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CLASSIFICATION THEORY FOR NON-ELEMENTARY CLASSES I: THE NUMBER OF UNCOUNTABLE MODELS OF $\psi \in L_{\omega_1,\omega}$. PART A

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ABSTRACT

Assuming that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$, we prove that every $\psi \in L_{\omega_1,\omega}$ has many non-isomorphic models of power $\aleph_n$ for some $n > 0$ *or* has models in all cardinalities. We can conclude that every such ψ has at least $2^{\aleph_1}$ non-isomorphic uncountable models. As for the more vague problem of classification, restricting ourselves to the atomic models of some countable T (we can reduce general cases to this) we find a cutting line named "excellent." Excellent classes are well understood and are parallel to totally transcendental theories, have models in all cardinals, have the amalgamation property, and satisfy the Los conjecture. For non-excellent classes we have a non-structure theorem, e.g., if they have an uncountable model then they have many non-isomorphic ones in some $\aleph_n$ (provided $2^{\aleph_m} < 2^{\aleph_{m+1}}$).

§0. Introduction

In his list of questions H. Friedman [4] quoted the following question (due to Baldwin): Can a sentence $\psi \in L(Q)$ have exactly one uncountable model?

In [11] we answered this question for $\psi \in L_{\omega_1,\omega}(Q)$. We proved there that if $I(\aleph_1, \psi) < 2^{\aleph_1}$ then there exists a model for the sentence ψ of cardinality $\aleph_2$ ($I(\lambda, \psi)$ is the number of non-isomorphic models of ψ of cardinality λ). We proved this assuming $V = L$, or more exactly $\diamondsuit_{\aleph_1}$.

For $\psi \in L_{\omega_1,\omega}(Q)$ which satisfies $I(\aleph_1, \psi) < 2^{\aleph_1}$, we obtained a clear picture of the structure of the models in $\aleph_1$ assuming $\diamondsuit_{\aleph_1}$.

Our aim here is to continue the work we began in [11] in two directions. The

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first, and more important, is to check what happens in higher cardinalities; the second direction is to minimalize the set theoretical assumptions. We shall prove a classification theorem for $\psi \in L_{\omega_1,\omega}$ (the main result was announced in the notices and in [11], p. 147 and lectured on at the ASL meeting in Jerusalem, Fall 75). A preprint of this paper was circulated in 1977.

Everywhere in this paper we assume that the language L is countable.

We think that the non-structure proofs can be used separately in other contexts. Weak diamonds (introduced in Devlin and Shelah [3]) are used throughout. This includes Theorem 1.3 (where, assuming $2^{\aleph_0} < 2^{\aleph_1}$ and failure of $\aleph_0$-amalgamation of our class K, we show K has no universal model in $\aleph_1$ and has $2^{\aleph_1}$ non-isomorphic models in $\aleph_1$). This was continued in Avraham and Shelah [1] for Aronszajn trees D and in Grossberg and Shelah [6] for locally finite groups. This applies as well to Lemma 6.2 (generalized in [17] and then in [6] which deals with a similar, somewhat harder situation) and Theorems 6.4 and 6.14, where we build many models in λ^{++} using failure of amalgamation in λ. In all cases, if we weaken our interpretation of "many" the proofs would become easier.

Furthermore, we think that some other methods of this paper will be useful elsewhere. In particular, the way we analyze a model of power λ (an existing one, or one we want to construct) using n-dimensional diagrams seems to have a general flavour and really it was used in the classification of countable first order theory, together with a variant of the generalized symmetry lemma.

Our proofs can be modified to $\psi \in L_{\omega_1,\omega}(Q)$ as in [11]. In this paper we prove the theorem for $L_{\omega_1,\omega}$ only; for $L_{\omega_1,\omega}(Q)$ everything should be reproved. This is possible with a minor modification: in Theorem 1.1 required from K the additional requirement that small sets (in the sense of Q) will be countable, and instead of the usual elementary extension to work with elementary extensions which do not increase small sets, i.e.,

$$\text{if } M < N \quad \text{and} \quad M \models \neg Qx\phi(x,\bar{b}) \quad \text{then } N \models x\phi(a,\bar{b}) \text{ implies } a \in M.$$

Since our proofs here (for $\psi \in L_{\omega_1,\omega}$) are complicated enough, and in [17] we deal with a more general setting than $L_{\omega_1,\omega}(Q)$, we omitted the explicit proof from our present paper. In [17] we proved, for example: If for ψ there exists a unique model of cardinality $\aleph_1$, then there exists a model to ψ of cardinality $\aleph_2$; this theorem is proved in ZFC alone without any set theoretic assumptions. In [17] we also start to prove the main theorem from here for the more general context. We do the first step, i.e., we draw from $I(\aleph_1, K) < 2^{\aleph_1}$ (and $2^{\aleph_0} < 2^{\aleph_1}$) the suitable conclusions.

The cardinals $\mu(n)$ are defined in Theorem 6.4 but it suffices to say that for all practical matters $\mu(n)=2^{\aleph_n}$, e.g., $\mu(n)^{\aleph_0}=2^{\aleph_n}$, $\mu(1)=2^{\aleph_1}$, and a proof of the consistency of $\mu(n)<2^{\aleph_n}$ seems quite hard, and, of course, G.C.H or $\neg 0^{\#}$ implies $\mu(n)=2^{\aleph_n}$.

MAIN THEOREM 0.1. *Assume* $2^{\aleph_n}<2^{\aleph_{n+1}}$ *for all* $n<\omega$. *For each* $\psi\in L_{\omega_1,\omega}$ *which has an uncountable model at least one of the following holds:*

(a) *For some* $n>0$, $\mu(n)\leqq I(\aleph_n,\psi)$ (= *number of non-isomorphic models of* ψ *in* $\aleph_n$).

(b) ψ *has models in every infinite cardinality, and if it is categorical in some* $\lambda>\aleph_0$, *then it is categorical in every* $\mu\geqq\aleph_1$.

PROOF. First we shall prove a theorem: (assume $(\forall n<\omega)2^{\aleph_n}<2^{\aleph_{n+1}}$).

THEOREM 0.2. (1) *For every countable complete first order theory T, as in Theorem* 1.1 *let K be the class of atomic models of T, and assume that K has an uncountable member, then at least one of the following holds:*

(a) *For some* $0<n<\omega$, $I(\aleph_n,K)\geqq\mu(n)$,

(b) *K is an excellent class.*

(2) *Suppose K is an excellent class. It has models in every power, it has the amalgamation property (moreover, if* $M_0<M_1,M_2$ *are in stable amalgamation then over* $M_1\cup M_2$ *there is a primary model). Also if K is categorical in some uncountable cardinal then it is categorical in all uncountable cardinals.*

Theorem 0.2 is proved as follows: as we can assume $2^{\aleph_0}+I(\aleph_1,K)<2^{\aleph_1}$, by Theorem 1.4 we have some conclusions of [11].

Now if K is excellent (see Definition 5.4) then by Theorem 5.6(2) K has models in every cardinality, by Conclusion 5.8(2) the amalgamation property holds, and by Theorem 5.9 K is categorical in all uncountable cardinals, or in none of them.

On the other hand, if K is not excellent, by Claim 5.5(2) there exists a natural number $2\leqq n(K)<\omega$, and by Lemma 6.2 and Theorem 6.4, $I(\aleph_{n(K)},K)\geqq\mu(n(K))$. So we have finished proving the above Theorem 0.2, but our intention was to prove Theorem 0.1. Let K_1 be the class of models of ψ and assume (a) of Theorem 0.1 fails.

By Theorem 1.1, there are a class K of the atomic models of some complete countable first order T and an uncountable M^*, $M^*\in K_1$, such that K and $K_2=\{N:N\equiv_{\infty,\omega}M^*\}$ have the same number of models of each power, up to isomorphism. So $I(\aleph_n,K)\leqq I(\aleph_n,\psi)<\mu(n)$ for $0<n<\omega$, hence by Theorem 0.2 (1) K is excellent. Hence for every μ, $1\leqq I(\mu,K)=I(\mu,K_2)\leqq I(\mu,K_1)$. As for categoricity of K_1, if $I(\lambda,K_1)=1$, as we know $I(\lambda,K_2)\geqq 1$ and $K_2\subseteq K_1$

necessarily all models in K_1 of power λ are $L_{\infty,\omega}$-equivalent to M^*, hence by [10] this holds for every $\mu \geqq \lambda$, hence

$$\{M \in K_2 : |M| \geqq \lambda\} = \{M \in K_1 : |M| \geqq \lambda\},$$

hence for $\mu \geqq \lambda$, $I(\mu, K_1) = I(\mu, K_2) = I(\mu, K) = 1$.

If $\aleph_0 < \mu < \lambda$, $I(\mu, K_1) > 1$ then for some $M \in K_1 - K_2$, $\|M\| = \mu$. As easily K_2 is axiomatized by one sentence in $L_{\omega_1,\omega}$ (by [11] 6.1(A), p. 145) for some $\theta \in L_{\omega_1,\omega}$, $M \models \theta$, $M^* \models \neg\theta$. Allying the above to $\psi \wedge \theta$, it has a model of power λ. So K_1 is not categorical in λ, contradiction.

Conclusion 0.3. Suppose $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for each n. If $\psi \in L_{\omega_1,\omega}$ has at least one uncountable model, then it has at least $2^{\aleph_1}$.

Proof. Since $\mu(n) \geqq 2^{\aleph_1}$ for every $n \geqq 1$ (if (a) of Theorem 0.1 holds) and models of distinct power are not isomorphic (if (b) of Theorem 0.1 holds).

Remark. Since the main step in Theorem 0.1 is Theorem 0.2, i.e., the theorem for the atomic models of T, we shall refer to it as to the main theorem.

A central place in the proof is occupied by n-dimensional diagrams. We now explain how we can arrive at such diagrams naturally, when we approximate a model by structures of smaller cardinalities. Suppose we want to describe a model M of cardinality λ (for proving, e.g., existence or uniqueness). A reasonable way is by an increasing continuous elementary chain M_i $(i < \lambda)$, $M = \bigcup_{i<\lambda} M_i$, $\|M_i\| < \lambda$. Now instead of describing M, we now have to describe M_{i+1} over M_i (i.e., assuming M_i is given), and our gain is that $\|M_{i+1}\| < \lambda$. Now we have a pair (M^1, M^0) of models of cardinality λ_1, so we choose an increasing and continuous elementary chain (M_i^1, M_i^0) $(i < \lambda_1)$, $M^l = \bigcup_{i<\lambda_1} M_i^l$, $\|M_i^l\| < \lambda_1$. Now for each i we have to describe M_{i+1}^1 assuming M_i^1, M_i^0, M_{i+1}^0 are given; this is, essentially, an amalgamation problem. After n stages we should reconstruct M_n assuming M_w $(w \subseteq n, |w| < n)$ are given, $M_w \cap M_v = M_{w \cap v}$ (and more relevant conditions).

So the diagram becomes complicated, but the cardinality smaller. So for each λ and n we have a describing problem. The main point is that if for $n_0 + 1$ and each $\mu < \lambda$, we get an answer, we get it for λ and n_0, so it suffices to get an answer for one cardinality and for all n.

Remember that Morley in his categoricity theorem does not use such an inductive process; rather he proves that every model is saturated, which is a more global approach. We believe other problems can be attacked in this way. We used a similar method in [12].

Let us review the paper and explain the structure of the proof. In §1 we shall quote results from [11] and reprove the main theorem from there assuming a weaker set theoretical assumption than the $\diamondsuit_{\aleph_1}$. The main theorem is Theorem 1.3, which says if $I(\aleph_1, \psi) < 2^{\aleph_1}$ then the class K satisfies the $\aleph_0$ amalgamation property. In Theorem 1.4, assuming $2^{\aleph_0} < 2^{\aleph_1}$ and K has the $\aleph_0$ amalgamation property, relying heavily on [11] we introduce our basic stability machinery (i.e., we quote the needed facts on the rank and introduce a substitute to stationarization) which will be used in the rest of the paper. So after Theorem 1.3, $I(\aleph_1, \psi) < 2^{\aleph_1}$ is assumed. We also introduce a dimension of a type p for $(A, |M|, |M|)$ which will help us to measure a replacement for the saturatedness of M.

Note that there exists a sentence $\psi \in L_{\omega_1,\omega}$ which is categorical in every cardinality and the model is even not $(L_{\omega_1,\omega}, \aleph_1)$ homogeneous (see Marcus [9]). So we should find a replacement, and these are the full models over good sets which will be introduced in §2 and for their definition we need the above-mentioned dimension. This is continued in §2, where we define "(M, A) is λ-full". This is a generalization of $(M, a)_{a\in A}$ is λ-saturated. However, we restrict ourselves to good A (see below) and our existence and uniqueness proofs go by induction on the power (in §5) and we shall succeed in doing this in all powers only for excellent classes (note that even a first order T does not have a saturated model in every power, but a countable totally transcendental T has). We also look at other possible generalizations of stationarization (weak stationarization, triples in stable amalgamation and the analysis of $p \in D_A$ for good A).

