If S is a large or small creature, $c : S \to E$, then there is a creature $T \leq_{\mathrm{thin}} S$ such that $c{\restriction}\mathrm{Succ}_T(\eta)$ is constant for all $\eta \in T$.*
(2) *If S is a small creature, $c : \mathbf{ext}(S) \to E$, then there is a small creature $T \leq_{\mathrm{thin}} S$ such that $c{\restriction}\mathbf{ext}(T)$ is constant.*
(3) *If S is a large creature, $c : \mathbf{ext}(S) \to E$, then there are large creatures T and T' such that $T' \leq_{\mathrm{drop}} T \leq_{\mathrm{thin}} S$ and $c{\restriction}\mathbf{ext}(T')$ is constant.*

Proof of (1). We define T by upward induction, starting with $\mathbf{Root}(T) = \mathbf{Root}(S)$. Given $\eta \in T$, we find a set $A_\eta \subseteq \mathrm{Succ}_S(\eta)$, $A_\eta \in D^S(\eta)$ such that $c{\restriction}A_\eta$ is constant, and we let $\mathrm{Succ}_T(\eta) := A_\eta$. $\square$

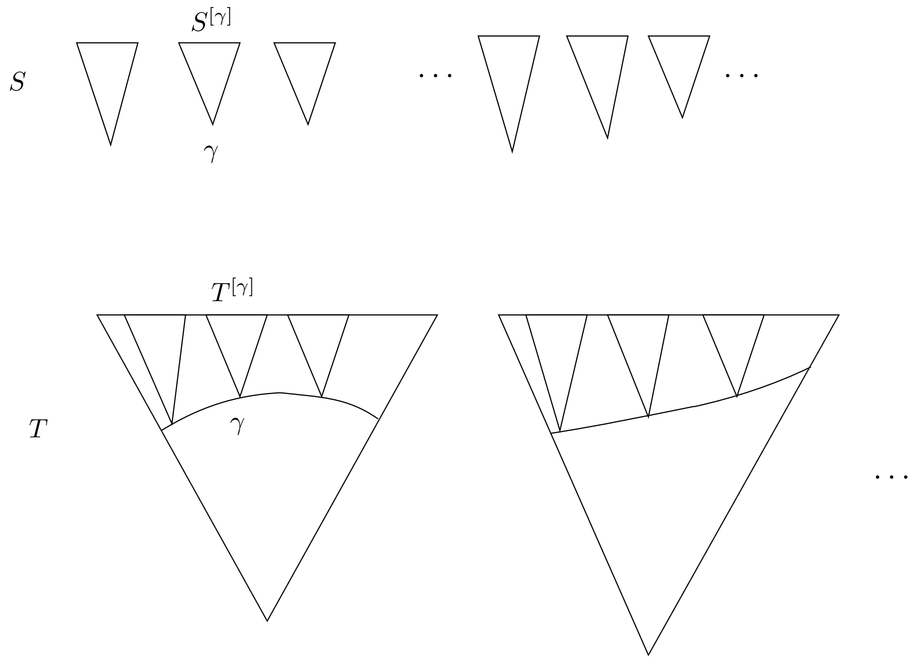


FIGURE 6. $T \leq S$

Proof of (2). We define a map $\bar{c} : S \to E$ by downward induction (see 2.7):

- ∗ For $\eta \in \mathbf{ext}(S)$: $\bar{c}(\eta) = c(\eta)$.
- ∗ For $\eta \in \mathbf{int}(S)$ we find a (unique) value $e_\eta \in E$ such that the set $\{\nu \in \mathrm{Succ}_S(\eta) : \bar{c}(\nu) = e_\eta\}$ is in D_η, and we set $\bar{c}(\eta) := e_\eta$.

Now we let $e_0 := \bar{c}(\mathbf{root}(S))$,

$$T := \{\nu \in S : \forall \eta \trianglelefteq \nu \, \bar{c}(\eta) = e_0\}.$$

Clearly $T \leq_{\text{thin}} S$, and $c\restriction\mathbf{ext}(T)$ is constant with value e_0. □

Proof of (3). We apply (2) to each $S^{[\gamma]}$, for all $\gamma \in \mathbf{Root}(S)$, to get a large $T \leq_{\text{thin}} S$ such that $c\restriction\mathbf{ext}(T^{[\gamma]})$ is constant, say with value e_γ, for all $\gamma \in \mathbf{Root}(T)$. Now find e_0 such that the set $\{\gamma : e_\gamma = e_0\}$ has cardinality λ, and let $T' := \bigcup_{e_\gamma = e_0} T^{[\gamma]}$. Then $T' \leq_{\text{drop}} T \leq_{\text{thin}} S$, and c is constant (with value e_0) on $\mathbf{ext}(T')$. □

2.6. **Comparing large creatures.** The constructions "glue", "drop" and "thin" are ways to get new, in some sense "stronger", large creatures from old ones. The following definition gives a common generalization of the above constructions.

Definition 2.26. Let S, T be creatures. We say $T \leq S$ iff there is a front $F \subseteq T$ such that

- $F \subseteq \mathbf{Root}(S)$,
- for each $\gamma \in F$: $T^{[\gamma]} \leq_{\text{thin}} S^{[\gamma]}$ (see Fact and Definition 2.17).

(See Figure 6.)

Remark 2.27. We usually consider this relation if both S and T are large or both are small, but we also allow the possibility that S is large and T is small. It is easy to see that if S is small and $T \leq S$, then also T must be small and $T \leq_{\text{thin}} S$.

Fact 2.28. Assume that $T \leq S$ are concrete creatures. Then:

(1) For all $\eta \in T \cap S$ we have $T^{[\eta]} \leq_{\text{thin}} S^{[\eta]}$.
(2) $\mathbf{ext}(T) \subseteq \mathbf{ext}(S)$, and S is downward closed in T.

The next fact is the main reason for our notational device of "concrete" creatures (in Definition 2.12): Thanks to Fact 2.14, we may just write $\eta \trianglelefteq \nu$ in the proof below rather than having to distinguish $\trianglelefteq^{S_1}$, $\trianglelefteq^{S_2}$, etc.

Fact 2.29 (Transitivity). If $S_3 \leq S_2 \leq S_1$ are concrete creatures, then $S_3 \leq S_1$.

Proof. Assume $S_3 \leq S_2 \leq S_1$, where $S_k \leq S_{k-1}$ is witnessed by a front $F_k \subseteq S_k$ for $k = 2, 3$. We claim that $F_2 \cap S_3$ witnesses $S_3 \leq S_1$. Clearly $F_2 \cap S_3 \subseteq \mathbf{Root}(S_1)$. To check that $F_2 \cap S_3$ is a front in S_3, consider any branch b in S_3. b is of the form $b = \{\eta \in S_3 : \eta \trianglelefteq \nu_0\}$ for some $\nu_0 \in \mathbf{ext}(S_3)$. The set $\{\eta \in S_2 : \eta \trianglelefteq \nu_0\}$ is also a branch in S_2, so it meets F_2 (hence $F_2 \cap S_3$, by Fact 2.28) in a singleton.

