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ITERATED FORCING AND CHANGING COFINALITIES

BY

SAHARON SHELAH[†]

ABSTRACT

We weaken the notion of proper to semi-proper, so that the important properties (e.g., being preserved by some iterations) are preserved, and it includes some forcing which changes the cofinality of a regular cardinal $>\aleph_1$ to $\aleph_0$. So, using the right iterations, we can iterate such forcing without collapsing $\aleph_1$. As a result, we solve the following problems of Friedman, Magidor and Avraham, by proving (modulo large cardinals) the consistency of the following with G.C.H.: (1) for every $S\subseteq\aleph_2$, S or $\aleph_2-S$ contains a closed copy of ω_1, (2) there is a normal precipitous filter D on $\aleph_2$, $\{\delta<\aleph_2:\operatorname{cf}\delta=\aleph_0\}\in D$, (3) for every $A\subseteq\aleph_2$, $\{\delta<\aleph_2:\operatorname{cf}\delta=\aleph_0,\ \delta$ is regular in $L(\delta\cap A)\}$ is stationary. The results can be improved to equi-consistency; this will be discussed in a future paper.

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§0. Introduction and Notation

$H(\lambda)$, for regular λ, is the set of sets hereditarily of cardinality $< \lambda$. If $\bar{a}$ is a sequence, i.e., a function from an ordinal, then $l(\bar{a})$ is its length.

On is the class of ordinals, Car the class of cardinals, ICar the class of infinite cardinals, UCar = ICar $- \{\aleph_0\}$ and RCar the class of infinite regular cardinals, SCar = RCar $\cup \{2\}$, RUCar = RCar $\cap$ UCar, and we let

$$S^\alpha_\beta = \{\delta < \aleph_\alpha : \mathrm{cf}\,\delta = \aleph_\beta\}.$$

Notation on Forcing

(1) P, Q denote forcing notions (i.e., partial orders) and p, q, r elements of forcing notions. Let $p \leqq q$ mean q gives more information. We make the convention that each P has a minimal element $\emptyset$ (which thus gives no information). Two elements of P are compatible if they have an upper bound. An antichain $I \subseteq P$ is a set of pairwise incompatible elements.

(2) $P \subseteq Q$ means P is a submodel of Q. $P \lessdot Q$ means $P \subseteq Q$, and any maximal antichain of P is a maximal antichain of Q (hence compatibility is preserved).

Remember that $G \subseteq P$ is generic if it is directed, closed downward and not disjoint to any maximal antichain (of course G is in a bigger universe, e.g., $V[G]$). Remember also that G has a canonical P-name: $\underset{\sim}{G}$ or $\underset{\sim}{G}_P$.

§1. Iterated forcing with RCS (revised countable support)

Iterated forcing with countable support is widely used. One of its definitions is that at the limit stage with cofinality $\aleph_0$ we take the inverse limit, and at the limit stage with cofinality $> \aleph_0$ we take the direct limit. Another formulation is given below (Definition 1.1). However, the applications, as far as I remember, are for forcing notions which preserve the property "the cofinality of δ is uncountable".

However, in our case we are interested just in forcing which does change some cofinality to $\aleph_0$. In such cases, we cannot break the iterated forcing into an initial segment and the rest (i.e., break $\langle P_i, Q_i : i < \alpha\rangle$ into $\langle P_i, Q_i : i < \beta\rangle$ and $\langle P_i/P_\beta, Q_i : \beta \leqq i < \alpha\rangle$, see Definition 1.1). The reason is that maybe the first forcing changes the cofinality of some δ, $\beta < \delta < \alpha$ to $\aleph_0$; hence P_δ/P_β is not the inverse limit of $\langle P_i/P_\beta, Q_i : \beta \leqq i < \alpha\rangle$.

Hence we suggest another iteration, RCS (revised countable support), which seems the reasonable solution to this dilemma.

1.1 Definition. We call $\langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ a CS iteration (CS means countable support) if:

(a) $\underset{\sim}{Q}_i$ is a P_i-name of a forcing notion,

(b) the set of elements of P_α is $\{p : p$ is a function, whose domain is a countable subset of α, such that for every $i \in \operatorname{Dom} p$, $p(i)$ is a P_i-name of a member of $\underset{\sim}{Q}_i\}$, (i.e. $\emptyset \Vdash_P$ "$p(i) \in \underset{\sim}{Q}_i$"},

(c) the partial ordering on P is defined by: $p \leqq q$ *iff* for every i in the domain of p, $q \restriction i \Vdash_{P_i}$ "$p(i) \leqq q(i)$".

(3) If $P \lessdot Q$, $G \subseteq P$ generic, we let $Q/G = \{q \in Q :$ for every $p \in G$, q is compatible with p (in Q)$\}$.

So it is well known that forcing with Q is equivalent to forcing first with P and then with Q/G. Also Q/G has a P-name which we should denote by $Q/\underset{\sim}{G}_P$, but denote by Q/P.

(4) If $\underset{\sim}{Q}$ is a P-name of a forcing notion, $P * \underset{\sim}{Q}$ is their composition, so $P \lessdot P * \underset{\sim}{Q}$, $\underset{\sim}{Q} = (P * \underset{\sim}{Q})/P$. Remember $P * \underset{\sim}{Q} = \{(p, \underset{\sim}{q}) : p \in P$, $\underset{\sim}{q}$ a P-name of a member of $\underset{\sim}{Q}\}$; $(p_1, \underset{\sim}{q}_1) \leqq (p_2, \underset{\sim}{q}_2)$ iff $p_1 \leqq p_2$ and $p_2 \Vdash_P \underset{\sim}{q}_1 \leqq \underset{\sim}{q}_2$.

Now if $P_1 = P_0 * \underset{\sim}{Q}_0$, $\underset{\sim}{q}_1$ a P_1-name, $G_0 \subseteq P_0$ generic, then in $V[G_0]$, $\underset{\sim}{q}_1$ can be naturally interpreted as a Q_0-name, called $\underset{\sim}{q}_1/G_0$, which has a P_0-name $\underset{\sim}{q}_1/\underset{\sim}{G}_0$ or q_1/P_0; but usually we do not care to make those fine distinctions.

(5) Using $\bar{Q} = \langle P_i, Q_i : i < \alpha\rangle$, P_α will mean $\operatorname{R Lim} \bar{Q}$ (see Definition 1.2).

(6) If D is a filter on a set J, $D \in V$, $V \subseteq V'$ (e.g., $V' = V[\underset{\sim}{G}]$) then in an abuse of notation, D will denote also the filter it generates (on J) in V'.

(7) D_κ is the closed unbounded filter on κ.

1.2 Definition. We define the following notions by *simultaneous induction on* α:

(A) $\bar{Q} = \langle P_i, Q_i : i < \alpha\rangle$ is an RCS iteration (RCS stands for revised countable support),

(B) a $\bar{Q}$-named ordinal (or $[j,\alpha)$-ordinal),

(C) a $\bar{Q}$-named condition (or $[j,\alpha)$-condition),

and we define $\underset{\sim}{q} \restriction \xi$, $\underset{\sim}{q} \restriction \{\xi\}$ for a $\bar{Q}$-named $[j,\alpha)$-condition $\underset{\sim}{q}$ and ordinal ξ,

(D) the RCS-limit of $\bar{Q}$, $\mathrm{R\,Lim}\,\bar{Q}$ which satisfies $P_i \lessdot \mathrm{R\,Lim}\,\bar{Q}$ for every $i<\alpha$. (We should write $\mathrm{R\,Lim}_{\aleph_0}$, but omit the $\aleph_0$ as we deal with countable support only.)

(A) *We define "$\bar{Q}$ is an RCS iteration"*

$\alpha=0$: no condition.

α is limit: $\bar{Q}=\langle P_i, \underset{\sim}{Q}_i : i<\alpha\rangle$ is an RCS iteration iff for every $\beta<\alpha$, $\bar{Q}\restriction\beta$ is one.

$\alpha=\beta+1$: $\bar{Q}$ is an RCS iteration iff $\bar{Q}\restriction\beta$ is one, $P_\beta = \mathrm{R\,Lim}(\bar{Q}\restriction\beta)$ and $\underset{\sim}{Q}_\beta$ is a P_β-name of a forcing notion.

(B) We define "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j,\alpha)$-ordinal of depth γ above r".

The intended meaning is a $(\mathrm{R\,Lim}\,\bar{Q})$-name of an ordinal of a special kind, however $\mathrm{R\,Lim}\,\bar{Q}$ is still not defined. So we use the part already known.

For $\gamma=0$: "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j,\alpha)$-ordinal of depth γ above r" means $\underset{\sim}{\zeta}$ is a (plain) ordinal in $[j,\alpha)$, i.e., $j\leqq \underset{\sim}{\zeta}<\alpha$: $r\in P_{\zeta+1}$.

For $\gamma>0$: "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j,\alpha)$-ordinal of depth γ above r" means that for some $\beta<\alpha$, $r\in P_{\beta+1}$, $\underset{\sim}{\zeta}$ is a $P_{\beta+1}$-name of a $[\max\{j,\beta\},\alpha)$-named ordinal above r, i.e., for some maximal antichain above r, $I=\{p_i : i<i_0\}\subseteq P_{\beta+1}$ (so $r\leqq p_i$), $\{\gamma_i : i<i_0\}$ and $\{\underset{\sim}{\zeta}_i : i<i_0\}$, we have: $\underset{\sim}{\zeta}_i$ is a $\bar{Q}$-named $[\max\{j,\beta\},\alpha)$-ordinal of depth γ_i above p_i, $\gamma_i<\gamma$, and $\underset{\sim}{\zeta}$ is $\underset{\sim}{\zeta}_i$ if p_i (i.e., if p_i will be in the generic set); (this is informal but clear).

Without γ: We say $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j,\alpha)$-ordinal above r, it if is such for some depth.

Without r: $r=\emptyset$.

Similarly, we omit "$[j,\alpha)$-" when $j=0$.

(C) *We define* "$\underset{\sim}{q}$ is a $\bar{Q}$-named $[j,\alpha)$-condition of depth γ above r" and also $\underset{\sim}{q}\restriction\{\xi\}$, $\underset{\sim}{q}\restriction\xi$ and the $\bar{Q}$-named $[j,\alpha)$-ordinal $\zeta(\underset{\sim}{q})$ associated with $\underset{\sim}{q}$.

The definition is similar to (B).

For $\gamma=0$: We say "$\underset{\sim}{q}$ is a $\bar{Q}$-named $[j,\alpha)$-condition of depth γ above r" if for some ordinal ζ, $j\leqq\zeta<\alpha$ and $\underset{\sim}{q}$ is a P_ζ-name of a member of $\underset{\sim}{Q}_\zeta$, $r\in P_{\zeta+1}$, $r\restriction\zeta \Vdash_{P_\zeta}$ "if $\rho\in r$ then $\rho\restriction\{\zeta\}\leqq \underset{\sim}{q}$". We let

$$\underset{\sim}{q}\restriction\xi=\begin{cases}\underset{\sim}{q} & \text{if } \xi>\zeta,\\ \emptyset & \text{if } \xi\leqq\zeta,\end{cases}$$

$$\underset{\sim}{q} \restriction \{\xi\} = \begin{cases} \underset{\sim}{q} & \text{if } \xi = \zeta, \\ \emptyset & \text{if } \xi \neq \zeta. \end{cases}$$

Lastly we let $\zeta(\underset{\sim}{q}) = \underset{\sim}{\zeta}$.

For $\gamma > 0$: We say $\underset{\sim}{q}$ is a $\bar{Q}$-named $[j,\alpha)$-condition of depth γ above r, if for some $\bar{Q}$-named $[j,\alpha)$-ordinal of depth γ above r, $\underset{\sim}{\zeta}$, defined by β, $\{p_i : i < i_0\} \subseteq P_{\beta+1}$, $\{\gamma_i : i < i_0\}$, $\{\underset{\sim}{\zeta}_i : i < i_0\}$, we have $\bar{Q}$-named $[\max\{\beta, j\}, \alpha)$-condition of depth γ above $r \cup p_i, \underset{\sim}{q}_i$ $(i < i_0)$ such that $\zeta(\underset{\sim}{q}_i) = \underset{\sim}{\zeta}_i$, and $\underset{\sim}{q}$ is $\underset{\sim}{q}_i$ if p_i. Also if $\gamma_i = 0$, $\zeta_i = \beta$ then $\underset{\sim}{q}_i \geqq p_i$ and if $\gamma_i > 0$, γ, $\{p_{i,j} : j < j_0\}$, γ' define $\underset{\sim}{q}_i$, $\gamma' = \beta$ then $p_i \leqq p_{i,j}$ or they are contradictory.

We then let $\zeta(\underset{\sim}{q}) = \underset{\sim}{\zeta}$, and $\underset{\sim}{q} \restriction \xi$ is defined similarly with $\underset{\sim}{q}_i \restriction \xi$, and lastly $\underset{\sim}{q} \restriction \{\xi\}$ is defined similarly with $\underset{\sim}{q}_i \restriction \{\xi\}$.

We omit γ and/or "$[j,\alpha)$-" if this holds for some γ and/or j.

(D) *We define* $\mathrm{R\,Lim}\,\bar{Q}$ as follows:

if $\alpha = 0$: $\mathrm{R\,Lim}\,\bar{Q}$ is trivial forcing with just one condition: $\emptyset$;

if $\alpha > 0$: we call $\underset{\sim}{q}$ an atomic condition of $\mathrm{R\,Lim}\,\bar{Q}$, if it is a $\bar{Q}$-named condition.

The set of conditions in $\mathrm{R\,Lim}\,\bar{Q}$ is

$\{p : p$ a countable set of atomic conditions; and for every $\beta < \alpha$, $p \restriction \beta =^{\mathrm{def}} \{r \restriction \beta : r \in p\} \in P_\beta$, and $p \restriction \beta \Vdash_{P_\beta}$ "$p \restriction \{\beta\} =^{\mathrm{def}} \{r \restriction \{\beta\} : r \in p\}$ has an upper bound"$\}$.

The order is inclusion.

Now we have to show $P_\beta \lessdot \mathrm{R\,Lim}\,\bar{Q}$ (for $\beta < \alpha$) which is obvious noting that any $\bar{Q}$-named $[j,\beta)$-ordinal (or condition) is a $\bar{Q}$-named $[j,\alpha)$-ordinal (or condition), and see 1.4(1).

REMARK. We can obviously define $\bar{Q}$-named sets; but for conditions (and ordinals for them) we want to avoid the vicious circle of using names which are interpreted only after forcing with them.

Now we point out some properties of RCS iteration.

1.4 CLAIM. *Let* $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ *be an* RCS *iteration,* $P_\alpha = \mathrm{R\,Lim}\,\bar{Q}$.