Last but not least, in §2 we introduce good sets, which essentially are sets over which amalgamation holds. (For countable sets this is the exact definition.) Be careful: not every set is good! Really the search for good sets is a central theme of this work [this may sound like a silly warning, and I insert it only because some excellent mathematicians succeed in reading the paper without realizing it]. So the $\aleph_0$-amalgamation property implies that the universe of a countable model is a good set. Note that for a saturated uncountable model M, the isomorphism type of $(M, a_n)_{n<\omega}$ is determined by the isomorphism types of $(M, a_n)_{n<k}$ for $k < \omega$, but nothing of this sort holds for models of $\psi \in L_{\omega_1,\omega}$. We believe that amalgamation bases (though not good sets as defined) will play a role in other parallel investigations.

In the third section we introduce the n-dimensional diagram $\{M_s : s \in I\}$ (usually $I = \mathcal{P}(m)$ or $I = \mathcal{P}^-(n)$). Looking at the way they arise (as explained above) the order on $\bigcup I$ is important. However, as for stable theories we have a symmetry lemma; generalizing it we get that the order is not important (see Lemma 3.5). Note also that for our machinery to work we demand (in good systems) various sets to be good and triples to be in stable amalgamation.

In the fourth section we define the properties for $(\lambda, \mathcal{P}^-(n))$-good systems which naturally arise from our purposes: existence, uniqueness and also non-uniqueness (i.e., $\bigcup_{s \in \mathcal{P}^-(n)} M_s$ is not an amalgamation base), and goodness ($\bigcup \{M_s : s \in \mathcal{P}^-(n)\}$ is a good set). We usually restrict ourselves to full models so that the system may become unique, hence uniqueness and non-uniqueness become complementary. We then describe how to decompose a (λ, I)-good system to an increasing continuous sequence of $(\mu, I \times \{0,1\})$-good systems for $\mu < \lambda$.

In the fifth section we define "excellency" (= having all positive properties in $\aleph_0$, i.e., goodness, existence and uniqueness). We prove the existence and uniqueness of full models in excellent classes; moreover, the notion of excellent classes is similar to the notion of elementary classes from the following points of view (see Theorem 5.9): K has the amalgamation property, for every λ it has a model. Moreover, it has a full model, and this full model is unique (also, it will be universal and homogeneous for K). Furthermore, K satisfies the Los conjecture: it will be categorical in all $\lambda > \aleph_0$ or none. So if K is excellent we have fulfilled possibility (b) in the Main Theorem. We are left with (a); we shall prove that the negation of (a) implies K is excellent, i.e., if for every $n < \omega$, $I(\aleph_n, \psi) < \mu(n)$, then K is excellent. How to prove this? The central idea in this paper is to transfer properties of models of cardinality $\aleph_n$ to properties of countable models, and then back from countable models to models of higher cardinalities. We do it to use the only method to construct atomic models we know, and this is the Henkin omitting types theorem which, unfortunately, holds only for countable models (this is also the reason why we assumed L countable); more specifically, for $\aleph_0$ the goodness property becomes the negation of non-uniqueness property and existence property becomes true. Using §4, in Theorems 5.1 and 5.2 we transfer the positive properties up. Now if the class is not excellent we want to prove (a), i.e., that there exist n, $0 < n < \omega$, such that $I(\aleph_n, \psi) \geqq \mu(n)$. If K is not excellent there is a first $n(K)$ such that, for $n(K)$-dimensional cubes of countable models, the "suitable" amalgamation property fails. For this case we have §6 using heavily the weak diamond. We transfer this property up, i.e, we prove that for $(n(K)-1)$-dimensional cubes of full models of power $\aleph_1$, the amalgamation fails. We continue to decrease the dimension and increase the power of the full models till we prove that the usual amalgamation fails for full models of power $\lambda \stackrel{\text{def}}{=} \aleph_{n(K)-2}$. Next, we prove first that if there exists M_0, M_1, M_2 full models of cardinality λ^+ such that M_1, M_2 cannot be amalgamated over M_0, then K has $2^{\lambda^{++}} = 2^{\aleph_{n(K)}}$ non-isomorphic models of power λ^{++}; this is not hard and is similar to what was done in Theorem 1.3.

For the case when the amalgamation property of full models of cardinality λ^+ holds, we are in a more complicated situation and the $2^{\lambda^{++}}$ non-isomorphic models are constructed by approximations, models of power λ (see the explanation after Theorem 6.4).

The main problems are:

PROBLEM 1. Prove Theorem 0.1 in ZFC; a beginning has been done in [17].

CONJECTURE 2. Suppose $\psi \in L_{\omega_1,\omega}$ is categorical in one $\lambda \geqq \beth_{\omega_1}$, then it is categorical in every cardinal $\geqq \beth_{\omega_1}$.

Notice that the last problem is interesting provided there is $n < \omega$ such that $I(\aleph_n, \psi) \geqq \mu(n)$; otherwise it has a positive answer by the Main Theorem (we are in case (2)).

Notice that in this case, even if $I(\aleph_1, \psi) = 2^{\aleph_1}$, Theorem 1.1 still holds. By remembering that $\beth_{\omega_1}$ is the Hanf number of $L_{\omega_1,\omega}$, which implies the existence of an Ehrenfeucht–Mostowski model of ψ of cardinality $\beth_{\omega_1}$ with dense skeleton, then this model realizes only countably many $L_{\omega_1,\omega}$-types, which was the only need of the assumption $I(\aleph_1, \psi) < 2^{\aleph_1}$ in Theorem 1.1.

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§1. Review of the results on $\aleph_1$, eliminating the diamond, and stationary types

As this paper is a continuation of [11], we quote here the required results, so that the reader need not refer back to [11]. One of these results is that w.l.o.g. we can replace the class of models of a sentence $\psi \in L_{\omega_1,\omega}$ by the class of atomic models of a first-order complete countable T, which has uncountable models (see Theorem 1.1).

In Theorem 1.3 we show that if the $\aleph_0$-amalgamation property fails, then K has $2^{\aleph_1}$ models of power $\aleph_1$ and has no universal model in $\aleph_1$, assuming $2^{\aleph_0} < 2^{\aleph_1}$ (and not the diamond). This is done using the weak diamond principle; and by it we can replace the diamond by $2^{\aleph_0} < 2^{\aleph_1}$ everywhere in [11]. Note that even "categoricity in $\aleph_1$ of K does not imply the $\aleph_0$-amalgamation property" is consistent with ZFC $+2^{\aleph_0}=2^{\aleph_1}$ (as explained in the introduction of [11]). Note that if we just want to prove "$I(\aleph_1, K)$ is large" the proof is considerably simpler, and the present proof is continued in §6.

The theorem most used later is Theorem 1.4. Remember that in [14] the notions "$\mathrm{tp}(\bar{a}, A)$ does not fork over $B \subseteq A$", "$\mathrm{tp}(\bar{b}, A)$ is the stationarization of $\mathrm{tp}(\bar{b}, B)$" where $B \subseteq A$ (which for superstable theories are equivalent to equality of rank) play important roles. We want a parallel in our context. As we already have a suitable rank (see [11], section 4) we can use equality of ranks. Unfortunately, our ranks do not have the extension property, so we restrict ourselves to such types (called stationary). Note $\mathrm{tp}(\bar{a}, M)$ is always stationary (remember we are always interested in $\mathrm{tp}(\bar{a}, A)$, only if $A \cup \bar{a}$ is atomic). This is done in Theorem 1.4 (1) under the assumption $I(\aleph_1, K)+2^{\aleph_0}<2^{\aleph_1}$. We also draw a conclusion which will be needed in §2: over any countable M there is a countable extension universal over M (see Theorem 1.4 (2)).

Definition 1.5 and Theorem 1.6 are from a previous work ([14], chapter IV) which deals with generalizations of prime models. Definition 1.7 introduces a concept of dimension of stationary type corresponding to a triple (A_1, A_2, A_3) when the interesting case is (A, M, M) when $A \subseteq M$. This is a measure to the "saturatedness" of the model M according to stationary types over M. This will be used in the definition of full model in the next section which will be our substitute to saturated models in elementary classes.

NOTATION. For a class K of models, let $I(\lambda, K)$ be the number of models in K of cardinality λ, up to isomorphism. Let K_1 be the class of models of $\psi \in L_{\omega_1,\omega}$.

THEOREM 1.1. *Assume* $2^{\aleph_0}+I(\aleph_1, \psi)<2^{\aleph_1}$, *then there is a complete countable first order theory T such that, letting K be the class of atomic models of T*:

(i) *if K_1 has an uncountable member, so does K*;

(ii) *if* $I(\beth_{\omega_1}, K_1) \geqq 1$ *then* $I(\beth_{\omega_1}, K) \geqq 1$;

(iii) *for every λ, $I(\lambda, K) \leqq I(\lambda, K_1)$ and for some uncountable $M \in K_1$, for every λ, $I(\lambda, K)=I(\lambda,\{N \in K_1 : N \equiv_{\infty,\omega} M\})$*;

(iv) *in T, every formula is equivalent to an atomic formula, and T has no function symbols*;

(v) *note that K is categorical in $\aleph_0$ and this model has a proper elementary extension in K.*

PROOF. By [11], to get the required conclusion using lemma 2.5 and lemma 3.1 in [11], p. 132, it is sufficient to find an uncountable model of ψ in which only countably many $L_{\omega_1,\omega}$-types are realized. This follows from theorem 2.3 and lemma 2.1 in [11], p. 129. So from now on we shall assume

HYPOTHESIS. K is the class of atomic models of a countable complete T which has uncountable models. For notational simplicity T has no function symbols and every formula is equivalent to a relation.

CONVENTION 1.2. Let $\mathfrak{C}$ be a $\bar{\kappa}$-saturated model of T (this model is not a member of our class); a model will be an atomic elementary submodel of $\mathfrak{C}$ (if not designated otherwise) (see [14], p. 7 as to why we do not lose generality) so $\models \varphi$ means $\mathfrak{C} \models \varphi$. Let A, B, C denote atomic subsets of $\mathfrak{C}$, a, b, c, d denote elements of such sets, and $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ finite sequence of elements from such sets (this is equivalent to: a $(\bar{a})$ realizes in $\mathfrak{C}$ over $\emptyset$ an atomic type). Models will be denoted by the letters M, N (perhaps with index) always in K. We shall not distinguish strictly between a sequence and its range, and write $\bar{a} \in A$ instead of $\bar{a} \subseteq A$.

Let $\mathrm{tp}(\bar{b}, A) = \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in A, \models \varphi[\bar{b}, \bar{a}]\}$.

For $A \subseteq B$ we say that $\mathrm{tp}(\bar{a}, B)$ does not split over A *if* for every $\bar{b} \in B$ and $\bar{c} \in B$, $\mathrm{tp}(\bar{b}, A) = \mathrm{tp}(\bar{c}, A)$ implies $\mathrm{tp}(\bar{a} {}^\wedge \bar{b}, A) = \mathrm{tp}(\bar{a} {}^\wedge \bar{c}, A)$.

K has the λ-amalgamation property *if* whenever $M < M_l$ $(l = 0, 1)$ $\|M\| = \|M_l\| = \lambda$ then there is N, and elementary mappings f_l, $f_l : M_l \to N$, $f_0 \restriction M = f_1 \restriction M$. Remember that by our notation M, M_l, N are from K.

In [11], §§1–5 we actually used the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ only, except in [11], 3.4, where we assumed $\diamondsuit_{\aleph_1}$; here we prove also this assuming $2^{\aleph_0} < 2^{\aleph_1}$ only.

THEOREM 1.3 ($2^{\aleph_0} < 2^{\aleph_1}$). *Suppose K does not have the $\aleph_0$-amalgamation property. Then $I(\aleph_1, K) = 2^{\aleph_1}$. Also K has no universal member in $\aleph_1$.*

PROOF. To clarify the proof, we first prove the second statement, hence $I(\aleph_1, K) > 1$. So let M^* be a model of cardinality $\aleph_1$, so w.l.o.g. $|M^*| = \omega_1$. By the hypothesis there are countable $M < M_l$ $(l = 0, 1)$ which exemplify the failure of the $\aleph_0$-amalgamation property. We shall show that M^* is not universal.

For this we define by induction on $\alpha < \omega_1$ models M_η for $\eta \in {}^\alpha 2$, such that:

(i) M_η is countable, $|M_\eta| = \omega(1 + l(\eta))$,

(ii) $\eta \triangleleft \nu$ implies $M_\eta < M_\nu$ ($\eta \triangleleft \nu$ stands for: η is an initial segment of ν),

(iii) for limit δ, and $\eta \in {}^\delta 2$, $M_\eta = \bigcup_{\alpha < \delta} M_{\eta \restriction \alpha}$.

For $\alpha = 0$ and α limit there is no problem. For successor $\alpha = \beta + 1$, for each $\eta \in {}^\beta 2$ choose an isomorphism f_η from M onto M_η (remember K is categorical in $\aleph_0$), and define for $l = 0, 1$ a function f^l_η and a model $M_{\eta {}^\wedge \langle l \rangle}$ such that f^l_η extends f_η and is an isomorphism from M_l onto $M_{\eta {}^\wedge \langle l \rangle}$.

