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ISOMORPHISM TYPES OF ARONSZAJN TREES

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ABSTRACT

We study the isomorphism types of Aronszajn trees of height ω_1 and give diverse results on this question (mainly consistency results).

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Introduction

The question of isomorphism of Aronszajn trees was dealt with by H. Gaifman and E. Specker [11] who proved that there are $2^{\aleph_1}$ non-isomorphic Aronszajn trees. It can be seen, however, that every two of their non-isomorphic trees are *isomorphic on a closed unbounded set*, i.e. there is a closed unbounded

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(club) set $C \subseteq \omega_1$ such that restriction of the trees to the levels in C gives two isomorphic trees. So we are led to the question about isomorphism of trees on a closed unbounded set. Saying that two trees are *really different* if they do not contain isomorphic restrictions to closed unbounded set of levels, the question is whether really different Aronszajn trees exist. It turned out that this question is interesting from a Set-Theoretical point of view. There are consistency results requiring new models, and different possible answers exist. Very roughly the situation is this: The continuum hypothesis (CH) implies that there are many really different Aronszajn trees, while on the other hand it is consistent that every two Aronszajn trees are isomorphic on a closed unbounded set. (When we say that φ is consistent we mean of course that if ZFC is consistent then ZFC + φ is consistent.) Following are some notations and then a detailed description of our results.

An *Aronszajn tree* is always a tree of height ω_1 without cofinal branches and with the usual normality properties. (Every point has extensions to every higher level, two points with the same predecessors are equal.) (See [13] or [9] or [18] for basic information on Aronszajn trees.) If T is a tree then T_α designates the α's level of T, and for $C \subseteq \omega_1$, $T \restriction C = \bigcup_{\alpha \in C} T_\alpha$ is a tree under the restriction of the partial ordering of T. A function $f: T^1 \rightarrow T^2$ is an *embedding* of the tree T^1 into the tree T^2 iff f is one-to-one and order preserving

$$x < y \Leftrightarrow f(x) < f(y).$$

f is an *isomorphism* if it is an embedding of T^1 onto T^2.

For $C \subseteq \omega_1$, two trees T^1 and T^2 are said to be isomorphic on C iff $T^1 \restriction C$ is isomorphic to $T^2 \restriction C$. It follows that if $C \subseteq \omega_1$ is unbounded and T^1, T^2 are isomorphic on C then they are isomorphic also on the closure of C (C with all its limit points).

We say T^1 *is embeddable into* T^2 *on a club set* iff for some (closed) unbounded $C \subseteq \omega_1$ there is an embedding of $T^1 \restriction C$ into $T^2 \restriction C$. We say that two trees are *isomorphic on a club set* iff for some (closed) unbounded set $C \subseteq \omega_1$ these trees are isomorphic on C. We say two trees are *near* each other if some tree is embeddable on a club set into both of them. From now on, embeddability for Aronszajn trees always means embeddability on a club set.

In order to understand our work it may be helpful to compare questions on Aronszajn trees to an apparently similar problem — the isomorphism questions of $\aleph_1$-dense real order. Say K is the class of $\aleph_1$-dense real orders. (A real order is $\aleph_1$-dense iff it has no first nor last element and between any two points there are $\aleph_1$ many points.)

As any order preserving function from M_1 to M_2 ($M_1, M_2 \in K$) is determined by its restriction to a countable set, it is easy to find, assuming CH, $M_i \in K$, $i < 2^{\aleph_1}$, such that no $M \in K$ is embeddable to two of them, i.e., every two orders in this family do not contain isomorphic subsets in K. Also, one can see that (assuming CH) there is no prime order in K (one which is embeddable in each other).

For Aronszajn trees (with embeddability always on a closed unbounded set) we can easily get similar results assuming $\diamondsuit_{\omega_1}$. But what can we say if CH alone is assumed? Now, by Devlin and Shelah [10] there is a weak form of the diamond which follows from CH (and in fact is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$) and which partially substitutes $\diamondsuit$. Using a combinatorial principle we derive from this weak diamond, we get $2^{\aleph_1}$ Aronszajn trees no one embeddable into another and even the only embedding of each tree into itself is the identity. It also follows from this principle that for every Aronszajn T^1 there is Aronszajn T^2 such that T^1 is not embeddable into T^2 (i.e. there is no prime Aronszajn tree). Now note that (unlike the $\aleph_1$-dense real order case) the results we get assuming $2^{\aleph_0} = \aleph_1$ are not as strong as those we can get with $\diamondsuit$. This is not incidental. We can prove the consistency with GCH of: There is a universal Aronszajn tree and for any two trees there is one embeddable into both.

What about the other direction — consistency results which negate the consequences of CH? J. Baumgartner proved the consistency of "Every two members of K are isomorphic" [7]. He starts with $V \models$ GCH and uses a c.c.c. forcing. The main lemma he needs is: *if* $V \models$ CH, $M_1, M_2 \in K$, *then for some forcing poset* $\mathcal{P}$ *satisfying the c.c.c. with* $|\mathcal{P}| = \aleph_1$, $\Vdash^{\mathcal{P}}$ "$M_1 \cong M_2$". Now the parallel assertion for Aronszajn trees is open, but we can find an $\aleph_1$-complete forcing poset $\mathcal{P}$ and a c.c.c. poset $\mathcal{Q} \in V^{\mathcal{P}}$ such that $\Vdash^{\mathcal{P} \times \mathcal{Q}}$ "T^1 and T^2 are isomorphic (on a club set)". At the time we found the proof it was not clear how to iterate these posets, and we had quite an involved argument and construction for doing it; but with the invention of proper forcing (see [16]) this becomes easy. However, if we want $2^{\aleph_0} > \aleph_2$, we still have to use some of the involved methods.

Our results can be divided into three parts:

1. *Consequences of* $2^{\aleph_0} < 2^{\aleph_1}$ (§1, §2)

If CH or even if $2^{\aleph_0} < 2^{\aleph_1}$ holds then:

(a) There are $2^{\aleph_1}$ pairwise really different Aronszajn trees. Not one of these tree is embeddable on a club set into the other.

(b) There is an Aronszajn tree T such that for every closed unbounded

$C \subseteq \omega_1$, $T \upharpoonright C$ is rigid (i.e., the only embedding of $T \upharpoonright C$ into $T \upharpoonright C$ is the identity, so T is a *really rigid* tree). We can combine (b) with (a).

(c) For every Aronszajn tree T^1 there is an Aronszajn tree T^2 such that T^1 is not embeddable into T^2 on a club set, i.e., there is no prime Aronszajn tree.

(d) We formulate a combinatorial principle that follows from the weak diamond (of [10] which itself is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$) and hopefully will serve to obtain results like (a)–(c).

2. *Consistency results with CH* (§3, §4)

(a) GCH $+ (a_1) + (a_2)$ is consistent:

(a_1) There is a universal Aronszajn tree T, i.e., a tree T such that for every Aronszajn tree T^*, $T^* \upharpoonright C$ is order embeddable into $T \upharpoonright C$ for some $C \subseteq \omega_1$. The universal tree T is a special Aronszajn tree. Hence in this model there are no Souslin trees. Thus (a_1) is a strengthening of Jensen's theorem ([9]) which says that CH is consistent with Souslin's hypothesis.

(a_2) Every two Aronszajn trees contain subtrees which are isomorphic on a club set. (We say two such trees are near.)

(b) We have some consistency result with CH concerning Souslin trees. For example, "CH + there is a Souslin tree" does not imply that there are $2^{\aleph_1}$ really different Souslin trees.

PROBLEM. Does the existence of two really different Souslin trees follow the existence of a Souslin tree?

3. *Consistency results with* $2^{\aleph_0} > \aleph_1$ (§5, §6, §7)

(a) Martin's Axiom $+ 2^{\aleph_0} = \aleph_2 +$ "Every two Aronszajn trees are isomorphic on a closed unbounded set" is consistent (in §3).

(b) Martin's Axiom $+ 2^{\aleph_0} > \aleph_1$ does not imply that every two Aronszajn trees are isomorphic on a closed unbounded set (in §4).

(c) Martin's Axiom $+ 2^{\aleph_0} = \kappa +$ "Every two Aronszajn trees are isomorphic on a closed unbounded set" is consistent (in §5). κ here is "any" regular cardinal such that $\kappa^{\xi} = \kappa$. (The difference between this item and (a) is that in (a) we get $2^{\aleph_0} = \aleph_2$ and here $2^{\aleph_0}$ is as big as we want; moreover the proofs are different: in (a) we use proper forcing and in (c) a technique of using generic reals.)

Chart 1 summarizes the situation. In this chart, bold type is for implication and italic type for consistency results. Embeddability and isomorphism in the Aronszajn tree case always mean on a closed unbounded set.

A related subject which can be treated similarly is Specker order (see e.g. Galvin and Shelah [12]). A Specker order is an uncountable order with no

Chart 1

	$\diamond$	Continuum Hypothesis	$2^{\aleph_0} < 2^{\aleph_1}$	$\mathrm{MA} + 2^{\aleph_0} > \aleph_1$
Aronszajn trees	**There are $2^{\aleph_1}$ Aronszajn trees, no two have isomorphic subtrees. There is no universal tree.**	*Every two trees have isomorphic subtrees. There is a universal Aronszajn tree* (§2).	**There are $2^{\aleph_1}$ trees no one embeddable into the other, and every one embeddable into itself only by the identity. There is no prime tree** (§1).	*Every two trees are isomorphic*: §3 *for* $2^{\aleph_0} = \aleph_2$ *and* §5 *for* $2^{\aleph_0} > \aleph_2$. *MA does not imply that every two trees are isomorphic* (§4).
$\aleph_1$-dense real orders (K)		**There are $2^{\aleph_1}$ orders no two have isomorphic suborders. $\mathbf{R}$ is a universal order. There is no prime order.**	**Like above, replace tree by order** [15]. *Every two orders have isomorphic suborders. There is a universal order* [2].	*Every two orders are isomorphic*: [7] *for* $2^{\aleph_0} = \aleph_2$ *and* [2] *for* $2^{\aleph_0} > \aleph_2$. *MA does not imply that every two orders are isomorphic* [5].

uncountable real sub-order nor uncountable well-order or anti-well-order subset. We call a Specker order I normal if $I = \bigcup_{\alpha<\omega_1} I_\alpha$, $\langle I_\alpha : \alpha < \omega_1 \rangle$ increasing and continuous, $|I_\alpha| = \aleph_0$, I is dense with no first or last element, and for every α and $x \in I - I_\alpha$,

$$J_{x,\alpha} = \{y \in I - I_\alpha : (\forall a \in I_\alpha)(a < x \leftrightarrow a < y)\}$$

has no first or last element. By [14] there are $2^{\aleph_1}$ pairwise non-isomorphic normal Specker orders. Let I be an order, $C \subseteq I \times I$ is a *chain* iff for $(x, y), (x', y') \in C$ either $x \leqq x'$ and $y \leqq y'$ or $x \geqq x'$ and $y \geqq y'$ hold. Say

$$K'_S = \left\{ I : I \text{ is a normal Specker order and } I \times I = \bigcup_{n<\omega} C_n \text{ each } C_n \text{ a chain} \right\},$$

i.e., K'_S is the family of normal Specker order whose square is the union of countably many chains.

We can represent every Specker order by an Aronszajn tree of sequences of non-zero rationals ordered lexicographically (where we identify η with $\eta {}^\wedge \langle 0, \dots \rangle$) and then prove similar results for K'_S. For example: Con (Any two members of K'_S are isomorphic or antiisomorphic). This cannot be improved to "Any two members of K'_S are isomorphic" (see [14]).

PROBLEM. It is not clear whether we can get the consistency of:

(*) Any Specker order contains a suborder in K'_S.

This is equivalent to the following consistency question.

(**) If T is an Aronszajn tree, $T = A_1 \cup A_0$, then there is an unbounded $B \subseteq T$ and $l \in \{0, 1\}$ such that $x, y \in B \Rightarrow \text{g.l.b.}\{x, y\} \in A_l$.

To the reader who wants to get some of the main ideas without too much work we suggest reading §1, where we prove that $2^{\aleph_0} < 2^{\aleph_1}$ implies that there are two really different Aronszajn trees. §5 can be read quite easily and gives the main result. §6 is an example of a method, which can be studied also in [5], in [2] and in [1] and gives some limitations to Martin's Axiom. Some of the ideas of §7 appear in [4] in a simpler context.

Although this paper can be read independently of other material, at some places the use of Devlin and Johansbråten's book [9] is necessary for a full understanding of some details.

Historically, this paper owes much to the proof of Jensen [9]; §4 in particular resembles his work. The study began by Avraham, who showed the consistency with CH of the existence of a universal Aronszajn tree. Shelah proved the consistency of "Martin's Axiom for stable posets + Every two Aronszajn trees are isomorphic on a club set" (see [6] for the definition of stable posets). Afterward Avraham used the generic reals to get the consistency of the full Martin's Axiom with the isomorphism of any two Aronszajn trees (§7). For the notion of proper forcing see [16]. The construction of a proper poset forcing two Aronszajn trees to be isomorphic on a closed unbounded set (§5) is due to Shelah. J. Baumgartner was the first to use finite conditions to force a closed unbounded set. The proof that Martin's Axiom $+ 2^{\aleph_0} > \aleph_1$ does not imply that every two Aronszajn trees are isomorphic on a club set is due to Avraham and is based on a method of Shelah (exposed in [5]). Shelah proved the consistency of "CH + Every two Aronszajn trees are near" (item (a_2) in 2). The result on Souslin trees in §4 is due to Avraham. In §1 we use the weak diamond of Devlin and Shelah ([10]). Shelah used the weak diamond to obtain non-isomorphic structures ([15]). Avraham applied this method to prove that $2^{\aleph_0} < 2^{\aleph_1}$ implies the existence of two really different Aronszajn trees and of a "really rigid" Aronszajn tree. (Rigid Aronszajn trees were constructed in ZFC by Baumgartner, Avraham and Todorcevic independently; [3] and [17].) The general combinatorial principle of §2, the definition and use of the "small sets" to obtain $2^{\aleph_1}$ really different Aronszajn trees, are due to Shelah. After this study was done it was brought to our knowledge that the question whether "every two trees are isomorphic in a closed unbounded set" is consistent was asked by J. Baumgartner.

Part I: Theorems in ZFC

§1. Consequences of $2^{\aleph_0} < 2^{\aleph_1}$: non-isomorphic trees

In this section we prove that $2^{\aleph_0} < 2^{\aleph_1}$ implies that there are $2^{\aleph_1}$ pairwise non-isomorphic on a club Aronszajn trees of height ω_1. We then present a combinatorial axiom that follows from $2^{\aleph_0} < 2^{\aleph_1}$, implies the above result, and hopefully will have other consequences as well. In order to present the ideas more slowly we first construct directly from the weak diamond *two* Aronszajn trees non-isomorphic on a club (below), then follows the axiom (§2) and finally we show how to use the axiom to get $2^{\aleph_1}$ pairwise non-isomorphic rigid trees.

Devlin and Shelah [10] proved that $2^{\aleph_0} < 2^{\aleph_1}$ implies the *weak diamond principle*: For each $F : {}^{\omega_1}(2^{\aleph_0}) \to 2$ there is $g \in {}^{\omega_1}2$ such that for every $f \in {}^{\omega_1}(2^{\aleph_0})$, $\{\alpha < \omega_1 : g(\alpha) = F(f \restriction \alpha)\}$ is stationary.

We remind the reader that in order to construct an Aronszajn tree one constructs it together with an order preserving embedding of the tree into the rationals (a special tree). At limit stages of the construction, one uses, and therefore has to keep, the following property: for every point x in the tree and rational $\varepsilon > 0$ there is a point above x, at every higher level, such that the rational numbers assigned to these two points differ by less than ε. As this is standard we will not mention it further.

Now to the construction of two Aronszajn trees not isomorphic on a club set. By induction on α we will attach to every $\eta \in {}^{\omega_1}2 (\eta : \alpha \to 2, \alpha < \omega_1)$ a tree $T(\eta)$ of height $\omega \cdot \alpha$ (together with its order preserving embedding into the rationals). The elements of our trees are chosen to be ordinals: the μ level consists of the ordinal interval $[\omega \cdot \mu, \omega \cdot (\mu + 1))$. If $\eta' \supseteq \eta$ then $T(\eta')$ is an end extension of $T(\eta)$.

For limit α and $\eta \in {}^{\alpha}2$, $T(\eta) = \bigcup_{\gamma<\alpha} T(\eta \restriction \gamma)$.

Suppose $T(\eta)$, $\eta \in {}^{\alpha}2$, is constructed, define $T(\eta \frown (0))$ and $T(\eta \frown (1))$ to be two trees extending $T(\eta)$ such that:

(a) The height of $T(\eta \frown (i)$ is $\omega \cdot \alpha + \omega$ for $i = 0, 1$.
(b) The set of cofinal branches of $T(\eta)$ determined by points of $T(\eta \frown (0))$ and the set of cofinal branches of $T(\eta)$ determined by points of $T(\eta \frown (1))$ are disjoint.

(The set of cofinal branches of $T(\eta)$ determined by $T(\eta \frown (0))$ are all branches of $T(\eta)$ of the form $\{x \in T(\eta) : x < a\}$ where $a \in T(\eta \frown (0))$ and a is of level $\omega \cdot \alpha$ there.)

Now for every $\eta \in {}^{\omega_1}2$, $T(\eta) = \bigcup_{\gamma<\omega_1} T(\eta \restriction \gamma)$ is a special Aronszajn tree.