For any $\eta \in F_2$, $S_2^{[\eta]} \leq S_1^{[\eta]}$. Let $\gamma \in F_3$, $\gamma \trianglelefteq \eta$. Then we have $S_3^{[\gamma]} \leq_{\text{thin}} S_2^{[\gamma]}$, so by Fact 2.19 also $S_3^{[\eta]} \leq_{\text{thin}} S_2^{[\eta]} \leq_{\text{thin}} S_1^{\eta}$. □

Examples 2.30. (1) For any $\gamma \in \mathbf{Root}(S)$ we have $S^{[\gamma]} \leq S$.
(2) $S \leq S$ is witnessed by the front $\mathbf{Root}(S)$.
(3) Assume that $T \leq_{\text{drop}} S$ or $T \leq_{\text{thin}} S$. Then again $\mathbf{Root}(T)$ witnesses $T \leq S$.
(4) Assume that T is obtained from S as in GLUE (2.24). Then the front $\bigcup_{\gamma \in \mathbf{Root}(T)} \mathrm{Succ}(\gamma)$ witnesses $T \leq S$.

Lemma 2.31. *Let S and T be large concrete creatures, $T \leq S$.*

(1) $\mathbf{ext}(T) \subseteq \mathbf{ext}(S)$.
(2) *If $F \subseteq S$ is a front of S, then $F \cap T$ is a front of T.*

Proof. (1) is clear.

For (2), note that nodes in $F \cap T$ are linearly ordered by $<^T$, because they were linearly ordered by $<^S$, and S and T are concrete, so $<^S$ and $<^T$ agree. Every external node of T is also an external node of S, so every branch of T contains a branch of S. Hence every branch of T meets F. □

3. Creatures and functions

3.1. **Weak and strong nodes.** In this section we will consider functions $f : \lambda^k \to \lambda$. We will write tuples $(x_1, \ldots, x_k) \in \lambda^k$ as $\vec{x}$. For $\alpha \in \lambda$ we write $\vec{x} < \alpha$ iff we have $\max(x_1, \ldots, x_k) < \alpha$, similarly for $\vec{x} \leq \alpha$.

However, the use of k-ary functions is only a technicality; the reader may want to consider only the case $k = 1$ and then conclude the general results either by analogy or by assuming that all clones under consideration are determined by their unary fragments (this is true if all clones contain a certain fixed 1-1 function $p : \lambda \times \lambda \to \lambda$). Also, to more easily visualize the results below it may be helpful to assume all functions under consideration are strictly increasing.

Definition 3.1 (Weak and strong nodes, strong creatures)**.** Let $f : \lambda^k \to \lambda$ be a k-ary function. Let $(S, \trianglelefteq, <, D)$ be a creature, $\eta \in S$.

(1) If $\eta \in \mathbf{ext}(S)$, then we say that η is f-weak.
(2) $\eta \in \mathbf{int}(S)$ is f-weak (in S) iff there is a $\vec{y} \in \lambda^k$, $\vec{y} \le \min[\eta]$, $f(\vec{y}) \ge \sup[\eta]$. (Alternatively, we may say that η is weaker than f or that f is stronger than η.)
(3) $\eta \in \mathbf{int}(S)$ is f-strong (in S) iff for all $\vec{y} \in \lambda^k$ with $\vec{y} < \sup[\eta]$ we have $f(\vec{y}) < \sup[\eta]$. (Alternatively, we may say that f is weaker than η or that η is stronger than f.)
(4) We say that T is f-strong iff each $\gamma \in \mathbf{Root}(T)$ is f-strong.

See Figure 7.

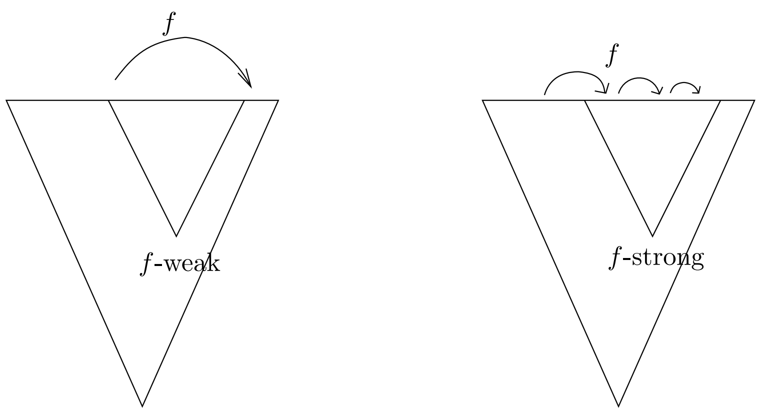


FIGURE 7. f-weak and f-strong nodes

Remark 3.2. If $\eta \trianglelefteq \nu$ and η is f-weak, then also ν is weak. So weakness is inherited "upwards". Strength is in general not inherited downwards, but the following holds:

> If F is a front above η and all $\nu \in F$ are f-strong, then also η is F-strong.

Fact 3.3. Let S and T be concrete creatures. Assume that $\eta \in S \cap T$ is f-strong (or f-weak) in S and $T \le S$. Then η is again f-strong (or f-weak, respectively) in T. Similarly, if S is f-strong, then (by Remark 3.2) so is T.

Proof. Assume η is f-strong in S, so $f(\vec{y}) < \sup_S[\eta]$ for all $\vec{y} < \sup_S[\eta]$. Since $\sup_T[\eta] = \sup_S[\eta]$, η will also be f-strong in T.

Assume η is f-weak in S, so $f(\vec{y}) \ge \sup_S[\eta] = \sup_T[\eta]$ for some $\vec{y} \le \min_S[\eta]$. Clearly $\min_S[\eta] \le \min_T[\eta]$, so also η will also be f-weak in T. $\square$

Fact 3.4. Let S be a (large or small) creature, $f \in \mathscr{O}$.

(1) There is $T \le_{\text{thin}} S$ such that each $\eta \in T$ is either f-strong or f-weak.
(2) Moreover, there is T as above such that also for each internal $\eta \in \mathbf{int}(T)$ either all $\nu \in \mathrm{Succ}_T(\eta)$ are f-strong or all $\nu \in \mathrm{Succ}_T(\eta)$ are f-weak.

Proof. We define $T \leq_{\text{thin}} S$ by upward induction, starting with $\mathbf{Root}(T) := \mathbf{Root}(S)$. Now for each $\eta \in T$ we consider two cases:

(1) η is f-strong (in S). In this case we define $\mathrm{Succ}_T(\eta) := \mathrm{Succ}_S(\eta)$. By Fact 3.3, η will also be f-strong in T.

(2) For some $\vec{y} < \sup_S[\eta]$ we have $f(\vec{y}) \geq \sup_S[\eta]$. Recall that (by Fact 2.16), $\sup_S[\eta] = \sup\{\min_S[\nu] : \nu \in \mathrm{Succ}_S(\eta)\}$. So we can find $\nu_0 \in \mathrm{Succ}_S(\eta)$ with $\vec{y} \leq \min_S[\nu_0]$.
Define $\mathrm{Succ}_T(\eta) := \{\nu \in \mathrm{Succ}_S(\eta) : \nu_0 < \nu\}$. (Note that this set is in D^T_η.) This ensures that η will be f-weak in T.