(1) *If* $\beta < \alpha$, *then not only* $P_\beta \lessdot P_\alpha$, *but if* $q \in P_\beta$, $p \in P_\alpha$, *then* q, p *are compatible iff* $q, p \restriction \beta$ *are compatible.*

(2) *If* $\underset{\sim}{\beta}, \underset{\sim}{\gamma}$ *are* $\bar{Q}$*-named* $[j, l(\bar{Q}))$*-ordinals, then* $\mathrm{Max}\{\underset{\sim}{\beta}, \underset{\sim}{\gamma}\}$ *(defined naturally) is a* $\bar{Q}$*-named* $[j, l(\bar{Q}))$*-ordinal.*

(3) *If* $\alpha = \beta_0 + 1$, *in Definition* 1.2, *part* (D), *in defining the set of elements of* P_α *we can restrict ourselves to* $\beta = \beta_0$. *Also in such a case,* $P_\alpha = P_{\beta_0} * \underset{\sim}{Q}_{\beta_0}$ *(essen-*

tially). More exactly, $\{p \cup \{\underset{\sim}{q}\} : p \in P_{\beta_0},\ \underset{\sim}{q}$ *a* P_{β_0}*-name of a member of* $\underset{\sim}{Q}_{\beta_0}\}$ *is a dense subset of* P_α, $p_1 \cup \{\underset{\sim}{q}_1\} \leqq p_2 \cup \{\underset{\sim}{q}_2\}$ *iff* $p_1 \leqq p_2$, $p_2 \Vdash \underset{\sim}{q}_1 \leqq \underset{\sim}{q}_2$.

(4) *The following set is dense in* P_α : $\{p \in P_\alpha$: *for every* $\beta < \alpha$, *if* $r_1, r_2 \in p$, *then* $\Vdash_{P_\beta}$ "*if* $r_1 \restriction \{\beta\} \neq \emptyset$, $r_2 \restriction \{\beta\} \neq 0$ *then they are equal*"$\}$.

(5) $|P_\alpha| \leqq (\Sigma_{i<\alpha} 2^{|P_i|})^{\aleph_0}$, *for limit* α.

(6) *If* $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| \leqq \kappa$", κ *a cardinal, then* $|P_{i+1}| \leqq 2^{|P_i|} + \kappa$.

PROOF. Easy.

1.5 THE ITERATION LEMMA

(1) *Suppose* F *is a function, then for every ordinal* α *there is a unique* RCS*-iteration* $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha' \rangle$, *such that*:

(a) *for every* i, $\underset{\sim}{Q}_i = F(\bar{Q} \restriction i)$,

(b) $\alpha' \leqq \alpha$,

(c) *either* $\alpha' = \alpha$ *or* $F(\bar{Q})$ *is not a* $(\mathrm{R\,Lim}\,\bar{Q})$*-name of a forcing notion.*

(2) *Suppose* $\beta < \alpha$, $G_\beta \subseteq P_\beta$ *is generic, then in* $V[G_\beta]$, $\bar{Q}/G_\beta = \langle P_i/G_\beta, \underset{\sim}{Q}_i : \beta \leqq i < \alpha \rangle$ *is an* RCS*-iteration and* $\mathrm{R\,Lim}\,\bar{Q} = P_\beta * (\mathrm{R\,Lim}\,\bar{Q}/G_\beta)$ *(essentially).*

(3) *The Associative Law*

If α_ξ $(\xi \leqq \xi(0))$ *is increasing and continuous,* $\alpha_0 = 0$, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha_{\xi(0)} \rangle$ *is a* RCS*-iteration* $P_{\xi(0)} = \mathrm{R\,Lim}\,\bar{Q}$, *then so are* $\langle P_{\alpha(\xi)}, P_{\alpha(\xi+1)}/P_{\alpha(\xi)} : \xi < \xi(0) \rangle$ *and* $\langle P_i/P_{\alpha(\xi)}, \underset{\sim}{Q}_i : \alpha(\xi) \leqq i < \alpha(\xi+1) \rangle$; *and vice versa.*

PROOF. Easy.

1.6 CLAIM. *If* κ *is regular, and* $|P_i| < \kappa$ *for every* $i < \kappa$, *and* $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ *is an* RCS*-iteration, then*

(1) *every* $\bar{Q}$*-named ordinal is in fact a* $(\bar{Q} \restriction i)$*-named ordinal for some* $i < \alpha$,

(2) *like* (1) *for* $\bar{Q}$*-named conditions,*

(3) $P_\kappa = \bigcup_{i<\kappa} P_i$.

PROOF. Easy.

1.7 CLAIM. *Suppose* $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta \rangle$ *is an* RCS*-iteration,* δ *limit and* $p \in P_\delta$, *and* $\underset{\sim}{\zeta}$ *is* $\bar{Q}$*-named ordinal. Then there are* $i < \delta$, *and* $p' \in P_{i+1}$, $p \restriction i+1 \leqq p'$ *such that* $p' \Vdash_{P_i}$ "$\underset{\sim}{\zeta} = i$". *The same holds for* $\bar{Q}$*-named condition (if* $\underset{\sim}{Q}_i \subseteq V$).

PROOF. Easy, by induction on the depth of $\underset{\sim}{\zeta}$.

§2. Proper forcing revisited

Properness is a property of forcing notions which implies that $\aleph_1$ is not collapsed by forcing with P, and is preserved by countable-support iteration (and also $\aleph_1$-free iteration, see [18]). In [16], [17] it was introduced, and many examples of forcing not collapsing $\aleph_1$ were shown to be proper ($\aleph_1$-complete, C.C.C., Sacks forcing, Laver forcing and more). It was argued that proper forcing is essentially the most general property implying $\aleph_1$ is not collapsed and preserved under iteration. So the forcing of shooting a closed unbounded set through a stationary subset S of $\aleph_1$ (see Baumgartner, Harrington and Kleinberg [4]), though not collapsing $\aleph_1$, is excluded as if $\aleph_1 = \bigcup_{n<\omega} S_n$, S_n pairwise disjoint stationary subsets of $\aleph_1$, and we shoot a closed unbounded subset through each $\omega_1 - S_n$, in the limit $\aleph_1$ is collapsed. Of course we can "kill" stationary sets in a fixed normal ideal of $\aleph_1$ (see e.g. [10]) and properness really demands somewhat more than not destroying stationary subsets of $\aleph_1$ (also stationary subsets of $S_{\aleph_0}(\lambda) = \{A \subseteq \lambda : |A| \leqq \aleph_0\}$ should not be destroyed); but those seemed technical points.

However, in [16], [17], [18] we were mainly interested in forcing of power $\aleph_1$, so another restriction of properness was ignored; if P is proper, any countable set of ordinals in V^P is included in a countable set of V. So forcing changing the cofinality of some λ, $\mathrm{cf}\,\lambda > \aleph_1$, to $\aleph_0$, are not included. In fact, there are such forcings which do not collapse $\aleph_1$, and moreover, do not add reals: Prikry forcing [15] (which changes the cofinality of a measurable cardinal to $\aleph_0$) and Namba [14] which change the cofinality of $\aleph_2$ to $\aleph_0$.

We suggest here a property of forcing, called semi-properness, such that the theorems proved for proper forcing hold (when we use RCS-iteration) and it includes Prikry forcing. We do not know whether there is a forcing changing the cofinality of $\aleph_2$ to $\aleph_0$ which is semi-proper (i.e., provably from ZFC), but we shall have an approximation to this. (See [19] for an answer.)

So in this section we introduce the notion, and prove the preservation under RCS-iteration. In this we weaken a little the assumptions: for limit δ, $\underset{\sim}{Q}_\delta$ is not necessarily semi-proper, only $P_{\delta+1}/P_{i+1}$ $(i < \delta)$ is semi-proper. This change does not influence the proof, but is very useful, as we can exploit the fact that δ was a large cardinal in V. Note that the useful result is Corollary 2.8.

2.1 Definition. A forcing notion P is proper if for any large enough regular χ, and well ordering $<$ of $H(\chi)$, and countable $N \prec (H(\chi), \in, <)$ such that $P \in N$ and for every $p \in P \cap N$ *there is* $q \in P$, $q \geqq p$ such that: for every maximal antichain I of P which belongs to N, $I \cap N$ is predense above q. Equivalently, for every P-name $\underset{\sim}{\beta}$ of an ordinal which belongs to N, $q \Vdash_P$ "$\underset{\sim}{\beta} \in N$".

We call q under such circumstances (N, P)-generic.

2.2 DEFINITION. A forcing notion P is *$\underset{\sim}{S}$-semi-proper* ($\underset{\sim}{S}$ a P-name of a class of cardinals) *if* for any large enough regular λ, and well-ordering $<$ of $H(\lambda)$, and countable $N \prec (H(\lambda), \in, <)$, such that $P \in N$, and for every $p \in P \cap N$ *there is* $q \in P$ such that: for every cardinal $\kappa \in N$ and P-name $\underset{\sim}{\beta} \in N$ of an element of κ,

$$q \Vdash_P \text{“if } \kappa \in \underset{\sim}{S} \text{ then there is } A \in N, |A| < \kappa, \underset{\sim}{\beta} \in A\text{”}$$

(equivalently, if $\underset{\sim}{S}$ consists of regular cardinals of V, $q \Vdash_P$ “if $\kappa \in \underset{\sim}{S}$ then $\operatorname{Sup} N \cap \kappa = \operatorname{Sup} N(G) \cap \kappa$”).

(Note we write A and not $\underset{\sim}{A}$, i.e., A is in V; also when κ is regular in V, w.l.o.g. $A = \gamma$ for some $\gamma < \kappa$; this is the main case.)

We call q, under such circumstances, $\underset{\sim}{S}$-semi (N, P)-generic. If $\underset{\sim}{S} = \{\kappa : \text{in } V^P, \kappa$ is a cardinal of cofinality $> \aleph_0\}$ then we omit it.

2.3 CLAIM. (1) *If P is* UCar*-semi-proper, or even* RUCar*-semi-proper then P is proper, and vice versa. Moreover q in Definition* 2.2 *is* (N, P)*-generic, which means: if $\underset{\sim}{\beta} \in N$ is a P-name of an ordinal then $q \Vdash_P$ “$\underset{\sim}{\beta} \in N$”.*

(2) *P is $\underset{\sim}{S}$-semi-proper iff the condition of Definition* 2.2 *holds for some $\lambda > 2^{|P|}$, and well-ordering $<$.*

(3) *P is $\underset{\sim}{S}$-semi-proper iff* $(B^P - \{0\}, \geqq)$ *is, where B^P is the complete Boolean algebra corresponding to P.*

(4) *In Definition* 2.2, *for $\kappa > \aleph_0$, and $\kappa > |P|$, the condition is trivially satisfied by any q, so only $\underset{\sim}{S} \cap \{\kappa : \aleph_0 < \kappa \leqq |P|\}$ is relevant.*

(5) *P is semi-proper iff P is* (RUCar^{V^P})*-semi-proper.*

PROOF. Easy.

2.4 DEFINITION. (1) A property is preserved by RCS-iteration, if for any RCS-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$, if $\underset{\sim}{Q}_i$ has the property (in V^{P_i}) for each i, then $\mathrm{R\,Lim}\, \bar{Q}$ has the property.

(2) A property is strongly preserved by RCS-iteration *if* $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is an RCS-iteration and for every $\gamma \leqq \beta < \alpha$, γ not a limit ordinal, $P_{\beta+1}/P_\gamma$ has the property then $\mathrm{R\,Lim}\, \bar{Q}$ has the property.

(3) We can replace RCS-iteration by any other kind of iteration in this definition.

REMARK. In [16] many properties were shown to be preserved by CS iteration. In fact the proofs show they are strongly preserved.

2.5 CLAIM. (1) *In Definition* 2.4(1), (2) *it suffices to consider* $\alpha = 2$ or α *a* *regular cardinal and $\gamma < \beta < \alpha$ implies P_β / P_γ has the property (where for 2.4(2) γ is zero or successor ordinal).*

(2) *If a property is strongly preserved by* RCS-*iteration* then *the property is preserved by* RCS-*iteration.*

PROOF. Easy; for (1) use 1.5(3).

2.6 THE SEMI-PROPERNESS ITERATION LEMMA

(1) *Semi-properness is strongly preserved by* RCS *iteration.*

(2) *Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is an* RCS *iteration, for any $j \leqq \alpha$ for arbitrarily large non-limit $i < j$, P_j / P_i is $\underset{\sim}{S}_{i,j}$-semi-proper ($\underset{\sim}{S}_{i,j}$ is a P_j-name). Let $S = \{\lambda : \lambda$ an uncountable regular cardinal, and $\Vdash_{P_j}$ "$\lambda \in \underset{\sim}{S}_{i,j}$" for any $i < j \leqq \alpha$, as above$\}$.*

Then $P_\alpha = \mathrm{R\,Lim}\, \bar{Q}$ is S-semi-proper provided that:

(C1)
$$\text{for every limit } \delta \geqq \alpha \text{ there is } \zeta < \delta, \text{ such that}$$
$$\Vdash_{P_\zeta} \text{"cf}\, \delta = \aleph_0 \text{ or for every } \zeta \leqq i < j < \delta,\ \underset{\sim}{S}_{i,j} \text{ defined} \Rightarrow \text{cf}\, \delta \in \underset{\sim}{S}_{i,j}\text{"}.$$

(3) *In* (2) *we can weaken* (C1) *by replacing ζ by a $(\bar{Q} \restriction \delta)$-named ordinal, and replace S by $\underset{\sim}{S} = \{\lambda :$ for some $i < j \leqq \alpha$, λ is an uncountable regular cardinal in $V[\underset{\sim}{G}_{P_i}]$ and belongs to $S_{i,j}[\underset{\sim}{G}_{P_j}]$.*

REMARK. For $i < \alpha$ non-limit clearly $\underset{\sim}{S}_{i,i+1}$ is defined, so $\underset{\sim}{Q}_i$ is $\underset{\sim}{S}_{i,i+1}$-semi-proper.

PROOF. (2) We prove the theorem by induction on α, for all $\bar{Q}$'s, and even for forcing extensions of V.

Let $T = \{(i,j) : \underset{\sim}{S}_{i,j}$ is defined$\}$.

Note that for any $\beta \leqq \gamma \leqq \alpha$, β non-limit, $\bar{Q} \restriction [\beta, \gamma) = \langle P_i / P_\beta, \underset{\sim}{Q}_i : \beta \leqq i < \gamma \rangle$ satisfies the hypothesis on $\bar{Q}$. Let λ be big enough, $<$ a well-ordering of $H(\lambda)$, $\bar{Q} \in H(\lambda)$, $N \prec (H(\lambda), \in, <)$ N countable, $P_\alpha \in N$ hence w.l.o.g. $\bar{Q} \in N$ [because $(H(\lambda), \in, <) \models$ "there is $\bar{Q}$, an RCS-iteration as in 2.4(2) such that $P_\alpha = \mathrm{R\,Lim}\, \bar{Q}$", so as $P_\alpha \in N \prec (H(\lambda), \in, <)$ there should be such $\bar{Q}$ in N]. Furthermore, let $p \in P_\alpha \cap N$.

Case A: α non-limit.

The cases $\alpha = 0$, $\alpha = 1$ are too trivial to consider. For $\alpha > 1$ by the induction hypothesis on α we can assume $\alpha = 2$.