Let $\sigma \in {}^{\omega_1}2$ be defined by $\sigma(\alpha) = 0$ for $\alpha < \omega_1$. We will find $h \in {}^{\omega_1}2$ such that $T(h)$ and $T(\sigma)$ are not isomorphic on a club set. The trees $T(\eta)$, $\eta \in {}^{\omega_1}2$, were constructed in such a way that the points of $T(\eta)$ for $\eta : \alpha \to 2$ consist of ordinals in $\omega \cdot \alpha$.

Now let $\eta \in {}^{\alpha}2$ and $C \subseteq \omega \cdot \alpha$ be closed unbounded in $\omega \cdot \alpha$, and let $i : T(\sigma \restriction \alpha) | C \to T(\eta)$ be an embedding on the levels in C of $T(\sigma \restriction \alpha)$ into $T(\eta) | C$ (i is the beginning of an order preserving isomorphism on C which is a beginning of a closed unbounded set). Define $F(\eta, i, C) \in \{0, 1\}$ as follows:

$F(\eta, i, C) = 0$ iff i can be extended to an embedding of $T(\sigma) \mid C \cup \{\omega \cdot (\alpha+1)\}$ into $T(\eta^\frown(0))$.

Note that as the $\omega \cdot \alpha$'s levels in $T(\eta^\frown(0))$ and $T(\eta^\frown(1))$ define disjoint sets of branches it cannot be the case that such i can be extended both into $T(\eta^\frown(0))$ and into $T(\eta^\frown(1))$.

Now, in case $\omega \cdot \alpha = \alpha$, under suitable encoding, F can be viewed as a function from ${}^{\omega_1}(2^{\aleph_0})$ into 2. So, by the weak diamond principle there is $g \in {}^{\omega_1}2$ such that for every $\eta \in {}^{\omega_1}2$, $C \subseteq \omega_1$, and $i : T(\sigma) \mid C \to T(\eta) \mid C$,

$$\{\alpha < \omega_1 : g(\alpha) = F(\eta \restriction \alpha,\ i \restriction \alpha,\ C \cap \alpha)\} \text{ is stationary.}$$

Let $g^* = 1 - g$ (for every α, $g(\alpha) + g^*(\alpha) = 1$).

CLAIM. *$T(\sigma)$ and $T(g^*)$ are not isomorphic on a club set.*

PROOF. Suppose on the contrary that $C \subseteq \omega_1$ is closed unbounded and $i : T(\sigma) \mid C \to T(g^*) \mid C$ is order preserving. There is some α, $\omega \cdot \alpha = \alpha$, $\alpha \in C$, such that

$$g(\alpha) = F(g^* \restriction \alpha,\ i \restriction \alpha,\ C \cap \alpha).$$

If $g(\alpha) = 1$ then $g^*(\alpha) = 0$, hence $i \restriction \alpha$ is extended by i to an embedding of $T(\sigma) \mid \alpha + \omega$ into $T((g^* \restriction \alpha)^\frown(0))$, so, by the definition of F, $F(g^* \restriction \alpha, i \restriction \alpha, C \cap \alpha) = 0$. A contradiction.

If $g(\alpha) = 0$ then $g^*(\alpha) = 1$ and it follows that $i \restriction \alpha$ can be extended on $T(\sigma)$ both to the left and to the right of $T(g^* \restriction \alpha)$ which is impossible as was mentioned above.

§2. A combinatorial principle about the small sets

2.1. DEFINITION. (1) Let $F : {}^{\omega_1}\mathfrak{c} \to 2$ be given ($\mathfrak{c}$ is $2^{\aleph_0}$). A is a subset of ω_1. We say that a function $g : \omega_1 \to 2$ is an *A-diamond* for F iff, for any $\eta \in {}^{\omega_1}\mathfrak{c}$,

$$\{\alpha \in A : F(\eta \restriction \alpha) = g(\alpha)\} \text{ is a stationary subset of } \omega_1.$$

(2) $A \subseteq \omega_1$ is called a *small* subset of ω_1 iff for some $F : {}^{\omega_1}\mathfrak{c} \to 2$ no function is an A-diamond for F.

(3) $I = \{A \subseteq \omega_1 : A$ is a small subset of $\omega_1\}$.

In [10] the following is proved.

2.2. THEOREM. *I is a countably complete normal ideal on ω_1 (which includes all non-stationary sets) which is a proper ideal ($\omega_1 \notin I$) if $2^{\aleph_0} < 2^{\aleph_1}$.*

Our aim in this section is to prove.

2.3. THEOREM. *Suppose $2^{\aleph_0} < 2^{\aleph_1}$, then there are $2^{\aleph_1}$ pairwise non-isomorphic* *on a club-set Aronszajn trees; each of these trees is club-set rigid (any embedding on a club-set of the tree into itself is the identity).*

First, observe the following facts:

(1) Let S be a tree of height α (a limit countable ordinal) and let S^0, S^1 be end extensions of S of height $\alpha+1$ such that the set of cofinal branches of S that are determined by the members of the α-level of S^0 is disjoint from the one determined by the α-level of S^1. Then for any club $C \subseteq \alpha$ and $f : T \mid \alpha \to S$ (where T is a tree of height ω_1) there is $l = 0, 1$ such that f cannot be extended to an isomorphism of $T \mid C^*$ into $S^* \mid C^*$ whenever C^* extends C and S^* extends S^l.

(2) If S is again of height α and $a, b \in S_\beta$, $\beta < \alpha$, and if S^0, S^1 are end extensions of S of height $\alpha+1$ which determine disjoint sets of branches cofinal in S and containing a, but determine the same set of branches which contain b, then for any $f : S \to S$ (such that $f(b) = a$) and club $C \subseteq \alpha$, there is $l = 0, 1$ such that f cannot be extended to an embedding of S^l on $C \cup \{\alpha\}$ into S^l.

Let us now define a relation: $R^0(S, T, f, C)$ iff f is an embedding on the club $C \subseteq \omega_1$ of the Aronszajn tree S into the Aronszajn tree T. Also, for any $\xi \in \omega_1 - 1$ define: $R^\xi(S, T, f, C)$ iff f is an embedding on the club $C \subset \omega_1$ of the Aronszajn tree T into itself, such that the ξ member, a, of T is in the range of f and $f^{-1}(a) \neq a$ (so S is here a dummy variable).

Quite naturally we can view a function $\eta : \alpha \to \mathfrak{c}$ as a countable tree of height α with an embedding into the rationals (having those properties that are required when a rational embedded tree is constructed). If δ is limit, then $\eta : \delta \to \mathfrak{c}$, as a tree, is the union of $\eta \restriction \alpha$, $\alpha \in \delta$. Now $R^\xi(S, T, f, C)$ is meaningful when $S, T \in {}^{\omega_1}\mathfrak{c}$ and f, C are as before.

Using facts (1) and (2), the relation R^ξ (for $\xi \in \omega_1$) satisfies this:

2.4. For any limit ordinal α and $\sigma : \alpha \to \mathfrak{c}$, there are extensions of σ, $\sigma^{0,\xi}$, $\sigma^{1,\xi} : \alpha + 1 \to \mathfrak{c}$ such that the following holds:

For any $\sigma, \tau : \alpha \to \mathfrak{c}$ and f and club $C \subseteq \alpha$, there is $l = 0, 1$ such that, whenever $f^* \supseteq f$ and C^* end-extends C, for any $S, T : \omega_1 \to \mathfrak{c}$ extending $\sigma^{0,\xi}$ and $\tau^{l,\xi}$, respectively, $R^\xi(S, T, f^*, C^*)$ does not hold.

To prove Theorem 2.3 we need $t_i : \omega_1 \to \mathfrak{c}$ for $i < 2^{\aleph_1}$, such that, for any $\xi \in \omega_1$, club C and function f and $i \neq j$, $R^\xi(t_i, t_j, f, C)$ does not hold (because then the t_i's give the desired trees).

2.5. Theorem. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let R^ξ, $\xi \in \omega_1$, be relations satisfying 2.4. Then there are $t_i : \omega_1 \to \mathfrak{c}$, for $i < 2^{\aleph_1}$, such that, for $i \neq j$ and for any f and club C, $R^\xi(t_i, t_j, f, C)$ does not hold.*

Proof. As I is a non-trivial countably complete ideal on ω_1 (Theorem 2.2), Ulam's theorem say that I is not saturated and so there are $S^\xi_\beta(\beta, \xi \in \omega_1)$

pairwise disjoint non-small subsets of ω_1. We assume that for every α there are unique β and ξ with $\alpha \in S^{\xi}_{\beta}$.

2.6. Definition. The function $F(\sigma, \tau, f, C)$ is defined for every $\sigma, \tau : \alpha \to \mathfrak{c}$ and f and club $C \subseteq \alpha$. $F(\sigma, \tau, f, C) = l$ iff $l = 0, 1$ is that given in 2.4 for the relation R^{ξ} where ξ is given by $\alpha \in S^{\xi}_{\beta}$.

Since S^{ξ}_{β} is non-small, there is a function $g^{\xi}_{\beta} : \omega_1 \to 2$ which is an S^{ξ}_{β}-diamond for F; in some natural way we can view F as defined on ${}^{\omega_1}\mathfrak{c}$.

Now, for any $I \subseteq \omega_1$ we define $t_I : \omega_1 \to \mathfrak{c}$. $t_I \restriction \alpha$ is defined by induction on limit α. If δ is limit of limit ordinals, then $t_I \restriction \delta = \bigcup_{\alpha < \delta} t_I \restriction \alpha$. If α is limit and $t_I \restriction \alpha$ is defined, we first find ξ and β such that $\alpha \in S^{\xi}_{\beta}$. And then, if $\beta \in I$,

$$t_I \restriction \alpha + 1 = (t_I \restriction \alpha)^{g^{\xi}_{\beta}(\alpha), \xi} \qquad \text{(see 2.4)}.$$

If $\beta \notin I$, then

$$t_I \restriction \alpha + 1 = (t_I \restriction \alpha)^{0, \xi}.$$

$t_I \restriction \alpha + 1$ is now extended to $t_I \restriction \alpha + \omega$ in some arbitrary way. ($t_I \restriction \omega$ is also arbitrarily chosen.)

It is easy to find $2^{\aleph_1}$ many subsets of ω_1 no one included in the other. Hence all we need is

2.7. Claim. *If $J \not\subseteq I$, $f \in {}^{\omega_1}\mathfrak{c}$ and $C \subseteq \omega_1$ is a club set, then $R^{\xi}(t_I, t_J, f, C)$ does not hold.*

Proof. Pick $\beta \in J - I$. By Definition 2.1, and since g^{ξ}_{β} is an S^{ξ}_{β} diamond for F, there is $\alpha \in S^{\xi}_{\beta}$ such that $\alpha \in C'$ (a limit point of C) and

$$F(t_I \restriction \alpha,\ t_J \restriction \alpha,\ f \restriction \alpha,\ C \cap \alpha) = g^{\xi}_{\beta}(\alpha), \tag{2.8}$$

as $\beta \notin I$, $t_I \restriction \alpha + 1 = (t_I \restriction \alpha)^{0,\xi}$. As $\beta \in J$, putting $g^{\xi}_{\beta}(\alpha) = l$, $t_J \restriction \alpha + 1 = (t_J \restriction \alpha)^{l,\xi}$. By (2.8) and the definition of F (2.6) we conclude (from $f \supseteq f \mid \alpha$, C extends $C \cap \alpha$ and t_I, t_J extending $(t_I \restriction \alpha)^{0,\xi}$ and $(t_J \restriction \alpha)^{l,\xi}$, respectively) that $R^{\xi}(t_I, t_J, f, C)$ does not hold.

Part H. Consistency Results with CH

§3. Preliminaries

We recall that embedding and isomorphism means on a club set.

In the previous section we saw that $2^{\aleph_0} < 2^{\aleph_1}$ implies that there are many Aronszajn trees which are not isomorphic on a club set. Similar arguments show that for any Aronszajn tree T the "weak diamond" can be used to construct an

Aronszajn tree T^*; such T is not embeddable into T^*. So $2^{\aleph_0} < 2^{\aleph_1}$ implies there is no prime Aronszajn tree. How about universal Aronszajn trees? (where U is a universal Aronszajn tree if any Aronszajn tree A can be embedded into U).

It is not difficult to show that $\diamond$ implies:

(a) There are $2^{\aleph_1}$ many Aronszajn trees such that no two are near and embedding of each one into itself can only be the identity.

(b) There is no universal Aronszajn tree.

Could one replace $\diamond$ above by the weaker CH? No, since the following is consistent.

(a*) Every two Aronszajn trees are near each other (i.e. contain isomorphic copies).

(b*) There exists a universal Aronszajn tree (which is a special Aronszajn tree).

We shall not prove this consistency result here, only a hint is given in 4.23. However, this hint should be sufficient to the reader of §3.4. In §4 we *do give* the details for the proof of the consistency of:

> CH + There exists a Souslin tree + There exists a special Aronszajn tree which is universal among all the Aronszajn trees which do not contain a Souslin subtree.

In this model the dichotomy of Souslin trees and special Aronszajn trees is sharp: There exists a Souslin tree and every Aronszajn tree either contains a Souslin tree or is already a special Aronszajn tree. (A tree embeddable on a club into the rationals is already a special tree. See [9], first page.) In contrast, in L, there is a non-special Aronszajn tree which is embeddable into the reals (and thus contains no Souslin subtree). See [8] for this. A further property of this model is that there are, up to a club set, only $\aleph_1$ many Souslin trees.

We do not know how to obtain a model with a unique Souslin tree.

3.1. Definition. (i) Let T^1, T^2 be trees, the *product* $T^1 \times T^2$ is the set of pairs $\{(a,b): a \in T^1_\gamma \text{ and } b \in T^2_\gamma \text{ for some } \gamma < \omega_1\}$ partially ordered by: $(a,b) < (\bar{a}, \bar{b})$ iff $a < \bar{a}$ and $b < \bar{b}$.

(ii) If T is a tree and $a \in T$ then $T_a = \{x \in T : x \geqq a\}$.

(iii) Let R be a tree, $a_1, \ldots, a_k \in R_\gamma$ for some $\gamma < \omega_1$ are different, then $R_{a_1} \times R_{a_2} \times \cdots \times R_{a_k}$ is called a *derived tree of* R and k is the *dimension* of that derived tree.

The rigid Souslin tree constructed in L ([9] p. 46) has the property that each of its derived trees is also a Souslin tree.

3.2. LEMMA. *Assume R and all its derived trees are Souslin. Let A be an Aronszajn tree and R' a derived tree of R such that, in the Boolean universe of R', A becomes non-Aronszajn; moreover the dimension of R' is minimal with respect to that property. Then R' is embeddable on a club set into A.*

PROOF. Let $(a_1, \dots, a_n)$ be an n-tuple of distinct elements of R_γ (for some $\gamma < \omega_1$) such that R' is $R_{a_1} \times \cdots \times R_{a_n}$. By the premise of the lemma there is a name b which is forced by every element of R' to be a cofinal branch of A. Now, for every $\alpha < \omega_1$, each $e \in R'$ has some extension $e' \in R'$ such that for some $a \in A_\alpha$, $e' \Vdash^{R'}$ "the intersection of b with A_α is a". Let $D_\alpha \subseteq R'$ be a maximal subset of incompatible members of R' which thus determine the value of the member of b at the α's level of A. D_α is countable as R' is Souslin. It follows that there exists a closed unbounded $E \subseteq \omega_1$ such that for $\alpha \in E$ and $e \in R'_\alpha$ there is $a \in A_\alpha$ such that $e \Vdash a$ is in b. Denote this unique a by $f(e)$. Now this function f is clearly an order preserving function from $R' \restriction E$ into a subtree of $A \restriction E$. We shall find closed unbounded $D \subseteq E$ such that the restriction of f to $R' \restriction D$ is one-to-one. Observe first that, as A is Aronszajn, every $e \in R' \restriction E$ has two extensions e' and e'' such that $f(e') \neq f(e'')$. A stronger property holds by the minimality of the dimension of R'.

CLAIM. *For every $e \in R' \restriction E$, $e = (e_1, \dots, e_n)$, and for every $h \subsetneqq \{1, \dots, n\}$, there are two extensions $e' = (e'_1, \dots, e'_n)$ and $e'' = (e''_1, \dots, e''_n)$ of e such that*

$$f(e') \neq f(e'') \quad \text{and} \quad e'_i = e''_i \qquad \text{for } i \in h.$$

PROOF. If not, if for some $h = \{h(1), \dots, h(k)\} \subsetneqq \{1, \dots, n\}$, $k < n$, every two extensions of e with the same restriction to h have the same value by f, then $R_{e_{h(1)}} \times \cdots \times R_{e_{h(k)}}$ contradicts the minimality of the dimension n.

Suppose now $d \subseteq R_\mu$, $\mu \in E$, is such that we can write $d = e_1 \cup e_2$ where $e_1, e_2 \in R'_\mu$ are distinct (but not necessarily disjoint), then there is an extension d' of d such that if we let e'_1 and e'_2 be the resulting extensions of e_1 and e_2 (i.e., $e'_1 \cup e'_2 = d'$), then $f(e'_1) \neq f(e'_2)$.

It follows that the set of d' in R^{2n} which satisfies $f(e'_1) \neq f(e'_2)$ whenever $d' \supseteq e'_1 \cup e'_2$ is a dense open set. Since R^{2n} is also Souslin, there is a club $D \subseteq \omega_1$ such that any tuple in $R^{2n} \restriction D$ is in that dense set. This means that for $e_1 \neq e_2$ in $R' \restriction D$, $f(e_1) \neq f(e_2)$.

We state now without proof a simple fact.

3.3. LEMMA. *Let T be a Souslin tree and R a tree. R is Souslin in V^T iff $T \times R$ is a Souslin tree.*

Next we review some definitions and lemmas from [9].