This completes the definition of T, proving (1). (2) now follows from (1) together with Lemma 2.25(1). $\square$

Fact 3.5. Let S be a large creature, $f : \lambda^k \to \lambda$. Then there is $T \leq S$ which is f-strong.

Proof. Using the regularity of λ, we can find a continuous increasing sequence of ordinals $(\xi_i : i < \lambda)$ with the following properties:

– For all $i < \lambda$, all $\vec{x} < \xi_i$: $f(\vec{x}) < \xi_i$.
– For all $i < \lambda$, all $\gamma \in \mathbf{Root}(S)$: If $\min_S[\gamma] < \xi_i$, then $\sup_S[\gamma] < \xi_i$, and moreover there is $\gamma' > \gamma$ in $\mathbf{Root}(S)$ with $\sup_S[\gamma'] < \xi_i$.
– For all $i < \lambda$, the set $[\xi_i, \xi_{i+1}) \cap \mathbf{ext}(S)$ is nonempty.

These conditions will ensure that for all $i < \lambda$ the set

$$\Gamma_i := \{\gamma \in \mathbf{Root}(S) : \xi_i \leq \min \mathbf{ext}(S^{[\gamma]}) < \sup \mathbf{ext}(S^{[\gamma]}) \leq \xi_{i+1}\}$$

is infinite with no last element.

Now obtain T from S by gluing together each set $\{S^{[\gamma]} : \gamma \in \Gamma_i\}$ (see Definition 2.24) for each $i < \lambda$. $\square$

3.2. **Gauging functions with creatures.** This section contains the crucial point of our construction: the close correspondence between the relation $f \in \langle g \rangle_{\max}$ and the relation $f <_S g$.

Definition 3.6. Let S be a large creature, $f : \lambda^k \to \lambda$, $F \subseteq S$ a front. We say that F gauges f (in S) if

- For all $\eta \in F$, η is f-strong.
- Whenever $\eta \triangleleft \nu$, $\eta \in F$, then ν is f-weak.

We say that S gauges f if there is a front $F \subseteq S$ gauging f.

Fact 3.7. Let $T \leq S$ be large concrete creatures. If S gauges f, then also T gauges f.

Proof. By Lemma 2.31, $F \cap T$ is a front in T. Let $F \subseteq S$ gauge f (in S). Then $F \cap T$ still gauges f (in T), witnessing that T gauges f. (Use Fact 3.3.) $\square$

Fact 3.8. For every function $f \in \mathscr{O}$ and every large creature S which is f-strong there is a large creature $T \leq_{\text{thin}} S$ which gauges f.

Proof. By Fact 3.4, we can first find $T \leq_{\text{thin}} S$ such that all nodes in T are f-strong or f-weak and all internal nodes have either only f-weak successors or only f-strong successors.

Now let F be the set of all $\eta \in \mathbf{int}(T)$ with the property

$$\eta \text{ is } f\text{-strong, but all } \nu \in \mathrm{Succ}(\eta) \text{ are } f\text{-weak.}$$

Every branch b of T contains an f-strong node (in $\mathbf{Root}(T)$) and an f-weak node (in $\mathbf{ext}(T)$) so b contains a highest strong node η_b. Since η_b has weak successors, all successors of η_b are weak; hence $\{\eta_b\} = b \cap F$. Hence F is a front, and clearly F gauges f. $\square$

Definition 3.9. Let S be a creature, $F \subseteq S$ a front. We let

$$\mathbf{lim}_F := \{\sup_S[\eta] : \eta \in F\}$$

and we write h_F for the function $h_{\mathbf{lim}_F}$.

Remark 3.10. In the special case that $F = \mathbf{ext}(S)$, we have $\mathbf{lim}_F = F$, so our (new) definition of h_F agrees with our (old) definition in Definition 1.4 of $h_{\mathbf{ext}(S)}$. However, we will usually only consider fronts $F \subseteq \mathbf{int}(S)$.

Remark 3.11. If F contains only internal nodes, then each point of $\mathbf{lim}_F$ is a limit point of $\mathbf{ext}(S)$. We will see below that h_F grows much faster than $h_{\mathbf{ext}(S)}$. In an informal sense, h_F is the smallest function that is still stronger than each $\eta \in F$. Lemmas 3.12 and 3.13 below capture a part of that intuition.

Lemma 3.12. *Let S be a large creature, $F \subseteq S$ a front. Let g be a function which is stronger than each $\eta \in F$. Then $h_F \leq_{\mathbf{ext}(S)} g$. (See Fact 1.6 for the definition of $\leq_A$.)*

Proof. Let $A := \mathbf{ext}(S)$. For each $\eta \in F$ fix $\vec{x}_\eta$ such that $\max(\vec{x}_\eta) \leq \min[\eta]$ and $g(\vec{x}_\eta) \geq \sup[\eta]$. (The existence of $\vec{x}_\eta$ follows from our assumption that g is η-strong.)

We will define a function $\vec{y} : \lambda \to \lambda^k$:

For each $\alpha \in A$ we can find $\eta = \eta_\alpha \in F$ with $\eta_\alpha \trianglelefteq \alpha$. Let $\vec{y}(\alpha) = \vec{x}_{\eta_\alpha}$.

For $\alpha \in \lambda \setminus A$ let $\vec{y}(\alpha) = \vec{0}$.

Clearly $\vec{y}(\alpha) \leq \alpha$, so the function $\vec{y}$ (i.e., each of its components) is in $\mathscr{C}_{\max}$.

For $\alpha \in \mathbf{ext}(S)$ we have

$$h_F(\alpha) = \sup[\eta_\alpha] \leq g(\vec{y}(\alpha)),$$

and for $\alpha \notin \mathbf{ext}(S)$ we have $h_F(\alpha) = h_F(h_A(\alpha))$. In any case we have

$$h_F(\alpha) \leq g(\vec{y}(h_A(\alpha)));$$

therefore $h_F \in \langle h_A, g\rangle_{\max}$. $\square$

Lemma 3.13. *Let S be a large creature, $F \subseteq S$ a front. Let f be a function weaker than all $\eta \in F$. Then $f \leq_{\mathbf{ext}(S)} h_F$.*

Proof. For any $\vec{x}$, let $\eta \in F$ be minimal such that $\sup[\eta] > \vec{x}$. Then $h_F(\max \vec{x}) = \sup[\eta]$, but (as η is f-strong), $f(\vec{x}) < \sup[\eta]$. Hence $f(\vec{x}) < h_F(\max \vec{x})$ for all $\vec{x}$, so $f \in \langle h_F\rangle_{\max}$. $\square$

Lemma 3.14. *Let S be a large creature, $F \subseteq \mathbf{int}(S)$ a front. Let f be a function which is weaker than each $\eta \in F$. Then $h_F \not\leq_{\mathbf{ext}(S)} f$.*

Proof. Pick any $\eta \in F$, and let $\xi := \sup[\eta]$. Let

$$\mathscr{D} := \{c \in \mathscr{O} : \forall \vec{x} < \xi\, (c(\vec{x}) < \xi)\}.$$