Now, for $\eta \in {}^{\omega_1} 2$, let $M_\eta = \bigcup_{\alpha < \omega_1} M_{\eta \restriction \alpha}$; if M^* is universal, for each $\eta \in {}^{\omega_1} 2$ there is an elementary embedding g_η of M_η into M^*. By [3], 6.1, as $2^{\aleph_0} < 2^{\aleph_1}$, there are distinct $\eta, \nu \in {}^{\omega_1}2$, and $\alpha < \omega_1$, $\omega\alpha = \alpha$ such that $\eta \restriction \alpha = \nu \restriction \alpha$, $0 = \eta(\alpha) \neq \nu(\alpha) = 1$ and $g_\eta \restriction M_{\eta \restriction \alpha} = g_\nu \restriction M_{\nu \restriction \alpha}$. This is a contradiction, as

$$(g_\eta \restriction M_{\eta \restriction (\alpha+1)}) f^0_{\eta \restriction \alpha} : M_0 \to M^*, \qquad (g_\nu \restriction M_{\nu \restriction (\alpha+1)}) f^1_{\eta \restriction \alpha} : M_1 \to M^*$$

show M, M_0, M_1 can be amalgamated. Contradiction.

Now let us prove $I(\aleph_1, K) = 2^{\aleph_1}$. By [11], 2.1(B) we can assume M_l are chosen such that

(*) $\{\text{tp}(\bar{a}, |M|) : \bar{a} \in |M_l|\}$ are maximal and distinct

[i.e., for any N, $M_l < N$, and $\bar{a} \in N$, for some $\bar{a}' \in M_l$, $\text{tp}(\bar{a}, |M|) = \text{tp}(\bar{a}', |M|)$; but $\{\text{tp}(\bar{a}, |M|) : \bar{a} \in |M_0|\} \neq \{\text{tp}(\bar{a}, |M|) : \bar{a} \in |M_1|\}$] (we could find M_0, M_1 such that $\{\text{tp}(\bar{a}, |M|) : \bar{a} \in |M_l|\}$ are incomparable).

We call a set $S \subseteq \omega_1$ small if there is a function F, such that for every $\eta \in {}^{\omega_1}2$ for some $h : \omega_1 \to 2^{\aleph_0}$, $\{\alpha < \omega_1 : \alpha \in S \Rightarrow F(h \restriction \alpha) = \eta(\alpha)\}$ contains a closed unbounded set. By [3], 3.1, 4.1(2) (essentially) the small subsets of ω_1 form a normal ideal, which is nontrivial (when $2^{\aleph_0} < 2^{\aleph_1}$).

So, in particular, this ideal is $\aleph_1$-complete (notice the analogy to the ideal of non-stationary subsets of ω_1). Hence by Ulam's theorem there are pairwise disjoint non-small sets $S_\alpha \subseteq \omega_1$ $(\alpha < \omega_1)$. We let M, M_l, M_η, f_η be as above. Now let us define a function F. If $\delta < \omega_1$, $\eta, \nu \in 2^\delta$, $\omega\delta = \delta$, $h : \delta \to \delta$, h an elementary embedding of M_η into M_ν, and the diagram

$$\begin{array}{l} M_{\eta\hat{}\langle 0\rangle} \\ \cup \\ M_\eta \xrightarrow{\ h\ } M_{\nu\hat{}\langle 0\rangle} \end{array}$$

can be amalgamated, then $F(\eta, \nu, h) = 1$, otherwise $F(\eta, \nu, h) = 0$.

As S_α is not small, there is a sequence $\rho_\alpha \in {}^{\omega_1}2$, such that for every $\eta, \nu \in {}^{\omega_1}2$, $h : \omega_1 \to \omega_1$, $\{i < \omega_1 : F(\eta \restriction i, \nu \restriction i, h \restriction i) = \rho_\alpha(i)\} \cap S_\alpha$ is stationary. (Formalistically, we should make F into a function from $\{h : h : {}^\alpha 2 \to 2 \times 2 \times \alpha$ for some $\alpha\}$ into $2 = \{0,1\}$.)

Now for every set $I \subseteq \omega_1$ we define $\eta_I \in {}^{\omega_1}2$:

$$\eta_I(i) = \begin{cases} \rho_\alpha(i), & i \in S_\alpha, \quad \alpha \in I, \\ 0, & i \notin \bigcup_{\alpha < \omega_1} S_\alpha \quad \text{or} \quad i \in S_\alpha, \quad \alpha \notin I. \end{cases}$$

We shall now show that M_{η_I} $(I \subseteq \omega_1)$ are pairwise non-isomorphic. So suppose

$h: M_{\eta_I} \to M_{\eta_J}$ is an elementary embedding, $\gamma \notin J$, $\gamma \in I$ and we shall get a contradiction.

Let $S = \{\delta < \omega_1 : h \restriction \delta \text{ is into } \delta,\ \delta = \omega\delta\}$. Clearly S is a closed unbounded subset of ω_1. For each $\delta \in S \cap S_\gamma$, $\eta_J(\delta) = 0$, $\eta_I(\delta) = \rho_\gamma(\delta)$; by ρ_α's definition

$$S' = \{\delta \in S_\gamma : F(\eta_I \restriction \delta, \eta_J \restriction \delta, h \restriction \delta) = \rho_\gamma(\delta)\}$$

is stationary, so we can choose $\delta \in S \cap S'$ and let $\eta = \eta_I \restriction \delta$, $\nu = \eta_J \restriction \delta$. If $\rho_\gamma(\delta) = 0$ then $\eta_I(\delta) = 0$, $F(\eta, \nu, h \restriction \delta) = 0$, and it is easy to check that

$$\begin{array}{l} M_{\eta_I \restriction (\delta+1)} \\ \cup \\ M_\eta \xrightarrow[h \restriction \delta]{} M_{\eta_J \restriction (\delta+1)} \end{array}$$

can be amalgamated (use $h \restriction \omega(\delta+1)$), contradicting the definition of F (remembering $\eta_I(\delta) = \eta_J(\delta) = 0$). So necessarily $\rho_\gamma(\delta) = 1$, so $\eta_I(\delta) = 1$, $F(\eta, \nu, h \restriction \delta) = 1$, hence by F's definition

$$\begin{array}{l} M_{\eta \frown \langle 0 \rangle} \\ \cup \\ M_\eta \xrightarrow[h \restriction \delta]{} M_{\nu \frown \langle 0 \rangle} \end{array}$$

can be amalgamated, but using $h \restriction \omega(\delta+1)$ we see that also

$$\begin{array}{l} M_{\eta \frown \langle 1 \rangle} \\ \cup \\ M_\eta \xrightarrow[h \restriction \delta]{} M_{\nu \frown \langle 0 \rangle} \end{array}$$

can be amalgamated, but this contradicts (*) for the following reason.

Let $N < M_\nu$, N_1 be the images of M_η, $M_{\eta \frown \langle 1 \rangle}$ by h resp. As the diagram before the last can be amalgamated (in K, of course) there is a model N^*, $M_{\nu \frown \langle 0 \rangle} < N^*$ and an elementary embedding $f: M_{\eta \frown \langle 0 \rangle} \to N^*$ extending $h \restriction \delta$; and let N_0 be the image of $M_{\eta \frown \langle 0 \rangle}$ by f. By (*), $\{\mathrm{tp}(\bar{a}, N) : \bar{a} \in N_l\}$ $(l = 0, 1)$ are maximal and distinct. Suppose there is $\bar{a}_0 \in N_0$ such that $p = \mathrm{tp}(\bar{a}_0, N) \notin \{\mathrm{tp}(\bar{a}, N) : \bar{a} \in N_1\}$. But by the choice of $M_{\nu \frown \langle 0 \rangle}$, $\mathrm{tp}(\bar{a}_0, M_\nu)$ is realized in $M_{\nu \frown \langle 0 \rangle}$, say by $\bar{a}_1$, so $\bar{a}_1$ realizes p. So for some $\alpha < \omega_1$, $N_1 \cup \{\bar{a}_1\} \subseteq M_{\eta_J \restriction \alpha}$. Hence, by the maximality of $\{\mathrm{tp}(\bar{a}, N) : \bar{a} \in N_1\}$, p belongs to it, contradicting the choice of p. If there is $\bar{a}_1 \in N_1$ such that $p = \mathrm{tp}(\bar{a}_1, N) \notin \{\mathrm{tp}(\bar{a}, N) : \bar{a} \in N_0\}$, we can get a similar contradiction.

REMARKS. (1) Trivially there is a family of $2^{\aleph_1}$ subsets of ω_1, no one a subset of another. So in fact we have proven there are in K $2^{\aleph_1}$ models of cardinality $\aleph_1$ no one elementarily embedded into another.

(2) We can similarly prove that any Σ_1-equivalence relation on ${}^{\omega_1}2$ which satisfies the parallel of the non-$\aleph_0$-amalgamation property has $2^{\aleph_1}$ equivalence classes.

For an abstract version and applications, see Avraham and Shelah [1]. Also, in the present paper, there are proofs which go deeper into the matter: in Lemma 6.2(3) (there we find a way to circumvent the maximality of the N_l in a similar situation) and really in Theorem 6.4 (there we build $2^{\lambda^{++}}$ models of cardinality λ^{++}, using non-uniqueness of amalgamation in λ). In both cases it is much harder to get the maximal number than to get quite many.

By [11] (omitting (γ) of definition 4.1 (c) on p. 135, which just simplifies matters),

THEOREM 1.4. *Suppose $I(\aleph_1, K) < 2^{\aleph_1}$, $2^{\aleph_0} < 2^{\aleph_1}$. Then:*

(1) (a) *For every complete type $p = \mathrm{tp}(\bar{a}, A)$ ($A \cup \bar{a}$ atomic, of course), an ordinal $R(p)$, called its rank, is defined. The rank is $< \omega_1$, is monotonic [i.e., for $A' \subseteq A$, $R(\mathrm{tp}(\bar{a}, A')) \geqq R(p)$], is preserved under automorphisms of $\mathfrak{C}$, and for every such p there is a finite $B \subseteq A$ such that $R(p) = R(\mathrm{tp}(\bar{a}, B))$. Note that this holds for any B', $B \subseteq B' \subseteq A$.*

(b) *We call p stationary if there are a finite set B and a model $N \supseteq B$ such that $B \subseteq A$ and $R(\mathrm{tp}(\bar{a}, B)) = R(p) = R(\mathrm{tp}(\bar{a}, |N|))$. Note that if p is stationary then p does not split over B and for any atomic $C \supseteq A$ there is $\bar{a}'$ realizing p such that $C \cup \bar{a}'$ is atomic and $R(\mathrm{tp}(\bar{a}', C)) = R(p)$, we call $\mathrm{tp}(\bar{a}', C)$ the stationarization of p over C, denote it by p_C, and it is unique.*

(c) *The symmetry property holds: If $\mathrm{tp}(\bar{a}_l, A)$, $l = 0,1$ are stationary then $R(\mathrm{tp}(\bar{a}_1, A \cup \bar{a}_0)) = R(\mathrm{tp}(\bar{a}_1, A))$ if and only if $R(\mathrm{tp}(\bar{a}_0, A \cup \bar{a}_1)) = R(\mathrm{tp}(\bar{a}_0, A))$.*

(2) *For every countable M, there is an N, $M < N$, $\|N\| = \|M\|$ such that: any N', $\|N'\| \leqq \|N\|$, $M < N'$, can be elementarily embedded into N over M, i.e., the embedding is the identity over M. An N as above is called universal over M (for not necessarily countable M).*

(3) *For any countable M, if $\bar{a}$, $\bar{b} \in M$, $p = \mathrm{tp}(\bar{a}, \bar{b})$ is stationary, there are $\bar{a}_n \in M$ ($n < \omega$) such that $\{\bar{a}_n : n < \omega\}$ is an indiscernible set over $\bar{b}$ in M based on p, where we use the following definition:*

We say $\boldsymbol{I}$ is an indiscernible set over B in A based on p if:

(a) *$\boldsymbol{I}$ is a set of sequences of a fixed finite length;*

(b) *p is a stationary type over some* $C \subseteq B$;

(c) $B \cup \boldsymbol{I} \subseteq A$;

(d) *for every* $\bar{c} \in \boldsymbol{I}$, $\mathrm{tp}(\bar{c}, B \cup (\boldsymbol{I} - \{\bar{c}\}))$ *is a stationarization of* p;

(e) *for every* $\bar{d} \in A$, *for some finite* $\boldsymbol{J} \subseteq \boldsymbol{I}$, *for every* $\bar{c} \in \boldsymbol{I} - \boldsymbol{J}$, $\mathrm{tp}(\bar{c}, B \cup \bar{d} \cup (\boldsymbol{I} - \{\bar{c}\}))$ *is a stationarization of p.*

(4) *Now if* M_i *is increasing* ($i < \alpha \geqq \omega$), $\bar{c}_i \in M_{i+1}$, $\mathrm{tp}(\bar{c}_i, M_i)$ *a stationarization of a fixed p (so* $p \in S^m(B)$, $B \subseteq M_0$), *then* $\{\bar{c}_i : i < \alpha\}$ *is an indiscernible set over B in* M_α *based on p.*

(5) *If* $\boldsymbol{I}$ *is an indiscernible set over B in A based on p* then $\boldsymbol{I}$ *is indiscernible over B, i.e., for each n, for all distinct* $\bar{c}_l \in I$ $(l = 0, \cdots, n)$, $\mathrm{tp}(\bar{c}_0{}^\frown \cdots {}^\frown \bar{c}_n, B)$ *is fixed.*

REMARK. It is not clear whether "in A" is necessary.