If T is a Souslin tree then for every dense open subset $D \subseteq T$ there is $\alpha < \omega_1$ such that $T_\alpha \subseteq D$. It follows that every cofinal branch of T is a generic branch. (See [9] p. 19.)

3.4. DEFINITION. (i) Let T, S be trees, $C \subseteq \omega_1$ a club set, $\pi : S \to T \restriction C$ is called a *projection* iff:

(1) π is order preserving ($s < s' \Rightarrow \pi(s) < \pi(s')$) and Range $(\pi) = T \restriction C$.

(2) If $t \in T \restriction C$, $t > \pi(s)$ for some $s \in S$ then there exists $s' > s$ such that $\pi(s') = t$.

(ii) We say *S is a refinement of T* iff there exists a closed unbounded set C and a projection $\pi : S \to T \restriction C$. (See [9] p. 85.)

3.5. LEMMA. *Let T^i, $i < \omega_1$, be Souslin trees and S a tree which is a refinement of every T^i with $\pi_i : S \to T^i \restriction C_i$ as projections. Let $D \subseteq S$ be a dense open set,* then *there is a closed unbounded set $C \subseteq \omega_1$ such that for every $\alpha \in C$, $i < \alpha$ and $a \in T^i_\alpha$, there exists $s \in S \restriction \alpha$, $s \in D$ with $\pi_i(s) < a$.* (See [9] p. 107, Lemma 7.)

PROOF. Let H_{ω_3} be the collection of all sets hereditarily of cardinality $< \omega_3$. $C \subseteq \omega_1$ is chosen such that for every $\alpha \in C$, there is an elementary countable submodel $N_\alpha < H_{\omega_3}$ such that $N_\alpha \cap \omega_1 = \alpha$, $D \in N_\alpha$ and, for $i < \alpha$, T^i, $\pi_i \in N_\alpha$.

Every $a \in T^i_\alpha$ defines a cofinal branch of $T^i \restriction \alpha$ and hence an N_α generic branch and the lemma follows.

3.6. DEFINITION. *The closed-set poset* and its properties. ([9] p. 97) The aim of this poset is to add a generic club subset of ω_1 such that each old club set contains the generic one — except for a countable set of ordinals

$$\mathscr{C} = \{\langle \nu, A \rangle : \nu < \omega_1 \text{ and } A \text{ is a club subset of } \omega_1\},$$

partially ordered by $\langle \nu', A' \rangle \geqq \langle \nu, A \rangle$ iff $\nu' \geqq \nu$ and $A' \subseteq A$ and $\nu \cap A' = \nu \cap A$.

$\mathscr{C}$ is σ-closed and satisfies the $\aleph_2$-c.c. (assuming $2^{\aleph_0} = \aleph_1$).

Let M be a model of ZFC and $\dot{\mathscr{C}}$ be an M-generic filter on the closed set poset $\mathscr{C}$.

Define $C = \bigcup \{\nu \cap A : \langle \nu, A \rangle \in \dot{\mathscr{C}}\}$, then C is called *an M-generic closed unbounded subset of ω_1*.

$\langle \nu, A \rangle$ is said to be *compatible* with C if $A \cap \nu = C \cap \nu$ and $A \supseteq C$. It is not difficult to see that $\dot{\mathscr{C}} = \{\langle \nu, A \rangle : \langle \nu, A \rangle$ is compatible with $C\}$. So $\dot{\mathscr{C}}$ is reconstructed from C.

C has the desired property: for any club subset $D \in M$ of ω_1, for some $\alpha < \omega_1$, $C - \alpha \subseteq D$.

NOTATIONS. Let $N < H_{\omega_3}$ be a countable elementary submodel of H_{ω_3}. $\bar{N}$ denotes the transitive collapse of N and $\pi_N : N \to \bar{N}$ is the collapsing function.

$$\alpha_n = \omega_1 \cap N = \pi_N(\omega_1) = \omega_1^{\bar{N}}.$$

3.7. LEMMA. *Let M be a model of* ZFC + GCH. *In M, let U be a function such that, for each countable $N < H^M_{\omega_3}$, U_N is a countable transitive model of ZF^- with $\bar{N} \in U$. Let C be an M-generic club subset of ω_1 (over $\mathscr{C}^M$). Let $X \in H^M_{\omega_3}$.* Then *there is a countable $N < H^M_{\omega_3}$ such that $X \in N$ and such that in $M[C]$ we have:*

(i) *$C \cap \alpha_N$ is a U_N-generic closed unbounded subset of α_N over $\pi_N(\mathscr{C}^M)$.*

(ii) *π_N^{-1}, the inverse map of π_N, can be extended to $\pi^{-1} : \bar{N}[C \cap \alpha] \to H^{M[C]}_{\omega_3}$, an elementary embedding.*

PROOF in [9] p. 99, Lemma 3. We give here a short sketch. Given any N as above and $\langle \nu, A \rangle \in N$, we can construct in ω-many steps an extension $\langle \nu', A' \rangle \in M$ which is U_N generic over $\Pi_N(\mathscr{C})$. For $\langle \nu', A' \rangle \in \mathscr{C}$, the conclusion of the lemma follows.

3.8. *About the diamond.* $\diamondsuit$ is the diamond on ω_1 and $\diamondsuit^*$ is the stronger property saying there is a sequence $\langle W_\alpha \mid \alpha \in \omega_1 \rangle$ with $W_\alpha \subseteq P(\alpha)$, countable, such that for any $X \subseteq \omega_1$, $\{\alpha \mid X \cap \alpha \in W_\alpha\}$ contains a club set.

There are some theorems which assure that the diamond propery is preserved is certain generic extensions.

(a) If $\diamondsuit^*$ holds then, in any generic extension via $\mathscr{C}$, $\diamondsuit^*$ still holds (Lemma 4, p. 79 in [9]). We will use the fact that $\diamondsuit^*$ holds in $L[A]$ for $A \subseteq \omega_1$.

(b) If $\diamondsuit^*$ holds and T^i $(i < \omega_1)$ are Souslin trees, then a $\diamondsuit^*$ sequence can be found which retains its $\diamondsuit^*$ property in any generic extension via any one of the T^i's (Lemma 5, p. 101 in [9]).

(c) Assume $\mathscr{P}$ is ω_1-closed, then any $\diamondsuit$ sequence retains its property after forcing with $\mathscr{P}$ (Lemma 6, p. 81 in [9]).

(d) Assume $\diamondsuit$ and let T^i, $i \in \omega_1$, be Souslin trees. Then there is a diamond sequence which stays a diamond sequence in any generic extension over T^i.

Let us sketch a proof for this last result (for one tree). So assume $\langle S_\alpha : \alpha \in \omega_1 \rangle$ is a given $\diamondsuit$ sequence, and let T be a Souslin tree. Now CH holds, so countable ordinals can encode a lot of information and we can look at $S_\alpha \subseteq \alpha$ as giving us a name in T and some $x \in T \restriction \alpha$. Then we just pick any $y \in T_\alpha$ extending x and interpret the name given by S_α using the branch determined by y. The outcome is called $\bar{S}_\alpha$. Now $\langle \bar{S}_\alpha : \alpha \in \omega_1 \rangle$ is the required diamond sequence. To see this, remark that T is a c.c.c. poset and hence any club subset of ω_1 in the extension contains an old club set. Now let $\underset{\sim}{A}$ be a name in T-forcing of a subset of ω_1; let

C be a club set; and let $x \in T$ be a condition. For some $\alpha \in C$, S_α encodes $\underset{\sim}{A} \restriction \alpha$ and x (and to develop $\underset{\sim}{A} \cap \alpha$ we need to know only the generic branch through $T \restriction \alpha$). At that stage $\bar{S}_\alpha$ was defined so that some extension of x forces $\underset{\sim}{A} \cap \alpha = \bar{S}_\alpha$.

3.9. *On Aronszajn trees.* (See [9] p. 63) Let T be an Aronszajn tree. The fact that any uncountable subtree of T contains an infinite level can be generalized as follows.

DEFINITIONS. (1) Let $\bar{a} = (a_1, \dots, a_n) \in T_\alpha^n$ (i.e., $\forall i$ $(a_i \in T_\alpha)$); for $\beta \leqq \alpha$ define $\pi_\beta(\bar{a}) = (b_1, \dots, b_n)$ iff $\bar{b} = (b_1, \dots, b_n) \in T_\beta^n$ and $\forall i$ $(b_i \leqq a_i)$. $\pi_\beta(\bar{a})$ is called the projection of the n-tuple $\bar{a}$ on the β-level of T. The a_i's and b_i's are not necessarily distinct.

(2) A set of n-tuples, S, of T will always be such that $\bar{a} \in S \Rightarrow \bar{a} \in T_\alpha$ for some α (and we then write $\bar{a} \in S_\alpha$).

(3) A set, S, of n-tuples as above is *downward closed* iff $\bar{a} \in S_\alpha$ and $\beta < \alpha$ imply $\pi_\beta(\bar{a}) \in S$.

(4) A set, Y, of n-tuples, all from the same level T_α is said to be well-distributed iff, for every finite $F \subseteq T_\alpha$, there is $\bar{a} \in Y$ disjoint from F. Equivalently: There is in Y an infinite, pairwise-disjoint subset.

3.10. LEMMA. ([9] Lemma 7, p. 63) *Let S be a downward closed set of n-tuples from an Aronszajn tree T. There is a club set $C \subseteq \omega_1$ such that, for all $\alpha, \beta \in C$ with $\alpha < \beta$, if $\bar{a} \in S_\alpha$ and if the set of $\bar{b} \in S_\beta$ such that $\pi_\alpha(\bar{b}) = \bar{a}$ is non-empty, then this set is well-distributed.*

§4. A model of CH with few Souslin trees in which every Aronszajn tree is either special or contains a Souslin tree

4.1. THEOREM. ZFC + GCH *and the following is consistent: There exists a Souslin tree R and a special Aronszajn tree U such that for every Aronszajn tree A one of the following holds:*

(1) *Either A is embeddable on a closed unbounded set into U (and in this case A is special), or*

(2) *A contains a Souslin tree (which is actually a derived tree of R).*

PROOF. First we construct, in L, a tree R such that every derived tree of $\underset{\sim}{R}$ is a Souslin tree. (See [9], p. 46 for construction of a rigid Souslin tree.) Next we pick any special Aronszajn tree U which will finally become our universal special tree.

The structure of the proof is like that of Jensen's ([9]); we firstly add ω_2 many generic closed unbounded subsets of ω_1 and then we iterate ω_2 Souslin trees with the aim to embed every possible Aronszajn tree into U, while keeping R and its derived trees Souslin. Of course, we cannot embed R itself into the special tree U and at the same time keep R Souslin. We will be able, however, to embed into U any Aronszajn tree A which remains Aronszajn in every generic extension by a derived tree of R.

The following theorem is the key step in the iteration of Souslin trees.

4.2. THEOREM. *Denote our universe by V and assume $\diamondsuit^*$ holds in V. Let U be a special Aronszajn tree. Let R and T be Souslin trees such that every derived tree of R is Souslin and remains Souslin in V^T (i.e. in every generic extension via T). Let A be a name of an Aronszajn tree in V^T which remains an Aronszajn tree in $V^{T\times R'}$ for every derived tree R' of R. Let $\mathscr{C}$ be the closed unbounded set forcing poset and let $\dot{\mathscr{C}}$ be a V generic filter over $\mathscr{C}$. Then in $V[\dot{\mathscr{C}}]$ there is a Souslin tree $\tilde{T}$ which is a refinement of T such that:*

(1) *$\tilde{T}\times R'$ is Souslin for every derived tree R' of R.*

(2) *In $V[\dot{\mathscr{C}}]^{\tilde{T}}$, A is embedded on a club set into U.*

PROOF. For any $\gamma<\omega_1$ there is a dense subset of T consisting of conditions which describe $A \restriction \gamma$. Since T is Souslin we can find in V a club set $F\subseteq\omega_1$ such that for every $\gamma\in F$ and $x\in T_\gamma$ there is a countable tree $A(x)$ of height $\gamma+1$ such that

(i) $x \Vdash^T$ "$A\restriction\gamma = A(x)\restriction\gamma$" and

(ii) for any branch b of $A(x)\restriction\gamma$, b is determined by some element of $A(x)$ of level γ iff there exists $z\in T$, $z\geqq x$, such that $z\Vdash b$ is determined by some element of A_γ.

(A point in a tree determines the branch of all its predecessors.) Thus the last level, γ, of $A(x)$ consists of those points which some extension of x forces to be in $A\gamma$. Since T is a c.c.c. poset, this last level is countable too.

Observe that if g is a generic branch of T then

$$A=\bigcup\{A(x)\restriction\gamma : x\in T_\gamma \quad\text{and}\quad x\in g\}.$$

Let C be the generic closed unbounded subset of ω_1 we get from $\dot{\mathscr{C}}$ and let γ_α, $\alpha<\omega_1$, be the monotonic and continuous enumeration of C. We assume that $\gamma_0=0$ and $\gamma_i\in F$ for $i>0$.

Let $W=\langle W_\alpha:\alpha<\omega_1\rangle$ be a $\diamondsuit^*$ sequence in V that remains a $\diamondsuit^*$ sequence in $V^{T\times R'}$ for every derived tree R' of R. Let R^i, $i<\omega_1$, be an enumeration in V of all derived trees of R. Let $W^*=\langle W^*_\alpha\supseteq W_\alpha:\alpha\in\omega_1\rangle$ be a $\diamondsuit^*$ sequence in $V[C]$ that remains a $\diamondsuit^*$ sequence in any $V[C]$ generic extension over R^i, $i<\omega_1$.

The elements of $\tilde{T}$ consist of pairs (x, f) where $x \in T$ and f is a function. In fact, $(x, f) \in \tilde{T}_\alpha$ implies $x \in T_{\gamma_\alpha}$ and $f: A(x) \restriction C \to U \restriction C$ is a level preserving, one-to-one order preserving embedding. (So f is defined also on the last level of $A(x)$.)

For the partial order of $\tilde{T}$ we let $(x, f) < (\bar{x}, \bar{f})$ iff $x < \bar{x}$ in T and $\bar{f}$ extends $f \restriction A(\bar{x})$. (Some of the members of the last level of $A(x)$ might be not in $A(\bar{x})$; that is why we cannot ask $\bar{f} \supseteq f$.)

The construction of $\tilde{T}$ is described in §4.5, but it is clear now that in $V[C]^{\tilde{T}}$, A is embedded into u (on the club C).

4.3. Definition. (i) Let Z be a tree. $\beta \leqq \alpha$. For $a_1, \ldots, a_n \in Z_\alpha$ the *projection* of $\langle a_1, \ldots, a_n \rangle$ on Z_β is the n-tuple $a'_1, \ldots, a'_n \in Z_\beta$ such that $a'_i \leqq a_i$.

(ii) Let $\bar{a} = a_1, \ldots, a_n \in A_\alpha$, $\bar{u} = u_1, \ldots, u_n \in U_\alpha$ where $a_i \neq a_j$ and $u_i \neq u_j$ for $i \neq j$, so that $\langle \bar{a}, \bar{u} \rangle$ can be viewed as a one-to-one function p $(p(a_i) = u_i)$. Let $f: A \to U$ be a partial function defined on some levels of A. Suppose $\beta_0 \leqq \alpha$ is the last level such that f is defined on A_{β_0}. We say $\langle \bar{a}, \bar{u} \rangle$ (or the function p) is *compatible* with the function f iff:

(1) $a'_i \neq a'_j$ and $u'_i \neq u'_j$ for $i \neq j$, whenever $(a'_1, \ldots, a'_n)$ and $(u'_1, \ldots, u'_n)$ are the projections of $\bar{a}$ and $\bar{u}$ on A_β and U_β for $\beta_0 < \beta \leqq \alpha$ $(\beta \in C)$.

(2) $f(a'_i) = u'_i$ where $a'_1, \ldots, a'_n$ and $u'_1, \ldots, u'_n$ are the projections of $\bar{a}$ and $\bar{u}$ on A_{β_0} and U_{β_0} respectively.

$\tilde{T}_\alpha$ is constructed by induction on $\alpha < \omega_1$. The following property of $\tilde{T}$ will be needed and therefore ensured inductively to hold:

(*) For every $\beta < \alpha$, $(x, f) \in \tilde{T}_\beta$, $x' > x$, $x' \in T_{\gamma_\alpha}$, and $\bar{a} = a_1, \ldots, a_n \in A(x')_{\gamma_\alpha}$ (the last level of $A(x')$), and every $\bar{u} = u_1, \ldots, u_n \in U_{\gamma_\alpha}$ where $\langle \bar{a}, \bar{u} \rangle$ is compatible with f, there is $f' \supseteq f$ with $(x', f') \in \tilde{T}_\alpha$ such that $f'(a_i) = u_i$ for $i \leqq n$.

At every step of the construction we have to take care of countably many missions; these are succinctly described by saying that we take N_α generic filters where N_α, $\alpha < \omega_1$, are defined as follows:

$$N_\alpha = L_\delta[\tilde{T} \restriction \alpha,\ U \restriction \gamma_\alpha + 1,\ T \restriction \gamma_\alpha + 1,\ A \restriction \gamma_\alpha + 1, \tag{4.4}$$
$$\{R^i \restriction \alpha + 1 : i < \alpha\},\ C \cap \gamma_\alpha,\ W^* \restriction \gamma_{\alpha+1}]$$

where $\delta < \omega_1$ is such that N_α is a model of ZF^- (ZF minus the power set axiom).