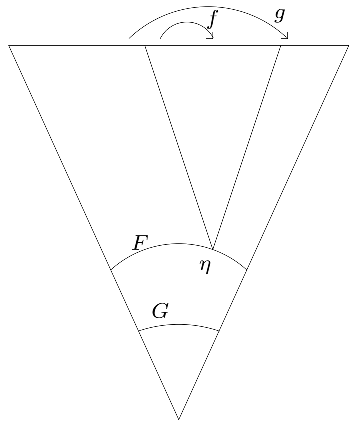


FIGURE 8. g is η-strong, f is η-weak

Then $\mathscr{D}$ is a clone containing f (as η is f-strong). As ξ is a limit point of $\mathbf{ext}(S)$, we also have $h_{\mathbf{ext}(S)}(\vec{x}) < \xi$ for all $\vec{x} < \xi$, so $h_{\mathbf{ext}(S)} \in \mathscr{D}$. Hence $\langle h_{\mathbf{ext}(S)}, f\rangle_{\max} \subseteq \mathscr{D}$, but $h_F \notin \mathscr{D}$, so $h_F \notin \langle h_{\mathbf{ext}(S)}, f\rangle_{\max}$. $\square$

Notation 3.15. If F is a front in S, $\nu \in \mathbf{ext}(S)$, then we write $F^{\cdot}(\nu)$ for the unique $\eta \in F$ with $\eta \trianglelefteq \nu$.

Recall from Remark 3.2 that "higher" nodes (in the sense of $\trianglelefteq$) are usually weaker (in the sense of f-weakness) than lower nodes. This apparent reversal of inequalities lies at the heart of the next definition.

Definition 3.16. Assume that S is a large creature gauging f and g, witnessed by fronts F and G. We write

$f <_S g$ iff: For all $\nu \in \mathbf{ext}(S)$, $F^{\cdot}(\nu)$ lies strictly *above* $G^{\cdot}(\nu)$: $G^{\cdot}(\nu) \triangleleft F^{\cdot}(\nu)$. (See Notation 3.15.)

$f \sim_S g$ iff: For all $\nu \in \mathbf{ext}(S)$, $F^{\cdot}(\nu) = G^{\cdot}(\nu)$.

We say that "S compares f and g" iff S gauges f and g and one of

$$f <_S g, \qquad f \sim_S g, \qquad g <_S f$$

holds.

Fact 3.17. If $f <_S g$ and $T \leq S$, then $f <_T g$. Similarly, if $f \sim_S g$ and $T \leq S$, then $f \sim_T g$.

The following lemma is the core of the whole proof.

Lemma 3.18. *Let S be a large creature gauging f and g. If $f <_S g$, then $f \in \langle h_{\mathbf{ext}(S)}, g\rangle_{\max}$, but $g \notin \langle h_{\mathbf{ext}(S)}, f\}\rangle_{\max}$. In other words:*

If $f <_S g$, then $f \leq_{\mathbf{ext}(S)} g$, but $g \not\leq_{\mathbf{ext}(S)} f$.

Proof. Let F gauge f. So every $\eta \in F$ is f-strong but g-weak. By Lemma 3.13, we have $f \leq_{\mathbf{ext}(S)} h_F$ and by Lemma 3.12 $h_F \leq_{\mathbf{ext}(S)} g$. So $f \leq_{\mathbf{ext}(S)} g$, as $\leq_{\mathbf{ext}(S)}$ is transitive (Fact 1.6).

If we had $g \leq_{\mathbf{ext}(S)} f$, then (as $h_F \leq_{\mathbf{ext}(S)} g$, by Lemma 3.12) we would get $h_F \leq_{\mathbf{ext}(S)} f$, contradicting Lemma 3.14. $\square$

Lemma 3.18 shows that if S can "see" that g grows faster than f, then together with $h_{\mathbf{ext}(S)}$, g dominates f, but not conversely. We can also read this as:

If $f <_S g$, then "on the set $\mathbf{ext}(S)$" g dominates f quite strongly.

But can we always find a creature S that can compare the different behaviors of f and g? This is answered in the next lemma.

Lemma 3.19. *Let $f, g \in \mathscr{O}$, and let S be a large creature. Then there is a large creature $T \leq S$ which compares f and g, i.e., $f <_T g$ or $f \sim_T g$ or $g <_T f$.*

Proof. By Fact 3.5 we can find $S_1 \leq S$ which is f-strong, and by Fact 3.8 we can find $S_2 \leq S_1$ gauging f, witnessed by a front F. Similarly we can find $S_3 \leq S_2$ gauging g, witnessed by G. $F \cap S_3$ still witnesses that S_3 also gauges f.

To each external node ν of S_3 we assign one of three colors, depending on whether

(1) $F^{\cdot}(\nu) = G^{\cdot}(\nu)$, or
(2) $F^{\cdot}(\nu) \lhd G^{\cdot}(\nu)$, or
(3) $F^{\cdot}(\nu) \rhd G^{\cdot}(\nu)$.

Using Lemma 2.25 we can find $T \leq S_3$ such that all branches of T get the same color. Now $T \leq S$, and one of $f \sim_T g$, $f <_T g$, or $g <_T f$ holds. $\square$

Fact 3.20. Assume $f \sim_S g$ or $f <_S g$. Let F and G be the fronts gauging f and g, respectively. Then:

(1) Every $\eta \in S$ which is g-strong is also f-strong.
(2) For all $\eta \in S$: η is g-strong iff η is $\max(f, g)$-strong.
(3) G gauges $\max(f, g)$.
(4) $\max(f, g) \sim_S g$.

Proof. (1) On every branch in S the g-strong nodes are exactly the nodes which are $\trianglelefteq G$; these nodes are $\trianglelefteq F$, hence f-strong.

(2) Let η be g-strong, so for $\vec{x} < \sup[\eta]$ we have $g(\vec{x}) < \sup[\eta]$. As η is also f-strong, we also have

$$\forall \vec{x} < \sup[\eta] : \max(f, g)(\vec{x}) < \sup[\eta].$$

(3) By (2).
(4) By (3). $\square$

4. Fuzzy creatures

Ideally, we would like to construct a decreasing sequence $(S_i : i < \lambda^+)$ of creatures such that the relations $\bigcup_i <_{S_i}$ and $\bigcup_i \sim_{S_i}$ can be used for the construction described in Section 1.1. However, the partial order $\leq$ on creatures is not even σ-closed; i.e., we can find a countable decreasing sequence with no lower bound.

We will now slightly modify the relation $\leq$ between large creatures to a relation $\leq^*$ which has better closure properties but still keeps the important properties described in Lemma 3.18.

4.1. By any other name: $\leq_{\text{thin}}$, $\leq_{\text{thin/short}}$, $\approx$.

Fact and Definition 4.1. Assume that S, S_1, S_2 are concrete creatures and:

either: S is small, and both S_1 and S_2 are $\leq_{\text{thin}} S$,
or: S is large, and both S_1 and S_2 are $\leq_{\text{thin/short}} S$.

We define a structure $T = (T, \trianglelefteq^T, <^T, D^T)$ (which we also call $S_1 \cap S_2$) as follows:

(1) $\mathbf{Root}(T) = \mathbf{Root}(S_1) \cap \mathbf{Root}(S_2)$, $T = S_1 \cap S_2$,
(2) $<^T = <^{S_1} \cap <^{S_2}$,
(3) $\triangleleft^T = \triangleleft^{S_1} \cap \triangleleft^{S_2}$,
(4) $D_\eta = D_\eta^{S_1} \cap D_\eta^{S_2}$ for all $\eta \in T$.