PROOF. (1) (a) This rank is defined in definition 4.2 in [11] (but we omit part (γ) of definition 4.1(c)). By lemma 2.1(B) in [11], K is $\aleph_0$-stable (see definition 3.5(B) in [11]), and by lemma 4.2 (A $\Rightarrow$ C) in [11] $R(p) < \infty$; by lemma 4.1(C) in [11] this implies $R(p) < \omega_1$. The other properties mentioned are easy to prove.

(b) First notice that being a stationary type is preserved by automorphisms of $\mathfrak{C}$. Assume that $p = \mathrm{tp}(\bar{a}, A)$ splits over B, i.e., there exist $\bar{c}, \bar{d} \in A$ such that $\mathrm{tp}(\bar{c}, B) = \mathrm{tp}(\bar{d}, B)$ and $\phi(\bar{x}; \bar{c}) \wedge \neg \phi(\bar{x}; \bar{d}) \in p$, therefore by monotonicity of rank and $R(p) = R(\mathrm{tp}(\bar{a}, B))$, say α, the following holds:

$$\alpha = R(p) = R(\mathrm{tp}(\bar{a}, B) \cup \{\phi(\bar{x}; \bar{c})\}) = R(\mathrm{tp}(\bar{a}, B) \cup \{\neg \phi(\bar{x}, \bar{d})\}).$$

Let f be an automorphism of $\mathfrak{C}$ such that $f \restriction B = \mathrm{id}_B$ and $f(\bar{d}) = \bar{c}$, hence

$$\alpha = R(f(p)) = R(\mathrm{tp}(\bar{a}, B) \cup \{\phi(\bar{x}; \bar{c})\}) = R(\mathrm{tp}(\bar{a}, B) \cup \{\neg \phi(\bar{x}; \bar{c})\});$$

now by definition of the rank function this implies $R(\mathrm{tp}(\bar{a}, B)) \geqq \alpha + 1$, contradiction.

Given C atomic assume $R(p) = \alpha$, let p_C be the set

$$p_C = \{\phi(\bar{x}; \bar{c}) : \phi(\bar{x}; \bar{y}) \in L, \bar{c} \in C, R(p \cup \{\phi(\bar{x}; \bar{c})\}) = \alpha\}.$$

Let B, N witness the stationarity of p, i.e., $B \subseteq A$ is finite, and $B \subseteq N$, and $R[p] = R[p \restriction B] = R[\mathrm{tp}(\bar{a}', N)]$ for some $\bar{a}'$. Define

$$p'_C = \{\phi(\bar{x}; \bar{c}) : \phi(\bar{x}; \bar{y}) \in L, \bar{c} \in C, R[(p \restriction B) \cup \{\phi(\bar{x}; \bar{c})\}] = \alpha\}.$$

We shall prove that p'_C is consistent, a complete type over C, $C \cup \bar{a}'$ is atomic for $\bar{a}'$ realizing p'_C and $R[p'_C] = \alpha$. By the finite character of the rank (see [11]) this implies $p_C = p'_C$. As the definition of rank implies that there is at most one stationarization, we finish.

Let us show that p'_C is consistent: For any $\Psi = \{\psi_i(\bar{x};\bar{c}_i): i < n\} \subseteq p'_C$ denote $\bar{c} = \bar{c}_0\hat{}\bar{c}_1\hat{}\cdots\hat{}\bar{c}_{n-1}$. Because $\bar{c} \cup B$ is atomic and N a model, there exists $\bar{c}' \in N$ such that $\operatorname{tp}(\bar{c}, B) = \operatorname{tp}(\bar{c}', B)$. As $R[p] = R[\operatorname{tp}(\bar{a}, B)]$ it follows that $\Psi' = \{\psi_i(\bar{x};\bar{c}'_i): i < n\} \subseteq \operatorname{tp}(\bar{a}, |N|)$, therefore it is consistent.

By acting on $\mathfrak{C}$ with an automorphism which fixes B and replaces $\bar{c}$ by $\bar{c}'$ also Ψ is consistent, hence p'_C is consistent. To show the completeness of p'_C is a similar argument: let $\phi(\bar{x};\bar{c})$ $(\bar{c} \in C)$ be such that $\phi(\bar{x};\bar{c}) \notin p'_C$ and $\neg\phi(\bar{x};\bar{c}) \notin p'_C$; as before there exists $\bar{c}' \in N$ satisfying $\operatorname{tp}(\bar{c}, B) = \operatorname{tp}(\bar{c}', B)$, finally we have $\phi(\bar{x};\bar{c}') \notin \operatorname{tp}(\bar{a}, |N|)$ and $\phi(\bar{x}, \bar{c}') \notin \operatorname{tp}(\bar{a}, |N|)$, contradiction.

Similarly to the consistency we can prove that $R[p'_C] = \alpha$. It is obvious that p'_C extends p. Now also if $\bar{a}'$ realizes p'_C then $C \cup \bar{a}'$ is atomic: as for every $\bar{c} \in C$ there are $\bar{c}'\hat{}\bar{a}'' \in N$ realizing $\operatorname{tp}(\bar{c}\hat{}\bar{a}', B)$, and N is atomic.

(c) In [11] theorems 5.1 (B $\Rightarrow$ C) and 5.4 we proved the symmetry property over models (i.e., when A is a model) and it is easy to check that symmetry over models implies symmetry over sets when we deal with stationary types.

(2) We shall define the universal model over M as a union of a countable elementary chain $\{M_n : n < \omega\}$, $M_0 = M$, assume $\{M_k : k < n\}$ is defined, and define M_n to be a countable model extending M_{n-1} and containing finite sequences realizing all the complete types over M_{n-1} (this set is countable because, as we said before, by the assumptions of this theorem K is $\aleph_0$-stable). Such M_n exist, as K has the $\aleph_0$-amalgamation and is closed under increasing unions. Now we shall prove that $N = \bigcup_{n<\omega} M_n$ is a countable universal model in K over M. That it is in K is obvious, because each M_n is from K and it is clear that K is closed under unions of elementary chains.

Universality: Let $N' \supseteq M$, $N' \in K$, $\|N'\| = \aleph_0$; we shall define an embedding of N' into N over M (= identity over M). Denote $\{a_n : n < \omega\} = |N'|$ and we shall define a corresponding sequence $\{b_n : n < \omega\}$ in N such that for every $n < \omega$

$$(*)_n \qquad \operatorname{tp}(\langle a_0 \cdots a_{n-1}\rangle, |M|) = \operatorname{tp}(\langle b_0 \cdots b_{n-1}\rangle, |M|).$$

We prove it by induction on n. Assume we have proved $(*)_n$ and define $b_n \in N$ such that $(*)_{n+1}$ will hold. Let M' be a prime model over $M \cup \{a_0, \cdots, a_{n-1}\}$ (the existence of M' is proved in lemma 4.4 in [11]). Because of its primeness, w.l.o.g. $M' \subseteq N'$ and there is $k < \omega$ such that M' is embedded, say, by f into M_k such that $M \cup \{b_0, \cdots, b_{n-1}\} \subseteq f(M') \subseteq M_k$, $f \restriction M = \mathrm{id}$ and $f(a_l) = b_l$ for $0 \leqq l < n$. Consider the image of the type $p = \operatorname{tp}(a_n, M')$ under $f : f(p)$; it is included in some $\{\operatorname{tp}(c, M_k) : M_k \cup c \text{ atomic}\}$ (this holds because of the amalgamation property); by the definition of M_{k+1}, $\operatorname{tp}(c, M_k)$ is realized in M_{k+1} by b_n. Now it is trivial to check that $(*)_{n+1}$ holds.

(3), (4) are easy.

(5) Just like [14], lemma I2.5 (p. 11).

Recall from [14], ch. IV.

DEFINITIONS 1.5. (1) We say $p = \operatorname{tp}(\bar{b}, A)$ is isolated if there is $\varphi = \varphi(\bar{x}, \bar{a}) \in p$, such that each $\bar{b}'$ satisfying φ realizes p. We say B is constructible over A if $A \subseteq B$, $B = A \cup \{\bar{a}_i : i < \alpha\}$, and for each i, $\operatorname{tp}(\bar{a}_i, A \cup \{\bar{a}_j : j < i\})$ is isolated. We say B is atomic over A if $\operatorname{tp}(\bar{b}, A)$ is isolated for each $\bar{b} \in B$. We say M is primary over A if M is constructible over A.

(2) We say $p = \operatorname{tp}(\bar{b}, A)$ is $\boldsymbol{F}^m$-isolated if it is isolated or for some $\bar{c} \in A$, $\operatorname{tp}(\bar{b}, \bar{c})$ is stationary, and $\operatorname{tp}(\bar{b}, A)$ is its stationarization. We define $\boldsymbol{F}^m$-constructible similarly.

THEOREM 1.6. (1) *If over A there is a primary model M it is unique (over A, of course) and it is prime over A, i.e., for every N, $A \subseteq N$, M can be elementarily embedded into N over A.*

(2) *if $\operatorname{tp}(\bar{c}, A)$ is isolated, A atomic, then $A \cup \bar{c}$ is atomic. If M is atomic over A, M countable, then M is primary over A. If M is primary over A, then it is atomic over A. If B is constructible over A, then B is atomic over A.*

DEFINITION 1.7. Let $p \in S^m(B)$ be stationary. The dimension of p for (A_1, A_2, A_3) is the first cardinal κ, such that for some $C \subseteq A_2$, $|C| = \kappa$, there is a stationarization $q \in S^m(A_1 \cup C)$ of p, but it is not realized in A_3 and we assume always that $A_1 \cup A_2 \cup A_3$ is atomic and $B \subseteq A_1 \cup A_2$. So we can replace A_2 by $A_1 \cup A_2$.

REMARK. This notion is used in Definition 2.7 in the next section; to understand it, look there.

§2. Goodness and fullness

We define and investigate here the notion of good sets (in Definition 2.1, Lemma 2.2, Claims 2.3 and 2.4, and Conclusion 2.5 (1)), which will be used extensively later. For a countable A, A is good iff over A there is a prime [primary] model iff over A there is a universal countable model (the definition for uncountable A has technical importance only). Remember that good A satisfies what our intuition (built on first-order model theory) many times tells us is always true.

In Definition 2.7, and Lemma 2.8 (3), (4) we deal with the situation when M is

λ-full over A, which is our substitute for $(M,a)_{a\in A}$ is λ-saturated, but we require A to be good. In Lemma 2.12 (2) we deal with the λ-fullness of the union of an increasing chain (compare with: (see [14] III, theorem 3.11) if M_i is increasing, (for $i<\delta$) each M_i is κ-saturated $\kappa(T)\leqq \mathrm{cf}\ \delta$ implies $\bigcup_{i<\delta} M_i$ is κ-saturated). To achieve this we deal again with "stationarization". The notion we gave in §1 is not totally satisfying as it does not always exist. We suggest "weak stationarization" q of $p\in D_A$ in D_C where (A,C) satisfies the Tarski–Vaught condition, if $p\subseteq q$, q does not split over some finite subset of A (see Definition 2.9 (1)). Now if A is good every $p\in D_A$ does not split over some finite subset of A, and then q exists, is unique, and the definition is compatible with the previous one, etc. (see Lemma 2.10(1)). A strongly related fact is Lemma 2.2(1) which says that if $\mathrm{tp}(\bar a,A)\in D_A$, A good, then for some $\bar b$, $\bar c$, $\bar a\subseteq \bar b{}^\wedge \bar c$, $\mathrm{tp}(\bar b,A)$ is isolated and $\mathrm{tp}(\bar c,A\cup \bar b)$ is the stationarization of $\mathrm{tp}(\bar c,\bar b)$. So every $p\in D_A$ is analyzed using two well-behaved cases. This also analyzes, for good A, the weak stationarization of $p\in D_A$.

We say A,B,C is in stable amalgamation if A is good, (A,C) satisfies the Tarski–Vaught condition, and for every $\bar b\in B$ $\mathrm{tp}(\bar b,C)$ is the weak stationarization of $\mathrm{tp}(\bar b,A)$. We almost always amalgamate in this way and we deal with this notion in Definition 2.9 (see Lemma 2.10 (2) (existence), Lemma 2.10 (3), (4) (connection with stationarization and dimension), Lemmas 2.11, 2.12 (1)).