So w.l.o.g. $P_2 = Q_0 * \underset{\sim}{Q}_1$, and let $p = (p_0, \underset{\sim}{p}_1) \in P_1 \cap N$. As clearly, $Q_0 \in N$, there is $q_0 \in Q_0$, $p_0 \leqq q_0$, which is $\underset{\sim}{S}_{0,1}$-semi(N, P)-generic. To help us in understanding let $G_0 \subseteq Q_0$ be generic, $q_0 \in G_0$. As $<$ is a well-ordering of $H(\lambda)$, $(H(\lambda)[G_0], \in, <)$ has defined Skolem functions, and a definable well-ordering

($H(\lambda)[G_0]$ is $H(\lambda)$ of the universe $V[G_0]$, we are assuming that any member of $H(\lambda)[G_0]$ has a name in $H(\lambda)$).

Now let $N[G_0]$ be the Skolem Hull of N in $(H(\lambda)[G_0], \in, <)$. So as $Q_1 = P_1/G_0$ is $\underset{\sim}{S}_{1,2}$-semi-proper, and $Q_1 \in N[G_0] \prec (H(\lambda)[G_0], \in, <)$, there is $q_1 \in Q_1$ $\underset{\sim}{S}_{1,2}$-semi $(N[G_0], Q_1)$-proper. Let $G_1 \subseteq Q_1$ be generic, $q_1 \in G_1$.

So if $\kappa \in N$, and $\kappa \in S$, then as q_0 is $\underset{\sim}{S}_{0,1}$-semi (N, Q_0)-generic, $q_0 \in Q_0$ clearly $\mathrm{Sup}(N \cap \kappa) = \mathrm{Sup}(N[G_0] \cap \kappa)$; and similarly $\mathrm{Sup}(N[G_0] \cap \kappa) = \mathrm{Sup}(N[G_0, G_1] \cap \kappa)$.

As G_0, G_1 were arbitrary except that $q_0 \in G_0$, $q_1 \in G_1$ clearly $(q_0, \underset{\sim}{q}_1)$ is S-semi (N, P_2)-generic.

Case B: α a limit ordinal and there are $\beta < \alpha$ and $p', p \restriction \beta \leqq p'$, $p' \Vdash_{P_\beta}$ "$\mathrm{cf}\, \alpha = \aleph_0$".

As $N \prec (H(\lambda), \in, <)$, $\bar{Q} \in N$, $p \in N$, we can assume $p' \in N$ hence w.l.o.g. $p \restriction \beta = p'$. Moreover by Case A it suffices to prove that P_α / P_β is S-semi-proper. By the induction hypothesis, w.l.o.g. $\mathrm{cf}\, \alpha = \aleph_0$, and as $\bar{Q} \in N$, $\alpha \in N$, so there are $\alpha_n < \alpha$, $\alpha_n < \alpha_{n+1}$, $\alpha = \bigcup_{n<\omega} \alpha_n$, and w.l.o.g. each α_n is a successor ordinal.

Now let $\{(\underset{\sim}{\beta}_n, \kappa_n) : n < \omega\}$ be a list of the pairs $(\underset{\sim}{\beta}, \kappa)$, where $\kappa \in N$, κ a cardinal in S and $\underset{\sim}{\beta}$ a P_α-name of an ordinal $< \kappa$, $\underset{\sim}{\beta} \in N$. We define by induction on $n < \omega$ conditions p_n, q_n such that:

(1) $p_n \in N \cap P_\alpha$, $p_0 = p$, $p_n \leqq p_{n+1}$, $p_{n+1} \restriction \alpha_n = p_n \restriction \alpha_n$,

(2) $q_n \in P_{\alpha_n}$, $q_{n+1} \restriction \alpha_n = q_n$, q_n is S_{α_n}-semi (N, P_{α_n})-generic,

(3) $p_n \leqq q_n$,

(4) $p_{n+1} \Vdash$ "$\mathrm{cf}\, \kappa_n = \aleph_0$ or $\underset{\sim}{\beta}_n < \underset{\sim}{\gamma}_n$" where $\underset{\sim}{\gamma}_n$ is a P_{α_n}-name of an ordinal $< \kappa_n$, $\underset{\sim}{\gamma}_n \in N$.

This is easy and $\bigcup_{n<\omega} q_n$ is as required.

Case C: α a limit ordinal and for no $\beta < \alpha$, $p' \in P_{\beta'}$, $p \restriction \beta \leqq p'$ does $p' \Vdash_{P_\beta}$ "$\mathrm{cf}\, \alpha = \aleph_0$".

Let $\alpha_n \in N$, $\alpha_n < \alpha_{n+1}$, $\bigcup \alpha_n = \mathrm{Sup}\, N \cap \alpha$ (as $\bar{Q} \in N$, $\alpha \in N$), α_n successor; and repeat the previous proof. Notice only that we can force any $\bar{Q}$-named ordinal $< \alpha$ to be $< \bigcup_{n<\omega} \alpha_n$ by 1.7 (like (4) above) and this insures $p_n \leqq \bigcup_{l<\omega} q_l$ for every n, and also that we are using condition (C1) from the lemma.

(3) A similar proof.

(1) Follows.

In fact we have proved also the following

2.7 Lemma. *If $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is an* RCS-*iteration (of semi-proper forc-*

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ing), δ a limit ordinal, and $\emptyset \Vdash_{P_i}$ "$\mathrm{cf}\,\delta > \aleph_0$", *for every* $i < \alpha$, *α limit* then $\emptyset \Vdash_{P_\alpha}$ "$\mathrm{cf}\,\delta > \aleph_0$.". *Moreover* $\bigcup_{i<\delta} P_i$ *is a dense subset of* P_δ.

Also note that the most useful case of 2.6 is

2.8 Corollary. *If* $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ *is an* RCS *iteration, and for every* $j < \alpha$ *for arbitrarily large non-limit* $i < j+1$, P_{j+1}/P_i *is* $\{\aleph_1\}$*-semi-proper, and for every successor* $i < \alpha$, $\Vdash_{P_{i+n}}$ "*the power of* P_i *is* $\aleph_1$" *for some* $n < \omega$ then P_α *is* $\{\aleph_1\}$*-semi-proper. If in addition* $|P_\alpha| = |\alpha|$ *and* P_α *satisfies the* $|P_\alpha|$*-chain condition or* $|P_i| < |\alpha|$, *α inaccessible then* P_α *is semi-proper.*

Remark. For iteration of proper forcings, there is really no difference between CS and RCS iterations.

§3. Pseudo-completeness

A widely used notion on forcing is $\aleph_1$-completeness, i.e., if $p_n \leqq p_{n+1} \in P$, then there is $p \in P$, $p_n \leqq p$ for every n. This is the simplest forcing which does not add reals, nor new ω-sequences of ordinals. In our perspective we want a condition parallel to this, including, e.g., Prikry forcing.

3.1 Definition. For a forcing P, a P-name $\underset{\sim}{S}$ of a set of cardinals of V, an ordinal α and condition p we define a game $G^{\alpha}_{\underset{\sim}{S}}(p, P)$ (or $G^{\alpha}(p, P, \underset{\sim}{S})$): in the ith move, player I chooses a λ_i and a P-name $\underset{\sim}{\beta}_i$ of an ordinal $< \lambda_i$, and player II has to find a condition p_i, and a set $A_i \subseteq \lambda_i$, $|A_i| < \lambda_i$, ($A_i \in V$) such that:

(A) $p_i \Vdash$ "$\underset{\sim}{\beta}_i \in A_i$ or $\lambda_i \notin \underset{\sim}{S}$"; and

(B) $p_i \geqq p$, $p_i \geqq p_j$ for $j < i$.

The play continues for α moves.

In a specific play, player II wins iff $\{p\} \cup \{p_i : i < \alpha\}$ has an upper bound (and loses otherwise).

A player wins the game if he has a winning strategy.

Notation. Writing RCar, SCar, etc. for $\underset{\sim}{S}$, we mean: as interpreted in V^P.

3.2 Claim. (1) *At most one player can win the game* $G^{\alpha}_{\underset{\sim}{S}}$.

(2) *If for every* $\lambda_i \in \underset{\sim}{S}$ *and* $\mu \in \mathrm{SCar}$, $\mu \leqq \lambda_i \Rightarrow \mu \in \underset{\sim}{S}$, *then in the definition of the game, it does not matter if we demand* $|A_i| = 1$ (*i.e., if one side has a winning strategy iff he has a winning strategy in the revised game*).

(3) *If for every cardinal* μ, $\mu_1 \leqq \mu \leqq \mu_0 \Rightarrow \mu \in \underset{\sim}{S}$ *then in the definition of the game, it does not matter if we demand, when* $\lambda_i = \mu_0$, *that* $|A_i| < \mu_1$.

(4) *Also we can replace* λ_i *by any set* $B \in N$, $|B| = \lambda_i$. *If* λ_i *is regular* (*even if only in* V) *we can demand* $A_i \in \lambda_i$ (*i.e., is an initial segment*).

(5) *If for every regular* $\mu \leqq \lambda$, $\mu \in \underset{\sim}{S}$ *and there is* $n \in \underset{\sim}{S}$, $1 < n < \aleph_0$ *and for every* p, *player II does not lose in the game* $G^{\alpha}_{\underset{\sim}{S}}(p, P)$, then *forcing by* P *does not introduce new* α*-sequences from* λ. (*Usually* $n = 2$; *for* $n > 2$ *we have to work somewhat more in the proof.*) *If* α *is* $> \omega$ *we can omit the* n.

(6) *If* $n \in \underset{\sim}{S}$, $n < \omega$, *adding* $\{m : n < m < \aleph_0\}$ *to* S *does not change anything*; *also if* $\operatorname{cf} \lambda \in \underset{\sim}{S}$ *adding* λ *does not change anything.*

3.3 Definition. The forcing P is $(\underset{\sim}{S}, \alpha)$-complete if player II wins in the game $G^{\alpha}_{\underset{\sim}{S}}(p, P)$ for every p.

We define $(\underset{\sim}{S}, < \beta)$-complete similarly.

P is pseudo κ-complete if it is (Car, μ)-complete for every (cardinal) $\mu < \kappa$.

3.4 Lemma. (1) *If* P *is* $|\alpha|^{+}$*-complete then it is* (Car, α)*-complete.*

(2) *If* P *is* $(\lambda \cap \mathrm{SCar}, \alpha)$*-complete,* $\alpha \leqq \lambda$, *and forcing by* P *does not change the cofinality of any* μ, $\aleph_0 < \mu \leqq \lambda$, *then forcing by* P *does not add new* α*-sequences from* λ (*remember* $\lambda = \{\beta : \beta < \lambda\}$).

(3) *In particular if* P *is* $(\{2\}, \omega)$*-complete* (*or even* $(\{n\}, \omega)$*-complete*) *then forcing by* P *does not add reals.*

(4) *If* P *is* $(\underset{\sim}{S}, \omega)$*-complete then* P *is* $\underset{\sim}{S}$*-semi-proper. In fact* (RUCar, ω)*-completeness suffices for semi-properness.*

(5) *If* p *is* (S_1, α_1)*-complete, then it is* (S_2, α_2)*-complete provided that* $(\forall \gamma \in S_2)(\exists \beta \in S_1) \operatorname{cf} \gamma = \operatorname{cf} \beta$, *and* $\alpha_2 \leqq \alpha_1$ (*for* γ *natural number,* $\gamma = \beta$).

(6) P *is* $(\underset{\sim}{S}, \alpha)$*-complete implies* $(B - \{0\}, \geqq)$ *is* $(\underset{\sim}{S}, \alpha)$*-complete,* B *the complete Boolean algebra corresponding to* P.

Proof. Easy.

3.5 Theorem. RCS *iteration strongly preserves* (SCar, ω)*-completeness,* (RCar, ω)*-completeness and* (RUCar, ω)*-completeness.*

Remark. We can also imitate 2.6, 2.8, and vice versa.

Proof. We use Claim 2.5(1), so have to deal only with iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ where $\alpha = 2$ or $\alpha = \lambda$ a regular cardinal.

Let $\underset{\sim}{S}$ be any one of those three classes of cardinals, and $\underset{\sim}{S}_0$, $\underset{\sim}{S}_{i,j}$ be the corresponding $\underset{\sim}{Q}_0$, P_j/P_i name (remember, the meaning of our S depends on which forcing it applies to).

Case A: $\alpha = 2$

Let $p = (p_0, \underset{\sim}{p}_1) \in Q_0 * \underset{\sim}{Q}_1$, and let $F_0, \underset{\sim}{F}_1$ be the winning strategies of player II in $G^{\omega}_{S}(p_0, Q_0)$, $G^{\omega}_{S}(p_1, Q_1)$ resp. By 3.2(4), we can assume F_l gives us an ordinal or $\in\{0,1\}$ if the corresponding λ is regular or 2 resp.

Let in the ith move player I choose λ_i and a P_2-name $\underset{\sim}{\beta}_i$ of an ordinal $<\lambda_i$, and player II choose $(p_{0,i}, \underset{\sim}{p}_{1,i}) \in P_2$, a P_1-name $\underset{\sim}{A}_{1,i}$, and a set $A_{0,i} \subseteq \lambda_i$. Player II preserves the following property:

(*)

(a) $p_{0,i} \Vdash_{Q_0}$"the following is an initial segment of the play of $G^{\omega}_{S}(p_1, \underset{\sim}{Q}_1)$ in which player II uses the strategy $\underset{\sim}{F}_1$: $\langle \cdots, \langle \lambda_j, \underset{\sim}{\beta}_j \rangle, \langle \underset{\sim}{p}_{1,j}, \underset{\sim}{A}_{1,j} \rangle \cdots \rangle_{j \leqq i}$".

(b) $p_{0,i} \Vdash_{Q_0}$"$\underset{\sim}{A}_{1,i}$ is an ordinal $\underset{\sim}{\alpha}_i < \lambda_i$ if $\lambda_i \geqq \aleph_0$ and a singleton $\{\underset{\sim}{\alpha}_i\} \subseteq \lambda_i$ if $\lambda_i = 2$ and $\underset{\sim}{A}_{1,i} \subseteq A_{0,i}$".

(c) $A_{0,i}$ is an ordinal $<\lambda_i$ if $\lambda_i \geqq \aleph_0$, and a singleton $\subseteq 2$ if $\lambda_i = 2$.

(d) The following is an initial segment of a play of the game $G^{\omega}_{\underset{\sim}{S}_0}(p_0, Q_0)$ in which player II uses his winning strategy F_0: in the jth move player I chooses λ_j, $\underset{\sim}{\alpha}_j$ and player II chooses $p_{0,j}, A_{0,j}$.

It is easy to see that player II can do this and that it is a winning strategy.

Case B: $\alpha = \lambda$ a regular cardinal and $p \in P_\lambda$ and there are $\beta < \lambda$, $p' \in P_\beta$, $p \restriction \beta \leqq p'$, $p' \Vdash_{P_\beta}$ "cf $\lambda = \aleph_0$".