Then T is a creature, and $T \leq_{\text{thin}} S_1$, $T \leq_{\text{thin}} S_2$ (or $T \leq_{\text{thin/short}} S_1, S_2$, respectively).

Proof. We first check that T is a planar tree. Clearly T is nonempty: If S is small, then T contains $\mathbf{Root}(S) = \mathbf{Root}(S_1) = \mathbf{Root}(S_2)$, and if S is large, then these equalities $\mathbf{Root}(S) = \mathbf{Root}(S_1) = \mathbf{Root}(S_2)$ hold modulo a set of size $< \lambda$. Hence we have Definition 2.1(A).

The orders $\trianglelefteq^{S_1}$ and $\trianglelefteq^{S_2}$ agree on T, as they both are restrictions of $\trianglelefteq^S$, and the same is true for $<^{S_1}$ and $<^{S_2}$. This implies Definition 2.1(B),(C),(D).

We now check that T is a creature. For any $\eta \in T$ and any $A \subseteq \mathrm{Succ}(\eta)$ we have

$$A \in D_\eta^T \quad \Leftrightarrow \quad A \in D_\eta^{S_1} \wedge A \in D_\eta^{S_2} \quad \Leftrightarrow \quad A \in D_\eta^S,$$

so D_η^T is indeed an ultrafilter, i.e., Definition 2.6(E). Using Fact 2.5 we see Definition 2.6(F),(G). $T \leq S_1, S_2$ is clear. □

Definition 4.2. Let S, S' be small or large creatures. We write $S \approx_{\text{thin}} S'$ for

$$\exists T: \ T \leq_{\text{thin}} S \text{ and } T \leq_{\text{thin}} S'.$$

Let S, S' be large creatures. We write $S \approx S'$ for

$$\exists T: \ T \leq_{\text{thin/short}} S \text{ and } T \leq_{\text{thin/short}} S'.$$

Note that $S \approx_{\text{thin}} S'$ implies that there is "union" creature $S^* \geq_{\text{thin}} S, S'$.

Fact 4.3. $\approx_{\text{thin}}$ and $\approx$ are equivalence relations.

Proof. If S, S', S'', T, T' are small (or large) creatures such that T witnesses $S \approx_{\text{thin}} S'$ and T' witnesses $S' \approx_{\text{thin}} S''$, then by Fact and Definition 4.1 we see that $T'' := T \cap T'$ is again a small (or large) creature, and T'' witnesses $S \approx_{\text{thin}} S''$.

The proof for $\approx$ is similar. □

Definition 4.4 (The relation $\leq^*$)**.** Let T and S be large concrete creatures. We say that $T \leq^* S$ if there is T' with $T \approx T' \leq S$.

Lemma 4.5 (Pullback lemma)**.** *If $T_1 \leq S_1 \approx S_0$ are large creatures, then there is a large creature T_0 such that $T_1 \approx T_0 \leq S_0$:*

$$\begin{array}{ccc} & & S_0 \\ & & \approx \\ T_1 & \leq & S_1 \end{array} \qquad \Longrightarrow \qquad \begin{array}{ccc} T_0 & \leq & S_0 \\ \approx & & \approx \\ T_1 & \leq & S_1 \end{array}$$

Proof. Let F witness $T_1 \leq S_1$, and let $\gamma_0 \in F$ be so large that for all $\gamma \in F$ with $\gamma > \gamma_0$ we have $S_1^{[\gamma]} \approx_{\text{thin}} S_0^{[\gamma]}$.

Let $F_0 := \{\gamma \in F : \gamma > \gamma_0\}$, and define

$$T_0 = \bigcup_{\gamma \in F_0} \{\eta \in T_1 : \eta \trianglelefteq \gamma\} \cup S_0^{[\gamma]}.$$

T_0 can be naturally equipped with a creature structure $(\trianglelefteq^{T_0}, <^{T_0}, D^{T_0})$ such that $T_0 \approx T_1$. For defining D^{T_0} we use the fact that for all $\eta \in T_0$ with $\eta \lhd \gamma \in F_0$ the set $\mathrm{Succ}_{T_0}(\eta)$ is either equal to $\mathrm{Succ}_{T_1}(\eta)$ or an end segment of this set, so in any case is in $D^{T_0}_\eta$.

Now clearly $T_0 \le S_0$ is witnessed by F_0. ☐

Corollary 4.6. *The relation $\le^*$ (between large creatures) is transitive.*

Proof. Let $T \le^* S \le^* R$. We use our "pullback lemma", Lemma 4.5:

$$\begin{array}{ccccc} & & S' & \le & R \\ & & \approx & & \\ T' & \le & S & & \\ \approx & & & & \\ T & & & & \end{array} \quad\Longrightarrow\quad \begin{array}{ccccc} T'' & \le & S' & \le & R \\ \approx & & \approx & & \\ T' & \le & S & & \\ \approx & & & & \\ T & & & & \end{array}$$

and then appeal to the transitivity of $\le$ and $\approx$. ☐

4.2. **Fusion.**

Lemma 4.7. *Let $\delta < \lambda$ be a limit ordinal. Assume that $(S_i : i < \delta)$ is a sequence of large concrete creatures satisfying $i < j \Rightarrow S_j \le^* S_i$. Then there is a large creature S_δ such that for all $i < \delta$, $S_\delta \le^* S_i$.*

A main idea in the proof is to divide λ into λ many pieces, each of length δ: $\lambda = \bigcup_{\xi<\lambda}[\delta\cdot\xi, \delta\cdot\xi+\delta)$.

Proof. By elementary ordinal arithmetic, for each $\zeta < \lambda$ there is a unique pair (ξ, i) with $\xi < \lambda$, $i < \delta$, and $\zeta = \delta\cdot\xi + i$.

Recall the definition of concrete creatures: each internal node η is a pair $(\alpha(\eta), \beta(\eta))$, and $\mathbf{ext}(S^{[\eta]})$ is a subset of the interval $[\alpha(\eta), \beta(\eta))$, with supremum $\beta(\eta)$. We choose (inductively) a sequence $r(\zeta)$ (for $\zeta < \lambda$) of roots such that for all $\xi < \lambda$, all $i < \delta$:

- $r(\delta\cdot\xi + i) \in \mathbf{Root}(S_i)$.
- For all $\zeta' < \zeta$: $r(\zeta') < r(\zeta)$.
  (If $\zeta' = \delta\cdot\xi' + i'$, $\zeta = \delta\cdot\xi + i$ with $i \ne i'$, then $r(\zeta') \in S_{i'}$ and $r(\zeta) \in S_i$ come from different creatures, but they can still be compared:
  $r(\zeta') < r(\zeta)$ means $\sup_{S_{i'}}[r(\zeta')] \le \min_{S_i}[r(\zeta)]$.)