4.5. *Construction of $\tilde{T}$ in $V[C]$*

Successor stages. Assume $\tilde{T}_\alpha$ is constructed. For every $(x, f) \in \tilde{T}_\alpha$ and $x' > x$, $x' \in T_{\gamma_{\alpha+1}}$, define the following poset:

$$\mathcal{P}_{x',f} = \{p : p \text{ is a finite one-to-one function, } p : A(x')_{\gamma_{\alpha+1}} \to U_{\gamma_{\alpha+1}} \text{ and } p \text{ is compatible with } f\}.$$

Observe that $\mathcal{P}_{x',f} \in N_{\alpha+1}$.

Now for every $(x,f) \in T_\alpha$, $x' > x$, $x' \in T_{\gamma_{\alpha+1}}$ and $p_0 \in \mathcal{P}_{x',f}$ pick some $\tilde{p}$ which is $N_{\alpha+1}$ generic over $\mathcal{P}_{x',f}$ such that $p_0 \in \tilde{p}$. Let $g = f \restriction A(x')$ and put $(x', g \cup \bigcup \tilde{p})$ into $\tilde{T}_{\alpha+1}$. Obviously $\tilde{T}_{\alpha+1}$ is countable and $(*)$ continues to hold.

Limit stages. Assume $\tilde{T} \mid \alpha$ is constructed, $\alpha < \omega_1$ a limit ordinal. We define $\tilde{T}_\alpha$. For $x' \in T_{\gamma_\alpha}$ define the poset:

$$\mathcal{P}_{x'} = \{(f,p) : (x,f) \in \tilde{T} \mid \alpha \text{ for some } x < x',\ p : A(x')_{\gamma_\alpha} \to U_{\gamma_\alpha},$$

$$p \text{ is a finite one-to-one function compatible with } f\}.$$

$$(f,p) \leqq (f',p') \text{ iff } f \restriction A(x') \subseteq f' \quad \text{and} \quad p \subseteq p'.$$

Now for every $(f,p) \in \mathcal{P}_{x'}$ pick some N_α generic filter $\dot{\mathcal{P}}_{x'}$ over $\mathcal{P}_{x'}$ such that $(f,p) \in \dot{\mathcal{P}}_{x'}$. Using $(*)$ it is seen that

$$f' = \bigcup \{f^* \restriction A(x') \cup p^* : (f^*, p^*) \in \dot{\mathcal{P}}_{x'}\}$$

is an order-preserving one-to-one function extending $f \restriction A(x')$ defined on $A(x') \mid C$; put (x', f') into $\tilde{T}_\alpha$. $\tilde{T}_\alpha$ is thus countable and $(*)$ holds.

This ends the definition of $\tilde{T}$. We show that $\tilde{T}_\alpha$ is Souslin and even.

4.6. Lemma. *$R^i \times \tilde{T}$ is Souslin for $i < \omega_1$. (This holds in $V[C]$.)*

Proof. Suppose on the contrary $R^i \times \tilde{T}$ is not Souslin for some $i < \omega_1$. This is equivalent to: $\tilde{T}$ is not Souslin in some generic extension $V[C][\dot{R}^i]$ where $\dot{R}^i$ is $V[C]$ generic over R^i. $\mathcal{C}$ and R^i are posets in V so we can change the order of forcing $V[C][\dot{R}^i] = V[\dot{R}^i][C]$. Moreover, as in a c.c.c. forcing extension every uncountable club set contains an old club set and as a Souslin tree is a c.c.c. poset which adds no new countable subsets to ω_1, we see that C is a generic closed unbounded set over the closed set forcing poset as defined in $V[\dot{R}^i]$. Let X be a maximal, uncountable pairwise-incompatible subset of $\tilde{T}$ and let $\underset{\sim}{X}$ be some name of X in $V[\dot{R}^i]^{\mathcal{C}}$.

Now in $V[\dot{R}^i]$, for every $N < H_{\omega_3}^{V[\dot{R}^i]}$, put $\alpha = N \cap \omega_1$ and let U_N be some transitive countable model of ZF^- such that $\bar{N}$, U_α, T_α, R^i_α, $\{A(x) : x \in T_\alpha\} \in U_N$. ($\bar{N}$ is the collapse of N.) By Lemma 3.7 (applied in $V[\dot{R}^i]$) there is some $N < H_{\omega_3}^{V[\dot{R}^i]}$ such that $\underset{\sim}{X}$, $\{R^i \mid i < \omega_1\}$, T, A, U and the diamond sequence W are in N and such that $C \cap \alpha$ is U_N generic over $\pi_N(\mathcal{C})$ where $\alpha = \alpha_N$ and π_N^{-1} (the inverse of the collapsing function) can be extended to

$$\pi_N^{-1} : \bar{N}[C \cap \alpha] \to H_{\omega_3}^{V[\dot{R}^i][C]}.$$

$\tilde{T} \restriction \alpha \in \bar{N}[C \cap \alpha]$ is a tree constructed as above and $X \restriction \alpha \in \bar{N}[C \cap \alpha]$ is a maximal subset of pairwise incompatible members of $\tilde{T} \restriction \alpha$.

4.7. CLAIM. *$X \restriction \alpha \in W_\alpha^*$ and hence $X \restriction \alpha \in N_\alpha$.*

PROOF. There is a club set $D \subseteq \omega_1$ in $V[\dot{R}^i][C]$ such that $\beta \in D \Rightarrow X \restriction \beta \in W_\beta^*$. So there is such a set D in Range($\bar{\pi}^1$). But $D \restriction \alpha$ is then unbounded in α, hence $\alpha \in D$ and $X \restriction \alpha \in W_\alpha^*$. As W_α^* is an element of N_α we get that $X \restriction \alpha \in N_\alpha$.

We want to prove that every $(f_0, p_0) \in \mathcal{P}_{x'}$ $(x' \in T_{\gamma_\alpha})$ has an extension $(f, p) \in \mathcal{P}_{x'}$ such that (x, f) is above an element of $X \restriction \alpha$ (where $x \in T$, $x < x'$ is such that $(x, f) \in \tilde{T}$). If we will prove that, then as $\tilde{T}_\alpha$ consists of elements obtained generically on N_α and as $X \restriction \alpha \in N_\alpha$, we will get that every $(x'f') \in \tilde{T}_\alpha$ is above some element of $X \restriction \alpha$ and hence $X \restriction \alpha$ is maximal in $\tilde{T}$ and $X = X \restriction \alpha$ is countable, proving Lemma 4.5

4.8. *Suppose on the contrary that for some $x' \in T\gamma_\alpha$ and $(f_0, p_0) \in \mathcal{P}_{x'}$, for every extension $(f, p) \in \mathcal{P}'_x$, (x, f) is not above an element of $X \restriction \alpha$.*

We will get a contradiction. Say $(x_0, f_0) \in \tilde{T}_\tau$, $\tau < \alpha$, $x_0 < x'$ and let k be the cardinality of p_0.

T is a Souslin tree in $V[\dot{R}^i]$ hence $T \restriction \alpha$ is a Souslin tree in $\bar{N}$ and remains so in $\bar{N}[C \cap \alpha]$ (a σ-closed extension). $\dot{x}' = \{x \in T \restriction \alpha : x < x'\}$ is cofinal and hence an $\bar{N}[C \cap \alpha]$ generic branch of the Souslin tree $T \restriction \alpha$. In $\bar{N}[C \cap \alpha][\dot{x}']$, $A \restriction \alpha$ is an Aronszajn tree which is in fact $A(x') \restriction \alpha$. (It is Aronszajn because A is the name of a tree which remains Aronszajn in $V^{R^i \times T}$.)

4.9. DEFINITION. Let $q : A(x')_\beta \to U_\beta$, $\gamma_\tau \leqq \beta < \alpha$, be a finite one-to-one partial function. We say *q is like p_0* iff

(1) The cardinality of q is k (that of p_0), and the projection of q on γ_τ equals the projection of p_0 on γ_τ.

(2) For every extension $(x, f) \in \tilde{T}$ of (x_0, f_0), where $x < x'$, and $\beta \geqq \gamma_{\text{level}(x,f)}$, if f is compatible with q, then (x, f) is not above an element of $X \restriction \alpha$.

Obviously, the projection of p_0 on any $\gamma_\tau \leqq \beta < \alpha$ is like p_0; and if q is like p_0 then any projection of q (above γ_τ) is like p_0. Note that the definition of "q is like p_0" does not mention p_0; the only parameters are $\tilde{T} \restriction \alpha$, $A(x') \restriction \alpha$, $U \restriction \alpha$, (x_0, f_0), $X \restriction \alpha$.

Remember that $C \cap \alpha$ is U_N generic over $\pi_N(\mathscr{C})$. $x', p_0 \in U_N$. So in $U_N[C \cap \alpha]$ it is true that for every β, $\gamma_\tau \leqq \beta < \alpha$, the projection of p_0 on β is like

p_0. This truth is forced by some $c \in \pi_N(\mathscr{C})$ which belongs to the generic filter generated by $C \cap \alpha$. So, for every q which is a projection of p_0 on β, $\gamma_\tau \leqq \beta < \alpha$, we get that in U_N

$$c \Vdash^{\pi_N(\mathscr{C})} q \text{ is like } p_0.$$

Observe that this forcing sentence is meaningful in $\bar{N}[\dot{x}']$ because all the parameters are (names) in $\bar{N}[\dot{x}]$. Moreover, an absoluteness argument gives us that

(4.10) For every q as above, in $\bar{N}[\dot{x}']$ it is true that $c \Vdash^{\pi_N(\mathscr{C})} q$ is like p_0

(i.e. $\bar{N}[\dot{x}']$ has replaced U_N).

Now we define in $\bar{N}[\dot{x}']$ the following set:

$$S_0 = \{q : c \Vdash q \text{ is like } p_0\}.$$

S_0 can be viewed as a subtree of $((A(x') \restriction \alpha) \times (U \restriction \alpha))^k$, and by 4.8, S_0 is uncountable in $\bar{N}[\dot{x}']$. In $\bar{N}[\dot{x}']$, $A(x') \restriction \alpha$ is an Aronszajn tree, as mentioned before. $U \restriction \alpha$ is a special Aronszajn tree so $(A(x') \restriction \alpha) \times (U \restriction \alpha)$ is an Aronszajn tree. Let S be the subset of S_0 consisting of all those tuples which have extensions in S_0 at every upper level. Then any tuple in S has extensions is X at every higher level below α. By Lemma 3.10 we have in $\bar{N}[\dot{x}']$ a closed unbounded $B \subseteq \alpha$ such that for all $\eta, \beta \in B$ with $\eta < \beta$ and $q \in S_\eta$ the set S^q_β (of all extensions of q in S of level β) is well-distributed. As W remains a diamond sequence in $\bar{N}[\dot{x}']$, there is a closed unbounded $B' \subseteq B$ such that, for $\eta \in B'$, $S \restriction \eta \in W_\eta$. It is known that in a generic extension made with a c.c.c. poset every club set contains an old club set, so we can assume that $B' \in \bar{N}$. We work in $\bar{N}[\dot{x}'][C \cap \alpha]$. Pick some $\eta \geqq \tau$ such that $\gamma_\eta = \eta$ and $C - \eta \subseteq B'$.

4.11. Remark. $S \restriction \gamma_\beta \in N_\beta$ for all $\beta \geqq \eta$.

Proof. $\gamma_\beta \in C - \eta \subseteq B'$ so $S \restriction \gamma_\beta \in W_{\gamma_\beta}$, but $W_{\gamma_\beta} \in N_\beta$ hence $S \restriction \gamma_\beta \in N_\beta$.

Fix now $q_0 \in S_\eta$ and some $(x, f) \in T_\eta$ extending (x_0, f_0) with $x \in \dot{x}'$ such that $f \supseteq q_0$.

4.12. Lemma. *For any $(z, g) \in \tilde{T}_\beta$, $\alpha > \beta \geqq \eta$, with $z < x'$, if $(z, g) \geqq (x, f)$, then there is some $q \in S_{\gamma_\beta}$ such that $q \subseteq g$.*

Proof. By induction on β — the level of (z, g). If $\beta = \eta$ then $g = f$ extends q_0. Assume the claim is true for some $\beta > \eta$. For a given $(z, g) \in \tilde{T}_{\beta+1}$ above (x, f) with $z < x'$, let $(z', g') \in \tilde{T}_\beta$, $(z', g') < (z, g)$. As the claim is true for β, there is some $q \in S_{\gamma_\beta}$ such that $g' \supseteq q$. $S^q_{\gamma_{\beta+1}}$ is well distributed as $\gamma_\beta, \gamma_{\beta+1} \in B$, and

$S^q_{\gamma_{\beta+1}} \in N_{\beta+1}$ by 4.11. g was constructed to be $N_{\beta+1}$ generic over $\mathcal{P}_{z,g'}$ and a density argument shows that g is compatible with some member of $S^q_{\gamma_{\beta+1}}$. (Here we use the fact that $S^q_{\gamma_{\beta+1}}$ is well distributed.)

Now to the limit case. Assume the claim is true for all $\beta' < \beta$, β is limit, and $(z, g) \in \tilde{T}_\beta$ is above (x, f) with $z < x'$. We want to show that g extends some member of S_{γ_β}. Recall that g was constructed generically over the poset $\mathcal{P}_z$, so we need a density argument. Take any $(f^*, p^*) \in \mathcal{P}_z$ where $(z^*, f^*) \in \tilde{T}_{\beta^*}$ is above (x, f), $\beta^* < \beta$, $z^* < z$. Then f^* contains some q from $S_{\gamma_{\beta^*}}$ (by the induction assumption). Now pick some $q^* > q$, $q^* \in S_{\gamma_{\beta^*+1}}$, q^* disjoint from the projection of p^* on γ_{β^*+1} (this is possible as $S^q_{\gamma_{\beta^*+1}}$ is well distributed). Let $z^{**} > z^*$, $z^{**} < z$ with $z^{**} \in T_{\gamma_{\beta^*+1}}$. Then by $(*)$ there is $f^{**} \supseteq f^*$ such that $(z^{**}, f^{**}) \geqq (z^*, f^*)$ and f^{**} is compatible both with q^* and p^*. Take some $q^{**} \in S^{q^*}_{\gamma_\beta}$; then $(f^{**}, p^* \cup q^{**}) \in \mathcal{P}_z$ is as required.

Now that the claim has been proved, the contradiction to 4.8 follows: Take $(z, g) \in \tilde{T} \restriction \alpha$ extending (x, f) with $z \in \dot{x}'$ and (z, g) above an element of $X \restriction \alpha$ (possible by Lemma 3.5). By Lemma 4.12, g extends some $q \in S_{\gamma_\beta}$ where β is the level of (z, g). But q is like p_0, a contradiction to (2) in Definition 4.9.

This ends the proof of Theorem 4.2.

4.13. Now we describe the last item of the proof, the Souslin iteration machinery that permits us to iterate Souslin trees ω_2 times. What we need is a reworking of Jensen's iteration theorem ([9] Ch. 8) while keeping R and all its derived trees Souslin.

4.14. THEOREM. *Assume $\diamondsuit$ and $\square$ hold in our universe W. Let R be a Souslin tree such that all its derived trees are Souslin. Suppose σ is a function defined on sequences of length $< \omega_2$ of Souslin trees, such that if $\langle T^\tau : \tau \leqq \nu \rangle$ $(\nu < \omega_2)$ is a sequence of Souslin trees, and in the Boolean universe of T^τ every derived tree of R is Souslin, then $\sigma(\langle T^\tau : \tau \leqq \nu \rangle)$ is a Souslin tree which is a refinement of T^ν and such that in the Boolean universe of that tree every derived tree of R is Souslin.* Then *there is a sequence $\langle T^\tau : \tau < \omega_2 \rangle$ of Souslin trees and projections of T^τ on T^γ for $\gamma < \tau$ such that*:

(i) *Every derived tree of R remains Souslin in the Boolean universe of T^τ for every $\tau < \omega_2$.*

(ii) $T^{\tau+1} = \sigma(\langle T^\nu : \nu \leqq \tau \rangle)$ *for every $\tau < \omega_2$.*

(iii) T^τ *is a refinement of T^γ for $\gamma < \tau$, and the projections commute.*

PROOF. Let $S_\alpha \subseteq \alpha$, $\alpha < \omega_1$, be a $\diamondsuit$ sequence that remains a diamond sequence in $W^{R'}$ for every derived tree R' of R. Let $\langle A_\lambda \mid \lambda \in \lim \omega_2 \rangle$ be the $\square$

sequence. For $\lambda \in \lim \omega_2$ we denote by $\langle \lambda(\nu) \mid \nu < \operatorname{otp}(A_\lambda)\rangle$ the increasing and continuous enumeration of the club $A_\lambda \subseteq \lambda$.

We define by induction on $\tau < \omega_2$ the trees T^τ as well as projections $h_{\gamma,\tau}$ and club sets $C_{\gamma,\tau}$ where $h_{\gamma,\tau} : T^\tau \to T^\gamma \mid C_{\gamma,\tau}$ is the projection function ($\gamma < \tau$).

In order to be able to use the diamond we assume the set of points of every tree is the set of countable ordinals.

4.15. *Construction of the trees.* At successors, $\tau + 1 < \omega_2$, we simply use the function σ to get $T^{\tau+1} = \sigma(\langle T^\nu \mid \nu \leqq \tau\rangle)$. T^0 is any tree satisfying (i).

At limit stages λ when the trees $\{T^i \mid i < \lambda\}$ have been constructed we cook up T^λ out of the trees $T^{\lambda(\nu)}$, $\nu < \operatorname{otp}(A_\lambda)$. To use the diamond we look at $\bigcup_{\nu<\operatorname{otp}(A_\lambda)}(T^{\lambda(\nu)} \times \{\nu\})$, which is a subset of $\omega_1 \times \omega_1$, and embed it into ω_1, using a fixed correspondence between ω_1 and $\omega_1 \times \omega_1$. So now the set S_α can be interpreted as a subset of the disjoint unions of the Souslin trees along the club set A_λ. This is an important but somewhat technical way of decoding the diamond and we will overlook it from now on.