By the previous case, it suffices to prove that $P_\alpha / P_{\beta+1}$ is $(\underset{\sim}{S}, \omega)$-complete, so w.l.o.g. cf $\lambda = \aleph_0$, and in fact $\lambda = \aleph_0$, and there are no problems. We leave the details as an exercise to the reader.

Case C: $\alpha = \lambda$ is regular and for every $\beta < \alpha$, $p \restriction \beta \Vdash_{P_\beta}$ "cf $\lambda > \aleph_0$".

We describe the winning strategy of player II. By a hypothesis, for every non-limit $\beta < \gamma$, player II has a winning strategy $\underset{\sim}{F}_{\beta,\gamma}[r]$ (a P_β-name) for winning the game $G^{\omega}_{S}(r, P_\gamma / P_\beta)$. In stage n, he has defined not only p_n, but $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \lambda$, no one of them a limit ordinal and for each l, $\langle p_{l+1} \restriction [\alpha_l, \alpha_{l+1}), p_{l+2} \restriction [\alpha_l, \alpha_{l+1}), \cdots, p_n \restriction [\alpha_l, \alpha_{l+1}) \rangle$ is an initial segment of a play of the game $G^{\omega}_{\underset{\sim}{S}_{\alpha_l \alpha_{l+1}}}(p_{l+1} \restriction [\alpha_l, \alpha_{l+1}), P_{\alpha_{l+1}} / P_{\alpha_l})$ in which player II uses the strategy $\underset{\sim}{F}_{\beta,\gamma}[p_{l+1} \restriction [\alpha_l, \alpha_{l+1})]$.

So this is similar to Case A, using n instead of 2, and even more similar to Case B. The difference is that here in the end, maybe for some $\bar{Q}$-named condition $\underset{\sim}{q} \in p_l$, $\zeta(\underset{\sim}{q}) > \bigcup_{n<\omega} \alpha_n$. So from time to time player II "let player I wait" and looked at a suitably chosen $\underset{\sim}{q} \in p_n$, and define p_{n+1} so that $\zeta(\underset{\sim}{q})$ is equal to a P_{α_n}-name (using 1.7) and $(p_{n+1} \restriction \alpha_n) \Vdash_{P_{\alpha_n}}$ "$\zeta(\underset{\sim}{q}) \leqq \alpha_{n+1}$".

3.6 Definitions. (1) For a forcing P, a P-name $\underset{\sim}{S}$ of a set of cardinals, an ordinal α and a condition p we define the games $EG^{\alpha}_{\underset{\sim}{S}}(p, P)$, $RG^{\alpha}_{\underset{\sim}{S}}(p, P)$ (or $EG^{\alpha}(p, P, \underset{\sim}{S})$, $RG^{\alpha}(p, P, \underset{\sim}{S})$). (E stands for essentially, R for really.)

(2) In a play of the game $\mathrm{E}G^{\alpha}_{\underset{\sim}{S}}(p,P)$ in the ith move, player I chooses a cardinal λ_i and a P-name $\underset{\sim}{\beta}_i$ of an ordinal $<\lambda_i$ and player II has to find a set $A_i \subseteq \lambda_i$, $|A_i| < \lambda_i$, $(A_i \in V)$.

The play continues for α moves. In the end player II wins if he can find a condition $p' \in P$, $p \leqq p'$ such that for every $i < \alpha$, $p' \Vdash$ "$\underset{\sim}{\beta}_i \in A_i$ or $\lambda_i \notin \underset{\sim}{S}$".

(3) In a play of the game $\mathrm{R}G^{\alpha}_{\underset{\sim}{S}}(p,P)$ in the ith move, player I choose a condition q_i, $q_i \geqq p$, $q_i \geqq p_j$ for every $j < i$, and a cardinal λ_i and a P-name $\underset{\sim}{\beta}_i$ of an ordinal $<\lambda_i$ and player II has to find a condition p_i and a set $A_i \subseteq \lambda_i$, $|A_i| < \lambda_i$, $(A_i \in V)$ such that

(A) $p_i \Vdash$ "$\underset{\sim}{\beta}_i \in A_i$ or $\lambda_i \notin \underset{\sim}{S}$",

(B) $p_i \geqq q_i$ (hence $p_i \geqq p$, $p_i \geqq p_j$ for $j < i$).

The play continues for α moves, and player II wins if $\{p\} \cup \{p_i : i < \alpha\}$ has an upper bound.

3.7 Definition. The forcing P is essentially $(\underset{\sim}{S}, \alpha)$-complete [really $(\underset{\sim}{S}, \alpha)$-complete] if player II wins in the game $\mathrm{E}G^{\alpha}_{\underset{\sim}{S}}(p,P)$ [$\mathrm{R}G^{\alpha}_{\underset{\sim}{S}}(p,P)$] for every $p \in P$.

3.8 Lemma. (1) *The parallels of* 3.2, 3.4 *hold.*

(2) *Let P be a forcing, B the corresponding Boolean algebra. Then P is essentially $(\underset{\sim}{S}, \alpha)$-complete iff $(B - \{0\}, \geqq)$ is $(\underset{\sim}{S}, \alpha)$-complete.*

Proof. Easy.

3.9 Theorem. (1) RCS *iteration strongly preserves "essential (S, ω)-completeness" for $S \in \{$*SCar, RCar, RUCar*$\}$.*

(2) *For example, $\aleph_1$-*RS *iteration preserves, e.g., "$(S, \aleph_1)$-completeness and real $(S, \aleph_0)$-completeness" for S as above (κ-*RS *means in* 2.2 *we replace "countable" by "of power $\leqq \kappa$").*

Proof. Similar to previous ones.

§4. Specific forcings

We prove here on various forcings that they are semi-proper and even (S, α)-complete; of course, otherwise our previous framework will be empty.

Prikry forcing (adding an unbounded ω-sequence to a measurable cardinal without adding bounded subsets) satisfies all we can expect. But for our purposes, more important are forcings which change the cofinality of $\aleph_2$ to $\aleph_0$, without adding reals (or at least not collapsing $\aleph_1$). Namba [14] has found such forcing, when CH holds.

However we do not know the answer to:

PROBLEM. Is Namba forcing $\{\aleph_1\}$-semi proper? (Not necessarily, see [19].)

However, it is not necessarily $(\{2\}, \omega)$-complete; this is equivalent to "$D_{\aleph_2}$ is Galvin" (see below).

We deal with a generalization of Namba forcing, $\mathrm{Nm}(\bar{D})$ ($\bar{D}$ a system of filters on sets of power $\aleph_2$), and prove the relevant assertion (4.7). Then we prove that if each filter in $\bar{D}$ has the $(\{\aleph_1, \aleph_2, 2\}, \omega)$-Galvin property, then $\mathrm{Nm}(\bar{D})$ is $\{\aleph_1, \aleph_0, 2\}$-semi-proper. The point is that when a large cardinal is collapsed to $\aleph_2$, if D was originally a normal ultrafilter, then after the collapse it may well have some largeness property like the one of Galvin.

4.1 DEFINITION. If D is a complete normal ultrafilter on κ, then the D-Prikry forcing, $\mathrm{PF}(D)$, is:

$$\{(f, A): f \text{ a function, with domain } n < \omega,\ f \text{ is increasing, } (\forall i < n) f(i) < \kappa, \text{ and } A \text{ belongs to } D\},$$

$(f_1, A_1) \leqq (f_2, A_2)$ iff $f_1 \subseteq f_2$, $A_1 \supseteq A_2$, and for $i \in \mathrm{Dom}\, f_2 - \mathrm{Dom}\, f_1$, $f_2(i) \in A_1$.

Prikry defined this notion and proved [15] in fact that:

4.2 THEOREM. *For any normal ultrafilter D over κ,* $\mathrm{PF}(D)$ *is* $(\mathrm{Car}^V - \{\kappa\}, \lambda)$-*complete for every $\lambda < \kappa$, and changes the cofinality of only one cardinal, κ (to $\aleph_0$).*

4.3 DEFINITION. (1) A filter-tagged tree is a pair (T, D) such that:

(a) T is a non-empty set of finite sequences of ordinals, closed under taking initial segments, and for some $\eta_0 \in T$, $\nu \in T$, $l(\nu) \leqq l(\eta_0) \Rightarrow \nu = \eta_0 \restriction l(\nu)$; we call η_0 the trunk of T.

(b) D is a function such that for every $\eta \in T$, $D_\eta = D(\eta)$ is a filter and if $\eta_0 \lessdot \eta \in T$ then $\mathrm{Suc}_T(\eta) = \{\nu \in T : l(\nu) = l(\eta) + 1,\ \nu \restriction l(\eta) = \eta\} \neq \emptyset \bmod D_\eta$.

(2) We call (T, D) normal if for every η, D_η is a filter over $\mathrm{Suc}_T(\eta)$. For $\eta \in T$, $(T, D)_{[\eta]} = (T_{[\eta]}, D) = (\{\nu \in T : \nu \leqq \eta \text{ or } \eta \leqq \nu\}, D)$.

4.4 DEFINITION. For filter-tagged trees (T_l, D_l):

(1) We define: $(T_1, D_1) \leqq (T_2, D_2)$ iff

(a) $T_2 \subseteq T_1$,

(b) for some $\eta_0 \in T_2$, $(T_2, D_2) = (T_2, D_2)_{[\eta_0]}$ and for every $\eta, \eta_0 \leqq \eta \in T_2$, $\mathrm{Suc}_{T_2}(\eta) \neq \emptyset \bmod D_1(\eta)$ and $D_1(\eta) \restriction \mathrm{Suc}_{T_2}(\eta) = D_2(\eta) \restriction \mathrm{Suc}_{T_2}(\eta)$ where for filter D over I, and $J \subseteq I$, $J \neq \emptyset \bmod D$,

$$D \restriction J = \{A \cap J : A \in D\}.$$

(2) We define: $(T_1, D_1) \leqq^* (T_2, D_2)$ if in addition $\{\eta_0 \restriction l : l\} = T_1 \cap \{\nu : l(\nu) \leqq l(\eta_0)\}$.

(3) We define: $(T_1, D_1) \leqq_n (T_2, D_2)$ if in addition (to (1)) for η of length $\leqq n$,

$$\eta \in T_1 \Leftrightarrow \eta \in T_2.$$

4.5 Claim. *For every (T, D) for a unique normal (T, D'), $(T, D) \leqq (T, D') \leqq (T, D)$.*

4.6 Lemma. *If (T, D) is a filter-tagged tree, which is λ^+-complete (i.e., each D_η is a λ^+-complete filter) and $H : T \to \lambda$, $\lambda^{\aleph_0} = \lambda$, then there is $(T', D') \leqq^* (T, D)$ such that $H(\eta)$ depend only on $l(\eta)$, for $\eta \in T'$.*

Proof. For any sequence $\bar\alpha = \langle \alpha_n : n < \omega \rangle$, $\alpha_n < \lambda$, we define a game $G_{\bar\alpha}$:

Let η_0 be the trunk of T.

In the first (= zeroth) move player I chooses $A_1 \subseteq \mathrm{Suc}_{\eta_0}(T)$, $A_1 = \emptyset \bmod D_{\eta_0}$, and player II chooses $\eta_1 \in \mathrm{Suc}_{\eta_0}(T) - A_1$.

In the nth move, player I chooses $A_{n+1} \subseteq \mathrm{Suc}_{\eta_n}(T)$, $A_{n+1} = \emptyset \bmod D_{\eta_n}$ and player II chooses $\eta_{n+1} \in \mathrm{Suc}_{\eta_n}(T) - A_{n+1}$.

In the end, player II wins the play if $H(\eta_n) = \alpha_n$. Now we prove

(*) For some $\bar\alpha = \langle \alpha_n : n < \omega \rangle$, $\alpha_n < \lambda$, player II wins the game (i.e., has a winning strategy).

Clearly the game is closed, hence it suffices to prove that for some $\bar\alpha$, player I does not have a winning strategy. So assume for every $\bar\alpha$ player I has a winning strategy $F_{\bar\alpha}$, and we shall get a contradiction. A winning strategy is a function which, given the previous moves of the opponent ($\eta_1, \cdots, \eta_n$ in our case), give a move to the player, so that in any play in which he uses the strategy he wins the play.

Now define by induction on n, $\eta_n \in T$ such that $l(\eta_n) = n$, $\eta_{n+1} \restriction n = \eta_n$:

$$\eta_0 = \langle\ \rangle,$$

$$\eta_{n+1} \in \mathrm{Suc}_{\eta_n}(T) - \bigcup_{\bar\alpha} F_{\bar\alpha}(\langle \eta_1, \cdots, \eta_n \rangle).$$

Why does η_{n+1} exist? For every $\bar\alpha$, $F_{\bar\alpha}(\langle \eta_1, \cdots, \eta_n \rangle) = \emptyset \bmod D_{\eta_n}$, D_{η_n} is λ^+-complete and the number of $\bar\alpha$'s is $\lambda^{\aleph_0} = \lambda < \lambda^+$. So $\bigcup_{\bar\alpha} F_{\bar\alpha}(\langle \eta_1, \cdots, \eta_n \rangle) = \emptyset \bmod D_{\eta_n}$, and so η_{n+1} exists as $\mathrm{Suc}_{\eta_n}(T) \neq \emptyset \bmod D_{\eta_n}$.

But let $\alpha^*_n = H(\eta_n)$, $\bar\alpha^* = \langle \alpha^*_n : n < \omega \rangle$, so

$$F_{\bar\alpha^*}(\ ), \eta_1, \cdots, F_{\bar\alpha^*}(\langle \eta_1, \cdots, \eta_n \rangle), \eta_{n+1}, \cdots$$

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is a play of $G_{\bar{\alpha}^*}$ in which player I uses his strategy $F_{\bar{\alpha}^*}$, but he lost: contradiction, hence (*) holds.

PROOF OF THE LEMMA FROM (*). Let $\langle \alpha_n : n < \omega \rangle$ be as in (*), and W be the winning strategy of player II and $\eta_0 = \langle \ \rangle$ for notational simplicity.

Let $T_0 = \{\eta \in T : l(\eta) = n$, and for some $A_1, \cdots, A_n$, for every $0 < l \leqq n$, $\eta \restriction l = W(\langle A_1, \cdots, A_l \rangle)\}$.

It is clear that T_0 is closed under initial segments. Now if $\eta \in T_0$, then $\mathrm{Suc}_{T_0}(\eta) \neq \emptyset \bmod D_\eta$, for otherwise if $n = l(\eta)$, and $A_1, \cdots, A_n$ are "witnesses for $\eta \in T_0$", then player I could have chosen $A_{n+1} = \mathrm{Suc}_{T_0}(\eta)$, and then by definition, $W(A_1, \cdots, A_{n+1}) \in T_0$ and also $W(A_1, \cdots, A_{n+1}) \notin A_{n+1} = \mathrm{Suc}_{T_0}(\eta)$ but $W(A_1, \cdots, A_{n+1}) \in \mathrm{Suc}_T(\eta)$ and $\mathrm{Suc}_{T_0}(\eta) = T_0 \cap \mathrm{Suc}_T(\eta)$, contradiction.