Considering the matrix $(S_i^{[r(\delta\cdot\xi+i)]} : i < \delta, \xi < \lambda)$ of small creatures, we first note that

$$T_{\langle 0\rangle} := \bigcup_{\xi<\lambda}\bigcup_{i<\delta} S_i^{r(\delta\cdot\xi+i)}$$

is a large concrete creature. (Whenever $\delta\cdot\xi' + i' < \delta\cdot\xi + i$, and $\eta' \in S_{i'}^{r(\delta\cdot\xi'+i')}$, $\eta \in S_i^{r(\delta\cdot\xi+i)}$, then $\eta' < \eta$.)

We also see that $T_{\langle 0\rangle} \le^* S_0$, because for each $\xi < \lambda$ and each $i < \delta$ there is a small creature X with

$$T_{\langle 0\rangle}^{[r(\delta\cdot\xi+i)]} = S_i^{[r(\delta\cdot\xi+i)]} \approx_{\text{thin}} X \le S_0.$$

(See Figure 9.)

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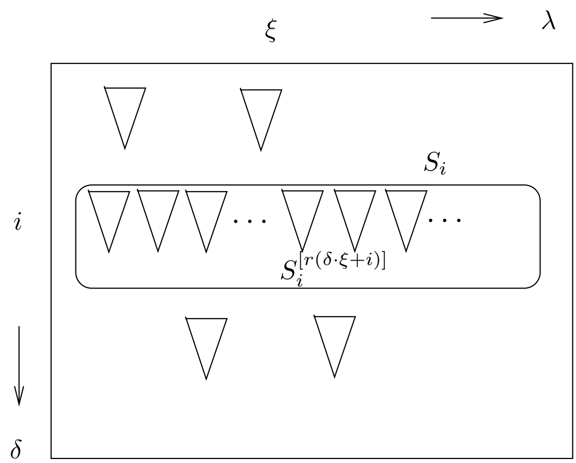

FIGURE 9

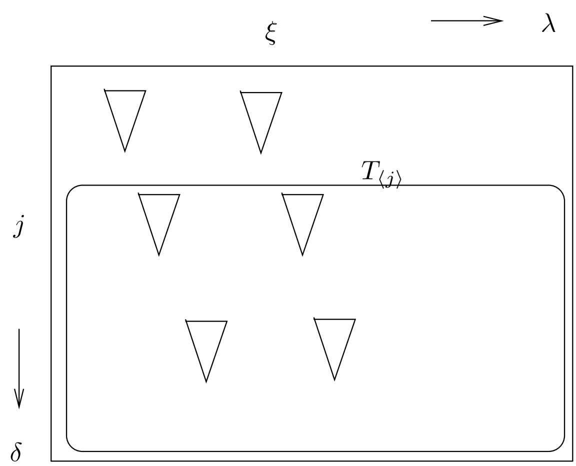

FIGURE 10. The creature $T_{\langle j\rangle}$

Similarly, we see that for every $j < \delta$,

$$T_{\langle j\rangle} := \bigcup_{\xi<\lambda} \bigcup_{j\leq i<\delta} S_i^{r(\delta\cdot\xi+i)}$$

is a large creature and $T_{\langle j\rangle} \leq^* S_j$ (see Figure 10).

It remains to define a large creature $\bar{T}$ such that $\bar{T} \leq^* T_{\langle j\rangle}$ for all $j < \delta$.

For each $\xi < \lambda$ the set

$$T_{\langle 0,\xi\rangle} := \bigcup_{i<\delta} S_i^{r(\delta\cdot\xi+i)}$$

is a medium creature (see Figure 11).

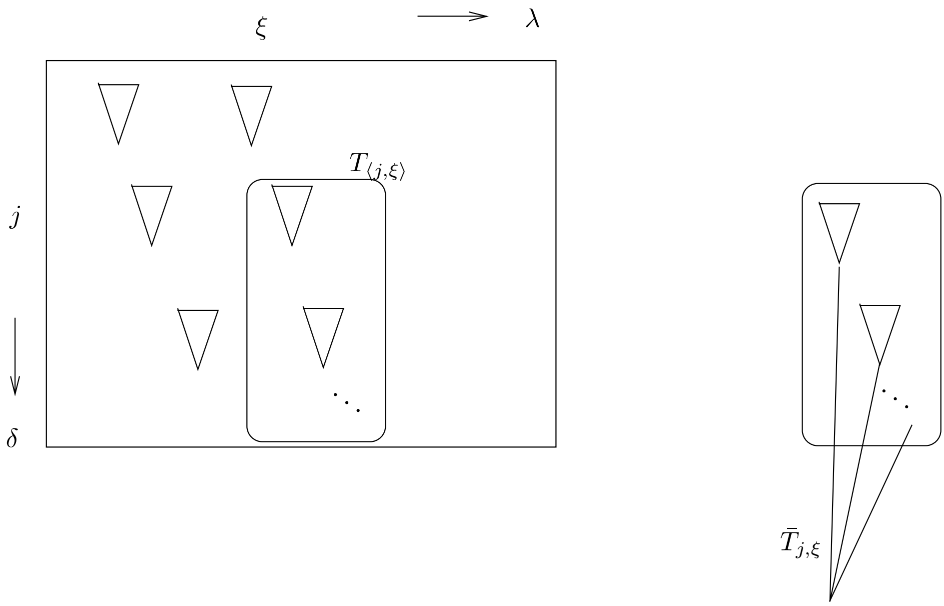

FIGURE 11. Obtaining $\bar{T}_{j,\xi}$ from $T_{\langle j,\xi\rangle}$

Let $U_{0,\xi}$ be an ultrafilter on $\mathbf{Root}(T_{\langle 0,\xi\rangle})$ which converges to $\sup \mathbf{Root}(T_{\langle 0,\xi\rangle})$, and let r_ξ be a new root. Then

$$\bar{T}_{0,\xi} := \sum_{U_{0,\xi}, r_\xi} T_{\langle 0,\xi\rangle}$$

is a small creature, and $\bar{T}_0 := \bigcup_{\xi<\lambda} \bar{T}_{0,\xi}$ is a large creature. By construction, $\bar{T}_0 \leq_{\text{glue}} T_{\langle 0\rangle}$.

We can similarly define

$$T_{\langle j,\xi\rangle} := \bigcup_{j\leq i<\delta} S_i^{r(\delta\cdot\xi+i)} \qquad \bar{T}_{j,\xi} := \sum_{U_{j,\xi}, r_\xi} T_{\langle j,\xi\rangle}$$

(where $U_{j,\xi}$ is the restriction of $U_{0,\xi}$ to $\mathbf{Root}(T_{\langle j,\xi\rangle})$, an end segment of $T_{\langle 0,\xi\rangle}$). Again, $\bar{T}_j := \bigcup_{\xi<\lambda} \bar{T}_{j,\xi}$ is a large creature satisfying $\bar{T}_j \leq T_{\langle j\rangle}$. But by definition we have $\bar{T}_0 \approx_{\text{thin}} \bar{T}_j$, so $\bar{T}_0 \leq^* T_{\langle j\rangle}$ for all $j < \delta$. (See also Figure 12.) $\square$

Lemma 4.8. *Assume that $(S_\xi : \xi < \lambda)$ is a sequence of large concrete creatures satisfying $\xi < \xi' \Rightarrow S_{\xi'} \leq^* S_\xi$. Then there is a large creature S_λ such that for all $\xi < \lambda$: $S_\lambda \leq^* S_\xi$.*