As we said in the introduction, we want to work with generalized amalgamation properties and using existence of models of cardinality less than λ in the class K to construct models of cardinality λ; for this we shall use existence of generalized amalgamation properties (we have not yet defined them). We know already from what was said in the introduction that our amalgamation properties will depend on two parameters: a cardinality κ and a natural number n (n is the dimension of the diagram, and κ the cardinality of the models in the diagram).

We shall want to find a condition on the models in the diagram which will imply existence of a model M and a possibility to embed the other models in the diagram into M. A sufficient condition is, for example, existence of a universal model over the union of the models in the diagram. For every cardinality κ we shall look for an assumption which will imply existence of such a model; fortunately, in §5 in Theorems 5.1 and 5.2, we shall prove that if generalized amalgamation holds for every $\kappa<\lambda$ and for every $k\leqq n+1$, then the property holds for λ and n. Therefore it suffices to prove the property for $\aleph_0$ and every natural number. So instead of taking care of existence of a universal model over an arbitrary set, it is enough to have a condition which implies existence of a universal model over countable sets; but this does not hold generally. For this goal we shall introduce in Definition 2.1 the suitable notion which we shall call a good set (not necessarily for countable sets). In Lemma 2.2 and Claims 2.3, 2.4 we shall prove some properties of good sets; for example, we conclude for a countable set A that: A is good iff there exists a countable model N_A, $N_A \supseteq A$ which is universal over A. Conclusion 2.5 (1) is another characterization of good sets (for countable sets); Conclusion 2.5(2) has technical value. Definition 2.7 is the important definition of a full model (the substitute for the use of saturated model in Morley's theorem), where the second requirement (homogenicity) follows from the others; this trivial fact is proved in Lemma 2.8.

What is our aim? We want to have theorems, for example, of the form: *If* there exists a model of cardinality λ (in the class K) + an additional assumption, *then* there exists a model of cardinality λ^+. For example, the main result of [11] is of this form; we proved it there for $\psi \in L_{\omega_1,\omega}$ If $I(\aleph_1, \psi) \geqq 1$ and $I(\aleph_1, \psi) < 2^{\aleph_1}$ then $I(\aleph_2, \psi) \geqq 1$. We shall reprove this result in Conclusion 2.13 (when, with the exception of demonstrating our method, we gain from the point of view of the set theoretical axioms we use — the only additional axiom to ZFC is $2^{\aleph_0} < 2^{\aleph_1}$); it is an open problem whether the assumption of $2^{\aleph_0} < 2^{\aleph_1}$ is necessary.

How shall we prove Conclusion 2.13? Since the class K is closed under the union of elementary chains, it is clear that it suffices to prove that every model of cardinality $\aleph_1$ has a proper elementary extension. Let M be the given model from K of cardinality $\aleph_1$, let $\{M_\alpha : \alpha < \omega_1\}$ be countable models from K such that $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and we want to construct a model $N \in K$ by constructing a continuous increasing elementary chain of countable models in K, N_α $(\alpha < \omega_1)$ such that N_0 is a proper elementary extension of M_0 not included in M, and choose by induction $N_{\alpha+1}$ to be an elementary extension of $M_{\alpha+1}$ and N_α.

Now we are left with just one problem: What implies the possibility of choosing the models N_α as above? To overcome this difficulty we define in Definition 2.9(2) under what conditions a triple of sets A, B, C is in stable amalgamation (think temporarily of these sets as models, A is an elementary submodel of C and $B \supseteq A$). In Lemma 2.10(2) we have the existence we wanted (substitute A by any atomic countable model which includes $M_{\alpha+1} \cup N_\alpha$ [exists by the amalgamation property]; for B choose a proper countable elementary extension of A in K; $N_{\alpha+1}$ will be $f(B)$ and C is M). Later we shall use this triple in more general situations. This is the reason for not requiring A, B, C to be models in the definition.

The other facts proved in this section make possible our induction on $\alpha < \omega_1$, to take care of limit ordinals, and we show the relation between full models and stable amalgamation which will be used in later sections.

From now on, for the rest of the paper, we assume

HYPOTHESIS. The conclusions of Theorem 1.4 hold.

DEFINITION 2.1. A set A ($\subseteq \mathfrak{C}$, of course, and atomic) is called *good* if $\bar{a} \in A$, $\models (\exists \bar{x})\varphi(\bar{x};\bar{a})$ implies $\varphi(\bar{x};\bar{a})$ belongs to a complete type over A which is isolated. For any A let $D_A = \{\mathrm{tp}(\bar{a}, A) : A \cup \bar{a}$ is atomic$\}$.

LEMMA 2.2. *Let A be good.*

(0) *If A is countable then there is a countable primary model over A.*

(1) $p \in D_A$ iff *there are $\bar{b}$, $\bar{c}$, $\mathrm{tp}(\bar{b}, A)$ isolated, $\mathrm{tp}(\bar{c}, A \cup \bar{b})$ is a stationarization of $\mathrm{tp}(\bar{c}, \bar{b})$ and $p = \mathrm{tp}(\bar{a}, A)$ for some $\bar{a} \subseteq \bar{b} \cup \bar{c}$. In fact we can have $\bar{a} = \bar{c}$. If $A \subseteq M$, $\bar{a} \in M$ realizes p, then we can choose $\bar{b}$, $\bar{c} \in M$. Also every $p \in D_A$ does not split over a finite subset of A.*

(2) *For each $\bar{a}$, if $A \cup \bar{a}$ is atomic, then $A \cup \bar{a}$ is good.*

(3) $|D_A| \leq |A| + \aleph_0$ *and D_A has the density and amalgamation properties (density — if $\models (\exists \bar{x})\, \varphi(\bar{x}, \bar{a})$ then $\varphi(\bar{x}, \bar{a}) \in p$ for some $p \in D_A$; amalgamation — if $\mathrm{tp}(\bar{a}_0{}^\wedge \bar{b}_0, A) \in D_A$, $\mathrm{tp}(\bar{a}_1{}^\wedge \bar{c}_1, A) \in D_A$, and $\mathrm{tp}(\bar{a}_0, A) = \mathrm{tp}(\bar{a}_1, A)$, then for some $\bar{b}_1$, $\mathrm{tp}(\bar{a}_1{}^\wedge \bar{b}_1{}^\wedge \bar{c}_1, A) \in D_A$ and $\mathrm{tp}(\bar{a}_0{}^\wedge \bar{b}_0, A) = \mathrm{tp}(\bar{a}_1{}^\wedge \bar{b}_1, A)$).*

(4) *If A is countable, there is a countable model N_A which is $(D_A, \aleph_0)$-homogeneous over A (i.e., each $p \in D_A$ is realized, and even for every $\bar{a} \in N_A$, each $p \in D_{A \cup \bar{a}}$ is realized). This model is unique, and universal over A (i.e., every N, $A \subseteq N$, $\|N\| \leq \aleph_0$, can be elementarily embedded into N_A over A). Of course for each $\bar{a} \in N_A$, $\mathrm{tp}(\bar{a}, A) \in D_A$.*

We first prove

CLAIM 2.3. *For a countable A, if over A there is a countable universal model N_A, then D_A is countable.*

CLAIM 2.4. *If D_A is countable then A is good.*

PROOF OF CLAIM 2.3. For every $p \in D_A$, choose $\bar{a}$ realizing it; $A \cup \bar{a}$ is countable and atomic, hence there is a countable N, $A \cup \bar{a} \subseteq N$, so N can be elementarily embedded into N_A over A, so N_A realizes p. Hence $|D_A| \leq \|N_A\| \leq \aleph_0$.

PROOF OF CLAIM 2.4. If A is not good and $\varphi(\bar{x}, \bar{a})$ exemplifies this, let $A = \{a_n : n < \omega\}$, and define $\varphi_\eta(\bar{x}, \bar{a}_\eta)$ for $\eta \in {}^{\omega >}2$ by induction on $l(\eta)$ such that $\varphi_{\langle\,\rangle}(\bar{x}; \bar{a}_{\langle\,\rangle}) = \varphi(\bar{x}; \bar{a})$, $\models (\exists \bar{x})\varphi_\eta(\bar{x}, \bar{a}_\eta)$, $\bar{a}_\eta \in A$,

$$\models (\forall \bar{x})[\varphi_{\eta^\wedge\langle l\rangle}(\bar{x}; \bar{a}_{\eta^\wedge\langle l\rangle}) \rightarrow \varphi_\eta(\bar{x}; \bar{a}_\eta)] \qquad \text{for } l = 0, 1$$

and $\models \neg(\exists \bar{x})[\varphi_{\eta\char`^\langle 0\rangle}(\bar{x}, \bar{a}_{\eta\char`^\langle 0\rangle}) \wedge \varphi_{\eta\char`^\langle 1\rangle}(\bar{x}, \bar{a}_{\eta\char`^\langle 1\rangle})]$, and $\varphi_\eta(\bar{x}, \bar{y}_\eta)$ isolate a type in $D_\varnothing$; and $a_l \in \bar{a}_\eta$ when $l < l(\eta)$. This is easy and if $\bar{a}_\eta$ realizes $\{\varphi_{\eta\restriction n}(\bar{x}, \bar{a}_{\eta\restriction n}) : n < \omega\}$, for $\eta \in {}^\omega 2$, then $\{\mathrm{tp}(\bar{a}_\eta, A) : \eta \in {}^\omega 2\}$ is a subset of D_A of cardinality $2^{\aleph_0}$, so $|D_A| = 2^{\aleph_0}$.

Proof of Lemma 2.2. First we assume that A is countable and at the end of the proof of this lemma we shall explain how to get rid of this assumption.

(0) Let T_A be the first-order theory T expanded by constants c_a for every $a \in A$.

Since A is good, T_A is an atomic theory; now apply Henkin's omitting types theorem and obtain a countable atomic model M'_A of the theory T_A. By the definition of T_A, the reduct M_A of M'_A to $L(T)$ is atomic over A. Since M_A is atomic over A and countable by Theorem 1.6(2) it is primary over A.

(1) Suppose $p \in D_A$, then for some $\bar{a}$, $p = \mathrm{tp}(\bar{a}, A)$ and $A \cup \bar{a}$ is atomic. As $A \cup \bar{a}$ is atomic there is a model M, $A \cup \bar{a} \subseteq M$. As A is good, by Lemma 2.2(0), there is a model M' primary over A. By Theorem 1.6(1), M' is prime over A hence can be embedded into M over A hence w.l.o.g. $A \subseteq M' \subseteq M$. Let $\bar{a} \subseteq \bar{c} \subseteq M$, then $M' \cup \bar{c}$ is atomic, so for the type $\mathrm{tp}(\bar{c}, M')$ the rank is defined and it is stationary. Choose a finite $B \subseteq M'$ such that $R[\mathrm{tp}(\bar{c}, M')] = R[\mathrm{tp}(\bar{c}, B)]$. By Theorem 1.6(2) we know that $\mathrm{tp}(\bar{b}, A)$ is isolated ($\bar{b}$ is an enumeration of B), and by Theorem 1.4(1) (b) $\mathrm{tp}(\bar{c}, \bar{b})$ is stationary. By the monotonicity of the rank (Theorem 1.4(2) (a)), $\mathrm{tp}(\bar{c}, A \cup \bar{b})$ is the stationarization of $\mathrm{tp}(\bar{c}, \bar{b})$. So we have the "only if" direction.

For the "if" part assume $\bar{a} \subseteq \bar{b} \cup \bar{c}$, $\mathrm{tp}(\bar{b}, A)$ isolated and $\mathrm{tp}(\bar{c}, A \cup \bar{b})$ is a stationarization of $\mathrm{tp}(\bar{c}, \bar{b})$. As $\mathrm{tp}(\bar{b}, A)$ is isolated, $A \cup \bar{b}$ is atomic. As $\mathrm{tp}(\bar{c}, A \cup \bar{b})$ is a stationarization of $\mathrm{tp}(\bar{c}, \bar{b})$, by Theorem 1.4(1) (b) (second sentence) $(A \cup \bar{b}) \cup \bar{c}$ is atomic. This implies $A \cup \bar{a} \subseteq A \cup (\bar{b} \cup \bar{c}) = (A \cup \bar{b}) \cup \bar{c}$ is atomic, hence $\mathrm{tp}(\bar{a}, A) \in D_A$.