So $(T_0, D) \leqq^* (T, D)$ is as required.

4.7 THEOREM. *Suppose* $2^{\aleph_0} = \aleph_1$, $T^* = \aleph_2^{<\omega}$, (T^*, D^*) *an* $\aleph_2$-*complete filter-tagged tree, and* $D^*_\eta \supseteq D^{\mathrm{cb}}_\eta = \{A \subseteq \mathrm{Suc}_{T^*}(\eta): \ |\mathrm{Suc}_{T^*}(\eta) - A| < \aleph_2\}$ (e.g. $\bar{D}^*_\eta = D^{\mathrm{cb}}_\eta$), (cb is for co-bounded).

Let $P = \mathrm{Nm}(T^*, D^*) = \{(T, D^*) : (T, D^*) \geqq (T^*, D^*)\}$ (we write $\eta \in (T, D^*)$ if $\eta \in T$, $p_l = (T_l, D^*)$, etc.) with the order $\leqq$.

Then P does not add reals and change the cofinality of $\aleph_2$ to $\aleph_0$.

REMARK. If we wave CH, P may add reals but it does not collapse $\aleph_1$; sometimes it satisfies the $\aleph_4$-c.c.

NOTATION. If $\mathrm{Dom}(\bar{D}^*) = T$ let $\mathrm{Nm}(\bar{D}^*) = \mathrm{Nm}(T, D^*)$, and if $T = \aleph_2^{<\omega}$, $\bar{D}^*(\eta) = \{\{\eta^\frown\langle \alpha \rangle : \alpha \in A\} : A \in D\}$, we let $\mathrm{Nm}(D) = \mathrm{Nm}(\bar{D}^*)$.

PROOF. If $G \subseteq P$ is generic, then $\bigcup \{\eta : \eta \in (T, D)$ for every $(T, D) \in G\}$ is a member of $\omega_2^{<\omega}$ (in $V[G]$) and as $D_\eta \supseteq D^{\mathrm{cb}}_\eta$, it is unbounded.

Now suppose $\underset{\sim}{\tau}$ is a name of an ω-sequence from ω_1, and let $(T, D) \in P$. We define by induction (T_n, D) such that:

(a) $(T_0, D) = (T, D)$,

(b) $(T_n, D) \leqq_n (T_{n+1}, D)$ (hence $(T_n, D) \leqq^* (T_{n+1}, D)$),

(c) for every $\eta \in T_{n+1}$, $l(\eta) = n + 1$, for some $\bar{\alpha}_\eta$ and $l \leqq n$

$$(T_{n+1}, D)_{[\eta]} \Vdash_P \text{“}\underset{\sim}{\tau} \restriction l = \bar{\alpha}_\eta\text{”, and } l \text{ is maximal.}$$

Clearly $(\bigcap_{n<\omega} T_n, D) \in P$, $(T_n, D) \leqq (\bigcap_{n<\omega} T_n, D)$.

Now use Lemma 4.6 on $(\bigcap_{n<\omega} T_n, D)$, and H, $H(\eta) = \bar{\alpha}_\eta$ and get (T', D'), $(T, D) \leqq^* (T', D')$, $H(\eta) = \bar{\alpha}^n$ for $\eta \in T$, $l(\eta) = n + 1$. Now for each l, there is (T'', D''), $(T', D') \leqq (T'', D'')$ and $\bar{\alpha}$ such that $(T'', D'') \Vdash_l$ "$\underset{\sim}{\tau} \restriction l = \bar{\alpha}$", and let

$\eta_0 \in T''$ be the trunk of T''; w.l.o.g. $l+1 < l(\eta_0)$. By the construction $l \leqq l(\bar{\alpha}_{\eta_0})$, hence $\bar{\alpha} = \bar{\alpha}^k \restriction l$ for $k = l(\eta_0)$, hence we can take $k = l+1$ (use (T', D') itself), so $(T', D') \Vdash$ "$\underset{\sim}{\tau} = \langle \bar{\alpha}^{m(n)}(n) : n < \omega \rangle$", for $m(n)$ large enough.

4.8 Problem. Is the forcing semi-proper? (See [19].)

4.9 Definition. A filter D on a set I, S a set of cardinals, we call D an (S, α)-Galvin filter (and the dual ideal a Galvin ideal) if player II has a winning strategy in the following game, for every $J \subseteq I$, $J \neq \emptyset \bmod D$: (we call the game the (S, α)-Galvin game for D, J).

In the ith move player I defines a function F_i from I to some $\lambda \in S$ and player II chooses $A_i \subseteq J \cap \bigcap_{j<i} A_j$ such that $|F_i(A_i)| < \lambda$. Player II wins if $\bigcap_{i<\alpha} A_i \neq \emptyset \bmod D$. For simplicity we can say J was chosen by player I in his first move.

Galvin suggests this game for $D^{\mathrm{cb}}_{\omega_2}$ = the co-bounded subset of λ for a cardinal λ, $\alpha = \omega$ and $S = \{2\}$. So for $\alpha = \omega$, $S = \{2\}$ we omit (S, α). Note that only $S \cap (|I| + 1)$ has any importance.

Galvin, Jech and Magidor [6] and Laver [11] independently proved the following (really in [6] a slightly weaker version is proved but the difference is immaterial for us).

4.10 Theorem. *If we start with a universe V, $V \models$ "G.C.H. $+ \kappa$ is measurable" and use Levi collapsing of κ to $\aleph_2$ (i.e., every λ, $\aleph_1 \leqq \lambda < \kappa$ now will have cardinality $\aleph_1$) then in the new universe $V[G]$, $D^{\mathrm{cb}}_{\omega_2}$ is a Galvin filter, in fact* (Car $- \{\aleph_2\}$, $\omega + 1$)*-Galvin filter. Moreover if $D \in V$ was a normal ultrafilter on κ, then in $V[G]$ there is a family W of subsets of κ, $A \in W \Rightarrow A \neq \emptyset \bmod D$, W is dense $[($i.e. $\forall A \subseteq \kappa)$ $A \neq \emptyset \bmod D \Rightarrow (\exists B \in W)(B \subseteq A)]$ and W is closed under intersection of countable descending chains. [We identify here D with the filter it generates in $V[G]$ which is normal.] We call this the $\aleph_1$-Laver property (omitting $\aleph_1$ usually).*

The relevance of this is:

4.11 Theorem. *Let $S \subseteq$* SCar.

(1) *Let D be an (S, α)-Galvin filter on I, which is $\aleph_2$-complete and $P =$* PP$(D) = \{A \subseteq I : A \neq \emptyset \bmod D\}$, *order by inverse inclusion. Then P is (S, α)-complete.*

(2) *We can replace the hypothesis in* (1) *above by "D is $|\alpha|^+$-Laver" and get even "real (S, α)-complete".*

(3) *If P is* Nm(T^*, D^*) *(see* 4.9*), each D^*_η is an (S, ω)-Galvin, $\aleph_2$-complete filter then P is (S, ω)-complete; and if $S \supseteq \{\aleph_1\}$, then P is semi-proper (as we can add all λ,* cf $\lambda > \aleph_2$ *to S). (Note we are not assuming* CH.)

PROOF. (1), (2) obvious.

(3) Also easy, but we shall do it. For simplicity let $S = \mathrm{SCar} - \{\aleph_2\}$. By 3.4(4) it suffices to prove (S, ω)-completeness. For every $\eta \in T^*$, let H_η be a winning strategy for player II in the (S, ω)-Galvin game for D^*_η. Now let us describe the winning strategy of player II in $G^\omega_S(p, P)$. For notational simplicity $2 \notin S$.

Let $p = (T_0, D^*)$; w.l.o.g. the trunk of T_0 is $\langle\ \rangle$.

In the first move player I chooses $\lambda_0 \in S$ and a P-name $\underset{\sim}{\beta}_0$ of an ordinal $< \lambda$.

Player II chooses $p_0 \in P$, $p \leqq^* p_0$, $p_0 \Vdash_P$ "$\underset{\sim}{\beta}_0 \leqq \beta_0$" (possible by the proof of 4.7).

However if player II continues to play like this, he will lose as maybe $\bigcap_n T_n$ ($p_n = (T_n, D^*)$) will be $\{\langle\ \rangle\}$.

So he is thinking how to make $\mathrm{Suc}_{\cap T_n}(\langle\ \rangle) \neq \emptyset \bmod D^*_{\langle\ \rangle}$. If he, on the other hand, will demand $p_0 \leqq_1 p_{n+1}$, he will have $\mathrm{Suc}_{\cap T_n}(\langle\ \rangle) \neq \emptyset \bmod D^*$, but it will be hard (and in fact impossible) to do what is required when, e.g., $\lambda_i = \aleph_1$. So what he will do is to decrease $\mathrm{Suc}_{T_n}(\langle\ \rangle)$, but do it using his winning strategy $H_{\langle\ \rangle}$ for the (S, ω)-Galvin game for $D_{\langle\ \rangle}$. So in the second move player I chooses a cardinal $\lambda_1 \in S$ and P-name $\underset{\sim}{\beta}_1$ of an ordinal $< \lambda_1$. Player II, first for each $\eta \in p_0$, $l(\eta) = 1$, chooses p^η_1, $(p_0)_{[\eta]} \leqq^* p^\eta_1$, $p^\eta_0 \Vdash_P$ "$\underset{\sim}{\beta}_1 \leqq \beta_\eta$". This defines a function from $\mathrm{Suc}_{T_0}(\langle\ \rangle)$ to λ_1, so player II consults the winning strategy $H_{\langle\ \rangle}$, gets $A^0_{\langle\ \rangle} \subseteq \lambda_1$, $|A^0_{\langle\ \rangle}| < \lambda_1$ and lets $T_1 = \bigcup \{T^\eta_q : \beta_\eta \in A^0_{\langle\ \rangle}\}$.

In the third move, player II tries also to insure that also $\{\eta \in \bigcap_n T_n : l(\eta) = 2\}$ will be as required. Now player I chooses $\lambda_2 \in S$ and P-name $\underset{\sim}{\beta}_2$. Player II chooses for every $\eta \in T_1$, $l(\eta) = 2$ a condition p^η_2, $(p_1)_\eta \leqq^* p^\eta_2$, $p^\eta_2 \Vdash_P$ "$\underset{\sim}{\beta}_2 \leqq \beta_\eta$". So for every $\eta \in T_1$, $l(\eta) = 1$, we have a function from $\mathrm{Suc}_\eta(T_1)$ to λ_2, so consulting the strategy H_η, player II chooses $A^1_\eta \subseteq \lambda$, $|A^1_\eta| < \lambda$. We can assume A^1_η is an initial segment, and for $\lambda < \aleph_0$ the number of possible A^1_η is finite. So now the function $\eta \rightarrow A^1_\eta$ ($\eta \in \mathrm{Suc}_{T_1}(\langle\ \rangle)$) is a function whose domain is in $\{\eta \in T_1 : l(\eta) = 1\}$ (remember, by 3.2(6), if $n \in S$, $n < \omega$ then w.l.o.g. $\{m : n \leqq m < \aleph_0\} \in S$). So player II can consult again the strategy $H_{\langle\ \rangle}$, and find $A^2_{\langle\ \rangle}$, and let $T_2 = \bigcup \{T^\eta_2 : l(\eta) = 2,\ \eta \in T_1,\ \beta_n \in A^1_{\eta\restriction 1},\ A^1_{\eta\restriction 1} \in A^2_{\eta\restriction 0} = A^2_{\langle\ \rangle}\}$.

The rest is very clear.

§5. Chain conditions and Avraham's problem

Chain conditions are very essential for iterated forcing. In Solovay and Tenenbaum [20] this is the point, but even when other conditions are involved, we have to finish the iteration and exhaust all possibilities, so some chain condition is necessary to "catch our tail." In our main line we want to collapse some large cardinal κ to $\aleph_2$, in an iterated forcing of length (and power) κ, each P_i of power $< \kappa$. So we want that κ is not collapsed, and the obvious way to do this is by the κ-chain condition. We prove it by the traditional method of the Δ-system. For general RCS iteration, we have to assume κ is Mahlo (i.e., $\{\lambda < \kappa$ strongly inaccessible$\}$ is stationary) and for iteration of semi-proper forcings we ask for less.

Now we are able to answer the following problem of U. Abraham:

PROBLEM. Suppose G.C.H. holds in V. Is there a set $A \subseteq \aleph_1$ so that every ω-sequence from $\aleph_2$ belongs to $L[A]$? (See [1] for partial positive results.)

For this we collapse some inaccessible κ which is the limit of measurable cardinals, to $\aleph_2$, changing the cofinalities of arbitrarily large measurables $< \kappa$ to $\aleph_0$.

5.1 DEFINITION. (1) For any iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$. We call $\bar{p} = \langle p_i : i \in S \rangle$ a Δ-system if, for $i < j$ in S, $p_i \restriction i = p_j \restriction j$ and $p_i \in P_j$. We call $p_i \restriction i$ the heart of the Δ-system, $\mathrm{hr}(\bar{p})$.

(2) For a forcing P, we call $\bar{p} = \langle p_i : i \in S \rangle$ a μ-weak Δ-system if $p_i \in P$, $\bigcup_{i \in S} i$ is a regular cardinal κ, and there is a condition $q = \mathrm{hr}(\bar{p})$ (the heart of $\bar{p}$) such that for every r, $q \leqq r \in P$ there is $\alpha < \kappa$ satisfying: if $\alpha < \alpha_i \in S$ for $i < \mu_1 < \mu$ then $\{r\} \cup \{p_{\alpha_i} : i < \mu_1\}$ has an upper bound in P.

5.2 CLAIM. *Any Δ-system in an* RCS *iteration as in Definition* 5.1, *is a $\aleph_1$-weak Δ-system.*

5.3 THE CHAIN CONDITIONS LEMMA. (1) *Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ is an* RCS *iteration, κ regular, $|P_i| < \kappa$ for $i < \kappa$ and let $A = \{\lambda < \kappa : \lambda$ strongly inaccessible$\}$.* Then *for every sequence $\bar{p} = \langle p_j : j \in B \subseteq A \rangle$, we can find a closed unbounded $C \subseteq \kappa$ and a pressing down function h on $C \cap B$ (i.e., $h(j) < j$) such that for any α, $\langle p_j : j \in B \cap C,\ h(j) = \alpha \rangle$ is a Δ-system.*

(2) *If every $\underset{\sim}{Q}_i$ is semi-proper we can replace A by $A' = \{i : \emptyset \Vdash_{P_i} \mathrm{cf}\, i > \aleph_0\}$, provided that κ is regular.*

Before we prove the lemma note

5.4 COROLLARY. (1) *If in* 5.3, *A is stationary, then $P_\kappa = \mathrm{R\,Lim}\, \bar{Q}$ satisfies the κ-chain condition.*

(2) *If D is a normal ultrafilter on κ, $B \in D$, B G A then (in* 5.3) *for some $B' \in D$, $\langle p_j : j \in B' \rangle$ is a Δ-system.*

PROOF OF 5.3. (1) If B is not stationary, the conclusion is trival, so suppose B

is stationary. So necessarily κ is strongly inaccessible (as every member of A is and $B \subseteq A$), hence by 1.6, $P_\kappa =^{\mathrm{df}} \mathrm{R\,Lim}\,\bar{Q} = \bigcup_{i<\kappa} P_i$. As $|P_i| < \kappa$ for every $i < \kappa$, there is a one to one function H from P_κ onto κ. Again as $|P_i| < \kappa$ for $i < \kappa$, clearly

$$C = \{i : H \text{ maps } \bigcup_{j<i} P_j \text{ onto } i \text{ and for } j < i, j \in B \text{ implies } p_j \in P_i\}$$

is a closed unbounded subset of κ. We now define the function h with domain B: $h(i) = H(p_i \restriction i)$.