Proof. We choose a fast enough increasing sequence $(r(\xi) : \xi < \lambda)$ with $r(\xi) \in \mathbf{Root}(S_\xi)$ such that

$$\forall \zeta < \xi : r(\zeta) < r(\xi).$$

Now let $T_0 := \bigcup_{\xi<\lambda} S_\xi^{[r(\xi)]}$, and similarly $T_\zeta := \bigcup_{\zeta\leq\xi<\lambda} S_\xi^{[r(\xi)]}$. It is easy to see that $T_0 \approx T_\zeta \leq S_\zeta$ for all ζ. Hence $T_0 \leq^* S_\zeta$ for all ζ. $\square$

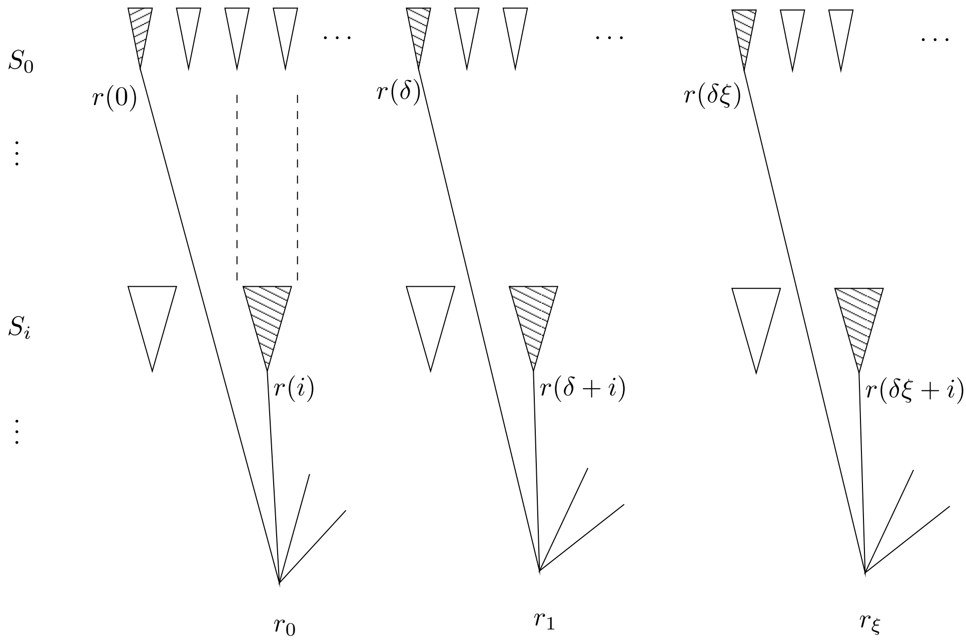

FIGURE 12. A fusion of δ many large creatures

Corollary 4.9. *Assume that* $\mathbf{S}$ *is a set of large concrete creatures which is linearly quasiordered by* $\leq^*$*, and assume that* $|\mathbf{S}| \leq \lambda$*. Then there is a large creature* T *such that* $\forall S \in \mathbf{S} : T \leq^* S$*.*

Proof. By Lemmas 4.7 and 4.8. Use induction on $|S|$. $\square$

5. THE FILTER D_S AND THE CLONE $\mathscr{C}_S$

Let S be a large concrete creature, and let f, g be operations on λ. Recall that $f <_S g$ iff there are fronts $F, G \subseteq S$ gauging f and g, respectively, such that F meets each branch of S above G.

Definition 5.1. We write $f <^*_S g$ if there is $S' \approx S$, $f <_{S'} g$; similarly for $\sim^*$.

Lemma 5.2. *If* $f <^*_S g$ *and* $T \leq^* S$*, then* $f <^*_T g$*.*

Proof. By the definition of $\leq^*$ (see Definition 4.4), there is T_0 such that $T \approx T_0 \leq S$. Let $S' \approx S$ be such that S' gauges f. Using the pullback lemma, Lemma 4.5, we find $T' \leq S'$, $T' \approx T_0$. So $T' \approx T$, $f <_{T'} g$ (by Fact 3.17), which implies $f <^*_T g$.

$$\begin{array}{ccc}
 & & S' \\
 & & \approx \\
T_0 & \leq & S \\
\approx & & \\
T & &
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{ccc}
T' & \leq & S' \\
\approx & & \approx \\
T_0 & \leq & S \\
\approx & & \\
T & &
\end{array}$$

$\square$

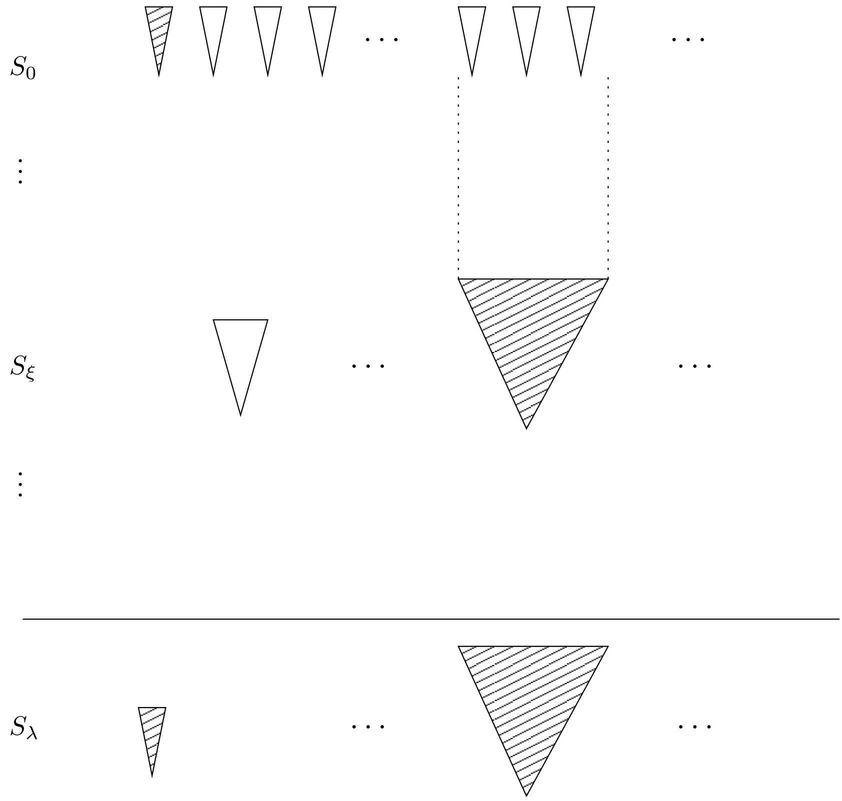

FIGURE 13. A fusion of λ many large creatures

Definition 5.3. Let S be a large creature. We define

$$D_S := \{A \subseteq \lambda : \exists S' \approx S, \mathbf{ext}(S') \subseteq A\}.$$

Fact 5.4. Let S be a large creature. Then $A \in D_S$ iff there is $T \leq_{\text{thin/short}} S$ with $\mathbf{ext}(T) \subseteq A$.