Now let $p \in D_A$, assume $p = \mathrm{tp}(\bar{a}, A)$ and we shall find a finite $B \subseteq A$ such that p does not split over B. Let M_A be primary over A, since $A \cup \bar{a}$ is atomic we can choose $M \supseteq A \cup \bar{a}$; without loss of generality we may assume that $M_A \subseteq M$. By Theorem 1.4(1)(b) there exist a finite $C \subseteq M_A$ such that $\mathrm{tp}(\bar{a}, M_A)$ is stationary over C; by the second part of Theorem 1.4(1)(b), $\mathrm{tp}(\bar{a}, M_A)$ does not split over C. Since $\bar{c} \in M_A$ ($\bar{c}$ an enumeration of C) by the last part of Theorem 1.6(2) there exist a finite $B \subseteq A$ and a formula $\varphi(\bar{x}, \bar{y})$ such that $\varphi(\bar{x}, \bar{b})$ isolates $\mathrm{tp}(\bar{c}, A)$, $\bar{b}$ is an enumeration of B; this clearly shows that $\mathrm{tp}(\bar{c}, A)$ does not split over $\bar{b}$. We claim that p does not split over B. Let $\overline{d_1}, \overline{d_2} \in A$ be such that $\mathrm{tp}(\overline{d_1}, B) = \mathrm{tp}(\overline{d_2}, B)$ and $\psi(\bar{x}, \overline{d_1})$, $\neg\psi(\bar{x}, \overline{d_2}) \in p$; since $\mathrm{tp}(\bar{c}, A)$ does not split over B we have $\mathrm{tp}(d_1, C) = \mathrm{tp}(d_2, C)$. But since p is stationary over C, p does not split over C hence $\psi(\bar{x}, \overline{d_1}) \in p\langle -- \rangle$, $\psi(\bar{x}, \overline{d_2}) \in p$, contradiction.

(2) We want to apply Claim 2.4 so we have to prove that $D_{A \cup \bar{a}}$ is countable, but first we prove the countability of D_A and this will follow from Claim 2.3 provided we will be able to prove the existence of a countable universal model over A. By goodness of A take M' countable primary over A; let N_A be a universal countable model over M' (exists by Theorem 1.4(2)). We claim that N_A is the model we want. Let $N \supseteq A$ be an arbitrary countable model. Using Theorem 1.6(1) we can find $N' \subseteq N$ primary over A which is isomorphic over A to M', say, by $f: N' \cong_A M'$. As N_A is universal over M' we can extend f to an embedding of N into N_A. So we have proved that D_A is countable but we are interested in the countability of $D_{A \cup \bar{a}}$, and this is true because we can define a one-to-one mapping from $D_{A \cup \bar{a}}$ into D_A as follows: By the atomicity of $A \cup \bar{a}$ and the definition of D_A for every $\bar{b}$ the following holds:

$$\mathrm{tp}(\bar{b}, A \cup \bar{a}) \in D_{A \cup \bar{a}} \Leftrightarrow \mathrm{tp}(\bar{a}^\wedge \bar{b}, A) \in D_A.$$

(3) As before assume that A is countable. $|D_A| \leqq |A| + \aleph_0$ follows from the proof of (2) and Claim 2.3. The density is easy also; if $\models \exists \bar{x} \varphi[\bar{x}, \bar{a}]$, $\bar{a} \in A$, by Definition 2.1 for some $\bar{b}$, $\models \varphi[\bar{b}, \bar{a}]$ and $\mathrm{tp}(\bar{b}, A)$ is isolated, but this implies $A \cup \bar{b}$ is atomic, hence $\mathrm{tp}(\bar{b}, A) \in D_A$. The amalgamation: as $\mathrm{tp}(\bar{a}_0{}^\wedge \bar{b}_0, A) \in D_A$ by D_A's definition, $A \cup \bar{a}_0{}^\wedge \bar{b}_0$ is atomic, A is good by assumption, so by (2) also $A \cup \bar{a}_0{}^\wedge \bar{b}_0$ is good, and let M_0 be the countable primary model over $A \cup \bar{a}_0{}^\wedge \bar{b}_0$; by the same argument let M_1 be the countable primary model over $A \cup \bar{a}_1{}^\wedge \bar{c}_1$. Let $N_l < M_l$ be primary models over $A \cup \bar{a}_l$ and let M_l^* be the countable universal models over M_l. Choose f_0 an elementary mapping over A taking $\bar{a}_0$ to $\bar{a}_1$ (this can be done by equality of types over A). As N_l is primary over $A \cup \bar{a}_l$ it is possible to extend f_0 to an isomorphism f_1 from N_0 onto N_1. Now use the fact that K has the $\aleph_0$-amalgamation property (by Theorem 1.4(2)): M_0 and M_1 can be amalgamated over f_1. By the universality of M_1^*, M_0 can be embedded by an extension of f_1 into M_1^*. Let $f_2 \supseteq f_1$ be the embedding of M_0 into M_1^*, denote $\bar{b}_1 = f_2(\bar{b}_0)$ and this sequence has the required properties.

(4) This follows directly from (3).

Up to now we have dealt with countable A. For uncountable A note that we can define on A $\aleph_0$ functions such that any $B \subseteq A$ closed under those functions is good and (B, A) satisfies the Tarski–Vaught condition (see Definition 2.6 below). We can also find for each pair of formulas $\varphi(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{z})$ functions $F_l(\bar{y})$ $(l < l(\bar{z}))$ such that if $\bar{b} \in A$, $\models (\exists \bar{x})\ \varphi(\bar{x}, \bar{b})$ and there is $\bar{b}$ such that $\models (\exists \bar{x})$

$(\varphi(\bar{x},\bar{a}) \wedge \psi(\bar{x},\bar{b}))$, $\models (\exists \bar{x})(\varphi(\bar{x},\bar{a}) \wedge \neg \psi(\bar{x},\bar{b}))$ then $\bar{b}' = \langle F_0(\bar{a}), \cdots, F_{l(\bar{z})-1}(\bar{a})\rangle$ satisfies this. If $A' \subseteq A$ is closed under all those functions A is not good, then A is not good. Similarly for the other properties. Now it is easy.

CONCLUSION 2.5. (1) For countable A, A is good iff D_A is countable.

(2) If p has dimension κ for (A_1, A_2, A_3), $A_1 \cup A_3 \subseteq A_2$ and A_2 is good and $\subseteq A$, *then* p has dimension κ for (A_1, A, A_3).

(3) Suppose $\boldsymbol{I}$ is indiscernible over B in A based on p. If $B \cup \boldsymbol{I} \subseteq A \subseteq C$ and A is a good set then $\boldsymbol{I}$ is indiscernible over B in C. Also if $\boldsymbol{J} \subseteq \boldsymbol{I}$, $B \cup \boldsymbol{J} \subseteq C \subseteq A$ and C is a good set, then $\boldsymbol{J}$ is indiscernible over B in C.

PROOF. (1) Easy, by Lemma 2.2(3) and Claim 2.4.

(2) So let $C \subseteq A$, $|C| < \kappa$, $p \in S^m(B)$, B be a finite subset of A_2, p stationary, and we should prove that the stationarization of p over $B \cup A_1 \cup C$ is realized in A_3 (this proves the dimension is $\geqq \kappa$; it is obvious that the dimension is $\leqq \kappa$). As A_2 is good, for every finite sequence $\bar{d} \in A$, we can find $\bar{d}_0 \in A_2$, $\bar{d}_1$, $\bar{d}_2$ such that: $\operatorname{tp}(\bar{d}_1, \bar{d}_0) \vdash \operatorname{tp}(\bar{d}_1, A_2)$, $\operatorname{tp}(\bar{d}_2, A_2 \cup \bar{d}_1)$ is a stationarization of $\operatorname{tp}(\bar{d}_2, \bar{d}_1)$, $\bar{d} \subseteq \bar{d}_1 \cup \bar{d}_2$ (and of course $A \cup \bar{d}_1 \cup \bar{d}_2$ is atomic).

Let C^* be $\bigcup\{\bar{d}_0 : \bar{d} \in C\}$, if $\kappa > \aleph_0$, and $C^* = \bar{d}_0$ where $C = \bar{d}$ (i.e., the set is equal to the range of the sequence). Now $C^* \subseteq A_2$, $|C^*| < \kappa$, hence by the hypothesis, the stationarization of p over $B \cup A_1 \cup C^*$, q, is realized by some $\bar{b} \in A_3$. We shall prove that $\operatorname{tp}(\bar{b}, A_1 \cup B \cup C^* \cup C)$ is a stationarization of p, thus finishing. It suffices to prove that if $\bar{d} \in C$, $\kappa = \aleph_0 \Rightarrow \bar{d} = \bar{d}^*$, then $\operatorname{tp}(\bar{b}, A_1 \cup B \cup C^* \cup \bar{d})$ is a stationarization of p, or equivalently, of q. As $\bar{d} \subseteq \bar{d}_1 \cup \bar{d}_2$, $\bar{d}_0 \subseteq C^*$, it suffices to prove for $l = 0, 1$ that $\operatorname{tp}(\bar{b}, A_1 \cup B \cup C^* \cup \bar{d}_l \cup \bar{d}_{l+1})$ is the stationarization of p. For $l = 0$, remember that $\operatorname{tp}(\bar{d}_1, \bar{d}_0) \vdash \operatorname{tp}(\bar{d}_1, A_2)$ hence (as $\bar{d}_0 \in C^*$, $\bar{b} \in A_3 \subseteq A_2$) $\operatorname{tp}(\bar{d}_1, \bar{d}_0) \vdash \operatorname{tp}(\bar{d}_1, A_1 \cup B \cup C^* \cup \bar{b})$ hence $\operatorname{tp}(\bar{d}_1, A_1 \cup B \cup C^*) \vdash \operatorname{tp}(\bar{d}_1, A_1 \cup B \cup C^* \cup \bar{b})$ hence $\operatorname{tp}(\bar{b}, A_1 \cup B \cup C^*) \vdash \operatorname{tp}(\bar{b}, A_1 \cup B \cup C^* \cup \bar{d}_1)$. So $\operatorname{tp}(\bar{b}, A_1 \cup B \cup C^*)$ has a unique extension in $S^m(A_1 \cup B \cup C^* \cup \bar{d}_1)$, but p has a stationarization there, r, so $r \restriction (A_1 \cup B \cup C^*)$ is also a stationarization of p (by monotonicity of rank), hence by uniqueness $q \subseteq r$, hence $r = \operatorname{tp}(\bar{b}, A_1 \cup B \cup C^* \cup \bar{d}_1)$ (we used Theorem 1.4). So $\operatorname{tp}(\bar{b}, A_1 \cup B \cup C^* \cup \bar{d}_1)$ is a stationarization of p.

For $l = 1$ the proof is by the symmetry property (see Theorem 1.4 (1) (c)).

(3) A similar proof.

DEFINITION 2.6. The pair (A, B) satisfies the Tarski–Vaught condition (or B satisfies the Tarski–Vaught condition over A) *if* for every $\bar{b} \in B$, $\bar{a} \in A$ if $\models \varphi[\bar{b}, \bar{a}]$ then for some $\bar{b}' \in A$, $\models \varphi[\bar{b}', \bar{a}]$ (when B is atomic this is equivalent to

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$\mathrm{tp}(\bar{b}\,^\wedge \bar{a}, \phi) = \mathrm{tp}(\bar{b}'\,^\wedge \bar{a}, \phi)$). Notice (A, B) satisfies the Tarski–Vaught condition iff $(A, A \cup B)$ satisfies it.

DEFINITION 2.7. M will be called full over A (or the pair (M, A) is full) if

(1) A is good, $A \subseteq M$;

(2) $(M, a)_{a \in A}$ is $(D_A, \aleph_0)$-homogeneous $=$ M is $(D_A, \aleph_0)$-homogeneous over A [i.e., if $\bar{a}_0, \bar{a}_1, \bar{b}_0 \in M$, $\mathrm{tp}(\bar{a}_0, A) = \mathrm{tp}(\bar{a}_1, A)$ then for some $\bar{b}_1 \in M$, $\mathrm{tp}(\bar{a}_0\,^\wedge \bar{b}_0, A) = \mathrm{tp}(\bar{a}_1\,^\wedge \bar{b}_1, A)$ and every $p \in D_A$ is realized in M];

(3) M is weakly full over A, where:

(a) M is weakly λ-full over A if $A \subseteq M$, and for every stationary $p \in S^m(B)$, $B \subseteq M$, B finite, the dimension of p for $(A, |M|, |M|)$ is $\geqq \lambda$,

(b) if $\lambda = \|M\|$ we omit it.

DEFINITION 2.7A. "M is λ-full over A" if M is weakly λ-full over A and (1), (2) above holds.

REMARK. Weakly λ-full is a substitute for λ-saturated; but we shall not pursue this notion for its own sake here.

LEMMA 2.8. (1) *If (A, B), (B, C) satisfies the Tarski–Vaught condition, and $A \subseteq B$ then (A, C) satisfies the Tarski–Vaught condition.*

(2) *If A_i $(i < \alpha)$ is increasing and (B, A_i) $[(A_i, B)]$ satisfies the Tarski–Vaught condition for each i then $(B, \bigcup_{i<\alpha} A_i)$ $[(\bigcup_{i<\alpha} A_i, B)]$ satisfies the Tarski–Vaught condition.*

(3) *(M, A) is full iff A is good and (M, A) is weakly full.*

(4) *If M is weakly full over A, $B \subseteq A$ then M is weakly full over B. If M is full over A, $B \subseteq A$, B good then M is full over B.*

PROOF. (1), (2), (4) trivial.

(3) The implication $\Rightarrow$ is trivial. As for $\Leftarrow$, from Definition 2.7 we miss only part (2) and this is easy by the characterization of D_A in Lemma 2.2 (1).

DEFINITION 2.9. (1) If (A, C) satisfies the Tarski–Vaught condition, $p \in D_C$ is the weak stationarization of $q \in D_A$ if $q \subseteq p$, and p does not split over some finite subset of A (see Lemma 2.2 (1)).