For limit $\lambda < \omega_2$, we have two cases: 4.16 and 4.20.

4.16. *Case I.* $\operatorname{cf}(\lambda) = \omega$. For a club C, let C^* denote the set of fixed points of C. Say $\theta = \operatorname{otp}(A_\lambda)$, so $\theta < \omega_1$. Let

$$C^\lambda = \Big(\bigcap_{\nu<\tau<\theta} C^*_{\lambda(\nu),\lambda(\tau)}\Big) - \theta \cup \{0\}.$$

$C^\lambda \subseteq \omega_1$ is a club set and we set $C_{\lambda(\nu),\lambda} = C^\lambda$. Let $\langle c_\nu : \nu < \omega_1\rangle$ enumerate C^λ in an increasing and continuous way.

T^* is defined to be the inverse limit of $T^{\lambda(\nu)} \mid C^\lambda$, i.e., T^* consists of all sequences $\langle x_\nu \mid \nu < \theta\rangle = \bar{x}$ such that, for some $\alpha \in C^\lambda$, $x_\nu \in T^{\lambda(\nu)}_\alpha$ for all $\nu < \theta$; and, for $\tau < \nu$, $x_\tau = h_{\lambda(\tau),\lambda(\nu)}(x_\nu)$.

T^* is ordered naturally: $\bar{x} < \bar{y}$ iff $x_\nu < y_\nu$ for all $\nu < \theta$. $\pi(\bar{x}) = x_\nu$ is a natural projection of T^* on $T^{\lambda(\nu)} \mid C^\lambda$.

The Souslin tree T^λ will be an initial subtree of T^*, constructed with the aid of the $\diamondsuit$, such that the following holds:

(*) For any $\bar{x} \in T^\lambda$, $y \in T^{\lambda(\nu)} \mid C^\lambda$ with $y \geqq x_\nu$ there is $\bar{x}' \in T^\lambda$, $\bar{x}' \geqq \bar{x}$, such that $x'_\nu = y$.

So condition (*) says that T^λ can be projected onto $T^{\lambda(\nu)} \mid C^\lambda$, and hence onto T^μ, $\mu < \lambda$.

The construction of T^λ is done inductively as follows:

Successors steps. Suppose $T^\lambda \mid \eta + 1$ is defined, $\eta > 0$. For every $\bar{x} \in T^\lambda_\nu$ and $y \in T^{\lambda(\nu)}_{c_\nu+1}$, $y > x_\nu$, pick some $\bar{x}' \in T^*_{\eta+1}$, $\bar{x}' > \bar{x}$, with $x'_\nu = y$, and put $\bar{x}' \in T^\lambda_{\eta+1}$.

In case $\eta = 0$ we use the $\diamondsuit$ to define T_1^λ if the following special property holds:

(4.17) $\theta = c_1$, $S_\theta \subseteq \bigcup_{\nu<\theta} T^{\lambda(\nu)} \times \{\nu\}$, and for any $y \in T_\theta^{\lambda(i)}$, $i < \theta$, there are $j < \theta$, $j > i$, and $y' \in T_\theta^{\lambda(j)}$ such that $y = h_{\lambda(i),\lambda(j)}(y')$ and y' is above an element of S_θ.

In that case, we construct T_1^λ such that, for every $\bar{z} \in T_1^\lambda$, there is $j < \theta$ with z_j above an element of S_θ.

Limit steps. Suppose $T^\lambda \restriction \alpha$ is defined and α is limit. $T_\alpha^\lambda \subseteq T_\alpha^*$ is defined such that (*) holds, as follows.

We show at first that, for every $\bar{x} \in T^\lambda \restriction \alpha$ and $y \in T_{c_\alpha}^{\lambda(\nu)}$, $y > x_\nu$, there is $\bar{y} \in T_\alpha^*$, $\bar{y} > \bar{x}$ such that $y_\nu = y$, and, for every $\bar{z} \in T^*$, if $\bar{z} < \bar{y}$, then $\bar{z} \in T^\lambda \restriction \alpha$. Let $\langle \alpha_i : i < \omega \rangle$ be an increasing sequence cofinal in α, and $\langle \theta_i : i < \omega \rangle$ an increasing sequence cofinal in θ with $\theta_0 = \nu$. Define now, inductively, an increasing sequence $\bar{x}^i \in T^\lambda \restriction \alpha$ above $\bar{x}, \bar{x}^i < \bar{x}^j$ for $i < j$, and a sequence $y_n \in T_{c_\alpha}^{\lambda(\theta_n)}$, $y_0 = y$, such that

$$y_n = h_{\lambda(\theta_n),\lambda(\theta_{n+1})}(y_{n+1}) \quad \text{and} \quad y_n > x_{\theta_n}^n.$$

(*) is used in this construction.

Next, to obtain T_α^λ, for every $\bar{x} \in T^\lambda \restriction \alpha$, $y \in T_{c_\alpha}^{\lambda(\nu)}$, $y > x_\nu$, we pick $\bar{y} \in T_\alpha^*$, $y_\nu = y$, as was shown above to exist, and put $\bar{y} \in T_\alpha^\lambda$. In the following case we use the $\diamondsuit$: Suppose $S_\alpha \subseteq T^\lambda \restriction \alpha$ is a dense set with the following additional property:

(4.18) For every $\bar{x} \in T^\lambda \restriction \alpha$ and $y \in T_{c_\alpha}^{\lambda(\nu)}$, $y > x_\nu$, there is $\bar{z} \in T^\lambda \restriction \alpha$, $\bar{z} \geqq \bar{x}$, $z_\nu < y$, such that $\bar{z}$ is above some element of S_α.

In that case we construct T_α^λ such that every $\bar{y} \in T_\alpha^\lambda$ is above an element of S_α.

4.19. Claim. *$R' \times T^\lambda$ is Souslin for every derived tree R' of R.*

Proof. We will show that, in every generic extension with a derived tree R', T^λ is a Souslin tree. Let $W[\dot{R}']$ be such a generic extension of the ground model W and $X \in W[\dot{R}']$, $X \subseteq T^\lambda$, be a maximal pairwise incompatible subset of T^λ. $\langle S_\alpha : \alpha \in \omega_1 \rangle$ is a diamond sequence in $W[\dot{R}']$ too, so $\{\alpha : X \cap \alpha = S_\alpha\} = S$ is a stationary subset of ω_1. The trees $T^{\lambda(\nu)}$, $\nu < \theta$, are Souslin trees in $W[\dot{R}']$, because $R' \times T^\tau$ is a Souslin tree for $\tau < \lambda$ (by the induction hypothesis). By Lemma 3.5 we have a closed unbounded set $B \subseteq \omega_1$ such that for $\alpha \in B \cap S$ the additional property (4.18) above holds. It follows by the usual argument given in the construction of a Souslin tree that X is countable.

4.20. *Case* II. $\mathrm{cf}(\lambda) = \omega_1$.

In this case we must have $\mathrm{otp}(A_\lambda) = \omega_1$. Let $\langle \lambda(\nu) : \nu < \omega_1 \rangle$ enumerate A_λ in an increasing and continuous manner. The tree T^λ is defined using the trees

$T^{\lambda(\nu)}$, $\nu < \omega_1$, as follows. We will define a sequence of indexes $i_\alpha < \omega_1$ for $\alpha < \omega_1$ and then set $T^\lambda_\alpha = T^{\lambda(\alpha)}_{i_\alpha}$. T^λ is partially ordered, thus, for $a \in T^\lambda_\alpha$, $b \in T^\lambda_\beta$,

$$a < b \text{ iff } \alpha < \beta \quad \text{and} \quad a < h_{\lambda(\alpha),\lambda(\beta)}(b).$$

i_α are defined inductively. If $\alpha = \beta + 1$ then i_α is the first i greater than all i_η, $\eta \leqq \beta$, which is a fixed point of $C_{\lambda(\eta),\lambda(\gamma)}$ for all $\eta < \gamma \leqq \beta + 1$. If α is limit put $i^* = \bigcup_{\delta<\alpha} i_\delta$, then $i^* \in (\bigcap_{\tau<\nu<\alpha} C^*_{\lambda(\tau),\lambda(\nu)}) - \alpha$, hence $i^* = c^{\lambda(\alpha)}_\tau$ for some $\tau \geqq 1$ (where $\langle c^{\lambda(\alpha)}_\tau : \tau < \omega_1 \rangle$ enumerates $C^{\lambda(\alpha)}$). Set now $i_\alpha = \tau$. We leave it to the reader to check that T^λ is indeed a tree and that the projections can be defined.

4.21 Claim. *If R' is a derived tree of R then $R' \times T^\lambda$ is a Souslin tree.*

Proof. Let $X \subseteq T^\lambda$ be a dense open set in $W[\dot{R}']$ (a W-generic extension over the derived tree R'). We know, by the induction hypothesis, that $T^{\lambda(\nu)}$, $\nu < \omega_1$, are Souslin trees in $W[\dot{R}']$. $\langle S_\alpha : \alpha < \omega_1 \rangle$ continues to be a diamond sequence in $W[\dot{R}']$, so $S = \{\alpha : X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1. We can get a club set, $B \subseteq \omega_1$, such that for $\alpha \in B$ there is a countable elementary submodel $N < H^{W[\dot{R}']}_{\omega_3}$ with $\alpha = \alpha_N$ and everything we spoke about is in N. $\langle T^{\lambda(\nu)} \restriction \alpha : \nu \in \alpha \rangle$ is a sequence of Souslin trees in $\bar{N}$ and $T^\lambda \restriction \alpha$ is their limit as was defined before. For $\alpha \in B \cap S$, $S_\alpha = X \cap (T^\lambda \restriction \alpha) \in \bar{N}$ is a dense open subset of $T^\lambda \restriction \alpha$. Say $\lambda(\alpha) = \bar{\lambda}$, then $A_{\bar{\lambda}} = A_\lambda \restriction \alpha = \{\lambda(i) : i < \alpha\}$.

By the arguments of Lemma 3.5 for every $y \in T^{\lambda(i)}_\alpha$, $i < \alpha$, there is $j < \alpha$, $j > i$, and $y' \in T^{\lambda(j)}_\alpha$ such that $y = h_{\lambda(i),\lambda(j)}(y')$ and y' is above some element of S_α. As $i_\alpha = 1$, it turns out that the special property in the successor case definition of $T^{\bar{\lambda}}_1$ holds and hence that every element of T^λ_α is above some point in X. So T^λ is a Souslin tree.

4.22. Proof of Theorem 4.1. Now that all parts of the proof are presented, we assemble them and give a general description of the proof of Theorem 4.1. Recall that R is a Souslin tree constructed in L such that all the derived trees of R are Souslin, U is some special Aronszajn tree which will become the universal special Aronszajn tree. To prepare the ground we iterate with countable support ω_2 many generic club subsets of ω_1, $\langle C_\alpha \restriction \alpha < \omega_2 \rangle$ such that $W = L[\langle C_\alpha : \alpha \in \omega_2 \rangle]$ is a generic extension of L made with a σ-closed, $\aleph_2$-c.c. poset. W and L have the same cardinals, GCH holds in W, the diamond and the $\square$ sequences of L still work in W. R as any other Souslin tree in L remains Souslin in W. Every subset of $\aleph_1$ in W is already in some intermediate extension $V = L[\langle C_\alpha : \alpha \in \gamma \rangle]$, $\gamma < \omega_2$, and C_γ is a V-generic closed unbounded subset of ω_1. As $V = L[G]$ for some $G \subseteq \omega_1$, it follows that $\diamondsuit^*$ holds in V. (See [9] p. 113 for details.)

Now in W, we construct inductively an iteration sequence of Souslin trees T^τ, $\tau < \omega_2$, using Theorem 4.14 for limit stages and Theorem 4.2 at successor stages. The induction hypothesis for T^τ, $\tau < \omega_2$, is that R and its derived trees are Souslin in W^{T^τ}. We take care, in turn, of all Aronszajn trees in every W^{T^τ}. At successor stages, when T^τ is already constructed, we are presented with a name of an Aronszajn tree, A, in W^{T^τ}; we pick some intermediate universe $V = L[\langle C_\alpha : \alpha \in \gamma\rangle]$ such that T^τ, $A \in V$. There are then two possibilities:

(1) A is an Aronszajn tree in $V^{T^\tau \times R'}$ for every derived tree R' of R. In this case we use Theorem 4.2 to get in $V[C_\gamma]$ the Souslin tree $T^{\tau+1}$ which is a refinement of T^τ such that (in $V[C_\gamma]$ and hence in W) $T^{\tau+1} \times R'$ is a Souslin tree for any derived trees R' and, in $W^{T^{\tau+1}}$, A is embedded on a club set into U.

(2) (1) does not hold. Then we simply set $T^{\tau+1} = T^\tau$ and pass to the next Aronszajn tree.

Finally, let T be the direct limit (union) of $\langle T^\tau \mid \tau \in \omega_2\rangle$ and then $W[\dot{T}]$ is the model promised by Theorem 4.1. Indeed $|T| = \aleph_2$ and T satisfies that the c.c.c. GCH holds in $W[\dot{T}]$; R and its derived trees are Souslin trees there. For every Aronszajn tree A in $W[\dot{T}]$, if A is Aronszajn in the Boolean universe of every derived tree of R, then at some stage A was embedded on a club set into U. On the other hand, if A is non-Aronszajn in some derived tree R', then, by Lemma 3.2, A contains an isomorphic image, on a club set, of a derived tree of R. It follows also that if A is Souslin in $W[\dot{T}]$ then, modulus a club set, A is the union countable many derived trees of R.

4.23. *Hints to further results.* To get the consistency of "GCH + there is a universal Aronszajn tree" is obviously simpler than Theorem 4.1, since there is no Souslin tree we wish to keep Souslin.

Let us remark now why the proof of Theorem 4.1 fails to give the consistency with GCH of "Every two Aronszajn trees are isomorphic on a club set". Of course, we proved that CH implies the negation of that statement, but where exactly does the consistency proof break? Well, suppose we are given a Souslin tree T (an intermediate stage of the iteration) and *names* in T-forcing of Aronszajn trees A_1 and A_2. For a club set of α's, $x \in T_\alpha$ "knows" what $A_1 \mid \alpha$ and $A_2 \mid \alpha$ are $(A_1(x) \mid \alpha$ and $A_2(x) \mid \alpha)$. But $x \in T_\alpha$ cannot guess what the α level of A_1 and A_2 is. Yet, any isomorphism of $A_1 \mid \alpha$ onto $A_2 \mid \alpha$ must take into consideration the branches defined by the α level. Hence we cannot refine T to a tree which makes A_1 and A_2 isomorphic. The proof of Theorem 4.1 (modeled after Jensen's proof) worked because T is a c.c.c. forcing which adds no reals, hence there are only countably many possible α-levels of A_1 and A_2 and we took care of all of them.

On the other hand, it is possible to get the consistency of CH + "any two trees have isomorphic subtrees". When we want to find a subtree of both $A_1(x)\restriction\alpha$ and $A_2(x)\restriction\alpha$, which is the beginning of a common tree, we enumerate all possible α-levels of A_1 and A_2. Splitting each possible α-level into countably many sets, we obtain $\{B_i \mid i \in \omega\}$ such that:

(i) B_i is a set of branches through $A_1(x)\restriction\alpha$ and $A_2(x)\restriction\alpha$ whose union is $(A_1(x) \cup A_2(x))\restriction\alpha$.

(ii) $B_i \cap B_j = \emptyset$ for $i \neq j$.

(iii) For any $y \in T$ describing the α-level of A_1 and A_2, there is i such that $y \Vdash B_i$ is included in the set of branches determined by the α-level of the trees.

Part III: Consistency Results with $\neg$ CH

§5. Every two Aronszajn trees are isomorphic on a club set $+ 2^{\aleph_0} = \aleph_2$

5.1. THEOREM. *The following is consistent with* ZFC;

Every two Aronszajn trees are isomorphic + *Martin's Axiom* $+ 2^{\aleph_0} = \aleph_2$.

PROOF. Given two Aronszajn trees T^1, T^2 we define a proper forcing notion $\mathcal{P} = \mathcal{P}(T^1, T^2)$ which makes the two trees isomorphic on a club set. For proper forcing look at [16]. At the present state of knowledge we do not know how to go with proper forcing beyond $2^{\aleph_0} = \aleph_2$, so the approach of §7 is of value.

5.2. DEFINITION OF $\mathcal{P}$. $p = (c, f) \in \mathcal{P}$ iff $c \subseteq \omega_1$ is a finite set (called the set of levels of p), and, if $\bar{c}$ denotes $\max c$, $f = (\bar{a}, \bar{b})$, $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$ where $a_i \in T^1_{\bar{c}}$, $b_i \in T^2_{\bar{c}}$, for $i \leqq n$, is a pair of n-tuples such that for $i, j \leqq n$ and $\alpha \in c$

$$\exists x \in T^1_\alpha(x \leqq a_i \,\&\, x \leqq a_j) \Leftrightarrow \exists y \in T^2_\alpha(y \leqq b_i \,\&\, y \leqq b_j).$$

In other words, the function $a_i \to b_i$ can be extended to an isomorphism of $\{x \in T^1 \mid x \in T^1_\alpha$ and $x \leqq a_i$ for some $\alpha \in c$ and $i \leqq n\}$ onto $\{y \in T^2 \mid y \in T^2_\alpha$ and $y \leqq b_i$ for some $\alpha \in c$ and $i \leqq n\}$. In this case there is only one such isomorphism and we call it the *isomorphism determined* by p.

We partially order $\mathcal{P}$ in a natural way: $(c, f) \leqq (c', f')$ iff $c \subseteq c'$ and the isomorphism determined by (c', f') extends the isomorphism determined by (c, f).