We first prove that h is pressing down. Clearly $p_i \restriction i \in P_i$, and if $i \in B \cap C$ then i is strongly inaccessible and $(\forall j < i)|P_j| < i$, hence by 1.6, $P_i = \bigcup_{j<i} P_j$, hence $p_i \restriction i \in \bigcup_{j<i} P_j$. So if $i \in B \cap C$, $p_i \restriction i \in \bigcup_{j<i} P_j$ hence $H(p_i \restriction i) < i$.

Now clearly $i < j \in B \cap C$, $h(i) = h(j)$ implies $p_j \restriction j = p_i \restriction i$, and by C's definition $p_i \in P_j$, so we finish.

(2) The proof is similar, using 2.7.

5.5 THEOREM. *Suppose* CON(ZFC + "*there is an inaccessible cardinal* κ *which is the limit of measurable cardinals*").

Then the following theory is consistent: ZFC + G.C.H. + $(\forall A \subseteq \aleph_1)(\exists \bar{\alpha})(\bar{\alpha}$ *an* ω*-sequence of ordinals* $< \aleph_2$, $\bar{\alpha} \notin L[A])$.

PROOF. We start with a model V of ZFC + "κ is inaccessible, and limit of measurables." W.l.o.g. V satisfies G.C.H. (see [8]), and we define an iterated forcing $\langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$, such that $|P_i| < \kappa$. We do it by induction on i, and clearly (see 1.4(5) for i limit) the induction hypothesis $|P_i| < \kappa$ continues to hold. If $\bar{Q}_i = \langle P_j, \underset{\sim}{Q}_j : j < i\rangle$ is defined, let κ_i be the first measurable cardinal $> |P_i|$, where $P_i = \mathrm{R\,Lim}\,\bar{Q}_i$. It is known (see e.g. [8]) that κ_i is measurable in V^{P_i}, and any normal ultrafilter on it from V is an ultrafilter (and normal) in V^{P_i} too. By a hypothesis $\kappa_i < \kappa$. So let $\underset{\sim}{Q}_{i,0}$ be $\mathrm{PF}(D_i)$, $D_i \in V$ any normal ultrafilter on κ_i, and $\underset{\sim}{Q}_{i,1}$ be the Levi collapse of κ_i^+ to $\aleph_1$ (i.e. $\underset{\sim}{Q}_{i,1} = \{f : \mathrm{Dom}(f)$ is an ordinal $< \aleph_1$, and $\mathrm{Range}(f) \subseteq \kappa_i^+\}$, with inclusion as order). We let $\underset{\sim}{Q}_i = \underset{\sim}{Q}_{i,0} * \underset{\sim}{Q}_{i,1}$.

Now by 4.2, $\underset{\sim}{Q}_{i,0} = \mathrm{PF}(D_i)$ is $(\mathrm{Car}^V - \{\kappa\}, \omega)$-complete, $\underset{\sim}{Q}_{i,1}$ is $(\mathrm{Card}^V, \omega)$-complete trivially (by 3.4(1)) hence by 3.5 $\underset{\sim}{Q}_i$ is $(\mathrm{Card}^V - \{\kappa\}, \omega)$-complete.

Hence by 3.5, 2.7, $P_\kappa = \mathrm{R\,Lim}\langle P_i, \underset{\sim}{Q}_i : i \in \kappa\rangle$ does not change the cofinality of $\aleph_1$ and is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete, hence it does not add reals. By 3.4(1) each $\underset{\sim}{Q}_i$ is semi-proper, so by 3.5 P_κ is semi-proper. By 5.3(2) P_κ have the κ-chain condition, so clearly if $G_\kappa \subseteq P_\kappa$ is generic, $\aleph_1^{V[G_\kappa]} = \aleph_1^V$, $\aleph_2^{V[G_\kappa]} = \kappa$, V, $V[G_\kappa]$ have the same reals, and $V[G_\kappa]$ satisfies G.C.H.

Now if $A \subseteq \omega_1$, then as P_κ satisfies the κ-chain condition, A is determined by

$G_i =^{\text{def}} G_\kappa \cap P_i$ for some $i < \kappa$. By 1.2, G_i is generic for P_i, so $L[A] \subseteq V[G_i]$, but in $V[G_i]$ an ω-sequence from $\aleph_2^{V[G_\kappa]}$ is missing: the Prikry sequence we shoot through κ_{i+1}, which was measurable in $V[G_i]$.

§6. Reflection properties of S_0^2: refining Avraham's problem and precipitous ideals

In the previous section we have collapsed a large cardinal κ to $\aleph_2$, such that to "many" measurable cardinals $< \kappa$ we add an unbounded ω-sequence. However, "many" was interpreted as "unbounded set". This is very weak, and, it seemed, will not usually suffice.

Notice that it is known that if we collapse a large cardinal by $\aleph_1$-complete forcing then $S_1^2 =^{\text{def}} \{\delta < \aleph_2 : \text{cf}\, \delta = \aleph_1\}$ has reflection and bigness properties, e.g., those from Theorem 4.10. However, for S_0^2, we get nothing as it is equal to $\{\delta < \aleph_2$: in the universe before the collapse, $\text{cf}\, \delta = \aleph_0\}$ and it is known, e.g., that on such a set there was no normal ultrafilter.

So we can ask whether S_0^2 can have some "large cardinal properties". The natural property to consider is precipitous filters D on $\aleph_2$ such that $S_0^2 \in D$. Such ultrafilters were introduced in Jech and Prikry [9].

Their important property is that if we force by $\text{PP}(D)$ (see 4.11), G is generic, the domain of E is I, and in $V[G]$, $E \supseteq D$ is the ultrafilter G generates (on old sets), then V^I/E (taking only old $f: I \to V$) is well-founded. Jech, Magidor, Mitchell and Prikry [10] proved that the existence of a precipitous filter on $\aleph_1$ is equiconsistent with the existence of a measurable cardinal, and also proved the consistency of "$D_{\aleph_1}$ (= the filter of closed unbounded sets) is precipitous". (Notice that the Laver property is stronger.) Magidor asked whether "ZFC + G.C.H. + there is a normal precipitous filter D on $\aleph_2$, $S_0^2 \in D$" is consistent.

We answer positively, by collapsing suitably some κ to $\aleph_2$, letting $D = D_{\aleph_2} + A$, $A = \{\lambda < \kappa :$ in the old universe λ is measurable$\}$. This works if A is stationary. This was proved previously and independently, using much larger cardinals, by Gitik.

We can also consider the following strengthening of Avraham's problem:

Problem. If V satisfies G.C.H., does there exist $A \subseteq \aleph_2$ such that, for every $\delta < \aleph_2$, every ω-sequence from δ belongs to $L(A \cap \delta)$?

Again we have to change the cofinality on a stationary set, and to iterate forcing such that stationarily often we change the cofinality of $\aleph_2$.

The first time, the collapse of some λ to $\aleph_2$ is Levi's collapse so by 4.11, 4.10 we have a (Card^V, ω)-complete forcing Q_λ doing this; but later the collapse is not even $\aleph_1$-complete. However, looking again at 2.7 and Theorem 3.5 on iterated (Car, ω)-complete forcing, we see that less is needed. If $\bar{Q} = \langle P_i, Q_i : i < \lambda \rangle$ collapses λ to $\aleph_2$, it suffices that $(\mathrm{R\,Lim}\,\bar{Q}/P_{i+1}) * Q_\lambda$ is $(\mathrm{Card}^{V^{P_{i+1}}}, \omega)$-complete. So this is what we shall do. But for clarity of exposition, we first prove a weaker lemma.

6.1 Lemma. *Suppose D is a normal ultrafilter on λ, $\bar{Q} = \langle P_i, Q_i : i < \lambda \rangle$ an* RCS *iteration, $|P_i| < \lambda$. Suppose further $P_\lambda = \mathrm{R\,Lim}\,\bar{Q}$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete and collapse λ to $\aleph_2$. Consider the following game $G(p_0, A_0)$, for $p_0 \in P_\lambda$, A_0 a P_λ-name of a subset of λ, $p_0 \Vdash_{P_\lambda}$ "$A_0 \neq \emptyset \bmod D$".*

Player I chooses P_λ-names β_1 (of an ordinal $< \aleph_1$) and F_1 (a function from λ to $\aleph_1$).

Player II has to choose $p_1 \in P_\lambda$, $p_0 \leqq p_1$ and $\gamma_1 < \omega_1$ (and $\beta_1 < \omega_1$) such that $p_1 \Vdash_{P_\lambda}$ "$A_1 = A_0 \cap F_1^{-1}(\{\gamma_1\}) \neq \emptyset \bmod D$, and $\beta_1 = \beta_1$".

In the n-th move, player I chooses P_λ-names β_n, F_n, and player II chooses p_n, $p_{n-1} \leqq p_n$ and $\gamma_n < \omega_1$ and $\beta_n < \omega_1$ such that $p_n \Vdash_{P_\lambda}$ "$A_n = A_{n-1} \cap F_n^{-1}(\{\gamma_n\}) \neq \emptyset \bmod D$, and $\beta_n = \beta_n$".

In the end, player II wins if $\bigcup_{n<\omega} p_n \in P_\lambda$ and $\bigcup_{n<\omega} p_n \Vdash_{P_\lambda}$ "$\bigcap_{n<\omega} A_n \neq \emptyset \bmod D$".

Our conclusion is that player II wins the game.

Proof. So let $p_0 \in P_\lambda$, A_0 a P_λ-name, $p_0 \Vdash_{P_\lambda}$ "$A_0 \neq \emptyset \bmod D$" and we shall describe the winning strategy of player II in the game $G(p_0, A_0)$. Let the winning strategy of player II in $G^{\omega}_{\{2,\aleph_0,\aleph_1\}}(p, P_\lambda)$ be $H[p]$. By 3.2(2), we can assume that player II really determined the value of the ordinals given to him. We can also assume player II is given by player I a pair of names of ordinals (instead of one).

Let $B_0 = \{i < \lambda$: there is $p \geqq p_0$, $p \Vdash_{P_\lambda}$ "$i \in A_0$"$\}$. Now $B_0 \in D$ because otherwise, as D is an ultrafilter in V, $B_0 = \emptyset \bmod D$, since $p_0 \Vdash_{P_\lambda}$ "$A_0 \subseteq B_0$" (by B's definition) we have $p_0 \Vdash_{P_\lambda}$ "$A_0 = \emptyset \bmod D$", contradiction.

Now for every $i \in B$, there is $p_{0,i} \in P_\lambda$, $p_0 \leqq p_{0,i}$ such that $p_{0,i} \Vdash_{P_\lambda}$ "$i \in B_0$".

So let player I first move in $G(p_0, A_0)$ by choosing β_1 a P_λ-name of an ordinal $< \aleph_1$, and $F_1 : \lambda \to \aleph_1$, F_1 a P_λ-name. Now for each $i \in B_0$, player II simulates a play of the game $G_i = G^{\omega}_{\{2,\aleph_0,\aleph_1\}}(p_{0,i}, P_\lambda)$. He plays $(\beta_1, F(i))$ (i.e., a pair of names of ordinals $< \aleph_1$) for player I_i, and by the strategy $H[p_{0,i}]$ gets a move for player II_i : $p_{1,i} \in P_\lambda$, $p_{0,i} < p_{1,i}$, and $\alpha_{1,i} < \aleph_1$, $\varepsilon_{1,i} < \aleph_1$ such that $p_{1,i} \Vdash_{P_\lambda}$ "$\beta_1 = \alpha_{1,i}$ and $F_1(i) = \varepsilon_{1,i}$". Now for some $B_1 \subseteq B_0$, $B_1 \in D$, and $\langle p_{1,i} : i \in B_1 \rangle$ is a Δ-system with heart p_1 (see 5.1), and we can also make $\langle \alpha_{1,i}, \varepsilon_{1,i} : i \in B_1 \rangle$ constant $(\alpha_1, \varepsilon_1)$ (for $i \in B_1$).

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Now player II can make his move in G: he chooses p_1, α_1 and ε_1. It is easy to check this is a legitimate move.

So player II continues to play such that after the nth move:

$(*)_n$ there are $B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1 \subseteq B_0$ all in D, $p_{l,i} \in P_\lambda$ for $0 \leqq l \leqq n$, $i \in B_l$, $p_{0,i} \leqq p_{1,i} \leqq \cdots \leqq p_{n,i}$, $\langle p_{l,i} : i \in B_l \rangle$ is a Δ-system with heart p_l (for l, $0 < l \leqq n$), $p_0 \leqq p_1 \leqq \cdots \leqq p_n$, and at the lth move player I chooses $\underset{\sim}{\beta}_l$, $\underset{\sim}{F}_l$, and player II chooses $p_l, \alpha_l, \varepsilon_l$ and (for $l = 1, n$ and $i \in B_l$) $p_{l,i} \Vdash$ "$\alpha_l = \underset{\sim}{\beta}_l$ and $\underset{\sim}{F}_l(i) = \varepsilon_l$". Also for each $l \leqq n$, $i \in B_l$, the following is an initial segment of a play of the game $G^{\omega}_{\{2,\aleph_0,\aleph_0\}}(p_{0,i}, P_\lambda)$, in which player II uses the winning strategy $H[p_{0,i}]$:

$$\langle \underset{\sim}{\beta}_1, \underset{\sim}{F}_1(i) \rangle, \quad \langle p_{1,i}, \alpha_1, \varepsilon_1 \rangle, \quad \langle \underset{\sim}{\beta}_2, \underset{\sim}{F}_2(i) \rangle, \quad \langle p_{2,i}, \alpha_2, \varepsilon_2 \rangle, \quad \cdots, \quad \langle p_{l,i}, \alpha_l, \varepsilon_l \rangle.$$

It is easy to check that player II can use this strategy; moreover, by the choice of $H[p_{0,i}]$, for every such play, $p_i = \bigcup_{l<\omega} p_{i,l} \in P_\lambda$, for every $i \in \bigcap_n B_n$; as $B_n \in D$, $\bigcap_{n<\omega} B_n \in D$ and clearly $\langle p_i : i \in \bigcap_{n<\omega} B_n \rangle$ is a Δ-system with heart $p = \bigcup_l p_l$, and so by 5.2 $p \Vdash_{P_\lambda}$ "the set of $i \in \bigcap_n B_n$ such that p_i is in the generic set, is $\neq 0 \bmod D$". Also $p \Vdash_{P_\lambda}$ "for every $i \in \bigcap_n B_n$, $F_l(i) = \varepsilon_l$". So clearly player II has won the play, hence the game.