(2) The triple A, B, C is in *stable amalgamation* if $A \cup B \cup C$ is atomic (but this follows from the rest), A is good, (A, C) satisfies the Tarski–Vaught condition, and for each $\bar{b} \in B$, $\mathrm{tp}(\bar{b}, C \cup A)$ is the weak stationarization of $\mathrm{tp}(\bar{b}, A)$.

REMARK. Note that A, B, C is in stable amalgamation iff A, $A \cup B$, $A \cup C$ is in stable amalgamation.

LEMMA 2.10. *If A is good and (A, C) satisfies the Tarski–Vaught condition, then*:

(1) *Any $q \in D_A$ has a unique weak stationarization $p \in D_{C \cup A}$, and if q is isolated so is p, if q is stationary then p is its stationarization over $A \cup C$. If $q = \mathrm{tp}(\bar{a} \,^\wedge \bar{b}, A)$, $\mathrm{tp}(\bar{a}, A)$ isolated, $\mathrm{tp}(\bar{b}, A \cup \bar{a})$ the stationarization of $\mathrm{tp}(\bar{b}, \bar{a})$, $p = \mathrm{tp}(\bar{a}' \,^\wedge \bar{b}', A \cup C)$, then $\mathrm{tp}(\bar{b}', A \cup \bar{a}')$ is the stationarization of $\mathrm{tp}(\bar{b}', \bar{a}')$. If q does not split over $B \subseteq A$, B finite, then p also does not split over B (i.e., we strengthen the "some" in Definition 2.9(1)).*

If $q_0 = \mathrm{tp}(\bar{a} \,^\wedge \bar{b}, A)$, $q_1 = \mathrm{tp}(\bar{a}, A)$, $p_l \in D_C$, their respective weak stationarizations, $p_0 = \mathrm{tp}(\bar{a}' \,^\wedge \bar{b}', C)$, then $p_1 = \mathrm{tp}(\bar{a}', C)$. If A_i $(i < \alpha)$ is increasing continuous, each A_i $(i + 1 < \alpha)$ is good, $p_i = \mathrm{tp}(\bar{a}, A_i)$, p_{i+1} a weak stationarization of p_i (so (A_i, A_{i+1}) satisfies the Tarski–Vaught condition), then $\bigcup_{i<\alpha} p_i$ is a weak stationarization of p_0.

(2) *For any B, $A \subseteq B$, there is an elementary mapping f, $f \restriction A = \mathrm{id}$, $B = \mathrm{Dom} f$, such that A, $f(B)$, C is in stable amalgamation.*

(3) *If A, M, C is in stable amalgamation $(A \subseteq M, A \subseteq C)$, B a finite subset of M, $p \in S^m(B)$ stationary, and the dimension of p for (A, M, M) is κ,* then *the dimension of p for $(C, M \cup C, M)$ is κ.*

(4) *For A, M, C as above, if $C^* \subseteq M$, $\bar{a} \in M$, $\mathrm{tp}(\bar{a}, A \cup C^*)$ is stationary, then $\mathrm{tp}(\bar{a}, A \cup C \cup C^*)$ is its stationarization.*

PROOF. (1) Let $q \in D_A$ and we shall define a type p with the required properties.

Existence: Since A is good by Lemma 2.2(1) there is $B \subseteq A$ finite such that q does not split over B. Define p to be the following set:

$$p = \{\phi(\bar{x}; \bar{c}) : \bar{c} \in C \cup A, \bar{a} \in A, \mathrm{tp}(\bar{c}, B) = \mathrm{tp}(\bar{a}, B), \phi(\bar{x}; \bar{a}) \in q\}.$$

The consistency of p is proved exactly in the same form as the consistency of p_C in Theorem 1.4(1) (b) using the fact that $(A, A \cup C)$ satisfies the Tarski–Vaught condition; it is obvious that p does not split over B so p is a weak stationarization of q over $A \cup C$.

Uniqueness: Let p_1, p_2 be two distinct weak stationarizations from $D_{A \cup C}$ of q; let $\phi(\bar{x}; \bar{c}) \in p_1$ and $\neg\phi(\bar{x}; \bar{c}) \in p_2$. By the choice of p_1, p_2 there exists B_1, $B_2 \subseteq A$ finite such that they do not split over them, respectively. Choose $\bar{a} \in A$ such that $\mathrm{tp}(\bar{a}, B_1 \cup B_2) = \mathrm{tp}(\bar{c}, B_1 \cup B_2)$. By $\phi(\bar{x}; \bar{c}) \in p_2$, $\neg\phi(\bar{x}; \bar{a})$ cannot belong to q (otherwise p_1 would split over $B_1 \cup B_2$), by the same argument

$\neg\phi(\bar{x};\bar{c})\in p_2 \Rightarrow \phi(\bar{x};\bar{a})\notin q$; together this is a contradiction to completeness of q.

Isolation: Assume that q is isolated by the formula $\phi_0(\bar{x};\bar{a})$ and we shall show that the same formula isolates the type p. Let $p=\operatorname{tp}(\bar{b},A)$ and $q=\operatorname{tp}(\bar{b},A\cup C)$, assume that ϕ_0 do not isolate p, hence there exists a $\phi(\bar{x};\bar{c})\in p$ such that the following holds:

$$(*)\qquad \vDash\exists\bar{x}[\phi_0(\bar{x};\bar{a})\wedge\phi(\bar{x};\bar{c})]\wedge\exists\bar{x}[\phi_0(\bar{x};\bar{a})\wedge\phi(\bar{x};\bar{c})].$$

As (A,C) satisfies the Tarski–Vaught condition, and $\bar{a}\in A$, $\bar{c}\in C$, there is $\bar{c}'\in A$ such that

$$(**)\qquad \vDash(\exists\bar{x})[\varphi_0(\bar{x},\bar{a})\wedge\varphi(\bar{x},\bar{c}')]\wedge(\exists\bar{x})[\varphi_0(\bar{x},\bar{a})\wedge\neg\varphi(\bar{x},\bar{c}')].$$

As $\bar{c}'\in A$ either $\varphi(\bar{x},\bar{c}')\in q$ [hence $\varphi(\bar{x},\bar{a})\vdash\varphi(x,c')$] or $\neg\varphi(\bar{x},\bar{c}')\in q$ [hence $\varphi_0(\bar{x},\bar{a})\vdash\neg\varphi(x,c')$], contradicting $(**)$ in both cases.

Stationarity: Now assume q is stationary and let p be weak stationarization over $A\cup C$. On the other hand, by Theorem 1.4(1) (b) the stationarization p_1 of q over $A\cup C$ exists and it does not split over some finite subset of A. Hence it is also a weak stationarization of q and by the uniqueness of the weak stationarization $p=p_1$.

If q does not split over finite $B\subseteq A$ then also p does not split over B: this follows from the uniqueness of weak stationarization and the first paragraph in this proof. The other claims are easy to prove.

(2) Let $B=\{b_i:i<\alpha\}$, and for every $i(0),\cdots,i(n)<\alpha$ let $q_{i(0),\cdots,i(n)}(x_{i(0)},\cdots,x_{i(n)})$ be the weak stationarization of $\operatorname{tp}(\langle b_{i(0)},\cdots,b_{i(n)}\rangle,A)$ over $A\cup C$. By (1), $\{i(0),\cdots,i(n)\}\subseteq\{j(0),\cdots,j(m)\}\subseteq\alpha$ implies

$$q_{i(0),\cdots,i(n)}(x_{i(0)},\cdots,x_{i(n)})\subseteq q_{j(0),\cdots,j(m)}(x_{j(0)},\cdots,x_{j(m)}).$$

Hence $\Gamma\overset{\text{df}}{=}\bigcup\{q_{i(0),\cdots,i(n)}(x_{i(0)},\cdots,x_{i(n)}):i(0),\cdots,i(n)<\alpha,n<\omega\}$ is a complete type in the variables $\{x_i:i<\alpha\}$ over $A\cup C$. Let the assignment $x_i\to b'_i$ satisfy Γ and let f be an elementary mapping which is the identity over A and maps $\bar{b}$ to $\bar{b}'$. It is clear that A, $f(B)$, C are in stable amalgamation.

(3) As the dimension of p for (A,M,M) is κ, clearly the dimension of p for $(C,M\cup C,M)$ is $\leqq\kappa$. For the other direction, let $C^*\subseteq M$, $B\subseteq C^*$, $|C^*|<\kappa$ and it suffices to prove that the stationarization of p over $C\cup C^*$ is realized in M.

As the dimension of p for (A,M,M) is κ, there is $\bar{b}\in M$ which realizes the stationarization of p over $A\cup C^*$. Suppose $\operatorname{tp}(\bar{b},C\cup C^*)$ is not a stationarization of p. Then there are $\bar{c}\in C$ and $\bar{c}^*\in C^*\cup A$ such that $\operatorname{tp}(\bar{b},\bar{c}{}^\frown\bar{c}^*)$ is not a stationarization of p, but $B \subseteq \bar{c}^*$. As A, M, C is in stable amalgamation, for some $\bar{a} \in A$, $\mathrm{tp}(\bar{b}^\wedge \bar{c}^*, C)$ does not split over $\bar{a}$. Also (A, C) satisfies the Tarski–Vaught condition hence there is $\bar{c}' \in A$ which realizes $\mathrm{tp}(\bar{c}, \bar{a})$, so by definition of non-splitting:

$$(*) \qquad \mathrm{tp}(\bar{b}^\wedge \bar{c}^{*\wedge} \bar{c}, \bar{a}) = \mathrm{tp}(\bar{b}^\wedge \bar{c}^{*\wedge} \bar{c}', \bar{a}).$$

As $\bar{c}' \in A$, $\mathrm{tp}(\bar{b}, A \cup C^*)$ is the stationarization of p, $B \subseteq \bar{c}^*$, also $\mathrm{tp}(\bar{b}, \bar{c}' \cup \bar{c}^*)$ is the stationarization of p, so by (*), $\mathrm{tp}(\bar{b}, \bar{c} \cup \bar{c}^*)$ is the stationarization of p, contradicting the choice of $\bar{c}$, $\bar{c}^*$; hence $\mathrm{tp}(\bar{b}, C \cup C^*)$ is the stationarization of p, so we finish.

(4) The same proof as (3).

LEMMA 2.11. (1) *If A_i is increasing and continuous $(i < \alpha)$ and the triple A_i, B, A_{i+1} is (when $i+1 < \alpha$) in stable amalgamation*, then *so is A_0, B, $\bigcup_{i<\alpha} A_i$.*

(2) *If A, B, C is in stable amalgamation $B' \subseteq B$, $C' \subseteq C$, then A, B', C' is in stable amalgamation.*

(3) *If A_i $(i < \alpha)$ is increasing and continuous, similarly B_i, C_i $(i < \alpha)$, and for each i the triple A_i, B_i, C_i is in stable amalgamation*, then *so is $\bigcup_{i<\alpha} A_i$, $\bigcup_{i<\alpha} B_i$, $\bigcup_{i<\alpha} C_i$, provided that $\bigcup_{i<\alpha} A_i$ is good.*

PROOF. (1) Let $\bar{b} \in B$. For each i, $\mathrm{tp}(\bar{b}, A_{i+1})$ is a weak stationarization of $\mathrm{tp}(\bar{b}, A_i)$. By Lemma 2.8(2), $(A_0, \bigcup_{i<\alpha} A_i)$ satisfies the Tarski–Vaught condition and by Lemma 2.10(1), $\mathrm{tp}(\bar{b}, \bigcup_{i<\alpha} A_i)$ is the weak stationarization of $\mathrm{tp}(\bar{b}, A_0)$ and A_0 is obviously good.

(2) Obvious.

(3) $\bigcup_{i<\alpha} A_i$ is good by hypothesis.

$(\bigcup_i A_i, \bigcup_i C_i)$ satisfies the Tarski–Vaught condition because if $\bar{a} \in \bigcup_i A_i$, $\bar{c} \in \bigcup_i C_i$, as A_i, C_i are increasing, for some i, $\bar{a} \in A_i$, $\bar{c} \in C_i$, so we can find $\bar{c}' \in A_i \subseteq A$ as required.

Let $\bar{b} \in \bigcup_i B_i$, so for some i_0, $\bar{b} \in B_{i_0}$, hence $i_0 \leqq j < \alpha$ implies: $\mathrm{tp}(\bar{b}, C_j \cup A_j)$ is a weak stationarization to $\mathrm{tp}(\bar{b}, A_j)$. As $\bigcup_j A_j$ is good by Lemma 2.2(1) for some finite $B^* \subseteq \bigcup_j A_j$, $\mathrm{tp}(\bar{b}, \bigcup_i A_i)$ does not split over B^*. By change of notation we can assume $B^* \subseteq A_{i_0}$; so $\mathrm{tp}(\bar{b}, A_j)$ does not split over $B^* \subseteq A_j$ $(i_0 \leqq j < \alpha)$ hence (by Lemma 2.10(1)) $\mathrm{tp}(\bar{b}, C_j \cup A_j)$ does not split over B^*. As this holds for every j, $\mathrm{tp}(\bar{b}, \bigcup_j C_j \cup \bigcup_j A_j)$ does not split over B^*, hence is a weak stationarization of $\mathrm{tp}(\bar{b}, \bigcup_j A_j)$.