5.3. DEFINITION. If $(\bar{a}, \bar{b})$ is a pair of n-tuples: $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$, $a_i \in T^1_\gamma$, $b_i \in T^2_\gamma$ (γ is then called the *level* of $(\bar{a}, \bar{b})$) and if $\alpha \leqq \gamma$, then the *projection*, $\pi_\alpha(\bar{a},\bar{b})$, of $(\bar{a},\bar{b})$ to α is the pair $(\bar{a}^*,\bar{b}^*)$ where $a_i^* \in T^1_\alpha$, $b_i^* \in T^2_\alpha$ and $a_i^* \leqq a_i$, $b_i^* \leqq b_i$ for $i \leqq n$. If $p=(c,f)\in \mathcal{P}$, $\alpha \leqq \bar{c} = \bigcup c$, then $\pi_\alpha(p)$ is the condition given by $(c \cap \alpha) \cup \{\alpha\}$ and the projection of f to α, if such a condition exists, i.e., $\pi_\alpha((c,f)) = (c \cap \alpha \cup \{\alpha\}, \pi_\alpha(\bar{a},\bar{b}))$ if this is a condition in $\mathcal{P}$ and the projection is undefined otherwise. If $\alpha \in c$, then $\pi_\alpha((c,f))$ is always defined. If $\alpha' < \alpha$ then $\pi_{\alpha'}(\pi_\alpha(p)) = \pi_{\alpha'}(p)$. If p is a condition, $p=(c,f)$, and g is a pair of n-tuples of level $\beta \geqq \bar{c}$ such that $(c \cup \{\beta\}, g)$ is a condition extending p, then we say p and g are compatible and denote $(c \cup \{\beta\}, g)$ by $p \cup g$.

It is quite clear that if $\dot{\mathcal{P}}$ is a generic filter over $\mathcal{P}$ then $C = \bigcup\{c \mid (c,f)\in \dot{\mathcal{P}}$ for some $f\}$ is an unbounded subset of ω_1 and $F = \bigcup\{f \mid (c,f) \in \dot{\mathcal{P}}$ for some $c\}$ is the desired isomorphism between $T^1 \mid C$ and $T^2 \mid C$. (We assume that our trees are normal and that every point has $\aleph_0$ successors.)

We prove now that $\mathcal{P}$ is proper in order to show that ω_1 is not collapsed and that we can iterate this forcing. What have we to prove in order to show that $\mathcal{P}$ is proper? Given a countable elementary substructure $N < H_{\aleph_2}$ (the collection of all sets of cardinality hereditarily less than $\aleph_2$) such that $\mathcal{P}$, T^1, $T^2 \in N$ and given $p_0 \in \mathcal{P} \cap N$, we must find $q^* \geqq p_0$, $q^* \in \mathcal{P}$, such that for any $D \in N$, a dense subset of $\mathcal{P}$, any $q \geqq q^*$ is compatible with some member of $D \cap N$.

So let N be as above, let $\alpha = N \cap \omega_1$, then $\alpha \subseteq N$. Given $p_0 = (c_0, f_0)$, define $c^* = c_0 \cup \{\alpha\}$ and let f^* be any pair $(\bar{a}^*,\bar{b}^*)$ of level α such that the projection of $(\bar{a}^*,\bar{b}^*)$ on $\bigcup c_0$ is f_0. $(c^*,f^*) = q^* \in \mathcal{P}$ is as required: Let $D \in N$ be a dense subset of $\mathcal{P}$ and $q \geqq q^*$ be given. First find $q' \geqq q$, $q' \in D$. $\pi_\alpha(q') = (c',f')$ is a condition extending q^*. Pick $\alpha' < \alpha$, $\alpha' \geqq \bigcup(c' \cap \alpha)$ such that f' and the projection of f' on α' have the same cardinality, i.e. the elements of f' do not meet above α'. Let $p = \pi_{\alpha'}(q')$, $p \in N \cap \mathcal{P}$. What we need is an extension $p^* \geqq p$, $p^* \in D \cap N$, p^* compatible with $(c',f') = \pi_\alpha(q')$ for then p^* will be compatible with q', hence with q. In order that p^* will be compatible with (c',f') it has to "respect" f', i.e. if $f' = (\bar{a}',\bar{b}')$, $\bar{a}' = (a_1,\ldots,a_l)$, $\bar{b}' = (b_1,\ldots,b_l)$, then p^* must take points below a_i which are in the levels of p^* to points below b_i. For any $\alpha' < \gamma < \alpha$, $\pi_\gamma(f') = f'_\gamma$ is compatible with p, we denote by $p \cup f'_\gamma$ the projection $\pi_\gamma(q')$. As $N < H_{\aleph_2}$, $p \cup f'_\gamma$ has the following property: there is $p' \geqq p \cup f'_\gamma$, $p' \in D \cap N$, such that p' has no levels between α' and γ and $\pi_\gamma(p') = p \cup f'_\gamma$. (The point is that q' gives such a condition and hence there is one in N.)

Now we define in N the following set H of pairs of l-tuples. $g \in H$ iff the level of g is $\gamma, \alpha' < \gamma < \alpha$, g is compatible with p and $p \cup g$ has an extension $g' \in D \cap N$ such that $\pi_\gamma(g') = p \cup g$. H is uncountable in N (as $f'_\gamma \in H$ for all $\alpha' < \gamma < \omega_1^N$).

By Lemma 3.10 there is $\alpha' < \gamma < \alpha$ such that H has $\aleph_0$ members of level γ with pairwise disjoint domain. Pick one $g \in H$ of level γ such that g has a domain disjoint from f'_γ. Now $p \cup g$ has an extension $g' \in D \cap N$ such that $\pi_\gamma(g') = p \cup g$; it follows that g' is compatible with (c', f') as required.

This has taken care of a single step in the iteration. Now an iteration of length ω_2 of proper forcing of size $\aleph_1$ gives the desired model.

§6. Martin's Axiom is not enough

We are going to prove the following theorem.

6.1. THEOREM. *Martin's Axiom & $2^{\aleph_0} > \aleph_0$ do not imply that every two Aronszajn trees are isomorphic on a club set.*

Let us give first a general and simplified description of the proof. We start with two Aronszajn trees T^1, T^2 which are not isomorphic, and proceed with an iteration of c.c.c. posets in order to get MA, while trying to keep the two trees non-isomorphic. What happens if a c.c.c. poset $\mathcal{P}$ appears in some intermediate stage of the iteration and in $V^{\mathcal{P}}$ the two trees are isomorphic? We must destroy the c.c.c.-ness of $\mathcal{P}$ if we want to get MA. This is the way to do it: Let f be a name in $V^{\mathcal{P}}$ of an isomorphism on a club C of T^1 and T^2. Pick conditions $p_i \in \mathcal{P}$ and $a_i \in T^1_i$, $b_i \in T^2_i$ for $i < \omega_1$ such that

$$p_i \Vdash f(a_i) = b_i.$$

It follows that if $i < j$ and the level of C where a_i and a_j meet is different from the level of C where b_i and b_j meet, then p_i and p_j are incompatible. (We say then that (a_i, b_i) and (a_j, b_j) are conflicting on C.)

So define a poset $\mathcal{Q} = \mathcal{Q}(\langle a_i, b_i \rangle : i \in \omega_1)$ by $q \in \mathcal{Q}$ if q is a finite set of pairs and any two such pairs are conflicting on C (and hence their respective conditions are incompatible in $\mathcal{P}$). Now, in $V^{\mathcal{Q}}$, $\mathcal{P}$ is no longer c.c.c. To show that $\mathcal{Q}$ itself is a c.c.c.-poset and, moreover, is not dangerous, we need a special property of the trees T^1, T^2.

6.2. NOTATIONS AND DEFINITIONS. T is an Aronszajn tree, $\bar{a} \in T_\alpha$ means that $\bar{a} = \langle a_1, \ldots, a_n \rangle$ and for $i \leqq n$, $a_i \in T_\alpha$ (the α'th level of T), α is called the *level* of $\bar{a}$. We write $\bar{a} \in T$ if $\bar{a} \in T_\alpha$ for some $\alpha < \omega_1$. If $\bar{a} \in T_\alpha$, $\beta \leqq \alpha$, then $\pi_\beta(\bar{a}) \in T_\beta$ — the *projection* of $\bar{a}$ on T_β — is defined naturally, in particular if $a \in T_\alpha$, $\beta \leqq \alpha$, then $\pi_\beta(a) \leqq a$. Let $C \subseteq \omega_1$ be a club set; we say $a, b \in T$ *meet* at level $\nu \in C$ on C if a, b are incompatible in T and ν is the maximal ordinal in C such that $\pi_\nu(a) = \pi_\nu(b)$.

We shall deal with sequences of n-tuples $\langle \bar{a}_\zeta \in T : \zeta \in \omega_1 \rangle$ such that:

(6.3) For some α and $\bar{a} \in T_\alpha$, $\bar{a}$ is an n-tuple of distinct elements such that $\bar{a} = \pi_\alpha(\bar{a}_\zeta)$ for all $\zeta \in \omega_1$. The level of $\bar{a}_\zeta$ is $\geqq \zeta$.

6.4. Definition. $\bar{a}, \bar{b} \in T$ are said to be *conflicting on C n-tuples* if, for all $i, j \leqq n$ with $i \neq j$, a_i and b_i are incompatible and the level where a_i and b_i meet on C is different than the level where a_j and b_j meet on C.

If $C^1 \subseteq C$, $\bar{a}$ and $\bar{b}$ are conflicting on C^1, then they are conflicting on C.

Claim. *The Soulin tree T built by Jensen in* [9, p. 46] *is readily seen to satisfy the following*:

(6.5) *For any $\langle \bar{a}_\zeta \in T : \zeta \in \omega_1 \rangle$ satisfying* (6.3) *and club C there are $\alpha, \beta \in \omega_1$ such that $\bar{a}_\alpha$ and $\bar{a}_\beta$ are conflicting on C.*

To remind the reader: at limit levels in the construction of T we add $\aleph_0$ branches "generically" (over the diamond); a condition is a bounded information of finitely many branches. So for some "elementary" δ, the diamond guesses $\langle \bar{a}_\xi : \xi \in \delta \rangle$ and $C \cap \delta$. T_δ is defined so that any n tuple above δ there is conflicting with some $\bar{a}_\xi$ below δ.

6.6. Lemma. *Assume T satisfies* (6.5). *Let C be a club set and, for $\xi \in \omega_1$, l_ξ is a finite set of m-tuples of T such that any two n-tuples in l_ξ are conflicting on C and, for any n-tuple $\bar{b} \in l_\xi$, the elements of $\bar{b}$ meet below ξ on C.* Then *there are ξ, ξ' such that any two n-tuples in $l_\xi \cup l_{\xi'}$ are conflicting.*

Proof. For any limit $\xi < \omega_1$ take all the tuples in l_ξ of level $\geqq \xi$ and look at their projection on T_ξ. We thus get a finite subset of T_ξ which we enumerate as $\bar{a}_\xi$. Now define a pressing down function, $g(\xi)$, giving

(i) the information on the tuples of l_ξ of level below ξ,

(ii) where the elements of $\bar{a}_\xi$ meet on C (below ξ), and

(iii) the finite information on the order type of the elements of $\bar{a}_\xi$ and the tuples of l_ξ.

g has a constant value on an uncountable subset $I \subseteq \omega_1$. We get α and $\bar{a} \in T_\alpha$ such that $\bar{a} = \pi_\alpha(\bar{a}_\xi)$ and no elements of $\bar{a}_\xi$ meet on C above α for $\xi \in I$. But as T satisfies (6.5), we get $\xi, \xi' \in I$ with $\bar{a}_\xi$ and $\bar{a}_{\xi'}$ conflicting on C. It is long to write, but easy to see that l_ξ and $l_{\xi'}$ are as required.

6.7. Definition. (i) A poset $\mathcal{Q}$ is *dangerous* (for T) if there is a club $C \subseteq \omega_1$ and a sequence $\langle (q_i, \bar{a}_i) : i < \omega_1 \rangle$ such that $\langle \bar{a}_i : i < \omega_1 \rangle$ satisfies (6.3), $q_i \in \mathcal{Q}$ and, for $i \neq j$, if q_i are compatible in $\mathcal{Q}$, then $\bar{a}_i$ and $\bar{a}_j$ are *not* conflicting on C.

(ii) $\mathcal{Q}$ is a *non-dangerous* poset if $\mathcal{Q}$ satisfies the c.c.c. and $\mathcal{Q}$ is not dangerous, i.e. for any club $C \subseteq \omega_1$ and any sequence $\langle (q_i, \bar{a}_i) : i \in \omega_1 \rangle$ as above, for some $i \neq j$, q_i and q_j are compatible and $\bar{a}_i$ and $\bar{a}_j$ are conflicting on C.

6.8. LEMMA. *If T satisfies* (6.5) *in V and $\mathcal{Q}$ is a non-dangerous poset in V, then in $V^{\mathcal{Q}}$, T still satisfies* (6.5).

PROOF. As $\mathcal{Q}$ satisfies the c.c.c. any club C in $V^{\mathcal{Q}}$ contains a club C^* in V. If T does not satisfy (6.5) in $V^{\mathcal{Q}}$ then we have a club C, which we assume to be in V, and a sequence of n-tuples. For any α we can find a condition q_α which forces $\bar{a}_\alpha$ to be the α'th member in the sequence. Now $(q_\alpha, \bar{a}_\alpha)$, $\alpha \in \omega_1$, shows that $\mathcal{Q}$ is dangerous.

6.9. COROLLARY. *If $\mathcal{Q}$ is non-dangerous then, in $V^{\mathcal{Q}}$, T is Aronszajn.*

PROOF. If a_α, $\alpha < \omega_1$, is a branch of T then clearly a_α, a_β are never in conflict. Hence (6.5) implies T is Aronszajn.

6.10. LEMMA. *It is possible to iterate non-dangerous posets*:

(a) *If $\mathcal{Q}$ is non-dangerous and $\mathcal{P} \in V^{\mathcal{Q}}$ is non-dangerous then the composition $\mathcal{Q} * \mathcal{P}$ is non-dangerous.*

(b) *The direct limit* (*iteration with finite support*) *of non-dangerous posets is non-dangerous.*

PROOF. Like Solovay and Tennenbaum.

6.11. DEFINITION. Let T be a tree satisfying (6.5) and $D \subseteq \omega_1$ be a club and $\langle \bar{b}_\eta \in T : \eta \in \omega_1 \rangle$ be a sequence of n-tuples of T satisfying (6.3). We define $\mathcal{Q}(\langle \bar{b}_\eta : \eta \in \omega_1 \rangle, D) = \mathcal{Q}$ to be the poset of all finite sets of pairwise conflicting on D n-tuples from $\{\bar{b}_\eta : \eta \in \omega_1\}$.

6.12. LEMMA. *$\mathcal{Q}$ as above is non-dangerous.*

PROOF. That $\mathcal{Q}$ satisfy the c.c.c. is just Lemma 6.6.

If C is a club and $(q_i, \bar{a}_i)$ are as in Definition 6.7, then $q_i \cup \{\bar{a}_i\}$ is a finite set of tuples. Applying the proof of Lemma 6.6 we get q_i and q_j compatible such that $\bar{a}_i$ and $\bar{a}_j$ are conflicting on C.

6.13. For $\alpha < \omega_1$ we define $\mathcal{Q}^\alpha$ like the definition of $\mathcal{Q}$, only now our conditions are conflicting n-tuples from $\{\bar{b}_\eta : \eta \in \omega_1 - \alpha\}$.

6.14. CLAIM. *For some $\alpha < \omega_1$, $\mathcal{Q}^\alpha$ satisfies the following*:

For any $\beta \in \omega_1$, $\{e \in \mathcal{Q}^\alpha : e$ contains an n-tuple of level above $\beta\}$ is dense in $\mathcal{Q}^\alpha$

PROOF. Otherwise we will contradict the c.c.c. of $\mathcal{Q}$.

6.15. PROOF OF THEOREM 6.1. We iterate $\aleph_2$ times (for example) non-dangerous posets and get $2^{\aleph_0} = \aleph_2 +$ Martin's Axiom for non-dangerous posets, i.e. we find a non-dangerous poset $\mathcal{P}$ such that in $V^{\mathcal{P}}$: for every non-dangerous poset $\mathcal{R}$ and every collection of $\aleph_1$ dense subsets of $\mathcal{R}$ there is a filter on $\mathcal{R}$ intersecting all the dense subsets of the collection. Now we claim that actually the full Martin's Axiom holds in $V^{\mathcal{P}}$: Let $\mathcal{R}$ be a dangerous poset; we want to show that it does not satisfy the c.c.c. As $\mathcal{R}$ is dangerous we have a club $C \subseteq \omega_1$ and a sequence $(r_i, \bar{a}_i)$ such that $\langle \bar{a}_i : i \in \omega_1 \rangle$ satisfies (6.3) and if $\bar{a}_i$ and $\bar{a}_j$ are conflicting, then r_i and r_j are incompatible in $\mathcal{R}$. As $\mathcal{P}$ is non-dangerous T satisfies (6.5) in $V^{\mathcal{P}}$. Hence by 6.12, $\mathcal{Q}((\bar{a}_i : i < \omega_1), C) = \mathcal{Q}$ is non-dangerous. By Claim 6.14, for some α, $\mathcal{Q}^\alpha$ satisfies the claim and hence, as Martin's Axiom holds for non-dangerous posets in $V^{\mathcal{P}}$, we have a filter on $\mathcal{Q}^\alpha$ which gives us $\aleph_1$ pairwise conflicting $\bar{a}_i$, hence $\aleph_1$ pairwise incompatible members of $\mathcal{R}$.