Remark. We could have used any S, $S \subseteq \{2, \aleph_0, \aleph_1\}$ instead of $\{2, \aleph_0, \aleph_2\}$ and get a parallel result.

6.2 Lemma. *Suppose λ is measurable, D a normal ultrafilter over λ, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \lambda \rangle$ an* RCS *iteration, P_λ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete and $|P_i| < \lambda$ for $i < \lambda$.*

Then $P_\lambda * \mathrm{Nm}(D)$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete.

Proof. Just combine the proofs of 6.1 and 4.11(3).

6.3 Definition. A filter D on a set I (in a universe V) is called precipitous if the following holds:

$$\Vdash_{\mathrm{PP}(D)} \text{"there are no } f_n : I \to \text{ordinals}, f_n \in V, \text{ such that } f_{n+1} <_{\underset{\sim}{E}} f_n \text{ for each } n\text{"}$$

where

(i) $\mathrm{PP}(D) = \{A \subseteq I : A \neq \emptyset \bmod D\}$ ordered by reverse inclusion,

(ii) $\underset{\sim}{E}$ is the filter generated by the generic set of $\mathrm{PP}(D)$,

(iii) $f <_{\underset{\sim}{E}} g$ means $\{\alpha \in I : f(\alpha) < g(\alpha)\} \in \underset{\sim}{E}$.

Remark. The following is an equivalent definition A: filter D over I is precipitous if player I does not have a winning strategy in the following game $\mathrm{PrGm}(D)$.

First move player I chooses $A_1 \subseteq I$, $A_1 \neq \emptyset \bmod D$,
player II chooses $B_1 \subseteq A_1$, $B_1 \neq \emptyset \bmod D$;

nth move player I chooses $A_n \subseteq B_{n-1}$, $A_n \neq \emptyset \bmod D$,
player II chooses $B_n \subseteq A_n$, $B_n \neq \emptyset \bmod D$.

Player II wins if $\bigcap_{n<\omega} A_n$ (which is $= \bigcap_{n<\omega} B_n$) is non-empty (not necessarily $\neq \emptyset \bmod D$).

See Jech and Prikry [9], and Jech, Magidor, Mitchell and Prikry [10].

6.4 Theorem. *Suppose* "ZFC + G.C.H. + κ *is strongly inaccessible and* $A = \{\lambda < \kappa : \lambda$ *measurable*$\}$ *is stationary*" *is consistent. Then*:

(1) *The following statement is consistent with* ZFC + G.C.H.:

for every $B \subseteq \aleph_2$ *for some* δ (*in fact* $\delta \in A$), $\operatorname{cf} \delta = \aleph_0$, *but in* $L[B \cap \delta]$, δ *is a regular cardinal* $> \aleph_1$.

(2) *If in the hypothesis* $A \in D$, D *is a normal ultrafilter on* κ, *then there is a normal precipitous ideal on* $\aleph_2$ *to which* S_0^2 *belongs.*

Proof. So let V be a model of ZFC + G.C.H., and let κ be a strongly inaccessible cardinal, such that $A = \{\lambda < \kappa : \lambda$ measurable$\}$ is stationary.

We now define by induction on $i < \kappa$ forcing notions $P_i \in V$, $\underset{\sim}{Q}_i \in V^{P_i}$ such that $|P_i| < \kappa$, $\langle P_j, \underset{\sim}{Q}_j : j < i\rangle$ is an RCS iteration. So by 1.5(1) it suffices to define $\underset{\sim}{Q}_i$ for a given $\langle P_j, \underset{\sim}{Q}_j : j < i\rangle$.

Case 1. $i = \lambda$ is a measurable cardinal, such that for every $j < \lambda$, $|P_j| < \lambda$.

In this case let D_λ be a normal ultrafilter over λ, and $Q_\lambda = \mathrm{Nm}(D_\lambda)$.

Case 2. Not case 1.

In this case let Q_i be the Levi collapse of $(2^{|P_i|})^V$ to $\aleph_1$, i.e., $\{f \in V^{P_i} : f$ a countable function from ω_1 to $2^{|P_i|}\}$.

Now by 3.5 and 6.2 it is easy to see that $P_\kappa = \mathrm{R\,Lim}\langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ is $(\{2, \aleph_0, \aleph_1\}, \omega)$-complete, and by 5.4 it satisfies the κ-chain condition.

So clearly in V^{P_λ} G.C.H. holds, every real is from V, and $\aleph_1 = \aleph_1^V$, $\aleph_2 = \kappa$. Also if $\lambda \in A$, then $(\forall i < \lambda)|P_i| < \lambda$ (prove by induction on i for each λ). Let $G \subseteq P_\lambda$ be generic, and we shall prove that $V[G]$ satisfies the requirements:

Part 1. So let $B \subseteq \aleph_2$, and let $\underset{\sim}{B} \in V$ be a P_λ-name for it. Then $C_1 = \{\delta < \kappa : G \cap P_\delta$ determine $B \cap \delta\}$ contains $C_0 = \{\delta : (\forall i < \delta)$, $B \cap \{i\}$ has a P_j-name for some $j < \delta\}$ which is closed unbounded in κ, and $C_0 \in V$, because P_κ satisfies the κ-chain condition.

Now if $\lambda \in C_0 \cap A$, then we know $|P_i| < \lambda$ for $i < \lambda$, so $Q_\lambda = \mathrm{Nm}(D_\lambda)$, hence in $V[G]$, $\operatorname{cf} \lambda = \aleph_0$. On the other hand, clearly $G \cap P_\lambda$ is a generic subset of P_λ, by 5.4 P_λ satisfies the λ-chain condition, so $\Vdash_{P_\lambda}$ "cf $\lambda = \lambda$". Hence in $V[G \cap P_\lambda]$, $A \cap \lambda$ is present, but λ is a regular cardinal $> \aleph_1$. So also in $L[A \cap \lambda]$, λ is a regular cardinal $> \aleph_1$.

Part 2. The following is essentially the same proof as [10] who do it for the Levi collapse; and it suffices for (2) of the theorem. It follows from Magidor [12] theorem 2.1, and is included for completeness only.

6.4 LEMMA. *Suppose κ is measurable, D a normal ultrafilter over κ, $\bar{Q} = \langle P_i, Q_i : i < \kappa \rangle$ an* RCS *iteration, $|P_i| < \kappa$ for $i < \kappa$, $P = P_\kappa = \mathrm{R\,Lim}\,\bar{Q}$.*

Then in V^P, D is a precipitous filter.

PROOF. If not, in V^P there is $A_0 \in \mathrm{PP}(D)$, $A_0 \Vdash_{\mathrm{PP}(D)}$ "$\langle \underset{\sim}{f}_n : n < \omega \rangle$ is an ω-sequence of functions from κ to ordinals which belong to V^P which is decreasing mod $\underset{\sim}{E}$, $\underset{\sim}{f}_n \in V^P$".

So there is $p \in P$ a P-name $\underset{\sim}{A}_0$, and $P * \mathrm{PP}(D)$-names $\underset{\sim}{f}'_n$ of the $\underset{\sim}{f}_n$ such that $p \Vdash_P$ "$\underset{\sim}{A}_0$, $\underset{\sim}{f}'_n$ are as above".

Let $B_0 = \{\lambda < \kappa : \lambda$ is strongly inaccessible and for some $p' \geqq p$, $p' \in P$, and $p' \Vdash_{P_\lambda}$ "$\lambda \in \underset{\sim}{A}_0$"$\}$.

Because D is normal, κ measurable, $\{\lambda < \kappa : \lambda$ strongly inaccessible$\} \in D$, hence $B_0 \in D$. For each $\lambda \in B$ choose $p_{\lambda,0}$, $p \leqq p_{\lambda,0} \in P$, $p_{\lambda,0} \Vdash$ "$\lambda \in \underset{\sim}{A}_0$". By 5.3 there is $B'_0 \subseteq B_0$, $B'_0 \in D$ such that $\langle p_{\lambda,0} : \lambda \in B'_0 \rangle$ is a Δ-system with heart p_0.

Now we define by induction on $n < \omega$, $p_{\lambda,n}$, p_n, p'_n, B_n, B'_n, $\underset{\sim}{A}_n$, g_n, $\alpha_{\lambda,n}$ such that

(1) $\langle p_{\lambda,n} : \lambda \in B'_n \rangle$ is a Δ-system of members of P with heart p'_n,

(2) $B_{n+1} \subseteq B'_n \subseteq B_n$, $B_{n+1} \in D$,

(3) $p_{n+1} \geqq p'_n \geqq p_n$ all in P,

(4) $p_{\lambda,n+1} \geqq p_{\lambda,n}$ both in P, $\underset{\sim}{g}_n$ a P-name of a function from κ to On,

(5) $p_{\lambda,n} \Vdash_P$ "$\lambda \in \underset{\sim}{A}_n$ and $\underset{\sim}{g}_n(\lambda) = \alpha_{\lambda,n}$", $\alpha_{\lambda,n} < \alpha_{\lambda,n-1}$ for $n > 0$,

(6) $\underset{\sim}{A}'_n = \{\lambda \in B'_n : p_{\lambda,n}$ is in the generic set of $P\}$,

(7) $p_{n+1} \Vdash_P$ "$\underset{\sim}{A}_{n+1} \in \mathrm{PP}(D)$ and $\underset{\sim}{A}_{n+1} \subseteq A'_n$ and $[A_{n+1} \Vdash_{\mathrm{PP}(D)}$ "$\underset{\sim}{f}_n = \underset{\sim}{g}_n$"$]$ and $\underset{\sim}{A}_{n+1} \subseteq \{i < \kappa : \underset{\sim}{g}_n(i) < \underset{\sim}{g}_{n-1}(i)\}$",

(8) $B_{n+1} = \{\lambda \in B_n :$ there is $p' \geqq p_{\lambda,n}$, $p' \geqq p_{n+1}$, such that for some α, $p' \Vdash$ "$\lambda \in \underset{\sim}{A}_{n+1}$ and $\underset{\sim}{g}_n(\lambda) = \alpha$"$\}$.

The definition is easy.

Now as $B'_n \in D$, $\bigcap_{n<\omega} B'_n \neq \emptyset$, and if λ belongs to the intersection, $\langle \alpha_{\lambda,n} : n < \omega \rangle$ is a strictly decreasing sequence of ordinals, contradiction.

REMARK. Really precipitousness of ideals on κ is preserved by κ-c.c. forcing.

§7. Strong preservation and properness

In this section we list various properties which are preserved under RCS iteration. The most important one is the weakest strengthening of "not adding a real" which is preserved by RCS iteration. As the proofs are as in [18], we do not repeat them.

7.1 Definition. (1) For $\alpha \leqq \omega_1$, forcing notion P, and a P-name set $\underset{\sim}{S}$ of cardinals we say that P is $\underset{\sim}{S}$-semi α-proper if for any large enough regular λ and well ordering $<$ of $H(\lambda)$, and increasing continuous sequence N_i $(i<\alpha)$ of elementary countable submodels of $(H(\lambda), \in, <)$ such that $\langle N_j : j \leqq i\rangle \in N_{i+1}$ for $i<\alpha$ and for any $p \in N_0 \cap P$ there is $q \in P$, $q \geqq p$, which is $\underset{\sim}{S}$-semi (N_i, P)-generic for each $i<\alpha$.

(2) We call P $\underset{\sim}{S}$-semi $(<\omega_1)$-proper if it is $\underset{\sim}{S}$-semi α-proper for every $\alpha<\omega_1$.

7.2 Definition. (1) The forcing notion P has the ω^ω-bounding property, if for every generic $G \subseteq P$ and function $f : \omega \to \omega$ from $V[G]$, for some $g : \omega \to \omega$ from V $(\forall n)\ f(n) \leqq g(n)$.

(2) The forcing notion P has the Sacks property, if for every generic $G \subseteq P$ and function $f : \omega \to \omega$ from $V[G]$, and function $h : \omega \to \omega$ which diverges to infinity [i.e. $(\forall n)(\exists m)(\forall k)(k \geqq m \to h(k) \geqq n)$] there is $g \in V$ a function from ω to finite subsets of ω, s.t. $(\forall n)|g(n)| \leqq h(n)$, and $(\forall n) f(n) \in g(n)$.

(3) The forcing notion P has the Laver property if for every generic $G \subseteq P$ and $f : \omega \to \omega$ from $V[G]$ and $h : \omega \to \omega$ from V which diverge to infinity, and function $f' : \omega \to \omega$ from V such that $(\forall n)\ f(n) \leqq f'(n)$, *there is* a function $g \in V$ from ω to finite subsets of ω such that $(\forall n)$ $|g(n)| \leqq h(n)$, and $(\forall n)$ $f(n) \in g(n)$.

Remark. The classical example of a P with the ω^ω-bounding property is adding a random real. Of course a forcing which is $\aleph_1$-complete, or even just does not add reals, has all those properties.

The Sacks property is satisfied by Sacks forcing and also by Silver's forcing.

The Laver property is satisfied by Laver forcing, and has a role in his proof of the consistency of the Borel Conjecture: *Every set of strong measure zero is countable.*

7.3 Theorem. *In 2.6 we can replace $\underset{\sim}{S}$-semi proper by each of the following properties (retaining the original $\underset{\sim}{S}$ in all substitutions):*

(1) *$\underset{\sim}{S}$-semi α-proper $(\alpha<\omega_1)$,*

(2) *$\underset{\sim}{S}$-semi ω-proper and ω^ω-boundedness,*

(3) *$\underset{\sim}{S}$-semi ω-proper and Sacks property,*
(4) *$\underset{\sim}{S}$-semi ω-proper and Laver property,*
(5) *$\underset{\sim}{S}$-semi ω-proper and P-property.*

PROOF. Like [16], [19].

REMARK. We can define suitable games; the property will be the existence of a winning strategy of the favorite player. But the preservation theorem is weaker. For example, the Sacks game for p, f, P ($p \in P$, $f \in {}^{\omega}\omega$) is as follows; in the nth move player I choose a P-name $\underset{\sim}{\zeta}_n$ of a natural number and player II a set $W_n \subseteq \omega$. In the end player II wins if $|W_n| \leqq f(n)$ and there is $q, p \leqq q \in P$, $q \Vdash_P$ "$\underset{\sim}{\tau}_n \in W_n$".

REMARK. Also the theorems from [16] on iterated forcing not adding reals, holds, provided that the sequence of completeness filters is in the ground model and each filter is generated by $\leqq 2^{\aleph_0}$ sets.