Proof. If $A \in D_S$, then there are S' and T such that $T \leq_{\text{thin/short}} S$, $T \leq_{\text{thin/short}} S'$, and $\mathbf{ext}(S') \subseteq A$. But then also $\mathbf{ext}(T) \subseteq \mathbf{ext}(S') \subseteq A$. □

Fact 5.5. Let S be a large creature. Then:

(1) D_S is a filter on λ, and all $A \in D_S$ are unbounded.
(2) If $S' \approx S$, then $D_S = D_{S'}$.
(3) If $T \leq S$, then $D_T \supseteq D_S$.
(4) If $T \leq^* S$, then $D_T \supseteq D_S$.

Proof. (1) D_S is clearly upward closed. Let $A_1, A_2 \in D_S$, witnessed by $S_1, S_2 \leq_{\text{thin/short}} S$; then $S_1 \cap S_2$ witnesses $A_1 \cap A_2 \in D_S$.

(2) Immediate from the definition.

(3) Follows from $\mathbf{ext}(T) \subseteq \mathbf{ext}(S)$ and the pullback lemma.

(4) By (2) and (3). □

Definition 5.6. For any large creature S we let

$$\mathscr{C}_S := \langle h_A : A \in D_S \rangle_{\max} = \bigcup_{A \in D_S} \langle h_A \rangle_{\max}.$$

As a corollary to Fact 5.5 and Lemma 1.7 we get:

Fact 5.7. Let S be a large creature. Then:

(1) $\mathscr{C}_S = \{f : \exists S' \approx S \, \exists k \, \forall \vec{x} \, (f(\vec{x}) \leq h^{(k)}_{\mathbf{ext}(S')}(\max(\vec{x})))\}$
(2) If $S' \approx S$, then $\mathscr{C}_S = \mathscr{C}_{S'}$.
(3) If $T \leq S$, then $\mathscr{C}_T \supseteq \mathscr{C}_S$.
(4) If $T \leq^* S$, then $\mathscr{C}_T \supseteq \mathscr{C}_S$.

Lemma 5.8. *Let S be a large creature, $f, g \in \mathscr{O}$, and assume $f <^*_S g$. Then $f \in \langle \mathscr{C}_S \cup \{g\} \rangle$, but $g \notin \langle \mathscr{C}_S \cup \{f\} \rangle$.*

Proof. There is $S' \approx S$ with $f <_{S'} g$. But $D_S = D_{S'}$ and $\mathscr{C}_S = \mathscr{C}_{S'}$, so we may as well assume $f <_S g$.

By Lemma 3.18, $f \in \langle h_{\mathbf{ext}(S)}, g \rangle_{\max} \subseteq \langle \{h_A : A \in D_S\} \cup \{g\} \rangle_{\max} = \langle \mathscr{D}_S \cup \{g\} \rangle$.

Assume that $g \in \langle \mathscr{C}_S \cup \{f\} \rangle$. Then there is $A \in D_S$ such that $g \in \langle h_A, f \rangle_{\max}$.

Let $S' \leq_{\text{thin/short}} S$ with $\mathbf{ext}(S') \subseteq A$. Then

$$g \in \langle h_A, f \rangle_{\max} \subseteq \langle h_{\mathbf{ext}(S')}, f \rangle_{\max}.$$

But $S' \leq S$, and $f <_S g$ implies $f <_{S'} g$. Hence (again by Lemma 3.18) we get $g \notin \langle h_{\mathbf{ext}(S')}, f \rangle_{\max}$, a contradiction. $\square$

6. Transfinite induction

Definition 6.1. We say that a sequence $(S_i : i < \lambda^+)$ of large creatures is "sufficiently generic" iff the sequence decreases with respect to $\leq^*$:

$$\forall i < j : S_j \leq^* S_i$$

and

$$\forall f, g \in \mathscr{O} \, \exists i < \lambda^+ : \; f <_{S_i} g \; \vee f \sim_{S_i} g \; \vee g <_{S_i} f.$$

Lemma 6.2. *Assume $2^\lambda = \lambda^+$. Then there is a sufficiently generic sequence.*

Proof. This is a straightforward transfinite induction: There are 2^λ many pairs $(f, g) \in \mathscr{O} \times \mathscr{O}$. By our assumption $2^\lambda = \lambda^+$ we can enumerate all these pairs as

$$\mathscr{O} \times \mathscr{O} = \{(f_i, g_i) : i < \lambda^+\}.$$

Using Corollary 4.9, we can now find a sequence $(S_i : i < \lambda^+)$ of large concrete creatures such that the following hold for all i:

- If i is a limit ordinal, then $S_i \leq^* S_j$ for all $j < i$.
- $S_{i+1} \leq S_i$.
- S_{i+1} gauges f_i and g_i.
- S_{i+1} compares f_i and g_i: $g_i <_{S_{i+1}} f_i$ or $g_i \sim_{S_{i+1}} f_i$ or $f_i <_{S_{i+1}} g_i$. $\square$

Conclusion 6.3. Let $(S_i : i < \lambda^+)$ be a sufficiently generic sequence. Define $\mathscr{C}_\infty := \bigcup_i \mathscr{C}_{S_i}$. This is an increasing union of clones, so $\mathscr{C}_\infty$ is also a clone.

Let $f <_\infty g$ iff there is i such that $f <_{S_i} g$, or equivalently, iff there is $i < \lambda^+$ such that $f <^*_{S_i} g$. Define $f \sim_\infty g$ analogously.

Then the properties (a)(b')(c') in section 1.1 are satisfied, so Section 1.1(1)(2)(3) hold. Moreover, for all $f \in \mathscr{O}$ there is g with $f <_\infty g$, so $[\mathscr{C}_\infty, \mathscr{O}]$ has no coatom.

Proof. (a) If $f <_\infty g$, then $f <^*_{S_i} g$ for some i. By Lemma 5.8, $f \in \langle \mathscr{C}_{S_i} \cup \{g\}\rangle$, so $f \in \langle \mathscr{C}_\infty \cup \{g\}\rangle$.

(b') If $g \in \langle \mathscr{C}_\infty \cup \{f\}\rangle$, then there is $i < \lambda^+$ such that $g \in \langle \mathscr{C}_{S_i} \cup \{f\}\rangle$, as the sequence $(\mathscr{C}_{S_i})$ is increasing, by Fact 5.7. Choose $j > i$ so large that S_j compares f and g, so one of $f <_{S_j} g$, $f \sim_{S_j} g$, $g <_{S_j} f$ holds. The first alternative is excluded by Lemma 5.8.

(c') follows from Fact 3.20.

Finally, let $f \in \mathscr{O}$. Find $i < \lambda^+$ such that S_i gauges f. Let $A := \{\sup_{S_i}[\gamma] : \gamma \in \mathbf{Root}(S_i)\}$, and let $g := h_A$. Then:

($*$) Each $\gamma \in \mathbf{Root}(S_i)$ is f-strong but g-weak.

Now find $j > i$ such that S_j compares f and g. The possibilities $g <_{S_j} f$ and $f \sim_{S_j} g$ are excluded by ($*$), so $f <_{S_j} g$; hence also $f <_\infty g$. $\square$

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Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8–10/104, 1040 Wien, Austria

E-mail address: martin.goldstern@tuwien.ac.at

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel — and — Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08854

E-mail address: shelah@math.huji.ac.il