In order to prove our theorem, that in $V^{\mathcal{P}}$ there are two Aronszajn trees not isomorphic on a club, we need the following lemma. Pick $a, b \in T$ incompatible; T_a is defined to be the subtree of all points in T above a, similarly T_b.

6.16. LEMMA. *If $\mathcal{P}$ is non-dangerous then, in $V^{\mathcal{P}}$, T_a and T_b are not isomorphic on a club set.*

PROOF. Suppose, on the contrary, that for some club C (which we assume to be in V) there is an isomorphism $f : T_a \restriction C \to T_b \restriction C$. Then in V one can find $p_\alpha \in \mathcal{P}$, $a_\alpha \in T_a \restriction C$, $b_\alpha \in T_b \restriction C$; a_α, b_α of level α, such that $p_\alpha \Vdash$ "$f(a_\alpha) = b_\alpha$", for $\alpha < \omega_1$. But then $(p_\alpha, (a_\alpha, b_\alpha))$, $\alpha < \omega_1$, clearly show that $\mathcal{P}$ is dangerous, because if p_α, p_β are compatible by p, then p forces $f(a_\alpha) = b_\alpha$, $f(a_\beta) = b_\beta$ and p forces that f is an isomorphism on C; hence (a_α, b_α) and (a_β, b_β) must agree on C.

§7. Making the continuum above $\aleph_2$

7.1. THEOREM. *Assume V satisfies* ZFC + GCH. *$\kappa \geqq \aleph_2$ is a regular cardinal. Then there is a generic extension of V in which*

Martin's Axiom $+ 2^{\aleph_0} = \kappa +$

Any two Aronszajn trees are isomorphic on a club set

holds.

PROOF. The generic extension has two parts: we do *first* a preparatory extension, and *then* iterate c.c.c. posets to get Martin's Axiom and the isomorphisms. The preparatory extension, like Jensen's adding closed unbounded sets, gives us a universe where it is possible to construct c.c.c. posets forcing an isomorphism on a club set for two given Aronszajn trees. We describe first the preparatory extension.

7.2. DEFINITION. $\mathcal{Q}$ denotes the poset for adding $\aleph_1$ many Cohen reals:

$$\mathcal{Q} = \{f : |f| < \aleph_0 \ \& \ \mathrm{Dom}(f) \subseteq \omega_1 \ \& \ \mathrm{Range}(f) \subseteq \{0,1\}\}.$$

$\mathscr{E} \in V^{\mathcal{Q}}$ is the name of the countably closed poset for adding a closed unbounded subset of ω_1, i.e. $\mathscr{E}$ is in $V^{\mathcal{Q}}$, the family of closed countable subsets of ω_1 ordered by end extension. $\mathscr{E}$ is not the closed-set poset of §3, but it serves a similar purpose.

Define $\mathscr{P} = \mathcal{Q} * \mathscr{E}$, i.e. $\mathscr{P}$ is the set of all pairs (f, c) where $f \in \mathcal{Q}$ and $\phi \Vdash^{\mathcal{Q}} c$ is a closed countable subset of ω_1. The name c can be chosen as a function defined on a countable subset of ω_1 such that, for $\alpha \in \mathrm{Dom}(c)$, $c(\alpha)$ is a (countable) set of pairwise incompatible members of $\mathcal{Q}$ — a maximal incompatible subset of those forcing $\alpha \in c$.

$\mathscr{P}$ is partially ordered as follows:

$$(f, c) \leqq (f^*, c^*) \text{ iff } f \leqq f^* \quad \text{and} \quad f^* \Vdash c = c^* \cap \mathrm{Sup}(c).$$

We define the projection of $\mathscr{P}$ on $\mathcal{Q}$ by $\pi_l(f, c) = f$. We also define $\pi_r(f, c) = c$

7.3. DEFINITION OF $\mathscr{P}_\gamma$ FOR $\gamma \leqq \kappa$. $\mathscr{P}_\gamma$ is the set of all countable partial functions $h : \gamma \to \mathscr{P}$ such that $\pi_l(h(\alpha)) \neq \varnothing$ only for finitely many α's. $\mathscr{P}_\gamma$ is partially ordered component-wise, i.e. $h \leqq h'$ iff $\mathrm{Dom}(h) \subseteq \mathrm{Dom}(h')$ and $h(\alpha) \leqq h'(\alpha)$ for $\alpha \in \mathrm{Dom}\, h$.

So $\mathscr{P}_\gamma$ is a mixed multiplication, with finite support on the left side and countable support on the right side.

7.4. DEFINITIONS.

(i) Denote $\mathcal{Q}_\gamma = \Pi_{\alpha<\gamma}\mathcal{Q}$, the multiplication with finite support of γ many copies of $\mathcal{Q}$. $\mathcal{Q}_\gamma$ is naturally embedded into $\mathscr{P}_\gamma$ and we can define a natural projection $\pi_l : \mathscr{P}_\gamma \to \mathcal{Q}_\gamma$ (by $\pi_l(h)(\alpha) = \pi_l(h(\alpha))$).

(ii) For $e \in \mathcal{Q}_\gamma$ and $p \in \mathscr{P}_\gamma$ we say *e and p are compatible* if e and $\pi_l(p)$ are compatible in $\mathcal{Q}_\gamma$. In case e and p are compatible $e \cup p \in \mathscr{P}_\gamma$ is defined naturally. $e \cup p \geqq p$. If $p^* \geqq p$, p^* and e are compatible, then $e \cup p \leqq e \cup p^*$.

7.5. LEMMA. *$\mathscr{P}_\gamma$ satisfies the $\aleph_2$-c.c.*

PROOF. Using $2^{\aleph_0} = \aleph_1$ the cardinality of $\mathscr{P}$ is $\aleph_1$. Now the proof is standard using a Δ-system.

For any $\delta < \gamma$, $\mathscr{P}_\gamma = \mathscr{P}_\delta \times \mathscr{P}^\delta$, where $\mathscr{P}_\delta$ is the set of countable functions in $\mathscr{P}_\gamma$ defined on δ, and $\mathscr{P}^\delta$ is the set of those functions defined on $\gamma - \delta$.

7.6. LEMMA. *$\mathcal{P}^\delta$ satisfy the $\aleph_2$-c.c. in $v^{\mathcal{P}_\delta}$.*

PROOF. Although in $V^{\mathcal{P}_\delta}$ the continuum hypothesis does not necessarily hold, we can form a Δ-system using the fact that the elements of $\mathcal{P}^\delta$ are functions in V and the $\aleph_2$-c.c. of $\mathcal{P}_\delta$.

7.7. LEMMA. *If $p_n \leqq p_{n+1}$, $p_n \in \mathcal{P}_\gamma$, and $\pi_l(p_n) = \pi_l(p_0)$, for $n < \omega$, then $p = \bigcup_{n<\omega} p_n \in \mathcal{P}_\gamma$.*

PROOF. Actually we have to define $\bigcup_{n<\omega} p_n$. Well, for $\alpha \in \bigcup_{n<\omega} \mathrm{Dom}(p_n)$, $p_n(\alpha)$ is an increasing sequence in $\mathcal{P}$. $\pi_l(p_n)(\alpha) = \pi_l(p_0)(\alpha)$ is some fixed condition in $\mathcal{Q}$, which forces $\pi_r(p_n(\alpha))$ to be an increasing sequence of closed countable subsets of ω_1. Now pick a name for the closure of the union of that sequence and thus define $p(\alpha)$. It is clear that p is the least upper bound of $\langle p_n \mid n \in \omega \rangle$.

7.8. LEMMA. *Let $g : \omega \to \mathrm{On}$ be a function in $V^{\mathcal{P}_\gamma}$; then $g \in V^{\mathcal{Q}_\gamma}$.*

PROOF. Using the projection π_l, we know that forcing with $\mathcal{P}_\gamma$ can be viewed as a two stage extension: the first with $\mathcal{Q}_\gamma$ and then with the remainder. The lemma says that in forcing with $\mathcal{P}_\gamma$ no new ω-sequences are added to the middle world obtained with $\mathcal{Q}_\gamma$. As $\mathcal{Q}_\gamma$ satisfies the c.c.c. the lemma shows that $\aleph_1$ is not collapsed in $V^{\mathcal{P}_\gamma}$.

We give a density argument for $\mathcal{P}_\gamma$; given $p \in \mathcal{P}_\gamma$ we have to find $p^* \geqq p$ such that $p^* \Vdash g \in V^{\mathcal{Q}_\gamma}$. So let $p \in \mathcal{P}_\gamma$, $p \Vdash g : \omega \to \mathrm{On}$, be given. We will define an increasing sequence $p_n \in \mathcal{P}_\gamma$, $n < \omega$, $p_0 \geqq p$, and a maximal set, E_n, of pairwise incompatible elements of $\mathcal{Q}_\gamma$ above $\pi_l(p)$, for $n \geqq 0$, such that:

(a) $\pi_l(p_n) = \pi_l(p)$,

(b) for every $e \in E_n$, e and p_n are compatible and $e \cup p_n \Vdash^{\mathcal{P}_\gamma} g(n) = \alpha(n, e)$, for some $\alpha(n, e) \in \mathrm{On}$.

Later (in 7.10) we will show how to define p_n and E_n; first let us end the proof. Define $\bigcup_{n<\omega} p_n = p^*$ (using Lemma 7.7). Now define a name G in $V^{\mathcal{Q}_\gamma}$ of a function on ω such that for every n and $e \in E_n$, $e \Vdash G(n) = \alpha(n, e)$.

CLAIM. *$p^* \Vdash^{\mathcal{P}_\gamma} G = g$. Hence $p^* \Vdash g \in V^{\mathcal{Q}_\gamma}$.*

PROOF OF THE CLAIM. E_n is a maximal set of pairwise incompatible elements above $\pi_l(p^*) = \pi_l(p)$, hence $p^* \Vdash G$ is a function. Suppose for some $\alpha \in \mathrm{On}$ and $p^{**} \geqq p^*$, $p^{**} \Vdash g(n) = \alpha$. Then $\pi_l(p^{**})$ is compatible with some $e \in E_n$, so $e \cup p^{**} \Vdash g(n) = \alpha(n, e)$, hence $\alpha = \alpha(n, e)$ and $e \cup p^{**} \Vdash G(n) = \alpha$. So $p^* \Vdash g \subseteq G$. Hence $p^* \Vdash g = G$. It remains to show how to get p_{n+1} from p_n and how to define E_n. For this we need the following.

7.9. LEMMA. *If $p_1, p_2 \in \mathcal{P}_\gamma$, $p_1 \leqq p_2$, then there is $p_2^* \in \mathcal{P}_\gamma$ such that $p_1 \leqq p_2^*$, $\pi_l(p_2^*) = \pi_l(p_1)$ and $\pi_l(p_2) \cup p_2^* = p_2$.*

PROOF. For every $\eta \in \operatorname{Dom} p_2$ we define $p_2^*(\eta)$ as follows. We know that $p_1(\eta) \leq p_2(\eta)$. Let $p_1(\eta) = (f_1, c_1)$, $p_2(\eta) = (f_2, c_2)$. Then we define $p_2^*(\eta) = (f_1, c_2^*)$, where c_2^* is the name of the subset which is c_2 if f_2 holds and c_1 otherwise.

Now we turn to the construction of p_n and E_n by induction on $n < \omega$. Assume p_{n-1} is constructed. We will define inductively $p_n^\alpha \in \mathcal{P}_\gamma$ and $q_\alpha \in \mathcal{Q}_\gamma$ for $\alpha \in \omega_1$ such that $\alpha < \beta$ implies

$$p_n^\alpha \leqq p_n^\beta, \quad \pi_l(p_n^\alpha) = \pi_l(p_n^\beta), \quad q_\alpha \text{ and } q_\beta \text{ are incompatible}$$
$$\text{or } q_\beta = \emptyset; \quad p_n^0 = p_{n-1} \text{ or } p_0^0 = p.$$

For limit δ we let $p_n^\delta = \bigcup_{\alpha<\delta} p_n^\alpha$, by Lemma 7.7. Suppose p_n^α is defined:

If there is $r \in \mathcal{P}_\gamma$, $r \geqq p_n^\alpha$ such that $r \Vdash g(n) = \eta$, for some η, and, for any $\beta < \alpha$, $\pi_l(r)$ and q_β are incompatible, then using Lemma 7.9 pick such r and find $p_n^{\alpha+1} \geqq p_n^\alpha$ with $\pi_l(p_n^{\alpha+1}) = \pi_l(p_n^\alpha)$ and $\pi_l(r) \cup p_n^{\alpha+1} = r$. Set $q_\alpha = \pi_l(r)$.

If there is no such r, let $p_n^{\alpha+1} = p_n^\alpha$ and $q_\alpha = \emptyset$. As $\mathcal{Q}_\gamma$ satisfies the c.c.c., for some $\alpha < \omega_1$, $p_n^\alpha = p_n^{\alpha+1}$, $q_\alpha = \emptyset$; then we define $p_n = p_n^\alpha$ and $E_n = \{q_\xi : \xi < \alpha\}$. E_n is a maximal anti-chain above $\pi_l(p)$. This ends the proof of Lemma 7.8

7.10. LEMMA. *Let $M < H_{\kappa^+}$ be a countable elementary submodel of H_{κ^+} (the familly of all sets of cardinality hereditarily less than κ^+) such that $\mathcal{P}_\gamma \in M$ for some $\gamma \leqq \kappa$. Let $p \in M \cap \mathcal{P}_\gamma$ be given, then there is $p^* \geqq p$ such that, for every $D \in M$ predense in $\mathcal{P}_\gamma$, $D \cap M$ is predense in $\mathcal{P}_\gamma$ above p^* (i.e., $\mathcal{P}_\gamma$ is proper).*

PROOF. Let $\langle D_n : n \in \omega \rangle$ be an enumeration of all predense subsets of $\mathcal{P}_\gamma$ which are elements of M. We will construct an increasing sequence $p_n \in \mathcal{P}_\gamma \cap M$ with $\pi_l(p) = \pi_l(p_n)$, $p_0 = p$, and define $E_n \in M$, a maximal above $\pi_l(p)$ pairwise incompatible subset of $\mathcal{Q}_\gamma$, such that, for $e \in E_n$, $e \cup p_n \geqq d$ for some $d \in D_n \cap M$. This suffices, for we can define $p^* = \bigcup_{n<\omega} p_n$. p^* is as required: let $p' \geqq p^*$ and $D \in M$ predense in $\mathcal{P}_\gamma$, then $D = D_n$ for some n, and $\pi_l(p')$ is compatible with some $e \in E_n$; but then

$$e \cup p' \geqq e \cup p^* \geqq e \cup p_n \geqq d \qquad \text{for some } d \in D_n \cap M.$$

The construction of p_n, E_n is as follows. Suppose p_{n-1} is constructed. Define — in M — an increasing sequence, $p_n^\alpha \in \mathcal{P}_\gamma$, $\alpha < \omega_1^M$, $p_n^0 = p_{n-1}$, $\pi_l(p_n^\alpha) = \pi_l(p)$ and define also $h_\alpha \in \mathcal{Q}_\gamma$ such that $\alpha < \beta \Rightarrow h_\alpha$ and h_β are incompatible or $h_\beta = \emptyset$. The definition is done as follows: $p_n^\delta = \bigcup_{\alpha<\delta} p_n^\alpha$ for limit δ. If p_n^α is constructed and $\{h_\beta : \beta < \alpha\}$ is not maximal above $\pi_l(p)$, then using Lemma 7.9 find $h_\alpha \in \mathcal{Q}_\gamma$ incompatible with any h_β, $\beta < \alpha$, and $p_n^{\alpha+1} \geqq p_n^\alpha$ such that $\pi_l(p_n^{\alpha+1}) = \pi_l(p_n^\alpha)$ and $h_\alpha \cup p_n^{\alpha+1} \geqq d$ for some $d \in D_n \cap M$. As $\mathcal{Q}$ satisfies the c.c.c. we stop at $\mu < \omega_1$ (in M) and get $p_n = \bigcup_{\alpha<\mu} p_n^\alpha$ in M. We set $E_n = \{h_\beta : \beta \in \mu\}$.

This ends the proof of Lemma 7.10, and the first part of our generic extension is $V^{\mathcal{P}_\kappa}$. In $V^{\mathcal{P}_\kappa}$, $2^{\aleph_0} = 2^{\kappa} = \kappa$ and cardinals are not collapsed. The second part is an iteration of length κ, with finite support, of c.c.c. posets of cardinality $< \kappa$, which finally gives a model of Martin's Axiom $+ 2^{\aleph_0} = \kappa +$ Every two Aronszajn trees are isomorphic on a club set.

So we place ourselves in $V[\dot{\mathcal{P}}_\kappa]$, where $\dot{\mathcal{P}}_\kappa$ is a V-generic filter over $\mathcal{P}_\kappa$ and assume $\mathcal{R} \in V[\dot{\mathcal{P}}_\kappa]$ is a c.c.c. poset of cardinality $< \kappa$. Moreover, in $V[\dot{\mathcal{P}}_\kappa][\dot{\mathcal{R}}]$ ($\dot{\mathcal{R}}$ is a $V[\dot{\mathcal{P}}_\kappa]$ generic filter over $\mathcal{R}$), we are faced with two Aronszajn trees A_1 and A_2 and we want to find a c.c.c. poset which forces the two trees to be isomorphic on a club set.

Since $\mathcal{P}_\kappa$ satisfies the $\aleph_2$-c.c., there is $\gamma < \kappa$ such that (1) $\mathcal{R} \in V[\dot{\mathcal{P}}_\gamma]$, where $\dot{\mathcal{P}}_\gamma = \dot{\mathcal{P}}_\kappa \cap \mathcal{P}_\gamma$, (2) A_1, A_2 are Aronszajn trees in $V[\dot{\mathcal{P}}_\gamma][\dot{\mathcal{R}}]$.