LEMMA 2.12. *Suppose A_i, M_i $(i < \delta)$ are increasing and continuous, and for each $i < \delta$, $A_i \subseteq M_i$, A_i, M_i, A_{i+1} is in stable amalgamation*, then:

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(1) $A = \bigcup_{i<\delta} A_i$ *is good (note that each* A_i *is good), and* A_i, M_i, A *is in stable amalgamation. If in addition* $A_{i+1} \cup M_i$ *is a good set for every* i *then* $A \cup M_i$ *is good as well as* $A_j \cup M_i$;

(2) $M = \bigcup_{i<\delta} M_i$ *is* λ*-full over* A ($\lambda \geqq \aleph_0$, *of course) provided that at least one of the following occurs*:

(i) *for arbitrarily large* $i < \delta$, M_i *is* λ*-full over* A_i *and for every* i, $A_{i+1} \cup M_i$ *is a good set,*

(ii) cf $\delta \geqq \lambda$ *and for every finite* $B \subseteq M$ *and stationary* $p \in S^m(B)$, *for arbitrarily large* $i < \delta$, *some sequence from* M *realizes the stationarization of* p *over* $A \cup M_i$ *(or some sequence from* M_{i+1} *realizes the stationarization of* p *over* $A_{i+1} \cup M_i$; *this implies the first possibility by Lemma* 2.10 (4)),

(iii) δ *is divisible by* λ *(as ordinals), for every* $i < \delta$, $A_{i+1} \cup M_i$ *is good, and for every* $i < \delta$, *and stationary type* p *over some* $B \subseteq M_i$, *the stationarization of* p *over* $A \cup M_i$ *is realized in* M_{i+1}.

PROOF. (1) By Lemma 2.11(2), for each $i < j < \delta$, A_j, M_i, A_{j+1} is in stable amalgamation; hence by Lemma 2.11(1), for each i, A_i, M_i, A is in stable amalgamation.

Let us prove A is good. Let $\bar{a} \in A$, $\models (\exists \bar{x})\, \varphi[\bar{x};\bar{a}]$: then for some i, $\bar{a} \in A_i$, hence for some $\bar{b} \in M_i$, $\models \varphi[\bar{b},\bar{a}]$ and $\mathrm{tp}(\bar{b},A_i)$ is isolated ($\bar{b}$ exists as A_i is good). Now $\mathrm{tp}(\bar{b},A)$ is a weak stationarization of $\mathrm{tp}(\bar{b},A_i)$ (by Definition 2.9(2)) hence $\mathrm{tp}(\bar{b},A)$ is isolated (by Lemma 2.10(1)) and this proves A is good.

As for the last phrase of (1) w.l.o.g. δ, A, A_i, M_i are countable and let $i < \delta$. Let $i(0) < \delta$ and we define by induction on j, $i(0) \leqq j < \delta$, $M^*_j \subseteq M_j$, M^*_{j+1} atomic over $M^*_j \cup A_{j+1}$, and M^*_j continuous, $M^*_{i(0)} = M_{i(0)}$. If we succeed in carrying out the induction the last phrase follows (note that necessarily any set atomic over $M^*_i \cup A_j$, $i(0) \leqq i < j \leqq \delta$, is atomic over $M^*_{i(0)} \cup A$).

For $j = i(0)$ and for j limit there are no problems.

For defining M^*_{j+1} it suffices to show that $A_{j+1} \cup M^*_j$ is good. Now over $A_{j+1} \cup M_j$ there is a countable universal model N (by Lemma 2.2(4) as it is a countable good set). By Claims 2.3 and 2.4, it suffices to show that N is universal over $A_{j+1} \cup M^*_j$. So suppose N' is a countable model $A_{j+1} \cup M^*_j \subseteq N'$. By the stable amalgamation applied to M^*_j, N', M_j, there is an elementary mapping f, $\mathrm{Dom}\, f = M_j$, $f \restriction M^*_j =$ the identity, and for every $\bar{c} \in M_j$, $\mathrm{tp}(f(\bar{c}),N')$ is the weak stationarization of $\mathrm{tp}(f(\bar{c}),M^*_j) = \mathrm{tp}(\bar{c},M^*_j)$ over N'. By Lemma 2.10(1), $\mathrm{tp}(f(\bar{c}),N')$ is also the stationarization of $\mathrm{tp}(\bar{c},M^*_j)$ over N', hence $\mathrm{tp}(f(\bar{c}),M^*_j \cup A_{j+1})$ is the stationarization of $\mathrm{tp}(\bar{c},M^*_j)$ over $M^*_j \cup A_{j+1}$. However by Lemma

2.10(4), for $\bar{c} \in M_j$, $\mathrm{tp}(\bar{c}, M_j^* \cup A_{j+1})$ is the stationarization of $\mathrm{tp}(\bar{c}, M_j^*)$ over $M_j^* \cup A_{j+1}$, hence $\mathrm{tp}(\bar{c}, M_j^* \cup A_{j+1}) = \mathrm{tp}(f(\bar{c}), M_j^* \cup A_{j+1})$. So $f^{-1} \cup \mathrm{id}_{A_{j+1}}$ is an elementary mapping and let g_0 be an elementary mapping extending it with domain $f(M_j) \cup N'$, so $g_0 \restriction (M_j^* \cup A_{j+1})$ is the identity and $M_j \cup g_0(N')$ is an atomic set, hence there is a countable model N'', $M_j \cup g(N') \subseteq N''$, hence N'' can be embedded by some g_1 into N over $M_j \cup A_j$. Now $g_0 g_1$ embeds N' into N over $M_j^* \cup A_{j+1}$, so N is really universal over $M_j^* \cup A_{j+1}$.

(2) The proof splits into three cases, according to which of the three alternative hypotheses hold.

First Case: (i) holds.

So let $\bar{b} \in M$, $p \in S^m(\bar{b})$ be stationary. So for every large enough i, $\bar{b} \in M_i$. Hence by the hypothesis, for some i, $\bar{b} \in M_i$, M_i is λ-full and by part (1) $A \cup M_i$ is a good set. So the dimension of p for (A_i, M_i, M_i) is $\geqq \lambda$ (as M_i is λ-full over A_i). Now, as mentioned above, the triple A_i, M_i, A is in stable amalgamation, hence by Lemma 2.10(3) the dimension of p for $(A, M_i \cup A, M_i)$ is $\geqq \lambda$. By a hypothesis, $M_i \cup A$ is good, hence by Conclusion 2.5(2) the dimension of p for (A, M, M_i) is $\geqq \lambda$ and by monotonicity the dimension of p for (A, M, M) is at least λ.

Second Case: (ii) holds.

Let $\bar{b} \in M$, $p \in S^m(\bar{b})$ be stationary, $C \subseteq M$, $|C| < \lambda$. As cf $\delta \geqq \lambda$, for some $i < \delta$, $C \cup \bar{b} \subseteq M_i$, so by the hypothesis, for some j, $i < j < \delta$, and the stationarization of p over $A \cup M_j$ is realized in M by some $\bar{a}$; so $\bar{a}$ realizes the stationarization of p over $A \cup C \cup \bar{b}$, so we finish.

Third Case: (iii) holds.

If cf $\delta \geqq \lambda$ the conclusion follows as we proved Lemma 2.12(2) (ii). So suppose cf $\delta < \lambda$, but δ is divisible by λ, hence for every regular $\mu < \lambda$, $(\forall \alpha < \delta)$ $(\alpha + \mu < \delta)$.

Now if $\zeta < \delta$ is a limit ordinal, by Lemma 2.12(2) (ii) M_ζ is (cf ζ)-full over A_ζ. Let $\mu \leqq \lambda$ be regular, then for every $\alpha < \delta$, $M_{\alpha+\mu}$ is μ-full (by the last sentence, remembering $\alpha + \mu < \lambda$) so by Lemma 2.12(2) (i) M_δ is μ-full over A. As this holds for every regular $\mu \leqq \lambda$, clearly M_δ is λ-full over A.

As a demonstration of our methods we shall now give a new proof of the result from [11].

Conclusion 2.13. (1) There exists a model $M \in K$ such that $\|M\| = \aleph_2$.

(2) For every $M \in K$ of cardinality $\aleph_1$ there is a model $N \in K$ such that $M \subsetneqq N$ and $\|N\| = \aleph_1$.

PROOF. (1) By an elementary chain argument it clearly follows from (2); notice that K is closed under union of elementary chains.

(2) Let $\{M_\alpha : \alpha < \omega_1\}$ be an increasing continuous elementary chain of countable models from K such that $M = \bigcup_{\alpha<\omega_1} M_\alpha$. We shall get the model N as a union of an increasing continuous elementary chain of countable models of length ω_1. Define this chain $\{N_\alpha : \alpha < \omega_1\}$ by induction on α such that M_α, N_α, M is in stable amalgamation. For $\alpha = 0$, by Theorem 1.1(v) there is $N_0' \in K$ such that $M_0 \lneqq N_0'$. By the remark after the proof of Lemma 2.2 M_0 is good, and by Lemma 2.10(2) there is an automorphism f (of $\mathfrak{C}$) such that $f \restriction M_0 = \mathrm{id}$, Dom $f = N_0'$, and M_0, $f(N_0')$, M is in stable amalgamation. By the requirement $N_0' \not\subseteq M_0$ it is clear that $f(N_0') \not\subseteq M$; finally define $N_0 = f(N_0')$. For $\alpha = \delta$ limit ordinal, define $N_\alpha = \bigcup_{\beta<\delta} N_\beta$ and the induction assumption is satisfied by Lemma 2.11(3) and Lemma 2.12(1). For $\alpha = \beta + 1$, as $N_\beta \cup M_{\beta+1}$ is countable and atomic choose a countable atomic model $M_0' \supseteq N_\beta \cup M_{\beta+1}$ and act on M_0' as we have done before for N_0' and get $N_\alpha = N_{\beta+1} \gneqq N_\beta$ countable. Now $N = \bigcup_{\alpha<\omega_1} N_\alpha$ is an atomic elementary extension of M of cardinality $\aleph_1$, and it is a proper extension of M since in the first stage we took $N_0 < N$ which is not contained in M.

REMARK. It is tempting to try to repeat the proof of the last conclusion for higher cardinalities, i.e., to prove existence of a model of cardinality $\aleph_3$, for example. But unfortunately we cannot do this by the same proof; there we used the fact that the sets $M_{\alpha+1} \cup N_\alpha$ ($\alpha < \omega_1$) can be extended to a model, and this is followed from the countability of the models (by the $\aleph_0$-amalgamation property of K).

In the following sections (to be published soon) we shall prove existence of models in higher cardinalities by assuming additional assumptions on K which will imply the goodness of more sets.

REFERENCES

1. U. Abraham and S. Shelah, *Isomorphism of Aronszajn trees, independence results*, Isr. J. Math., to appear.

2. K. Devlin, *Aspects of Constructibility*, Springer-Verlag, 1973.

3. K. Devlin and S. Shelah, *A weak form of diamond which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Isr. J. Math. **29** (1978), 239–247.

4. H. Friedman, *Hundred and two problems in mathematical logic*. J. Symb. Logic **40** (1975), 113–129.

5. J. Gregory, *High Souslin trees and the greneralized continuum hypothesis*, J. Symb. Logic **41** (1976), 663–671.

6. R. Grossberg and S. Shelah, *On locally finite groups*, Isr. J. Math. **44** (1983), 289–302.

7. H. J. Keisler, *Infinitary Languages*, North-Holland Publishing Company, 1971.

8. J. Makowsky, *Compactness embedding and definability*, in *The Model Theoretic Logics* (J. Barwise and S. Feferman, eds.), Springer–Verlag, in press.

9. L. Marcus, *A prime minimal model with an infinite set of indiscernibles*, Isr. J. Math. **11** (1972), 180–183.

10. S. Shelah, *The number of non-almost isomorphic models of T in a power*, Pac. J. Math. **36** (1971), 473–487.

11. S. Shelah, *Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$*, Isr. J. Math. **20** (1975), 127–148.

12. S. Shelah, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Isr. J. Math. **21** (1975), 319–349.

13. S. Shelah, *Generalized quantifiers and compact logic*, Trans. Am. Math. Soc. **204** (1975), 362–364.

14. S. Shelah, *Classification Theory and the Number of Non-isomorphic Models*, North-Holland Publishing Co., 1978.

15. S. Shelah, *Models with second order properties III, omitting types for $L(Q)$ in λ for $L(Q)$*, Proc. Berlin Symp., July 1977, Arch. Math. Logic **21** (1980), 1–11.

16. S. Shelah, *Proper Forcing*, Springer-Verlag Lecture Notes **940**, 1982.

17. S. Shelah, *Classification theory for non-elementary classes II, Abstract elementary classes*, to appear.

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