§8. Friedman's problem

Friedman [5] asked the following

PROBLEM. Is there for every $S \subseteq S_0^{\alpha}$ a closed set of order type ω_1, included in S or $S_0^{\alpha} - S$? We call this statement $\mathrm{Fr}(\aleph_\alpha)$.

Van Liere proved that $\mathrm{Fr}(\aleph_2)$ implies $\aleph_2$ is a Mahlo strongly inaccessible cardinal in L; and $\mathrm{Fr}(\aleph_\alpha) + \mathrm{not}\,\mathrm{Fr}(\aleph_2)$ ($\aleph_\alpha$ regular $> \aleph_2$) implies $0^{\#}$ exists. We prove the consistency of $\mathrm{Fr}(\aleph_2) + \mathrm{G.C.H.}$ with ZFC, modulo the consistency of some measurable cardinal (of order 1).

8.1 DEFINITION. We define by induction on n what are a measurable cardinal of order n and a normal ultrafilter of order n. For $n = 0$ those are the usual notions. For $n+1$, D is a normal ultrafilter of order $n+1$ on κ if $\{\lambda < \kappa : \lambda$ is measurable of order $n\} \in D$ and it is a normal ultrafilter. We call κ measurable of order $n+1$ if there is an ultrafilter of order $n+1$ on it.

8.2 LEMMA. *Suppose D is a normal ultrafilter on κ, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ an* RCS *iteration and $|P_i| < \kappa$ for every $i < \kappa$, and $A = \{\lambda < \kappa : \Vdash_{P_\kappa}$ "$\mathrm{cf}\,\lambda = \aleph_0$"$\}$ belongs to D.*

Suppose further that $G \subseteq P_\kappa$ is generic, $S \subseteq \kappa$, $S \in V[G]$, $S \neq \emptyset \bmod D$, and (in $V[G]$) let

$$Q_\kappa = \{f : \text{the domain of } f \text{ is some successor ordinal } \alpha < \aleph_1,\ f \text{ is into } S \text{ and it is increasing and continuous}\}.$$

So let $\underset{\sim}{S}, \underset{\sim}{Q}_\kappa$ *be* P_κ*-names for them and* w.l.o.g. $\Vdash_{P_\kappa}$ "$\underset{\sim}{S} \neq \emptyset \bmod D$". *We then conclude*:

(1) *if* P_κ *is* $\{\aleph_1\}$*-semi-proper, then so is* $P_\kappa * \underset{\sim}{Q}_\kappa$,

(2) *if* P_κ *is essentially* $(\{2, \aleph_0, \aleph_1\}, \omega)$*-complete, then so is* $P_\kappa * \underset{\sim}{Q}_\kappa$.

PROOF. (1) The problem is that Q_κ destroys a stationary set, so it is not proper, though it obviously does not add ω-sequences. So let $\underset{\sim}{S}$, $\underset{\sim}{Q}_\kappa$ be P_κ-names for S, Q_κ.

Let λ be regular, big enough, $\bar{Q}, Q_\kappa$, $\underset{\sim}{S} \in H(\lambda)$, let $<$ be a well ordering of $H(\lambda)$ and let $N \prec (H(\lambda), \in, <)$ be countable, p, q, $\bar{Q}$, $\underset{\sim}{S}$, $\underset{\sim}{Q}_\kappa$, $\in N$, $(p, \underset{\sim}{q}) \in P_\kappa * \underset{\sim}{Q}_\kappa$, and we shall prove the existence of an $\{\aleph_1\}$-semi $(N, P_\kappa * \underset{\sim}{Q}_\kappa)$-generic condition $\geqq (p, q)$. In V (hence in $H(\lambda)$), we let

$$S_0 = \{\lambda \in A : \text{there is } p' \in P_\kappa, p \leqq p', p' \Vdash \text{“}\lambda \in S\text{”}\}.$$

As in previous cases $S_0 \in D$, and for each $\lambda \in S_0$ let $p_{\lambda,0} \in P_\kappa$, $p_{\lambda,0} \Vdash$ "$\lambda \in S$" and for some $S_1 \subseteq S_0$, $S_1 \in D$, and $\langle p_{\lambda,0} : \lambda \in S_1 \rangle$ is a Δ-system (see 5.3). As N was an elementary submodel we can assume S_0, S_1, $\langle p_{\lambda,0} : \lambda \in S_1 \rangle$ and its heart p_0 belongs to N (but of course not all included in N). Let $S_2 = \bigcap \{S' : S' \in D$ and $S' \in N\}$, so clearly $S_2 = \{\alpha_i : i < \kappa\} \subseteq S_1$ is an indiscernible sequence over $N \cup \omega_1$ and it belongs to D. Clearly, $p \leqq p_0$.

Let $N \cap P_\kappa \subseteq P_\mu$, $S_3 = S_2 - (\mu + 1)$ ($\mu < \kappa$, of course).

Let $\chi \in S_3$, and let N^* be the Skolem Hull (in $(H(\lambda), \in, <)$) of $|N| \cup \{\chi\}$, as $\{\alpha_i : i \in S_2\}$ is indiscernible over $|N| \cup \omega_1$. Clearly

$$N^* \cap \omega_1 = N \cap \omega_1.$$

Clearly $P_\chi \in N^*$ (as $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle \in N^*$, $\chi \in N^*$) and P_χ is $\{\aleph_1\}$-semi proper (as $P_\chi \lessdot P_\kappa$), and $p_0 \in N^*$. Hence there is $p_1 \in P_\chi$, $p_1 \geqq p_0$, which is $\{\aleph_1\}$-semi (N^*, P_χ)-generic. As $|N^*| \cap \omega_1 = |N| \cap \omega_1$, p_1 is also $\{\aleph_1\}$-semi (N, P_χ)-generic, hence $\{\aleph_1\}$-semi (N, P_κ)-generic.

As $p_{0,\chi} \restriction \chi = p_0 \leqq p_1$, $p_{0,\chi} \cup p_1 \in P_\kappa$; and for simplifying the thinking, let $G \subseteq P_\kappa$ be generic, $p_{0,\chi} \cup p_1 \in G$. Clearly $\underset{\sim}{Q}_\kappa[G] \in N^*[G] \prec (H(\lambda)[G], \in, <)$, we can choose $f_n \in \underset{\sim}{Q}_\kappa[G]$ $(n < \omega)$ increasing, $f_0 = \underset{\sim}{q}[G]$, so that for every dense $D \subseteq \underset{\sim}{Q}_\kappa[G] \cap N^*[G]$ some f_n belongs to it. Now $f = \bigcup_{n<\omega} f_n \cup \{\langle \delta, \chi \rangle\} \in \underset{\sim}{Q}_\kappa[G]$ $(\delta = |N^*| \cap |N_1|)$ as $p_{0,\chi} \in G$. So clearly

$$(p_{0,\lambda} \cup p_1, \underset{\sim}{f}) \in P_\kappa * \underset{\sim}{Q}_\kappa$$

is as required.

(2) By 3.8(2) w.l.o.g. P is (S, ω)-complete, where S will be $\{2, \aleph_0, \aleph_1\}$; let $P = P_\kappa$, $\underset{\sim}{Q} = \underset{\sim}{Q}_\kappa$.

Let $(p, q) \in P * Q$ and we shall describe the winning strategy of player II in $EG_S^\omega(p, P * Q)$.

Suppose in the nth move, player I chooses the $P * Q$-name β_n of an ordinal $< \aleph_1$, and player II will choose β_n. Player II will do the following: after the nth move he will have $(p_\eta, q_\eta) \in P * Q$ for every increasing sequence η of ordinals $< \kappa$ of length $\leqq n$ such that:

(1) $(p_{\langle\ \rangle}, q_{\langle\ \rangle}) = \langle p, q \rangle$,

(2) $(p_{\eta \restriction l}, q_{\eta \restriction l}) \leqq (p_\eta, q_\eta)$,

(3) $(p_\eta, q_\eta) \Vdash$ "$\beta_{l(\eta)} = \beta_\eta$",

(4) for some $A_n \in D$, for every increasing $\eta \subseteq A$, $\beta_\eta = \beta_n$,

(5) $p_\eta \Vdash_P$ "Sup Rang $q_\eta >$ Max$_l\, \eta(l)$",

(6) let $\beta(n, \eta)$ be the P-name of the first $\beta < \aleph_1$ such that some $q: q_\eta \leqq q \in Q$ force (in Q) $\beta_n = \beta$,

then $\beta(0, \langle\ \rangle)$, $p_{\eta \restriction 1}$, $\beta(1, \eta \restriction 1)$, $p_{\eta \restriction 2}, \cdots$, $\beta(l-1, \eta \restriction (l-1))$, $p_{\eta \restriction l}$, is a beginning of a play of $G_S^\omega(p, f)$ in which player II uses his winning strategy.

Clearly player II can do the above and it gives him a strategy. We have to prove that he wins by it. So let $A = \bigcap_n A_n$, and for $\eta \in {}^\omega A$ increasing, we know that $\{p_{\eta \restriction l} : l < \omega\}$ has an upper bound (by 6) so let it be p_η.

Let P be $(B - \{0\}, \geqq)$, B a complete Boolean algebra.

Let $K = \{T : T$ a tree of increasing sequences from A, closed under initial segments, $\langle\ \rangle \in T$ and for every $\eta \in T$, $\{i \in A : \eta^\frown\langle i \rangle \in T\} \in D\}$ (we can replace D by $D_\kappa + A$ or $D_\kappa^{\mathrm{cb}} + A$ in this context). Let Lim $T = \{\eta : l(\eta) = \omega$, $\eta \restriction k \in T$ for every $k < \omega\}$. So K is closed under intersection of $< \kappa$ elements. For each $T \in K$, $\eta \in T$, let a_η^T be Sup$\{p_\nu :\ \nu \in \mathrm{Lim}\, T,\ \eta = \nu \restriction l(\eta)\}$ (in the Boolean sense). Clearly a_η^T decreases with T, so as B satisfies the κ-chain condition, for some T, $a_{\langle\ \rangle}^T$ is minimal (i.e., $T' \subseteq T$, $T' \in K$ implies $a_{\langle\ \rangle}^{T'} = a_{\langle\ \rangle}^T$), and similarly for every $\eta \in T$.

Obviously,

(1) $a_\eta^T = \mathrm{Sup}_{\eta^\frown\langle i \rangle \in T}\, a_{\eta^\frown\langle i \rangle}^T$ (hold for any tree),

(2) $0 < b < a_\eta^T$ implies $\{i : b \cap a_{\eta^\frown\langle i \rangle}^T \neq 0\} \neq \varnothing \bmod D$ (by T's minimality).

Let $T^* = \{\eta : a_\eta^T$ belong to the generic set of $P\}$. Hence

(3) $a_{\langle\ \rangle}^T \Vdash_P$ "for any $\eta \in T^*$ for κ, i, $\eta^\frown\langle i \rangle \in T^*$".

Now if $G \subseteq P$ is generic, $a_{\langle\ \rangle}^T \in G$, $S[G]$ is a stationary subset of S_0^2, and $C = \{\delta :$ if $\eta \in {}^{\omega >}\delta$, then Range $q_\eta[G] \subseteq \delta\}$ is closed unbounded. Hence for some η, δ, the following holds: $\delta \in S[G] \cap C$, $(\forall k)\eta \restriction k \in T^*$, and $\bigcup_{l<\omega} \eta(l) = \delta$, and let $q^* = \bigcup_{l<\omega} q_{\eta \restriction l} \cup \{\langle \mathrm{Sup\,Dom}\, q, \delta \rangle\} \in Q$. Let q^* be the P-name of q^*. It is easy to check $(a_{\langle\ \rangle}^T, q^*)$ is as required.

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8.3 THEOREM. *If "ZFC + G.C.H. + there is a measurable of order 1" is consistent, then so is "ZFC + G.C.H. + every stationary subset of* $\aleph_2$*, or its complement, contains a closed copy of* ω_1*".*

REMARK. We do not try to get the weakest (it is enough that $\{\lambda < \kappa : \lambda$ measurable of order $0\}$ is weakly compact). It will be interesting to find an equi-consistency result.

PROOF. So let V satisfy G.C.H., $B \subseteq \kappa$ the set of measurables of order 0, not 1, and for every $\mu \in B$, let D_μ be a normal ultrafilter on μ and $\diamondsuit_B$ holds, and $\langle S_\mu : S_\mu \subseteq H(\mu), \mu \in B\rangle$ exemplifies it. Moreover, if $S \subseteq H(\kappa)$, φ a π^1_1 sentence, $(H(\kappa), \in, S) \models \varphi$ then $\{\mu \in B : S \cap H(\mu) = S_\mu, (H(\mu), \in, S_\mu) \models \varphi\} \neq \emptyset$.

We define an RCS iterated forcing $\langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ by induction on i, such that $|P_i| < \kappa$, and for every measurable $\mu < \kappa$, $i < \mu \Rightarrow |P_i| < \mu$.

If we have defined $\underset{\sim}{Q}_j$ for $j < i$ then P_j ($j \leqq i$) is defined. If $i \in \kappa - B$, $\underset{\sim}{Q}_i$ is $\{f : f$ a countable function from $\aleph_1$ to $|P_i|^+ + \aleph_2\}$.

If $i \in B$, $S_i = \langle p, \underset{\sim}{S}\rangle$, $p \in P_i$, $\underset{\sim}{S}$ a P_i-name, $p \Vdash_{P_i}$ "$\underset{\sim}{S}$ is a subset of S^i_0, and $\underset{\sim}{S}$ is stationary". Then we let Q_i be as in 8.2 if p is in the generic set, and trivial otherwise.

We leave the checking, that the forcing works, to the reader. In fact we get every stationary $S \subseteq S^2_0$ contains a closed copy of ω_1.

8.4 THEOREM. *Suppose "ZFC + there are two supercompact cardinals" is consistent. Then so is* ZFC + G.C.H. + "Fr($\aleph_\alpha$) *for every regular* $\aleph_\alpha$" ($\alpha > 1$).

PROOF. Let $V \models$ G.C.H. $+ \kappa < \lambda + \kappa$, λ are supercompact.

By a theorem of Laver [11] we can assume no κ-complete forcing will destroy the supercompactness of κ. The following is known:

FACT. If $\aleph_\alpha \geqq \lambda$ is regular, $S \subseteq S^\alpha_0$ is stationary, then for some μ, $\kappa < \mu < \lambda$, $\delta < \aleph_\alpha$, cf $\delta = \mu$ and $S \cap \delta$ is stationary.

Let P be the Levi collapse of λ to κ^+. By Baumgartner [2], in V^P, for every stationary $S \subseteq \lambda \cap S^\infty_0$, for some $\delta < \lambda$, cf $\delta = \kappa$ (in V^P), $S \cap \delta$ is stationary. Even more easily, if in V, $\kappa < \aleph_\beta \leqq \lambda$, $\aleph_\beta$ regular, $S \subseteq S^\beta_0$ stationary, it remains stationary in V^P. Let Q be the forcing from 6.3, V^{P*Q} is as required.

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THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL
AND
THE OHIO STATE UNIVERSITY
COLUMBUS, OH 43210 USA