If we let $\mathcal{P}^{(\gamma)} = \{h \in \mathcal{P}_\kappa : \gamma \notin \operatorname{Dom} h\}$, and $\dot{\mathcal{P}}^{(\gamma)} = \dot{\mathcal{P}}_\kappa \cap \mathcal{P}^{(\gamma)}$, then $\dot{\mathcal{P}} = \{h(\gamma) : h \in \dot{\mathcal{P}}_\kappa \text{ and } \gamma \in \operatorname{Dom}(h)\}$ is $V[\dot{\mathcal{P}}^{(\gamma)}]$ generic over $\mathcal{P}$, and $V[\dot{\mathcal{P}}^{(\gamma)}][\dot{\mathcal{P}}] = V[\dot{\mathcal{P}}_\kappa]$. Since $\mathcal{P}$ and $\mathcal{R}$ are in $V[\dot{\mathcal{P}}^{(\gamma)}]$, a well-known lemma about product of forcing says that $V[\dot{\mathcal{P}}_\kappa][\dot{\mathcal{R}}] = V[\dot{\mathcal{P}}^{(\gamma)}][\dot{\mathcal{R}}][\dot{\mathcal{P}}]$ and $\dot{\mathcal{P}}$ is $V[\dot{\mathcal{P}}^{(\gamma)}][\dot{\mathcal{R}}]$-generic over $\mathcal{P}$. Let $W = V[\dot{\mathcal{P}}^{(\gamma)}][\dot{\mathcal{R}}]$, then what we need is just

7.11. Theorem. *Suppose $V \subseteq W$ are transitive models of* ZFC. *V satisfies* CH (*but W might not*). *Let A_1, A_2 be Aronszajn trees in W which stay Aronszajn trees in $W[\dot{\mathcal{P}}]$* (*where $\dot{\mathcal{P}}$ is a W-generic filter over $\mathcal{P}$*). *Then there is in $W[\dot{\mathcal{P}}]$ a c.c.c. poset $\mathcal{S}$ which forces A_1 and A_2 to be isomorphic on a club set.*

Proof. To describe our poset we need a few definitions.

7.12. Definitions. (i) Let T be a tree; we write $\bar{a} = (a_1, \ldots, a_k) \in T$ to mean that $\forall i$ ($a_i \in T_\alpha$), for some α. This α is called the *level* of $\bar{a}$ and we write $\bar{a} \in T_\alpha$ (the a_i are not necessarily distinct).

(ii) If $\bar{a} \in T_\alpha$ is a k-tuple and $\beta \leqq \alpha$, then $\pi_\beta(\bar{a})$ — the projection of $\bar{a}$ in T_β — is the k-tuple $\bar{b} \in T_\beta$ such that $b_i \leqq a_i$ for all $1 \leqq i \leqq k$.

(iii) If $\bar{a} \in T$ then

$$T^{\bar{a}} = \{x \in T : \text{for some } i, \quad x \leqq a_i\}.$$

Thus, $T^{\bar{a}}$ is a subtree of T, an initial segment in fact. Similarly, for $C \subseteq \omega_1$, $(T \restriction C)^{\bar{a}} = \{x \in T \restriction C : \text{for some } i, x \leqq a_i\}$; we do not require that the level of $\bar{a}$ is in C.

(iv) We write $(\bar{a}, \bar{b}) \in A_1 \times A_2$ if for some γ (the *level* of $(\bar{a}, \bar{b})$) $\bar{a} \in A_1$ and $\bar{b} \in A_2$ are of level γ and both are k-tuples for some k. For $\beta \leqq \gamma$, $\pi_\beta(\bar{a}, \bar{b})$ is $(\pi_\beta(\bar{a}), \pi_\beta(\bar{b}))$.

(v) Let $E \subseteq \omega_1$ be a club set. Let $(\bar{a}, \bar{b}) \in A_1 \times A_2$. If there is an epimorphism $f : (A_1 | E)^{\bar{a}} \to (A_2 | E)^{\bar{b}}$ such that for any $x \in A_1 | E$ and any i,

$$x \leqq a_i \Leftrightarrow f(x) \leqq b_i$$

then there is a unique such isomorphism. We denote this fact by $f = (\bar{a}, \bar{b})$ and we call f "the isomorphism determined by $(\bar{a}, \bar{b})$ on E".

7.13. Definition of the forcing poset $\mathcal{S}(E, A_1, A_2)$. Let E be a club subset of ω_1 and A_1, A_2 Aronszajn trees.

$$\begin{aligned} \mathcal{S} &= \mathcal{S}(E, A_1, A_2) \\ &= \{(\bar{a}, \bar{b}) \in A_1 \times A_2 : (\bar{a}, \bar{b}) \text{ determines an isomorphism } f(\bar{a}, \bar{b}) \text{ on } E\}. \end{aligned}$$

$\mathcal{S}$ is partially ordered by $(\bar{a}, \bar{b}) \leqq (\bar{a}', \bar{b}')$ iff $f = (\bar{a}, \bar{b}) \subseteq f' = (\bar{a}', \bar{b}')$.

Thus the conditions are really those isomorphisms determined by tuples, but we found it more convenient — from a notational point of view — to use this definition.

Note that if $(\bar{a}, \bar{b}) \in \mathcal{S}$, then $\pi_\beta(\bar{a}, \bar{b}) \in \mathcal{S}$. Observe also that if σ and σ^* are in $\mathcal{S}$ of level γ and γ^* (with $\gamma \leqq \gamma^*$, say), then σ and σ^* are compatible iff σ and $\pi_\gamma(\sigma^*)$ are.

It is easy to see that in any generic extension via $\mathcal{S}$, A_1 and A_2 are isomorphic on E (we can assume that any point in our trees has infinitely many successors).

Which club set should we choose to define $\mathcal{S}$? Recall that $\mathcal{P} = \mathcal{Q} * \mathcal{E}$ (Definition 7.2) and so $W[\dot{\mathcal{P}}] = W[\dot{\mathcal{Q}}][\dot{\mathcal{E}}]$ where $\dot{\mathcal{Q}}$ is a W-generic filter over $\mathcal{Q}$, and $\dot{\mathcal{E}}$ is a $W[\dot{\mathcal{Q}}]$-generic filter over $\mathcal{E}$ (= countable closed subsets of ω_1 which are in $V[\dot{\mathcal{Q}}]$). Let $C = \bigcup \dot{\mathcal{E}}$. We define $\mathcal{S} = \mathcal{S}(C, A_1, A_2)$. The proof of Theorem 7.1 now depends only on the following.

7.14. Main Lemma. *In $W[\dot{\mathcal{Q}}][\dot{\mathcal{E}}]$, $\mathcal{S}$ satisfies the c.c.c.*

Proof. Let $D \in W[\dot{\mathcal{Q}}, \dot{\mathcal{E}}]$ be a maximal antichain of $\mathcal{S}$. Assume, by way of contradiction, that D is uncountable. Let $\underset{\sim}{D} \in W[\dot{\mathcal{Q}}]$ be a name in $\mathcal{E}$ forcing of D.

In $W[\dot{\mathcal{Q}}]$, for each countable $N \prec H_{\kappa^+}$, let $u(N)$ be a countable transitive model of ZF^- (i.e., minus the power-set axiom) such that $\bar{N} \in u(N)$ (where $\pi_N : N \to \bar{N}$, $\bar{N}$ is transitive).

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7.15. SUBLEMMA. *There is, in $W[\dot{\mathcal{Q}}]$, a countable $N \prec H_{\kappa^+}$ such that A_1, A_2, $\underset{\sim}{D}$, etc. are in N, and:*

(1) $\dot{\mathscr{C}} \cap \bar{N}$ is a $u(N)$-generic filter over $\pi_N(\mathscr{C})$.

(2) The elementary embedding, $\pi_N^{-1} : \bar{N} \to H_{\kappa^+}^{W[\dot{\mathcal{Q}}]}$, can be extended to an elementary embedding

$$\pi^{-1} : \bar{N}[\dot{\mathscr{C}} \cap \bar{N}] \to H_{\kappa^+}^{W[\dot{\mathcal{Q}}][\dot{\mathscr{C}}]}.$$

PROOF. (2) is a standard consequence of (1): Given any $x \in \bar{N}[\dot{\mathscr{C}} \cap N]$ let $\underset{\sim}{x} \in \bar{N}$ be a name of x. $\pi_N^{-1}(\underset{\sim}{x})$ is a name in $\mathscr{C}$-forcing. Let π^{-1} be defined as the interpretation of $\pi_N^{-1}(\underset{\sim}{x})$ in $W[\dot{\mathcal{Q}}][\dot{\mathscr{C}}]$. π^{-1} is well-defined and the range of π^{-1} is an elementary substructure of $H_{\kappa^+}^{W[\dot{\mathcal{Q}}]}[\dot{\mathscr{C}}] = H_{\kappa^+}^{W[\dot{\mathcal{Q}},\dot{\mathscr{C}}]}$, which we call $N[\dot{\mathscr{C}}]$.

To prove (1) we use a density argument in $W[\dot{\mathcal{Q}}]$ for $\mathscr{C}$ forcing. Let $c \in \mathscr{C}$ be any condition. Pick some countable $N \prec H_{\kappa^+}$ such that A_1, A_2, $\underset{\sim}{D}$, c, etc. are in N. As $u(N) \in W[\dot{\mathcal{Q}}]$ is a countable transitive set, for some $\mu < \omega_1$:

$$u(N) \in W[\dot{\mathcal{Q}} \mid \mu], \qquad \text{where } \dot{\mathcal{Q}} \mid \mu = \{f \in \dot{\mathcal{Q}} : \mathrm{Dom}(f) = \mu\}.$$

Let $q' = \bigcup\{f \restriction [\mu, \mu + \omega) : f \in \dot{\mathcal{Q}}\}$. Then q' is defined on the interval $[\mu, \mu + \omega)$. Let $q(n) = q'\ (\mu + n)$, then $q : \omega \to \omega$ is a $W[\dot{\mathcal{Q}} \mid \mu]$ Cohen generic real.

$\pi_N : N \to \bar{N}$ is the collapsing function. It is easy to see that $\pi_N(\mathscr{C}) = \mathscr{C} \cap \bar{N}$. So $\pi_N(\mathscr{C})$ is a countable set in V. (To see this, let $\eta : \omega_1 \to \mathscr{C}$, $\eta \in V$, be onto. We can assume $\eta \in N$, and then $\pi_N(\eta) = \eta \restriction \alpha_N$ ($\alpha_N = \pi_N(\omega_1)$). So $\pi_N(\mathscr{C}) = \eta''\alpha_N$.) Let $\{x_n : n \in \omega\} = \mathscr{C} \cap \bar{N}$ be an enumeration in V.

We define inductively, in $V[q]$, an increasing sequence, $\langle c_i : i \in \omega \rangle$, in $\mathscr{C}$. $c_0 = c$ is the given condition. Suppose c_n is defined by

$$c_{n+1} = \begin{cases} x_{q(n)} & \text{if } x_{q(n)} \geqq c_n, \\ c_n & \text{otherwise.} \end{cases}$$

Let c^* be the closure of $\bigcup_{n \in \omega} c_n$ (i.e. $\bigcup_{n \in \omega} c_n$ together with its supremum). As the sequence is defined in $V[\dot{\mathcal{Q}}]$, c^* is in $\mathscr{C}$

7.16. CLAIM. *The set $\{x \in \pi_N(\mathscr{C}) : x \leqq c_n$ for some $n\}$ is a $u(N)$ generic filter over $\pi_N(\mathscr{C})$.*

PROOF. Let $F \in u(N)$ be a dense open subset of $\pi_N(\mathscr{C})$. We want n such that $c_n \in F$. As q is a $W[\dot{\mathcal{Q}} \mid \mu]$-generic real, it is enough to find a dense set of conditions in ${}^{\omega}\omega$ which force the above. Well, given any condition $f : k \to \omega$, there is $b \in \pi_N(\mathscr{C})$ such that $f \Vdash c_k = b$. Now pick $d \in F$ which extends b and find $i \in \omega$ such that $d = x_i$. Define $f^* = f \cup \{\langle k, i \rangle\}$, then $f^* \Vdash d = c_{k+1}$.

Claim 7.16 implies that

$$c^* \Vdash \dot{\mathscr{E}} \cap \bar{N} = \{x \in \pi(\mathscr{E}) : x \leqq c_n \text{ for some } n\}.$$

Indeed it is easy to see that c^* forces the $\supseteq$ inclusion, and since the right side is a maximal filter and the left side is a filter, we get equality.

Returning to the proof of Lemma 7.14, let $N \prec H_{\kappa^+}$ be as in Sublemma 7.15. Denote by $N[\dot{\mathscr{E}}] \prec H_\kappa^{W[\mathscr{Q}]}$ the image of $\bar{N}[\dot{\mathscr{E}} \cap \bar{N}]$ under π^{-1}. As $\underset{\sim}{D} \in N$, $D \in N[\dot{\mathscr{E}}]$. Letting $\alpha_N = \pi(\omega_1)$, $\pi(A_i) = A_i \restriction \alpha_N$, $i = 1, 2$. $\pi(\mathscr{S}) = \mathscr{S}(C \cap \alpha_N, A_1 \restriction \alpha_N, A_2 \restriction \alpha_N)$ is the subset of conditions in $\mathscr{S}$ of level $< \alpha_N$. $\pi(D) = D \cap \pi(\mathscr{S})$ is a maximal antichain in $\pi(\mathscr{S})$.

Since D is uncountable, there is $(\bar{a}_0, \bar{b}_0) \in D$ of level $\geqq \alpha_N$. Let $(\bar{a}_1, \bar{b}_1) = \pi_{\alpha_N}(\bar{a}_0, \bar{b}_0)$. Note that $(\bar{a}_1, \bar{b}_1) \in u(N)$.

7.17. Definition. Let $\varphi(\bar{x}, \bar{y})$ have the meaning:

$(\bar{x}, \bar{y}) \in \mathscr{S}$ and $\forall \xi <$ level $(\bar{x}, \bar{y})$, if $(\bar{u}, \bar{v}) \in D$ is of level ξ, then $(\bar{x}, \bar{y})$ and $(\bar{u}, \bar{v})$ are incompatible in $\mathscr{S}$. (When appropriate, $\mathscr{S}$ and D are replaced by $\pi(\mathscr{S})$ and $\pi(D)$.)

Not only does $\varphi(\bar{a}_1, \bar{b}_1)$ hold, but, for any $\mu < \alpha_N$, $\varphi(\pi_\mu(\bar{a}_1, \bar{b}_1))$. This truth about $u(N)$ is forced by some $c \in \dot{\mathscr{E}} \cap \bar{N}$. Hence for every $\mu < \alpha_N$, in $u(N)$, letting $(\bar{a}, \bar{b}) = \pi_\mu(\bar{a}_1, \bar{b}_1)$,

$$c \Vdash^{\pi(\mathscr{E})} \varphi(\bar{a}, \bar{b}). \tag{7.18}$$

An analysis of the forcing relation and absoluteness arguments show that (7.18) holds also in $\bar{N}$. Define, in $\bar{N}$, the following subset of $\mathscr{S}$:

$$\mathscr{S}_0 = \{(\bar{a}, \bar{b}) \in \mathscr{S} : c \Vdash \varphi(\bar{a}, \bar{b})\}.$$

By (7.18) $\mathscr{S}_0$ is uncountable in $\bar{N}$. $\mathscr{S}_0$ is an initial segment of $\mathscr{S}$. Let $\mathscr{S}_1$ be the set of all $(\bar{a}, \bar{b}) \in \mathscr{S}_0$ such that uncountably many members of $\mathscr{S}_0$ are projected to $(\bar{a}, \bar{b})$. Again, $\mathscr{S}_1$ is an uncountable initial segment of $\mathscr{S}$ and every $(\bar{a}, \bar{b}) \in \mathscr{S}_1$ has uncountably many members of $\mathscr{S}_1$ projected to $(\bar{a}, \bar{b})$. This holds in $\bar{N}$ of course. Lemma 3.10 and a density argument ($\dot{\mathscr{E}} \cap \bar{N}$ is an $\bar{N}$-generic filter) give two ordinals $\beta_0, \beta_1 \in \mathring{C} \cap \alpha_N$ such that β_1 is the successor of β_0 in C and for some $(\bar{a}, \bar{b}) \in \mathscr{S}_1$ of level β_0 there is an infinite well-distributed set of $(\bar{a}', \bar{b}') \in \mathscr{S}_1$ of level β_1 which are projected to $(\bar{a}, \bar{b})$.

As $\pi(D)$ is a maximal antichain of $\pi(\mathscr{S})$ in $\bar{N}[\dot{\mathscr{E}} \cap \bar{N}]$, there is $d \in D$ which is compatible with $(\bar{a}, \bar{b})$; so let $d' \in \pi(\mathscr{S})$ extend both d and $(\bar{a}, \bar{b})$. Let $d^* = \pi_{\beta_1}(d')$, then we can find $(\bar{a}', \bar{b}',) \in \mathscr{S}_1$ of level β_1 which is disjoint from d^* and is projected to $(\bar{a}, \bar{b})$. It follows that d^* and $(\bar{a}', \bar{b}')$ are compatible in $\mathscr{S}$. Hence any

$(\bar{a}^*, \bar{b}^*) \in \mathcal{S}$ which is projected to $(\bar{a}', \bar{b}')$ (by π_{β_1}) is compatible with d^* and hence with d' and with d. This is a contradiction to $\varphi(\bar{a}^*, \bar{b}^*)$ when the level of $(\bar{a}^*, \bar{b}^*)$ is above that of d.

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